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Generalized Bäcklund Transformations for Toda Field Theories

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Resumo

Nesta tese fazemos um profundo estudo sobre as Transformações de Bäcklund (BT). Essas deixam as equações de uma teoria específica invariantes. Escolhemos o modelo de Toda generalizado dos quais obteremos as BT, essa escolha não é arbitrária pois podemos derivar as equações de Toda generalizadas a partir da equação de curvatura nula, ponto chave dessa tese. Podemos obter essas transformações a partir de dois formalismos diferentes, transformações de gauge e lagrangiano. Devido a dificuldade da generalização via lagrangiana damos uma ênfase maior no formalizamo de gauge onde conseguimos obter BT das equações de Toda generalizadas. Essa transformações não são únicas, sendo classificadas em dois tipos: tipo I e II. No entanto mostramos que ambos tipos podem ser derivados de um mesmo princípio.

No capítulo 1 introduzimos todas as ferramentas necessárias para entendimento dos capítulos subsequentes. Introduzimos o conceito de álgebra e como obter as equações da hierarquia Toda. Também falamos sobre o conceito de integrabilidade e a derivação de infinitas cargas conservadas. No capítulo 2 falamos sobre como obter as transformações do tipo I em ambos os formalismos (gauge e lagrangiano). Mostramos que a integrabilidade da teoria não é quebrada pelas transformações, propriedade fundamental, já que estamos no estudo de teoria de campos não lineares integráveis. No capítulo 3 estendemos essas ideias para as transformações do tipo II. No capítulo 4 generalizamos o algoritmo que permitiu obter BT do tipo I e II, mostramos a não unicidade das transformações. No capítulo 5 discutimos consequências dos resultados dos capítulos anteriores e problemas em abertos deixados por essa tese.

Palavras Chaves: transformações de Bäcklund; transformações de gauge; teoria Toda; hierarquia integrável; equações diferenciais não lineares; defeitos; integrabilidade; curvatura nula;

Áreas do conhecimento: Física; Física Matemática; Integrabilidade.

Abstract

In this thesis we make a profound study on the subject of Bäcklund Transformation (BT). These are transformations that leave a set of differential equations invariant. We shall consider the construction of Bäcklund transformation for the Toda field theories. The reason is that this theory can be described in terms of zero curvature condition which is a key point of this subject. We shall see that these transformations can be obtained from two different formalism, the Lagrangian and Gauge transformation applied to the zero curvature condition. These transformations are shown not to be unique, and can be classified in two types namely, type I and II.

In chapter 1 we introduce all tools needed for understanding subsequent chapters. We take a shallow dive into Lie algebras, enough to derive Toda hierarchy. Also, we discuss about important concepts of integrability such as the existence of infinitely many conserved charges and how to obtain them. In chapter 2 we derive type I BT in both Lagrangian and Gauge formalism. Also we show that the integrability of the theory is preserved which it is a fundamental property. In chapter 3 we extend the same ideas for type II BT. Here new tools are developed given the difficulty to obtain these transformations. In Chapter 4 we generalize the algorithm to obtain BT for both type I and II showing the non uniqueness of these transformations. In Chapter 5 we discuss consequences of the previous results and comment on further open problems.

Key Words: Bäcklund transformation; gauge transformation; Toda hierarchy; integrable hierarchy; nonlinear differential equations; defects; integrability; zero curvature;

Knowledge Area: Physics; Mathematical Physics; Integrability.

Contents

Chapter 1

Classical Integrable Models

Nonlinear integrable field theory remains an active field of research since the dawn era in mid-1960s. At the time, computational simulations were in the early stages and even though computation power was limited compared to what we have now, important discoveries were made. A famous experiment led by Fermi in the study of non-linearity in physical system unleashed further discovery in nonlinear integrable field and soliton theories. This experiment was a simulation of 64 particles in a one-dimensional system with nonlinear forces between first neighbors [25]. It was expected thermalization in a long run since nonlinearity. Instead the experiment showed an apparent paradox, the modes of oscillation exhibited a remarkable result having a quasi-periodic behavior showing that nonlinearity is not enough to guarantee the equipartition of energy and system seemed to break ergodic hypothesis [58]. This famous problem, which was one of the first digital computer simulations, was conducted by Enrico Fermi, John Pasta, Stanislaw Ulam, and Mary Tsingou and became to known as FPU problem (the lady's name was omitted). Mary Tsingou's contributions to the FPU problem were largely ignored by the community until a published paper [19] with additional information regarding contribution. Now properly named, the FPUT problem stands as an example of the importance on the studies of nonlinearity in physics.

It was in 1965 that Kruskal and Zabusky published a paper [59] showing that solutions of the Korteweg-de Vries (KdV) equation have qualitatively the same behavior presented in FPUT problem, resolving the paradox. In the same paper, the authors coined for the first time the term *soliton*. The KdV equation have spatially localized solutions and this solitary waves showed to have particle-like behavior; the shape remain unaltered after a "collision". The KdV equation itself was known before the computational era. The first reported sight of a soliton was done in 1834, where the Scottish engineer John Scott Russell observed a hump-shaped wave in water tank experiments and reported to the British Association in 1844 [51]. At the end of the same century two mathematicians, Korteweg and de Vries, derived the nonlinear equation

$$u_t + u_{xxx} + 6uu_x = 0,$$

which models wave propagation in a rectangular channel and now has its name. In fact, this nonlinear equation in one dimension finds applications in several areas of physics such as shallow-water waves with weakly non-linear restoring forces, ion acoustic waves in a plasma, acoustic waves on a crystal lattice. This equation reappeared Kruskal and Zabusky's paper [59] and further investigation showed that it has several conserved quantities beyond energy and momentum. At the time, Miura was interested in equations of the type

$$v_t + v_{xxx} + 6v^n v_x = 0,$$

and showed that only for values $n = 1$ and $n = 2$ there exists several conservation laws [44, 26]. For the case $n = 2$ the equation became known as modified Korteweg and de Vries (mKdV). Actually, mKdV and KdV equations are related to each other by the famous Miura transformation [27, 44]. At first, the investigations of conservation laws for these equations was done by hard calculation. Latter it was proved these equations have in fact infinitely many conservation laws. Beyond KdV and mKdV there are several important equation in physics that have several conserved charges, such as sine/sinh-Gordon, short-pulse, Boussinesq, nonlinear Schrödinger equation etc. In fact, they can be classified algebraically, belonging to some hierarchy. In this chapter we will develop a method to deriving some of these equations, and formalize the concept of integrable hierarchy.

When a physicist talks about conservation laws, Nother's theorem always comes in mind. The statement of this theorem, relating symmetries and conserved charges, seems a good path to seek for conservation laws. However, this theorem has some limitations where it can only apply to continuous and smooth symmetries over physical space. Reflection, parity are examples of symmetries excluded from the theorem's applicability. Many hamiltonian systems have conservation laws that go beyond those predicted by Nother's theorem. Some have infinitely many of this and we will derive a method to obtain they by using an algebraic formalism. The interest in conservation laws is due to its connection with integrability given by Liouville theorem. In simple words, the theorem says that if the number of conserved charges is half the number of degrees of freedom one might be able to completely integrate the equation of motion. We will not go in much detail since there are extensive literature in this subject. Liouville's statement says about finite degrees of freedom (for discrete particle system), but the result can be extended to field theory as well.

In principle, just assuring integrability does not provide a method to find explicit solutions. There are several methods to find solutions for linear differential equations, a system being linear makes things a lot easier. For nonlinear though things are a bit trickier. There isn't a general method for solving them. But the theories studied here are so special that even though being nonlinear they have infinitely many soliton solutions and we are able to derive the infinity set of conserved charges. All this power is a consequence of deriving these nonlinear equations by algebraic methods where we define a Lax pair and equation of motion in zero curvature form. The benefits goes even further where writing equation of motion in this terms makes possible to explore gauge invariance. Gauge invariance can then be used to derive Bäcklund Transformations, the main subject of this thesis. Until now these transformation are only known for a set of special equations as the ones studied here. For some ordinary equation there isn't yet a systematic way of obtaining them.

This thesis is organized as follows. In chapter 1 we develop all tools needed for constructing integrable hierarchies. The introduction of affine Lie algebras is brief, but enough for our purposes. Our focus will be the generalized Toda hierarchy which we will obtain from zero curvature condition. Also we develop a method for deriving the infinity set of conserved charges. In chapter 2 we derive de type I BT for generalized Toda theory in both algebraic and Lagrangian formalism. We discuss further consequences of these transformations and connection with defects and preservation of integrability after imposing internal boundary conditions. In chapter 3 we extend the results for type II BT. In chapter 4 we present other possible solutions for BT, giving a general algorithm to obtain these transformations. In chapter 5 we discuss extensions, generalizations, and open problems left by the content of this thesis.

1.1 Linear Associated Problem and Zero Curvature Condition

In order to develop classical field theories where integrability is ensured, we will get rid of Lagrangian formalism and obtain equation of motion in another fashion. Let us start by defining what is called the Linear Associated Problem. Consider a set of of fields $\{\phi_i\}$ satisfying the following equation

$$\begin{cases} \partial_x \Phi &= A_x \Phi \\ \partial_t \Phi &= A_t \Phi \end{cases}, \quad \Phi = \begin{pmatrix} \phi_1 \\ \vdots \\ \phi_n \end{pmatrix} \tag{1.1}$$

where $A_x,\ A_t$ are $n \times n$ matrices called potentials. The immediate interest in equations written in the form of (1.1) is its complete solvability. This is a consequence of the possibility to obtain an infinite set of conservation laws. We will prove this later. Actually, there are several important equations of motion that can be written in this way. Sadly there does not exist a general method to construct any equations of motion in the form of (1.1) [11]. There does exist, however, methods to obtain matrices A_μ that give rise to very interesting models that are largely studied in the integrable systems and physics.

Let us write the same linear associated equations in a different fashion, which is called Zero Curvature Condition. To write the same equation in a different way seems to be worthless but in fact, it brings clarity to the algebraic structure that rules these class of equations. The compatibility condition $\partial_x\partial_t\Phi = \partial_t\partial_x\Phi$ of the equation (1.1) allow us to write

$$\begin{aligned}
&\partial_t\partial_x\Phi = \partial_t A_x\,\Phi + A_x A_t\Phi,\\
&\partial_x\partial_t\Phi = \partial_x A_t\,\Phi + A_t A_x\Phi,\\
&\partial_x\partial_t\Phi - \partial_t\partial_x\Phi = 0 \;\Rightarrow\; \left(\partial_x A_t - \partial_t A_x + [A_x,\, A_t]\right)\Phi = 0,
\end{aligned}$$

which can be written as

$$[\partial_x + A_x,\, \partial_t + A_t] = 0. \tag{1.2}$$

Equation (1.2) is known as Zero Curvature equation. The potentials A_μ are to be thought as a linear combination of elements from some Lie algebra $\mathfrak{g}$. An arbitrary set of linear combinations most likely won't give some equations of motion. There is, however, an algebraic construction of the zero curvature equation that allows one to obtain nontrivial integrable ones. In the next section we talk about the algebra tools needed to obtain those potentials.

Another interest on having equation written in the form of (1.2) is its gauge invariance. Suppose g is some group element. Then we can gauge transform the equation obtaining another set of potentials that also satisfy the zero curvature equation:

$$\begin{aligned}
&g\,[\partial_x + A_x,\, \partial_t + A_t]\,g^{-1} = 0\\
\Rightarrow\quad &\left[\partial_x + A'_x,\, \partial_t + A'_t\right] = 0,
\end{aligned}$$

where

$$A'_\mu = gA_\mu g^{-1} - \partial_\mu g\, g^{-1}.$$

This property can be used to generalize several transformations such as Miura and Bäcklund [20, 21, 32]. We're going to explore this property extensively in this thesis.

1.2 Lie Algebra

A Lie Algebra is a vector space endowed with a bilinear map $\mathfrak{g} \times \mathfrak{g} \to \mathfrak{g}$ satisfying the Jacobi identity. To go deeper into this subject good references can be found [37, 29, 24]. For now, it is enough to say that if $T_i \in \mathfrak{g}$ then

$$\left[T_i, T_j\right] = f_{ijk} T_k \tag{1.3a}$$

$$\left[\left[T_i, T_j\right], T_k\right] + \left[\left[T_k, T_i\right], T_j\right] + \left[\left[T_j, T_k\right], T_i\right] = 0 \tag{1.3b}$$

where f_{ijk} are a set of numbers called *structure constant of the algebra* obeying $f_{ijk} = -f_{jik}$ and $[\,\cdot\,,\,\cdot\,]$ is the usual commutator. Relation (1.3b) is known as Jacobi identity.

For this thesis, we will work with models that are written in terms of the $\mathfrak{sl}(n+1)$ algebra since the most famous integrable systems are built on top of it. Nevertheless, nothing prevents one from obtaining different models for other existing algebras like $\mathfrak{sp}(n)$, $\mathfrak{so}(2n)$, $\mathfrak{so}(2n+1)$ etc. [42, 52, 43].

It goes without saying that all the theories studied here are independent of any specific representation. For the sake of simplicity, we will choose the adjoint matrix representation to work with. If we consider E_{ij} an element of $\mathfrak{sl}(n+1)$ its k, l entry can be defined as

$$\begin{aligned} (E_{i,i})_{kl} &= \delta_{i,k}\delta_{l,i} - \delta_{i,k-1}\delta_{l-1,i} \\ (E_{i,j})_{kl} &= \delta_{i,k}\delta_{l,j}, \end{aligned} \quad \text{where} \quad k, l = 1, \cdots, n+1 \tag{1.4}$$

The set $\{E_{i,j}\}$ of $(n+1) \times (n+1)$ matrices has $n(n+2)$ elements and caries important properties. First, this linear space can be split into two orthogonal spaces, $\mathfrak{sl} = \mathcal{H} \oplus \mathcal{P}$. $\mathcal{H}$ is called the Cartan subalgebra, containing the maximum number of commuting operators. The set $\mathcal{P}$ contains the others, called step operators. The notation used in (1.4) does not bring clarity about the algebraic structure and the commutation relation. To do so we will introduce the concept of root system of the Lie algebra, which is a set of vectors $\{\alpha_i\}$ belonging to some vector space $\mathbb{R}^n$. In fact there is a map between $E_{i,j} \leftrightarrow E_{\{\alpha\}}$, for $i \neq j$. Also, the convention is to use h_i to denote the diagonal elements, then follow the notation

$$\begin{aligned} E_{i,i} &\to h_i \\ E_{i,j} &\to E_{F(i,j)} \end{aligned} \tag{1.5}$$

where

$$F(i,j) = (-1)^{\theta(i,j)} \sum_{k=\min\{i,j\}}^{\min\{i,j\}+|i-j|-1} \alpha_k$$

and θ is the discrete Heaviside function

$$\theta(i,j) = \begin{cases} 0 & \text{if } i < j \\ 1 & \text{if } i > j \end{cases} \tag{1.6}$$

Hence a better notation to an $\mathfrak{sl}(n+1)$ algebra is

$$\begin{aligned} \mathfrak{sl}(n+1) &= \mathcal{H} \oplus \mathcal{P} \\ &= \{h_1, \cdots, h_n\} \oplus \{E_{\alpha_1}, E_{\alpha_2}, \cdots\} \end{aligned}$$

Definition (1.4) and the map (1.5) is too specific. A more formal and general approach would be to formalize all of it in terms of root system getting rid of any matrix representation. The advantage to bring lights into root system of the algebra is the easier obtainment of commutation relation between its elements. Indeed, by properly setting the vectors $\{\alpha_i\}_{i=1}^{n}$ we can define what is called Cartan Matrix

$$K_{ij} = \frac{2\,\alpha_i \cdot \alpha_j}{\alpha_j^2} \tag{1.7}$$

By this definition the matrix encodes the angle relation between the simple roots[1] of the algebra. Therefore, if one has the Cartan matrix associated to some algebra, it is possible to obtain the commutation relation for all elements

$$\begin{aligned} \left[h_i, h_j\right] &= 0 \\ \left[h_i, E_\alpha\right] &= n(\alpha)^j K_{ji} E_\alpha \\ \left[E_\alpha, E_\beta\right] &= \begin{cases} \text{sign}(\alpha,\beta) E_{\alpha+\beta}, & \text{if } \alpha+\beta \text{ is a root} \\ n(\alpha)^j h_j, & \text{if } \alpha+\beta = 0 \\ 0 & \text{otherwise} \end{cases} \end{aligned} \tag{1.8}$$

where $n(\alpha)$ are the coefficients of the linear combination of simple roots that gives α and $\text{sign}(\alpha,\beta)$ gives the sign of permutation between roots. Since in this thesis

[1]Simple roots are the ones which are not a linear combination of other roots.

we will only deal with $\mathfrak{sl}(n+1)$, the associated Cartan matrix for this algebra is

$$K = \begin{pmatrix} 2 & -1 & 0 & \cdots & 0 & 0 \\ -1 & 2 & -1 & \cdots & 0 & 0 \\ 0 & -1 & 2 & \cdots & 0 & 0 \\ & \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & \cdots & 0 & -1 \\ 0 & 0 & 0 & \cdots & -1 & 2 \end{pmatrix} \tag{1.9}$$

This matrix is strongly related to generalized Toda Theory as we will see later.

1.2.1 Affine Lie Algebra

Even though simple Lie Algebra is a powerful tool it is need to do an extension on it in order to properly model the integrable theories studied here. This extension is a generalization called Kac-Moody algebra, which is an entirely field of research on its own. But to not get lost and accomplish our purpose we can simply say the extension adds a grading structure, a sort of infinite layers repeating the algebra pattern. Conventionally the notation for affine lie algebra is $\hat{\mathfrak{sl}} := \mathfrak{sl} \otimes \mathbb{C}[\lambda]$ where operators get mapped as following

$$\lambda^n h_i \to h_i^{(n)}$$
$$\lambda^n E_\alpha \to E_\alpha^{(n)}$$

Formally the algebra extension comes from a generalization of the Cartan matrix (1.7). This has deep consequences on the algebra, even commutation relation (1.3a) get extended

$$\left[T_i^{(m)}, T_j^{(n)}\right] = f_{ijk} T_k^{(m+n)} + c\, m\, \delta_{m+n,0}\, \kappa \tag{1.10}$$

where an additional operator κ was added. For those interested in going a deeper in the subject we let references [30, 40, 56]. Kac-Moody algebra is essential for requiring conformal invariance or for constructing soliton solutions; for example the dressing method makes use of it to obtain infinitely many soliton solutions (n-soliton solution) [9, 31]. For our purposes, there is no problem in disregarding the central term in (1.10) taking the limit $c \to 0$ and doing so we end up with what is called *loop - algebra*. This "simpler version" of Kac-Moody algebra preserves the main property needed to the algebraic construction of integrable hierarchies which is the grading structure.

The extension in the algebra allow us to define a grading operator

$$d = \lambda \frac{d}{d\lambda}, \qquad \text{where we define its action in the algebra as } \left[d,\, T^{(n)}\right] = nT^{(n)}$$

With this operator we can define another one that splits the algebra $\hat{\mathfrak{g}}$ into graded subsets $\mathcal{G}_m$. There are many ways to define this new operator Q and how it is defined might affect the graded decomposition of the algebra. In this thesis we will use the *principal gradation* where the corresponding Q is defined as

$$Q = (n+1)d + \sum_{a=1}^{n} \mu_a \cdot H \tag{1.11}$$

where μ_a are the fundamental weights, which is a set of dual vectors to the simple roots of $\hat{\mathfrak{sl}}(n+1)$; they satisfy the property $\mu_a \cdot \alpha_b = \delta_{a,b}, \quad a, b = 1, \cdots n$. H is the diagonal operator in the Cartan subalgebra. Another common grading structure that is used for constructing several nonlinear integrable field theories is the *homogeneous gradation*[2] but there are others as well [5, 22]. By defining some grading operator Q it allows to decompose the affine algebra $\hat{\mathfrak{g}}$ into graded subsets, $\mathcal{G}_a$, satisfying

$$\hat{\mathfrak{g}} = \sum_{a \in Z} \mathcal{G}_a, \qquad [Q,\, \mathcal{G}_a] = a\mathcal{G}_a, \qquad [\mathcal{G}_a,\, \mathcal{G}_b] \in \mathcal{G}_{a+b}$$

Our focus is on the algebra $\hat{\mathfrak{g}} = \hat{sl}(n+1) = \sum_{a \in Z} \mathcal{G}_a$ where employing the principal gradation (1.11) gives a finer decomposition where each subset will be

$$\begin{aligned}
\mathcal{G}_{m(n+1)} &= \{h_1^{(m)}, \cdots, h_n^{(m)}\}, \\
\mathcal{G}_{m(n+1)+1} &= \{E_{\alpha_1}^{(m)}, \cdots, E_{\alpha_n}^{(m)}, E_{-\alpha_1 \cdots -\alpha_n}^{(m+1)}\}, \\
\mathcal{G}_{m(n+1)+2} &= \{E_{\alpha_1+\alpha_2}^{(m)}, E_{\alpha_2+\alpha_3}^{(m)}, \cdots, E_{\alpha_{n-1}+\alpha_n}^{(m)}, E_{-\alpha_1 \cdots -\alpha_{n-1}}^{(m+1)}, E_{-\alpha_2 \cdots -\alpha_n}^{(m+1)}\}, \\
&\vdots \\
\mathcal{G}_{m(n+1)+n} &= \{E_{-\alpha_1}^{(m+1)}, \cdots, E_{-\alpha_n}^{(m+1)}, E_{\alpha_1 \cdots \alpha_n}^{(m)}\}.
\end{aligned}$$

1.3 Construction of Integrable Hierarchies

Much of what is presented in this section can be found in more details in a tons of references [23, 7, 22, 5]. There are in fact infinitely many equations that can be written in the form of (1.2). Some of these equation are connected under some transformation like Miura transformation [20, 27]. The name integrable hierarchy

[2]For the homogeneous gradation the Q operator is simply defined as $Q = d$

is another name for a system of commuting Hamiltonian flows; several hamiltonian systems sharing the same class of solution and number of conservation laws.

