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On Bifurcation and Symmetry of Solutions of Symmetric Nonlinear Equations with Odd-Harmonic Forcings*

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In this work we study existence, bifurcation, and symmetries of small solutions of the nonlinear equation $Lx = N(x, p, \varepsilon) + \mu f$, which is supposed to be equivariant under the action of a group OH_m, and where f is supposed to be OH_m-invariant. We assume that L is a linear operator and $N(\cdot, p, \varepsilon)$ is a nonlinear operator, both defined in a Banach space X, with values in a Banach space Z, and p, μ, and ε are small real parameters. Under certain conditions we show the existence of symmetric solutions and under additional conditions we prove that these are the only feasible solutions. Some examples of nonlinear ordinary and partial differential equations are analyzed.

1. INTRODUCTION

We are concerned with equations of the form

$$Lx = N(x, p, \varepsilon) + \mu f, \tag{1.1}$$

where X and Z are Banach spaces with $X \subset Z$; L is a linear operator and

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$N(\cdot, p, \varepsilon)$ is a nonlinear operator, both defined in X with values in Z; f is fixed element in Z; and p, μ, and ε are small real parameters.

We assume that the group $O(2) \times \mathbb{Z}_2$ acts on Z and that the operators L and N commute with the actions of the subgroup $G = D_{2m} \times \mathbb{Z}_2$ and that f is invariant under the induced actions of the subgroup $OH_m = [(R_{\pi/m}, -I), (\tau, I)]$ of G on the space Z. Here $m \geq 2$ is a fixed integer, $O(2)$ is the group generated by the planar rotations $R_\theta = \begin{pmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{pmatrix}$, $\forall\theta \in \mathbb{R}$ and by the reflexion $\tau = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$. For $k \geq 2$ integer, $\mathbb{Z}_k$ is the cyclic subgroup of order k, of the group $O(2)$, generated by $R_{2\pi/k}$, and D_k is the dihedral subgroup of $O(2)$, of order $2k$, generated by $R_{2\pi/k}$ and τ. The brackets indicate generated subgroup.

Under certain conditions, we show that, corresponding to each maximal isotropy subgroup $\tilde{\Sigma}$ of OH_m, there exists a unique subharmonic branch solution of Eq. (1.1), such that the isotropy subgroup of each solution is $\tilde{\Sigma}$. Also, under certain additional conditions, these solutions are the only feasible ones.

Similar results for Eq. (1.1), with L and N D_m-equivariant and f D_m-invariant, were considered by Galante [3] and Galante and Rodrigues [4]. Many authors have considered specific cases of Eqs. (1.1). Hale and Rodrigues [6] studied Duffing's Equations,

$$\ddot{x} + x = px + x^3 + \mu f(t) \tag{1.2}$$

with f even, 2π-periodic such that $\int_0^{2\pi} f(s) \cos s ds \neq 0$. They proved that the only 2π-*periodic* solutions of (1.2) are even functions of t. Using these results they studied in [7] the damped Duffing's Equation:

$$\ddot{x} + x = px + x^3 + \delta\dot{x} + \mu \cos t.$$

Rodrigues and Vanderbauwhede [8] generalized these results for abstract nonlinear equations in Banach spaces and more applications were provided.

Fürkotter and Rodrigues [1, 2] considered the case where f is π/m-odd-harmonic. Under a generic condition, they proved that the small 2π-periodic solutions of (1.2) maintain some symmetry properties of the forcing term f, when $\mu \neq 0$.

In Section 2 we present the hypotheses and the issues related to the action of the group $O(2) \times \mathbb{Z}_2$ and its subgroups G and OH_m, on the spaces Z, X and Ker L. We also determine the maximal isotropy subgroups of OH_m with respect to its actions on Ker L and the corresponding fixed spaces.

In Section 3, using the equivariant Liapunov–Schmidt method and implicit function theorem, we obtain the reduced complex equation and its normal form.

In Section 4 we analyze the reduced bifurcation equation. We determine the bifurcation points and show the existence of a unique branch solutions corresponding to each maximal isotropy subgroup of OH_m.

In Section 5 we show that, under certain conditions, the only nontrivial solutions of Eq. (1.1) are those whose isotropy subgroups are maximal isotropy subgroups of OH_m, relative to its actions on Ker L. We also determine a criterion for the genericity of this result.

In Section 6 we show how to apply the foregoing theory to equations of the form (1.1) with a reversal symmetry. In Section 7 we present some applications of our results to nonlinear ordinary and partial differential equations.

To our knowledge the main abstract results of this paper we new. They are stated in Theorem 4.6, Theorem 4.7, and Theorem 5.8. The interpretation of these results on the examples of Section 7 are also new.

The results of this paper indicate that when one truncates the Taylor series of the non-linearity, it is important to consider terms up to a sufficient large order. Otherwise the information in our theorems may be lost. For example, if one intends to apply our results to the pendulum equation $\ddot{v} + (g/L) \sin v = \mu \cos(mwt)$, for w close to $w_0 = \sqrt{g/L}$, one should use all the terms of the Taylor expansion of $\sin v$, around $v = 0$, up to the order v^m if m is odd, and up to the order v^{2m+1} if m is even.

2. Hypotheses and Group Actions

Throughout this paper we suppose that $m \geq 2$ is a fixed integer. We will be concerned with the equation

$$Lx = N(x, p, \varepsilon) + \mu f, \tag{2.1}$$

where $\lambda = (p, \mu, \varepsilon) \in \mathbb{R}^3$ varies in an open neighborhood Λ of the origin in $\mathbb{R}^3$, X and Z are real Banach spaces, $f \in Z$, $L \in \mathcal{L}(X, Z)$, and $N \in BC^{k+3}(B_r \times \Omega, Z)$, where $k = m$ if m is odd and $k = 2m$ if m is even. Here we indicate by B_r the open ball centered in the origin in X, Ω is an open neighborhood of the origin in $\mathbb{R}^2$, and $BC^{k+3}(B_r \times \Omega, Z)$ is the space of all functions defined in $B_r \times \Omega$ with values in Z, which have bounded and continuous derivatives up to the order $k + 3$ in $B_r \times \Omega$.

As a basic reference for notations and for some general results that will be mentioned in this section, we indicate Golubitsky [5] and Vanderbauwhede [9].

Consider the following hypotheses:

(H_1) *$L : X \mapsto Z$ is a Fredholm operator with zero index and* ker L = span$[u_1, u_2]$.

(H_2) *$N(x, p, \varepsilon) = (pA + \varepsilon B)x + Q_3(x^3) + R(x, p, \varepsilon)$, where $A, B \in \mathcal{L}(X, Z)$, $A_{|\text{Ker } L} = I_{\text{Ker } L}$, $Q_3 \in \mathcal{L}(X^3, Z)$, $R(x, p, \varepsilon) = o(|px| + |\varepsilon x| + |x|^3)$ as $(x, p, \varepsilon) \to (0, 0, 0)$, and N is odd in x.*

(H_3) *There exists a representation Γ of the group $O(2)$ on $Z : (g, x) \in O(2) \times Z \mapsto g \cdot x := \Gamma(g)x$, such that:*

(i) $\Gamma(g)X \subset X, \forall g \in O(2)$;

(ii) $\Gamma(\phi)_{|\text{Ker } L} := \Gamma(R_\phi)_{|\text{Ker } L} = R_\phi$ *and* $\Gamma(\tau)_{|\text{Ker } L} = \tau$, *where*

$$R_\phi = \begin{pmatrix} \cos\phi & -\sin\phi \\ \sin\phi & \cos\phi \end{pmatrix}, \forall \phi \in \mathbb{R}, \qquad \text{and} \qquad \tau = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}.$$

In order to describe the symmetries of (2.1), we will consider the group $O(2) \times \mathbb{Z}_2$ and their subgroups $G = D_{2m} \times \mathbb{Z}_2$ and $OH_m = [(\xi, -I), (\tau, I)] \subset G$, where $\xi = R_{\pi/m}$, and $\mathbb{Z}_2 = \{I, -I\}$ and $D_{2m} = [\xi, \tau]$ are respectively the cyclic group of order two and the dihedral group of order $4m$. The brackets indicate the generated subgroup. Let the action of $O(2) \times \mathbb{Z}_2$ on Z and X be defined by

$$(R_\phi, \pm I) \cdot w = \pm\Gamma(\phi) \cdot w, \tag{2.2a}$$

$$(\tau, \pm I) \cdot w = \pm\Gamma(\tau) \cdot w, \qquad \forall w \in Z, \tag{2.2b}$$

and the corresponding induced actions of the subgroups G and OH_m on Z and X.

(H_4) *The operators L and $N(x, p, 0)$ are $(O(2) \times \mathbb{Z}_2)$-equivariant, $N(x, p, \varepsilon)$ is G-equivariant if $\varepsilon \neq 0$, and f is OH_m-invariant.*

It follows from (H_4) that Eq. (2.1) is OH_m-equivariant. In particular, it is G-equivariant if $\mu = 0$ and $O(2) \times \mathbb{Z}_2$-equivariant if $\varepsilon = \mu = 0$.

Let Q: Ker $L \times \mathbb{R}^3 \mapsto$ Ker L be a C^{k+3}, OH_m-equivariant function. In the next section, $Q(u, \lambda) = 0$ will play the role of the bifurcation equation obtained via the Liapunov–Schmidt method.

Consider the action of D_{2m} on $\mathbb{C}$, defined by

$$R_{\pi/m} \cdot z = -e^{i\pi/m} z, \tag{2.3a}$$

$$\tau \cdot z = \bar{z}. \tag{2.3b}$$

The linear isomorphism $\mathcal{K}$: Ker $L \mapsto \mathbb{C}$, defined by $\mathcal{K}(au_1 + bu_2) = a + bi$, commutes with the induced actions (2.2a), (2.2b) of OH_m on Ker L and of D_{2m} on $\mathbb{C}$; that is, the following hold:

TABLE I

Isotropy Subgroups of D_{2m} Acting on $\mathbb{C}$ by (2.3a), (2.3b)

Orbit type	Isotropy subgroup	Fixed-space
$\{0\}$	D_{2m}	$\{0\}$
$A = \{z \neq 0 \in \mathbb{C}/\mathrm{Im}(z^m) = 0\}$	$[\tau, R_\pi]$ if m is odd	$\mathbb{R}$
	$\mathbb{Z}_2(\tau)$ if m is even	$\mathbb{R}$
$B = \{z \neq 0 \in \mathbb{C} - A/\mathrm{Im}(z^{2m}) = 0\}$	$\mathbb{Z}_2(R_\pi)$ if m is odd	$\mathbb{C}$
	$\mathbb{Z}_2(R_{(m+1)\pi/m} \circ \tau)$, if m is even	$\mathbb{R}\{e^{i\pi/2m}\}$
$C = \{z \in \mathbb{C}/z \neq 0 \text{ and } \mathrm{Im}(z^{2m}) \neq 0\}$	$Z_2(R_\pi)$ if m is odd	$\mathbb{C}$
	$\{I\}$ if m is even	$\mathbb{C}$

$$-e^{i\pi/m} \cdot \mathscr{X}(u) = \mathscr{X}((R_{\pi/m}, -I)u) \tag{2.4a}$$

$$\overline{\mathscr{X}(u)} = \mathscr{X}(\overline{u}). \tag{2.4b}$$

Let $M(z, \lambda) := \mathscr{X}Q(\chi^{-1}z, \lambda): \mathbb{C} \times \mathbb{R}^3 \mapsto \mathbb{C}$. The following lemma holds:

LEMMA 2.1. *Under the above assumptions, $M(\cdot, \lambda)$ is a C^{k+3}, D_{2m}-equivariant function, relative to action of D_{2m} on $\mathbb{C}$ defined by* (2.3a), (2.3b).

