



Elimination and recursions in the scattering equations



Carlos Cardona^a, Chrysostomos Kalousios^{b,*}

^a Physics Division, National Center for Theoretical Sciences, National Tsing-Hua University, Hsinchu, 30013 Taiwan, ROC

^b ICTP South American Institute for Fundamental Research, Instituto de Física Teórica, UNESP – Universidade Estadual Paulista, R. Dr. Bento T. Ferraz 271 – Bl. II, 01140-070, São Paulo, SP, Brazil

ARTICLE INFO

Article history:

Received 12 January 2016

Accepted 1 March 2016

Available online 4 March 2016

Editor: M. Cvetič

Keywords:

Scattering amplitudes

Scattering equations

Elimination theory

ABSTRACT

We use the elimination theory to explicitly construct the $(n-3)!$ order polynomial in one of the variables of the scattering equations. The answer can be given either in terms of a determinant of Sylvester type of dimension $(n-3)!$ or a determinant of Bézout type of dimension $(n-4)!$. We present a recursive formula for the Sylvester determinant. Expansion of the determinants yields expressions in terms of Plücker coordinates. Elimination of the rest of the variables of the scattering equations is also presented.

© 2016 The Authors. Published by Elsevier B.V. This is an open access article under the CC BY license (<http://creativecommons.org/licenses/by/4.0/>). Funded by SCOAP³.

1. Introduction

The last decade a great amount of progress has been made in our understanding of scattering amplitudes for massless theories starting with the revolutionary work of Witten [1] on four dimensional scattering amplitudes in twistor space. In 2013 a new formulation for the computation of the tree level S -matrix of massless particles in any dimension was proposed in a series of papers by Cachazo, He and Yuan (CHY) [2–6]. The formalism was proven for Yang–Mills and scalars in [7]. By defining a map from the space of kinematic invariants of the scattering of n massless particles to the space of punctures over a Riemann sphere, CHY found the so-called scattering equations,

$$f_a \equiv \sum_{b \neq a}^n \frac{k_a \cdot k_b}{\sigma_a - \sigma_b} = 0, \quad a = 1, 2, \dots, n, \quad (1)$$

where k_a denotes the momentum of the a th particle and σ_b the position of the b th puncture on the sphere. Those equations have appeared previously in different contexts, particularly in the work of Fairlie and Roberts [8–10] and in the study of the high energy behavior of scattering of strings by Gross and Mende [11] (see also [12–16]). In this work we use the polynomial form of (1) that was presented in [17] and which can be written as

$$\sum_{S \subset A, |S|=m} k_S^2 \sigma_S = 0, \quad 2 \leq m \leq n-2, \quad (2)$$

where $A = 1, 2, \dots, n$ and the sum runs over all subsets of S with m elements. Furthermore,

$$k_S = \sum_{a \in S} k_a, \quad \sigma_S = \prod_{b \in S} \sigma_b. \quad (3)$$

According to CHY, massless scattering at tree level in arbitrary dimensions can in general be described by the contour integral

$$\mathcal{A}_n = \int \frac{d^n \sigma}{\text{vol } SL(2, \mathbb{C})} \sigma_{ij} \sigma_{jk} \sigma_{ki} \prod_{a \neq i, j, k} \delta(f_a) I_n(k, \epsilon, \sigma), \quad (4)$$

* Corresponding author.

E-mail addresses: carlosandres@mx.nthu.edu.tw (C. Cardona), ckalousi@ift.unesp.br (C. Kalousios).

which is completely localized over the solutions of the scattering equations, and $I_n(k, \epsilon, \sigma)$ depends on the theory. So far $I_n(k, \epsilon, \sigma)$ has been proposed for several different theories in various dimensions, including Einstein gravity, pure Yang–Mills [3], ϕ^3 [4], Einstein–Yang–Mills [5], massive scalar-gravity [18] and lately for Dirac–Born–Infeld and $U(N)$ non-linear sigma model [6].

Since the CHY original formulation, many papers have surfaced investigating several aspects of the scattering equations, such as its relation to momentum twistor formalism [19], solutions at particular kinematics (for example [4,17,20,19,21]), its geometrical interpretation [22], soft limits [23–26], generalizations to include massive particles (for example [18,27–30]) and formulations in terms of string-like models (for example [31–34]).

Although at first sight the CHY formalism seems quite simple and straightforward, when it comes to the study of solutions of the scattering equations they rapidly become difficult to manage since the number of solutions grows factorially as a function of the number of particles. However the amplitudes involve a sum over the solutions which can be found without even solving the equations. With that in mind, several interesting methods have been developed lately. In [35] the elimination method was used in order to write the scattering amplitude in terms of the solutions of a polynomial in a single variable, σ_a , and then use the Vieta formulas to get the sums over roots from the coefficients of the one variable polynomial in σ_a . This idea was enhanced in [36] by using the companion matrix method, which allows to encode the single variable polynomial in a matrix T_{σ_a} such that its eigenvalues correspond to the roots of the polynomial and hence the sum over the roots is reduced to taking the trace of the given matrix. In [37] we further elaborated on the equivalence of [35] and [36]. A related approach has been recently considered in [38] where the authors use the Bézoutian matrix to compute the amplitudes with emphasis to numerics. A parallel method was developed in [39] where known results for ϕ^3 theory are used as building blocks for the computation of other amplitudes. A different approach was considered in [40,41], where based on some results from [33] a direct matching between Feynman diagrams and integration measures in the scattering equation formalism of CHY was found.

This work is partly motivated by some ideas discussed in [35] and then in [36] and [37]. If one can eliminate the variables in the scattering equations, then using simple algebra one can evaluate any rational expression of the variables of the scattering equations summed over roots. Of most importance are of course the scattering amplitudes. The elimination theory was first used in the $n = 5$ and $n = 6$ case in the work of [17]. Here we are concerned with the general case.

We know from Bézout theorem that the number of solutions of (2) is $(n - 3)!$. This means that in the general case each of the variables appearing in the scattering equations can be expressed as one variable polynomials of order $(n - 3)!$. In the following we will show how this can be done for the chosen variable σ_{n-1} . We will then express the rest of the variables of the scattering equations as functions of σ_{n-1} only. In the mathematical literature this goes by the name of elimination theory. We will study the elimination theory of (2) following two separate approaches, by Sylvester and Bézout. The results obtained are the same and can be expressed in various determinantal forms. On our way we will also discover an on-shell recursion for the scattering equations.