There are standard ways to construct integrable hierarchies. The Zakharov-Shabat construction unleashed formal mathematical ways to obtain Lax pairs that give rise to nonlinear integrable systems [60, 11]. The connection between sine-Gordon and mKdV equation was fully understood in Drinfeld-Sokolov paper [23]. In the past few decades, this method was exhaustively studied and the connection with Lie algebras got stronger. Nowadays we can define some hierarchy by fixing the potential A_x while A_t is different for each equation of the hierarchy. To do so we associate to each equation a unique t_N where N labels the highest (or lowest) graded element in A_t. The zero curvature then will be written as

$$[\partial_x + A_x, \partial_{t_N} + A_{t_N}] = 0 \tag{1.12}$$

The various kind of A_x potential generate KdV (Korteweg-de Vries), NLS (Nonlinear Schrödinger equation), Dym hierarchies and much more [4, 23, 8]. Here we will focus on the Toda Hierarchy which for $\mathfrak{sl}(2)$ is the same as mKdV (modified Korteweg-de Vries) and for $n \geq 2$ is a multi-component generalization of mKdV hierarchy. We start by defining

$$A_x = A^{(0)} + E^{(1)}$$

where they are defined as

$$A^{(0)} = \partial_x \phi_i h_i^{(0)} \tag{1.13a}$$

$$E^{(1)} = E_{\alpha_1}^{(0)} + \cdots + E_{\alpha_n}^{(0)} + E_{-(\alpha_1+\cdots+\alpha_n)}^{(1)} \tag{1.13b}$$

While dealing with A_{t_N} we treat positive and negative flows separately, despite that it is possible to find solutions from a mixture of both [33]. By positive flows we mean $N > 0$ where A_{t_N} will contain only positive graded elements, negative flows where $N < 0$.

1.3.1 Positive Hierarchy

As said before what defines the hierarchy is the potential A_x. While A_x is fixed we can propose A_{t_N} as a linear combination of the algebra elements belonging to $\mathcal{G}_m$ (where $m \geq 0$). The zero curvature condition allow to solve all coefficients and to obtain the equation of motion. This is possible due to the grading structure that allow us to break down equation (1.12) in several equations. If we propose

$$A_{t_N} = D^{(N)} + D^{(N-1)} + \cdots + D^{(1)} + D^{(0)}$$

where each $D^{(k)}$ is a linear combination of elements in $\mathcal{G}_k$, the zero curvature decomposes by graded equations

$$\begin{aligned}
&\textbf{grade N+1:} && [E^{(1)}, D^{(N)}] = 0, \\
&\textbf{grade N:} && \left[E^{(1)},\, D^{(N-1)}\right] + \left[A_0,\, D^{(N)}\right] + \partial_x D^{(N)} = 0 \\
&\vdots && \vdots \qquad \vdots \\
&\textbf{grade 0:} && \left[A_0,\, D^{(0)}\right] + \partial_x D^{(0)} - \partial_{t_N} A_0 = 0,
\end{aligned}$$

where we have $N+2$ equation for $N+1$ unknown variables. So it is possible to solve from the highest to lowest graded equations finding all $D's$ elements while the zeroth grade is the desired equation of motion, since it is the only one with time derivative. The positive flows produces important equations like mKdV equation for $\mathfrak{sl}(2)$ and $N = 3$. Higher order equations can easily be found by following the same procedure.

1.3.2 Negative Hierarchy

Negative hierarchy follows the same reasoning as positive one does. The way of solving graded equations is from lowest to highest instead. Another difference is in the $A_{t_{-N}}$ potential where now it is defined as

$$A_{t_{-N}} = D^{(-N)} + D^{(-N+1)} + \cdots + D^{(-1)}$$

The absence of $D^{(0)}$ for negative flows is trivial, solution of (1.12) implies $D^{(0)} = 0$. Again by using this potential in the zero-curvature equation (1.12) it can be decomposed in several graded equations

$$\begin{aligned}
&\textbf{grade -N:} && \partial_x D^{(-N)} + \left[A_0,\, D^{(-N)}\right] = 0 \\
&\textbf{grade -N + 1:} && \partial_x D^{(-N+1)} + \left[A_0,\, D^{(-N+1)}\right] + \left[E^{(1)},\, D^{(-N)}\right] = 0 \\
&\vdots && \vdots \qquad \vdots \\
&\textbf{grade 0:} && \partial_{t_{-N}} A^{(0)} - [E^{(1)}, D^{(-1)}] = 0
\end{aligned}$$

and here we have $N+1$ equations for just N variables. As before the zeroth graded equation yields equation of motion.

1.3.3 Generalized Toda Field Theory

A particular interesting case is for $t_{-N} = t_{-1}$. This time flow is know as Generalized Toda Model, a conformal invariant field theory of high interest in physics. By taking $A_{t_{-1}} = D^{(-1)}$ the zero curvature equation splits into

$$\partial_x D^{(-1)} + \left[A^{(0)}, D^{(-1)}\right] = 0, \tag{1.14a}$$

$$\partial_{t_{-1}} A^{(0)} - \left[E^{(1)}, D^{(-1)}\right] = 0. \tag{1.14b}$$

The -1 grade equation can be easily solved if we write $A^{(0)} = B^{-1}\partial_x B$ where

$$B = e^{\sum_{=1}^{n} \phi_a h_a} \tag{1.15}$$

and the solution of (1.14a) is

$$D^{(-1)} = B^{-1}E^{(-1)}B,$$

where $E^{(-1)}$ is the adjoint of (1.13b). The associated time evolution (1.14b) is

$$\partial_{t_{-1}}\left(B^{-1}\partial_x B\right) = \left[E^{(1)}, B^{-1}E^{(-1)}B\right] \tag{1.16}$$

This equation is well known in the literature; x and t_{-1} must be seen as parameters only and properly set to be the space coordinates depending on each case. For $N = -1$ flow the correspondence is with the light-cone coordinates where we set

$$x \to \frac{1}{2}(x+t) = x_+$$
$$t_{-1} \to \frac{1}{2}(x-t) = x_-$$

Given these definitions the following set o potentials generates the generalized Toda equations

$$A_x = E^{(1)} + B^{-1}\partial_+ B \tag{1.17a}$$

$$A_{t_{-1}} = B^{-1}E^{(-1)}B \tag{1.17b}$$

By writing Toda Fields equations as a zero-curvature condition we can assure its integrability. There is also a Lagrangian description of it, equation (1.16) can be obtained from Euler-Lagrange equations where the Lagrangian is defined as

$$\mathcal{L} = \frac{1}{2}\sum_{\mu,i,j}\left(K_{ij}\partial_\mu\phi_i\partial_\mu\phi_j\right) + \frac{2m^2}{\beta^2}\left(e^{-\sum_{i,j}K_{ij}\phi_j} + \sum_i e^{\sum_j K_{ij}\phi_j}\right) \tag{1.18}$$

where K is the Cartan matrix defined in (1.9) and μ labels the space time coordinates x and t. We are interested in aspects such as integrability and there is no problem setting the constants m and β to one. The Lagrangian formalism will be important when dealing with defects later but our main focus will be the algebraic description. Euler-Lagrange gives

$$\partial^2\phi_i = e^{\sum_j K_{ij}\phi_j} - e^{-\sum_{jl}K_{jl}\phi_l} \tag{1.19}$$

where ∂^2 is the Laplacian operator. The equations (1.19) are same as (1.16) but written in a different way.

1.4 Conserved Quantities

In the mid-1960s KdV and mKdV equations had gained attention given the famous report of the FPUT problem [25]. These equations were shown to have conserved charges beyond those predicted by Nother's theorem. Zbusky, Miura, Gardner, and Kruskal obtained by direct calculation several conserved densities [58, 39]. Meanwhile, Morikazu Toda proposed a chain with first neighbors interaction with a nonlinear force (named as "Toda" lattice), that showed to have equally interesting properties[55]. In this section we will develop a method for obtaining conserved charges for generic pair of potentials A_x, A_t that satisfy zero curvature condition. By properly setting up this potential one can derive conserved charges for generalized KdV and mKdV hierarchy, for example. The algorithm allows finding infinitely many conservation laws as long as the given pair of potential satisfy the zero curvature condition. This property coins the notion of integrability.

There is more than one way to seek conserved charges. In the appendix, we present other possibilities to enrich the advantage of the algebraic construction of integrable models. Here we start by defining the following quantity

$$\Gamma_{ij} = \phi_i\phi_j^{-1}, \qquad \text{for } j \neq i$$

By just taking x, t derivative and using (1.1) we can obtain

$$\partial_\mu\Gamma_{ij} = (A_\mu)_{ij} - (A_\mu)_{jj}\Gamma_{ji} + \sum_{k\neq j}\left((A_\mu)_{ik} - \Gamma_{ji}(A_\mu)_{jk}\right)\Gamma_{kj} \tag{1.20}$$

for $\mu = x, t$. From the compatibility condition $\partial_x \partial_t \Gamma_{ij} = \partial_t \partial_x \Gamma_{ij}$ it is possible to derive the continuity equation

$$\partial_t \left((A_x)_{ii} + \sum_{j\neq i} (A_x)_{ij} \Gamma_{ji} \right) - \partial_x \left((A_t)_{ii} + \sum_{j\neq i} (A_t)_{ij} \Gamma_{ji} \right) = 0 \tag{1.21}$$

The continuity equation relates current and charge density. As long as current density satisfy certain boundary condition, it is possible to integrate and obtain some conservation law. If we require that

$$\left((A_t)_{ii} + \sum_{j\neq i} (A_t)_{ij} \Gamma_{ji} \right) \Bigg|_{-\infty}^{+\infty} = 0$$

It follows from (1.21) that the quantity

$$Q = \int_{-\infty}^{\infty} \left[(A_x)_{ii} + \sum_{j\neq i} (A_x)_{ij} \Gamma_{ji} \right] dx \tag{1.22}$$

is time invariant. The equation (1.20) for $\mu = x$ is a system of Riccati equations that can be solved with the ansatz

$$\Gamma_{ij}(\lambda) = \sum_{k=0}^{\infty} \frac{\Gamma_{ij}^{(k)}}{\lambda^k} \tag{1.23}$$

By solving the system of Riccati equations one finds all $\Gamma_{ij}^{(k)}$ as polynomials in the fields and their derivatives. [16, 2]

1.4.1 $\mathfrak{sl}(3)$ mKdV example

It is easy to find on literature derivation of conserved charges for the sine/sinh-Gordon equation (Toda $\mathfrak{sl}(2)$ field theory). Here we will derive a couple conserved charges for $\mathfrak{sl}(3)$ algebra and time flow $t_N = t_2$, which is the same as a multi field mKdV equation. These equations can be obtained by setting $N = 2$ in

(1.12), for positive hierarchy. At the end we have the following potentials

$$A_x = \begin{pmatrix} u_1 & \lambda & 0 \\ 0 & -u_1+u_2 & \lambda \\ \lambda & 0 & -u_2 \end{pmatrix} \tag{1.24a}$$

$$A_{t_2} = \frac{1}{3}(-\partial_x u_1 + 2\partial_x u_2 + u_1^2 + 2u_2u_1 - 2u_2^2)h_1 + \frac{1}{3}(-2\partial_x u_1 + \partial_x u_2 + 2u_1^2 - 2u_2u_1 - u_2^2)h_2 \tag{1.24b}$$

$$+\lambda(u_2 E_{\alpha_1} - u_1 E_{\alpha_2} + (u_1-u_2)E_{-\alpha_1-\alpha_2}) + \lambda^2(E_{-\alpha_1} + E_{-\alpha_2} + E_{\alpha_1+\alpha_2})$$

together with the following motion equations

$$\partial_t u_1 = \frac{1}{3}\partial_x(-\partial_x u_1 + 2\partial_x u_2 + u_1^2 + 2u_2u_1 - 2u_2^2)$$

$$\partial_t u_2 = \frac{1}{3}\partial_x(-2\partial_x u_1 + \partial_x u_2 + 2u_1^2 - 2u_2u_1 - u_2^2)$$

where $u_i = \partial_x\phi_i$ and $x \to x$ and $t_2 \to t$. The potentials in (1.24a)-(1.24b) are given in the homogeneous gradation to simplify the definition of Γ in (1.23); otherwise it would have to take in account power of λ for certain sub-indices. Follows that the set of equations (1.20) for $\mu = x$ is

$$\partial_x\Gamma_{12} = (A_x)_{12} + ((A_x)_{11} - (A_x)_{22})\Gamma_{12} + (A_x)_{13}\Gamma_{32} - (A_x)_{21}(\Gamma_{12})^2 - (A_x)_{23}\Gamma_{12}\Gamma_{32}$$

$$\partial_x\Gamma_{32} = (A_x)_{32} + ((A_x)_{33} - (A_x)_{22})\Gamma_{32} + (A_x)_{31}\Gamma_{12} - (A_x)_{21}\Gamma_{12}\Gamma_{32} - (A_x)_{23}(\Gamma_{32})^2$$

$$\partial_x\Gamma_{23} = (A_x)_{23} + ((A_x)_{22} - (A_x)_{33})\Gamma_{23} + (A_x)_{21}\Gamma_{13} - (A_x)_{32}(\Gamma_{23})^2 - (A_x)_{31}\Gamma_{13}\Gamma_{23}$$

$$\partial_x\Gamma_{13} = (A_x)_{13} + ((A_x)_{11} - (A_x)_{33})\Gamma_{13} + (A_x)_{12}\Gamma_{23} - (A_x)_{32}\Gamma_{13}\Gamma_{23} - (A_x)_{31}(\Gamma_{13})^2$$

$$\partial_x\Gamma_{21} = (A_x)_{21} + ((A_x)_{22} - (A_x)_{11})\Gamma_{21} + (A_x)_{23}\Gamma_{31} - (A_x)_{12}(\Gamma_{21})^2 - (A_x)_{13}\Gamma_{21}\Gamma_{21}$$

$$\partial_x\Gamma_{31} = (A_x)_{31} + ((A_x)_{33} - (A_x)_{11})\Gamma_{31} + (A_x)_{32}\Gamma_{21} - (A_x)_{12}\Gamma_{21}\Gamma_{31} - (A_x)_{13}(\Gamma_{31})^2$$

which by taking the mKdV potentials (1.24a) gives

$$\partial_x\Gamma_{12} = (-2u_1 + u_2)\Gamma_{12} - \lambda + \lambda\Gamma_{12}\Gamma_{32} \tag{1.25}$$
$$\partial_x\Gamma_{32} = (-u_1 + 2u_2)\Gamma_{32} - \lambda\Gamma_{12} - \lambda(\Gamma_{32})^2 \tag{1.26}$$

$$\partial_x\Gamma_{23} = (u_1 - 2u_2)\Gamma_{23} - \lambda + \lambda\Gamma_{13}\Gamma_{23} \tag{1.27}$$
$$\partial_x\Gamma_{13} = (-u_1 - u_2)\Gamma_{13} - \lambda\Gamma_{23} + \lambda(\Gamma_{13})^2 \tag{1.28}$$

$$\partial_x\Gamma_{21} = (2u_1 - u_2)\Gamma_{21} - \lambda\Gamma_{31} + \lambda(\Gamma_{21})^2 \tag{1.29}$$
$$\partial_x\Gamma_{31} = (u_1 + u_2)\Gamma_{31} - \lambda + \lambda\Gamma_{21}\Gamma_{31} \tag{1.30}$$

Now taking the expansion (1.23) into (1.25)-(1.26) yields

$$\lambda^1 : \quad \begin{cases} 0 &= -1 + \Gamma_{12}^{(0)}\Gamma_{32}^{(0)} \\ 0 &= \Gamma_{12}^{(0)} - \Gamma_{32}^{(0)2} \end{cases} \tag{1.31}$$

$$\lambda^0 : \quad \begin{cases} \partial_x\Gamma_{12}^{(0)} &= (-2u_1 + u_2)\Gamma_{12}^{(0)} + \Gamma_{32}^{(0)}\Gamma_{12}^{(1)} + \Gamma_{12}^{(0)}\Gamma_{32}^{(1)} \\ \partial_x\Gamma_{32}^{(0)} &= (-u_1 + 2u_2)\Gamma_{32}^{(0)} - \Gamma_{12}^{(1)} + 2\Gamma_{32}^{(0)}\Gamma_{32}^{(1)} \end{cases} \tag{1.32}$$

$$\lambda^{-1} : \quad \begin{cases} \partial_x\Gamma_{12}^{(1)} &= (-2u_1 + u_2)\Gamma_{12}^{(1)} + \Gamma_{32}^{(1)}\Gamma_{12}^{(2)} + \Gamma_{12}^{(1)}\Gamma_{32}^{(2)} \\ \partial_x\Gamma_{32}^{(1)} &= (-u_1 + 2u_2)\Gamma_{32}^{(1)} - \Gamma_{12}^{(2)} + 2\Gamma_{32}^{(1)}\Gamma_{32}^{(2)} \end{cases} \tag{1.33}$$

$$\vdots$$

The structure is the same for the other pairs $\Gamma_{32},\ \Gamma_{13}$ and $\Gamma_{21},\ \Gamma_{31}$. The solution for $\Gamma_{ij}^{(0)}$ will always be a constant which we will define as

$$\Gamma_{ij}^{(0)} = c_{ij} \equiv \omega$$

with ω satisfying the equation $\omega^3 = 1$. The equations for negative powers of λ can be written in a recursive way. Let as take the pair $\Gamma_{32},\ \Gamma_{12}$ as an example:

$$\begin{cases} 2c_{32}\Gamma_{32}^{(n)} - \Gamma_{12}^{(n)} = \partial_x\Gamma_{32}^{(n-1)} - (-u_1 + 2u_2)\Gamma_{32}^{(n-1)} - \sum_{k=1}^{n-1}\Gamma_{32}^{(k)}\Gamma_{32}^{(n-k+1)} \\ c_{12}\Gamma_{32}^{(n)} + c_{32}\Gamma_{12}^{(n)} = \partial_x\Gamma_{12}^{(n-1)} + (2u_1 - u_2)\Gamma_{12}^{(n-1)} - \sum_{k=1}^{n-1}\Gamma_{12}^{(k)}\Gamma_{32}^{(n-k+1)} \end{cases}$$

