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Szegő polynomials: some relations to L-orthogonal and orthogonal polynomials☆

C.F. Bracciali*, A.P. da Silva, A. Sri Ranga

Departamento de Ciências de Computação e Estatística, Instituto de Biociências, Letras e Ciências Exatas, Universidade Estadual Paulista (UNESP), Rua Cristóvão Colombo 2265, 15054-000 São José do Rio Preto, SP, Brazil

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Abstract

We consider the real Szegő polynomials and obtain some relations to certain self inversive orthogonal L-polynomials defined on the unit circle and corresponding symmetric orthogonal polynomials on real intervals. We also consider the polynomials obtained when the coefficients in the recurrence relations satisfied by the self inversive orthogonal L-polynomials are rotated.

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1. Introduction

Let $\mathrm{d}v(z)$ be a positive measure on the unit circle C. This means $v(\mathrm{e}^{\mathrm{i}\theta})$ is a real, bounded and non-decreasing function for $0 \leqslant \theta \leqslant 2\pi$. We consider the Szegő polynomials $\{S_n\}$ associated with the measure $\mathrm{d}v(z)$ defined by $\int_C S_n(z)\overline{S_m(z)}\,\mathrm{d}v(z) = 0$, $n \neq m$. These polynomials were introduced by Szegő (see for example [8]). For a good source for some basic information on these polynomials we refer to [10].

Since $\bar{z} = 1/z$ on the unit circle, the polynomials S_n can also be defined by $\int_C z^{-n+s} S_n(z) z\,\mathrm{d}v(z) = 0$, $0 \leqslant s \leqslant n-1$. Hence, the Szegő polynomials also satisfy the L-orthogonality property on the unit

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* Corresponding author.
E-mail address: cleonice@dcce.ibilce.unesp.br (C.F. Bracciali).

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circle in relation to $z\,\mathrm{d}\nu(z)$. Polynomials satisfying the L-orthogonality property on the positive real axis were introduced in [5]. For a study of these polynomials on the unit circle see for example [4].

The Szegő polynomials (given here in their monic form) are known to satisfy the system of recurrence relations

$$\begin{aligned} &S_{n+1}(z) = zS_n(z) + a_{n+1}S_n^*(z), \\ &(1-|a_{n+1}|^2)zS_n(z) = S_{n+1}(z) - a_{n+1}S_{n+1}^*(z) \end{aligned} \tag{1.1}$$

for $n \geqslant 0$. Here $S_n^*(z) = z^n\bar{S}_n(1/z)$ are the reciprocal polynomials. The numbers $a_n = S_n(0)$, $n \geqslant 1$, which are less than one in modulus, are known as the reflection coefficients of the Szegő polynomials.

In this manuscript, we consider the Szegő polynomials with real reflection coefficients and take a look at the polynomials $\{S_n(z)+S_n^*(z)\}$, $\{S_n(z)-S_n^*(z)\}$ and their relations to certain symmetric orthogonal polynomials on the interval $[-1,1]$. These relations, found in [3], were very nicely explored in [12]. Zhedanov uses the information contained in the relations associated with $S_n(z)+S_n^*(z)$ (or $S_n(z)-S_n^*(z)$) to derive information about S_n from the corresponding orthogonal polynomials and vice versa. However, different to Zhedanov, we look at how one can use simultaneously the relations associated with both $S_n(z)+S_n^*(z)$ and $S_n(z)-S_n^*(z)$ to do the same. We also give information on the polynomials obtained when rotating the coefficients of the three term recurrence relation satisfied by the polynomials $S_n(z)+S_n^*(z)$.

2. The para-orthogonal polynomials

In [4], Jones et al. considered the polynomials $S_n(z)+\omega_nS_n^*(z)$, where $|\omega_n|=1$. They called these para-orthogonal polynomials and showed that their zeros are all distinct and lie on the unit circle. Their proof is based on the self-inversive properties of these polynomials and the conditions

$$\int_C z^{-n+s}[S_n(z)+\omega_nS_n^*(z)]\,\mathrm{d}\nu(z) = 0, \quad 1 \leqslant s \leqslant n-1. \tag{2.1}$$

Here, restricting our selves to only real Szegő polynomials, we consider the two special cases of para-orthogonal polynomials

$$S_n^{(1)}(z) = \frac{S_n(z)+S_n^*(z)}{1+S_n(0)} \quad \text{and} \quad S_n^{(2)}(z) = \frac{S_n(z)-S_n^*(z)}{1-S_n(0)}, \quad n \geqslant 1.$$

The denominators are chosen in order to make the polynomials monic.

The Szegő polynomials are real if and only if the measure $\mathrm{d}\nu(z)$ satisfies the symmetry $\mathrm{d}\nu(1/z) = -\mathrm{d}\nu(z)$. The following results up to the recurrence relations in Theorem 2.1 were first given in [3].

From (1.1), the polynomials $S_n^{(i)}$, $i=1,2$, can be shown to satisfy the simple three-term recurrence relations

$$\left.\begin{aligned} &S_{n+1}^{(1)}(z) = (z+1)S_n^{(1)}(z) - (1+a_{n-1})(1-a_n)zS_{n-1}^{(1)}(z), \\ &S_{n+1}^{(2)}(z) = (z+1)S_n^{(2)}(z) - (1-a_{n-1})(1+a_n)zS_{n-1}^{(2)}(z) \end{aligned}\right\} \quad n \geqslant 1$$

with $a_0 = 1$, $S_0^{(1)}(z) = 1$, $S_0^{(2)}(z) = 1$, $S_1^{(1)}(z) = z+1$ and $S_1^{(2)}(z) = z-1$.