In the whole body of this paper we think of (2) as a special case of the multilinear system of equations

$$g_m'' \equiv \sum_{i_j \in \{0,1\}} b_{i_3 i_4 \dots i_{n-1} m} \sigma_3^{i_3} \sigma_4^{i_4} \dots \sigma_{n-1}^{i_{n-1}} = 0, \quad m = 1, 2, \dots, n - 3, \quad (5)$$

where b are arbitrary constants and we have fixed the $SL(2, \mathbb{C})$ invariance of (1) by specifying arbitrary values to $\sigma_1, \sigma_2, \sigma_n$. It is notationally convenient for what follows to consider σ_{n-1} as a parameter and rewrite (5) as

$$g_m \equiv \sum_{i_j \in \{0,1\}} c_{i_3 i_4 \dots i_{n-2} m} \sigma_3^{i_3} \sigma_4^{i_4} \dots \sigma_{n-2}^{i_{n-2}} = 0, \quad m = 1, 2, \dots, n - 3, \quad (6)$$

where c now depends linearly on σ_{n-1} . This defines a system of $n - 3$ equations in $n - 4$ unknowns. We now turn to the elimination via Sylvester type determinants.

2. Elimination in the scattering equations via Sylvester type determinants

In this part we will express the advertised relations in terms of Sylvester type determinants. Before focusing on the general case, we will first discuss the $n = 6$, $n = 7$ and $n = 8$ case. The $n = 5$ case is trivial.

2.1. $n = 6$

The number of scattering equations is three in this case. Let $S(a_1, a_2, \dots, a_i)$ denote the vector space of all polynomials of degree less than or equal to a_i in x_i . Then the dimension of $S(a_1, a_2, \dots, a_i)$ is $\prod_{j=1}^i (a_j + 1)$. We consider the linear map

$$\phi_3 : S(0, 1)^3 \rightarrow S(1, 2), \quad (f_1, f_2, f_3) \mapsto f_1 g_1 + f_2 g_2 + f_3 g_3, \quad (7)$$

where g_i are given by (6) for $n = 6$. Both the range and image of ϕ_3 have dimension 6. We choose to use the canonical order for the monomial bases for both vector spaces. We choose σ_3 and σ_4 to be the variables of the scattering equations and we consider σ_6 to be a parameter. Then ϕ_3 is given by a 6×6 matrix whose determinant is the resultant of the three scattering equations. By setting the resultant to zero we get the desired sixth order polynomial in σ_5 . We proceed with the explicit construction of the resultant.

From (6) the three scattering equations are

$$g_i = c_{00i} + c_{10i} \sigma_3 + c_{01i} \sigma_4 + c_{11i} \sigma_3 \sigma_4 = 0, \quad i = 1, 2, 3. \quad (8)$$

The monomial basis of $S(0, 1)$ is $\{1\} \otimes \{1, \sigma_4\} = \{1, \sigma_4\}$ and that of $S(1, 2)$ is $\{1, \sigma_3\} \otimes \{1, \sigma_4, \sigma_4^2\} = \{1, \sigma_3, \sigma_4, \sigma_3 \sigma_4, \sigma_4^2, \sigma_3 \sigma_4^2\}$. We multiply each of the three scattering equations with the elements of the monomial basis $S(0, 1)$ and we consider the following system of six equations

$$g_1 = g_2 = g_3 = \sigma_4 g_1 = \sigma_4 g_2 = \sigma_4 g_3 = 0. \quad (9)$$

Then (9) is a linear system in the variables of the monomial basis $S(1, 2)$. The desired sixth order polynomial in σ_5 is then given by the determinant of (9) and it is

$$\begin{vmatrix} c_{001} & c_{002} & c_{003} & 0 & 0 & 0 \\ c_{101} & c_{102} & c_{103} & 0 & 0 & 0 \\ c_{011} & c_{012} & c_{013} & c_{001} & c_{002} & c_{003} \\ c_{111} & c_{112} & c_{113} & c_{101} & c_{102} & c_{103} \\ 0 & 0 & 0 & c_{011} & c_{012} & c_{013} \\ 0 & 0 & 0 & c_{111} & c_{112} & c_{113} \end{vmatrix} = \begin{vmatrix} g_1 & g_2 & g_3 & 0 & 0 & 0 \\ g_{1,3} & g_{2,3} & g_{3,3} & 0 & 0 & 0 \\ g_{1,4} & g_{2,4} & g_{3,4} & g_1 & g_2 & g_3 \\ g_{1,34} & g_{2,34} & g_{3,34} & g_{1,3} & g_{2,3} & g_{3,3} \\ 0 & 0 & 0 & g_{1,4} & g_{2,4} & g_{3,4} \\ 0 & 0 & 0 & g_{1,34} & g_{2,34} & g_{3,34} \end{vmatrix} = 0, \quad (10)$$

where $g_{i,i_1 i_2 \dots i_r} = \frac{\partial^r g_i}{\partial \sigma_{i_1} \partial \sigma_{i_2} \dots \partial \sigma_{i_r}}$. We can also rewrite (10) compactly as

$$\begin{vmatrix} A_0 & 0 \\ A_1 & A_0 \\ 0 & A_1 \end{vmatrix} = 0, \quad A_{j_1} = \begin{pmatrix} c_{0j_1 1} & c_{0j_1 2} & c_{0j_1 3} \\ c_{1j_1 1} & c_{1j_1 2} & c_{1j_1 3} \end{pmatrix}, \quad j_1 = 0, 1. \quad (11)$$

There are many ways to expand (10), for example we can consider the Laplace expansion of the first three columns to get

$$(10) = [00, 10, 01][10, 01, 11] - [00, 10, 11][00, 01, 11] = \begin{vmatrix} [00, 10, 01] & [00, 10, 11] \\ [00, 01, 11] & [10, 01, 11] \end{vmatrix}, \quad (12)$$

where we have used the n -bracket notation

$$\begin{aligned} & [i_{11} i_{12} \dots i_{1(n-1)}, i_{21} i_{22} \dots i_{2(n-1)}, \dots, i_{n1} i_{n2} \dots i_{n(n-1)}] = \\ & \begin{vmatrix} c_{i_{11} i_{12} \dots i_{1(n-1)} 1} & c_{i_{11} i_{12} \dots i_{1(n-1)} 2} & \dots & c_{i_{11} i_{12} \dots i_{1(n-1)} n} \\ c_{i_{21} i_{22} \dots i_{2(n-1)} 1} & c_{i_{21} i_{22} \dots i_{2(n-1)} 2} & \dots & c_{i_{21} i_{22} \dots i_{2(n-1)} n} \\ \vdots & \vdots & \ddots & \vdots \\ c_{i_{n1} i_{n2} \dots i_{n(n-1)} 1} & c_{i_{n1} i_{n2} \dots i_{n(n-1)} 2} & \dots & c_{i_{n1} i_{n2} \dots i_{n(n-1)} n} \end{vmatrix}. \end{aligned} \quad (13)$$