This is just a linear system that can be easily solved. Doing the same for the others equations (1.27) - (1.30) we find

$$\Gamma_{21}^{(n)} = \frac{1}{2c_{21}^2 + c_{31}} \left(c_{21} A_1^{(n-1)} + B_1^{(n-1)} \right), \qquad \Gamma_{31}^{(n)} = \frac{1}{2c_{21}^2 + c_{31}} \left(-c_{31} A_1^{(n-1)} + c_{21} B_1^{(n-1)} \right)$$

$$\Gamma_{32}^{(n)} = \frac{1}{2c_{32}^2 + c_{12}} \left(c_{32} A_2^{(n-1)} + B_2^{(n-1)} \right), \qquad \Gamma_{12}^{(n)} = \frac{1}{2c_{32}^2 + c_{12}} \left(-c_{12} A_2^{(n-1)} + c_{32} B_2^{(n-1)} \right)$$

$$\Gamma_{13}^{(n)} = \frac{1}{2c_{13}^2 + c_{23}} \left(c_{13} A_3^{(n-1)} + B_3^{(n-1)} \right), \qquad \Gamma_{23}^{(n)} = \frac{1}{2c_{13}^2 + c_{23}} \left(-c_{23} A_3^{(n-1)} + c_{13} B_3^{(n-1)} \right).$$

where

$$\begin{aligned}
A_1^{(n)} &= \partial_x \Gamma_{21}^{(n)} - (2u_1 - u_2)\Gamma_{21}^{(n)} - \sum_{k=1}^{n} \Gamma_{21}^{(k)} \Gamma_{21}^{(n-k+1)} \\
B_1^{(n)} &= \partial_x \Gamma_{31}^{(n)} - (u_1 + u_2)\Gamma_{31}^{(n)} - \sum_{k=1}^{n} \Gamma_{21}^{(k)} \Gamma_{31}^{(n-k+1)}
\end{aligned}$$

$$\begin{aligned}
A_2^{(n)} &= \partial_x \Gamma_{32}^{(n)} - (-u_1 + 2u_2)\Gamma_{32}^{(n)} - \sum_{k=1}^{n} \Gamma_{32}^{(k)} \Gamma_{32}^{(n-k+1)} \\
B_2^{(n)} &= \partial_x \Gamma_{12}^{(n)} + (2u_1 - u_2)\Gamma_{12}^{(n)} - \sum_{k=1}^{n} \Gamma_{12}^{(k)} \Gamma_{32}^{(n-k+1)}
\end{aligned}$$

$$\begin{aligned}
A_3^{(n)} &= \partial_x \Gamma_{13}^{(n)} + (u_1 + u_2)\Gamma_{13}^{(n)} - \sum_{k=1}^{n} \Gamma_{13}^{(k)} \Gamma_{13}^{(n-k+1)} \\
B_3^{(n)} &= \partial_x \Gamma_{23}^{(n)} + (-u_1 + 2u_2)\Gamma_{23}^{(n)} - \sum_{k=1}^{n} \Gamma_{13}^{(k)} \Gamma_{23}^{(n-k+1)}
\end{aligned}$$

Finally we can write the conservation laws as

$$Q_i(\lambda) = \int_{-\infty}^{+\infty} dx \left[U_{ii} + \sum_{j \neq i} U_{ji} \Gamma_{ij}(\lambda) \right]$$

$$\Rightarrow \quad \begin{cases} Q_1(\lambda) &= -\int_{-\infty}^{\infty} dx \{ u_1 + \lambda \Gamma_{13}(\lambda) \} \\ Q_2(\lambda) &= -\int_{-\infty}^{\infty} dx \{ -u_1 + u_2 + \lambda \Gamma_{21}(\lambda) \} \\ Q_3(\lambda) &= -\int_{-\infty}^{\infty} dx \{ -u_2 + \lambda \Gamma_{32}(\lambda) \} \end{cases}$$

Each power of λ in Q is a conservation law. For simplicity choosing $c_{ij} = 1$ we write down explicitly some powers of Q_1:

$$Q_1^{(0)} = -\int_{-\infty}^{\infty} dx\ u_1$$

$$Q_1^{(1)} = -\frac{1}{3}\int_{-\infty}^{\infty} dx \left[-\partial_x u_1 + 2\partial_x u_2 + u_1^2 - u_2 u_1 + u_2^2\right]$$

$$Q_1^{(2)} = -\frac{1}{3}\int_{-\infty}^{\infty} dx \left[-\partial_x^2 u_1 + \partial_x^2 u_2 + u_1^2 u_2 - u_1 u_2^2 - u_1 \partial_x \left(-2u_1 + u_2\right)\right.$$
$$\left. + u_2 \partial_x \left(u_2 - 2u_1\right)\right]$$

$$\vdots$$

Thus, we have shown an example of how the charges of an integrable model can be constructed based on the zero curvature condition.

Chapter 2

Bäcklund Transformations

It was in nineteenth century that the fundamentals of differential geometry was developed and Bäcklund Transformation (BT) has its origin. Bäcklund found a transformation that recursively generates pseudospherical surfaces (surfaces with constant and negative curvature) which now has its name. This kind of surfaces got attention of important names like Bianchi, Lie, Darboux and others. At the time, it wasn't know that pseudospherical surfaces has strong relation with soliton theory. It took a while until the soliton to be "rediscovered" and the connection with geometry be made middle of twenties century. When talking about soliton, the sine/sinh-Gordon equation always comes to mind

$$\partial^2\phi = 2\sin(2\phi) \tag{2.1}$$

This equation has an infinite class of soliton solutions and appear in many applications such as Josephson junction, short pulse equation etc. This same equation can also be derived from Gauss-Mainardi-Codazzi system for pseudospherical surfaces parametrised in terms of asymptotic coordinates u and v. The reference [51] explores both geometrical and soliton approaches showing the connection between. In a geometric context Bäcklund obtained the following set of equations

$$\begin{cases} \partial_u(\phi - \psi) = 2\sigma \sin(\phi + \psi) \\ \partial_v(\phi + \psi) = \frac{2}{\sigma} \sin(\phi - \psi) \end{cases} \tag{2.2}$$

which was noticed to leave equation (2.1) invariant. This system is known as the BT for the sine-Gordon equation. Biachi noticed that transformations such as (2.2) can be recursively applied, having a commutative property. He developed the permutability theorem which allows finding solutions of (2.1) by algebraic means (more on that later). Thus having BT for some system makes it easier to integrate the equations of motion, given that it is a lower order differential equation. Powered by Biachi's permutability theorem, it is possible to generate an infinity class of solutions. In fact BT describes soliton interaction giving the rules of composing/decomposing soliton solutions. There are plenty application in physics [50]. As we will see later BT also describes defects in a medium while keeping the model integrable; in general defects do not preserve integrability.

Among several definitions, one can say that BT is a transformation that takes one partial differential equations into another. When they map between the same differential equations they are called *Auto Bäcklund Transformation*. When the equations are not the same it has names like *Gardner*, *Miura Transf.* etc. Here we understand by BT as the one between the same set of differential equations, no need to refer as *Auto BT*. A short definition follows:

> *A transformation that takes a set of partial differential equation and leaves it invariant by switching the set of fields. In other words the transformation leaves the associated equations invariant.*

What we will study in this chapter is a way to obtain these BT, now called Type I BT, for the generalized Toda Hierarchy. These result was published in our paper [21]. Nevertheless, the first generalization was made by Fordy and Gibbons [28] and later a generalized nonlinear soliton composition formula was developed in [41]. Here we take a algebraic and gauge approach to obtain the same results. There are clues that this same procedure can be applied to find BT for others hierarchies as well [20, 46].

2.1 Type I BT as a gauge transformation

Bäcklund transformation plays an important rule in soliton theory but until now it wasn't known by first principles how to obtain this set of transformations. The key point is to take advantage of the zero-curvature condition (1.2) being invariant under gauge transformation. We propose a group element $U = U(\phi, \psi)$ (a matrix representation of) and gauge transform the zero-curvature where potentials A_μ transforms like

$$A_\mu(\phi) \longrightarrow A_\mu(\psi) = U\, A_\mu(\phi)\, U^{-1} - \partial_\mu U U^{-1} \tag{2.3}$$

It follows that

$$\begin{aligned} U(\phi,\psi)\, [\partial_x + A_x(\phi),\, \partial_t + A_t(\phi)]\, U^{-1}(\phi,\psi) \\ &= \left[\partial_x + U\, A_x(\phi)\, U^{-1} - \partial_x U U^{-1},\, \partial_t + U\, A_t(\phi)\, U^{-1} - \partial_t U U^{-1}\right] \\ &= [\partial_x + A_x(\psi),\, \partial_t + A_t(\psi)] = 0 \end{aligned}$$

If such matrix U exist, since $A_\mu(\phi)$ and $A_\mu(\psi)$ have the same algebraic structure they describe the same model for different fields. It is worth mentioning that $\mu = x,\, t$ is just parameters rather then the standard x, t space-time, but should be properly set as such when needed.

The transformation (2.3) will be the main equation for finding the desired U. When convenient we rather prefer to use

$$A'_\mu(\psi)U = UA_\mu(\phi) - \partial_\mu U \tag{2.4}$$

to avoid dealing with the inverse matrix. Properties of the inverse matrix will be important later. Before we dive deeper, it is worth mentioning some important properties about the matrix U.

Proposition 1. *Since the potential A_μ belongs to some traceless algebra $\mathfrak{g}$ and U performs a gauge transformation taking $A_\mu : \mathfrak{g} \mapsto \mathfrak{g}$ then*

$$\partial_\mu \, det\{U\} = 0$$

Proof. The trace applied into equation (2.3) gives

$$\begin{aligned} \mathrm{Tr}\{A'_\mu\} &= \mathrm{Tr}\{UA_\mu U^{-1}\} - \mathrm{Tr}\{\partial_\mu U\, U^{-1}\} \\ \Rightarrow\ \mathrm{Tr}\{\partial_\mu U\, U^{-1}\} &= 0 \end{aligned} \tag{2.5}$$

It follows from Jacobi's formula that

$$\partial_\mu \ \det\{U\} = \ \det\{U\}\ \mathrm{Tr}\,\{\partial_\mu U\, U^{-1}\}$$

which from (2.5) we conclude that

$$\det\{U\} = \ const. \tag{2.6}$$

□

It is reasonable to expect that U belong to the group $SL(n+1)$ since $A_\mu \in \hat{\mathfrak{sl}}(n+1)$. In that case, property (2.6) would be trivially satisfied. Therefore, the natural choice would be to propose U as an exponential map of the algebra elements. Solving (2.4) in this approach seems more difficult. Instead, we will propose a general matrix U giving by

$$U(\phi,\psi) = \mathcal{U}^{(0)} + \sigma\mathcal{U}^{(-1)} + \sigma^2\mathcal{U}^{(-2)} + \sigma^3\mathcal{U}^{(-3)} \cdots \tag{2.7}$$

The matrices $\mathcal{U}^{(-m)}$ are defined as

$$\left(\mathcal{U}^{(-m)}\right)_{kl} = \frac{1}{\lambda^{\theta(i,i+m)}} \sum_{i=1}^{n+1} u_{i+m}^{(-m)} \delta_{i,k}\delta_{l,i+m}$$

where all indices follows the $\mathbf{mod}_{n+1}$ pattern, λ is the same grading parameter of the algebra, θ is the discrete Heaviside function defined in (1.6) and $u_i^{(-m)}$ is just a function of the fields ϕ and ψ. We will not require unitary determinant for

$U(\phi,\psi)$ since gauge transformation (2.4) is invariant under scaling factor. σ is called *Bäcklund parameter* and for now we will not attribute any meaning to it.

The gauge transformation has to be valid for any value of σ. So by taking $\sigma \to 0$ it is easy to see that $\mathcal{U}^{(0)}$ should be constant and proportional to the identity. Firstly let us solve a truncated version of (2.7), first order in σ, let us say. Then

$$U(\phi,\psi) = \begin{pmatrix} 1 & 0 & \cdots & 0 \\ 0 & 1 & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & 1 \end{pmatrix} + \begin{pmatrix} 0 & \cdots & 0 & \lambda^{-1}\sigma u_{n+1}^{(-1)} \\ \sigma u_1^{(-1)} & \cdots & 0 & 0 \\ \vdots & \ddots & \vdots & \vdots \\ 0 & \cdots & \sigma u_n^{(-1)} & 0 \end{pmatrix} \tag{2.8}$$

where $u_j^{(-1)} = u_j^{(-1)}(\phi,\psi)$. Given the potential A_x in (1.17a)

$$A_x(\phi) = \partial_x\phi_i h_i + E^{(1)} = \begin{pmatrix} \partial_x\phi_1 & 1 & 0 & \cdots & 0 \\ 0 & \partial_x(\phi_2-\phi_1) & 1 & & \\ & & & & \vdots \\ 0 & 0 & & \cdots & \\ \vdots & \vdots & & \partial_x(\phi_n-\phi_{n-1}) & 1 \\ \lambda & \cdots & & 0 & -\partial_x\phi_n \end{pmatrix},$$

gauge transform it (see eq. (2.4) for $\mu = x$) we get a set of $n+1$ linear partial differential equations for $u_k^{(-1)}$ where its solutions is

$$u_k^{(-1)} = c_k\, e^{-\phi_{k-1}+\phi_k+\psi_k-\psi_{k+1}}, \quad k = 1,\cdots,n$$

without loss of generality we can set all constant c_k to one, but also found by gauge transforming the potential A_t. In truth we have already completely solved $U(\phi,\psi)$ but if we chose to gague transform the $A_{t_{-1}}$ potential showed in (1.17b)

$$A_{t_{-1}}(\phi) = B^{-1}E^{(-1)}B = \begin{pmatrix} 0 & 0 & \cdots & 0 & \lambda^{-1}e^{-\phi_1-\phi_n} \\ e^{2\phi_1-\phi_2} & 0 & \cdots & 0 & 0 \\ 0 & e^{-\phi_1+2\phi_2-\phi_3} & & \vdots & \vdots \\ \vdots & \vdots & \ddots & 0 & 0 \\ 0 & 0 & & e^{-\phi_{n-1}+2\phi_n} & 0 \end{pmatrix}$$

we also find $c_1 = \cdots = c_{n+1} = 1$. The remaining equation of the transformation

(2.4) for $\mu = x, t$ are

$$\partial_x(\phi_i - \psi_i) = \sigma(e^{-\phi_{i-1}+\phi_i+\psi_i-\psi_{i+1}} - e^{-\phi_n-\psi_1}), \tag{2.9a}$$

$$\partial_{t_{-1}}(-\phi_{i-1} + \phi_i + \psi_i - \psi_{i+1}) = \frac{1}{\sigma}(e^{\phi_i-\phi_{i+1}-\psi_i+\psi_{i+1}} - e^{\phi_{i-1}-\phi_i-\psi_{i-1}+\psi_i}), \tag{2.9b}$$

where the sub-indices follows the mod $n+1$ pattern where $\phi_{i+n+1} = \phi_i$, $\psi_{i+n+1} = \psi_i$ with $\phi_0 = \psi_0 = 0$. This set of differential equations are the desired Type I BT for Generalized Toda Theory. These set of transformations was first found in [28]. We reproduced the same results in a gauge approach and published [20]. It can easily be checked that taking time derivative of equation (2.9a) or x derivative of (2.9b) and recursively use the same set of transformation will lead to equation of motion (1.19) for both fields. This are called compatibility which is the same as saying that *equations of motion are invariant under BT*.

We finally have what is called a gauge matrix representation of the Type I Bäcklund generator for Toda $\mathfrak{sl}(n+1)$ theory

$$U(\phi,\psi) = \begin{pmatrix} 1 & 0 & \cdots & \lambda^{-1}\sigma e^{-\phi_n-\psi_1} \\ \sigma e^{\phi_1+\psi_1-\psi_2} & 1 & \cdots & 0 \\ 0 & \ddots & \cdots & \vdots \\ \vdots & & 1 & 0 \\ 0 & \cdots & \sigma e^{\phi_n-\phi_{n-1}+\psi_n} & 1 \end{pmatrix} \tag{2.10}$$

and the set of equation (2.4) are the so desired Bäcklund Transformation.

By just using gauge transformation and proposing (2.7) we were able to completely solve U for first order expansion in σ. The type I transformation is quite simple to find, since the many zeros in matrix entries proposed in (2.8). But before we try to find U for a higher order in σ one could ask what would happen if we switch $\phi \leftrightarrow \psi$. None of the fields has privilege so it is natural to expect that everything stays the same. Let us propose $U'(\psi,\phi)$ having the same structure as (2.7). Then we can write

$$A_x(\phi) = U'(\psi,\phi)A_x(\psi)U'(\psi,\phi)^{-1} - \partial_x U'(\psi,\phi)U'(\psi,\phi)^{-1} \tag{2.11}$$

and we already know that

$$A_x(\psi) = U(\phi,\psi)A_x(\phi)U(\phi,\psi)^{-1} - \partial_x U(\phi,\psi)U(\phi,\psi)^{-1} \tag{2.12}$$

It is easy to see form (2.11) - (2.12) that

$$U'(\psi,\phi) \propto U(\phi,\psi)^{-1}$$

at least proportional to, since there is invariance under scaling factor. It is expected that $U'(\psi,\phi)$ to have the same order of expansion in σ as (2.10) does, with the field interchanged. However when inverting the matrix what we get is a full filed matrix with higher order in σ. To better illustrate we will take the $\mathfrak{sl}(3)$ example where the Type I matrix is

$$U(\phi,\psi) = \begin{pmatrix} 1 & 0 & \frac{\sigma e^{-\phi_2-\psi_1}}{\lambda} \\ \sigma e^{\phi_1+\psi_1-\psi_2} & 1 & 0 \\ 0 & \sigma e^{-\phi_1+\phi_2+\psi_2} & 1 \end{pmatrix} \tag{2.13}$$

and the inverse matrix is

$$\begin{aligned} U(\phi,\psi)^{-1} &= \frac{1}{c}U'(\psi,\phi) \\ &= \frac{1}{\det\{U\}}\begin{pmatrix} 1 & \frac{\sigma^2 e^{-\psi_1+\psi_2-\phi_1}}{\lambda} & -\frac{\sigma e^{-\psi_1-\phi_2}}{\lambda} \\ -\sigma e^{\psi_1-\psi_2+\phi_1} & 1 & \frac{\sigma^2 e^{-\psi_2+\phi_1-\phi_2}}{\lambda} \\ \sigma^2 e^{\psi_1+\phi_2} & -\sigma e^{\psi_2-\phi_1+\phi_2} & 1 \end{pmatrix} \end{aligned} \tag{2.14}$$

where c is a constant and without loss of generality we can set $c = \det\{U\}$ given *Proposition 1* and re-scaling invariance of gauge transformations. Even full filed matrix like $U'(\psi,\phi)$ and matrix like (2.13) being so unlike, and also having different order of σ, they both are transformation of the same type after all would be an absurd expect it being other way. The sign in σ due to the asymmetry of BT.

In fact the Type I BT is not unique. For a Toda $\mathfrak{sl}(n+1)$ theory we can propose at least two matrices

$$U = \sum_{k=0}^{1} \sigma^k \mathcal{U}^{(-k)} = \mathcal{U}^{(0)} + \sigma\mathcal{U}^{(-1)}$$

$$U = \sum_{k=0}^{n} \sigma^k \mathcal{U}^{(-k)} = \mathcal{U}^{(0)} + \sigma\mathcal{U}^{(-1)} + \cdots + \sigma^n \mathcal{U}^{(-n)}$$

that generates type I BT. The first one gives the already found matrix (2.10). The second one is a bit trickier to solve. Gauge transformation (2.4) for $\mu = x$ will produce $(n+1)^2$ equations for $n(n+1)$ variables. Not all equations are independent though given A_x being traceless. From those equations n will give the x

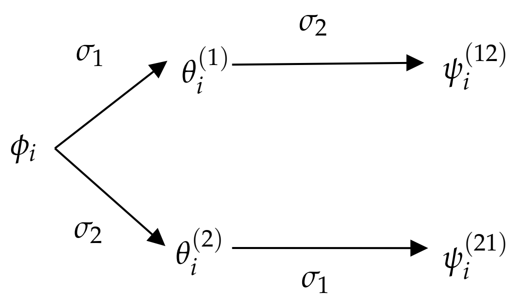
part of desired Type I BT. The other ones are highly coupled system of first order differential equations for variables $u_j^{(-k)}$. The best way to find these U entries are from the inverse matrix analysis that will be studied in chapter 3 and 4.

What we will see latter is that different order in σ in the ansatz (2.7) will produce other types of transformation. Specially if the sum in (2.7) index is exactly $n+1$ for $\mathfrak{sl}(n+1)$ Toda theory we obtain the Type II BT. For higher n and index of summation going up between 1 and $n+1$ we obtain type II like transformations, but this is subject for another chapter.

2.2 Nonlinear composition formula (Bianchi Theorem)

The equations (2.9a)-(2.9b) provide a systematic way of obtaining soliton solutions. To prove a system of nonlinear equation to be integrable is a great deal but it does not provide a way to find solutions. For the theories we study here there are many methods to this end such as Hirota method, tau functions, dressing method and Bäcklund transformations [6, 36, 35]. One way BT might be helpful for solving Toda equations is right straightforward by just solving the *pde*s (partial differential equations). By plugging one possible solution in (2.9a) for the field ϕ, for example, we have a *pde* whose solution for the field ψ also satisfy the Toda equations. Certainly if ϕ is a very complex function the *pde*s for ψ might be too difficult to solve and direct integration of BT would not be of much help. Actually it is possible get around this problem by taking, conventionally, the x equations (2.9a) and derive an algebraic formula allowing to compose different soliton solutions to get a new one.