Observe from the recurrence relation that $S_n^{(2)}(1)=0$ for $n \geqslant 1$. Thus, by letting $R_n^{(1)}(z)=S_n^{(1)}(z)$, $n \geqslant 0$ and $R_n^{(2)}(z)=(z-1)^{-1}S_{n+1}^{(2)}(z)$, $n \geqslant 0$, we obtain $2S_n(z)=(1+a_n)R_n^{(1)}(z)+(1-a_n)(z-1)R_{n-1}^{(2)}(z)$, $n \geqslant 1$, or equivalently,

$$2zS_{n-1}(z)=R_n^{(1)}(z)+(z-1)R_{n-1}^{(2)}(z), \quad n \geqslant 1. \tag{2.2}$$

Theorem 2.1. *The monic polynomials* $R_n^{(i)}$, $i=1,2$, *satisfy* $R_0^{(i)}=1$, $R_1^{(i)}(z)=z+1$ *and*

$$R_{n+1}^{(i)}(z)=(z+1)R_n^{(i)}(z)-4\alpha_{n+1}^{(i)}zR_{n-1}^{(i)}(z), \quad n \geqslant 1,$$

with $4\alpha_{n+1}^{(1)}=(1+a_{n-1})(1-a_n)>0$ *and* $4\alpha_{n+1}^{(2)}=(1-a_n)(1+a_{n+1})>0$, $n \geqslant 1$.

Moreover, these polynomials satisfy the L-orthogonality relations

$$\int_C z^{-n+s}R_n^{(1)}(z)\frac{z}{z-1}\,\mathrm{d}v(z)=0, \quad 0 \leqslant s \leqslant n-1 \tag{2.3}$$

and

$$\int_C z^{-n+s}R_n^{(2)}(z)(z-1)\,\mathrm{d}v(z)=0, \quad 0 \leqslant s \leqslant n-1. \tag{2.4}$$

Proof. The recurrence relations follows from above. The reason for choosing the multiplier 4 in the recurrence relation will become apparent after Theorem 3.1. The recurrence relations also confirm the self inversive property $z^nR_n^{(i)}(1/z)=R_n^{(i)}(z)$.

Now we give the proof of (2.4). Since

$$R_n^{(2)}(z)=\frac{S_{n+1}(z)-S_{n+1}^*(z)}{(z-1)(1-S_{n+1}(0))},$$

from (2.1)

$$\int_C z^{-(n+1)+s}R_n^{(2)}(z)(z-1)\,\mathrm{d}v(z)=0, \quad 1 \leqslant s \leqslant n.$$

Clearly, this is equivalent to (2.4). Now to prove (2.3), again from (2.1)

$$\int_C z^{-n+s}R_n^{(1)}(z)\frac{z-1}{z-1}\,\mathrm{d}v(z)=0, \quad 1 \leqslant s \leqslant n-1. \tag{2.5}$$

This is equivalent to

$$\int_C z^{-n}[P_{n,s}(z)]R_n^{(1)}(z)\frac{z}{z-1}\,\mathrm{d}v(z)=0, \quad 1 \leqslant s \leqslant n-1, \tag{2.6}$$

where $P_{n,s}(z)=(z-1)z^{s-1}$, $s=1,2,\ldots,n-1$. We now show that if we take $P_{n,n}(z)=z+z^{n-1}$ then the relation (2.6) also holds for $s=n$. From (2.5),

$$\int_C z^{-n+s}R_n^{(1)}(z)\frac{z}{z-1}\,\mathrm{d}v(z)=\int_C z^{-n+s}R_n^{(1)}(z)\frac{1}{z-1}\,\mathrm{d}v(z), \quad 1 \leqslant s \leqslant n-1.$$

Substituting z by $1/z$ on the right-hand side and using the symmetry of the measure and the self inversive property of $R_n^{(1)}$, we obtain

$$\int_C z^{-n}[z^s + z^{n-s}]R_n^{(1)}(z)\,\frac{z}{z-1}\,\mathrm{d}v(z) = 0, \quad 1 \leqslant s \leqslant n-1.$$

Hence, $P_{n,n}(z)$ is obtained when letting $s = 1$ in the expression $z^s + z^{n-s}$.

One can verify that the set of polynomials $P_{n,s}$, $s = 1, 2, \ldots, n$, of degree $\leqslant n-1$, forms a linearly independent set. In particular, for each of the monomials z^s, $0 \leqslant s \leqslant n-1$, we get a unique linear combination of the type $z^s = d_1^{(s)}P_{n,1}(z) + d_2^{(s)}P_{n,2}(z) + \cdots + d_{n-1}^{(s)}P_{n,n-1}(z) + \frac{1}{2}P_{n,n}(z)$. Consequently from (2.6) the result (2.3) of the theorem follows. □

3. Relation to orthogonal polynomials on the real line

The results in the previous section lead to consider what one can say about a sequence of polynomials $\{R_n\}$ satisfying the recurrence relation

$$R_{n+1}(z) = (z+1)R_n(z) - 4\alpha_{n+1} z R_{n-1}(z), \quad n \geqslant 1 \tag{3.1}$$

with $R_0(z) = 1$, $R_1(z) = z+1$ and $\alpha_{n+1} > 0$. We have the following theorem:

Theorem 3.1. *Let $\{R_n\}$ be the sequence of monic polynomials generated by the recurrence relation (3.1). Then the zeros of R_n are distinct (except for a possible double zero at $z = 1$) and lie on $C \cup (0, \infty)$. In particular, if $\{\alpha_{n+1}\}$ is a chain sequence then all the zeros are distinct and lie on the open unit circle $\{z : z = \mathrm{e}^{\mathrm{i}\theta}, 0 < \theta < 2\pi\}$. In this case, there exists a positive measure $\mathrm{d}v(z)$ on the unit circle such that*

$$\int_C z^{-n+s}R_n(z)\,\frac{z}{z-1}\,\mathrm{d}v(z) = 0, \quad 0 \leqslant s \leqslant n-1. \tag{3.2}$$

Proof. From the recurrence relation (3.1), $z^n R_n(1/z) = R_n(z)$. Let $x = x(z) = \frac{1}{2}(z^{1/2} + z^{-1/2})$. Here, given $z = r\mathrm{e}^{\mathrm{i}\theta}$ then $z^{1/2}$ is understood as $r^{1/2}\mathrm{e}^{\mathrm{i}\theta/2}$. Hence, the polynomials $P_n(x) = (4z)^{-n/2}R_n(z)$, $n \geqslant 0$, satisfy $P_0(x) = 1$, $P_1(x) = x$ and

$$P_{n+1}(x) = xP_n(x) - \alpha_{n+1}P_{n-1}(x), \quad n \geqslant 1. \tag{3.3}$$

From this it is well known that the zeros of P_n are real, distinct and lie symmetrically about the origin. If we write, $P_{2n}(x) = \prod_{k=1}^n (x^2 - x_{2n,k}^2)$, and $P_{2n+1}(x) = x\prod_{k=1}^n (x^2 - x_{2n+1,k}^2)$, then from $R_n(z) = (4z)^{n/2}P_n(x(z))$ we see that

$$R_{2n}(z) = \prod_{k=1}^{n} ((z - z_{2n,k})(z - 1/z_{2n,k})),$$

$$R_{2n+1}(z) = (z+1)\prod_{k=1}^{n} ((z - z_{2n+1,k})(z - 1/z_{2n+1,k})),$$

where $z_{n,k} = (2x_{n,k}^2 - 1) + 2\sqrt{x_{n,k}^2(x_{n,k}^2 - 1)}$. Hence, if $x_{n,k}^2 < 1$ then $z_{n,k}$ and $1/z_{n,k}$ are a conjugate pair of zeros of R_n on the unit circle and if $x_{n,i}^2 > 1$ then they are two positive zeros of R_n inverse to each other. If $z=1$ is a zero of R_n then it is a zero of multiplicity 2. Hence, we conclude that the zeros of R_n are either on the unit circle or on the positive real line.

Now if we assume $\{\alpha_{n+1}\}$ to be a chain sequence. That is, there exists a second sequence $\{g_n\}$, where $0 \leqslant g_0 < 1$ and $0 < g_n < 1$, $n \geqslant 1$, such that $\alpha_{n+1} = (1-g_{n-1})g_n$, $n \geqslant 1$. Then, it is well known (see for example [2]) that all the zeros of P_n are inside $(-1,1)$. Hence, in this case, all the zeros of R_n are on the open unit circle $\{z\colon z = e^{i\theta}, 0<\theta<2\pi\}$.

When the zeros of P_n, $n>1$, are within $(-1,1)$ then from Favard's Theorem it follows that these polynomials form a sequence of orthogonal polynomials in relation to a symmetric positive measure $d\phi$ with support inside $[-1,1]$. From the binomial expansion and the symmetry of P_n this is equivalent to (see also [7])

$$\int_{-1}^{1} \{x + i\sqrt{1-x^2}\}^{-(n-1)+2s} \frac{P_n(x)}{\sqrt{1-x^2}}\, d\phi(x) = 0, \quad 0 \leqslant s \leqslant n-1.$$

Letting $x = \frac{1}{2}(z^{1/2} + z^{-1/2})$, that is letting $z^{1/2} = x + i\sqrt{1-x^2}$, we obtain

$$\int_{-C} z^{-n+s} R_n(z) \frac{z}{z-1}\, d\phi(x(z)) = 0, \quad 0 \leqslant s \leqslant n-1.$$

Since $x(z)$ is a decreasing function of $z = e^{i\theta}$ as θ varies from 0 to 2π, we obtain the positive measure $d\nu(z) = -d\phi(x(z))$. □

In the recurrence relations of $R_n^{(1)}$ and $R_n^{(2)}$, the coefficients $\{\alpha_{n+1}^{(1)}\}$ and $\{\alpha_{n+1}^{(2)}\}$ are chain sequences with the respective parameter sequences $\{g_n^{(1)} = (1-a_n)/2\}$ and $\{g_n^{(2)} = (1+a_{n+1})/2\}$. That is,

$$(1-g_{n-1}^{(1)})g_n^{(1)} = \alpha_{n+1}^{(1)} \quad [\text{also } g_n^{(1)}(1-g_{n+1}^{(1)}) = \alpha_{n+1}^{(2)}],$$

$$(1-g_{n-1}^{(2)})g_n^{(2)} = \alpha_{n+1}^{(2)} \quad [\text{also } g_{n-2}^{(2)}(1-g_{n-1}^{(2)}) = \alpha_{n+1}^{(1)}] \tag{3.4}$$