In order to complete the elimination we need to express σ_3 and σ_4 as function of σ_5 . This can be achieved in various ways. We can consider any five of the six equations in (9), for example the first five, and use Cramer's rule to solve the corresponding linear system for σ_3 and σ_4 . We respectively find

$$\begin{vmatrix} c_{001} + c_{101}\sigma_3 & c_{002} + c_{102}\sigma_3 & c_{003} + c_{103}\sigma_3 & 0 & 0 \\ c_{011} & c_{012} & c_{013} & c_{001} & c_{002} \\ c_{111} & c_{112} & c_{113} & c_{101} & c_{102} \\ 0 & 0 & 0 & c_{011} & c_{012} \\ 0 & 0 & 0 & c_{111} & c_{112} \end{vmatrix} = \begin{vmatrix} c_{001} + c_{011}\sigma_4 & c_{002} + c_{012}\sigma_4 & c_{003} + c_{013}\sigma_4 & c_{001}\sigma_4 & c_{002}\sigma_4 \\ c_{101} & c_{102} & c_{103} & 0 & 0 \\ c_{111} & c_{112} & c_{113} & c_{101} & c_{102} \\ 0 & 0 & 0 & c_{011} & c_{012} \\ 0 & 0 & 0 & c_{111} & c_{112} \end{vmatrix} = 0 \quad (14)$$

The two determinants in (14) came from (10) by removing the sixth column which corresponds to the sixth scattering equation in (9) and by multiplying the second (third) row by σ_3 (σ_4) and adding that to the first row, finally eliminating the second (third) row.

In the case of $n = 6$ it is easy to simplify the problem even further by noticing that the first three equations in (9) only depend on three variables ($\sigma_3, \sigma_4, \sigma_3\sigma_4$). Therefore, it is sufficient to only consider the top left 4×3 block of (10) and apply the same operations that led to (14). Equivalently we can only consider the top left 3×3 blocks of (14). More specifically we find

$$\begin{aligned} & \begin{vmatrix} c_{001} + c_{101}\sigma_3 & c_{002} + c_{102}\sigma_3 & c_{003} + c_{103}\sigma_3 \\ c_{011} & c_{012} & c_{013} \\ c_{111} & c_{112} & c_{113} \end{vmatrix} = \begin{vmatrix} g_1 & g_2 & g_3 \\ g_{1,4} & g_{2,4} & g_{3,4} \\ g_{1,34} & g_{2,34} & g_{3,34} \end{vmatrix} = 0, \\ & \begin{vmatrix} c_{001} + c_{011}\sigma_4 & c_{002} + c_{012}\sigma_4 & c_{003} + c_{013}\sigma_4 \\ c_{101} & c_{102} & c_{103} \\ c_{111} & c_{112} & c_{113} \end{vmatrix} = \begin{vmatrix} g_1 & g_2 & g_3 \\ g_{1,3} & g_{2,3} & g_{3,3} \\ g_{1,34} & g_{2,34} & g_{3,34} \end{vmatrix} = 0. \end{aligned} \quad (15)$$

2.2. $n = 7$

We consider the map

$$\phi_4 : S(0, 1, 2)^4 \rightarrow S(1, 2, 3), (f_1, f_2, f_3, f_4) \mapsto f_1 g_1 + f_2 g_2 + f_3 g_3 + f_4 g_4. \quad (16)$$

In this case the range and image of ϕ_4 have dimension 24. The monomial basis of $S(0, 1, 2)$ is $\{1\} \otimes \{1, \sigma_4\} \otimes \{1, \sigma_5, \sigma_5^2\} = \{1, \sigma_4, \sigma_5, \sigma_4\sigma_5, \sigma_5^2, \sigma_4\sigma_5^2\}$ and that of $S(1, 2, 3)$ is $\{1, \sigma_3\} \otimes \{1, \sigma_4, \sigma_4^2\} \otimes \{1, \sigma_5, \sigma_5^2, \sigma_5^3\} = \{1, \sigma_3, \sigma_4, \sigma_3\sigma_4, \sigma_4^2, \sigma_3\sigma_4^2, \sigma_5, \sigma_3\sigma_5, \sigma_4\sigma_5, \sigma_3\sigma_4\sigma_5, \sigma_4^2\sigma_5, \sigma_3\sigma_4^2\sigma_5, \sigma_5^2, \sigma_3\sigma_5^2, \sigma_4\sigma_5^2, \sigma_3\sigma_4\sigma_5^2, \sigma_4^2\sigma_5^2, \sigma_3\sigma_4^2\sigma_5^2, \sigma_5^3, \sigma_3\sigma_5^3, \sigma_4\sigma_5^3, \sigma_3\sigma_4\sigma_5^3, \sigma_4^2\sigma_5^3, \sigma_3\sigma_4^2\sigma_5^3\}$.

From (6) the four scattering equations are

$$g_i = c_{000i} + c_{100i}\sigma_3 + c_{010i}\sigma_4 + c_{001i}\sigma_5 + c_{110i}\sigma_3\sigma_4 + c_{101i}\sigma_3\sigma_5 + c_{011i}\sigma_4\sigma_5 + c_{111i}\sigma_3\sigma_4\sigma_5 = 0, \quad i = 1, 2, 3, 4. \quad (17)$$

We consider the following set of 24 equations

$$g_i = \sigma_4 g_i = \sigma_5 g_i = \sigma_4 \sigma_5 g_i = \sigma_5^2 g_i = \sigma_4 \sigma_5^2 g_i = 0, \quad i = 1, 2, 3, 4 \quad (18)$$

and we expand them in the monomial basis of $S(1, 2, 3)$. Then the sought 24 degree polynomial in σ_6 is given by the 24×24 determinant

$$\begin{vmatrix} B_0 & 0 & 0 \\ B_1 & B_0 & 0 \\ 0 & B_1 & B_0 \\ 0 & 0 & B_1 \end{vmatrix} = 0, \quad B_{j_1} = \begin{pmatrix} A_{0j_1} & 0 \\ A_{1j_1} & A_{0j_1} \\ 0 & A_{1j_1} \end{pmatrix}, \quad A_{j_1 j_2} = \begin{pmatrix} c_{0j_1 j_2 1} & c_{0j_1 j_2 2} & c_{0j_1 j_2 3} & c_{0j_1 j_2 4} \\ c_{1j_1 j_2 1} & c_{1j_1 j_2 2} & c_{1j_1 j_2 3} & c_{1j_1 j_2 4} \end{pmatrix}. \quad (19)$$

The range of j_i is 0,1.