The idea is to consider two successive BT with different Bäcklund parameter σ. To better illustrate consider the following diagram



We start from a known solution ϕ_i and integrate the Bäcklund equations with parameters σ_1 and σ_2 where we obtain solutions $\theta_i^{(1)}$ and $\theta_i^{(2)}$. The upper indices indicates from which σ parameter the solution came from. Then we integrate the Bäcklund equations again, but relating θs solution with a new one named as ψ. Now the σs parameter must be interchanged. The result is either $\psi^{(12)}$ or $\psi^{(21)}$ depending on the order from which solution came from. Writing down the equations (see eq. (2.9a)):

$$\partial_x(\phi_i - \theta_i^{(1)}) = \sigma_1 \left(e^{-\phi_{i-1}+\phi_i+\theta_i^{(1)}-\theta_{i+1}^{(1)}} - e^{-\phi_n-\theta_1^{(1)}} \right) \tag{2.15a}$$

$$\partial_x(\phi_i - \theta_i^{(2)}) = \sigma_2 \left(e^{-\phi_{i-1}+\phi_i+\theta_i^{(2)}-\theta_{i+1}^{(2)}} - e^{-\phi_n-\theta_1^{(2)}} \right) \tag{2.15b}$$

$$\partial_x(\theta_i^{(1)} - \psi_i^{(12)}) = \sigma_2 \left(e^{-\theta_{i-1}^{(1)}+\theta_i^{(1)}+\psi_i^{(12)}-\psi_{i+1}^{(12)}} - e^{-\theta_n-\psi_1^{(12)}} \right) \tag{2.15c}$$

$$\partial_x(\theta_i^{(2)} - \psi_i^{(21)}) = \sigma_1 \left(e^{-\theta_{i-1}^{(2)}+\theta_i^{(2)}+\psi_i^{(21)}-\psi_{i+1}^{(21)}} - e^{-\theta_n-\psi_1^{(21)}} \right) \tag{2.15d}$$

The order of integration shouldn't matter and we claim that

$$\psi_i^{(12)} = \psi_i^{(21)} \equiv \psi_i$$

By taking (2.15a) - (2.15b) + (2.15c) - (2.15d) we get zero on the left side of equation where we end up with

$$e^{\psi_i-\psi_{i-1}} + \left(\frac{\sigma_1 e^{-\theta_n^{(2)}} - \sigma_2 e^{-\theta_n^{(1)}}}{\sigma_2 e^{\theta_i^{(1)}-\theta_{i-1}^{(1)}} - \sigma_1 e^{\theta_i^{(2)}-\theta_{i-1}^{(2)}}} \right) e^{-\psi_1} = \frac{\sigma_1 \left(e^{-\phi_{i-1}+\phi_i+\theta_i^{(1)}-\theta_{i+1}^{(1)}} - e^{\phi_n-\theta_1^{(1)}} \right) - \sigma_2 \left(e^{-\phi_{i-1}+\phi_i+\theta_i^{(2)}-\theta_{i+1}^{(2)}} - e^{\phi_n-\theta_1^{(2)}} \right)}{\sigma_1 e^{-\theta_{i-1}^{(2)}+\theta_i^{(2)}} - \sigma_2 e^{-\theta_{i-1}^{(1)}+\theta_i^{(1)}}} \tag{2.16}$$

This forms a system of nonlinear equations. For the $\mathfrak{sl}(2)$ algebra we have only one equation whose solution leads to

$$\psi = \phi + \ln \left(\frac{\sigma_1 e^{\theta^{(1)}} - \sigma_2 e^{\theta^{(2)}}}{\sigma_1 e^{\theta^{(2)}} - \sigma_2 e^{\theta^{(1)}}} \right) \tag{2.17}$$

This formula is well known in the literature [41, 28, 10, 51, 50] having names such as Bianchi composition formula, tangent rule etc. The interpretation is that whenever we have three solutions ϕ, $\theta^{(1)}$ and $\theta^{(2)}$ satisfying BT we can compose them to generate a new soliton solution ψ by using (2.17). It is convenient to find solutions of a nonlinear *pde* algebraically. To generalize this formula for $\mathfrak{sl}(n+1)$, it means solving the nonlinear system above, which is challenging. We can choose ϕ_i to be the vacuum solution, i.e. $\phi_i = 0$, with this choice the system (2.16) is easier to solve and we find

$$\psi_j = \ln\left(\prod_{i=1}^{n+1-j} \frac{\sigma_1 e^{\theta_i^{(1)}-\theta_{i+1}^{(1)}} - \sigma_2 e^{\theta_i^{(2)}-\theta_{i+1}^{(2)}}}{\sigma_1 e^{\theta_i^{(2)}-\theta_{i-1}^{(2)}} - \sigma_2 e^{\theta_i^{(1)}-\theta_{i-1}^{(1)}}}\right). \tag{2.18}$$

Since we have chose ϕ_i to be the vacuum solution θ_i needs to be a 1-soliton solution with $\theta_0^k = \theta_{n+1}^k = 0$. This formula was first obtained in [41] and it is a generalization Bianchi's composition formula for Toda $\mathfrak{sl}(n+1)$ theory. By taking θs as 1-soliton solution

$$\theta_i^{(k)} = \ln\left(\frac{1+\rho_k}{1+\omega^i\rho_k}\right) \tag{2.19}$$

This solution can be obtained by direct integration of (2.15a) - (2.15b) (or by dressing method) where $\rho_k = e^{\eta\left(\sigma_k x - \sigma_k^{-1} t_{-1}\right)}$ with η being a constant depending on the soliton mass and coupling constant, and ω is a complex number satisfying $\omega^{n+1} = 1$ (with $\omega \neq 1$). Here we see that Bäcklund parameters acquires a meaning of soliton momentum. Using (2.19) into (2.18) we find 2-soliton solution for the generalized Toda $\mathfrak{sl}(n+1)$ theory:

$$\psi_j = \ln\left(\frac{\sigma_1(1+\rho_1)(1+\omega^n\rho_2) - \sigma_2(1+\rho_2)(1+\omega^n\rho_1)}{\sigma_1(1+\omega^j\rho_1)(1+\omega^{j-1}\rho_2) - \sigma_2(1+\omega^j\rho_2)(1+\omega^{j-1}\rho_1)}\right) \tag{2.20}$$

It is important to emphasize that this solution was found algebraically, but also can be obtained by dressing or Hirota's method [6, 36, 35] showing consistence.

2.2.1 The limit $\sigma_2 \to \sigma_1$ - a new soliton solution

The expressions (2.18) or (2.20) should be valid for any value of σ_j since no restriction was imposed into Bäcklund parameters once ϕ_i is set to be the vacuum ($\phi_i = 0$). Taking the limit $\sigma_2 \to \sigma_1$ we find

$$\lim_{\sigma_2\to\sigma_1} \psi_j = \ln\left(\frac{\sigma\,(1+\rho)\,(1+\omega^n\rho) + \eta\,(\omega^n - 1)\left(t_{-1} - x\sigma^2\right)\rho}{\sigma + \omega^{j-1}\left((\omega+1)\sigma - \eta(\omega-1)\left(t_{-1} - x\sigma^2\right)\right)\rho + \sigma\omega^{2j-1}\rho^2}\right) \tag{2.21}$$

where we have set $\sigma_1 = \sigma$ in the right hand side of (2.21); it satisfies equations (1.19)

2.3 BT as Defects

In the past few decades, the study of field theories with defects has been an interesting topic, perhaps due to the, pioneering work of Josephson [38] which awarded him the Nobel Prize. Josephson studied two superconductors separated by a thin non-superconducting material. Supper current when pass throughout by tunneling effect generates what now is called *Josephson Effect*. Surprisingly though is the appearance of a sine-Gordon equation and BT in this context. There is where theories developed hire find its application.

BT naturally emerges in field theories with defects. What now is known as a purely transmitting defect was first studied in [13] and named Type I Bäcklund Transformation. It consists of two field theories modeled by two Lagrangian (not necessarily the same), one living in the domain $-\infty < x < x_0$, the other living in $x_0 < x < \infty$. The contact point x_0 is completely arbitrary and we will choose $x_0 = 0$, but general cases including moving defects were studied [57]. In this section we deal with theories with a single field. There are works generalizing these ideas to multiple fields in a Lagrangian approach [14][15].

Here we will focus in the case where the Lagrangian in both domains describe the same theory. We can write

$$\mathcal{L} = \theta(-x)\mathcal{L}_\phi + \delta(x)\mathcal{L}_D + \theta(x)\mathcal{L}_\psi \tag{2.22}$$

where $\delta(x)$ is the Dirac delta, $\theta(x)$ is the Heaviside step function and $\mathcal{L}_D$ is the defect contribution. In principle, there is no restriction on $\mathcal{L}_D$. Since we are working with Integrable Models, the goal is to find internal boundary constraints that keep the model integrable. Hence, we need to show that the number of conservation laws is maintained (which we will do so later), following the ideas of [16, 2, 47]. By Lagrangian approach it is easy to verify that the energy and momentum suffers contribution due to the defect. Generalizing this idea to other charges is more challenging, so we will use an alternative approach.

To begin with we will consider the Lagrangian $\mathcal{L}_\phi$ and $\mathcal{L}_\psi$ in (2.22), depending on ϕ and ψ, which are scalar fields in their respective domains $(-\infty, 0)$ and $(0, \infty)$.

Also, for the defect, let us use

$$\mathcal{L}_\phi = \frac{1}{2}\left((\partial_t\phi)^2 + (\partial_x\phi)^2\right) - V(\phi)$$
$$\mathcal{L}_\psi = \frac{1}{2}\left((\partial_t\psi)^2 + (\partial_x\psi)^2\right) - V(\psi)$$
$$\mathcal{L}_D = \frac{1}{2}(\phi\partial_t\psi - \psi\partial_t\phi) + \mathcal{D}(\phi,\psi)$$

where V and $\mathcal{D}$ are the external and defect potential, respectively. By now the only restriction imposed on V is that it vanishes as $x \to \pm\infty$. Also, we will assume $\mathcal{D}$ depends only on the fields, not their derivatives. Euler-Lagrange equations give

$$\partial^2\phi = -\frac{\delta V(\phi)}{\delta\phi}, \text{ for } x < 0 \tag{2.23a}$$

$$\partial^2\psi = -\frac{\delta V(\psi)}{\delta\psi}, \text{ for } x > 0 \tag{2.23b}$$

$$\partial_x\phi - \partial_t\psi = -\frac{\delta\mathcal{D}(\phi,\psi)}{\delta\phi}, \text{ for } x = 0 \tag{2.23c}$$

$$\partial_x\psi - \partial_t\phi = \frac{\delta\mathcal{D}(\phi,\psi)}{\delta\psi}, \text{ for } x = 0 \tag{2.23d}$$

From (2.22) it is easy to see that time translation is not broken so we expect energy conservation. Taking time derivative of E we get

$$\frac{dE}{dt} = \frac{1}{2}\frac{d}{dt}\left(\int_{-\infty}^{0}(\partial_t\phi)^2 + (\partial_x\phi)^2 + 2V(\phi)\,dx + \int_0^\infty(\partial_t\psi)^2 + (\partial_x\psi)^2 + 2V(\psi)\,dx\right)$$
$$= (\partial_x\phi\partial_t\phi - \partial_x\psi\partial_t\psi)\Big|_{x=0},$$

but it is easy to show from (2.23c),(2.23d) that

$$\frac{d\mathcal{D}}{dt}\Big|_{x=0} = (-\partial_x\phi\partial_t\phi + \partial_x\psi\partial_t\psi)\Big|_{x=0}$$

By redefining the energy including the defect contribution we can say that

$$\tilde{E} = E + \mathcal{D}$$

is conserved. The same idea can be extended to all infinity conservation laws of the theory, but before going further let us consider another example. The conservation of momentum P is less trivial since the spatial translation invariance of (2.22) is broken. We will show that if some certain conditions of defect contribution $\mathcal{D}$ is satisfied, momentum can also be conserved. By taking its time derivative of P we obtain

$$\begin{aligned}\frac{dP}{dt} &= \frac{d}{dt}\left(\int_{-\infty}^{0} \partial_t\phi\partial_x\phi \; dx + \int_{0}^{\infty} \partial_t\psi\partial_x\psi \; dx\right) \\ &= \frac{1}{2}\left((\partial_t\phi)^2 + (\partial_x\phi)^2 + 2V(\phi)\right)\Bigg|_{x=0} - \frac{1}{2}\left((\partial_t\psi)^2 + (\partial_x\psi)^2 + 2V(\psi)\right)\Bigg|_{x=0}.\end{aligned} \tag{2.24}$$

The two last terms in (2.24) can be entirely written in terms of $\mathcal{D}$, V and fields derivative. By squaring both eqs. (2.23c), (2.23d) and subtracting them from each other, after a bit of algebra, we can show that

$$-\partial_x\phi\frac{\delta\mathcal{D}}{\delta\phi} - \partial_x\psi\frac{\delta\mathcal{D}}{\delta\psi} = \frac{1}{2}\left[(\partial_t\phi)^2 + (\partial_x\phi)^2 - (\partial_t\psi)^2 - (\partial_x\psi)^2 + \left(\frac{\delta\mathcal{D}}{\delta\phi}\right)^2 - \left(\frac{\delta\mathcal{D}}{\delta\psi}\right)^2\right]. \tag{2.25}$$

Multiplying (2.23c), (2.23d) by $(\partial_x\phi + \partial_t\psi)$, $(\partial_x\psi + \partial_t\phi)$, respectively and subtracting them we get

$$(\partial_t\phi)^2 + (\partial_x\phi)^2 - (\partial_t\psi)^2 - (\partial_x\psi)^2 = -\partial_x\phi\frac{\delta\mathcal{D}}{\delta\phi} - \partial_x\psi\frac{\delta\mathcal{D}}{\delta\psi} - \partial_t\phi\frac{\delta\mathcal{D}}{\delta\psi} - \partial_t\psi\frac{\delta\mathcal{D}}{\delta\phi} \tag{2.26}$$

Using equations (2.25),(2.26) we can show that (2.24) can be written as

$$\frac{dP}{dt} = -\partial_t\psi\frac{\delta\mathcal{D}}{\delta\phi} - \partial_t\phi\frac{\delta\mathcal{D}}{\delta\psi} + \frac{1}{2}\left(\left(\frac{\delta\mathcal{D}}{\delta\phi}\right)^2 - \left(\frac{\delta\mathcal{D}}{\delta\psi}\right)^2\right) - V(\phi) + V(\psi) \tag{2.27}$$

We can impose some constraints into $\mathcal{D}$ such that terms without time derivative in the left hand side of (2.27) be equal zero, and the time derivatives become a total derivative. In doing so we have the following equations for defect term

$$\frac{\delta^2\mathcal{D}}{\delta\phi^2} = \frac{\delta^2\mathcal{D}}{\delta\psi^2} \tag{2.28a}$$

$$\left(\frac{\delta\mathcal{D}}{\delta\phi}\right)^2 - \left(\frac{\delta\mathcal{D}}{\delta\psi}\right)^2 = 2\left(V(\phi) - V(\psi)\right) \tag{2.28b}$$

So we can say that there exists some functional $\mathcal{P}_D$ such that $\bar{P} = P + \mathcal{P}_D$ where $\bar{P}$ is conserved. We conclude that the wave solution of ϕ traveling in the medium exchange both energy and momentum when passing thought defect.

We hadn't been specific yet about which theories in Lagrangian description with defect preserve energy and momentum-like quantities. Even though equations (2.28a),(2.28b) are highly constraining, as long as one is able to find V and $\mathcal{D}$ such that they are satisfied, E and P might be conserved if they properly include defect contribution. However, this doesn't guarantee integrability of the system. In order to do that, we first need a theory that is integrable and further require that internal boundary conditions does not break integrability. In the appendix we explore another method to seek for further conservation laws. Equation (A.6) gives a hint which theories are good candidates to obey this property. By imposing momentum like conservation charge for a single relativistic field theory it is required that the potential satisfies the following equation (see Appendix A)

$$V'''(u) - \lambda^2 V'(u) = 0.$$

This equation is very restrictive to which potentials have momentum like conserved charges. Solution of it includes sine/sinh-Gordon and Liouville equation (also free massive filed for $\lambda = 0,\ V = m\phi^2/2$ and a free massless field $\lambda = 0,\ V = 0$ which we are not interest on). Even the Tzitzéica equation is excluded although it has a zero curvature formulation. There is a way to obtain Tzitzéica's theory with a defect. For that, we need to generalize $\mathcal{D}$ including a Lagrangian multiplier or, in gauge formalism, expand in higher order (2.8). Doing so yields another sort of BT named Type II. We will study this case in more detail later.

Solving equations (2.28a)-(2.28b) for Toda $\mathfrak{sl}(2)$ potential we find

$$\mathcal{D} = -\sigma\left(e^{(\phi+\psi)} + e^{(\phi+\psi)}\right) - \sigma^{-1}\left(e^{\lambda(\phi-\psi)} + e^{-\lambda(\phi-\psi)}\right)$$

and equations (2.23a)-(2.23d) yield

$$\partial_x\phi - \partial_t\psi = \sigma\left(e^{\phi+\psi} - e^{-\phi-\psi}\right) + \sigma^{-1}\left(e^{\phi-\psi} - e^{-\phi+\psi}\right), \text{ for } x = 0$$
$$\partial_x\psi - \partial_t\phi = -\sigma\left(e^{\phi+\psi} - e^{-\phi-\psi}\right) + \sigma^{-1}\left(e^{\phi-\psi} - e^{-\phi+\psi}\right), \text{ for } x = 0$$

This set of equations are precisely the ones in (2.9a)-(2.9b) for the case $n = 1$ if one properly set $x \to x + t_{-1}$ and $t \to x - t_{-1}$. This is because parameters in zero curvature equation, for $t_N = t_{-1}$, are actually light-cone coordinates.

2.4 Defects that keep the model integrable

While seeing BT as a frozen defect it is crucial to verify if with the new internal boundary condition the integrability is maintained. In the previous section, we have seen that the energy E and total momentum P are kept constant if we consider the defect contribution. To verify this property for all infinity many charges following the Lagrangian formalism seems a bit challenging. However, if we use the associated linear problem for the equations it is easier to show integrability and it is possible to obtain the defect contribution to all infinite charges.