for $n \geqslant 1$, with initial parameters $g_0^{(1)} = 0$ and $0 < g_0^{(2)} = 1 - \alpha_2^{(1)} = (1+a_1)/2 < 1$. Here we have let $g_{-1}^{(2)} = 1$. The polynomials $P_n^{(1)}(x) = (4z)^{-n/2} R_n^{(1)}(z)$ and $P_n^{(2)}(x) = (4z)^{-n/2} R_n^{(2)}(z)$ satisfy the recurrence relations

$$P_{n+1}^{(i)}(x) = xP_n^{(i)}(x) - \alpha_{n+1}^{(i)} P_{n-1}^{(i)}(x), \quad n \geqslant 1 \tag{3.5}$$

and are orthogonal polynomials on $[-1,1]$ related to the measures $d\phi^{(1)}(x) = -d\nu(z)$ and $d\phi^{(2)}(x) = -(1-x^2)\, d\nu(z)$, respectively.

Earlier investigations (see [1]) have shown us that given two positive measures $d\phi^{(1)}(x)$ and $d\phi^{(2)}(x)$ on the real line such that $d\phi^{(2)}(x) = (1-q^2x^2)\, d\phi^{(1)}(x)$, then there exists a sequence of real numbers $\{\ell_n\}$, with $\ell_0 = 1$, such that for the associated monic orthogonal polynomials $P_n^{(1)}$ and $P_n^{(2)}$ the following hold:

$$P_{n+1}^{(1)}(x) = xP_n^{(1)}(x) - \tfrac{1}{4q^2}(1-\ell_n)(1+\ell_{n-1})P_{n-1}^{(1)}(x),$$

$$P_{n+1}^{(2)}(x) = xP_n^{(2)}(x) - \tfrac{1}{4q^2}(1-\ell_n)(1+\ell_{n+1})P_{n-1}^{(2)}(x),$$

$$P_{n+1}^{(1)}(x) = P_{n+1}^{(2)}(x) - \tfrac{1}{4q^2}(1-\ell_n)(1-\ell_{n+1})P_{n-1}^{(2)}(x)$$

for $n \geqslant 1$. Here q^2, different from zero, can take any appropriate value including negative. Clearly, if $q = 0$ then the two sets of polynomials are the same. The choice $q = 1$, so that the measures have their support within $[-1, 1]$, gives results (3.5) with $a_n = \ell_n$, $n \geqslant 0$. Hence we can state the following theorem.

Theorem 3.2. *Let* $\mathrm{d}\phi^{(1)}$ *and* $\mathrm{d}\phi^{(2)}$ *be two positive measures on* $[-1, 1]$ *such that* $\mathrm{d}\phi^{(2)}(x) = (1-x^2)\,\mathrm{d}\phi^{(1)}(x)$. *Let the respective monic orthogonal polynomials* $P_n^{(1)}$ *and* $P_n^{(2)}$ *associated with these measures satisfy* $P_{n+1}^{(i)}(x) = xP_n^{(i)}(x) - \alpha_{n+1}^{(i)}P_{n-1}^{(i)}(x)$, $n \geqslant 1$. *Let*

$$2zS_{n-1}(z) = R_n^{(1)}(z) + (z-1)R_{n-1}^{(2)}(z), \quad n \geqslant 1,$$

where $R_n^{(1)}(z) = (4z)^{n/2}P_n^{(1)}(x(z))$ *and* $R_n^{(2)}(z) = (4z)^{n/2}P_n^{(2)}(x(z))$. *Then* S_n *are the monic Szegő polynomials associated with* $\mathrm{d}v(z) = -\mathrm{d}\phi^{(1)}(x(z))$. *Furthermore, the reflection coefficients* $a_n = S_n(0)$ *can be generated by*

$$a_n = 1 - 4\alpha_{n+1}^{(1)}/(1 + a_{n-1}) \quad \textit{and} \quad a_{n+1} = -1 + 4\alpha_{n+1}^{(2)}/(1 - a_n), \;\; n \geqslant 1$$

with $a_0 = 1$. *Given explicitly (with* $\mu_0^{(i)}$, $i = 1, 2$, *as the respective moments of order zero),*

$$a_{2n-1} = 2\,\frac{\alpha_{2n-1}^{(2)}\alpha_{2n-3}^{(2)}\cdots\alpha_3^{(2)}\mu_0^{(2)}}{\alpha_{2n-1}^{(1)}\alpha_{2n-3}^{(1)}\cdots\alpha_3^{(1)}\mu_0^{(1)}} - 1 \quad \textit{and} \quad a_{2n} = 2\,\frac{\alpha_{2n}^{(2)}\alpha_{2n-2}^{(2)}\cdots\alpha_2^{(2)}}{\alpha_{2n}^{(1)}\alpha_{2n-2}^{(1)}\cdots\alpha_2^{(1)}} - 1, \;\; n \geqslant 1.$$

4. Examples of Szegő polynomials

We consider some examples to show how the above results can be used to obtain information about real Szegő polynomials.