We now need to express $\sigma_3, \sigma_4, \sigma_5$ as functions of σ_6 . The construction is identical to the one in the $n = 6$ case. We can consider any of the 24 equations in (17) and solve the corresponding linear system for the desired σ_i . We arbitrarily choose the first 23 equations. Then the desired 23×23 determinant in σ_3 is constructed from the 24×24 determinant in (19) by removing the last column, multiplying the second row by σ_3 and adding that to the first row and finally by deleting the second row. For the σ_4 polynomial we do the same but now instead of the second row we consider the third row and for the σ_5 polynomial we consider the seventh row according to the canonical order of our basis.

It is possible to expand (19) in terms of 4-brackets using its Laplace expansion. We consider the first four columns of (19) and view this as a 24×4 matrix. We then form all possible 4-brackets made out of any non-zero four rows of the aforementioned 24×4 matrix. There are 70 possibilities. We next form all possible non-zero 4-brackets made out of the columns 5, 6, 7, 8 of (19). We repeat this until we exhaust all columns of (19). We end up with six groups of 70 4-brackets each. We then form all the products that contain one 4-bracket of each group such that each row of any 4-bracket appears only once. Of course in doing the Laplace expansion an appropriate sign needs to be considered. We end up with an expression that contains 3274 terms of products of six 4-brackets each. It is possible to express an expansion of (19) in terms of 4-brackets as a 6×6 determinant with elements 4-brackets. This construction will be presented in the next section.

2.3. $n = 8$

In order to easier understand the general case it is instructive to study the $n = 8$ case that requires the construction of a 120×120 determinant. We consider the map

$$\phi_5 : S(0, 1, 2, 3)^5 \rightarrow S(1, 2, 3, 4), (f_1, f_2, f_3, f_4, f_5) \mapsto \sum_{i=1}^5 f_i g_i. \quad (20)$$

The construction is similar to the previous cases. We have worked it out and found that the 120 degree polynomial in σ_7 is given by the 120×120 determinant

$$\begin{vmatrix} C_0 & 0 & 0 & 0 \\ C_1 & C_0 & 0 & 0 \\ 0 & C_1 & C_0 & 0 \\ 0 & 0 & C_1 & C_0 \\ 0 & 0 & 0 & C_1 \end{vmatrix} = 0, \quad (21)$$

where

$$C_{j_1} = \begin{pmatrix} B_{0j_1} & 0 & 0 \\ B_{1j_1} & B_{0j_1} & 0 \\ 0 & B_{1j_1} & B_{0j_1} \\ 0 & 0 & B_{1j_1} \end{pmatrix}, \quad B_{j_1 j_2} = \begin{pmatrix} A_{0j_1 j_2} & 0 \\ A_{1j_1 j_2} & A_{0j_1 j_2} \\ 0 & A_{1j_1 j_2} \end{pmatrix} \quad (22)$$

and

$$A_{j_1 j_2 j_3} = \begin{pmatrix} c_{0j_1 j_2 j_3 1} & c_{0j_1 j_2 j_3 2} & c_{0j_1 j_2 j_3 3} & c_{0j_1 j_2 j_3 4} & c_{0j_1 j_2 j_3 5} \\ c_{1j_1 j_2 j_3 1} & c_{1j_1 j_2 j_3 2} & c_{1j_1 j_2 j_3 3} & c_{1j_1 j_2 j_3 4} & c_{1j_1 j_2 j_3 5} \end{pmatrix}, \quad j_i = 0, 1. \quad (23)$$

An expansion in terms of 5-brackets can be constructed by considering groups of five columns of (21), forming all possible 5-brackets and then forming products of the 5-brackets. The final answer is a sum and difference of terms each one containing the product of 24 5-brackets.

2.4. General case

For general n we consider the map

$$\phi_{n-3} : S(0, 1, \dots, n-5)^{n-3} \rightarrow S(1, 2, \dots, n-4), (f_1, f_2, \dots, f_{n-3}) \mapsto \sum_{i=1}^{n-3} f_i g_i. \quad (24)$$

The dimension of the range and image of ϕ_{n-3} is $(n-3)!$. We introduce the notation

$$A_{j_1 j_2 \dots j_{n-5}}^{(2)} = \begin{pmatrix} c_{0j_1 j_2 \dots j_{n-5} 1} & c_{0j_1 j_2 \dots j_{n-5} 2} & \cdots & c_{0j_1 j_2 \dots j_{n-5} (n-3)} \\ c_{1j_1 j_2 \dots j_{n-5} 1} & c_{1j_1 j_2 \dots j_{n-5} 2} & \cdots & c_{1j_1 j_2 \dots j_{n-5} (n-3)} \end{pmatrix} \quad (25)$$

and

$$A_{j_1 j_2 \dots j_{n-m-3}}^{(m)} = \underbrace{\begin{pmatrix} A_{0j_1 j_2 \dots j_{n-m-3}}^{(m-1)} & 0 & \cdots & 0 \\ A_{1j_1 j_2 \dots j_{n-m-3}}^{(m-1)} & A_{0j_1 j_2 \dots j_{n-m-3}}^{(m-1)} & \cdots & 0 \\ 0 & A_{1j_1 j_2 \dots j_{n-m-3}}^{(m-1)} & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & A_{1j_1 j_2 \dots j_{n-m-3}}^{(m-1)} \end{pmatrix}}_{m-1}, \quad m \geq 2. \quad (26)$$

Then the $(n-3)!$ order polynomial in σ_{n-1} is given by the determinant

$$\det A^{(n-3)} = 0. \quad (27)$$

It is easy to solve the recursion (25), (26) and explicitly write down $A^{(n-3)}$. The final answer is a repeated structure isolated from the non-zero rows of the first $(n-3)$ columns. The elimination of the rest of the variables as well as the expansion on $A^{(n-3)}$ in terms of n -brackets follows from the discussion of the $n=7$ case. We should also comment that the determinant of $A^{(n-3)}$ can be associated to the Cayley hyperdeterminant of a hypermatrix of the boundary format with entries the coefficients of (6).