We remark that the order of element $(\xi, -I) = (R_{\pi/m}, -I) \in OH_m$ is $2m$. Moreover, $(\tau, I)^2 = (I, I)$ and then OH_m is isomorphic to D_{2m}. The group isomorphism F: $OH_m \mapsto D_{2m}$ is defined by

$$F(\xi, -I) = R_{\pi/m}, \tag{2.5a}$$

$$F(\tau, I) = \tau. \tag{2.5b}$$

In Table I we give the isotropy subgroups of D_{2m}, acting on $\mathbb{C}$ by (2.3a), (2.3b). They are determined following the ideas of Golubitsky [5].

The maximal isotropy subgroups of D_{2m} are the ones that belong to the conjugacy class:

$$\widehat{[\tau, R_\pi]} = \left\{\sum_j = [R_{j2\pi/m} \circ \tau, R_\pi], j = 0 \cdots m - 1\right\}$$

when m is odd,

$$\widehat{\mathbb{Z}_2(\tau)} = \{E_j = \mathbb{Z}_2(R_{j2\pi/m} \circ \tau), j = 0 \cdots m - 1\},$$

and

$$\widehat{\mathbb{Z}_2(R_{\pi+\pi/m} \circ \tau)} = \{D_j = \mathbb{Z}_2(R_{\pi+(2j+1)\pi/m} \circ \tau), j = 0 \cdots m - 1\}$$

when m is even.

Using the isomorphism (2.5a), (2.5b) we determine the conjugacy class of the corresponding maximal isotropy subgroups of OH_m, acting on Ker L by the induced action (2.2a), (2.2b). They are given by

$$\widehat{[(\tau, I), (R_\pi, -I)]} = \left\{\tilde{\sum_j} = [(R_{j2\pi/m} \circ \tau, I), (R_\pi, -I)], j = 0, ..., m - 1\right\}$$

when m is odd;

$$\widehat{\mathbb{Z}_2(\tau, I)} = \{\tilde{E}_j = \mathbb{Z}_2(R_{j2\pi/m} \circ \tau, I), j = 0, ..., m - 1\},$$

and

$$\widehat{\mathbb{Z}_2(R_{\pi+\pi/m} \circ \tau, -I)} = \{\tilde{D}_j = \mathbb{Z}_2(R_{\pi+(2j+1)\pi/m} \circ \tau, -I), j = 0, ..., m - 1\}$$

when m is even.

The corresponding fixed-spaces on Ker L are

$$\mathrm{Fix}(\tilde{\Sigma}_j) = \mathrm{span}\{(R_{j\pi/m}, I)u_1\},$$

$$\mathrm{Fix}(\tilde{E}_j) = \mathrm{span}\{(R_{j\pi/m}, I)u_1\},$$

and

$$\mathrm{Fix}(\tilde{D}_j) = \mathrm{span}\{(R_{j\pi/m+\pi/2m}, I)u_1\},$$

$j = 0, ..., m - 1$.

In Section 6 we will consider Eq. (2.1) having symmetries which are described by the subgroup $\widetilde{OH}_m = [(\xi, -I), (\tau, -I)]$ of $O(2) \times \mathbb{Z}_2$. This subgroup is isomorphic to D_{2m} and will be considered to be acting on Z by the induced action (2.2a), (2.2b) and OH_m will be acting on X.

We suppose that Eq. (2.1) is $(\widetilde{OH}_m, OH_m)$-equivarient. Under the above conditions, the reduced Liapunov–Schmidt function $\tilde{Q}(u, \lambda)$: Ker $L \times \mathbb{R}^3 \mapsto$ Ker L will be $(\widetilde{OH}_m, OH_m)$-equivariant.

The linear isomorphism q: Ker $L \mapsto$ Ker L, defined by $q(au_1 + bu_2) = -au_2 + bu_1$, commutes with the actions of OH_m on X and with those of

$\widetilde{OH}_m$ on Z; that is, q is an $(OH_m, \widetilde{OH}_m)$-isomorphism of Ker L. It follows that $q \circ \tilde{Q}$: Ker $L \times \mathbb{R}^3 \mapsto$ Ker L is OH_m-equivariant.

From Lemma 1.2 we have

LEMMA 2.2. *If $\tilde{Q}(u, \lambda)$: Ker $L \times \mathbb{R}^3 \mapsto$ Ker L is $(\widetilde{OH}_m, OH_m)$-equivariant, then the function $M(z, \lambda) := \mathscr{X}(q \circ \tilde{Q})(\mathscr{X}^{-1}z, \lambda)$: $\mathbb{C} \times \mathbb{R}^3 \mapsto \mathbb{C}$ is D_{2m}-equivariant, relative to action* (2.3a)(2.3b) *of D_{2m} on $\mathbb{C}$.*

3. Liapunov–Schmidt Reduction and Normal Form

Under Hypotheses (H_1)–(H_4), there exists a continuous projection $P \in \mathscr{L}(Z)$, such that

$$X = \text{Ker } L \oplus (I - P)X,$$

$$Z = \text{Ker } L \oplus \mathscr{R}(L).$$

Without loss of generality, we can assume that P commutes with the action of $O(2)$ on Z.

The operator L restricted to $(I - P)X$ space has continuous inverse K: $\mathscr{R}(L) \mapsto (I - P)X$, satisfying $K(Lx) = (I - P)x \ \forall x \in X$ and $LK(I - P)w = (I - P)w \ \forall w \in \mathbb{Z}$.

By the equivariant Liapunov–Schmidt method, Vanderbauwhede [9], the equation (2.1) is reduced to the system:

$$v = K(I - P)[N(u + v, p, \varepsilon) + \mu f], \tag{3.1a}$$

$$0 = PN(u + v, p, \varepsilon), \tag{3.1b}$$

where $x = u + v \in$ Ker $L \oplus (I - P)X$.

From the implicit function theorem, Eq. (3.1a) defines a smooth function $v = v^*(u, \lambda)$ defined in a small neighborhood $U \times \Lambda_1 \subset$ Ker $L \times \Omega$ valued in a neighborhood $V \subset (I - P)X$, such that $v^*(0, 0) = 0$. The neighborhoods U and V can be chosen in such a way that $g \cdot U = U, g \cdot V = V, \forall g \in OH_m$, and the function $v^*(u, \lambda)$ is OH_m-equivariant.

If we substitute $v = v^*(u, \lambda)$ in (3.1b) we obtain the bifurcation equation

$$Q(u, \lambda) := PN(u + v^*(u, \lambda), p, \varepsilon) = 0, \tag{3.2}$$

where Q: $U \times \Lambda_1 \subset$ Ker $L \times \Omega \mapsto$ Ker L is OH_m-equivariant.

It follows from Lemma 2.1 and formula (3.2) that

$$M(z, \lambda) := \mathscr{X}PN(\mathscr{X}^{-1}z + v^*(\mathscr{X}^{-1}z, \lambda), p, \varepsilon) = \mathscr{X}Q(\mathscr{X}^{-1}z, \lambda): \mathbb{C} \times \mathbb{R}^3 \mapsto \mathbb{C}$$

is a C^{k+3}- and D_{2m}-equivariant function, relative to the action of D_{2m} on $\mathbb{C}$ defined by (2.3a), (2.3b).

In the next lemmas we determine the general form of a C^k function $M(z)$, defined in a neighborhood of origin in $\mathbb{C}$ and valued in $\mathbb{C}$, which is D_{2m}-equivariant relative to the above-mentioned action of D_{2m} on $\mathbb{C}$.

In the following we will denote by B_r^k the open ball of radius r, centered in the origin, in $\mathbb{R}^k$.

LEMMA 3.1. *Let* $0 < \varepsilon < r$, $H \in BC^2(B_r^{m+n}, \mathbb{R})$, *and* $f \in C^1(B_\varepsilon^n, \mathbb{R}^m)$ *with* $f(0) = 0$ *and such that*

$$\mathrm{Gr}(f) = \{(x, f(x))/x \in B_\varepsilon^n\} \subset H^{-1}(\{0\}).$$

Then there exists $\tilde{H}$: $B_\varepsilon^{m+n} \mapsto \mathcal{L}(\mathbb{R}^m, \mathbb{R})$ *of class* C^1, *such that*

$$H(x, y) = \tilde{H}(x, y)(y - f(x)), \qquad \forall (x, y) \in B_\varepsilon^{m+n}.$$

Proof. The stated results follow essentially from the equalities:

$$\begin{aligned} H(x, y) &= \int_0^1 \frac{d}{d\theta} H(x, f(x) + \theta(y - f(x)))\, d\theta \\ &= \int_0^1 \frac{\partial H}{\partial y}(x, f(x) + \theta(y - f(x)))\, d\theta \cdot [y - f(x)]. \end{aligned}$$

COROLLARY 3.2. *If* $H \in BC^2(B_r^2, \mathbb{R})$ *and* $\varphi \in \mathcal{L}(\mathbb{R}^2, \mathbb{R})$ *is nontrivial and satisfies* $\ker \varphi \cap B_r^2 \subset H^{-1}(\{0\})$, *then there exist* $0 < \varepsilon < r$ *and* $\tilde{H} \in BC^1(B_\varepsilon^2, \mathbb{R})$ *such that*

$$H(x, y) = \tilde{H}(x, y) \cdot \varphi(x, y), \qquad \forall (x, y) \in B_\varepsilon^2.$$

LEMMA 3.3. *If* $M(z)$: $B_r^2 \subset \mathbb{C} \mapsto \mathbb{C}$ *is a* D_{2m}*-equivariant function of class* C^{k+3}, *relative to action* (2.3a), (2.3b), *then there exist real functions* f *and* g *of class* C^3 *and* D_k*-invariant, defined in* B_ε^2 *with* $0 < \varepsilon < r$, *such that*

$$M(z) = f(z)z + g(z)(\bar{z})^{k-1}, \qquad \forall z \in B, \tag{3.3}$$

where $k = m$ *if* m *is odd and* $k = 2m$ *if* m *is even.*

Proof. Let $H(z) \stackrel{\text{def}}{=} \mathrm{Im}(M(z)\bar{z})$. From the D_{2m}-equivariance of $M(z)$, the function $H(z)$ satisfies

$$H(-e^{i\pi/m}) = H(z) \tag{3.4a}$$

$$H(\bar{z}) = -H(z). \tag{3.4b}$$

From relations (3.4a), (3.4b), $H(z)$ vanishes on the spaces $\operatorname{span}\{e^{i(j\pi/m)}\}$, $j = 0, \ldots, m-1$, if m is odd and on the spaces $\operatorname{span}\{e^{j\pi/2m}\}$, $j = 0, \ldots, 2m-1$, if m is even.