1. *Gegenbauer–Szegő polynomials*: We consider $\mathrm{d}\phi^{(1)}(x) = (1-x^2)^{\lambda-1/2}\,\mathrm{d}x$ and $\mathrm{d}\phi^{(2)}(x) = (1-x^2)^{\lambda+1/2}\,\mathrm{d}x$ in $[-1, 1]$, where $\lambda > -1/2$. Then $P_n^{(1)} = P_n^{(\lambda)}$ and $P_n^{(2)} = P_n^{(\lambda+1)}$ are the respective monic Gegenbauer polynomials. Hence,

$$\alpha_{n+1}^{(1)} = \frac{n(n+2\lambda-1)}{4(n+\lambda)(n+\lambda-1)} \quad \text{and} \quad \alpha_{n+1}^{(2)} = \frac{n(n+2\lambda+1)}{4(n+\lambda+1)(n+\lambda)}, \;\; n \geqslant 1.$$

The polynomials $S_n^{(\lambda)}$ defined by

$$2zS_n^{(\lambda)}(z) = R_{n+1}^{(1,\lambda)}(z) + (z-1)R_n^{(2,\lambda)}(z), \quad n \geqslant 0,$$

where $R_n^{(1,\lambda)}(z) = (4z)^{n/2}P_n^{(\lambda)}(x(z))$ and $R_n^{(2,\lambda)}(z) = (4z)^{n/2}P_n^{(\lambda+1)}(x(z))$, are the well known monic Gegenbauer–Szegő polynomials (see for example [10]) associated with the measure

$$\mathrm{d}v(z) = \left[\frac{(z-1)^2}{-4z}\right]^{\lambda} \frac{\mathrm{d}z}{2\mathrm{i}z} = \frac{1}{2}\,|\sin(\theta/2)|^{2\lambda}\,\mathrm{d}\theta$$

with $z = \mathrm{e}^{\mathrm{i}\theta}$. The reflection coefficients are $a_n^{(\lambda)} = \lambda/(n+\lambda)$, $n \geqslant 1$.

When $\lambda = 0$, then $a_n^{(0)} = 0$ and we have the results associated with the Chebyshev polynomials. In this case, we have $R_n^{(1,0)}(z) = z^n + 1$ and $R_n^{(2,0)}(z) = (z^{n+1} - 1)/(z-1)$. Thus, we obtain $S_n^{(0)}(z)=$

$(1/2z)[R^{(1,0)}_{n+1}(z)+(z-1)R^{(2,0)}_n(z)]=z^n$, which are the monic Szegő polynomials associated with the Lebesgue measure $\mathrm{d}v(z)=-\mathrm{d}\phi^{(1)}(x(z))=(2\mathrm{i}z)^{-1}\mathrm{d}z$.

2. *Koornwinder–Szegő polynomials*: For $\lambda>-3/2$, $M\geqslant 0$ and $0<\tau\leqslant 1$, let $\mathrm{d}\phi^{(1)}(x)$, defined on $[-1,1]$, be such that

$$\int_{-1}^{1} f(x)\,\mathrm{d}\phi^{(1)}(x)=\frac{1}{2M+1}\left\{M[f(-1)+f(1)]+c^{-1}\int_{-\tau}^{\tau} f(x)\frac{(\tau^2-x^2)^{\lambda+1/2}}{1-x^2}\,\mathrm{d}x\right\}.$$

Here

$$c=\int_{-\tau}^{\tau}\frac{(\tau^2-x^2)^{\lambda+1/2}}{1-x^2}\,\mathrm{d}x$$

is such that the total mass of the measure $\mathrm{d}\phi^{(1)}$ is 1 (a probability measure). Hence, the orthogonal polynomials $P_n^{(2)}$ associated with the measure $\mathrm{d}\phi^{(2)}(x)=[(2M+1)c]^{-1}(\tau^2-x^2)^{\lambda+1/2}\,\mathrm{d}x$ are the Gegenbauer polynomials of parameter $(\lambda+1)$ scaled downed to the interval $[-\tau,\tau]$. Hence, $P^{(2)}_{n+1}(x)=xP_n^{(2)}(x)-\alpha^{(2)}_{n+1}P^{(2)}_{n-1}(x)$, $n\geqslant 1$, where

$$\alpha^{(2)}_{n+1}=\frac{\tau^2}{4}\frac{n(n+2\lambda+1)}{(n+\lambda+1)(n+\lambda)}.$$

When $\tau=1$, in this case λ should be greater than $-\frac{1}{2}$, then $P_n^{(1)}$ are the symmetric Koornwinder polynomials [6].

Let us denote by $S_n^{(\lambda,M,\tau)}$ the Szegő polynomials associated with the measure $\mathrm{d}v(\lambda,\ M,\tau;z)=-\mathrm{d}\phi^{(1)}(x(z))$, which can be given by

$$\int_0^{2\pi} f(\mathrm{e}^{\mathrm{i}\theta})\,\mathrm{d}v(\lambda,M,\tau;\mathrm{e}^{\mathrm{i}\theta})=\frac{1}{2M+1}\left\{2Mf(1)+\frac{c(f,\lambda,\tau)}{c(1,\lambda,\tau)}\right\},$$

where $c(f,\lambda,\tau)=\int_{\theta(\tau)}^{2\pi-\theta(\tau)} f(\mathrm{e}^{\mathrm{i}\theta})[\tau^2-\cos^2(\theta/2)]^{\lambda+1/2}[\sin(\theta/2)]^{-1}\,\mathrm{d}\theta$, with $\theta(x)=2\arccos(x)$. For the reflection coefficients $S_n^{(\lambda,M,\tau)}(0)$ we then have $\{g_n(\lambda,M,\tau)=[1+S^{(\lambda,M,\tau)}_{n+1}(0)]/2\}_{n=0}^{\infty}$ is a parameter sequence of the chain sequence $\{\alpha^{(2)}_{n+1}\}$. That is

$$[1-g_{n-1}(\lambda,M,\tau)]g_n(\lambda,M,\tau)=\frac{\tau^2}{4}\frac{n(n+2\lambda+1)}{(n+\lambda+1)(n+\lambda)},\quad n\geqslant 1.$$