2.5. How to evaluate the amplitudes

One of the motivations of this work was to find a way to explicitly evaluate the scattering amplitudes using the scattering equations. We have succeeded in doing something more. In fact one can in principle explicitly evaluate any expression of the form

$$\sum_{\text{roots}} f(\sigma_i), \quad (28)$$

where the sum runs over the solutions of the generalized scattering equations (5) and $f(\sigma_i)$ is an arbitrary rational function of the variables that appear in the scattering equations. The idea is simple to understand and it was first presented in [35], enhanced in [36] and further elaborated in [37]. One uses the elimination to express the function $f(\sigma_i)$ in terms of σ_{n-1} only. We call the resulted function $g(\sigma_{n-1})$. Then, the sum of roots of (28) is given purely in terms of the coefficients of the $(n-3)!$ order polynomial in σ_{n-1} using the Vieta formulae for the roots of a polynomial. Furthermore, it is a rational function. A way to efficiently do this is via the help of the companion matrices ([36] and [37]). One replaces σ_{n-1} in $g(\sigma_{n-1})$ by its companion matrix, $T(\sigma_{n-1})$, performs the multiplication and summation of the matrices and then takes the trace of the final matrix. Schematically we have

$$\sum_{\text{roots}} f(\sigma_i) = \sum_{\text{roots}} g(\sigma_{n-1}) = \text{Tr}[g(T(\sigma_{n-1}))]. \quad (29)$$

For more details and for some examples we refer the reader to the aforementioned literature.

3. Elimination in the scattering equations via Bézout type determinants

As we have seen in the previous section, the Sylvester determinant is an elegant way to compute the resultant of a set of multivariate polynomials. In the multilinear case we considered we ended up with determinants of size $(n-3)!$. The Sylvester method is by no means the only way to use elimination. In this section we consider an alternative to Sylvester and we discuss the construction of the so-called Bézout formula for the scattering equations. The resultant coming from the determinant of the Bézout formula has dimension $(n-4)!$ and it is equivalent to the one obtained in the previous section. The elements of the Bézout formula are immediately given in terms of $(n-3)$ -brackets. Other mixed approaches of intermediate dimensionality can also be considered but we will not discuss them here.

Although the construction of the Bézout formula applies straightforwardly to general n , with a factorially increasing difficulty, for the sake of simplicity we are going to explicitly consider only the particular cases $n=6$ and $n=7$. The general idea is simple and was originally considered in [42] although in this section we will follow [43].

For a polynomial set of the general form (6) we define the matrix whose entries are given by

$$M_{1m} = g_m(\sigma_3, \dots, \sigma_{n-2}), \quad m = 1, 2, \dots, n-3, \quad i = 2, \dots, n-3, \\ M_{im} = \frac{g_m(\tilde{\sigma}_3, \dots, \tilde{\sigma}_{i+1}, \sigma_{i+2}, \dots, \sigma_{n-2}) - g_m(\tilde{\sigma}_3, \dots, \tilde{\sigma}_i, \sigma_{i+1}, \dots, \sigma_{n-2})}{\tilde{\sigma}_i - \sigma_i}. \quad (30)$$

The determinant of this matrix is a polynomial known as the affine bézoutian, which we denote as $B(\sigma, \tilde{\sigma})$. It can be proved that the monomials

$$\sigma_3^{\alpha_3} \dots \sigma_{n-2}^{\alpha_{n-2}} \tilde{\sigma}_3^{\beta_3} \dots \tilde{\sigma}_{n-2}^{\beta_{n-2}}, \quad (31)$$

composing the affine bézoutian satisfy the following relation,

$$\alpha_i < i-2, \quad \beta_i < (n-2)-i, \quad (32)$$

and hence,

$$B \in S(0, 1, \dots, n-5) \otimes S(n-5, n-6, \dots, 0)^*. \quad (33)$$

This in turn implies that the affine bézoutian is independent of σ_3 and $\tilde{\sigma}_{n-2}$. In the formulae that follow we subtract σ_3 times the second row from the first row in (30) in order to explicitly eliminate σ_3 . This is possible because $\det M = 0$ upon use of the scattering equations. More importantly, the affine bézoutian can be interpreted as a linear map from the dual vector space $S(n-5, n-6, \dots, 0)^*$ to $S(0, 1, \dots, n-5)$, which is represented by a $(n-4)! \times (n-4)!$ matrix B , i.e., the coefficients of the affine bézoutian are the components of the aforementioned matrix B , which in the mathematical literature is called the Bézout formula. Since the components of the Bézout formula come from the computation of a determinant, namely $\det M$, they can be written in terms of brackets. Here we sketch one way to obtain them.

First choose an order for the monomial basis and think of it as a vector. Take into account that there are two such vectors, one in the basis built out of σ s for $S(0, 1, \dots, n-5)$ and the other in the dual basis built out of $\tilde{\sigma}$ s for the dual space $S(n-5, n-6, \dots, 0)^*$.

Let us consider the bracket composition of the (l, m) th element of the matrix B , namely B_{lm} . The l th $\otimes$ m th component of the basis $S(0, 1, \dots, n-5) \otimes S(n-5, n-6, \dots, 0)^*$ is given by an element in the set (31) with values for α_i and β_j that depend on the chosen order or our monomial basis. Then we have

$$B_{lm} = \left[\prod_{i=4}^{n-2} \frac{1}{\alpha_i!} (\partial_{\sigma_i})^{\alpha_i} \prod_{j=3}^{n-3} \frac{1}{\beta_j!} (\partial_{\tilde{\sigma}_j})^{\beta_j} \right] \det(M)|_{(\sigma, \tilde{\sigma})=0}. \quad (34)$$

The notation $|_{(\sigma, \tilde{\sigma})=0}$ means we set all the σ s and $\tilde{\sigma}$ s to zero after taking the derivatives. Let us now move to some particular examples.