It follows that $H(z)$ vanish on the set $\{z \in \mathbb{C}/\operatorname{Im}(z)^k = 0\}$, where $k = m$ if m is odd and $k = 2m$ if m is even. Then $H(z)$ vanish on the lines $\operatorname{Ker} \phi_j$ where ϕ_j: $\mathbb{R}^2 \mapsto R$ is the linear functional:

$$\phi_j(x, y) = \begin{cases} \sin(j\pi/m)x - \cos(j\pi/m)y, j = 0, \ldots, m-1 & \text{if } m \text{ is odd,} \\ \sin(j\pi/2m)x - \cos(j\pi/2m)y, j = 0, \ldots, 2m-1 & \text{if } m \text{ is even.} \end{cases}$$

From Corollary 3.2 it follows that, if φ: $\mathbb{R}^2 \mapsto \mathbb{R}$ is a nontrivial linear functional and h: $B_r^2 \mapsto R$ is a smooth function vanishing on Ker L, then there exists a ball $B_{\bar{\varepsilon}}^2$ with $0 < \bar{\varepsilon} < r$ and a smooth function $\tilde{h}$: $B_{\varepsilon}^2 \mapsto \mathbb{R}$ such that $h(x, y) = \phi(x, y)\, \tilde{h}(x, y)$.

Applying these results k-times on the function $H(z) = H(x, y)$, we find a function $\tilde{H}$ defined in a ball $B_{\varepsilon_1}^2$, with $0 < \varepsilon_1 < r$, such that

$$H(x, y) = \prod_{j=0}^{k-1} \phi_j(x, y)\tilde{H}(x, y).$$

The functions $p(z) \overset{\text{def}}{=} \prod_{j=0}^{k-1} \phi_j(z)$ and $w(z) \overset{\text{def}}{=} \operatorname{Im}(\bar{z})^k$ are homogeneous of degree k. By the foregoing procedure there exists $\tilde{w}$: $\mathbb{R}^2 \mapsto \mathbb{R}$ such that $w(z) = p(z)\tilde{w}(z)$. Then $\tilde{w}(z)$ is constant and

$$H(x, y) = g(x, y) \operatorname{Im}(\bar{z})^k \tag{3.5}$$

where $g(x, y) = c\tilde{H}(x, y)$. From (3.4a), (3.4b) the function $g(z)$ is D_{2m}-invariant.

Now let

$$F(z) := M(z) - g(z)(\bar{z})^{k-1} = W_1(z) + W_2(z)i. \tag{3.6}$$

From (3.6) we have:

$$\operatorname{Im}(F(z) \cdot \bar{z}) = (W_1(z) + iW_2(z))(x - iy) = W_2(z)x - W_1(z)y = 0. \tag{3.7}$$

This implies that $W_1(z)$ vanishes on Ker ϕ where ϕ is the linear functional $\phi(x, y) = x$. Then there exists a ball $B^2_{\varepsilon_2}$, with $0 < \varepsilon_2 < r$, and a function $f: B^2_{\varepsilon_2} \mapsto \mathbb{R}$ smooth, such that $W_1(z) = xf(z)$.

By (3.7) we have $W_2(z) = yf(z)$ and then $F(z) = f(z)z$. If we let $\varepsilon = \min\{\varepsilon_1, \varepsilon_2\}$, it follows from (3.6) that

$$M(z) = f(z)z + g(z)(\bar{z})^{k-1}, \qquad \forall z \in B^2_\varepsilon.$$

Since $F(z)$ is D_k-invariant it follows from (3.6) that $f(z)$ is D_k-invariant.

4. Analysis of the Bifurcation Equation - Main Results

In the following we determine subharmonic branches of solutions of Eq. (2.1), whose symmetries are given by maximal isotropy subgroups of OH_m. We summarize our main results in Theorems 4.6 and 4.7.

The branch solutions of Eq. (2.1) and maximal isotropy subgroups of OH_m are related to the branch solutions of reduced equation $M(z, \lambda) = 0$ and maximal isotropy subgroups of D_{2m}, by the Liapunov–Schmidt method and by the isomorphism $\mathscr{X}$: Ker $L \mapsto \mathbb{C}$ defined in Section 2.

Therefore, we will analyze the restriction of the equation $M(z, \lambda) = 0$ to the fixed spaces Fix(Σ), where Σ is a maximal isotropy subgroup of D_{2m}.

In Lemma 4.1 we determine the Taylor expansion of the restriction of $M(z, \lambda)$ to Fix$(\Sigma) \times \mathbb{R}^3$ near $(z, \lambda) = (0, 0)$. In Lemma 4.2 we prove the existence of a set of singular points to $M(z, \lambda)_{|\mathrm{Fix}(\Sigma)\times\mathbb{R}^3}$; in Lemma 4.4 we prove the existence of a bifurcation surface and determine branches of solutions of $M(z, \lambda)_{|\mathrm{Fix}(\Sigma)\times\mathbb{R}^3}$ whose isotropy subgroups are maximal isotropy subgroups of OH_m, relative to its actions on Ker L.

Let Σ be a maximal isotropy subgroup of D_{2m}, acting on $\mathbb{C}$ by (2.3a), (2.3b). From Table I we have dim Fix$(\Sigma) = 1$.

Let Fix$(\Sigma) = $ span$\{z_0\}$ and suppose $|z_0| = 1$.

Lemma 4.1. *Under hypotheses* (H_1)–(H_4), *the reduced equation* $M(z, \lambda)$ *restricted to the space* Fix$(\Sigma) \times R^3$ *has the representation*

$$M(rz_0, \lambda) = [r(p + a\varepsilon + b\mu^2 + cr^2 + dr\mu + \cdots)]z_0,$$

where $a = \mathscr{X}PBu_1$, $b = 3\mathscr{X}PQ_3(u_1, Kf^2)$, $c = \mathscr{X}PQ_3(u_1^3)$, *and*

$$d = \begin{cases} 0 & \text{if } m \neq 3, \\ 3\mathscr{X}PQ_3(u_1^2, Kf) & \text{if } m = 3 \text{ and } z_0 \in \{1, e^{i2\pi/3}\}, \\ -3\mathscr{X}PQ_3(u_1^2, Kf) & \text{if } m = 3 \text{ and } z_0 = e^{i\pi/3}. \end{cases}$$

Proof. The stated results follow from the invariance property $M(\mathrm{Fix}(\Sigma) \times \mathbb{R}^3) \subset \mathrm{Fix}(\Sigma)$ using Taylor's expansion.

From Lemma 4.1 we have

$$M(rz_0, \lambda) = [rJ(rz_0, \lambda)]z_0, \tag{4.1}$$

where

$$J(rz_0, \lambda) = p + a\varepsilon + cr^2 + b\mu^2 + dr\mu + \cdots. \tag{4.2}$$

Consider the system of equations $J(rz_0, \lambda) = 0$, $J_r(rz_0, \lambda) = 0$. Since $J(rz_0, \lambda)$ and $J_r(rz_0, \lambda)$ vanish and $\det(\partial(J, J_r)/\partial(r, p)) = 2c \neq 0$, for $r = 0$, $\lambda = 0$, we obtain via implicit function theorem that $r = r(\varepsilon, \mu)$ and $p = p(\varepsilon, u)$, in a neighborhood B of origin.

Some calculations show that $p = -a\varepsilon + (d^2/4c - b)\mu^2 + \cdots$ and $r = (d/2c)\mu + \cdots$. We have proved the following:

LEMMA 4.2. *Suppose hypotheses* (H_1)–(H_4) *and* $c \neq 0$ *are satisfied. Then the real functions* $p = p(\varepsilon, \mu) = -a\varepsilon + (d^2/4c - b)\mu^2 + \cdots$ *and* $r = r(\varepsilon, \mu) = (d/2c)\mu + \cdots$, *defined in a small neighborhood B of origin in* $\mathbb{R}^2$, *are such that the points* $\{(r(\varepsilon, \mu)z_0, \lambda(\varepsilon, \mu)) : (\varepsilon, \mu) \in B\}$, *where* $\lambda(\varepsilon, \mu) = (p(\varepsilon, \mu), \varepsilon, \mu)$, *are singular points of the restriction* $M(rz_0, \lambda)$ *of* $M(z, \lambda)$ *to* $\mathrm{Fix}(\Sigma) \times \mathbb{R}^3$.

Remark 4.3. From Lemma 3.3 we have

$$J(rz_0, \lambda) = \begin{cases} f(rz_0, \lambda) + g(rz_0, \lambda)r^{m-2}\,\mathrm{Re}(\bar{z}_0^m) & \text{if } m \text{ is odd,} \\ f(rz_0, \lambda) + g(rz_0, \lambda)r^{2m-2}\,\mathrm{Re}(\bar{z}_0^{2m}) & \text{if } m \text{ is even.} \end{cases} \tag{4.3}$$

Differentiating (4.3) and using the D_{2m}-invariance of f and g we obtain $f_r(0, \lambda) = 0$ and $g_r(0, \lambda) = 0$. Therefore $r = 0$ solves $J_r(rz_0, \lambda) = 0$ when $m \neq 3$ and so $r(\varepsilon, \mu) = 0$ when $m \neq 3$.

LEMMA 4.4. *Suppose Hypotheses* (H_1)–(H_4) *and* $c \neq 0$ *are satisfied and* $(\varepsilon_0, \mu_0) \in B$. *For each maximal isotropy subgroup Σ of D_{2m} there exists a unique branch solution of* $M(z, p, \varepsilon_0, \mu_0) = 0$, *such that the isotropy subgroup of each nontrivial solution is Σ. The solutions of the branch bifurcate from the solution* $(r_0 z_0, \lambda_0)$, *where* $r_0 = r(\varepsilon_0, \mu_0)$, $\lambda_0 = \lambda(\varepsilon_0, \mu_0)$, *and* $\mathrm{Fix}\{\Sigma\} = \mathrm{span}\{z_0\}$.

Proof. Corresponding to (ε_0, μ_0) we have the singular point (r_0, λ_0) of $M(rz_0, \lambda)$, where $r_0 = r(\varepsilon_0, \mu_0)$ and $\lambda_0 = (p(\varepsilon_0, \mu_0), \varepsilon_0, \mu_0)$. In particular, we have $J(r_0z_0, \lambda_0) = J_r(r_0z_0, \lambda_0) = 0$. The Taylor expansion of $M(rz_0, p, \varepsilon_0, \mu_0)$ in a small neighborhood of (r_0, p_0) is given by

$$M(rz_0, p, \varepsilon_0, \mu_0) = \begin{cases} r[J_p(0, \lambda_0)(p - p_0) + \frac{1}{2}J_{rr}(0, \lambda_0)r^2 + \cdots]z_0 & \text{if } m \neq 3, \\ r_0[J_p(r_0z_0, \lambda_0)(p - p_0) + \frac{1}{2}J_{rr}(r_0, \lambda_0)(r - r_0)^2 + \cdots]z_0, & \text{if } m = 3, \end{cases} \tag{4.4}$$

where $\cdots$ are higher order terms.