In chapter 1 we developed a systematic way to obtain conservation laws for two given matrices A_x, A_t. The overall setup are two field theories where $\Phi = (\phi_1, \cdots, \phi_n)$ and $\Psi = (\psi_1, \cdots, \psi_n)$ are sitting in their respectively domains $x < 0$ and $x > 0$. Then following (1.22), the total charge is

$$Q_i = \int_{-\infty}^{0} \left(A_x(\phi)_{ii} + \sum_{k \neq i} A_x(\phi)_{ik} \Gamma_{ki}(\phi) \right) dx + \int_{0}^{\infty} \left(A_x(\psi)_{ii} + \sum_{k \neq i} A_x(\psi)_{ik} \Gamma_{ki}(\psi) \right) dx$$

If we take time derivative of the above expression and make use of equation (1.21) it is possible to show that

$$\frac{d\,Q_i}{dt} = \left[A_t(\phi)_{ii} + \sum_{k \neq i} A_t(\phi)_{ik} \Gamma_{ki}(\phi) \right] \Bigg|_{x=0} + \left[A_t(\psi)_{ii} + \sum_{k \neq i} A_t(\psi)_{ik} \Gamma_{ki}(\psi) \right] \Bigg|_{x=0} \tag{2.29}$$

In fact we can write the quantity $\Gamma_{ki}(\psi)$ entirely in terms of $U(\phi, \psi)$ and its sibling $\Gamma_{ij}(\phi)$. It follows that

$$\Gamma_{ij}(\psi) = \left[\frac{U_{ij} + \sum_{k \neq j} U_{ik} \Gamma_{kj}(\phi)}{U_{jj} + \sum_{k \neq j} U_{jk} \Gamma_{kj}(\phi)} \right] \tag{2.30}$$

Given the gauge invariance of the linear system (1.1) one can easily check that the fields $\Phi = (\phi_1, \cdots, \phi_n)$ and $\Psi = (\psi_1, \cdots, \psi_n)$ satisfy the following relation

$$\Psi = U(\phi, \psi)\Phi \tag{2.31}$$

To prove (2.30) we just take the $\Gamma_{ij}(\psi)$ definition and conveniently use (2.31) multiplying and dividing the whole expression by ϕ_j

$$\begin{aligned}\psi_i\psi_j^{-1} &= \left(\sum_k U_{ik}\phi_k\right)\left(\sum_k U_{jk}\phi_k\right)^{-1}\\ &= \frac{\sum_k U_{ik}\phi_k}{\sum_k U_{jk}\phi_k}\cdot\frac{\phi_j^{-1}}{\phi_j^{-1}}\end{aligned}$$

Then using (2.30) in (2.29) with a bit o algebra manipulation we obtain

$$\frac{dQ_i}{dt} = \frac{1}{U_{jj} + \sum_{k\neq j} U_{jk}\Gamma_{kj}} \left[\left(A_t^{(\phi)}{}_{ii} - A_t^{(\phi)}{}_{ii} + \sum_{k\neq i} A_t^{(\phi)}{}_{ik}\Gamma_{ki}\right)\left(U_{jj} + \sum_{k\neq j} U_{jk}\Gamma_{kj}\right) - \left(\sum_{k\neq i} A_t^{(\psi)}{}_{ik}\right)\left(U_{ij} + \sum_{k\neq j} U_{ik}\Gamma_{kj}\right)\right] \tag{2.32}$$

The right hand side of (2.32) can be written as a total derivative if we recall from (1.20) and (2.4) that

$$\begin{aligned}\partial_t\Gamma_{ij}^{\phi} &= (A_t^{\phi})_{ij} - (A_t^{\phi})_{jj}\Gamma_{ji} + \sum_{k\neq j}\left((A_t^{\phi})_{ik} - \Gamma_{ji}(A_t^{\phi})_{jk}\right)\Gamma_{kj}\\ \partial_t U_{ij} &= \sum_k U_{ik}(A_t^{\phi})_{kj} - (A_t^{\phi})_{ik}U_{kj}\end{aligned}$$

Finally we can write

$$\frac{dQ_i}{dt} = -\frac{d\mathcal{D}}{dt}$$

where

$$\mathcal{D}_i = \ln\left[U_{ii} + \sum_{k\neq i} U_{ik}\Gamma_{ji}(\phi)\right] \tag{2.33}$$

So if we define a modified charge $\tilde{Q}$ that includes defect contribution $\mathcal{D}$ we can say that the quantity

$$\tilde{Q}_i = Q_i + \mathcal{D}_i$$

is conserved.

This outcome is one of the most relevant in integrable field theories with defect. The importance of this result is that BT respect integrability, which is a fragile property that can easily be broken. A generic internal boundary condition would certainly wipe out the infinite set of conservation laws. It can be saw in equation (2.29) that are strictly hard to be satisfied. It only does due to relation (2.30) and so appears the role of BT.

2.5 Soliton Scattering for Toda $\mathfrak{sl}(\mathbf{n}+\mathbf{1})$

All the tools developed so far suggest that BT describes a process when a soliton scatter into a defect. In fact Type I BT are known as purely transmitting scattering [17] and it does not depend on its location. Example with multiple defects located at $x = x_1, x_2, \cdots$ has been already studied [53]. It will be classically exemplified here where we will study a soliton passing through a type I defect. In this process a soliton can be delayed (or advanced), absorbed or flipped to an anti-soliton. Each case will depend only in the relative values of soliton momentum and the Bäcklund parameter.

There are already many sine/sinh-Gordon scattering examples in the literature, thus consider the generalized $\mathfrak{sl}(n+1)$ Toda Theory. Let us study two processes, one is a 1-0 soliton scattering case where the soliton is absorbed, the other is a 1-1 soliton that passes through a defect acquiring a phase R. The model equation was pointed in (1.19) and the overall picture here is described by

$$\begin{cases} \partial_t\partial_x\phi_i = e^{K_{ij}\phi_j} - e^{-(\sum_j K_{lj})\phi_l}, \quad \text{for } x<0 \\ \partial_t\partial_x\psi_i = e^{K_{ij}\psi_j} - e^{-(\sum_j K_{lj})\psi_l}, \quad \text{for } x>0 \\ \partial_x(\phi_i-\psi_i) = \sigma\left(e^{-\phi_{i-1}+\phi_i+\psi_i-\psi_{i+1}} - e^{-\phi_n-\psi_1}\right), \quad \text{for } x=0 \end{cases} \tag{2.34}$$

We could use t rather than x BT equation but only one is needed to provide enough constraint. These equation are trivially satisfied for vacuum solution $\phi_i = \psi_i = 0$. Things start to be interesting by taking a soliton traveling in $x+$ direction. For that we will consider the following τ functions

$$\tau_k = 1 + \omega^k\rho \tag{2.35}$$

where $\rho = e^{\eta(kx-k^{-1}t)}$ and ω is any complex number satisfying the equation $\omega^{n+1} = 1$ for $\omega \neq 1$. The constant η depends on the soliton mass and the coupling constant (see (1.18)). This τ function can be found by the dressing method while obtaining a first class of soliton solution for Toda models [6, 10], but can also be obtained by integrating (2.9a)-(2.9b).

2.5.1 1-0 soliton scattering

By taking ψ_i as the vacuum solution and ϕ as 1-soliton which solution comes from dressing method case we set

$$\phi_i = \ln\left(\frac{\tau_0}{\tau_i}\right) = \ln\left(\frac{1+\rho}{1+\omega^i\rho}\right), \quad \psi_i = 0$$

Imposing this solution in the system (2.34) we find the relation

$$k = \frac{(1-\omega^n)\sigma}{\eta}.$$

This is just the condition for which value of soliton momentum the wave is completely absorbed by the defect.

2.5.2 1-1 soliton scattering

This process is a purely transmitting wave scattering. The phase shift acquired by the soliton when passing through the defect can be obtained by solving equation (2.34) where

$$\phi_i = \ln\left(\frac{\tau_0}{\tau_i}\right) = \ln\left(\frac{1+\rho}{1+\omega^i\rho}\right), \quad \psi_i = \ln\left(\frac{\bar{\tau}_0}{\bar{\tau}_i}\right) = \ln\left(\frac{1+R\rho}{1+\omega^i R\rho}\right),$$

where $\bar{\tau}_k = 1 + \omega^k R\rho$ and R is just a complex number. It follows from BT condition at $x = 0$ that

$$R = \frac{\eta k + (\omega^n - 1)\sigma}{\eta k + (1-\omega)\sigma}$$

This solution agrees with the previous case by taking $R \to 0$.

Chapter 3

Type II Bäcklund Transformations

In the previous chapter, we have analyzed the Type I BT, exploiting its features in both gauge formalism and the Lagrangian approach. It was noticed that some potentials have the property of having Type I BT while others do not. At first glance, there is no reason why Tzitzéica's potential does not allow the model to have Type I BT since it is an integrable model having a description in terms of zero-curvature equation and so being gauge invariant. On the other hand if this model had transformations like (2.9a) we could use it to obtain a first class of 1-soliton solution in which tau functions are linearly in ρ (see eq. (2.35)) although the first class of solution for Tizitzeica equation obtained by dressing method has tau functions quadratically in ρ. This is no longer a proof for nonexisting Type I BT for such model but rather indicates inconsistency giving evidence to the existence of another sort of Bäcklund like transformation.

In 2002 a paper published [12] proposed a BT for Tzitzéica model, this transformation includes an auxiliary field. In 2009 the pioneer work [18] where E. Corrigan and C. Zambon found a new kind of BT where Tizitzeica potential is included as well, and showed consistence with the previous results [12]. Not just but sine/sinh-Gordon was also verified to contemplate this kind of transformation named Type II BT. In this paper, the authors followed the Lagrangian approach where differently from type I it included a Lagrange multiplier for the defect term. This result showed that internal boundary conditions for integrable field theories with defects are not unique, opening new possibilities for the defect condition. The main difference between the results found in the previous chapter to this new kind of defect is the existence of an auxiliary field that plays no role in the dynamics but is essential to the transformation. Latter, the same result was found in an algebraic description where a matrix representation of Type II was obtained for both sine/sinh-Gordon and Tizitzeica model [2]. Following this approach, it was later found type II for the N = 1 super sinh-Gordon model [3] and extend for mKdV, KdV, and super-symmetric versions [1, 54]

Although the algebraic formalism facilitates obtaining of supersymmetric versions of sine/sinh-Gordon and other flows like mKdV, the algebraic structure of the gauge matrix representation remained unknown and the formalism was limited to the $\mathfrak{sl}(2)$ and $\mathfrak{sl}(3)$ cases. It was only in 2021 that the result was extended to the whole class of $\mathfrak{sl}(n+1)$ and finally obtaining Type II BT for generalized Toda field theory [20]. This chapter is devote to show this result which is the main contribution of our research.

3.1 Type II BT as a gauge Transformation

To obtain the matrix representation for type II we will follow the same standard ansatz proposed in (2.7). The difference though is the order of summation where now it goes from zero until $n+1$, for $\mathfrak{sl}(n+1)$. This implies some extra diagonal terms with gradation $-n-1$. Given this new setup, we propose the following matrix

$$U = \sum_{q=0}^{n+1} \sigma^q \mathcal{U}^{(-q)} = \mathcal{U}^{(0)} + \sigma \mathcal{U}^{(-1)} + \cdots + \sigma^{n+1} \mathcal{U}^{(-n-1)}, \tag{3.1}$$

or more explicit

$$U(\phi,\psi) = \begin{pmatrix} u_1^{(0)} & 0 & \cdots & 0 \\ \sigma u_1^{(-1)} & u_2^{(0)} & \cdots & 0 \\ \vdots & & \ddots & \vdots \\ \sigma^n u_1^{(-n)} & \cdots & \sigma u_n^{(-1)} & u_{n+1}^{(0)} \end{pmatrix} + \frac{1}{\lambda} \begin{pmatrix} \sigma^{n+1} u_1^{(-n-1)} & \sigma^n u_2^{(-n)} & \cdots & \sigma u_{n+1}^{(-1)} \\ 0 & \sigma^{n+1} u_2^{(-n-1)} & \cdots & \sigma^2 u_{n+1}^{(-2)} \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & \sigma^{n+1} u_{n+1}^{(-n-1)} \end{pmatrix} \tag{3.2}$$

The first challenge is that this matrix has too many variables i.e, $(n+1)(n+2)$ in total. Nonetheless we can solve the zeroth graded variables by taking the limit $\sigma \to 0$. Since the gauge transformation must be satisfied for any value of σ we find that in this limit $u_i^{(0)} = const$ where we set it to 1; the identity matrix also has to be a solution after all. At first glance, one can think the easier way to solve it follows the procedure of gauge transform by taking $A_x(\phi) \to A_x(\psi)$ (see eq. (2.4)). Following this reasoning we would have $(n+1)^2$ variables for $(n+1)^2$ equations. Unfortunately, since A_x is traceless, not all equations are independent and we end up with 1 less equation. Considering

$$A_x(\psi)U(\phi,\psi) = U(\phi,\psi)A_x(\phi) - \partial_x U(\phi,\psi) \tag{3.3}$$

we have a matrix equation whose diagonal terms have two expressions in powers of λ since algebra grading structure. Then, for each diagonal entry, we obtain two equations

$$\lambda^0 : \quad \partial_x(\phi_i - \phi_{i-1} - \psi_i + \psi_{i-1}) - \sigma\left(u_i^{(-1)} - u_{i-1}^{(-1)}\right) = 0 \tag{3.4a}$$

$$\lambda^{-1} : \quad \partial_x u_i^{(-n-1)} - u_i^{(-n-1)}(\phi_i - \phi_{i-1} - \psi_i + \psi_{i-1}) = 0 \tag{3.4b}$$

where $\phi_0 = \psi_0 = \phi_{n+1} = \psi_{n+1} = 0$ and $u_{k+n+1}^{(-1)} = u_k^{(-1)}$. Equation (3.4b) can be easily solved:

$$u_i^{(-n-1)} = c_i e^{\phi_i - \phi_{i-1} - \psi_i + \psi_{i-1}} \equiv \epsilon_i \tag{3.5}$$

The ϵ_i variable will make our expressions shorter and satisfy the property

$$\epsilon_1 \epsilon_2 \cdots \epsilon_{n+1} = 1 \tag{3.6}$$

Also, we can set all constants of integration to $c_i = 1$. Equation (3.4a) is the desired Type II BT, therefore we need to find who is $u_i^{(-1)}$ variables. The big challenge is to solve all $u_i^{(k)} = u_i^{(k)}(\phi, \psi)$ but differently from (3.4b) the remaining equations coming from (3.3) are highly coupled and difficult to solve.

To get around this problem we will propose an analysis in terms of the inverse matrix. We know $U(\phi, \psi)$ can take $A_x(\phi) \to A_x(\psi)$ or $A_x(\psi) \to A_x(\phi)$, then

$$A_x(\psi) = U(\phi, \psi) A_x(\phi) U(\phi, \psi)^{-1} - \partial_x U(\phi, \psi) U(\phi, \psi)^{-1} \tag{3.7a}$$

$$A_x(\phi) = U(\phi, \psi)^{-1} A_x(\psi) U(\phi, \psi) - U(\phi, \psi)^{-1} \partial_x U(\phi, \psi) \tag{3.7b}$$

Let us propose some matrix $U'(\psi, \phi)$ having the same structure as (3.1) with fields interchanged. Then we can write

$$A_x(\phi) = U'(\psi, \phi) A_x(\psi) U'(\psi, \phi)^{-1} - \partial_x U'(\psi, \phi) U'(\psi, \phi)^{-1} \tag{3.8}$$

By comparing (3.7b) with (3.8) we can see that

$$U'(\psi, \phi) \propto U(\phi, \psi)^{-1},$$

since there is invariance under scaling factor. By pure observation we found that the proportionality constant and the equality of above the equation holds with $\sigma \to -\sigma$ into $U'(\psi, \phi)$, then we should write

$$U(\psi, \phi; -\sigma) = \det\{U(\phi, \psi)\} \left(\frac{\lambda}{\lambda - \sigma^{n+1}}\right)^{n-1} U(\phi, \psi; \sigma)^{-1} \tag{3.9}$$

Proposition 1 in chapter 2 states that the determinant of U is constant and actually it is a polynomial in σ^{n+1}/λ. Since we are relaxing unit determinant, given the gauge invariance under scaling factor, the constant of proportionality in (3.9) is necessary, otherwise a inverting the matrix would loose consistency while stating the determinant being a polynomial in σ^{n+1}/λ. Given all the considerations, equation (3.9) for some Toda $\mathfrak{sl}(n+1)$ model is

$$\begin{pmatrix} 1+\frac{(-\sigma)^{n+1}\epsilon_1^{-1}}{\lambda} & \cdots \\ \vdots & \ddots \end{pmatrix} =$$

$$\left(\frac{\lambda}{\lambda-\sigma^{n+1}}\right)^{n-1} \begin{pmatrix} 1+\frac{\sigma^{n(n+1)}\epsilon_2\cdots\epsilon_{n+1}}{\lambda^n}+\frac{\sigma^{(n-1)(n+1)}\left(\epsilon_2+\cdots+\epsilon_{n+1}+u_2^{(-1)}u_3^{(-n)}\cdots+\cdots\right)}{\lambda}+\cdots & \cdots \\ \vdots & \ddots \end{pmatrix}$$

For large values of n the above matrix equation is a complete mess although comparing diagonal terms on both sides furnishes the same amount of equations for the unsolved variables $u_i^{(k)}$. The best way to proceed in this investigation is to take lower values of n and look for a pattern.

- $\mathfrak{sl}(\mathbf{2})$: Given definitions equation (3.9) is

$$\begin{pmatrix} 1+\frac{\sigma^2\epsilon_1^{-1}}{\lambda} & -\frac{\sigma u_2^{(-1)}}{\lambda} \\ -\sigma u_1^{(-1)} & 1+\frac{\sigma^2\epsilon_2^{-1}}{\lambda} \end{pmatrix} = \begin{pmatrix} 1+\frac{\sigma^2\epsilon_2}{\lambda} & -\frac{\sigma u_2^{(-1)}}{\lambda} \\ -\sigma u_1^{(-1)} & 1+\frac{\sigma^2\epsilon_1}{\lambda} \end{pmatrix}$$

The equation is already satisfied since property (3.6) says $\epsilon_2 = \epsilon_1^{-1}$ for $\mathfrak{sl}(2)$ case. We also need to impose a constant determinant

$$\det\{U\} = 1+\frac{\sigma^4}{\lambda^2}+\frac{\sigma^2}{\lambda}\left(\epsilon_1+\epsilon_2-u_1^{(-1)}u_2^{(-1)}\right)$$

$$\Rightarrow\ u_2^{(-1)} = \frac{\gamma+\epsilon_1+\epsilon_2}{u_1^{(-1)}}$$

where γ is a constant. Redefining the variable $u_1^{(-1)} = e^{\Lambda}$ we end up with

$$U(\phi,\psi) = \begin{pmatrix} 1+\frac{\sigma^2\epsilon_1}{\lambda} & \frac{\sigma(\gamma+\epsilon_1+\epsilon_2)e^{-\Lambda}}{\lambda} \\ \sigma e^{\Lambda} & 1+\frac{\sigma^2\epsilon_2}{\lambda} \end{pmatrix}$$

which are precisely the matrices found in [2]. Notice that the inverse property and constant determinant are satisfied with no necessity to define who are the last variable $u_2^{(-1)}$ (which re renamed to be written in terms of Λ). This new variable is called auxiliary field and has some equation related to it although it does not play any role in the field dynamics. The choice which

$u_k^{(-1)}$ will play the auxiliary field is completely arbitrary, we chose $u_2^{(-1)}$ as a convention. It follow that given the matrix representation of $U(\phi, \psi)$ the BT for Toda $\mathfrak{sl}(2)$ theory is

$$\partial_x(\phi - \psi) = \sigma \left(e^{\Lambda} - (\gamma + \epsilon_1 + \epsilon_2)e^{-\Lambda}\right) \tag{3.10a}$$

$$\partial_{t_{-1}}(\phi - \psi) = \frac{1}{\sigma} \left(e^{\Lambda - \phi - \psi} - (\gamma + \epsilon_1 + \epsilon_2)e^{-\Lambda + \phi + \psi}\right) \tag{3.10b}$$

$$\partial_x(\Lambda - \phi - \psi) = \sigma \left(e^{-\phi + \psi} - e^{\phi - \psi}\right) e^{-\Lambda} \tag{3.11a}$$

$$\partial_{t_{-1}}\Lambda = \frac{1}{\sigma} \left(e^{2\phi} - e^{2\psi}\right) e^{-\Lambda} \tag{3.11b}$$

Surely the compatibility taking second order derivative of above transformation leads to the equation of motion (1.19) in the case $n = 1$.

