



RNS derivation of N -point disk amplitudes from the revisited S-matrix approach

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Abstract

Recently, in [7] we proposed a revisited S-matrix approach to efficiently find the bosonic terms of the open superstring low energy effective lagrangian (OSLEEL). This approach allows to compute the α'^N terms of the OSLEEL using open superstring n -point amplitudes in which n is considerably lower than $(N + 2)$ (which is the order of the required amplitude to obtain those α'^N terms by means of the conventional S-matrix approach). In this work we use our revisited S-matrix approach to examine the structure of the scattering amplitudes, arriving at a closed form for them. This is a RNS derivation of the formula first found by Mafra, Schlotterer and Stieberger [21], using the pure spinor formalism. We have succeeded doing this for the 5, 6 and 7-point amplitudes. In order to achieve these results we have done a careful analysis of the kinematical structure of the amplitudes, finding as a by-product a purely kinematical derivation of the BCJ relations (for $N = 4, 5, 6$ and 7). Also, following the spirit of the revisited S-matrix approach, we have found the α' expansions for these amplitudes up to α'^6 order in some cases, by only using the well known open superstring 4-point amplitude, cyclic symmetry and tree level unitarity: we have not needed to compute any numerical series or any integral involving polylogarithms, at any moment.

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1. Introduction

In the recent years there has been a considerable progress in perturbative string theory (considering D-brane systems as well). Achievements have been going on in the determination of the couplings of D-branes with Ramond–Ramond and other massless states [1–4], the closed and the open sector of the superstring low energy effective lagrangian [5–7],¹ higher loop calculations of closed and open superstring scattering [9–11], Mellin correspondence between supergravity and superstring amplitudes [12,13] and a deeper understanding of the α' expansion of tree level open and closed superstrings [14–17], among many others.

With respect to tree level scattering amplitudes in open superstring theory, after a closed expression was found for the 5 and the 6-point gauge boson amplitudes (in [18] and [19], respectively),² one of the very important results that has been achieved in the literature is Mafra–Schlotterer–Stieberger’s formula for the N -point amplitude [21]:

$$A(1, \dots, N) = \sum_{\sigma_N \in S_{N-3}} F^{(\sigma_N)}(\alpha') A_{SYM}(1, \{2_\sigma, 3_\sigma, \dots, (N-2)_\sigma\}, N-1, N), \quad (1)$$

where $A(1, \dots, N)$ is the subamplitude and

$$\begin{aligned} \mathcal{A}_N &= i(2\pi)^{10} \delta^{(10)}(k_1 + \dots + k_N) \\ &\times \left[\text{tr}(\lambda^{a_1} \lambda^{a_2} \dots \lambda^{a_N}) A(1, 2, \dots, N) + \left(\begin{array}{c} \text{non-cyclic} \\ \text{permutations} \end{array} \right) \right] \end{aligned} \quad (2)$$

is the complete N -point open superstring (tree level) scattering amplitude (where $N \geq 3$).

In (1) $\sigma_N = \{2_\sigma, 3_\sigma, \dots, (N-2)_\sigma\}$ denotes a permutation of indexes $\{2, 3, \dots, (N-2)\}$ and the $F^{(\sigma_N)}(\alpha')$ ’s are the momentum factors which contain the α' information of the scattering amplitude³ (see Eqs. (32) and (33)). There are $(N-3)!$ terms in the sum in (1).

Formula (1) considers all possible scattering processes involving external gauge bosons and their fermionic massless superpartners. In it, A_{SYM} denotes the tree level scattering subamplitude of this process in $D = 10$ super Yang–Mills theory and A (on the left-hand side) is the corresponding scattering subamplitude in open superstring theory. In (2) the λ^a ’s are the gauge group matrices in the adjoint representation (see Eq. (A.2)).

Formula (1) has the merit of clarifying that the kinematics of the open superstring N -point amplitude, at any α' order, is governed by the super Yang–Mills kinematics, that is, from the low energy theory. It also has the virtue of identifying an explicit general integral formula, valid for any $N \geq 3$, for all the $F^{(\sigma_N)}(\alpha')$ momentum factors (see Eq. (32)).

During the last years quite nontrivial results have been obtained for the α' expansion of the $F^{(\sigma_N)}(\alpha')$ momentum factors, in Eq. (1). These factors are disk worldsheet integrals which can be found, but in terms of non-elementary functions so, besides the cases of $N = 3$ and $N = 4$, their α' expansion is a non-trivial thing to compute.⁴

¹ Also, in Ref. [8] a series of closed relations involving the tree level scattering of an arbitrary number of open and closed strings have been found.

² In Ref. [20] a closed expression for the 7-point formula was also found in the case of spacetime dimension $D = 4$.

³ Besides α' , the $F^{(\sigma_N)}(\alpha')$ ’s depend on the $k_i \cdot k_j$ scalar products, which can be written in terms of the independent Mandelstam variables of this N -point process.

⁴ They are $(N-3)$ multiple integrals (see Eq. (32)). In the case of the 5-point amplitude one finds the ${}_3F_2$ Hypergeometric function (see, for example, [22] and [23]); in the case of 6-point amplitudes one finds a double series of ${}_4F_3$ Hypergeometric functions (see, for example, [19] and [24]); and for $N \geq 7$ one has to deal with even more complicated

Besides the fact of improving the α' order of expansions of the past [27,18,19] in various orders [28,14,15], a conjecture has been established for the general form of the α' expansion of the $F^{(\sigma_N)}(\alpha')$'s (for arbitrary N and arbitrarily high order in α' [14])^{5,6} and, also, a recursive formula (in N) for these α' expansions has been proved in [16] (by means of a generalization of the work in Ref. [29]).⁷

All these results have as a common starting point the Mafra–Stieberger–Schlotterer (MSS) formula, given in Eq. (1). This formula was derived using the pure spinor formalism [30], which is manifestly supersymmetric right from the beginning. Formula (1) is the final (and simple) result of an elaborated study involving pure spinor superspace and its cohomology structure [31], first applied in the calculation of super Yang–Mills amplitudes and afterwards extended to the corresponding calculations in open superstring theory [21].

The purpose of the present work is two-fold. On one side, we show that it is possible to arrive to MSS's formula in (1) working only in the Ramond–Neveu–Schwarz (RNS) formalism [32].⁸ For the moment, we only have a proof for $3 \leq N \leq 7$ within this approach, but we think that a deeper understanding of our procedure can, eventually, lead to the proof for arbitrary N . On the other side, we shed light in how the α' expansion of the $F^{(\sigma_N)}(\alpha')$ momentum factors in (1) can be obtained, order by order in α' , not by doing the explicit computations of the coefficients of its expansion (which are given in terms of multiple zeta values (MZV's) [34,35,28,14]), but by demanding tree level unitarity and cyclic invariance of the amplitudes and (presumably) using only the α' expansion of the 5-point amplitude.⁹

The basic tool in which our findings are supported is the ‘revisited S-matrix approach’, found by us in the past year [7]. This method was initially proposed as an efficient tool to determine, order by order in α' , the bosonic part of the open superstring low energy effective lagrangian (OSLEEL) but, as we will see in the present work, it has a direct counterpart in the determination of the scattering amplitudes of the theory, allowing us to arrive to (1) and also to the α' expansion version of it. This method is intrinsically kinematic and supersymmetric, although it is not *manifestly* supersymmetric. It deals, first, with the pure external gauge boson interactions and only at the end, it incorporates the interactions between external gauge bosons and its fermionic superpartners.

The kinematics is present right from the beginning since the main statement of the method has to do with the kinematical structure of the N -point amplitudes of gauge bosons: in open

expressions. In all these cases the coefficients of the α' expansion are given in terms of harmonic (or Euler–Zagier) sums and/or polylogarithmic integrals: all of them are nowadays known how to be calculated (see, for example, Refs. [25] and [26]), but the required calculations to find them grow enormously with the α' order.

⁵ See Appendix F.2 for this explicit form.

⁶ The mentioned conjecture of Ref. [14] has been checked in [15] up to α'^{21} order for $N = 5$, up to α'^9 order for $N = 6$ and up to α'^7 order for $N = 7$.

⁷ The mathematical framework used in Refs. [29,15] is related, but is not directly the same one of Ref. [14].