Since $J_p(rz_0, \lambda) = 1$ when $r = 0$, $\lambda = 0$, it follows that $J_p(rz_0, \lambda_0) \neq 0$ when r_0 and λ_0 are small. In particular, $J_p(0, \lambda_0) \neq 0$ when λ_0 is small.

It follows from the implicit function theorem that the equation $M(rz_0, p, \varepsilon_0, \mu_0) = 0$ defines a unique function $p = p^*(r)$, in a small neighborhood of (r_0, p_0) such that $M(rz_0, p(r), \varepsilon_0, \mu_0) = 0$. From (4.4) we have:

$$p = p^*(r) = \begin{cases} p_0 + \dfrac{J_{rr}(0, \lambda_0)}{2J_p(0, \lambda_0)} r^2 + \cdots & \text{if } m \neq 3, \\ p_0 + \dfrac{J_{rr}(r_0, \lambda_0)}{2J_p(r_0, \lambda_0)} (r - r_0)^2 + \cdots & \text{if } m = 3. \end{cases}$$

The branch $(rz_0, \lambda(r))$, where $\lambda(r) := (p^*(r), \varepsilon_0, \mu_0)$, lies in $\mathrm{Fix}(\Sigma) \times \mathbb{R}^3$ and so $\Sigma_{rz_0} = \Sigma$ when $r \neq 0$.

From (4.4) it follows that (r_0z_0, λ_0) is a bifurcation point of $M(rz_0, p, \varepsilon_0, \mu_0)$. If $c < 0$, the number of nontrivial solutions of $M(z, p, \varepsilon_0, \mu_0)$ in $\mathrm{Fix}(\Sigma)$, when $m \neq 3$, is zero for $p \leq p_0$ and two for $p > p_0$. If $m = 3$ and $d \neq 0$ we have $r_0 \neq 0$ and so there are no nontrivial solutions in $\mathrm{Fix}(\Sigma)$ for $p < p_0$, one for $p = p_0$, and two for $p > p_0$.

A similar result holds if $c > 0$.

Remark 4.5. From formula (4.1) we see that $(0, \lambda)$ is a solution of $M(rz_0, \lambda) = 0$ for every λ. This solution will be called *the trivial solution.* Note that, if $m \neq 3$, the branch contains the trivial solution $(0, \lambda(0))$, but if $m = 3$ and $d \neq 0$ then $r_0 \neq 0$, and therefore the branch $(rz_0, \lambda(r))$, for r in a small neighborhood of r_0, does not contain trivial solutions.

The Liapunov–Schmidt method establishes a one-to-one (local) correspondence between solutions of Eq. (2.1) and solutions of the reduced equation $M(z, \lambda) = 0$.

The linear isomorphism $\mathcal{X}(au_1 + bu_2) = a + bi$: $\mathrm{Ker}\, L \mapsto \mathbb{C}$ and the isomorphism group F: $OH_m \mapsto D_{2m}$ defined in (2.5a), (2.5b) preserves the symmetric properties of the solutions.

If we let $z = re^{i\phi}$, where $r \in \mathbb{R}$ and $\phi \in [0, \pi]$, we have

$$u = \mathscr{X}^{-1}z = r\cos\phi u_1 + r\sin\phi u_2 = r\Gamma(\phi)u_1 = (R_\phi, I)ru_1.$$

Suppose that $(z, \lambda) \in \mathrm{Fix}(\Sigma) \times \mathbb{R}^3$ is solution of $M(z, \lambda) = 0$. In this case, the corresponding solution of (2.1) is given by

$$\begin{aligned} x(re^{i\phi}, \lambda) &= \mathscr{X}^{-1}(re^{i\phi}) + v^*(\mathscr{X}^{-1}re^{i\phi}, \lambda) \\ &= (R_\phi, I)ru_1 + v^*((R_\phi, I)ru_1, \lambda) \end{aligned}$$

and its isotropy subgroup is $\tilde{\Sigma} = F^{-1}(\Sigma) = \tilde{\Sigma}_\nu \subset OH_m$, where $\nu = (R_\phi, I)u_1 \in \mathrm{Ker}\, L$.

Fixing $(\varepsilon_0, \mu_0) \in B$, the branch solutions of $M(z, p, \varepsilon_0, \mu_0) = 0$, corresponding to the isotropy subgroups

$$\sum_{e^{ij\pi/m}} = \begin{cases} [R_{2j\pi/m} \circ \tau, R_\pi] & \text{if } m \text{ is odd,} \\ \mathbb{Z}_2(R_{2j\pi/m} \circ \tau) & \text{if } m \text{ is even} \end{cases}$$

and

$$\sum_{e^{i(\pi/2m + j(\pi/m))}} = \mathbb{Z}_2(R_{\pi+(2j+1)(\pi/m)} \circ \tau) \qquad \text{if } m \text{ is even,}$$

are respectively $(re^{ij\pi/m}, \lambda(r))$ and $(re^{i(\pi/2m + j\pi/m)}, \lambda(r))$, $j = 0, \ldots, m-1$, where $\lambda(r) = (p^*(r), \varepsilon_0, \mu_0)$.

It follows that

$$x(re^{(ij\pi/m)}, \lambda(r)) = (R_{j\pi/m}, I)ru_1 + v^*((R_{j\pi/m}, I)ru_1, \lambda(r)),$$

and

$$x(re^{i(\pi/2m + j\pi/m)}, \lambda(r)) = (R_{\pi/2m + j\pi/m}, I)ru_1 + v^*((R_{\pi/2m+j\pi/m}, I)ru_1, \lambda(r))$$

are the branches solutions of Eq. (2.1), corresponding respectively to the maximal isotropy subgroups of OH_m:

$$\tilde{\sum_{(R_{j\pi/m}, -I)u_1}} = \begin{cases} [(R_{2j\pi/m} \circ \tau, I), (R_\pi, -I)] & \text{if } m \text{ is odd,} \\ \mathbb{Z}_2((R_{2j\pi/m} \circ \tau, I)) & \text{if } m \text{ is even,} \end{cases}$$

and

$$\tilde{\sum_{(R_{\pi/2m+j\pi/m},-I)u_1}} = \mathbb{Z}_2((R_{\pi+(2j+1)\pi/m} \circ \tau, -I)) \qquad \text{if } m \text{ is even,}$$

$j = 0, ..., m - 1$.

Therefore, if we consider Eq. (2.1) with m even, $\varepsilon = \varepsilon_0$, $\mu = \mu_0$, where $(\varepsilon_0, \mu_0) \in B$, there exist $2m$ branches solutions, namely, m whose isotropy subgroup of each nontrivial solution is in the conjugacy class $\widehat{\mathbb{Z}_2((\tau, I))}$ and m whose isotropy subgroup of each nontrivial solution is in the conjugacy class $\mathbb{Z}_2((R_{(m+1)\pi/m} \circ \tau, -I))$.

If m is odd, there exist only m branches of solutions of Eq. (2.1), whose isotropy subgroup of each nontrivial solution is in the conjugacy class $\widehat{[(\tau, I), (R_\pi, -I)]}$.

Suppose $c < 0$ and $\delta > 0$. Let $\lambda = (\tilde{p}, \varepsilon_o, \mu_0)$, where $\tilde{p} \in (p_0 - \delta, p_0 + \delta)$, $(\varepsilon_0, \mu_0) \in B$, and $p_0 = p^*(\varepsilon_0, \mu_0)$.

Let m be even and $\tilde{p} > p_0$. Since we have $2m$ maximal isotropy subgroups, it follows from Lemma 4.4 that Eq. (2.1) with $\lambda = (\tilde{p}, \varepsilon_0, \mu_0)$ has $4m + 1$ solutions, namely, $2m$ whose isotropy subgroup is in the conjugacy class $\widehat{\mathbb{Z}_2((\tau, I))}$, $2m$ whose isotropy subgroup is in the conjugacy class $\widehat{\mathbb{Z}_2((R_{(m+1)\pi/m} \circ \tau, -I))}$, and one OH_m-invariant. If $\tilde{p} \leq p_0$ the equation (2.1) has only the trivial solution $x(0, \tilde{p}, \varepsilon_0, \mu_0)$, which is OH_m-invariant.

If $m = 3$ and $\tilde{p} < p_0$ we have only the trivial solution $x(0, \tilde{p}, \varepsilon_0, \mu_0)$. If $\tilde{p} = p_0$ we have four solutions, one trivial and one nontrivial in each branch. If $\tilde{p} > p_0$ we have seven solutions, one trivial and two nontrivial in each branch.

In a similar way if $m \neq 3$ is odd and $\tilde{p} > p_0$ we have $2m$ subharmonic solutions whose isotropy subgroup of each nontrivial solution is in the class $\widehat{[(\tau, I), (R_\pi, -I)]}$ and one trivial solution OH_m-invariant. If $\tilde{p} \leq p_0$ we have only the trivial solution.

We summarize some of our main results in the next two theorems:

THEOREM 4.6. *If Eq.* (2.1) *verifies hypotheses* (H_1)–(H_4) *with* $c \neq 0$, *then there exists a small neighborhood B of origin in* $\mathbb{R}^2$ *such that*:

(a) *The points* $\{(r(\varepsilon, \mu)z_0, \lambda(\varepsilon, \mu))/(\varepsilon, \mu) \in B\}$, *where* $r(\varepsilon, \mu)$ *and* $\lambda(\varepsilon, \mu)$ *are defined in Lemma* 4.2, *are bifurcations points of Eq.* (2.1);

(b) *Let* (ε_0, μ_0) *be fixed in B and* $\tilde{\Sigma}$ *a maximal isotropy subgroup of* OH_m *relative to its actions on* Ker L. *Corresponding to* $\tilde{\Sigma}$, *there exists a unique branch of subharmonic solutions of Eq.* (2.1), *such that the isotropy subgroup of each nontrivial solution is* $\tilde{\Sigma}$. *The solutions of the branch bifurcate from the solution* $x(r_0z_0, \lambda_0)$, *where* $\lambda_0 = (p_0, \varepsilon_0, \mu_0)$,

$$r_0 = r(\varepsilon_0, \mu_0), \qquad p_0 = p(\varepsilon_0, \mu_0), \qquad \textit{and} \qquad \mathrm{Fix}\{\tilde{\Sigma}\} = \mathrm{span}\{\mathcal{X}^{-1}z_0\}.$$

THEOREM 4.7. *Let* $\delta > 0$, $\lambda = (\tilde{p}, \varepsilon_0, \mu_0) \in (p_0 - \delta, p_0 + \delta) \times B$, $r_0 = (\varepsilon_0, \mu_0)$, $p_0 = p(\varepsilon_0, \mu_0)$, *and* $\lambda(r) = (p^*(r), \varepsilon_0, \mu_0)$, *where* $p = p^*(r)$ *is the function defined in Lemma* 4.4. *Under hypotheses* (H_1)–(H_4) *and* $c \neq 0$, *we have*:

(a) *If m is odd the maximal isotropy subgroups of OH_m relative to its action on* Ker L *are* $\widehat{[(\tau, I), (R_\pi, -I)]} = \{\tilde{\Sigma}_j = [(R_{2j\pi/m} \circ \tau, I), (R_\pi, -I)], j = 0, ..., m-1\}$. *The branch corresponding to $\tilde{\Sigma}_j$ indicated in Theorem* 4.6 *is* $x(re^{i(j\pi/m)}, \lambda(r)) = (R_{2\pi/m}, -I)ru_1 + v^*((R_{2\pi/m}, -I)ru_1, \lambda(r)), j = 0, ..., m-1$.