- $\mathfrak{sl}(\mathbf{3})$: For this case, the inverted matrix as shown in equation (3.9) is too big to fit on the page, but as said earlier only diagonal terms furnish enough equations. That said diagonal entries give us the following

$$1 + \frac{(-\sigma)^3 \epsilon_1^{-1}}{\lambda} = \frac{\lambda}{\lambda - \sigma^3} \left(1 + \frac{\sigma^3 \left(-u_2^{(-1)} u_3^{(-2)} + \epsilon_2 + \epsilon_3\right)}{\lambda} + \frac{\sigma^6 \epsilon_2 \epsilon_3}{\lambda^2}\right)$$

$$1 + \frac{(-\sigma)^3 \epsilon_2^{-1}}{\lambda} = \frac{\lambda}{\lambda - \sigma^3} \left(1 + \frac{\sigma^3 \left(-u_3^{(-1)} u_1^{(-2)} + \epsilon_1 + \epsilon_3\right)}{\lambda} + \frac{\sigma^6 \epsilon_1 \epsilon_3}{\lambda^2}\right)$$

$$1 + \frac{(-\sigma)^3 \epsilon_3^{-1}}{\lambda} = \frac{\lambda}{\lambda - \sigma^3} \left(1 + \frac{\sigma^3 \left(-u_1^{(-1)} u_2^{(-2)} + \epsilon_1 + \epsilon_2\right)}{\lambda} + \frac{\sigma^6 \epsilon_1 \epsilon_2}{\lambda^2}\right)$$

The property (3.6) for $\mathfrak{sl}(3)$ reads $\epsilon_{k+1}\epsilon_{k+2} = \epsilon_k^{-1}$ and then follows the solutions for above equations

$$u_1^{(-2)} = \frac{(1+\epsilon_1)(1+\epsilon_3)}{u_3^{(-1)}}$$
$$u_2^{(-2)} = \frac{(1+\epsilon_1)(1+\epsilon_2)}{u_1^{(-1)}}$$
$$u_3^{(-2)} = \frac{(1+\epsilon_2)(1+\epsilon_3)}{u_2^{(-1)}}$$

Again we need to assure a constant determinant and for that we choose to write one variable in terms of the other two. Again here, the choice is completely arbitrary where we have chosen $u_1^{(-1)} = e^{\Lambda_1}$, $u_2^{(-1)} = e^{\Lambda_2}$ to be the auxiliary fields writing it in terms of exponential. With that we find the last unknown variable

$$u_3^{(-1)} = (1+\epsilon_1)(1+\epsilon_2)(1+\epsilon_3)e^{-\Lambda_1-\Lambda_2}$$

and the desired matrix representation for the Type II BT for Toda $\mathfrak{sl}(3)$

$$U(\phi,\psi) = \begin{pmatrix} 1+\frac{\sigma^3\epsilon_1}{\lambda} & \frac{e^{-\Lambda_1}\sigma^2(1+\epsilon_1)(1+\epsilon_2)}{\lambda} & \frac{e^{-\Lambda_1-\Lambda_2}\sigma(1+\epsilon_1)(1+\epsilon_2)(1+\epsilon_3)}{\lambda} \\ e^{\Lambda_1}\sigma & 1+\frac{\sigma^3\epsilon_2}{\lambda} & \frac{e^{-\Lambda_2}\sigma^2(1+\epsilon_2)(1+\epsilon_3)}{\lambda} \\ \frac{e^{\Lambda_1+\Lambda_2}\sigma^2}{1+\epsilon_2} & e^{\Lambda_2}\sigma & 1+\frac{\sigma^3\epsilon_3}{\lambda} \end{pmatrix}, \tag{3.12}$$

which generates the following BT

$$\partial_x(\phi_1-\psi_1) = \sigma\left(e^{\Lambda_1} - (1+\epsilon_1)(1+\epsilon_2)(1+\epsilon_3)\,e^{-\Lambda_1-\Lambda_2}\right) \tag{3.13a}$$

$$\partial_x(\phi_2-\psi_2) = \sigma\left(e^{\Lambda_2} - (1+\epsilon_1)(1+\epsilon_2)(1+\epsilon_3)\,e^{-\Lambda_1-\Lambda_2}\right) \tag{3.13b}$$

$$\partial_{t_{-1}}(\phi_1-\psi_1) = \frac{1}{\sigma}\left((1+\epsilon_1)(1+\epsilon_2)e^{\phi_1-\phi_2+\psi_1-\Lambda_1} - \frac{e^{-\phi_1-\psi_2+\Lambda_1+\Lambda_2}}{1+\epsilon_2}\right) \tag{3.13c}$$

$$\partial_{t_{-1}}(\phi_2-\psi_2) = \frac{1}{\sigma}\left((1+\epsilon_2)(1+\epsilon_3)e^{\phi_2-\psi_1+\psi_2-\Lambda_2} - \frac{e^{-\phi_1-\psi_2+\Lambda_1+\Lambda_2}}{1+\epsilon_2}\right) \tag{3.13d}$$

$$\partial_x(\Lambda_1 - \phi_1 - \psi_1 + \psi_2) = \sigma\left((1+\epsilon_2)(1+\epsilon_3)e^{-\Lambda_1-\Lambda_2} - \frac{e^{\Lambda_2}}{1+\epsilon_2}\right)$$
$$\partial_x(\Lambda_2 + \phi_1 - \phi_2 - \psi_2) = \sigma\left((1+\epsilon_1)(1+\epsilon_2)e^{-\Lambda_1-\Lambda_2} - \frac{e^{\Lambda_1}}{1+\epsilon_2}\right)$$
$$\partial_{t_{-1}}\Lambda_1 = \frac{1}{\sigma}\left(e^{2\phi_1-\phi_2} - e^{2\psi_1-\psi_2}\right)e^{-\Lambda_1}$$
$$\partial_{t_{-1}}\Lambda_2 = \frac{1}{\sigma}\left(e^{-\phi_1+2\phi_2} - e^{-\psi_1+2\psi_2}\right)e^{-\Lambda_2}$$

Surely the compatibility taking second order derivatives of above transformation leads to the equations of motion (1.19) in the case $n = 2$. Also notice that the number of auxiliary fields increased by one compared to the previous case.

Studying $\mathfrak{sl}(2)$ and $\mathfrak{sl}(3)$ gives enough evidence in a path towards generalization. What we found is the following generalized gauge matrix BT for Toda $\mathfrak{sl}(n+1)$ theory which i, j matrix entry is

$$U(\phi,\psi)_{ij} = \left(1 + \frac{\sigma^{n+1}\epsilon_i}{\lambda}\right)\delta_{i,j} + \frac{\sigma^{\text{mod}_{n+1}\{i-j\}}}{\lambda^{\theta(i,j)}} \frac{\prod_{k=j}^{i-1} e^{\Lambda_k}}{\prod_{k=i}^{j-1} e^{\Lambda_k}} \frac{\prod_{k=1}^{j-i+1}(1+\epsilon_{k+i-1})}{\prod_{k=1}^{i-j-1}(1+\epsilon_{k+j})} \tag{3.15}$$

where θ is de discrete Heaviside function defined in (1.6) and all sub-indices follow the $\text{mod}_{\{n+1\}}$ pattern i.e, $\epsilon_{k+n+1} = \epsilon_k$, $\Lambda_{k+n+1} = \Lambda_k$.

3.1.1 The Tzitzeica-Bullough-Dodd limit

Among the benefits of taking integrable field theories in an algebraic description, there are still limiting cases that can be studied. If one just take the limit of (1.16) equation by imposing $\phi_1 = \phi_2 = \phi$ it is possible to obtain the Tzitzeica-Bullough-Dodd equation

$$\partial^2\phi = e^{\phi} - e^{-2\phi} \tag{3.16}$$

Formal description of this model can be done by folding the algebra $\mathfrak{sl}(3)^{(2)}$ (also denoted as $A_2^{(2)}$). We will comment folding defects latter. For now it is worth mentioning that just taking the limit $\phi_i = \phi$, $\psi_i = \psi$ and $\Lambda_i = \Lambda$ in (3.12) it is possible to obtain the matrix

$$U(\phi,\psi) = \begin{pmatrix} 1+\frac{\sigma^3 e^{\phi-\psi}}{\lambda} & \frac{2e^{-\Lambda}\sigma^2\left(1+e^{\phi-\psi}\right)}{\lambda} & \frac{2e^{-2\Lambda}\sigma\left(1+e^{\phi-\psi}\right)\left(1+e^{\psi-\phi}\right)}{\lambda} \\ e^{\Lambda}\sigma & 1+\frac{\sigma^3}{\lambda} & \frac{2e^{-\Lambda}\sigma^2\left(1+e^{\psi-\phi}\right)}{\lambda} \\ \frac{1}{2}e^{2\Lambda}\sigma^2 & e^{\Lambda}\sigma & 1+\frac{\sigma^3 e^{\psi-\phi}}{\lambda} \end{pmatrix} \tag{3.17}$$

which is the gauge representation of BT for Tzitzeica model found in [12, 2, 21]. Performing the gauge transformation of the potentials (1.17a)-(1.17b) (considering properly the limit case) we can find the following equation

$$\partial_x(\phi-\psi) = \sigma\left(e^{\Lambda} - 2(1+e^{\phi-\psi})(1+e^{-\phi+\psi})e^{-2\Lambda}\right) \tag{3.18a}$$

$$\partial_{t_{-1}}(\phi-\psi) = \frac{1}{\sigma}\left(2(e^{\phi}+e^{\psi})e^{-\Lambda} - \frac{1}{2}e^{-\phi-\psi+2\Lambda}\right) \tag{3.18b}$$

$$\partial_x(\Lambda - \frac{1}{2}(\phi+\psi)) = 2\sigma\,(e^{-\phi+\psi} - e^{\phi-\psi})e^{-2\Lambda} \tag{3.19a}$$

$$\partial_{t_{-1}}\Lambda = \frac{1}{\sigma}(e^{\phi}-e^{\psi})e^{-\Lambda} \tag{3.19b}$$

which is the BT for this model. This set of transformation gives the equation of motion (3.16) by compatibility, taking second order derivatives. It was firs found in [18] by Lagrangian terms. The same limit can not be taken for Type I matrix (2.13) otherwise it would imply $\phi = \psi$ therefore leading to a trivial transformation.

3.2 Type II BT as Defects

The results found in the previous section are remarkable and leave open questions for further discoveries. This was just a little step forward in the path of generalizing integrable field theories with defects. These findings were only possible given the inspiration of [18] where authors obtained by the first time another Bäcklund transformation by Lagrangian formalism and called it Type II. In this section we shall reproduce what was done in this paper and properly interpret type II BT equations in a defect context.

Although the description in algebraic terms is powerful, it lacks a proper interpretation in terms of internal boundary conditions. The Bäcklund equations found for Tzitzeica, sine/sinh-Gordon, etc, are valid in the entire domain. In contrast, it is evident in a Lagrangian fashion that we need to impose which fields live in which domain and the contact point as part of defect condition. Since gauge descriptions are valid elsewhere in the domain it is also valid at the defect location. To derive a Type II BT we take the following Lagrangian

$$\mathcal{L} = \theta(-x)\mathcal{L}_\phi + \delta(x)\mathcal{L}_D + \theta(x)\mathcal{L}_\psi \tag{3.20}$$

where now we introduce another field Λ in the defect part, this can be seen as a Lagrange multiplier

$$\mathcal{L}_\mathcal{D} = \frac{\phi\,\psi_t - \phi_t\psi}{2} + (\phi-\psi)_t\Lambda - (\phi-\psi)\Lambda_t - \mathcal{D}(\phi,\psi,\Lambda) \tag{3.21}$$

The introduction of this auxiliary field make a difference, taking the limit $\Lambda \to 0$ we should recover Type I transformations though. Euler-Lagrange coming from (3.20) give us

$$\phi_x - \psi_t = -2\Lambda_t - \frac{\delta\mathcal{D}}{\delta\phi} \tag{3.22a}$$

$$\psi_x - \phi_t = -2\Lambda_t + \frac{\delta\mathcal{D}}{\delta\psi} \tag{3.22b}$$

$$\phi_t - \psi_t = \frac{1}{2}\frac{\delta\mathcal{D}}{\delta\Lambda} \tag{3.22c}$$

Type II defects should not break integrability otherwise would be useless for this thesis. Here we deal with integrable fields and want things to remain that way. The presence of additional field Λ in (3.21) contributes to the total energy E but as done in type I case we can define another quantity $\bar{E}$ that includes defect contribution and assure energy like conservation. The energy can be written as

$$E = \frac{1}{2}\left(\int_{-\infty}^{0}\Big((\partial_t\phi)^2 + (\partial_x\phi)^2 + 2V(\phi)\Big)\,dx + \int_{0}^{\infty}\Big((\partial_t\psi)^2 + (\partial_x\psi)^2 + 2V(\psi)\Big)\,dx\right)$$

Taking a time derivative of E

$$\frac{dE}{dt} = (\partial_t\phi\partial_x\phi - \partial_t\psi\partial_x\psi)\Big|_{x=0}$$

from equations (3.22a)-(3.22c) we can show that

$$-\partial_t\mathcal{D} = \partial_t\phi\partial_x\phi - \partial_t\psi\partial_x\psi \qquad \text{for } x=0$$

and $\bar{E} = E + \mathcal{D}$ is conserved. Te same procedure can be done to show total momentum conservation where P is written as

$$P = \int_{-\infty}^{0} (\partial_t\phi\partial_x\phi)\; dx + \int_{0}^{\infty} (\partial_t\psi\partial_x\psi)\; dx$$

Taking time derivative of P

$$\frac{dP}{dt} = \frac{1}{2}\left((\partial_t\phi)^2 + (\partial_x\phi)^2 + 2V(\phi)\right)\Bigg|_{x=0} - \frac{1}{2}\left((\partial_t\psi)^2 + (\partial_x\psi)^2 + 2V(\psi)\right)\Bigg|_{x=0}$$

it is possible to show using equations (3.22a)-(3.22c) that the left side of above equation can be written entirely in terms of V, $\mathcal{D}$ and its derivative like so

$$\begin{aligned}\frac{dP}{dt} &= -\psi_t\frac{\delta\mathcal{D}}{\delta\phi} - \phi_t\frac{\delta\mathcal{D}}{\delta\psi} \\ &\quad + 2\Lambda_t\left(\frac{\delta\mathcal{D}}{\delta\phi} + \frac{\delta\mathcal{D}}{\delta\psi} + \frac{1}{2}\frac{\delta\mathcal{D}}{\delta\Lambda}\right) + \frac{1}{2}\left(\left(\frac{\delta\mathcal{D}}{\delta\phi}\right)^2 + \left(\frac{\delta\mathcal{D}}{\delta\psi}\right)^2\right) - V(\phi) + V(\psi)\end{aligned} \tag{3.23}$$

This expression is quite similar to the one found in chapter 2 (2.27). The difference though is the presence of terms related to Λ. To proceed in proving P can be written as total time derivative is a bit trickier. For that we will assume there is a function $F = F(\phi, \psi, \Lambda)$ designed to satisfy

$$\frac{1}{2}\frac{\delta\mathcal{D}}{\delta\Lambda}F = (\phi - \psi)_t F$$

and we will impose the following relation

$$\left(\frac{\delta\mathcal{D}}{\delta\phi}\right) + \left(\frac{\delta\mathcal{D}}{\delta\psi}\right) - 2(V(\phi) - V(\psi)) = \frac{\delta\mathcal{D}}{\delta\Lambda}F \tag{3.24}$$

By clamming the right side of (3.23) equation is a total time derivative of some quantities called $-\Omega$ we have

$$-\phi_t\frac{\delta\Omega}{\delta\phi} - \psi_t\frac{\delta\Omega}{\delta\psi} - \Lambda_t\frac{\delta\Omega}{\delta\Lambda} = -\psi_t\frac{\delta\Omega}{\delta\phi} - \phi_t\frac{\delta\Omega}{\delta\psi} + 2\Lambda_t\left(\frac{\delta\mathcal{D}}{\delta\phi} + \frac{\delta\mathcal{D}}{\delta\psi} + \frac{1}{2}\frac{\delta\mathcal{D}}{\delta\Lambda}\right) + (\phi - \psi)_t F$$

from where we can obtain the following system

$$\begin{cases} \frac{\delta\Omega}{\delta\phi} = \frac{\delta\mathcal{D}}{\delta\psi} - F \\ \frac{\delta\Omega}{\delta\psi} = \frac{\delta\mathcal{D}}{\delta\phi} + F \\ \frac{\delta\Omega}{\delta\Lambda} = 2\left(\frac{\delta\Omega}{\delta\phi} + \frac{\delta\mathcal{D}}{\delta\psi} + \frac{1}{2}\frac{\delta\mathcal{D}}{\delta\Lambda}\right) \end{cases}$$

Conveniently we change the basis $p = \frac{\phi+\psi}{2}$ and $q = \frac{\phi-\psi}{2}$ where system above can be written as

$$\begin{cases} \Omega_p = \mathcal{D}_p \\ \Omega_q = -\mathcal{D}_q - 2F \\ \Omega_\Lambda = -\mathcal{D}_\Lambda - 2\mathcal{D}_p \end{cases}$$

By taking second derivatives we can eliminate Ω and write the system as

$$\begin{cases} \mathcal{D}_{pq} = -F_p \\ \mathcal{D}_{p\Lambda} = -\mathcal{D}_{pp} \\ F_\Lambda = -F_p \end{cases} \tag{3.25}$$

If we say that

$$\mathcal{D} = f + g, \qquad F = -f_q \tag{3.26}$$

where $f = f(q, p - \Lambda)$ and $g = g(q, \Lambda)$ the system (3.25) is satisfied. In terms of f and g the expression (3.24) can be written as

$$\frac{\delta f}{\delta q}\frac{\delta g}{\delta\Lambda} - \frac{\delta f}{\delta\Lambda}\frac{\delta g}{\delta q} = 2\left(V(\phi) - V(\psi)\right) \tag{3.27}$$

So as long we are able to find f and g satisfying (3.27) for some set of potentials $V(\phi)$,$V(\psi)$ we have a well-defined BT of type II.

3.2.1 sine/sinh-Gordon model

One of the most simplest example we can take for a single field theory is the sine/sinh-Gordon whose potential in both domains $x < 0$ and $x > 0$ are respectively

$$V(\phi) = \frac{1}{2}\left(e^{2\phi} + e^{-2\phi}\right), \qquad \text{and} \qquad V(\psi) = \frac{1}{2}\left(e^{2\psi} + e^{-2\psi}\right)$$

For those potentials one can check that

$$f = \frac{\left(\gamma + e^{\phi-\psi} + e^{\psi-\phi}\right) e^{-\Lambda+\psi+\phi} + e^{\Lambda-\psi-\phi}}{\sigma}, \qquad g = \sigma\left(e^{-\Lambda}\left(\gamma + e^{\phi-\psi} + e^{\psi-\phi}\right) + e^{\Lambda}\right)$$

satisfies relation (3.27), and

$$\mathcal{D} = \frac{\left(\gamma e^{\psi+\phi} + e^{2\psi} + e^{2\phi}\right) e^{-\Lambda} + e^{\Lambda-\psi-\phi}}{\sigma} + \sigma\left(\left(\gamma + e^{\phi-\psi} + e^{\psi-\phi}\right) e^{-\Lambda} + e^{\Lambda}\right)$$

It is worth noticing that is not clear how many solution of (3.27) has for sine/sinh-Gordon potential. We don't know if it is possible to find another $\mathcal{D}$ functionally independent of the one above. But given the one we have the type II BT equations (3.22a)-(3.22c) becomes

$$\phi_x - \psi_t + 2\Lambda_t = \frac{e^{\Lambda-\phi-\psi} - \left(\gamma + 2e^{\phi-\psi}\right) e^{\phi+\psi-\Lambda}}{\sigma} - \sigma\left(e^{\phi-\psi} - e^{\psi-\phi}\right) e^{-\Lambda} \tag{3.28a}$$

$$\psi_x - \phi_t + 2\Lambda_t = \frac{\left(\gamma + 2e^{-\phi+\psi}\right) e^{\phi+\psi-\Lambda} - e^{\Lambda-\phi-\psi}}{\sigma} + \sigma\left(e^{\psi-\phi} - e^{\phi-\psi}\right) e^{-\Lambda} \tag{3.28b}$$

$$(\phi - \psi)_t = \frac{e^{\Lambda-\psi-\phi} - \left(\gamma e^{\psi+\phi} + e^{2\psi} + e^{2\phi}\right) e^{-\Lambda}}{2\sigma} + \frac{1}{2}\left(e^{\Lambda} - \left(\gamma + e^{\phi-\psi} + e^{\psi-\phi}\right) e^{-\Lambda}\right) \tag{3.28c}$$

These equations were already obtained in (3.10a)-(3.11b), but in light cone coordinates instead, showing consistence between lagrangian and algebraic approach.

3.2.2 Tzitzéica

Tzitzéica is another interesting single field theory that contemplates transformations of type II. It follows that Tzitzéica potential in both domains $x < 0$ and $x > 0$ are respectively

$$V(\phi) = e^{\phi} - e^{-2\phi}, \qquad \text{and} \qquad V(\psi) = e^{\psi} - e^{-2\psi}$$

Again solving (3.27) for Tzitzéica potential we find

$$f = \frac{4\left(e^{\frac{\phi-\psi}{2}} + e^{\frac{\psi-\phi}{2}}\right) e^{\frac{\psi+\phi}{2}-\Lambda} + e^{2\Lambda-\psi-\phi}}{2\sigma}, \qquad g = \sigma\left(e^{-2\Lambda}\left(e^{\frac{\phi-\psi}{2}} + e^{\frac{\psi-\phi}{2}}\right)^2 + e^{\Lambda}\right)$$

which follows that

$$\mathcal{D} = \frac{4\left(e^{\frac{\phi-\psi}{2}} + e^{\frac{\psi-\phi}{2}}\right) e^{\frac{\psi+\phi}{2}-\Lambda} + e^{2\Lambda-\psi-\phi}}{2\sigma} + \sigma\left(e^{-2\Lambda}\left(e^{\frac{\phi-\psi}{2}} + e^{\frac{\psi-\phi}{2}}\right)^2 + e^{\Lambda}\right)$$

again we can not assure uniqueness of this solution. Thanks to that we can find the following BT

$$\phi_x - \psi_t + 2\Lambda_t = \frac{4e^{\phi-\Lambda} - e^{2\Lambda-\psi-\phi}}{2\sigma} + \sigma\left(e^{\phi-\psi} - e^{\psi-\phi}\right)e^{-2\Lambda} \tag{3.29a}$$

$$\psi_x - \phi_t + 2\Lambda_t = \frac{e^{2\Lambda-\psi-\phi} - 4e^{\psi-\Lambda}}{\sigma} + \sigma\left(e^{\phi-\psi} - e^{\psi-\phi}\right)e^{-2\Lambda} \tag{3.29b}$$

$$(\phi-\psi)_t = \frac{2\left(e^{\frac{\phi-\psi}{2}} + e^{\frac{\psi-\phi}{2}}\right)e^{\frac{\psi+\phi}{2}-\Lambda} - e^{2\Lambda-\psi-\phi}}{2\sigma} + \sigma\left(\left(e^{\phi-\psi} + e^{\psi-\phi} + 2\right)e^{-2\Lambda} - \frac{e^{\Lambda}}{2}\right) \tag{3.29c}$$

These equations were already obtained in (3.18a)-(3.19b), but in light-cone coordinate instead, showing consistence between lagrangian and algebraic approach.