⁸ The subtlety is that we do not deal with fermion vertex operators at all. We only work with the N -point gauge boson amplitude, $A_b(1, \dots, N)$, given in Eq. (31), which comes from only gauge boson vertex operators in the RNS formalism [33]. See Section 4.2 for more details about this.

⁹ This statement seems to be in contradiction with the final part of the abstract of this work, where we said that we would only use the α' information from the 4-point amplitude. The clarifying statement is that, for the calculation purposes of this work, in which at most we have obtained α'^6 order results, it will be enough to use the 4-point amplitude information. As explained in Section 2.2, the 5-point amplitude information will become important from α'^8 order onwards and, as argued in that section, we claim that the *pure* 5-point amplitude is enough to find the whole α' terms of the $F^{(\sigma_N)}(\alpha')$ momentum factors.

superstring theory they should not contain $(\zeta \cdot k)^N$ terms, at any α' order [7]. With respect to supersymmetry, it is also present right from the beginning since (we believe that) it is the reason for the absence of those kinematical terms in the amplitudes.¹⁰

The structure of this work is as follows. In Section 2 we give a brief review of the revisited S-matrix method. We explain in it why we claim that demanding the absence of the $(\zeta \cdot k)^N$ terms in the N -point amplitude of gauge bosons and, based on the conjecture of Ref. [14] and the main result of Ref. [16], using only the α' expansion of the 5-point amplitude, that is enough information to find the complete (bosonic part of the) OSLEEL. We claim there, then, that using similar arguments there should be a direct analog of this situation from the perspective of the scattering amplitudes, that is, at a given order in α' , knowing the 5-point amplitude is enough information to know any higher N -point amplitude.¹¹

We begin the elaboration of these ideas in Section 3 by examining the space of N -point kinematic expressions which are on-shell gauge invariant and which do not contain $(\zeta \cdot k)^N$ terms. We find that this space is $(N - 3)!$ -dimensional (at least for $3 \leq N \leq 7$). We then check that a BCJ basis for Yang–Mills subamplitudes (see Eq. (5)) [37] can indeed be chosen as a basis for this space. In light of this important kinematical result, the determination of the explicit expressions of the open superstring subamplitudes (for gauge bosons) and of the BCJ relations themselves become simply a linear algebra problem: we know a vector of the space (that is, a given subamplitude for which we want an expression) and what is left to do is to find the components of this vector with respect to the basis of the vector space. We do this calculation in Section 4 for the open superstring subamplitudes (for gauge bosons), arriving precisely to the bosonic part of (1), and in Appendix E, for the BCJ relations themselves (arriving to the same result of Refs. [38,8]). Then, in the last part of Section 4, we briefly explain why once we have found the gauge boson amplitudes in this manifestly gauge invariant way, the corresponding amplitudes involving fermions are immediate, and thus leading to MSS result in Eq. (1) (this time considering there all possible scattering processes involving external gauge bosons and their supersymmetric partners).

In Section 5 we apply our revisited S-matrix method to find the α' expansion of the $F^{\{\sigma_N\}}(\alpha')$ momentum factors of (1), in the case of $N = 5, 6, 7$.¹² In order to achieve this, besides the requirements of the revisited S-matrix method, we demand cyclicity and tree level unitarity to be obeyed by the subamplitudes. We end Section 5 describing some subtleties that arise in the calculations of the revisited S-matrix method and also examining our conjecture of Section 2 about the 5-point amplitude being enough information to know any higher N -point amplitude at any α' order.

We end in Section 6 by summarizing our results and conclusions.

Throughout this work all scattering amplitudes are tree level ones.

Since, at some points, we have needed to deal with huge calculations and formulas, we have considered only the simplest ones in the main body of this work and we have left the more

¹⁰ It is well known that the $(\zeta \cdot k)^N$ terms are indeed present in the case of 3 and 4-point amplitudes of massless states in open bosonic string theory [33,36] and, from the general integral formula for the N -point amplitude [33], it is also believed that they are present in this general case. Therefore, it is quite reasonable to conjecture that the absence of the $(\zeta \cdot k)^N$ terms, in the case of supersymmetric open strings, is a consequence of spacetime supersymmetry.

¹¹ See footnote 9.

¹² Because of computer limitations, we have done the explicit calculations only up to α'^6 order in the first two cases and up to α'^4 order, in the last case. But it is clear that the method can be used to obtain higher order α' contributions. See more details in Section 5.

Table 1

Comparison of the number of coefficients to be determined, considering and not considering the constraint in Eq. (4).

p	Dimension of the general basis at order α'^p	Dimension of the constrained basis at order α'^p
1	1	0
2	4	1
3	13	1
4	96	0
$\vdots$	$\vdots$	$\vdots$

extensive ones for the appendices.¹³ These last ones usually do not offer any new conceptual insight, but they have played an important role checking our main statements.

2. Review of the revisited S-matrix method

2.1. Finding terms of the OSLEEL in an efficient way

Let $\mathcal{L}_{\text{eff}}$ be the general low energy effective lagrangian (LEEL) for nonabelian gauge bosons in (either bosonic or supersymmetric) open string theory. It has the following form:

$$\mathcal{L}_{\text{eff}} = \frac{1}{g^2} \text{tr} \left[-\frac{1}{4} F^2 + (2\alpha') F^3 + (2\alpha')^2 F^4 + (2\alpha')^3 (F^5 + D^2 F^4) + (2\alpha')^4 (F^6 + D^2 F^5 + D^4 F^4) + \mathcal{O}((2\alpha')^5) \right]. \quad (3)$$

Each of the F^n and the $D^{2p} F^q$ terms in (3) is an abbreviation of the sum of different contractions of Lorentz indexes for those sort of terms. For example, “ F^4 ” really denotes an abbreviation of $(b_1 F^{\mu\lambda} F^{\nu\lambda} F_{\mu}{}^{\rho} F_{\nu\rho} + b_2 F^{\mu}{}_{\lambda} F_{\nu}{}^{\lambda} F^{\nu\rho} F_{\mu\rho} + b_3 F^{\mu\nu} F_{\mu\nu} F^{\lambda\rho} F_{\lambda\rho} + b_4 F^{\mu\nu} F^{\lambda\rho} F_{\mu\nu} F_{\lambda\rho})$ [39], where $\{b_1, b_2, b_3, b_4\}$ are the coefficients to be determined.

In the second column of Table 1¹⁴ we have written the number of coefficients that the general LEEL contains at the first orders in α' .¹⁵ These coefficients are the ones that the conventional S-matrix approach usually finds by computing the open string N -point amplitudes (from $N = 4$ up to $N = p + 2$, at least) at α'^p order. In the third column of Table 1 we have written the number of coefficients (which is extremely small!) that the revisited S-matrix approach really needs to find in order to determine the OSLEEL at a given α' order. The reason for the smallness of these numbers (in relation to the corresponding ones in the second column) is that in the revisited S-matrix method (only applicable to the case of the supersymmetric string) the N -point amplitudes satisfy the constraint [7]

$$\text{absence of } (\zeta \cdot k)^N \text{ terms.} \quad (4)$$

¹³ Moreover, there are some extremely long expressions that we have preferred not to include them in the text of this work and only to attach them as ‘txt’ files, in the version that we have submitted to the hep-th arXiv preprint basis.

¹⁴ This table has been taken from Eq. (3.4) of Ref. [7].

¹⁵ The terms which are being taken into account in the LEEL are only the ones which remain invariant under field redefinitions.

This constraint implies further linear restrictions that the b_j coefficients of the LEEL in (3) should satisfy (see Section 4 of [7] for more details about these restrictions). These restrictions are so strong that only a small number of coefficients remain free: this number is precisely the ‘dimension of the constrained basis’ appearing in the third column of Table 1. These same type of constraints had correctly been found about ten years before, by Koerber and Sevrin, using the method of BPS configurations (which is not directly a string theory one) [40]. In [7] we pointed out that the (probable) reason for the constraint in (4) is spacetime supersymmetry.