Since $g_0(\lambda,M,\tau)=1-\alpha_2^{(1)}$, the initial parameter takes the value

$$g_0(\lambda,M,\tau)=\frac{1}{(2M+1)c(1,\lambda,\tau)}\int_0^{2\pi}[\tau\sin(\theta/2)]^{2\lambda+2}\,\mathrm{d}\theta.$$

Clearly, when $M\rightarrow\infty$ then $\{g_n(\lambda,M,\tau)\}$ moves towards the minimal parameter sequence and if $M=0$ then one can also show that $g_n(\lambda,M,\tau)$ represents the maximal parameter sequence.

3. *Szegő polynomials associated with twin periodic recurrence relations*: It was shown in [1] that given the recurrence relation (3.3) with $\alpha_2=p\alpha_0$, $\alpha_{2n-1}=\alpha_1$ and $\alpha_{2n}=\alpha_0$, $n\geqslant 2$, where p, α_0, α_1 are all positive, then the polynomials P_n are the monic orthogonal polynomials in relation to the

probability measure $\mathrm{d}\phi$ given by

$$\phi(x) = \frac{\Delta(\alpha_1 - \alpha_0)}{A(p)} U(x) + \frac{1}{2\pi A(p)} \int_{-\infty}^{x} \frac{\sqrt{b^2 - t^2}\sqrt{t^2 - a^2}}{|t|(1 + q(p)t^2)} I(t)\,\mathrm{d}t + \frac{\Delta((p-1)^2\alpha_0 - \alpha_1)}{2(p-1)A(p)} [U(x + \xi(p)) + U(x - \xi(p))],$$

where $b = \sqrt{\alpha_0} + \sqrt{\alpha_1}$, $a = |\sqrt{\alpha_0} - \sqrt{\alpha_1}|$, $A(p) = (p-1)\alpha_0 + \alpha_1$, $q(p) = -1/[\xi(p)]^2 = (1-p)/pA(p)$, $\Delta(x) = xU(x)$, $I(x) = U(x+b) - U(x+a) + U(x-a) - U(x-b)$ and the function $U(x)$ is equal to 0 for $x < 0$ and is equal to 1 for $x \geqslant 0$.

Clearly, the two jumps at $\pm\xi(p)$ have effect only if $p > 1 + \sqrt{\alpha_1/\alpha_0}$ or, when possible, $0 < p < 1 - \sqrt{\alpha_1/\alpha_0}$. If $p \geqslant 1 + \sqrt{\alpha_1/\alpha_0}$ then $[\xi(p)]^2 \geqslant b^2$. Hence, taking $b \leqslant 1$ and choosing the value of p such that $[\xi(p)]^2 = 1$ we obtain the following results. Let $\alpha_0 > 0$, $\alpha_1 > 0$ and let

$$b = \sqrt{\alpha_0} + \sqrt{\alpha_1} \leqslant 1, \quad a = |\sqrt{\alpha_0} - \sqrt{\alpha_1}| \quad \text{and} \quad B(\alpha_0, \alpha_1) = \sqrt{1 - b^2}\sqrt{1 - a^2}.$$

Then, with

$$p = p(\alpha_0, \alpha_1) = \frac{1 + (\alpha_0 - \alpha_1) + B(\alpha_0, \alpha_1)}{2\alpha_0},$$

the monic polynomials $P_n^{(1)}$ generated by the recurrence relation $P_{n+1}^{(1)}(x) = xP_n^{(1)}(x) - \alpha_{n+1}^{(1)} P_{n-1}^{(1)}(x)$, $n \geqslant 1$, where $\alpha_2^{(1)} = p(\alpha_0, \alpha_1)\alpha_0$, $\alpha_{2n-1}^{(1)} = \alpha_1$, $\alpha_{2n}^{(1)} = \alpha_0$, $n \geqslant 2$, $P_0^{(1)}(x) = 1$ and $P_1^{(1)}(x) = x$, are the orthogonal polynomials in relation to the measure $\mathrm{d}\phi^{(1)}$ given by

$$\int_{-1}^{1} f(x)\mathrm{d}\phi^{(1)}(x) = \frac{B(\alpha_0, \alpha_1)}{2A(\alpha_0, \alpha_1)} f(-1) + \frac{1}{2\pi A(\alpha_0, \alpha_1)} \int_{-b}^{-a} f(x) \frac{\sqrt{b^2 - x^2}\sqrt{x^2 - a^2}}{|x|(1 - x^2)}\,\mathrm{d}x + \frac{\Delta(\alpha_1 - \alpha_0)}{A(\alpha_0, \alpha_1)} f(0) + \frac{1}{2\pi A(\alpha_0, \alpha_1)} \int_{a}^{b} f(x) \frac{\sqrt{b^2 - x^2}\sqrt{x^2 - a^2}}{|x|(1 - x^2)}\,\mathrm{d}x + \frac{B(\alpha_0, \alpha_1)}{2A(\alpha_0, \alpha_1)} f(1).$$

Here $A(\alpha_0, \alpha_1) = A(p) = [1 - (\alpha_0 - \alpha_1) + B(\alpha_0, \alpha_1)]/2$. It is easily verified that, since $p = p(\alpha_0, \alpha_1)$, the sequence $\{a_n\}$ obtained from the relation $a_n = 1 - 4\alpha_{n+1}^{(1)}/(1 + a_{n-1})$, $n \geqslant 1$, where $a_0 = 1$, satisfies $a_{2n-1} = -(\alpha_0 - \alpha_1) - B(\alpha_0, \alpha_1)$ and $a_{2n} = (\alpha_0 - \alpha_1) - B(\alpha_0, \alpha_1)$, for $n \geqslant 1$.