3.1. $n = 6$

In the case of six particles, the polynomial form of the scattering equations is given by (8). For this polynomial set, the i th column of the M matrix is given by,

$$M_6 = \begin{pmatrix} c_{00i} + c_{01i}\sigma_4 \\ c_{10i} + c_{11i}\sigma_4 \\ c_{01i} + c_{11i}\tilde{\sigma}_3 \end{pmatrix}. \quad (35)$$

The determinant of M is given by a polynomial in $S(0, 1) \otimes S(1, 0)^*$ and the linear map from $S(1, 0)^*$ to $S(0, 1)$ in the basis $\{1, \sigma_4\} \otimes \{1, \tilde{\sigma}_3\}$ can be conveniently written as,

$$B_6 = \begin{pmatrix} [00, 10, 01] & [00, 10, 11] \\ [00, 01, 11] & [10, 01, 11] \end{pmatrix}, \quad (36)$$

where the 3-brackets were defined in (13). The determinant of B_6 produces the same resultant as the one given in (10).

3.2. $n = 7$

The scattering equations are given by (17). For this polynomial set, the i th column of the M matrix is given by,

$$M_7 = \begin{pmatrix} c_{000i} + c_{001i}\sigma_5 + c_{010i}\sigma_4 + c_{011i}\sigma_4\sigma_5 \\ c_{100i} + c_{101i}\sigma_5 + c_{110i}\sigma_4 + c_{111i}\sigma_4\sigma_5 \\ c_{010i} + c_{011i}\sigma_5 + c_{110i}\tilde{\sigma}_3 + c_{011i}\tilde{\sigma}_3\sigma_5 \\ c_{001i} + c_{011i}\tilde{\sigma}_4 + c_{101i}\tilde{\sigma}_3 + c_{111i}\tilde{\sigma}_3\tilde{\sigma}_4 \end{pmatrix}. \quad (37)$$

We compute the matrix of the linear map from $S(2, 1, 0)^*$ to $S(0, 1, 2)$ in the basis $(1, \sigma_5, \sigma_5^2, \sigma_4, \sigma_4\sigma_5, \sigma_4\sigma_5^2) \otimes (1, \tilde{\sigma}_4, \tilde{\sigma}_3, \tilde{\sigma}_4\tilde{\sigma}_3, \tilde{\sigma}_3^2, \tilde{\sigma}_4\tilde{\sigma}_3^2)$, to be

$$B_7 = \begin{pmatrix} A_{4 \times 4} & B_{4 \times 2} \\ C_{2 \times 4} & D_{2 \times 2} \end{pmatrix} \quad (38)$$

with

$$\begin{aligned}
A &= \begin{pmatrix} [000, 001, 010, 100] & [000, 010, 011, 100] & [000, 100, 101, 110] & [000, 100, 110, 111] \\ -[000, 001, 011, 101] & [001, 010, 011, 101] & [001, 100, 101, 111] & [001, 101, 110, 111] \\ -[000, 001, 010, 110] & [000, 010, 011, 110] & [010, 100, 101, 110] & [010, 100, 110, 111] \\ -[000, 001, 011, 111] & [001, 010, 011, 111] & [011, 100, 101, 111] & [011, 101, 110, 111] \end{pmatrix}, \\
B &= \begin{pmatrix} [000, 001, 100, 110] - [000, 010, 100, 101] & [000, 011, 100, 110] - [000, 010, 100, 111] \\ [000, 001, 101, 111] - [001, 011, 100, 101] & [001, 011, 101, 110] - [001, 010, 101, 111] \\ [000, 010, 101, 110] - [001, 010, 100, 110] & [010, 011, 100, 110] - [000, 010, 110, 111] \\ [000, 011, 101, 111] - [001, 011, 100, 111] & [010, 011, 101, 111] - [001, 011, 110, 111] \end{pmatrix}, \\
C &= \begin{pmatrix} -[000, 001, 010, 101] & [000, 010, 011, 101] & [000, 100, 101, 111] & [000, 101, 110, 111] \\ -[000, 001, 011, 100] & +[001, 010, 011, 100] & +[001, 100, 101, 110] & +[001, 100, 110, 111] \\ -[000, 001, 010, 111] & [000, 010, 011, 111] & [010, 100, 101, 111] & [010, 101, 110, 111] \\ -[000, 001, 011, 110] & +[001, 010, 011, 110] & +[011, 100, 101, 110] & +[011, 100, 110, 111] \end{pmatrix}, \\
D &= \begin{pmatrix} [000, 001, 100, 111] + [000, 001, 101, 110] & -[000, 010, 101, 111] + [000, 011, 101, 110] \\ -[000, 011, 100, 101] - [001, 010, 100, 101] & -[001, 010, 100, 111] + [001, 011, 100, 110] \\ [000, 010, 101, 111] + [000, 011, 101, 110] & -[000, 011, 110, 111] - [001, 010, 110, 111] \\ -[001, 010, 100, 111] - [001, 011, 100, 110] & +[010, 011, 100, 111] + [010, 011, 101, 110] \end{pmatrix},
\end{aligned}$$

where we have used the bracket notation (13).

The expansion of the determinant above gives us the 24 degree polynomial in σ_6 corresponding to the resultant of the system (17).

4. Discussion

In this work we applied the classical elimination theory in the context of the scattering equations. We chose to use a generalized set of the scattering equations in their polynomial form which leads to more compact expressions. We have achieved to construct the one variable polynomial of degree $(n-3)!$ in one of the variables of the scattering equations. The answer is given compactly by the determinant of a $(n-3)! \times (n-3)!$ matrix of Sylvester type. Then we expressed the rest of the variables of the scattering equations as function of one of the variables. The determinant we found satisfies on-shell recursion relations and it would be interesting to study its relation to BCFW recursions [44,45].

One of the features of our relations is that one can interestingly expand our expressions in terms of n -brackets that can be viewed as Plücker coordinates on the variety of lines in P^n . One way to see this is to use a particular Laplace expansion of the Sylvester determinant. Then the Sylvester determinant has the interpretation of the Chow form of the subvariety $P^1 \times P^1 \times \dots \times P^1$ in the Segre embedding [46].

There is an alternative way to use the elimination, namely the Bézout construction. In this case one ends up with a $(n-4)! \times (n-4)!$ determinant with elements constructed out of $(n-3)$ -brackets. The expansion of the Bézout determinant gives the same polynomials as the Sylvester determinant after making use of Plücker relations. Mixed determinantal expressions of intermediate dimensionality should also exist but we did not investigate this here.