Due to the highly constrained form that the OSLEEL lagrangian adopts after demanding the requirement in Eq. (4), in order to determine the α'^p order terms of it, a $(p+2)$ -point amplitude calculation, in open superstring theory, is no longer needed (as in the conventional S-matrix method): very much lower N -point amplitudes (expanded at α'^p order) are expected to be enough for this purpose. In fact, in [7] we saw that the α' expansion of the 4-point momentum factor (given in Eq. (F.3)) is enough to determine explicitly the OSLEEL, at least up to $\mathcal{O}(\alpha'^4)$ terms.

2.2. Using the 4 and the 5-point α' information to obtain N -point information

So, a main idea that arises naturally from the revisited S-matrix method is the fact that the α' expansion of the 4-point momentum factor (whose coefficients are completely known in terms of integer zeta values, at any α' order, see Eq. (F.2)) is enough information to obtain completely the α' expansion of higher N -point amplitudes, at least up to a certain order in α' . For example, with the calculations that we did in [7] it is clear that the 5 and the 6-point amplitudes (and *any* higher N -point amplitude) can be completely determined at least up to α'^4 order, because we found the OSLEEL explicitly up to that order, bypassing 5 and 6-point worldsheet integral calculations.

In fact, in [7] we raised the possibility that the OSLEEL could be determined to any α' order by means of the revisited S-matrix method plus the known α' expansion of the 4-point momentum factor (see Eqs. (F.2) and (F.3)), but this can hardly happen since there are higher order coefficients given in terms of multiple zeta values (MZV's),¹⁶ like $\zeta(3, 5)$, $\zeta(3, 7)$, $\zeta(3, 3, 5)$, etc., which already show up in the α' expansion of the 5-point amplitude (at α'^8 , α'^{10} and α'^{11} order, respectively [14,28]).¹⁷ These coefficients are not expected to be given by linear combinations of products of $\zeta(n)$'s (in which the coefficients of these linear combinations are rational numbers)¹⁸ [41], so they are not present in the α' expansion of the 4-point amplitude.

Since α'^8 is the first order at which these *non-trivial* MZV's arise, we do not have a proof, but we believe that up to α'^7 order any N -point amplitude can be found by means of the revisited S-matrix method plus only the known α' expansion of the 4-point momentum factor, $F^{(2)}$ (see Eqs. (34), (F.2) and (F.3)).

From α'^8 order onwards we expect the 4-point amplitude to only give a partial (but still important) information for the determination of the OSLEEL terms.¹⁹ For example, the $\alpha'^p D^{2p-4} F^4$

¹⁶ See F.2.1 for an extremely short review on MZV's.

¹⁷ We thank Rutger Boels for calling our attention to this point.

¹⁸ In this work we will refer to these peculiar MZV's as *non-trivial* MZV's, in opposition to the *trivial* ones, which are known to be given as rational linear combinations of products of $\zeta(n)$'s [42]. The *non-trivial* MZV's that we will be referring to are *only* the ones that appear in the MZV basis of this last reference. A few examples of *trivial* MZV's can be found in formulas (F.6), (F.7) and (F.8) of Appendix F.

¹⁹ Curiously enough, from Eqs. (F.15)–(F.19) of Appendix F we see that at α'^9 order the 5-point amplitude does not contain any *non-trivial* MZV's and, therefore, we suspect that the OSLEEL can eventually be completely determined at this α' order, by only using 4-point amplitude information.

terms of the OSLEEL (for $p = 2, 3, 4, \dots$) are still going to be *completely* determined by the 4-point amplitude [43,44].

One might ask if new MZV's, besides the ones that already appear in the 5-point amplitude, will eventually appear at a given (highly enough) α' order. This would be a signal, from this α' order onwards, that α' information from a 6-point (or eventually higher N -point) amplitude would be required, in order to determine those α' terms of the OSLEEL. But this does not seem to be the case. It has been recently discovered that the $F^{(\sigma_N)}(\alpha')$'s can be iteratively obtained (from the N -point of view), to any order in α' , from a unique and same Drinfeld associator (which is a generating series for the MZV's) [16]. So, the MZV's that already appear in the α' expansion of the 5-point amplitude, will be the same ones that will appear for higher N -point amplitudes.²⁰

So we are left with the conjecture that our revisited S-matrix method plus the α' expansion of the 5-point amplitude, are enough informations to find *all* the α' corrections coming from open superstring theory to the Yang–Mills lagrangian.²¹ In Subsection 5.6 we will elaborate a bit more about this conjecture.

In spite of this conjecture, for the calculations that we do in this work, which at most go to α'^6 order (see Section 5), we will still use only the known 4-point α' expansion (see Appendix F.1).

3. Basis for open superstring and Yang–Mills subamplitudes

In this section we prove that, from a kinematical structure perspective, open superstring and Yang–Mills (tree level) subamplitudes belong to a same space of kinematical N -point expressions (where $N \geq 3$) and that a possible basis for this space is given by the $(N - 3)!$ Yang–Mills subamplitudes²²

$$\mathcal{B}_N = \{A_{YM}(1, \{2_\sigma, 3_\sigma, \dots, (N-2)_\sigma\}, N-1, N), \sigma \in S_{N-3}\}, \quad (5)$$

where $\{2_\sigma, 3_\sigma, \dots, (N-2)_\sigma\}$ denotes a σ permutation of indexes $\{2, 3, \dots, (N-2)\}$.²³

In order to achieve this, in Subsection 3.1 we review some important facts about the structure of gauge boson scattering subamplitudes. Then, based on this previous background, in the following subsections we argue, case by case (from $N = 3$ up to $N = 7$), that the set of subamplitudes in (5) is indeed a basis for the corresponding space.²⁴

²⁰ Shortly speaking, the reason of why not all the MZV's of the α' expansion of the 5-point function do appear in the 4-point case (see Eqs. (F.2) and (F.3)), is that the kinematic polynomial that would go multiplying them is *zero*, in the case of $N = 4$. See Refs. [14] and [16] for more details of this explanation.

²¹ Even the $\alpha'^p D^{2p-4} F^4$ terms (for any $p = 2, 3, 4, \dots$), that were found in [44] using the 4-point amplitude, could in principle be determined using *only* the 5-point amplitude α' expansion.

²² The arguments that we present in this work have been only proved for $N = 3, 4, 5, 6, 7$, but we suspect that they can be generalized for an arbitrary N .

²³ All the kinematical proof that we deal with does not depend on the spacetime dimension D , as it also happens with the BCJ relations [37], for example. In spite of this, evidence has been found in Ref. [45] that the basis of this space might indeed depend on D . It will happen that this subtlety can be kept aside from the kinematical analysis that we do. The implications of our results will still be valid for *any* D , as it happens with the BCJ relations. See the introductory paragraph in Appendix D.3, for a few more details about this.

²⁴ The details of the computations that support our claim, when $N = 5, 6, 7$, are given in Appendix D.

3.1. Some general facts about the structure of scattering amplitudes of gauge bosons

If we compare open superstring and Yang–Mills (tree level) N -point subamplitudes for gauge bosons, both theories being treated in the Lorentz gauge, from the point of view of the structure of its kinematical terms they have in common the constraint in Eq. (4), namely, the absence of $(\zeta \cdot k)^N$ terms. In the first case this has been recently emphasized in [7], together with the strong implications that it has for determining the bosonic terms of the low energy effective lagrangian of the theory in a very simplified way and, in the second case, the claim in (4) can easily be confirmed by considering Feynman rules in the construction of tree level scattering amplitudes.