Consider the probability measure $\mathrm{d}v(z) = -\mathrm{d}\phi^{(1)}(x(z))$ on C which can be given by

$$\int_{0}^{2\pi} f(\mathrm{e}^{\mathrm{i}\theta})\mathrm{d}v(\mathrm{e}^{\mathrm{i}\theta}) = \frac{B(\alpha_0, \alpha_1)}{A(\alpha_0, \alpha_1)} f(1) + \frac{1}{2\pi A(\alpha_0, \alpha_1)} \int_{\theta(b)}^{\theta(a)} f(\mathrm{e}^{\mathrm{i}\theta}) \frac{\sqrt{b^2 - \cos^2(\theta/2)}\sqrt{\cos^2(\theta/2) - a^2}}{|\sin(\theta)|}\,\mathrm{d}\theta + \frac{\Delta(\alpha_1 - \alpha_0)}{A(\alpha_0, \alpha_1)} f(-1) + \frac{1}{2\pi A(\alpha_0, \alpha_1)} \int_{2\pi - \theta(a)}^{2\pi - \theta(b)} f(\mathrm{e}^{\mathrm{i}\theta}) \frac{\sqrt{b^2 - \cos^2(\theta/2)}\sqrt{\cos^2(\theta/2) - a^2}}{|\sin(\theta)|}\,\mathrm{d}\theta.$$

Here $\theta(x) = 2\arccos(x)$. Then the Szegő polynomials $S_n^{(\alpha_0,\alpha_1)}$ associated with $\mathrm{d}\nu(z)$ satisfy

$$S_{2n-1}^{(\alpha_0,\alpha_1)}(0) = -B(\alpha_0,\alpha_1) - (\alpha_0 - \alpha_1) \quad \text{and} \quad S_{2n}^{(\alpha_0,\alpha_1)}(0) = -B(\alpha_0,\alpha_1) + (\alpha_0 - \alpha_1), \quad n \geqslant 1.$$

If $\alpha_1 > \alpha_0$ there is a jump in the measure at the point $z = -1$. Note that $\sqrt{\alpha_0} + \sqrt{\alpha_1}$ can not exceed 1 and if $\sqrt{\alpha_0} + \sqrt{\alpha_1} < 1$ then there is also a jump at $z = 1$. This jump vanishes when $\sqrt{\alpha_0} + \sqrt{\alpha_1} = 1$.

5. Rotating the coefficients

Let $\{R_n\}$ be the sequence of monic polynomials generated by the recurrence relation (3.1). Hence by Theorem 3.1, there exists a positive measure $\mathrm{d}\nu$ on the unit circle such that (3.2) holds. We now give some information regarding the sequence of monic polynomials $\{R_n(\tau,z)\}$ given by the recurrence relation

$$R_{n+1}(\tau,z) = (z+1)R_n(\tau,z) - 4\alpha_{n+1}\mathrm{e}^{2\mathrm{i}\tau} z R_{n-1}(\tau,z), \quad n \geqslant 1.$$

For fixed τ such that $0 < \tau < \pi$, let $\Lambda(\tau)$ represent the curve (path)

$$\Lambda(\tau) \equiv \{z = z(t) = u(\tau,t) + iv(\tau,t):\ t_2(\tau) \geqslant t \geqslant t_1(\tau)\},$$

where

$$u(\tau,t) = t\,\frac{(t-1)^2\cot^2(\tau) - (t+1)^2}{(t-1)^2\cot^2(\tau) + (t+1)^2} \quad \text{and} \quad v(\tau,t) = \frac{2t(t^2-1)\cot(\tau)}{(t-1)^2\cot^2(\tau) + (t+1)^2}.$$

Here, $t_1(\tau) < t_2(\tau)$, such that $t_1(\tau)t_2(\tau) = 1$, are the two positive solutions of

$$\frac{(t-1)^2[\cot^2(\tau)+1]}{(t-1)^2\cot^2(\tau) + (t+1)^2} = \frac{4t}{(t+1)^2}.$$

Note that $\Lambda(\pi/2)$ is the real interval from $-3-2\sqrt{2}$ to $-3+2\sqrt{2}$. We also let $\Lambda(0)$ to be the unit circle.