(b) *If m is even the maximal isotropy subgroups of OH_m are* $\widehat{\mathbb{Z}_2(\tau, I)} = \{\tilde{E}_j = ((R_{2j\pi/m} \circ \tau, I)), j = 0, ..., m-1\}$ *and* $\widehat{\mathbb{Z}_2((R_{\pi+\pi/m} \circ \tau, -I))} = \{\tilde{D}_j = \mathbb{Z}_2(R_{\pi+(2j+1)\pi/m} \circ \tau, -I), j = 0, ..., m-1\}$. *The branches corresponding to $\tilde{E}_j$ and $\tilde{D}_j$ are respectively*

$$x(re^{i(j\pi/m)}, \lambda(r)) = (R_{j\pi/m}, I)ru_1 + v^*((R_{j\pi/m}, I)ru_1, \lambda(r))$$

and

$$x(re^{i(\pi/2m+j\pi/m)}, \lambda(r)) = (R_{\pi/2m+j\pi/m}, I)ru_1 + v^*((R_{\pi/2m+j\pi/m}, I)ru_1, \lambda(r)),$$

$j = 0, ..., m-1$.

(c) *Let* $c < 0$ *and* $n(\lambda, \tilde{\Sigma})$ *be the number of solutions of Eq.* (2.1) *with* $\lambda = (\tilde{p}, \varepsilon_0, \mu_0)$, *whose isotropy subgroup is $\tilde{\Sigma}$ or OH_m, where $\tilde{\Sigma}$ is a maximal isotropy subgroup of OH_m. We have*

$$\sum_{j=0}^{m-1} n(\lambda, \tilde{\Sigma}_j) = \begin{cases} 1, & \text{if } \tilde{p} \leq p_0, \\ 2m+1, & \text{if } \tilde{p} > p_0 \end{cases} \qquad \text{for } m \neq 3 \text{ odd},$$

$$\sum_{j=0}^{m-1} n(\lambda, \tilde{E}_j) + \sum_{j=0}^{m-1} n(\lambda, \tilde{D}_j) = \begin{cases} 1, & \text{if } \tilde{p} \leq p_0, \\ 4m+1, & \text{if } \tilde{p} > p_0 \end{cases} \qquad \text{for } m \text{ even},$$

and

$$\sum_{j=0}^{2} n(\lambda, \tilde{\Sigma}_j) = \begin{cases} 1, & \text{if } \tilde{p} < p_0, \\ 4, & \text{if } \tilde{p} = p_0, \\ 7, & \text{if } \tilde{p} > p_0 \end{cases} \qquad \text{when } m = 3.$$

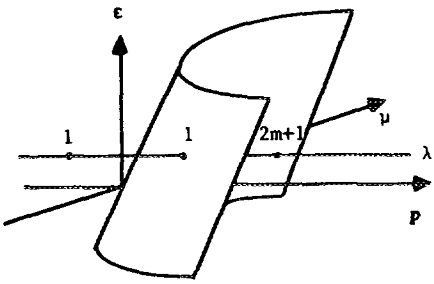

FIG 1. $m > 3$ odd.

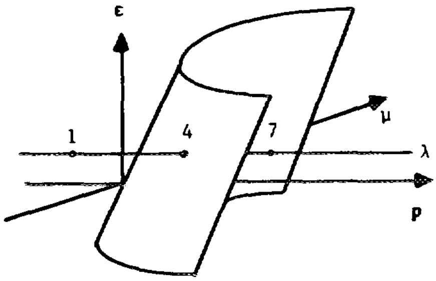

FIG 2. $m = 3$.

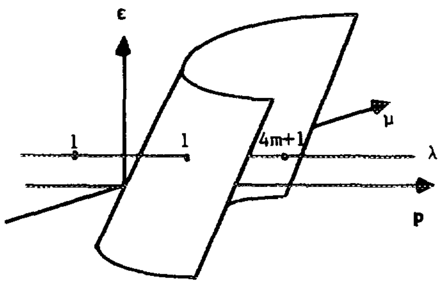

FIG 3. $m \geq 2$ even.

The total number of solutions of Eq. (2.1) with $\lambda = (\tilde{p}, \varepsilon_0, \mu_0)$, whose isotropy subgroup is OH_m or which is a maximal isotropy subgroup of OH_m, are shown in Fig. 1–3, with $a < 0$ and $(d^2/4c - b)) > 0$.

5. GENERICITY FOR "AUTONOMOUS" NONLINEARITIES

In the following, we consider Eq. (2.1) with hypotheses (H_1)–(H_4) and $\varepsilon = 0$. We see that in this case the operators $N(x, p) := N(x, p, 0)$ and L are $(O(2) \times \mathbb{Z}_2)$-equivariant.

From our previous results, for each fixed μ_0 sufficiently small and for each maximal isotropy subgroup $\tilde{\Sigma}$ of OH_m there exists a unique branch solutions of Eq. (2.1), such that the isotropy subgroup of each nontrivial solution is $\tilde{\Sigma}$.

Our main purpose in this chapter is to determine a generic condition under which Eq. (2.1) does not have other solutions. Theorem 5.8 is the main result of this chapter.

Consider the reduced function

$$M(z, \lambda) := f(z, \lambda)z + g(z, \lambda)(\bar{z})^{k-1}, \tag{5.1}$$

where $k = m$ if m is odd and $k = 2m$ if m is even, associated to Eq. (2.1).

LEMMA 5.1. *The function $g(z, \lambda)$ vanishes when $\mu = 0$, where $\lambda = (p, \mu, 0)$. Moreover, if m is even then $g(z, \lambda)$ satisfies*

$$g(z, p, -\mu) = g(z, p, \mu).$$

Proof. From (5.1) we have $\operatorname{Im}(M(z, \lambda)\cdot \bar{z}) = g(z, \lambda)\cdot \operatorname{Im}(\bar{z})^k$. Since $\varepsilon_0 = 0$, the function $M(z, \lambda)$ is $(O(2) \times \mathbb{Z}_2)$-equivariant when $\mu = 0$. Then $M(z, \lambda)\bar{z} = M(r, \lambda)r$, where $z = re^{i\phi}$. Since $\mathbb{R} = \operatorname{Fix}(\mathbb{Z}_2(\tau))$ it follows that $M(r, \lambda)r \in \mathbb{R}$ and this implies that $g(z, p, 0, 0) = 0$.

Now, since $N(x, p, 0)$ is an odd function of x, we have

$$\begin{aligned} -v^*(-u,p,-\mu,0) &= -K(I-P)[N(-u+v^*(-u,p,-\mu,0),p,0)-\mu f] \\ &= K(I-P)[N(u-v^*(-u,p,-\mu,0),p,0)+\mu f]. \end{aligned}$$

It follows that $-v^*(-u, p, -\mu, 0) = v^*(u, p, \mu, 0)$. If we let $u = \mathscr{X}^{-1}(z)$ we have

$$\begin{aligned} M(-z,p,-\mu,0) &= \mathscr{X}PN(-u+v^*(-u,p,-\mu,0),p,0) \\ &= \mathscr{X}PN(-u-v^*(u,p,\mu,0),p,0) \\ &= -\mathscr{X}Q(\mathscr{X}^{-1}z,\lambda) \\ &= -M(z,p,\mu,0). \end{aligned}$$

This implies that

$$\operatorname{Im}[M(-z, p, -\mu, 0)(-\bar{z})] = \operatorname{Im}[M(z, p, \mu, 0)\bar{z}]$$

and then, if m is even, we have

$$g(-z, p, -\mu, 0) = g(z, p, \mu, 0). \tag{5.2}$$

Now, we have from the D_{2m}-invariance of $g(z, \lambda)$

$$g(-z, \lambda) = g((-e^{i\pi/m})^m z, \lambda) = g(R_{m\pi/m} \cdot z, \lambda) = g(z, \lambda),$$

and from (5.2) we have $g(z, p, -\mu, 0) = g(z, p, \mu, 0)$.

Taking the Taylor expansion of $g(z, \lambda)$, with respect to μ, from Lemma 5.1 we obtain the following result:

COROLLARY 5.2. *With the above hypotheses we have*:

$$g(z, \lambda) = \begin{cases} \rho\mu + \cdots & \textit{if } m \textit{ is odd}, \\ \rho^2\mu^2 + \cdots & \textit{if } m \textit{ is even}, \end{cases}$$

were

$$\rho = \rho(N, f) = \begin{cases} \dfrac{\partial g}{\partial \mu}(0, 0) & \textit{if } m \textit{ is odd}, \\ \dfrac{1}{2}\dfrac{\partial^2 g}{\partial \mu^2}(0, 0) & \textit{if } m \textit{ is even}. \end{cases} \tag{5.3}$$

We claim that if $\rho = \rho(N, f) \neq 0$ and $\mu \neq 0$, then there are no solutions (z, λ) of the reduced equation with $\operatorname{Im}(z^k) \neq 0$.

Indeed, if we let $z = re^{i\phi}$, from (5.1) and Corollary 5.3 the condition $M(z, \lambda) = 0$ implies

$$0 = \operatorname{Im}[M(z, \lambda)e^{-i\phi}] = \begin{cases} \mu r^{m-1}\sin(m\phi)(\rho + \cdots) & \text{if } m \text{ is odd}, \\ \mu^2 r^{2m-1}\sin(2m\phi)(\rho + \cdots) & \text{if } m \text{ is even}. \end{cases} \tag{5.4}$$

Since $\mu \neq 0$, $z \neq 0$, and $\sin(k\phi) \neq 0$, it follows that $(\rho + \cdots) = 0$, which is impossible because $\rho \neq 0$. We have proved the following:

LEMMA 5.3. *If* $\rho(N, f) \neq 0$, *every nontrivial small solution of* (2.1) *with* $\varepsilon = 0$ *and* $\mu \neq 0$ *has isotropy subgroup maximal in* OH_m.