So, let us consider the space of all scalar N -point kinematical expressions constructed with the polarizations ζ_i and the momenta k_i of N external gauge bosons in a nonabelian theory (like open superstring theory or Yang–Mills theory, for example). The momenta and polarizations should satisfy:

$$\text{Momentum conservation: } k_1^\mu + k_2^\mu + \cdots + k_N^\mu = 0. \quad (6)$$

$$\text{Mass-shell condition: } k_1^2 = k_2^2 = \cdots = k_N^2 = 0. \quad (7)$$

$$\text{Transversality (Lorentz gauge) condition: } \zeta_i \cdot k_i = 0 \quad (i = 1, \dots, N). \quad (8)$$

Let us denote this space by $\mathcal{V}_N$. We further restrict $\mathcal{V}_N$ such that its elements $T(1, 2, \dots, N)$ obey the following conditions:

1. They are multilinear in the polarizations ζ_i ($i = 1, 2, \dots, N$).
 2. They do not contain $(\zeta \cdot k)^N$ terms.
 3. (On-shell) Gauge invariance: whenever any $\zeta_i \rightarrow k_i$ ($i = 1, 2, \dots, N$) then $T(1, 2, \dots, N)$ becomes *zero*.
- (9)

These three requirements are simply properties of tree level gauge boson subamplitudes in open superstring and Yang–Mills theories and we want the elements of $\mathcal{V}_N$ to also satisfy them. Since these elements, the $T(1, 2, \dots, N)$'s, are Lorentz scalars, they can only be constructed from linear combinations of²⁵

$$(\zeta \cdot \zeta)^1 (\zeta \cdot k)^{N-2}, (\zeta \cdot \zeta)^2 (\zeta \cdot k)^{N-4}, \dots, (\zeta \cdot \zeta)^{[N/2]} (\zeta \cdot k)^{N-2[N/2]}, \quad (10)$$

where $[p]$ denotes the integer part of p and $N \geq 3$.

Besides the terms excluded in (4) and the ones that we have mentioned in (10), there are no more possibilities of kinematical terms that can be constructed from the polarizations ζ_i and the momenta k_i of the external gauge bosons. The only place where some extra dependence in the momenta could be considered is in the scalar coefficients which in $T(1, 2, \dots, N)$ go multiplying the kinematical terms in (10): these coefficients are allowed to be given in the terms of the $k_i \cdot k_j$ factors (or equivalently, in terms of the Mandelstam variables) and, eventually, in terms of a length scale (for example $\sqrt{\alpha'}$, in the case of string theory).

For example, in the case of $N = 3$, the Yang–Mills subamplitude contains only $(\zeta \cdot \zeta)^1 (\zeta \cdot k)^1$ terms. It is given by [33]

$$A_{YM}(1, 2, 3) = 2g[(\zeta_1 \cdot k_2)(\zeta_2 \cdot \zeta_3) + (\zeta_2 \cdot k_3)(\zeta_3 \cdot \zeta_1) + (\zeta_3 \cdot k_1)(\zeta_1 \cdot \zeta_2)]. \quad (11)$$

²⁵ Some care must be taken with the expression “linear combination” because, as we will see immediately after (10), the coefficients that go multiplying the terms that appear in it will, in general, depend in the momenta k_i^μ .

Table 2

Number of independent allowed kinematical terms present in the space $\mathcal{V}_N$.

N	Element of $\mathcal{V}_N$	Number of independent allowed kinematical terms
3	$T(1, 2, 3)$	$3 (\zeta \cdot \zeta)^1 (\zeta \cdot k)^1$ terms
4	$T(1, 2, 3, 4)$	$24 (\zeta \cdot \zeta)^1 (\zeta \cdot k)^2, 3 (\zeta \cdot \zeta)^2$ terms
5	$T(1, 2, 3, 4, 5)$	$270 (\zeta \cdot \zeta)^1 (\zeta \cdot k)^3, 45 (\zeta \cdot \zeta)^2 (\zeta \cdot k)^1$ terms
6	$T(1, 2, 3, 4, 5, 6)$	$3840 (\zeta \cdot \zeta)^1 (\zeta \cdot k)^4, 720 (\zeta \cdot \zeta)^2 (\zeta \cdot k)^2, 15 (\zeta \cdot \zeta)^3$ terms
7	$T(1, 2, 3, 4, 5, 6, 7)$	$65\,625 (\zeta \cdot \zeta)^1 (\zeta \cdot k)^5, 13\,125 (\zeta \cdot \zeta)^2 (\zeta \cdot k)^3, 525 (\zeta \cdot \zeta)^3 (\zeta \cdot k)^1$ terms

It is easy to see that it is an element of $\mathcal{V}_3$. In Subsection 3.2.1 we will prove that the only elements of $\mathcal{V}_3$ are multiples of $A_{YM}(1, 2, 3)$ (see Eq. (21)), so the fact that $A_{YM}(1, 2, 3) \in \mathcal{V}_3$ will turn out to be an immediate thing.

In order to do a counting of all the possible independent kinematical terms that can be taken from the list in (10) to construct the expression for an element $T(1, 2, \dots, N) \in \mathcal{V}_N$, respecting the kinematic conditions in (6), (7) and (8) and also the requirement in (4), we need first to analyze the $(\zeta \cdot k)$ terms. In principle, for each i there are N possible $(\zeta_i \cdot k_j)$ terms (because j runs from 1 to N). But taking into account the transversality condition (8) and also momentum conservation (6), this implies that for each i ($= 1, 2, \dots, N$) we have:

$$\sum_{j \neq i}^N (\zeta_i \cdot k_j) = 0. \quad (12)$$

So, at the end, for each i there are only $(N - 2)$ independent $(\zeta_i \cdot k_j)$ terms.

With this information we are in conditions to do a counting of the different independent terms, specified by structure in Eq. (10), that in principle are allowed to appear in $T(1, 2, \dots, N)$. This leads us to the following table, for the number of independent allowed kinematical terms of $T(1, 2, \dots, N)$.

It is not difficult to prove that the number of independent $(\zeta \cdot \zeta)^j (\zeta \cdot k)^{N-2j}$ terms (where $j = 1, 2, \dots, [N/2]$), for an arbitrary N , is given by

$$d_{N,j} = \frac{N(N-1)(N-2) \dots (N-(2j-1))}{2^j j!} (N-2)^{N-2j}. \quad (13)$$

This is the formula that has been used to compute the corresponding number of kinematical terms in Table 2.²⁶ We have only worried to specify in this table the cases $N = 3, 4, 5, 6, 7$, because these are the ones that we will consider in the present work.

In the next subsections, when writing all the allowed independent kinematical terms in $T(1, \dots, N)$, in particular when choosing the $(N - 2)$ independent $(\zeta_i \cdot k_j)$ terms, our choice will be the following list of $N(N - 2)$ terms:

²⁶ The number of independent kinematical terms mentioned in Table 2 should not be confused with the number of independent coefficients that will appear multiplying each of those terms. It might even happen that some of these coefficients will become zero after demanding the requirement of on-shell gauge invariance and, consequently, the corresponding kinematical terms referred to in Table 2 will not be present in the final expression of $T(1, \dots, N)$.

$$\begin{aligned}
& \{(\zeta_1 \cdot k_2), & (\zeta_1 \cdot k_3), & \dots, & (\zeta_1 \cdot k_{N-1}), \\
& (\zeta_2 \cdot k_1), & (\zeta_2 \cdot k_3), & \dots, & (\zeta_2 \cdot k_{N-1}), \\
& \vdots & \vdots & \vdots & \vdots \\
& (\zeta_{N-1} \cdot k_1), & (\zeta_{N-1} \cdot k_2), & \dots, & (\zeta_{N-1} \cdot k_{N-2}), \\
& (\zeta_N \cdot k_1), & (\zeta_N \cdot k_2), & \dots, & (\zeta_N \cdot k_{N-2})\},
\end{aligned} \tag{14}$$

that is, for each i ($= 1, 2, \dots, N-1$) we have used the restriction in (12) to eliminate $(\zeta_i \cdot k_N)$ in terms of the remaining $(N-2)$ $(\zeta_i \cdot k_j)$ terms ($j = 1, 2, \dots, N-1$, with $j \neq i$) and, in the last line of (14), we have eliminated $(\zeta_N \cdot k_{N-1})$ in terms of the remaining $(\zeta_N \cdot k_j)$ terms ($j = 1, 2, \dots, N-2$).

3.2. Finding a basis for $\mathcal{V}_N$, when $3 \leq N \leq 7$

In this subsection we will explicitly present the derivation of a basis for $\mathcal{V}_N$ only in the case $N = 3$ and $N = 4$, which involve simple calculations and illustrate the procedure to arrive at that basis. For $N = 5, 6, 7$ we will just mention the final result and we will leave the details of the main calculations to [Appendix D](#).