Theorem 5.1. *The zeros of $R_n(\tau,z)$ are distinct and lie on the curve $\Lambda(\tau)$. Furthermore, if $-\mathrm{d}\phi(w(z)) = \mathrm{d}\nu(z)$ and $\mathrm{d}\nu(\tau,z) = -\mathrm{d}\phi(w(z)\mathrm{e}^{-\mathrm{i}\tau})$, where $2w(z) = z^{1/2} + z^{-1/2}$, then*

$$\int_{\Lambda(\tau)} z^{-n+s} R_n(\tau,z)\frac{z}{z-1}\,\mathrm{d}\nu(\tau,z) = 0, \quad 0 \leqslant s \leqslant n-1.$$

Proof. We start with the polynomials $P_n(\tau,w) = (4z)^{-n/2}R_n(\tau,z)$, which satisfy $P_{n+1}(\tau,w) = wP_n(\tau,w) - \alpha_{n+1}\mathrm{e}^{2\mathrm{i}\tau}P_{n-1}(\tau,w)$, $n \geqslant 1$. Clearly, they satisfy the orthogonality property

$$\int_{L(\tau)} w^s P_n(\tau,w)\mathrm{d}\phi(w\mathrm{e}^{-\mathrm{i}\tau}) = 0, \quad 0 \leqslant s \leqslant n-1,$$

where $L(\tau)$ is the path represented by the straight line from $-\mathrm{e}^{\mathrm{i}\tau}$ to $\mathrm{e}^{\mathrm{i}\tau}$. Equivalently,

$$\int_{L(\tau)} \{w + \mathrm{i}\sqrt{1-w^2}\}^{-(n-1)+2s}\frac{P_n(\tau,w)}{\sqrt{1-w^2}}\,\mathrm{d}\phi(w\mathrm{e}^{-\mathrm{i}\tau}) = 0, \quad 0 \leqslant s \leqslant n-1.$$

This gives the required result as the line represented by $L(\tau)$ is the image of the curve represented by $\Lambda(\tau)$ under the transformation $2w(z) = z^{1/2} + z^{-1/2}$. $\square$

These results, when $\tau = \pi/2$, lead to information given in [11].

As an example we consider the monic polynomials $R_n^{(\lambda)}(\tau, z)$ given by

$$R_{n+1}^{(\lambda)}(\tau, z) = (z+1)R_n^{(\lambda)}(\tau, z) - 4\alpha_{n+1}^{(\lambda)} e^{2i\tau} z R_{n-1}^{(\lambda)}(\tau, z), \quad n \geqslant 1,$$

where

$$\alpha_{n+1}^{(\lambda)} = \frac{n(n+2\lambda-1)}{4(n+\lambda)(n+\lambda-1)}.$$

Note that the monic Gegenbauer polynomials are $P_n^{(\lambda)}(w) = (4z)^{-n/2} R_n^{(\lambda)}(0, z)$. We obtain from the above theorem

$$\int_{\Lambda(\tau)} z^{-n+s} R_n^{(\lambda)}(\tau, z) z^{-\lambda} [(b^{(\tau)} - z)(z - 1/b^{(\tau)})]^{\lambda - 1/2} \, dz = 0, \quad 0 \leqslant s \leqslant n-1,$$

where $b^{(\tau)} = (2e^{2i\tau} - 1) + 2e^{i\tau}\sqrt{e^{2i\tau} - 1}$. When $\lambda = 1$ and $\tau = \pi/2$ we obtain the polynomials which are the denominator polynomials of the classical positive T-fraction (see [9])

$$\frac{z}{z+1} \, {}_{+} \, \frac{z}{z+1} \, {}_{+} \, \frac{z}{z+1} \, {}_{+} \, \frac{z}{z+1} \, {}_{+} \cdots .$$

References

[1] A.C. Berti, A. Sri Ranga, Companion orthogonal polynomials: some applications, Appl. Numer. Math. 39 (2001) 127–149.

[2] T.S. Chihara, An Introduction to Orthogonal Polynomials, Mathematics and its Applications Series, Gordon and Breach, New York, 1978.

[3] P. Delsarte, Y. Genin, The split Levinson algorithm, IEEE Trans. Acoust. Speech Signal Process. 34 (1986) 470–478.

[4] W.B. Jones, O. Njåstad, W.J. Thron, Moment theory, orthogonal polynomials, quadrature, and continued fractions associated with the unit circle, Bull. London Math. Soc. 21 (1989) 113–152.

[5] W.B. Jones, W.J. Thron, H. Waadeland, A strong Stieltjes moment problem, Trans. Amer. Math. Soc. 261 (1980) 503–528.

[6] T.H. Koornwinder, Orthogonal polynomials with weight function $(1-x)^\alpha(1+x)^\beta + M\delta(x+1) + N\delta(x-1)$, Canad. Math. Bull. 27 (1984) 205–214.

[7] A. Sri Ranga, Symmetric orthogonal polynomials and the associated orthogonal L-polynomials, Proc. Amer. Math. Soc. 123 (1995) 3135–3141.

[8] G. Szegő, Orthogonal Polynomials, 4th Edition, in: American Mathematical Society Colloquium Publications, Vol. 23, American Mathematical Society, Providence, RI, 1975.

[9] W.J. Thron, Some properties of continued fractions $1 + d_0 + K(z/(1 + d_n z))$, Bull. Amer. Math. Soc. 54 (1948) 206–218.

[10] W. Van Assche, Orthogonal polynomials in the complex plane and on the real line, in: M.E.H. Ismail, et al., (Eds.), Field Institute Communications 14: Special Functions, q-Series and Related Topics, American Mathematical Society, Providence, RI, 1997, pp. 211–245.

[11] L. Vinet, A. Zhedanov, Szegő polynomials on the real axis, Integral Transforms Spec. Functions 8 (1999) 149–164.

[12] A. Zhedanov, On some classes of polynomials orthogonal on arcs of the unit circle connected with symmetric orthogonal polynomials on an interval, J. Approx. Theory 94 (1998) 73–106.