Our derivation plays the role of the construction of a Gröbner basis of the scattering equations. One of the applications of our results is the evaluation of the sum over the solutions of the generalized scattering equations of any rational expression of the variables that appear in the scattering equation. This can now be achieved by pure algebraic manipulations. The idea is based on the fact that the sum over roots is given by particular combinations of the coefficients of the scattering equations via simple algebra. Therefore, one does not need to explicitly know the solutions of the equations. In fact one cannot know explicitly the solutions in view of the Abel–Ruffini theorem. In the case of the scattering amplitudes, we hope that specialized versions of the elimination could lead to huge calculational simplifications.

There is further a connection of the Sylvester determinant and the Cayley hyperdeterminant of a particular hypermatrix with elements the coefficients of the scattering equations.

One could hope that our construction can lead to a further investigation of the scattering equations and amplitudes and even perhaps connections with different geometrical and combinatorial structures.

Acknowledgements

It is a pleasure to thank Humberto Gomez and Francisco Rojas for useful comments and discussions. The work of C.C. is supported in part by the National Center for Theoretical Sciences (NCTS), Taiwan. The work of C.K. is supported by the São Paulo Research Foundation (FAPESP) under grants 2011/11973-4 and 2012/00756-5.

References

- [1] E. Witten, Perturbative gauge theory as a string theory in twistor space, Commun. Math. Phys. 252 (2004) 189–258, <http://dx.doi.org/10.1007/s00220-004-1187-3>, arXiv:hep-th/0312171.