Suppose m even. Consider the following Banach spaces, with the induced norms:

$$\mathcal{N} = \{N \in BC^{k+3}(B_r \times Z)/N \text{ satisfies } H_2 \text{ and } H_4\},$$

$$\mathcal{F} = \{f \in Z/f \text{ is } OH_m\text{-invariant}\}.$$

Consider the trilinear continuous function $G := \mathscr{L}(X^{2m+1}, Z) \times \mathscr{F}^2 \mapsto \mathbb{R}$ defined by

$$G(h,f,g) = -\frac{4m^2+2m}{2^{2m}}\left\{\sum_{j=0}^{m-1}\binom{2m-1}{2j}(-1)^j\mathscr{K}Ph(u_1^{2m-1-2j}, u_2^{2j}, Kf, Kg)\right.$$
$$\left.+\, i\sum_{j=0}^{m}\binom{2m-1}{2j+1}(-1)^j\mathscr{K}Ph(u_1^{2m-2-2j}, u_2^{2j+1}, Kf, Kg)\right\}.$$

We will show that the coefficient $\rho = \rho(N, f)$ defined in (5.3) has the representation $\rho = G(Q_{2m+1}, f, f) + \delta(N, f)$, where $\delta(N, f) = \delta(Q_3, ..., Q_{2m-1})$ is a real continuous function defined in $\mathcal{N} \times \mathscr{F}$, which does not depend on the derivatives $Q_k = (\partial N^k/\partial x^k)(0, 0)$ for $k > 2m - 1$.

Our objective is to show that, if there exist $(h_0, f_0) \in \mathscr{L}(X^{2m+1}, Z) \times \mathscr{F}$ such that $G(h_0, f_0, f_0) \neq 0$, then the condition $\rho(N, f) \neq 0$ is generic on the space $\mathcal{N} \times \mathscr{F}$; that is, the set $\{(N,f) \in \mathcal{N} \times \mathscr{F}/\rho(N,f) \neq 0\}$ is open and dense.

LEMMA 5.4. *Suppose m is even and hypotheses* (H_1)–(H_4) *are satisfied. The solution of auxiliary equation* $v^* = v^*(u, \lambda)$, *when* $p = \varepsilon = 0$ *and* $u = r\Gamma(\phi)u_1$, *has the representation*

$$v^* = \mu Kf + W_1 + W_2 + O(|r| + |\mu|)^{2m+3},$$

where

$$W_1 = W_1(Q_3, ..., Q_{2m-1}, f, r, \phi, \mu) = O(|r| + |\mu|)^3,$$
$$W_2 = W_2(Q_{2m+1}, f, r, \phi, \mu) = O(|r| + |\mu|)^{2m+1}.$$

Proof. The result can be proved just by using induction on m.

LEMMA 5.5. *Under hypotheses of Lemma* 5.4, *the coefficient* $\rho = \rho(N, f)$ *is continuous and has the representation*

$$\rho(N,f) = G(Q_{2m+1}, f, f) + \delta(N,f), \tag{5.5}$$

where

$$\delta(N,f) = \frac{1}{\pi}\int_0^{2\pi}\frac{\partial^{2m+1}}{\partial r^{2m-1}\,\partial\mu^2}$$
$$\left\{\mathrm{Im}[e^{-i\phi}\sum_{j=0}^{m-1}\mathscr{K}PQ_{2j+1}(r\Gamma(\phi)u_1 + \mu Kf + W_1)^{2j+1}]\sin(2m\phi)\,d\phi\right\}.$$

Proof. Let $W(r, \phi, \mu) := \operatorname{Im}[M(z, \lambda)e^{-i\phi}]$, where $u = \mathscr{K}^{-1}z = r\Gamma(\phi)u_1$ and $\lambda = (0, \mu, 0)$. By (5.4) we have

$$\rho(N,f) = \frac{1}{\pi}\int_0^\pi \frac{\partial W^{2m+1}}{\partial r^{2m-1}\,\partial\mu^2}(0, \phi, 0)\sin(2m\phi)\,d\phi. \tag{5.6}$$

From Lemma 5.4 we can write

$$\begin{aligned} W(r, \phi, \mu) &= \operatorname{Im}[e^{-i\phi}\mathscr{K}PN(r\Gamma(\phi)u_1 + v^*(r, \phi, \mu), 0)] \\ &= \operatorname{Im}[e^{-i\phi}\mathscr{K}PN(r\Gamma(\phi)u_1 + \mu Kf + W_1 + W_2 + W, 0)] \\ &= \operatorname{Im}[e^{-i\phi}\mathscr{K}P\sum_{j=0}^{m-1} Q_{2j+1}(r\Gamma(\phi)u_1 + \mu Kf + W_1)^{2j+1}] \\ &\quad + \operatorname{Im}[e^{-i\phi}\mathscr{K}PQ_{2m+1}((r\Gamma(\phi)u_1 + \mu Kf)^{2m+1})] + O(|\mu| + |r|)^{2m+3}, \end{aligned} \tag{5.7}$$

where $W = O(|r| + |\mu|)^{2m+3}$.

The coefficient of the term of order $r^{2m-1}\mu^2$ in the Taylor expansion of $\operatorname{Im}[e^{-i\phi}\mathscr{K}PQ_{2m+1}(r\Gamma(\phi)u_1 + \mu Kf)^{2m+1}]$, with respect to r, μ, is given by

$$G(Q_{2m+1}, f, f)\left[\sin(2m\phi) + (-1)^j 2^{2m-1}\sum_{k=1}^{2m-1} a_k \sin k\phi\right]. \tag{5.8}$$

Therefore, the identity (5.5) follows from (5.6), (5.7), and (5.8). The continuity of $\rho(N, f)$ follows from (5.5).

LEMMA 5.6. *Suppose $G(h_0, f_0, f_0) \neq 0$ for some $(h_0, f_0) \in \mathscr{L}(X^{2m+1}, Z) \times \mathscr{F}$ and that the homogeneous polynomial $x \in X^{2m+1} \mapsto h_0(x^{2m+1}) \in Z$ is $O(2) \times \mathbb{Z}_2$-equivariant. Fixing (N, f) in $\mathscr{N} \times \mathscr{F}$ we have:*

(a) *there exists an integer k_0 such that for $k > k_0$ we have $G(h_0, f + (1/k)f_0, f + (1/k)f_0) \neq 0$.*

(b) *The condition $\rho(N, f + (1/k)f_0) = 0$ implies $\rho(N + (1/k)h_0, f + (1/k)f_0) \neq 0$ for $k > k_0$.*

Proof. (a) $G(h_0, f + (1/k)f_0, f + (1/k)f_0) = G(h_0, f, f) + (2/k)G(h_0, f, f_0) + (1/k)^2G(h_0, f_0, f_0)$. Since $G(h_0, f_0, f_0) \neq 0$ the trinommy in $(1/k)$ does not vanish identically on the set $S = (-\varepsilon, \varepsilon) - \{0\} \subset \mathbb{R}$ where $\varepsilon > 0$ is small. It follows that there exist k_0, such that $k > k_0$ implies $1/k \in S$ and that $G(h_0, f + (1/k)f_0, f + (1/k)f_0) \neq 0$ for $k > k_0$.

(b) The condition $\rho(N, f + (1/k)f_0) = 0$ implies

$$\rho\left(N + \frac{1}{k}h_0, f + \frac{1}{k}f_0\right) = \rho\left(N, f + \frac{1}{k}f_0\right) + \frac{1}{k}G\left(h_0, f + \frac{1}{k}f_0, f + \frac{1}{k}f_0\right) \neq 0.$$

COROLLARY 5.7. *Under hypotheses of Lemma 5.7, with fixed (N, f) in $\mathcal{N} \times \mathcal{F}$, the sequence $(N_k, f_k) \subset \mathcal{N} \times \mathcal{F}$, defined by*

$$(N_k, f_k) = \begin{cases} \left(N, f + \frac{1}{k} f_0\right) & \text{if } \rho\left(N, f + \frac{1}{k} f_0\right) \neq 0, \\ \left(N + \frac{1}{k} h_0, f + \frac{1}{k} f_0\right) & \text{if } \rho\left(N, f + \frac{1}{k} f_0\right) = 0, \end{cases}$$

satisfies $\rho(N_k, f_k) \neq 0$ and converges to (N, f) as $k \to \infty$.

THEOREM 5.8. *If $G(h_0, f_0, f_0) \neq 0$ for some $(h_0, f_0) \in \mathcal{L}(X^{2m+1}, Z) \times \mathcal{F}$ then the set $\{(N, f) \in \mathcal{N} \times \mathcal{F}\colon \rho(N, f) \neq 0\}$ is open and dense in $\mathcal{N} \times \mathcal{F}$.*

Proof. It follows from Lemma 5.6 and Corollary 5.7.

Remark 5.9. Suppose m is odd and consider the trilinear function $H\colon \mathcal{L}(X^m, Z) \times \mathcal{F} \mapsto \mathbb{R}$ defined by

$$H(h, f, g) = -\frac{m}{2^{2m}} \left\{ \sum_{j=0}^{(m-1)/2} \binom{m-1}{2j} (-1)^j \mathscr{X} Ph(u_1^{m-1-2j}, u_2^{2j}, Kf, Kg) \right.$$
$$\left. + i \sum_{j=0}^{(m-1)/2} \binom{m-1}{2j+1} (-1)^j \mathscr{X} Ph(u_1^{m-2-2j}, u_2^{2j+1}, Kf, Kg) \right\}.$$

Following the above ideas we can prove that the coefficient $\rho(N, f) = H(Q_m, f, f) + \tilde{\delta}(N, f)$, where $\tilde{\delta}(N, f) = \tilde{\delta}(Q_3, \ldots, Q_{m-2}, f)$ is a continuous function which is independent of $Q_k := (\partial^k N/\partial x^k)(0, 0)$ for $k > m - 2$. The following corresponding result holds and the proof follows the above ideas.

THEOREM 5.10. *If $H(h_0, f_0, f_0) \neq 0$ for some $(h_0, f_0, f_0) \in \mathcal{L}(X^m, Z) \times \mathcal{F}^2$, then the set $\{(N, f) \in \mathcal{N} \times \mathcal{F} / \rho(N, f) \neq 0\}$ is open and dense in $\mathcal{N} \times \mathcal{F}$.*

6. EQUATIONS WITH A REVERSAL SYMMETRY

In this section we consider the equation (2.1) with hypotheses (H_1), (H_2), with $A_{|\mathrm{Ker}\,L} = \left(\begin{smallmatrix} 0 & -1 \\ 1 & 0 \end{smallmatrix}\right)$ instead of $A_{|\mathrm{Ker}\,L} = I_{|\mathrm{Ker}\,L}$, (H_3), and the following:

(H_5) (i) *The operator L and N are $(s/O(2) \times \mathbb{Z}_2)$-equivariant if $\varepsilon = 0$ and are $(\mathbb{Z}_{2m} \times \mathbb{Z}_2)$-equivariant if $\varepsilon \neq 0$.*

(ii) *$\Gamma(\tau) \in \mathcal{L}(Z)$ is a reversal symmetry for operators L and N, that is $-\Gamma(\tau) L(x) = L(\Gamma(\tau)x)$ and $-\Gamma(\tau) N(x, p, \varepsilon) = N(\Gamma(\tau)x, p, \varepsilon)$,*

(iii) *f is* $\widetilde{OH}_m := [(R_{\pi/m}, -I), (\tau, -I)]$*-invariant.*

Under the above assumptions, the operator $M(x, \lambda) = Lx - N(x, p, \varepsilon) - \mu f: X \times \mathbb{R}^3 \mapsto Z$ is $(OH_m, \widetilde{OH}_m)$-equivariant.