3.3 Type II Defects That Keep the model integrable

To show type II defect contribution for all infinite charges by the same way we did for E and P would be difficult. Rather we can do the same procedure done in chapter 2 but for type II Bäcklund matrix representation instead. There, all calculations were done independent of the matrix representation and the same result should apply here. So the expression for the defect contribution is exactly the one found previously (2.33) where U is either type I or type II, depending on the case.

3.4 Soliton Scattering

Consider the equations (3.13a)-(3.13d) which can be rearranged in a form of two systems of algebraic eqns. for variables e^{Λ_1} and e^{Λ_2} namely,

$$a_1 e^{\Lambda_1} + b_1 = c_1 e^{-\Lambda_2}, \qquad a_2 e^{\Lambda_1} + b_2 = c_2 e^{-\Lambda_2},. \tag{3.30}$$

$$a_3 e^{\Lambda_2} + b_3 = c_3 e^{-\Lambda_1}, \qquad a_4 e^{\Lambda_2} + b_4 = c_4 e^{-\Lambda_1}. \tag{3.31}$$

where,

$$a_1 = \frac{1}{\sigma}\partial_x q_2 D\, e^{-\phi_1-\psi_2}, \qquad b_1 = \sigma\partial_t q_2, \qquad c_1 = A_2 e^{\phi_2-\psi_1+\psi_2} - CD\, e^{-\phi_1-\psi_2},$$
$$a_2 = 1 - \frac{CD}{A_1} e^{-2\phi_1+\phi_2-\psi_1-\psi_2}, \qquad b_2 = -\frac{1}{\sigma}\partial_x q_1, \qquad c_2 = \sigma\partial_t q_1 \frac{C}{A_1} e^{-\phi_1+\phi_2-\psi_1},$$
$$a_3 = \frac{1}{\sigma}\partial_x q_1 D\, e^{-\phi_1-\psi_2}, \qquad b_3 = \sigma\partial_t q_1, \qquad c_3 = A_1 e^{\phi_1-\phi_2+\psi_1} - CD\, e^{-\phi_1-\psi_2},$$
$$a_4 = 1 - \frac{CD}{A_2} e^{-\phi_1-\phi_2+\psi_1-2\psi_2}, \qquad b_4 = -\frac{1}{\sigma}\partial_x q_2, \qquad c_4 = \sigma\partial_t q_2 \frac{C}{A_2} e^{-\phi_2+\psi_1-\psi_2}.$$

and

$$\begin{aligned} A_1 &= (1+\epsilon_1)(1+\epsilon_2) \\ A_2 &= (1+\epsilon_2)(1+\epsilon_3) \\ C &= (1+\epsilon_1)(1+\epsilon_2)(1+\epsilon_3) \\ D &= \frac{1}{(1+\epsilon_2)} \end{aligned}$$

The two eqns. in (3.30) is a system of equations for the two variables, $X_1 \equiv e^{\Lambda_1}$ and $Y_1 \equiv e^{-\Lambda_2}$ which have solution given by,

$$X_1 = \frac{b_1c_2 - b_2c_1}{a_2c_1 - a_1c_2}, \qquad Y_1 = \frac{a_1b_2 - a_2b_1}{a_1c_2 - a_2c_1}.$$

Likewise (3.31) is another system for variables $X_2 \equiv e^{-\Lambda_1}$ and $Y_2 \equiv e^{\Lambda_2}$ leading to

$$X_2 = \frac{a_3b_4 - a_4b_3}{a_3c_4 - a_4c_3}, \qquad Y_2 = \frac{b_3c_4 - b_4c_3}{a_4c_3 - a_3c_4}.$$

Consistency of these four expressions is given by the compatibility relations,

$$X_1X_2 = e^{\Lambda_1}e^{-\Lambda_1} = 1 \qquad \text{and} \qquad Y_1Y_2 = e^{\Lambda_2}e^{-\Lambda_2} = 1 \tag{3.32}$$

3.4.1 Vacuum-one Soliton solution

The first example to be considered is the *vacuum → one soliton* solution where we set the fields to be

$$\phi_i = 0, \qquad \psi_1 = \ln\left(\frac{1+\rho}{1+\omega\rho}\right), \qquad \psi_2 = \ln\left(\frac{1+\rho}{1+\omega^2\rho}\right). \tag{3.33}$$

This soliton solutions satisfies the equation (1.19) (for $n = 2$) whith $\rho = e^{\eta(kx+k^{-1}t)}$, $\eta^2 = 3$ and ω solution of $\omega^3 = 1$ (where $\omega \neq 1$). The compatibility (3.32) show that Backlund parameter and the momentum are not all independent, instead they satisfy

$$k^6 + \sigma^6 = 0. \tag{3.34}$$

3.4.2 Scattering of one-Soliton solutions

The *one soliton* $\to$ *one soliton* case is a more interesting since it is possible to find the phase shift for the soliton scattering. Let us define

$$\phi_1 = \ln\left(\frac{1+R\rho}{1+\omega R\rho}\right), \quad \phi_2 = \ln\left(\frac{1+R\rho}{1+\omega^2 R\rho}\right),$$
$$\psi_1 = \ln\left(\frac{1+\rho}{1+\omega\rho}\right), \quad \psi_2 = \ln\left(\frac{1+\rho}{1+\omega^2\rho}\right)$$

where R is some complex number namely, the phase shift. From the compatibility conditions (3.32) it follows that

$$\eta k^6 (R-1)^3 + 36k^3 R(R+1)\sigma^3 + \eta(R-1)^3\sigma^6 = 0$$

which admits 3 solutions:

$$R_1 = \frac{\alpha - 6\,\beta\,\gamma^{-1/3} + 6\,\gamma^{1/3}}{\eta(k^6+\sigma^6)},$$
$$R_2 = \frac{\alpha - 6\,\omega^2\beta\gamma^{-1/3} + 6\,\omega\gamma^{1/3}}{\eta(k^6+\sigma^6)},$$
$$R_3 = \frac{\alpha - 6\,\omega\beta\gamma^{-1/3} + 6\,\omega^2\gamma^{1/3}}{\eta(k^6+\sigma^6)},$$

where

$$\alpha = \eta\sigma^6 + \eta k^6 - 12k^3\sigma^3,$$
$$\beta = \eta k^9\sigma^3 - 4k^6\sigma^6 + \eta k^3\sigma^9,$$
$$\gamma = \left(\sqrt{k^6\sigma^6\left(k^{12}-\sigma^{12}\right)^2} - k^3\sigma^3\left(k^{12} - 6\eta k^9\sigma^3 + 18k^6\sigma^6 - 6\eta k^3\sigma^9 + \sigma^{12}\right)\right)/2.$$

It can be verified that $R_1 \cdot R_2 \cdot R_3 = 1$.

Chapter 4

Type II Like Transformations

We have seen that either Type I or Type II *Bäcklund Transformation* can be derived from Lagrangian formalism or in more algebraic style by means of gauge transformation. The Lagrangian fashion seems a bit difficult to generalize but equations (2.28a)-(2.28b) and (3.27) left open the possibility to find other kinds of BT. On the other hand, the algebraic approach appears to be more straightforward, even though it is not possible to assure uniqueness of the solutions. For small cases like $\mathfrak{sl}(2)$, $\mathfrak{sl}(3)$ the inverse matrix analysis together with determinant property of U seems to exhaust the possibilities but for higher cases there are further solutions, different from the ones showed before. This is what we will develop in this chapter.

4.1 The Algorithm

The way we solved $U(\phi, \psi)$ in the previous chapter followed some standard algorithm and it will not be different here. Basically for a given set of potentials $A_x,\ A_t\ \in \mathfrak{sl}(n+1)$ satisfying the zero curvature equation we propose a matrix representation of BT as

$$U_m(\phi, \psi) = \sum_{k=0}^{m} \sigma^k \mathcal{U}^{(-k)} \tag{4.1}$$

where $m \leq n+1$. The m index on the matrix is just to identify the order of summation. It is possible to completely solve all $\mathcal{U}^{(-k)}$ following three steps:

- Gauge transform one of the potentials, usually A_x and solve the uncoupled first order differential equations

$$A_x(\psi) U_m = U_m\, A_x(\phi) - \partial_x U_m. \tag{4.2}$$

- When the previous step let unsolved matrix entries we take the inverse method. Here we need treat differently the cases where $m < n+1$ and

$m = n + 1$. It follows

$$\text{if } m < n+1 \Rightarrow U_{n+1-m}(\psi, \phi; -\sigma) = \left(\frac{\lambda}{\lambda + (-1)^m(-\sigma)^{n+1}}\right)^{m-1} \det\{U_m(\phi, \psi)\} U_m(\phi, \psi; \sigma)^{-1}$$
$$\text{if } m = n+1 \Rightarrow U_{n+1}(\psi, \phi; -\sigma) = \left(\frac{\lambda}{\lambda - \sigma^{n+1}}\right)^{n-1} \det\{U_{n+1}(\phi, \psi)\} U_{n+1}(\phi, \psi; \sigma)^{-1}.$$

Usually by just comparing diagonal terms of both left and right side of the above equation is enough to produce equations for solving $u_j^{(-k)}$ variables, but nothing prevents to pick up and compare any matrix entries. Notice the sign of σ in the left side of equation, this is due BT change its sign under field interchange.

- Constant determinant is the final check and must always be satisfied. Sometimes this can be used to solve the last variables by imposing

$$\det\{U(\phi, \psi)\} = const.$$

  when the previous steps wasn't enough.

- Since $U(\phi, \psi)$ are completely solved there are still remaining equations from gauge transformation (4.2), those are the *Bäcklund transformations*. By gauging transforming potential A_t we get t part of transformation. The compatibility of equations should generate the equation of motion or in other words leaves it invariant.

The index m in (4.1) can be chosen in the interval $0 \leq m \leq n+1$ although we didn't studied for cases where $m > n+1$. The whole process of solving $U(\phi, \psi)$ might let auxiliary fields usually labeled as e^{Λ_j}. Those fields do not contribute to the dynamics and can't appear in the equation of motion while resolving compatibility, but they are necessary for the whole set and fundamental to the transformations.

4.2 $\mathfrak{sl}(4)$ example

We are looking for transformations that leaves the Toda equations invariant, that is basically the definition of BT given in chapter 2. We already know two of them for Toda $\mathfrak{sl}(4)$. For $m = 1$ (or $m = 3$) we get Type I BT[1]. For $m = 4$ we get the type II (see chapter 3). We are left with one more possibility which is $m = 2$. Let

[1]As pointed in chapter 2 BT are not unique and for Toda $\mathfrak{sl}(n+1)$ theory it is possible to find type I BT in cases $m = 1$ or $m = n$ given the ansats (4.1).

us dive into that case. We propose

$$U_2(\phi,\psi)=\sum_{k=0}^{2}\sigma_k\mathcal{U}^{(-k)}=\begin{pmatrix} 1 & 0 & \frac{\sigma^2 u_3^{(-2)}}{\lambda} & \frac{\sigma u_4^{(-1)}}{\lambda} \\ \sigma u_1^{(-1)} & 1 & 0 & \frac{\sigma^2 u_4^{(-2)}}{\lambda} \\ \sigma^2 u_1^{(-2)} & \sigma u_2^{(-1)} & 1 & 0 \\ 0 & \sigma^2 u_2^{(-2)} & \sigma u_3^{(-1)} & 1 \end{pmatrix}$$

that takes $A_x(\psi)$ into $A_x(\phi)$ and vice versa. We already have set $u_j^{(0)}=1$ because gauge tranf. must be valid for all values of σ including $\sigma\to 0$. These potentials was described in (1.17a)-(1.17b) for $n=3$ where the zero curvature equation give the following set of equations of motion:

$$\begin{cases} \partial^2\phi_1 = e^{2\phi_1-\phi_2}-e^{-\phi_1-\phi_3} \\ \partial^2\phi_2 = e^{-\phi_1+2\phi_2-\phi_3}-e^{-\phi_1-\phi_3} \\ \partial^2\phi_3 = e^{-\phi_2+2\phi_3}-e^{-\phi_1-\phi_3} \end{cases} \tag{4.3}$$

$$\begin{cases} \partial^2\psi_1 = e^{2\psi_1-\psi_2}-e^{-\psi_1-\psi_3} \\ \partial^2\psi_2 = e^{-\psi_1+2\psi_2-\psi_3}-e^{-\psi_1-\psi_3} \\ \partial^2\psi_3 = e^{-\psi_2+2\psi_3}-e^{-\psi_1-\psi_3} \end{cases} \tag{4.4}$$

Following the algorithm proposed in the previous section we find

$$U(\phi,\psi)=\begin{pmatrix} 1 & 0 & \frac{\sigma^2 e^{-\psi_1-\phi_2+\phi_3}}{\lambda} & \frac{\sigma\left(e^{\psi_1-\psi_2-\phi_3}+e^{-\psi_3+\phi_1-\phi_2}\right)e^{-\Lambda}}{\lambda} \\ \sigma e^{\Lambda} & 1 & 0 & \frac{\sigma^2 e^{\psi_1-\psi_2-\phi_3}}{\lambda} \\ \sigma^2 e^{\psi_2-\psi_3+\phi_1} & \sigma\left(e^{\psi_1+\phi_2-\phi_3}+e^{\psi_2-\psi_3+\phi_1}\right)e^{-\Lambda} & 1 & 0 \\ 0 & \sigma^2 e^{\psi_3-\phi_1+\phi_2} & \sigma e^{\Lambda-\psi_1+\psi_3-\phi_1+\phi_3} & 1 \end{pmatrix} \tag{4.5}$$

and the following BT

$$\partial_x(\phi_1 - \psi_1) = \sigma\left(e^{\Lambda} - \left(e^{\psi_1-\psi_2-\phi_3} + e^{-\psi_3+\phi_1-\phi_2}\right)e^{-\Lambda}\right) \tag{4.6a}$$

$$\partial_x(\phi_2 - \psi_2) = \sigma\left(\left(e^{\psi_1+\phi_2-\phi_3} + e^{\psi_2-\psi_3+\phi_1}\right)e^{-\Lambda} - \left(e^{\psi_1-\psi_2-\phi_3} + e^{-\psi_3+\phi_1-\phi_2}\right)e^{-\Lambda}\right) \tag{4.6b}$$

$$\partial_x(\phi_3 - \psi_3) = \sigma\left(e^{\Lambda-\psi_1+\psi_3-\phi_1+\phi_3} - \left(e^{\psi_1-\psi_2-\phi_3} + e^{-\psi_3+\phi_1-\phi_2}\right)e^{-\Lambda}\right) \tag{4.6c}$$

$$\partial_{t_{-1}}(\phi_1 + \psi_2 - \psi_3) = \frac{1}{\sigma}\left((e^{\psi_1-\psi_2+\psi_3+\phi_1-\phi_3} + e^{2\phi_1-\phi_2})e^{-\Lambda} - e^{\Lambda-\psi_1+\psi_2-\phi_1}\right) \tag{4.6d}$$

$$\partial_{t_{-1}}(\phi_2 - \phi_3 + \psi_1) = \frac{1}{\sigma}\left((e^{2\psi_1-\psi_2} + e^{\psi_1-\psi_3+\phi_1-\phi_2+\phi_3})e^{-\Lambda} - e^{\Lambda-\psi_1-\phi_1+\phi_2}\right) \tag{4.6e}$$

$$\partial_{t_{-1}}(\phi_3 - \psi_1 + \psi_2) = \frac{1}{\sigma}\left(\left(e^{2\psi_1-\psi_2} + e^{\psi_1-\psi_3+\phi_1-\phi_2+\phi_3}\right)e^{-\Lambda} - e^{\Lambda-\psi_1+\psi_2-\phi_1}\right) \tag{4.6f}$$

together with auxiliary equation

$$\partial_x(e^{\Lambda} - \phi_1 - \psi_1 + \psi_2) = \sigma\left(e^{\psi_1-\psi_2-\phi_3} - e^{\psi_2-\psi_3+\phi_1}\right) \tag{4.7a}$$

$$\partial_{t_{-1}}(e^{\Lambda}) = \frac{1}{\sigma}\left(e^{2\phi_1-\phi_2} - e^{2\psi_1-\psi_2}\right) \tag{4.7b}$$

It is possible to check that the set of transformation (4.6a)-(4.7b) transforms equations (4.3) into (4.4) and vice versa. Also, by taking the inverse of (4.5) one obtains the same matrix with ϕ, ψ interchanged times some overall constant.

There is a subtle difference between the transformation obtained here and type II shown in (3.15). The number of auxiliary fields for standard type II is equal to the algebra rank which for $\mathfrak{sl}(4)$ is three. In contrast the newest BT has only one auxiliary field, leading us to a question: would this be type I, type II transformation, or something else? The idea of "another type" of BT arouse in [18] in a claim to include Tizitzeica model since it does not have type I BT. To do that a Lagrange multiplier was inserted in defect part of Lagrangian which led to the existence of auxiliary fields in transformation. Following this idea is quite reasonable say (4.5) generates another type II transf for Toda $\mathfrak{sl}(4)$ theory.

Chapter 5

Conclusions and further developments

In this chapter we shall discuss further consequences of the results found in the previous chapters. Rather than giving formal mathematical description, we will focus on the main ideas. Also we leave some open questions for further research.

5.1 BT for others times flows

Given we have obtained Bäcklund Transformation for generalized Toda theory a natural question is if this result can be extended to others times flows of the same hierarchy. Fortunately the answer is yes and there is no need to do all the work again to obtain BT for mKdV equation for example. In this section we will argue why this is true. In a sense when writing equation of motion in a zero curvature way (eq. (1.2)) we are splitting the responsibility of generating them in two objects A_x and A_t. Then we choose one to be fixed, A_x for example, and allow the other, A_t, to vary in gradation, generating several equations motion. We rename $A_t \rightarrow A_{t_N}$ where N represent the highest (or lowest) graded element (see chapter 1). The choice which potential will be fixed is completely arbitrary, A_x is just convention. This is the essence of a integrable hierarchy.

Gauge invariance comes handy. Since the zero curvature equation is invariant under gauge transformation we have two matrix equations to work with, one for each potential.

$$U\left[\partial_x + A_x(\phi), \partial_{t_N} + A_{t_N}(\phi)\right]U^{-1} \quad \Rightarrow \quad \begin{array}{l} UA_x(\phi)U^{-1} - \partial_x U U^{-1} = A_x(\psi) \\ \\ UA_{t_N}(\phi)U^{-1} - \partial_{t_N} U U^{-1} = A_{t_N}(\psi) \end{array}$$

If one has the x part of BT and the motion equation then it is possible to derive the t part. The opposite is also true. And if one has the x and t part of the transformation it is possible to derive the motion equation. Then for obtaining the BT transformation for *mKdV* equation we need just to conjugate A_{t_3} (or A_{t_2} generalized one). The same procedure was done for super-symmetric version of these equations [1, 54]. Notice that to solve $U(\phi, \psi)$ we needed one of the potentials, then following the reasoning solving the matrix representation of Bäcklund transformation using A_x we are able to find the BT for the whole hierarchy.

5.2 BT for other hierarchies

All transformations obtained so far work for $\mathfrak{sl}(n+1)$ Toda hierarchy. How BT would be for NLS or KdV hierarchy for example? The biggest change is the fixed (conventionally) potential A_x which is different for each hierarchy. Hence a gauge matrix representation of BT for the respective equation would be completely different, but some principles keeps the same. Constant determinant and inverse matrix property does not depend on which basis or matrix representation elements A_μ are being written. Therefore we might be able to find BT for different hierarchies.