- [2] F. Cachazo, S. He, E.Y. Yuan, Scattering equations and Kawai–Lewellen–Tye orthogonality, *Phys. Rev. D* 90 (6) (2014) 065001, <http://dx.doi.org/10.1103/PhysRevD.90.065001>, arXiv:1306.6575.
- [3] F. Cachazo, S. He, E.Y. Yuan, Scattering of massless particles in arbitrary dimensions, *Phys. Rev. Lett.* 113 (17) (2014) 171601, <http://dx.doi.org/10.1103/PhysRevLett.113.171601>, arXiv:1307.2199.
- [4] F. Cachazo, S. He, E.Y. Yuan, Scattering of massless particles: scalars, gluons and gravitons, *J. High Energy Phys.* 1407 (2014) 033, [http://dx.doi.org/10.1007/JHEP07\(2014\)033](http://dx.doi.org/10.1007/JHEP07(2014)033), arXiv:1309.0885.
- [5] F. Cachazo, S. He, E.Y. Yuan, Einstein–Yang–Mills scattering amplitudes from scattering equations, *J. High Energy Phys.* 1501 (2015) 121, [http://dx.doi.org/10.1007/JHEP01\(2015\)121](http://dx.doi.org/10.1007/JHEP01(2015)121), arXiv:1409.8256.
- [6] F. Cachazo, S. He, E.Y. Yuan, Scattering equations and matrices: from Einstein to Yang–Mills, DBI and NLSM, arXiv:1412.3479.
- [7] L. Dolan, P. Goddard, Proof of the formula of Cachazo, He and Yuan for Yang–Mills tree amplitudes in arbitrary dimension, *J. High Energy Phys.* 1405 (2014) 010, [http://dx.doi.org/10.1007/JHEP05\(2014\)010](http://dx.doi.org/10.1007/JHEP05(2014)010), arXiv:1311.5200.
- [8] D. Fairlie, D. Roberts, Dual models without tachyons – a new approach (unpublished Durham preprint PRINT-72-2440, 1972).
- [9] D. Roberts, Mathematical structure of dual amplitudes, Durham PhD thesis, 1972, p. 73 f. [available at Durham E-Theses online].
- [10] D.B. Fairlie, A coding of real null four-momenta into world-sheet co-ordinates, *Adv. Math. Phys.* 2009 (2009) 284689, <http://dx.doi.org/10.1155/2009/284689>, arXiv:0805.2263.
- [11] D.J. Gross, P.F. Mende, String theory beyond the Planck scale, *Nucl. Phys. B* 303 (1988) 407, [http://dx.doi.org/10.1016/0550-3213\(88\)90390-2](http://dx.doi.org/10.1016/0550-3213(88)90390-2).
- [12] E. Witten, Parity invariance for strings in twistor space, *Adv. Theor. Math. Phys.* 8 (2004) 779–796, <http://dx.doi.org/10.4310/ATMP.2004.v8.n5.a1>, arXiv:hep-th/0403199.
- [13] P. Caputa, S. Hirano, Observations on open and closed string scattering amplitudes at high energies, *J. High Energy Phys.* 1202 (2012) 111, [http://dx.doi.org/10.1007/JHEP02\(2012\)111](http://dx.doi.org/10.1007/JHEP02(2012)111), arXiv:1108.2381.
- [14] P. Caputa, Lightlike contours with fermions, *Phys. Lett. B* 716 (2012) 475–480, <http://dx.doi.org/10.1016/j.physletb.2012.09.006>, arXiv:1205.6369.
- [15] Y. Makeenko, P. Olesen, The QCD scattering amplitude from area behaved Wilson loops, *Phys. Lett. B* 709 (2012) 285–288, <http://dx.doi.org/10.1016/j.physletb.2012.02.032>, arXiv:1111.5606.
- [16] F. Cachazo, Fundamental BCJ relation in $N = 4$ SYM from the connected formulation, arXiv:1206.5970.
- [17] L. Dolan, P. Goddard, The polynomial form of the scattering equations, *J. High Energy Phys.* 1407 (2014) 029, [http://dx.doi.org/10.1007/JHEP07\(2014\)029](http://dx.doi.org/10.1007/JHEP07(2014)029), arXiv:1402.7374.
- [18] S.G. Naculich, Scattering equations and BCJ relations for gauge and gravitational amplitudes with massive scalar particles, *J. High Energy Phys.* 1409 (2014) 029, [http://dx.doi.org/10.1007/JHEP09\(2014\)029](http://dx.doi.org/10.1007/JHEP09(2014)029), arXiv:1407.7836.
- [19] S. Weinzierl, On the solutions of the scattering equations, *J. High Energy Phys.* 1404 (2014) 092, [http://dx.doi.org/10.1007/JHEP04\(2014\)092](http://dx.doi.org/10.1007/JHEP04(2014)092), arXiv:1402.2516.
- [20] C. Kalousios, Massless scattering at special kinematics as Jacobi polynomials, *J. Phys. A* 47 (2014) 215402, <http://dx.doi.org/10.1088/1751-8113/47/21/215402>, arXiv:1312.7743.
- [21] C. Lam, Permutation symmetry of the scattering equations, *Phys. Rev. D* 91 (4) (2015) 045019, <http://dx.doi.org/10.1103/PhysRevD.91.045019>, arXiv:1410.8184.
- [22] Y.-H. He, C. Matti, C. Sun, The scattering variety, *J. High Energy Phys.* 10 (2014) 135, [http://dx.doi.org/10.1007/JHEP10\(2014\)135](http://dx.doi.org/10.1007/JHEP10(2014)135), arXiv:1403.6833.
- [23] B.U.W. Schwab, A. Volovich, Subleading soft theorem in arbitrary dimensions from scattering equations, *Phys. Rev. Lett.* 113 (10) (2014) 101601, <http://dx.doi.org/10.1103/PhysRevLett.113.101601>, arXiv:1404.7749.
- [24] F. Cachazo, S. He, E.Y. Yuan, New double soft emission theorems, *Phys. Rev. D* 92 (6) (2015) 065030, <http://dx.doi.org/10.1103/PhysRevD.92.065030>, arXiv:1503.04816.
- [25] N. Afkhami-Jeddi, Soft graviton theorem in arbitrary dimensions, arXiv:1405.3533.
- [26] C. Kalousios, F. Rojas, Next to subleading soft-graviton theorem in arbitrary dimensions, *J. High Energy Phys.* 01 (2015) 107, [http://dx.doi.org/10.1007/JHEP01\(2015\)107](http://dx.doi.org/10.1007/JHEP01(2015)107), arXiv:1407.5982.
- [27] S.G. Naculich, CHY representations for gauge theory and gravity amplitudes with up to three massive particles, arXiv:1501.03500.
- [28] S.G. Naculich, Amplitudes for massive vector and scalar bosons in spontaneously-broken gauge theory from the CHY representation, *J. High Energy Phys.* 09 (2015) 122, [http://dx.doi.org/10.1007/JHEP09\(2015\)122](http://dx.doi.org/10.1007/JHEP09(2015)122), arXiv:1506.06134.
- [29] S. Weinzierl, Fermions and the scattering equations, arXiv:1412.5993.
- [30] L. de la Cruz, A. Kniss, S. Weinzierl, The CHY representation of tree-level primitive QCD amplitudes, arXiv:1508.06557.
- [31] L. Mason, D. Skinner, Ambitwistor strings and the scattering equations, *J. High Energy Phys.* 1407 (2014) 048, [http://dx.doi.org/10.1007/JHEP07\(2014\)048](http://dx.doi.org/10.1007/JHEP07(2014)048), arXiv:1311.2564.
- [32] N. Berkovits, Infinite tension limit of the pure spinor superstring, *J. High Energy Phys.* 1403 (2014) 017, [http://dx.doi.org/10.1007/JHEP03\(2014\)017](http://dx.doi.org/10.1007/JHEP03(2014)017), arXiv:1311.4156.
- [33] N.E.J. Bjerrum-Bohr, P.H. Damgaard, P. Tourkine, P. Vanhove, Scattering equations and string theory amplitudes, *Phys. Rev. D* 90 (10) (2014) 106002, <http://dx.doi.org/10.1103/PhysRevD.90.106002>, arXiv:1403.4553.
- [34] E. Casali, Y. Geyer, L. Mason, R. Monteiro, K.A. Roehrig, New ambitwistor string theories, *J. High Energy Phys.* 11 (2015) 038, [http://dx.doi.org/10.1007/JHEP11\(2015\)038](http://dx.doi.org/10.1007/JHEP11(2015)038), arXiv:1506.08771.
- [35] C. Kalousios, Scattering equations, generating functions and all massless five point tree amplitudes, *J. High Energy Phys.* 05 (2015) 054, [http://dx.doi.org/10.1007/JHEP05\(2015\)054](http://dx.doi.org/10.1007/JHEP05(2015)054), arXiv:1502.07711.
- [36] R. Huang, J. Rao, B. Feng, Y.-H. He, An algebraic approach to the scattering equations, arXiv:1509.04483.
- [37] C. Cardona, C. Kalousios, Comments on the evaluation of massless scattering, arXiv:1509.08908.
- [38] M. Sogaard, Y. Zhang, Scattering equations and global duality of residues, arXiv:1509.08897.
- [39] F. Cachazo, H. Gomez, Computation of contour integrals on $\mathcal{M}_{0,n}$, arXiv:1505.03571.
- [40] C. Baadsgaard, N.E.J. Bjerrum-Bohr, J.L. Bourjaily, P.H. Damgaard, Integration rules for scattering equations, *J. High Energy Phys.* 09 (2015) 129, [http://dx.doi.org/10.1007/JHEP09\(2015\)129](http://dx.doi.org/10.1007/JHEP09(2015)129), arXiv:1506.06137.
- [41] C. Baadsgaard, N.E.J. Bjerrum-Bohr, J.L. Bourjaily, P.H. Damgaard, Scattering equations and Feynman diagrams, *J. High Energy Phys.* 09 (2015) 136, [http://dx.doi.org/10.1007/JHEP09\(2015\)136](http://dx.doi.org/10.1007/JHEP09(2015)136), arXiv:1507.00997.
- [42] A. Dickstein, I. Emiris, Multihomogeneous resultant matrices, ISSAC, <http://mate.dm.uba.ar/~alidick>, 2002.
- [43] B. Sturmfels, Solving Systems of Polynomial Equations, CBMS Regional Conference Series in Mathematics, American Mathematical Society, 2002, <https://math.berkeley.edu/~bernd/cbms.pdf>.
- [44] R. Britto, F. Cachazo, B. Feng, New recursion relations for tree amplitudes of gluons, *Nucl. Phys. B* 715 (2005) 499–522, <http://dx.doi.org/10.1016/j.nuclphysb.2005.02.030>, arXiv:hep-th/0412308.
- [45] R. Britto, F. Cachazo, B. Feng, E. Witten, Direct proof of tree-level recursion relation in Yang–Mills theory, *Phys. Rev. Lett.* 94 (2005) 181602, <http://dx.doi.org/10.1103/PhysRevLett.94.181602>, arXiv:hep-th/0501052.
- [46] I.M. Gelfand, M.M. Kapranov, A.V. Zelevinsky, Discriminants, Resultants, and Multidimensional Determinants, Birkhäuser, Boston, 1994.