Using the Liapunov–Schmidt method, we obtain the reduced function

$$\tilde{Q}(u, \lambda) = PN(u + v^*(u, \lambda, p, \varepsilon): \text{Ker } L \times \mathbb{R}^3 \mapsto \text{Ker } L$$

which is $(OH_m, \widetilde{OH}_m)$-equivariant.

The linear isomorphism $q: \text{Ker } L \mapsto \text{Ker } L$, defined by $q(au_1 + bu_2) = -au_2 + bu_1$ is $(\widetilde{OH}_m, OH_m)$-equivariant. Therefore the function $q \circ \tilde{Q}$: $\text{Ker } L \times \mathbb{R}^3 \mapsto \text{Ker } L$ is OH_m-equivariant.

In the foregoing sections we found that the bifurcation problem $(q \circ \tilde{Q})(u, \lambda) = 0$ is equivalent to $M(z, \lambda) = 0$, where

$$M(z, \lambda) = \mathscr{X} \circ (q \circ \tilde{Q})(\mathscr{X}^{-1}z, \lambda): \mathbb{C} \times \mathbb{R}^3 \mapsto \mathbb{C}$$

is D_{2m}-equivariant, relative to its action on $\mathbb{C}$ defined by (2.4a), (2.4b). Since $\tilde{Q}(u, \lambda) = 0$ if and only if $(q \circ \tilde{Q})(u, \lambda) = 0$ and v^* is OH_m-equivariant, it follows that all the results of Section 5 can be established for Eq. (2.1) with the hypotheses stated in this section.

7. Examples

Example 7.1. Consider the nonlinear scalar equation

$$\ddot{x} + x = [p + \varepsilon H(t)]x + g(x) + h(x, t, p, \varepsilon) + \mu f(t), \tag{7.1}$$

where $m \geq 2$ is an even fixed integer and

(a) p, μ, and ε are small parameters;

(b) f, $H: \mathbb{R} \mapsto \mathbb{R}$ are continuous and even functions, f is π/m-odd-harmonic (that is, $f(t + \pi/m) = -f(t)$) and H is π/m-periodic;

(c) $g: \mathbb{R} \mapsto \mathbb{R}$ is an odd C^{2m+3} function with $g'(0) = 0$ and $g^{(3)}(0) \neq 0$;

(d) $h: \mathbb{R}^4 \mapsto \mathbb{R}$ is a C^{2m+3} function which is odd in x, π/m-periodic, and even in t and it is of order $o(|px| + |\varepsilon x| + |x|^3)$ as $(x, p, \varepsilon) \to (0, 0, 0)$.

Equation (7.1) can put into the abstract form (2.1) by taking $X = C^2_{2\pi}(\mathbb{R}, \mathbb{R})$, $Z = C_{2\pi}(\mathbb{R}, \mathbb{R})$ with the C^2 and C^0 topologies, respectively, and $(Lx)(\cdot) := \ddot{x}(\cdot) + x(\cdot)$, $L: X \mapsto Z$, $N(x, p, \varepsilon)(\cdot) := [pA + \varepsilon B]x(\cdot) + \alpha_3 x^3(\cdot) + R(x, p, \varepsilon)(\cdot)$ where $R(x, p, \varepsilon)(\cdot) := g(x(\cdot)) - \alpha_3 x^3(\cdot) + h(x(\cdot),$

$(\cdot), t, p, \varepsilon)$, A and B belong to $\mathscr{L}(X, Z)$ and are given by $Ax(\cdot) = x(\cdot)$, $Bx(\cdot) = H(\cdot)x(\cdot)$, and $\alpha_3 = (1/3!)g^{(3)}(0)$.

Consider the representation of $O(2)$ on $\mathbb{Z}$, defined by $\Gamma(\phi)w(t) = w(t - \phi)$ and $\Gamma(\tau)w(t) = w(-t)$. The hypotheses (H_1)–(H_3) are satisfied with $u_1(t) = \cos t$ and $u_2(t) = \sin t$.

If we consider the group $O(2) \times \mathbb{Z}_2$ and its subgroups $G = D_{2m} \times \mathbb{Z}_2$ and $OH_m = [(R_{\pi/m}, -I), (\tau, I)]$ acting on Z and X by

$$(R_\phi, \pm I)w(t) = \pm\Gamma(\phi)w(t) = \pm w(t - \phi),$$
$$(\tau, \pm I)w(t) = \pm\Gamma(\tau)w(t) = \pm w(t - \phi),$$

$\forall\ w(\cdot) \in \mathbb{Z}$ and $\phi \in \mathbb{R}$; it follows that Hypothesis (H_4) is satisfied. The projection $P \in \mathscr{L}(X)$ is defined by

$$Pf(\cdot) := \frac{1}{\pi}\sum_{j=1}^{2}\int_0^{2\pi} f(s)u_j(s)\,ds \cdot u_j(\cdot).$$

In the special case where $f(t) = \cos mt$, $H(t) = 1 + \cos 2m\,t$, we have $c = \frac{3}{4}\alpha_3 \neq 0$ and the bifurcations points are given by $r(\varepsilon, \mu) = 0$ and $p(\varepsilon, \mu) = -\varepsilon + (3\alpha_3/2(1 - m^2)^2)\mu^2 + \cdots$.

Given (ε_0, μ_0) small, the $2m$-branches of solutions indicated in Theorems 4.6 and 4.7 are

$$x(re^{i(j\pi/m)}, \lambda(r)) = r\cos(t - j\pi/m) + v^*(r\cos(t - j\pi/m), \lambda(r))$$

and

$$x(re^{i(2j+1)\pi/2m}, \lambda(r)) = r\cos(t - (2j + 1)\pi/2m)$$
$$+ v^*(r\cos(t - (2j + 1)\pi/2m)), \lambda(r))$$

$j = 1, \ldots, m - 1$.

For $r \neq 0$ and $0 \leq j \leq m - 1$ fixed, the isotropy subgroup of the solution $x(t) = x(re^{i(j\pi/m)}, \lambda(r))(t)$ is $\tilde{E}_j = \mathbb{Z}_2((R_{2j\pi/m} \circ \tau, I))$ and this is equivalent to saying that $x(t + j\pi/m)$ is an even function of t. Also, for $r \neq 0$ the isotropy subgroup of the solution $x(t) = x(re^{i(2j+1)\pi/2m}, \lambda(r))(t)$ is $\tilde{D}_j = \mathbb{Z}_2((R_{(m+1+2j)\pi/m} \circ \tau, -I))$, and this is equivalent to saying that $x(t + (2j + 1)\pi/2m - \pi/2)$ is an odd function of t.

Concerning the genericity described in Theorem 5.8 for $\varepsilon = 0$ and $h \equiv 0$, if we take (h_0, f_0) where $f_0(t) = \cos mt$ and $h_0 : X^{2m+1} \mapsto Z$ defined by $h_0(x_1(\cdot), \ldots, x_{2m+1}(\cdot)) = \prod_{j=1}^{2m+1} x_j(\cdot)$ we have $G(h_0, f_0, f_0) = (m(2m + 1))/(2^{2m}(1 - m^2)^2) \neq 0$. Then, generically, the only nontrivial small solutions of Eq. (7.1) with $\varepsilon = 0$ and $\mu \neq 0$ are those whose isotropy subgroups are $\tilde{E}_j$ or $\tilde{D}_j$, $j = 0, ..., m - 1$.

EXAMPLE 7.2. Consider the nonlinear system

$$\dot{W} = A_0 W + [p + \varepsilon H(t)]AW + g(W) + h(W, t, p, \varepsilon) + \mu F(t), \quad (7.2)$$

where $m \geq 2$ is a even fixed integer and

(a) $W \in \mathbb{R}^{2n}$, p, μ, and ε are small real parameters;

(b) $A_0 = \text{diag}(A_1^0, ..., A_n^0)$, $A_1^0 = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}$, $A_k^0 = \begin{pmatrix} 0 & w \\ -w & 0 \end{pmatrix}$ where w is a postive real number which is not an integer, $k = 2, ..., n$;

(c) $F, H : \mathbb{R} \mapsto \mathbb{R}^{2n}$ are continuous, F is π/m-odd-harmonic, H is even and π/m-periodic; and $A = \text{diag}(A_1, ..., A_n)$ where $A_i = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}$, $i = 1 ..., n$.

(d) $g : \mathbb{R}^{2n} \mapsto \mathbb{R}^{2n}$ is a C^{2m+3} odd function with $g'(0) = 0$ and $g^{(3)}(0) \neq 0$;

(e) $h : \mathbb{R}^4 \mapsto \mathbb{R}^{2n}$, $(W, t, p, \varepsilon) \mapsto h(W, t, p, \varepsilon)$ is a C^{2m+3} function which is odd in W, π/m-periodic and even in t, and is of order $o(|pW| + |\varepsilon W|)$ as $(W, p, \varepsilon) \mapsto (0, 0, 0)$;

(f) for $S = \text{diag}(A_1, ..., S_n)$ where $S_k = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$, $k = 1, ..., n$, one has: $-SF(-t) = F(t)$, $-Sh(W, -t, p, \varepsilon) = h(SW, t, p, \varepsilon)$ and $-Sg(W)) = g(SW)$.

Equation (7.2) can be put into the abstract form (2.1) by taking $X := C^1_{2\pi}(\mathbb{R}, \mathbb{R}^{2n})$, $Z := C_{2\pi}(\mathbb{R}, \mathbb{R}^{2n})$, with C^1 and C^0 topologies respectively, and

$$(LW)(\cdot) = \dot{W}(\cdot) - A_0 W(\cdot) : X \mapsto Z,$$

$N(W, p, \varepsilon) := [pA + \varepsilon B]W + Q_3(W(\cdot)^3) + R(W, p, \varepsilon)(\cdot)$, where $R(W, p, \varepsilon)(\cdot) := g(W(\cdot)) - Q_3(W(\cdot)^3) + h(W(\cdot), (\cdot), p, \varepsilon)$, $B \in \mathcal{L}(X, Z)$ is given by $BW(\cdot) = H(\cdot)W(\cdot)$, and $Q_3 = (1/3!)g^{(3)}(0)$.

Consider the representation Γ of $O(2)$ on Z defined by

$$\Gamma(\phi)W(\cdot) = W(\cdot - \phi), \qquad \Gamma(\rho)(W(\cdot)) = SW(-\cdot),$$

$\forall\ W(\cdot) \in Z$.

Hypotheses (H_1), (H_2), and (H_3) are satisfied for system (7.2) with

$$u_1(t) = (\cos t, -\sin t, 0, ..., 0) \qquad u_2(t) = (\sin t, \cos t, 0, ..., 0),$$

and $A_{|\text{Ker L}} = A_1$.

The projection $P \in \mathscr{L}(Z)$ can be defined by

$$Pf(\cdot) := \frac{1}{2\pi} \sum_{j=1}^{2} \int_0^{2\pi} (f(s) \cdot u_j(s))\, ds \cdot u_j(\cdot),$$

where $(\cdot, \cdot)$ is the usual inner product in $\mathbb{R}^{2n}$.