3.2.1. Case of $N = 3$

It is well known that in the case of 3-point amplitudes of massless states, momentum conservation and the mass-shell condition (7) imply that the momenta obey the relations

$$k_i \cdot k_j = 0 \quad (i, j = 1, 2, 3). \tag{15}$$

According to [Table 2](#), $T(1, 2, 3)$ has only three independent $(\zeta \cdot \zeta)^1 (\zeta \cdot k)^1$ terms, which, following our prescription in (14), leads to:

$$T(1, 2, 3) = \lambda_1 (\zeta_1 \cdot k_2) (\zeta_2 \cdot \zeta_3) + \lambda_2 (\zeta_2 \cdot k_1) (\zeta_3 \cdot \zeta_1) + \lambda_3 (\zeta_3 \cdot k_1) (\zeta_1 \cdot \zeta_2). \tag{16}$$

Demanding on-shell gauge invariance for gauge boson 1, $A(1, 2, 3)$ should become 0 when $\zeta_1 \rightarrow k_1$. Using the mass-shell condition (7) for k_1 and the condition in (15), then

$$\begin{aligned}
T(1, 2, 3)|_{\zeta_1=k_1} &= 0 \\
\Rightarrow \lambda_2 (\zeta_2 \cdot k_1) (\zeta_3 \cdot k_1) + \lambda_3 (\zeta_3 \cdot k_1) (\zeta_2 \cdot k_1) &= 0 \\
\Rightarrow (\lambda_2 + \lambda_3) (\zeta_2 \cdot k_1) (\zeta_3 \cdot k_1) &= 0 \\
\Rightarrow \lambda_2 &= -\lambda_3.
\end{aligned} \tag{17}$$

Similarly, demanding on-shell gauge invariance for gauge bosons 2 and 3 we arrive, respectively, to the conditions

$$\lambda_3 = \lambda_1, \tag{18}$$

$$\lambda_1 = -\lambda_2. \tag{19}$$

To arrive to (18) and (19) we have needed to use, in accordance to Eq. (14), that $(\zeta_3 \cdot k_2) = -(\zeta_3 \cdot k_1)$, $(\zeta_1 \cdot k_3) = -(\zeta_1 \cdot k_2)$ and $(\zeta_2 \cdot k_3) = -(\zeta_2 \cdot k_1)$.

The three conditions $\{(17), (18), (19)\}$ are linearly dependent. From them we conclude that $\lambda_1 = -\lambda_2 = \lambda_3$ and, for convenience, we choose them to be $2g\lambda$. So, finally, (16) can be written as²⁷

$$T(1, 2, 3) = \lambda \cdot 2g [(\zeta_1 \cdot k_2)(\zeta_2 \cdot \zeta_3) + (\zeta_2 \cdot k_3)(\zeta_3 \cdot \zeta_1) + (\zeta_3 \cdot k_1)(\zeta_1 \cdot \zeta_2)], \quad (20)$$

where λ is an arbitrary dimensionless factor and g is the open string coupling constant (which agrees with the one from the Yang–Mills lagrangian).

Eq. (20) can equivalently be written as

$$T(1, 2, 3) = \lambda \cdot A_{YM}(1, 2, 3), \quad (21)$$

where $A_{YM}(1, 2, 3)$ has been given in Eq. (11).

So $\mathcal{B}_3 = \{A_{YM}(1, 2, 3)\}$ is a basis for $\mathcal{V}_3$ and the dimension of $\mathcal{V}_3$ is 1 ($\dim(\mathcal{V}_3) = 1$).

In this case, the constant λ cannot have any momentum dependence due to the conditions (15), so it can only be a numerical constant.

Two trivial, but immediate applications of (21) are the case of $T(1, 2, 3)$ being any of the Yang–Mills 3-point subamplitudes, for which this equation becomes the cyclic or the reflection (or combination of both) properties (where $\lambda = 1$ or $\lambda = -1$) and the case of $T(1, 2, 3)$ being the open superstring subamplitude, $A(1, 2, 3)$, for which Eq. (21) implies that $A(1, 2, 3)$ receives no α' corrections and that $\lambda = 1$.

3.2.2. Case of $N = 4$

In this case, according to Table 2, the open superstring 4-point subamplitude has a kinematical expression which consists of twenty four $(\zeta \cdot \zeta)^1 (\zeta \cdot k)^2$ and three $(\zeta \cdot \zeta)^2$ terms, all of them being independent, so, following the prescription in (14), leads to:

$$\begin{aligned} T(1, 2, 3, 4) = & \lambda_1(\zeta_1 \cdot \zeta_2)(\zeta_3 \cdot k_1)(\zeta_4 \cdot k_1) + \lambda_2(\zeta_1 \cdot \zeta_2)(\zeta_3 \cdot k_1)(\zeta_4 \cdot k_2) \\ & + \lambda_3(\zeta_1 \cdot \zeta_2)(\zeta_3 \cdot k_2)(\zeta_4 \cdot k_1) + \lambda_4(\zeta_1 \cdot \zeta_2)(\zeta_3 \cdot k_2)(\zeta_4 \cdot k_2) \\ & + \lambda_5(\zeta_1 \cdot \zeta_3)(\zeta_2 \cdot k_1)(\zeta_4 \cdot k_1) + \lambda_6(\zeta_1 \cdot \zeta_3)(\zeta_2 \cdot k_1)(\zeta_4 \cdot k_2) \\ & + \lambda_7(\zeta_1 \cdot \zeta_3)(\zeta_2 \cdot k_3)(\zeta_4 \cdot k_1) + \lambda_8(\zeta_1 \cdot \zeta_3)(\zeta_2 \cdot k_3)(\zeta_4 \cdot k_2) \\ & + \lambda_9(\zeta_1 \cdot \zeta_4)(\zeta_2 \cdot k_1)(\zeta_3 \cdot k_1) + \lambda_{10}(\zeta_1 \cdot \zeta_4)(\zeta_2 \cdot k_1)(\zeta_3 \cdot k_2) \\ & + \lambda_{11}(\zeta_1 \cdot \zeta_4)(\zeta_2 \cdot k_3)(\zeta_3 \cdot k_1) + \lambda_{12}(\zeta_1 \cdot \zeta_4)(\zeta_2 \cdot k_3)(\zeta_3 \cdot k_2) \\ & + \lambda_{13}(\zeta_2 \cdot \zeta_3)(\zeta_1 \cdot k_2)(\zeta_4 \cdot k_1) + \lambda_{14}(\zeta_2 \cdot \zeta_3)(\zeta_1 \cdot k_2)(\zeta_4 \cdot k_2) \\ & + \lambda_{15}(\zeta_2 \cdot \zeta_3)(\zeta_1 \cdot k_3)(\zeta_4 \cdot k_1) + \lambda_{16}(\zeta_2 \cdot \zeta_3)(\zeta_1 \cdot k_3)(\zeta_4 \cdot k_2) \\ & + \lambda_{17}(\zeta_2 \cdot \zeta_4)(\zeta_1 \cdot k_2)(\zeta_3 \cdot k_1) + \lambda_{18}(\zeta_2 \cdot \zeta_4)(\zeta_1 \cdot k_2)(\zeta_3 \cdot k_2) \\ & + \lambda_{19}(\zeta_2 \cdot \zeta_4)(\zeta_1 \cdot k_3)(\zeta_3 \cdot k_1) + \lambda_{20}(\zeta_2 \cdot \zeta_4)(\zeta_1 \cdot k_3)(\zeta_3 \cdot k_2) \\ & + \lambda_{21}(\zeta_3 \cdot \zeta_4)(\zeta_1 \cdot k_2)(\zeta_2 \cdot k_1) + \lambda_{22}(\zeta_3 \cdot \zeta_4)(\zeta_1 \cdot k_2)(\zeta_2 \cdot k_3) \\ & + \lambda_{23}(\zeta_3 \cdot \zeta_4)(\zeta_1 \cdot k_3)(\zeta_2 \cdot k_1) + \lambda_{24}(\zeta_3 \cdot \zeta_4)(\zeta_1 \cdot k_3)(\zeta_2 \cdot k_3) \\ & + \rho_1(\zeta_1 \cdot \zeta_2)(\zeta_3 \cdot \zeta_4) + \rho_2(\zeta_1 \cdot \zeta_3)(\zeta_2 \cdot \zeta_4) + \rho_3(\zeta_1 \cdot \zeta_4)(\zeta_2 \cdot \zeta_3). \end{aligned} \quad (22)$$

²⁷ In Eq. (20) we have substituted back the relation $(\zeta_2 \cdot k_1) = -(\zeta_2 \cdot k_3)$ in order to obtain the familiar expression of $A_{YM}(1, 2, 3)$.