Besides $U(\phi, \psi)$ that transforms between potentials of the same hierarchy there is a gauge matrix representation of transformation that relates A_μ of different hierarchies. The best example we can give is the Miura transformation. Miura proposed a relation between fields the *KdV* and *mKdV* equation []. Later these results were found in matrix gauge description [34] and generalization of it was developed in [20]. Let us consider u, v[1] being fields of *mKdV* and I, J from *KdV* equations. Then it is possible to find $S(u, I)$ that satisfies the following relation

$$A_\mu^{KdV} = S(u, I) A_\mu^{mKdV} S(u, I)^{-1} - \partial_\mu S(u, I) S(u, I)^{-1}$$

Due to the possibility of writing both Bäcklund and Miura in some matrix gauge representation we can find BT for *KdV* hierarchy as well. Suppose there is a matrix $K(I, J)$ that generates BT for *KdV* hierarchy. Then

$$A_\mu^{KdV}(J) = K(I, J) A_\mu^{KdV}(I) K(I, J)^{-1} - \partial_\mu K(I, J) K(I, J)^{-1}$$

It follows that we can write $K(I, J)$ in terms of $U(\phi, \psi)$ and $S(I, J)$; a pictorial description is in terms of the following diagram

[1]Notice the correlation of the fields of mKdV and generalized Toda equations: $u_i = \partial_x \phi_i$ and $v_i = \partial_x \psi_i$

$$
\begin{array}{ccc}
 & U(\phi,\psi) & \\
A_x^{mKdV}(u) & \longleftrightarrow & A_x^{mKdV}(v) \\
S(u,J)\ \Big\updownarrow & & \Big\updownarrow\ S(v,I) \\
A_x^{KdV}(I) & \longleftrightarrow & A_x^{KdV}(I) \\
 & K(I,J) &
\end{array}
$$

This allow us to infer

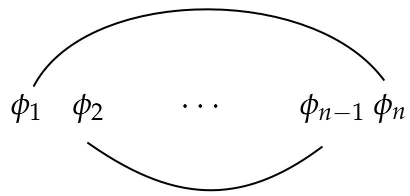
$$K(I,J) = S(u,I)U(\phi,\psi)S(v,J)^{-1}$$

The above relation is a bit trickier given the fact that Miura transformation is not unique. All calculations and representation for $S(I,J)$ can be found with much more detail in [20].

5.3 Folding

The idea of folding started in [45] where it was noticed that a generalization of Bullough-Dodd equations could be obtained as a "reduction" of $\mathfrak{sl}(n+1)$ models. The symmetries of the equations was investigated by looking at Dynkin diagrams that in a way encode the main structure of the model. Those symmetries allow a limiting case by identifying fields of the original theory and obtaining different equations of motion which also can be obtained from another algebra. After obtaining this new equations, by matching fields of the theory, they remain integrable in the same sense as before, also having a Lax pair representation. In this section we will barely touch in this subject, pointing out the main ideas of the extension that can be done from the results found in chapter 3. For more details good references can be found [48, 49].

What the folding process really means is obtaining a different linear combination of algebra elements or setting another basis of the same algebra $\mathfrak{g}$. This process allows to decompose $\mathfrak{g}$ in a way that another grading structure is possible. Usually to identify different basis for different gradation we set a upper-indice in algebra. But there is a shortcut to obtain these reduced theories. Having the affine Toda theory which is built in top of $\mathfrak{sl}(n+1)$ if we identify the fields like $\phi_1 = \phi_n$, $\phi_2 = \phi_{n-1}$, $\cdots$, as the diagram below, we end up with the generalized Tzitzeica-Bullough-Dodd model



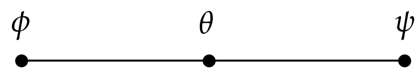
This was done for $\mathfrak{sl}(3)$ in the previous chapter but also can be extended for higher algebras. The same reasoning apply perfectly while dealing with defects where just equating fields in the type II BT matrix for Toda $\mathfrak{sl}(n+1)$ is enough to obtain the defect gauge matrix representation of (see (3.17)). This method seems a bit informal and the reason why this works is that equating fields are just an implication of folding the algebra $\mathfrak{sl}(n+1)$.

These are one possible reduction but is not the only existing one. In [45] there is a more complete description of the possibilities. Also, defects in these theories were studied [48, 49], where until then integrability could no be ensured. Now with a fully algebraic description of equation of motion and defect expressed as a gauge transformation integrability is guaranteed.

5.4 Composition of Bäcklund Transformation

When the idea of type II defects came up in [18] there was some evidence that this kind of defect could be a sort of two type I "fused" defects. The term fused defect had gain popularity henceforth. When dealing with this transformation in an algebraic context the idea to construct type II by fusing two type I defects is even more evident.

To better illustrate this let us take the $\mathfrak{sl}(3)$ example. Suppose we are doing two successive type I BT taking the field ϕ into θ and then to ψ.



In terms of matrix conjugation it reads

$$A_\mu(\theta) = U(\phi,\theta)A_\mu(\phi)U(\phi,\theta)^{-1} - \partial_\mu U(\phi,\theta)U(\phi,\theta)^{-1}$$
$$A_\mu(\psi) = U(\theta,\psi)A_\mu(\theta)U(\theta,\psi)^{-1} - \partial_\mu U(\theta,\psi)U(\theta,\psi)^{-1}$$

By comparison we can write a matrix $\tilde{U}$ leading ϕ to ψ

$$\tilde{U}(\phi,\psi;\sigma) = U(\theta,\psi;\sigma_1)\,U(\phi,\theta;\sigma_2)$$

In fact by performing the matrix multiplication and setting $\theta_1 = \phi_1 - \phi_2 + \psi_2$, $\theta_2 = \phi_1 - \psi_1 + \psi_2$, we can get rid of $\theta's$ field. The result is

$$\tilde{U}(\phi,\psi) = \begin{pmatrix} 1 & \frac{\sigma_1\sigma_2 e^{-\psi_1-\phi_1+\phi_2}}{\lambda} & \frac{(\sigma_1+\sigma_2)e^{-\psi_2-\phi_1}}{\lambda} \\ (\sigma_1+\sigma_2)\, e^{\psi_1+\phi_1-\phi_2} & 1 & \frac{\sigma_1\sigma_2 e^{\psi_1-\psi_2-\phi_2}}{\lambda} \\ \sigma_1\sigma_2 e^{\psi_2+\phi_1} & (\sigma_1+\sigma_2)\, e^{-\psi_1+\psi_2+\phi_2} & 1 \end{pmatrix}$$

which is exactly the second possible solution for type I BT showed in (2.14) with the proper σ parameter and field configuration. For that it is needed to set $\sigma_1 + \sigma_2 = \sigma$ and $\sigma_1\sigma_2 = \sigma^2$ where the solution is $\sigma_1 = \omega\sigma,\ \sigma_2 = \omega^2\sigma$ where ω is a complex number satisfying $\omega^3 = -1$ ($\omega \neq -1$). In fact, composing transformation like (2.13) is another tool for solving U of higher order in σ.

Surely a product of two matrix like (2.13) couldn't give some extra terms on diagonal beyond identity therefore to achieve some type II like matrix we need to multiply once again, or multiply two different kinds of type I BT. Doing so we have

$$U_{II}(\phi,\psi;\sigma) = U(\theta,\psi;\sigma_1)\,\tilde{U}(\phi,\theta;\sigma_2)$$

where the left side is supposed to be type II. The right side of above equation reads

$$U_{II}(\phi,\psi) =$$

$$\begin{pmatrix} 1+\frac{\sigma_1\sigma_2^2 e^{\phi_1-\psi_1}}{\lambda} & \frac{\sigma_1\sigma_2 e^{-\theta_1-\psi_1+\phi_2}}{\lambda} + \frac{\sigma_2^2 e^{-\theta_1-\phi_1+\phi_2}}{\lambda} & \frac{\sigma_1 e^{-\theta_2-\psi_1}}{\lambda} + \frac{\sigma_2 e^{-\theta_2-\phi_1}}{\lambda} \\ \sigma_1 e^{\theta_1+\psi_1-\psi_2} + \sigma_2 e^{\theta_1+\phi_1-\phi_2} & \frac{\sigma_1\sigma_2^2 e^{\psi_1-\psi_2-\phi_1+\phi_2}}{\lambda} + 1 & \frac{\sigma_1\sigma_2 e^{\theta_1-\theta_2+\psi_1-\psi_2-\phi_1}}{\lambda} + \frac{\sigma_2^2 e^{\theta_1-\theta_2-\phi_2}}{\lambda} \\ \sigma_1\sigma_2 e^{\theta_2+\psi_2+\phi_1-\phi_2} + \sigma_2^2 e^{\theta_2+\phi_1} & \sigma_1 e^{-\theta_1+\theta_2+\psi_2} + \sigma_2 e^{-\theta_1+\theta_2+\phi_2} & 1+\frac{\sigma_1\sigma_2^2 e^{\psi_2-\phi_2}}{\lambda} \end{pmatrix}$$

where comparing with type II matrix (3.12) we find the solution

$$\begin{cases} \theta_1 &= \Lambda_1 + \phi_2 + \psi_2 - \log\left(e^{\psi_1+\phi_2} + e^{\psi_2+\phi_1}\right) \\ \theta_2 &= \Lambda_1 + \Lambda_2 + \phi_2 + \psi_2 - \log\left(\left(e^{\psi_2} + e^{\phi_2}\right)\left(e^{\psi_1+\phi_2} + e^{\psi_2+\phi_1}\right)\right) \\ \sigma_1 &= \sigma_2 = \sigma \end{cases}$$

5.5 Conclusion

Describing classical integrable field theories in zero the curvature formalism brings outstanding advantages. Empowered by Lie algebras we were able to derive a whole hierarchy of equation of motion (Toda hierarchy), assure its integrability and even being able to obtain a set of infinite conservation laws; all these results were already know. The contribution of this thesis goes further, where we explored the gauge invariance to derive Bäcklund transformations for the whole hierarchy. We noticed that this transformations are not unique; actually there are two types: type I where transformations are only functionally depending on the fields of the theory, and type II which depends on auxiliary fields. We where able to derive both kinds of transformation by proposing a matrix representation of BT named as U. The ansatz is to propose the matrix U as polynomially expanded in σ and λ (see (2.7)). These two parameters play different roles where λ have to do with the algebra gradation and σ is the Bäcklund parameter. Starting from first principles such as gauge invariance together with some matrix properties such as constant determinant and inverse matrix we where able to completely solve U. The starting point is to gauge transform one of the potentials, usually A_x, allowing to solve the uncoupled linear differential equations. This suffices to solve one of the possible type I BT since it if of lower order expansion in σ and consequently many matrix entries are zero. For cases where powers of σ are higher, one can use the inverse matrix and constant determinant properties to solve the remaining unknowns matrix entries; this was developed in both chapter 3 and 4.

After obtaining the set of BT one might wonder what is the meaning of such transformations. Their applicability are wide and interpretation might vary depending on the context. We have seen that the same transformation can be obtained from both gauge context and from a Lagrangian view point, which describes a field theory with defects. In the gauge approach, BTs are valid in the whole domain and it is a bit hard to see whether these equations are describing defects. On the other hand, thinking about defects in a Lagrangian approach is natural given the discontinuity due to the presence of $\delta(x)$ (see (3.20)). The trade-off between Lagrange and gauge formalism goes in favor of the second one. For instance, we were able to prove integrability which would be quite hard from a Lagrangian perspective.

Applicability of these transformations go beyond field theory with internal boundary condition (defects). We can use BT to find a whole class of soliton solutions by Bianchi's formula (see (2.17) and (2.18)). This formula was obtained by interchanging successive BT with different Bäcklund parameters, σ_1 and σ_2, giving rise to an algebraic way of finding soliton solutions. The idea behind Binachi's formula is to find a way to, given a seed solution ϕ, compose other two solutions $\theta^{(1)}$, $\theta^{(2)}$ to generate a forth one. This is basically a way to say that "Bäcklund Transformations describe soliton interaction". The interchange of the σs parameters is crucial to find the formula and it should be valid for any values, including the limit $\sigma_2 \to \sigma_1$. From this limit, we were able to find a new class of soliton solutions (see (2.21)).

Appendix A

Conservation Laws

In this appendix we will follow the Lagrangian approach to find conserved quantities. This aims to completeness of the set up in the study of integrable field theories. To start with let us define a quantity called Lorentz spin

Definition 1. *Label it by a number s this quantity is defined as*

$$s = (\textit{number of } \partial_+ \textit{ derivatives}) - (\textit{number of } \partial_- \textit{ derivatives}) \tag{A.1}$$

Bare in mind the word *spin* here has nothing to do with the notion of spin in Quantum Mechanics, but just order of derivation give by above definition. Here we $\partial_+ = \frac{\partial}{\partial x_+}$, $\partial_- = \frac{\partial}{\partial x_-}$ and the variables x_+, x_- are the usual light cone coordinates

$$\begin{cases} x_+ = \frac{1}{2}(t+x) \\ x_- = \frac{1}{2}(t-x) \end{cases}$$

By following (A.1) definition the quantities $\partial_+ u$, $\partial_+^3 u \partial_- u$ and $(\partial_- u)^2$ would have spins 1, 2 and -2 respectively. For simplicity we will deal with a single field theory whose lagrangian is defined in light cone coordinate as

$$\mathcal{L} = \frac{1}{2}\partial_+ u \partial_- u - V(u)$$

which leads to the well known equation of motion

$$\partial_+ \partial_- u = -V'(u)$$

where $V'(u)$ means functional derivative with respect field u. The key point to find conserved quantities in this approach is by searching for a pair of quantities (T_{s+1}, X_{s-1}) that obeys the following continuity equation

$$\partial_- T_{s+1} = \partial_+ X_{s-1} \tag{A.2}$$

We will define T_{s+1} as a polynomial expression in $\partial_+ u$ (for $s > 0$). Then if one is able to find X_{s-1} such that (A.2) holds true then

$$Q_s = \int_{\infty}^{\infty} (T_{s+1} - X_{s-1})\, dx$$

is a conserved charge. The reason for that is that we can look to the equation (A.2) in the x, t coordinates as

$$\partial_t \left(T_{s+1} - X_{s-1}\right) - \partial_x \left(T_{s+1} + X_{s-1}\right) = 0 \tag{A.3}$$

which is nothing but the continuity equation in x, t coordinates. If we define

$$\rho_s = T_{s+1} - X_{s-1} \text{ and } j_s = T_{s+1} + X_{s-1}$$

with ρ_s and j_s being the charge and current density, respectively, we might say that the charge $Q_s = \int \rho_s \, dx$ is conserved if j_s vanish at the boundary

$$\lim_{x\to\infty} j_s - \lim_{x\to-\infty} j_s = 0. \tag{A.4}$$

Integrating (A.3) in x we have

$$\begin{aligned} &\int_{-\infty}^{\infty} \partial_t(T_{s+1} - X_{s-1}) \, dx - \int_{-\infty}^{\infty} \partial_x \left(T_{s+1} + X_{s-1}\right) dx = 0 \\ \Rightarrow \quad & \partial_t \int_{-\infty}^{\infty} \rho_s \, dx - \int_{-\infty}^{\infty} \partial_x \, j_s \, dx = 0 \\ \Rightarrow \quad & \partial_t \, Q_s - j_s|_{-\infty}^{\infty} = 0 \\ \Rightarrow \quad & \partial_t \, Q_s = 0 \end{aligned}$$

if (A.4) is satisfied.

Now that we have a well defined quantity Q_s let us try to find it by direct calculation:

- $\mathbf{s = 0}$: Since T_1 need to be polynomially in $\partial_+ u$ the only option is

  $$T_1 = \partial_+ u$$

  Taking a ∂_- derivative of T_1 we can write

  $$\partial_- \partial_+ u = \partial_+ (\partial_- u)$$

  which is exactly in the form of equation (A.2) where we can say that $X_{-1} = \partial_- u$. Then we find the charge

  $$Q_0 = \int_{-\infty}^{\infty} \partial_x u dx = u \Big|_{-\infty}^{\infty}$$

  which is called the topological charge.

- $\mathbf{s} = \mathbf{1}$: Doing the same procedure all options for T_2 is being a linear combination of $(\partial_+ u)^2$ and $\partial_{++}u$. It easy to see that the conserved charge is independent of a total derivative where we are left with

$$T_2 = (\partial_+ u)^2$$

By taking a ∂_- derivative of T_2 we can write

$$2u_+\partial_{-+}u = -2u_+V'(u) = \partial_+\left(-2V(u)\right)$$

which is exactly the shape of equation (A.2) by setting $X_0 = -2V(u)$. Then we find the charge

$$Q_1 = 2\int_{-\infty}^{\infty} (\partial_+ u)^2 + V(u)\, dx$$

It worth noticing that by switching x_+ to x_- we will obtain another conserved quantities for $s \to -s$. It implies we have another charge

$$Q_{-1} = 2\int_{-\infty}^{\infty} (\partial_- u)^2 + V(u)\, dx$$

It is not straightforward to see how both Q_1 and Q_{-1} are related to standard physics quantities. Considering

$$\begin{aligned} Q_1 + Q_{-1} &= 2\int_{-\infty}^{\infty} (\partial_+ u)^2 + (\partial_- u)^2 + 2V(u)\, dx \\ &= 4\int_{-\infty}^{\infty} \frac{1}{2}(\partial_x u)^2 + \frac{1}{2}(\partial_t u)^2 + V(u)\, dx \end{aligned}$$

We can see how this quantities is related to the energy E. Also the combination $Q_1 - Q_{-1}$ are associated to the momentum P

$$P = \frac{1}{4}(Q_1 - Q_{-1})$$

- $\mathbf{s} = \mathbf{2}$: For this case T_3 should be a linear combination of $\partial_+^3 u,\ \partial_+^2 u\partial_+ u$ and $(\partial_+ u)^3$. The first two ones can be disregarded since they are total derivatives which do not contribute to the conserved charge. So we are left with the last one where the only possibility is $T_3 = (\partial_+ u)^3$. It follow from eq. (A.2) that

$$\partial_- T_3 = -3(\partial_+ u)^2 V'(u) \tag{A.5}$$

where we have used equation of motion. The right hand side of (A.5) can not be written as a x_+ total derivative because $(\partial_+ u)$ does not appears linearly. We conclude that there is no polynomially conserved charge of $s = 2$

- **s = 3** : For this case T_4 can be a linear combination of $\partial_+^4 u,\ \partial_+^2 u(\partial_+ u)^2,\ \partial_+^3 u \partial_+ u,\ (\partial_+^2 u)^2,\ (\partial_+ u)^4$. From these options only the last two are not a total derivatives and we should write a most general expression for T_4 as the linear combination

$$T_4 = (\partial_+^2 u)^2 + \frac{1}{4}\lambda^2 (\partial_+ u)^4$$

where we have normalized the coefficient of the first term to 1 and λ is constant to be determined later (the 1/4 is for convenience). Taking a ∂_- in T_4 we have

$$\begin{aligned}
\partial_- T_4 &= 2(\partial_+^2 u)(\partial_+^2 \partial_- u) + \lambda^2 (\partial_+ u)^3 (\partial_{+-} u) \\
&= -2(\partial_+^2 u)(V'(u))_+ - \lambda^2 (\partial_+ u)^3 V'(u) \\
&= \partial_+(-(\partial_+ u)^2 V''(u)) + (\partial_+ u)^3 \left(V'''(u) - \lambda^2 V'(u) \right)
\end{aligned}$$

Then we can have a polynomially conserved charge for $s = 3$ if the following equation is satisfied

$$V'''(u) - \lambda^2 V'(u) = 0 \tag{A.6}$$

where $X_2 = -(\partial_+ u)^2 V''(u)$ and Q_3 is defined as

$$Q_3 = \int_{-\infty}^{\infty} T_4 - X_2 \, dx = \int_{-\infty}^{\infty} \left((\partial_+^2 u)^2 + \frac{1}{4}\lambda^2 (\partial_+ u)^4 + (\partial_+ u)^2 V''(u) \right) dx$$

Notice that until now we have not been specific about any shape of the potential $V(u)$. As long it satisfies the boundary condition (A.4). For $s = 3$ we have a restriction and not all potentials satisfy equation (A.6).

By proposing T_{s+1} as a quantity polynomially expanded in $\partial_\pm$ and solving (A.2) seems to be hard to find conserved charge. The difficulty grows as we choose higher values of s. However, this approach is reasonable because these conservation laws matches the one predicted by Noether's Theorem. The equation (A.6) is very restrictive to what potentials V would satisfay this equation allowing us to predict which theories could have the property infinity many charge guaranteed.

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