If we consider the group $O(2) \times \mathbb{Z}_2$ and their subgroups $\mathbb{Z}_n \times \mathbb{Z}_2$, $OH_m = [(R_{\pi/m}, -I), (\tau, I)]$ and $\widetilde{OH}_m = [(R_{\pi/m}, -I), (\tau, -I)]$ acting on Z (and X) by the action

$$(R_\phi, \pm I) \cdot w(t) = \pm \Gamma(\phi) w(t) = \pm w(t - \phi),$$
$$(\tau, \pm I) \cdot w(t) = \pm \Gamma(\tau) w(t) = \pm S w(-t)$$

$\forall w \in Z$; it follows that Hypothesis (H_5) of Section 6 is satisfied.

If $c = (1/2\pi) \int_0^{2\pi} Q_3(u_1^3(s)) u_2(s)\, ds \neq 0$ then, given (ε_0, μ_0) small, the $2m$-branches solutions pointed in Theorems 4.6 and 4.7 are

$$x(re^{i(j\pi/m)}, \lambda(r)) = ru_1(t - j\pi/m) + v^*(ru_1(t - j\pi/m), \lambda(r))$$

and

$$\begin{aligned} x(re^{i(2j+1)\pi/2m}, \lambda(r)) = {} & ru_1(t - (2j+1)\pi/2m) \\ & + v^*(ru_1(t - (2j+1)\pi/2m), \lambda(r)), \end{aligned}$$

$j = 0, \ldots, m - 1$.

For $r \neq 0$ the isotropy subgroup of $x(re^{i(j\pi/m)}, \lambda(r))$ and $x(re^{i(2j+1)\pi/2m}, \lambda(r))$ are respectively $\tilde{E}_j = \mathbb{Z}_2((R_{2j\pi/m} \circ \tau, I))$ and $\tilde{D}_j = \mathbb{Z}_2((R_{(m+1+2j)\pi/m} \circ \tau, -I))$, $j = 0, \ldots, m - 1$. This is equivalent to saying that $x(re^{i(j\pi/m)}, \lambda(r))(t + j\pi/m)$ and $x(re^{i(2j+1)\pi/2m}, \lambda(r))(t + (2j+1)\pi/2m - \pi/2)$ are invariants respectively by the operators $\Gamma(\tau)$ and $-\Gamma(\tau)$.

Concerning generality described in Theorem 5.8 for $\varepsilon = 0$ and $h \equiv 0$, if we take (h_0, F_0) with $F_0(t) = F(t) = \mathrm{col}((1 - m)\sin mt, (1 - m)\cos mt)$ and $h_0 : X^{2m+1} \mapsto Z$ defined by $h_0(W_1(\cdot), \ldots, W_{2m+1}(\cdot)) = (-\prod_{j=1}^{2m+1} y_j(\cdot), \prod_{j=1}^{2m+1} x_j(\cdot))$ where $W_j(\cdot) = \mathrm{col}(x_j(\cdot), y_j(\cdot))$, we have $G(h_0, F_0, F_0) = (-m(2m + 1)/2^{2m+1}) \neq 0$.

If follows that in the two-dimensional case, generically, the nontrivial small solutions of Eq. (7.1) with $\varepsilon = 0$ and $\mu \neq 0$ are those whose isotropy subgroups are $\tilde{E}_j$ or $\tilde{D}_j$.

EXAMPLE 7.3. Let $m \geq 2$ be a fixed integer. We analyze the existence and bifurcation of classical solutions of the boundary value problem on the disc $B = \{w \in \mathbb{R}^2 / |w| < 1\}$, defined by

$$\begin{cases}(\Delta + \lambda)u = pu + g(u,p) + h(u,p) + \mu f(w), & w \in B \\ u(w) = 0, & w \in \partial B;\end{cases} \tag{7.3}$$

where

(a) p and μ are small parameters;

(b) $f: \overline{B} \to \mathbb{R}$ is α-Holderian with index $\alpha \in (0,1), f(R_{\pi/m}w) = -f(w)$, and $f(Sw) = f(w)$ where $S = \operatorname{diag}(1, -1) \in M_2(\mathbb{R})$;

(c) $g: \mathbb{R} \to \mathbb{R}$ is a C^{2m+3} odd function with $g'(0) = 0$ and $g^{(3)}(0) \neq 0$;

(d) $h: \mathbb{R}^2 \mapsto \mathbb{R}$, $(u, p) \mapsto h(u, p)$ is a C^{2m+3} function which is odd in u and of order $o(|px|)$ as $(u, p) \to (0, 0)$;

(e) $\lambda = \xi^2$, where ξ is the first positive zero of the Bessel function $J_1(x)$.

Let $C^{k+\alpha}(\overline{B})$ be the space of the functions $u: B \to \mathbb{R}$ having partial derivatives up to order k, which can be continuously extended up to $\overline{B}$, and the kth derivatives are uniformly Hölder continuous in $\overline{B}$.

Consider on $C^{k+\alpha}(\overline{B})$ the norm:

$$\|u\|_{k+\alpha} = \|u\|_k + \sup_{|\beta|=k} [D^\beta u]_\alpha,$$

where

$$\|u\|_k = \sum_{j=0}^{k} \sup_{|\beta|=j} \sup_{w\in\overline{B}} |D^\beta u(w)|$$

and

$$[u]_\alpha = \sup_{\substack{w\neq v \\ w,v\in\overline{B}}} \frac{|u(w) - u(v)|}{|w - v|^\alpha}.$$

The boundary value problem (7.3) can be put into the abstract form (2.1) by taking $X := C_0^{2+\alpha}(\overline{B}) = \{u \in C^{2+\alpha}(\overline{B})/u(w) = 0 \text{ if } w \in \partial B\}$ and $Z := C^\alpha(\overline{B})$, with norms $\|\,\|_{2+\alpha}$ and $\|\,\|_\alpha$ respectively, and $(Lu)(\cdot) := (\Delta + \lambda)u(\cdot): X \mapsto Z$, $N(u, p) := pu(\cdot) + \alpha_3 u^3(\cdot) + R(u, p)(\cdot)$ where $R(u,p)(\cdot) := g(u(\cdot)) - \alpha_3 u^3(\cdot) + h(u), p)$ and $\alpha_3 = (1/3!)g^{(3)}(0)$.

Let Γ be the representation of $O(2)$ over Z defined by $\Gamma(\phi)u(w) := u(R_\phi w)$, $\Gamma_* u(w) := u(Sw)$. The hypotheses (H_1), (H_2), and (H_3) are satisfied. In polar coordinates u_1 and u_2 are defined by $u_1(s, \theta) = J_1(\sqrt{\lambda}s) \cos\theta$ and $u_2(s, \theta) = J_1(\sqrt{\lambda}s) \sin\theta$.

The projections $P \in \mathcal{L}(Z)$ are defined by

$$Pf(\cdot) := \frac{1}{\ell} \sum_{j=1}^{2} \int_{\bar{B}} f(u) u_j(w)\, du\, u_j(\cdot)$$

where

$$\ell := \pi \int_0^1 r J_1^2(\sqrt{\lambda_1} s)\, ds.$$

If we consider the group $O(2) \times \mathbb{Z}_2$ and their subgroups $OH_m = [(R_{\pi/m}, -I), (\tau, I)]$ acting on Z by the action:

$$(R_\phi, \pm I) \cdot u(w) = \pm\Gamma(\phi) u(w) = \pm u(R_\phi \cdot w),$$
$$(\tau, \pm I) \cdot u(w) = \pm\Gamma(\tau) u(w) = \pm u(Sw),$$

$\forall u \in Z$, we have that Hypothesis (H_4) is satisfied.

For the special case $f(w) = f(s, \theta) = J_m(\sqrt{\nu s}) \cos m\theta(\nu - \lambda)$, where $\sqrt{\nu}$ is the first positive zero of the Bessel function $J_m(x)$, the bifurcation occurs in the points (r, p, μ) where $r = r(0, \mu) = 0$ and $p = p(0, \mu) = -b\mu^2 + o(\mu^2)$ with $b = (3\alpha_3\pi/2\ell) \int_0^{2\pi} J_m^2(\sqrt{\nu s})\, J_1^2(\sqrt{\lambda s})\, ds$.

Since $c = (3\pi\alpha_3/4\ell) \int_0^1 s J_1^4(\sqrt{\lambda s})\, ds \neq 0$, for μ_0 small and $m \geq 2$ even, Eq. (7.3) has the following $2m$-branches of solutions:

$$x(re^{i(j\pi/m)}, \lambda(r))(s, \theta) = ru_1(s, \theta - j\pi/m) + v^*(ru_1(s, \theta - j\pi/m), \lambda(r))$$

and

$$\begin{aligned} x(re^{i(2j+1)\pi/2m}, \lambda(r))(s, \theta) = {} & ru_1(s, \theta - (2j+1)\pi/2m) \\ & + v^*(ru_1(s, \theta - (2j+1)\pi/m), \lambda(r)), \end{aligned}$$

$j = 0, \ldots, m-1$, where $\lambda(r) = (p_0 + (J_{rr}(0, \lambda_0))/(2J_p(0, \lambda_0))r^2 + \ldots, \mu_0)$ and $p_0 = p(0, \mu_0)$.

For $r \neq 0$ sufficiently small the isotropy subgroups of these solutions are, respectively, $\tilde{E}_j = \mathbb{Z}_2((R_{2j\pi/m} \circ \tau, I))$ and $\tilde{D}_j = \mathbb{Z}_2((R_{(m+1+2j)\pi/m} \circ \tau, -I))$, $j = 0, \ldots, m-1$. This is equivalent to saying that $x(re^{i(j\pi/m)}, \lambda(r))(s, \theta + j\pi/m)$ is an even and $x(re^{i(2j+1)\pi/2m}, \lambda(r))(s, \theta + (2j+1)\pi/2m - \pi/2)$ is an odd function of θ.

Concerning the generality, if we take $f_0(s, \theta) = f(s, \theta) = J_m(\sqrt{\nu}\, s) \cos m\theta(\nu - \lambda)$ and $h_0 : X^{2m+1} \mapsto Z$, defined by $h_0(x_1(\cdot), \ldots, X_{2m+1}(\cdot)) = \prod_{j=1}^{2m+1} x_j(\cdot)$, we have

$$G(h_0, f_0, f_0) = \frac{m(2m+1)}{2^{2m+1}l} \int_0^1 r J_1^{2m}(\sqrt{\nu}\, s) J_m^2(\sqrt{\nu}\, s)\, ds \neq 0.$$

Therefore, generically, the only nontrivial small solutions of Eq. (7.3) have isotropy subgroups $\tilde{E}_j$ or $\tilde{D}_j$, $j = 0, ..., m - 1$.

Remark 7.4. With the same adaptations we can establish similar results for the case m odd. An interesting case to be consider in our future work is when the kernel of L has dimension 4. In that case we need more parameters and it appears that new technical difficulties will arise in obtaining corresponding results.

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