In [Appendix C](#) we find that demanding on-shell gauge invariance of the subamplitude when $\zeta_1 \rightarrow k_1$ we can arrive to the following thirteen linearly independent relations:

$$\begin{aligned} \lambda_4 &= 0, & \lambda_8 &= 0, & \lambda_{12} &= 0, & \lambda_3 + \lambda_{10} &= 0, & \lambda_7 + \lambda_{11} &= 0, \\ \lambda_1 + \lambda_5 + \lambda_9 &= 0, & \lambda_6 + \lambda_2 &= 0, \\ 2\rho_1 + \lambda_{21}s - \lambda_{23}(s+t) &= 0, & 2\rho_2 + \lambda_{17}s - \lambda_{19}(s+t) &= 0, \\ 2\rho_3 + \lambda_{13}s - \lambda_{15}(s+t) &= 0, \\ \lambda_{14}s - \lambda_{16}(s+t) &= 0, & \lambda_{18}s - \lambda_{20}(s+t) &= 0, & \lambda_{22}s - \lambda_{24}(s+t) &= 0, \end{aligned} \quad (23)$$

where

$$s = (k_1 + k_2)^2 = 2k_1 \cdot k_2 \quad \text{and} \quad t = (k_1 + k_4)^2 = 2k_1 \cdot k_4 \quad (24)$$

are two of the three Mandelstam variables that appear in the 4-point scattering.^{28,29}

The set of equations in (23) is the 4-point analog to Eq. (17), found for the 3-point subamplitude when $\zeta_1 \rightarrow k_1$.

In the same way, demanding on-shell gauge invariance of the $T(1, 2, 3, 4)$, when $\zeta_2 \rightarrow k_2$, $\zeta_3 \rightarrow k_3$ and $\zeta_4 \rightarrow k_4$, we can arrive to a set of thirteen linearly independent equations in each case. The explicit expression of these additional equations and the details of its solution can be found in [Appendix C](#).

The important thing is that in the whole set of fifty two equations that come from demanding on-shell gauge invariance (Eqs. (23), (C.3), (C.6) and (C.9)) only half of them are linearly independent as a whole. The solution of this system is given in [Appendix C](#), in Eq. (C.10): it consists of 7 null coefficients (λ_1 , λ_4 , λ_8 , λ_{12} , λ_{15} , λ_{19} and λ_{21}) and 20 which are given in terms of the Mandelstam variables s and t and a *unique* arbitrary parameter, which, for convenience, we have chosen to be λ_{24} , written as $4g^2\lambda^{[2]}/t$ (where $\lambda^{[2]}$ is arbitrary). After substituting this solution in (22) we finally arrive to³⁰

$$\begin{aligned} T(1, 2, 3, 4) &= \lambda^{[2]} \cdot 8g^2 \frac{1}{st} \left\{ -\frac{1}{4} [ts(\zeta_1 \cdot \zeta_3)(\zeta_2 \cdot \zeta_4) + su(\zeta_2 \cdot \zeta_3)(\zeta_1 \cdot \zeta_4) \right. \\ &\quad \left. + ut(\zeta_1 \cdot \zeta_2)(\zeta_3 \cdot \zeta_4)] \right. \\ &\quad - \frac{1}{2} s [(\zeta_1 \cdot k_4)(\zeta_3 \cdot k_2)(\zeta_2 \cdot \zeta_4) + (\zeta_2 \cdot k_3)(\zeta_4 \cdot k_1)(\zeta_1 \cdot \zeta_3) \\ &\quad \left. + (\zeta_1 \cdot k_3)(\zeta_4 \cdot k_2)(\zeta_2 \cdot \zeta_3) + (\zeta_2 \cdot k_4)(\zeta_3 \cdot k_1)(\zeta_1 \cdot \zeta_4)] \right. \\ &\quad - \frac{1}{2} t [(\zeta_2 \cdot k_1)(\zeta_4 \cdot k_3)(\zeta_3 \cdot \zeta_1) + (\zeta_3 \cdot k_4)(\zeta_1 \cdot k_2)(\zeta_2 \cdot \zeta_4) \\ &\quad \left. + (\zeta_2 \cdot k_4)(\zeta_1 \cdot k_3)(\zeta_3 \cdot \zeta_4) + (\zeta_3 \cdot k_1)(\zeta_4 \cdot k_2)(\zeta_2 \cdot \zeta_1)] \right\} \end{aligned}$$

²⁸ The other one is $u = (k_1 + k_3)^2 = 2k_1 \cdot k_3 = -s - t$.

²⁹ Notice that our convention for the 4-point Mandelstam variables has a different sign than the common one in very cited references, like [33,46,47]. We have followed this convention in order to be compatible with the one that we use for higher N -point Mandelstam variables. See [Appendix D.1](#).

³⁰ Substituting Eqs. (23), (C.3), (C.6) and (C.9), together with $\lambda_{24} = 4g^2\lambda^{[2]}/t$, in Eq. (22), leads to an expression which is only on-shell equivalent to the one in Eq. (25): it is necessary to use Eqs. (C.12) and (C.13) in order to check the equivalence between both formulas.

$$\begin{aligned}
& -\frac{1}{2}u\left[(\zeta_1 \cdot k_2)(\zeta_4 \cdot k_3)(\zeta_3 \cdot \zeta_2) + (\zeta_3 \cdot k_4)(\zeta_2 \cdot k_1)(\zeta_1 \cdot \zeta_4) \right. \\
& \left. + (\zeta_1 \cdot k_4)(\zeta_2 \cdot k_3)(\zeta_3 \cdot \zeta_4) + (\zeta_3 \cdot k_2)(\zeta_4 \cdot k_1)(\zeta_1 \cdot \zeta_2)\right], \quad (25)
\end{aligned}$$

or equivalently,

$$T(1, 2, 3, 4) = \lambda^{\{2\}} \cdot A_{YM}(1, \{2\}, 3, 4), \quad (26)$$

where $A_{YM}(1, \{2\}, 3, 4) = A_{YM}(1, 2, 3, 4)$ is the familiar Yang–Mills 4-point subamplitude.^{31,32}

So $\mathcal{B}_4 = \{A_{YM}(1, 2, 3, 4)\}$ is a basis for $\mathcal{V}_4$ and $\dim(\mathcal{V}_4) = 1$.

In (26) $\lambda^{\{2\}}$ is a dimensionless factor which may depend in the dimensionless variables α' s and $\alpha't$, where, at this point, α' could be any squared length scale (that in the case of string theory would be the string fundamental constant). $\lambda^{\{2\}}$ corresponds to what it is usually called the “momentum factor”.

3.2.3. Case of $N = 5, 6, 7$

The procedure for $N > 4$ is exactly the same one presented in Subsections 3.2.1 and 3.2.2, but for the corresponding N -point subamplitudes whose general kinematical structure we have mentioned in Table 2. Since for these cases the calculations are much more involved than the ones that we have already done for $N = 4$, we leave the details of them for Appendix D. In all these cases it is possible to arrive to an $(N - 3)!$ -dimensional basis for $\mathcal{V}_N$ ($\dim(\mathcal{V}_N) = (N - 3)!$) and it is always possible to choose this basis in terms of only Yang–Mills subamplitudes, in particular the one indicated in at the beginning of this section, in Eq. (5). That set of subamplitudes precisely contains $(N - 3)!$ elements. That set constitutes one of the possible basis that have appeared in the literature as basis for the space of Yang–Mills subamplitudes [37,38,12] and also for the space of open superstring subamplitudes [21], which is the purpose of this whole section to prove.

In Appendix D we see that in the cases for $N > 4$ our calculations have not lead directly to the basis in (5), but only after having verified that the dimension of $\mathcal{V}_N$ is $(N - 3)!$ and that the set of amplitudes mentioned in (5) is linearly independent.³³

So, summarizing, with the calculations and proofs that we have done in Appendix D we can write an arbitrary element of $\mathcal{V}_5$, $\mathcal{V}_6$ and $\mathcal{V}_7$, respectively as³⁴

$$T(1, 2, 3, 4, 5) = \lambda^{\{23\}} A_{YM}(1, \{2, 3\}, 4, 5) + \lambda^{\{32\}} A_{YM}(1, \{3, 2\}, 4, 5), \quad (27)$$

$$\begin{aligned}
T(1, 2, 3, 4, 5, 6) = & \lambda^{\{234\}} A_{YM}(1, \{2, 3, 4\}, 5, 6) + \lambda^{\{324\}} A_{YM}(1, \{3, 2, 4\}, 5, 6) \\
& + \lambda^{\{243\}} A_{YM}(1, \{2, 4, 3\}, 5, 6) + \lambda^{\{342\}} A_{YM}(1, \{3, 4, 2\}, 5, 6) \\
& + \lambda^{\{423\}} A_{YM}(1, \{4, 2, 3\}, 5, 6) + \lambda^{\{432\}} A_{YM}(1, \{4, 3, 2\}, 5, 6) \quad (28)
\end{aligned}$$

³¹ This explicit expression for $A_{YM}(1, 2, 3, 4)$ can be found in many places in the literature, for example, in Section 3 of [23]. But there are some sign differences due to the different convention that we have used for the 4-point Mandelstam variables in Eq. (24).

³² We have used the curly brackets, ‘{ }’, in the index 2, just as a remainder of the rule, mentioned in Eq. (5), to choose the Yang–Mills subamplitudes of $\mathcal{B}_N$.

³³ Our proposal of basis in Eq. (5) is based in the previously known results of the BCJ relations for $N = 5, 6, 7$ [37, 38,12]. Except for the case of $N = 4$, without these previously known results it would have been quite difficult to guess this basis and, as seen in Appendix D, we would have had only $(N - 3)!$ known (and very long) N -point kinematical expressions for which we would have no interpretation at all.

³⁴ In Eqs. (27), (28) and (29) we are following the same sort of notation that the authors of [21] used to write the linear combinations of the subamplitudes, that is, the momentum factors are labeled by a superscript denoting the corresponding σ_N permutation.

and

$$\begin{aligned} T(1, 2, 3, 4, 5, 6, 7) = & \lambda^{\{2345\}} A_{YM}(1, \{2, 3, 4, 5\}, 6, 7) + \lambda^{\{2354\}} A_{YM}(1, \{2, 3, 5, 4\}, 6, 7) \\ & + \lambda^{\{2435\}} A_{YM}(1, \{2, 4, 3, 5\}, 6, 7) \\ & + \dots + \lambda^{\{5432\}} A_{YM}(1, \{5, 4, 3, 2\}, 6, 7), \end{aligned} \quad (29)$$

where the $\lambda^{\{\sigma_N\}}$'s are the corresponding momentum factors, which may depend in the dimensionless Mandelstam variables ($\alpha' s_i$ and $\alpha' t_j$) of each N -point scattering process of gauge bosons.

An independent set of Mandelstam variables for $N = 5, 6, 7$ is given in [Appendix D](#).

In the right-hand side of (29) there are $4! = 24$ terms that are being summed, in accordance to Eq. (5) for $N = 7$.

The same as in Eq. (26), in Eqs. (27), (28) and (29) we have inserted curly brackets, ' $\{\}$ ', just as a remainder of the rule, mentioned in Eq. (5), to choose the Yang–Mills subamplitudes of $\mathcal{B}_5$, $\mathcal{B}_6$ and $\mathcal{B}_7$, respectively.

3.3. Finding the components of an element of $\mathcal{V}_N$ with respect to the basis $\mathcal{B}_N$

Once we have accepted that the set $\mathcal{B}_N$ is indeed a basis for $\mathcal{V}_N$, at least for $3 \leq N \leq 7$, the next step consists in finding the components of an element of $\mathcal{V}_N$ with respect to $\mathcal{B}_N$. We will present here a procedure which we expect to be valid for any $N \geq 3$ (even for $N > 7$).³⁵

Let $T(1, \dots, N) \in \mathcal{V}_N$. To find the components of $T(1, \dots, N)$ with respect to the basis $\mathcal{B}_N$ we have to find the momentum factors, $\lambda^{\{\sigma_N\}}$'s, such that

$$T(1, \dots, N) = \sum_{\sigma_N \in \mathcal{S}_{N-3}} \lambda^{\{\sigma_N\}} A_{YM}(1, \{2_\sigma, 3_\sigma, \dots, (N-2)_\sigma\}, N-1, N), \quad (30)$$

where $\sigma_N = \{2_\sigma, 3_\sigma, \dots, (N-2)_\sigma\}$ denotes the same permutation of indexes $\{2, 3, \dots, (N-2)\}$ that we referred to in Eq. (5).

The natural (but tedious!) way to find the $\lambda^{\{\sigma_N\}}$'s is by writing down the expression of $T(1, \dots, N)$, and of each Yang–Mills subamplitude in $\mathcal{B}_N$, in terms of the kinematical terms listed in Eq. (10), with the convention (14) for the $(\zeta \cdot k)$ terms, for example. Then, a linear system for the $\lambda^{\{\sigma_N\}}$'s arises from demanding that the coefficient of each $(\zeta \cdot \zeta)^j (\zeta \cdot k)^{N-2j}$ term (where $j = 1, \dots, [N/2]$), in both sides of (30), is the same. Since this linear system is overdetermined, it is not necessary to consider *all* the equations of it in order to find the $(N-3)! \lambda^{\{\sigma_N\}}$'s.

For small values of N it is easy to see that considering in (30) the kinematical terms with the smallest number of $(\zeta \cdot k)$ terms (that is, the ones for $j = [N/2]$), that is enough information to find the $\lambda^{\{\sigma_N\}}$'s. Consider, for example, the cases of $N = 3, 4, 5$. From [Table 2](#) we see that the 3-point amplitude contains 3 $(\zeta \cdot \zeta)^1 (\zeta \cdot k)^1$ terms, the 4-point amplitude contains 3 $(\zeta \cdot \zeta)^2$ terms and the 5-point amplitude contains 45 $(\zeta \cdot \zeta)^2 (\zeta \cdot k)^1$ terms. It is clear, then, that considering these particular kinematical terms in (30), that would be enough information to find the $\lambda^{\{\sigma_N\}}$'s, since for $N = 3$ and $N = 4$ there is only one component and for $N = 5$ there are two components to determine.

³⁵ At least in this subsection we will work with the hypothesis that $\mathcal{B}_N$, as specified in (5), is a basis of $\mathcal{V}_N$ for any $N \geq 3$.

But, for sufficiently large N ,³⁶ the number of kinematical terms of the type mentioned above (the ones for $j = [N/2]$) is less than $(N - 3)!$ and, then, considering only those type of terms will not be enough information to find *all* the $\lambda^{\{\sigma_N\}}$'s.

Since, at this point, we are worried about a strategy to find the components of $T(1, \dots, N)$, for an arbitrary N , our proposal will be to consider the kinematical terms for $j = 1$, namely, the $(\zeta \cdot \zeta)^1 (\zeta \cdot k)^{N-2}$ terms. Those are the ones, among all the possible types of kinematical terms in (10), which show up in more amount in the most general expression for $T(1, \dots, N)$.³⁷

So, in the next two sections, when we will look for the momentum factors that are present in the BCJ relations and in the open superstring formula, we will consider the $(\zeta \cdot \zeta)^1 (\zeta \cdot k)^{N-2}$ terms in our kinematical analysis.

4. Closed expression for the N -point disk amplitude using the RNS formalism

An immediate and natural application of the analysis done in Subsection 3.3 is to find the momentum factors in the case that $T(1, \dots, N)$ is *any* of the possible Yang–Mills or open superstring N -point subamplitudes, because they are gauge invariant and the $(\zeta \cdot k)^N$ are absent in them.³⁸ In the first case, the result becomes one of the BCJ relations and in the second case, the result becomes a closed formula for the open superstring subamplitudes. We do the exercise for the BCJ relations in Appendix E and in this section we do the corresponding calculation for the open superstring, arriving to MSS's result, in Eq. (1), by means of a RNS formalism (for $3 \leq N \leq 7$).

We will first derive MSS formula in the case of pure gauge boson scattering in Subsection 4.1 and then, in Subsection 4.2, we will quickly see that there is no need to deal with fermion vertex operators (at least for the tree level amplitudes) in order to find the scattering amplitudes involving fermions (once the closed formula has been found for gauge bosons).

4.1. For gauge bosons only

Using the RNS formalism, it is known that the N -point subamplitude for gauge bosons in open superstring theory is given by the following integral formula [46]³⁹:

$$\begin{aligned}
 A_b(1, 2, \dots, N) = & 2 \frac{g^{N-2}}{(2\alpha')^2} (x_{N-1} - x_1)(x_N - x_1) \int_0^{x_{N-1}} dx_{N-2} \int_0^{x_{N-2}} dx_{N-3} \dots \int_0^{x_3} dx_2 \\
 & \times \int d\theta_1 \dots d\theta_{N-2} \prod_{p < q}^N (x_q - x_p - \theta_q \theta_p)^{2\alpha' k_p \cdot k_q} \times \int d\phi_1 \dots d\phi_N \\
 & \times \exp \left(\sum_{i \neq j}^N \frac{(2\alpha')^1 (\theta_j - \theta_i) \phi_j (\zeta_j \cdot k_i) - 1/2 (2\alpha')^1 \phi_j \phi_i (\zeta_j \cdot \zeta_i)}{x_j - x_i - \theta_j \theta_i} \right).
 \end{aligned} \tag{31}$$

³⁶ Consider $N \geq 12$, for example.

³⁷ It is not difficult to see, by considering formula (13), that the number of independent $(\zeta \cdot \zeta)^1 (\zeta \cdot k)^{N-2}$ terms is bigger than $(N - 3)!$.

³⁸ In the case of Yang–Mills subamplitudes, it was already mentioned in the beginning of Subsection 3.1 that they do not contain $(\zeta \cdot k)^N$ terms.

³⁹ In Eq. (31) we have used an index b in the N -point subamplitude as a remainder that it corresponds to the scattering process involving only bosons.

In this formula, the θ_i 's and the ϕ_j 's are Grassmann variables, while the x_k 's are common real variables, where $x_1 < x_2 < \dots < x_N$. Briefly speaking, the result in Eq. (31) has been obtained by averaging over the ground state of the theory, the product of N gauge boson vertex operators, localized at positions $x_1, x_2, \dots, x_N$. The residual symmetry of the integrand can be gauge-fixed, for example, by demanding $\{x_1 = 0, x_{N-1} = 1, x_N = +\infty\}$ and $\{\theta_{N-1} = \theta_N = 0\}$.⁴⁰

In this section we will prove that in the case of $T(1, \dots, N) = A_b(1, 2, \dots, N)$, given in Eq. (31), then the momentum factors of Eq. (30) are given precisely by the ones appearing in MSS formula, Eq. (1). The integral formula that Mafra, Schlotterer and Stieberger find for them is [21]

$$F^{\{23\dots N-2\}}(\alpha') = \int_0^1 dx_{N-2} \int_0^{x_{N-2}} dx_{N-3} \dots \int_0^{x_3} dx_2 \left(\prod_{i>j\geq 1}^{N-1} (x_i - x_j)^{2\alpha' k_i \cdot k_j} \right) \times \left\{ \prod_{p=2}^{N-2} \sum_{q=1}^{p-1} \left(\frac{2\alpha' k_p \cdot k_q}{x_p - x_q} \right) \right\}, \quad (32)$$

where $x_1 = 0$ and $x_{N-1} = 1$.⁴¹

Formula (32) is the one for the momentum factor which in Eq. (1) goes multiplying the subamplitude $A_{YM}(1, \{2, 3, \dots, N-2\}, N-1, N)$. The MSS prescription for the remaining $F^{\{\sigma_N\}}(\alpha')$ momentum factors consists in interchanging the indices $\{2, 3, \dots, N-2\}$, according to the σ_N permutation, *only* in the curly brackets of the right-hand side of Eq. (32). This interchange of indices should be done in *both*, the k_j 's momenta and the x_j 's variables inside the curly brackets. This will become more clear in the case-by-case study that we will consider in the following subsections.

Before going into the derivation of the momentum factors we have two remarks:

1. As they stand, formulas (31) and (32) are not applicable to the case of $N = 3$. Since it is very well known that in this case $A_b(1, 2, 3) = A_{YM}(1, 2, 3)$, the 3-point momentum factor is simply *defined* as being 1.
2. There is an equivalent expression that Mafra, Schlotterer and Stieberger give for the momentum factor in Eq. (32), for $N \geq 5$, and it comes by integrating by parts on it [35]:

$$F^{\{23\dots N-2\}}(\alpha') = \int_0^1 dx_{N-2} \int_0^{x_{N-2}} dx_{N-3} \dots \int_0^{x_3} dx_2 \left(\prod_{i>j\geq 1}^{N-1} (x_i - x_j)^{2\alpha' k_i \cdot k_j} \right) \times \left\{ \prod_{p=2}^{[N/2]} \sum_{q=1}^{p-1} \left(\frac{2\alpha' k_p \cdot k_q}{x_p - x_q} \right) \right\} \left\{ \prod_{p=[N/2]+1}^{N-2} \sum_{q=p+1}^{N-1} \left(\frac{2\alpha' k_p \cdot k_q}{x_p - x_q} \right) \right\}, \quad (33)$$

In the following subsections we will see that we reproduce formula (32) in the case of $N = 4$ and formula (33) in the cases of $N = 5, 6, 7$.

⁴⁰ In (31) we have already kept $x_1, x_{N-1}, x_N, \theta_{N-1}$ and θ_N fixed, but we have not yet chosen the peculiar values mentioned in the text. See Section 7.3 of Ref. [46], for more details.

⁴¹ In the expression in (32) x_N has already been taken to $+\infty$: that is why it does not appear on it.

In the case of $N = 4$ we will present the derivation with enough details to see the consistency of the relations that arise when we consider the $(\zeta \cdot \zeta)^1(\zeta \cdot k)^{N-2}$ terms of the subamplitudes. For $N = 5, 6, 7$ the procedure will be the same one, but we will not present so many details because the open superstring and the Yang–Mills subamplitudes become too large. For $N = 5$ we will explain how to arrive at the known expression of the two momentum factors (Eq. (33)), together with some evidence of the self consistency of the calculations, and for $N = 6, 7$ we will just explain how to arrive to the corresponding momentum factors.

4.1.1. Case of $N = 4$

In this case the relation (26) guarantees that choosing $T(1, 2, 3, 4) = A_b(1, 2, 3, 4)$ it is possible to write down

$$A_b(1, 2, 3, 4) = F^{[2]}(\alpha') A_{YM}(1, 2, 3, 4), \quad (34)$$

for a certain momentum factor $F^{[2]}(\alpha')$ that we want to find.

$A_{YM}(1, 2, 3, 4)$ has already been given in (25) and the expression for $A_b(1, 2, 3, 4)$ is not difficult to be obtained from (31) for $N = 4$ (after expanding the exponential and integrating over the two Grassmann θ_i 's and the four Grassmann ϕ_j 's).

Following the procedure proposed in Subsection 3.3, after writing all $(\zeta \cdot k)$ terms in the basis given in (14) and equating the $(\zeta \cdot \zeta)^1(\zeta \cdot k)^2$ terms of both sides of (34), we arrive to

$$\begin{aligned} & 2g^2 \{ (\zeta_1 \cdot \zeta_2) [- (4\alpha') I_1^{[4]}(\zeta_3 \cdot k_1)(\zeta_4 \cdot k_2) + (4\alpha') I_3^{[4]}(\zeta_3 \cdot k_2)(\zeta_4 \cdot k_1)] \\ & + (\zeta_1 \cdot \zeta_3) [(4\alpha') I_1^{[4]}(\zeta_2 \cdot k_1)(\zeta_4 \cdot k_1) + (4\alpha') I_1^{[4]}(\zeta_2 \cdot k_1)(\zeta_4 \cdot k_2) \\ & - (4\alpha') I_2^{[4]}(\zeta_2 \cdot k_3)(\zeta_4 \cdot k_1)] \\ & + (\zeta_1 \cdot \zeta_4) [- (4\alpha') I_1^{[4]}(\zeta_2 \cdot k_1)(\zeta_3 \cdot k_1) - (4\alpha') I_3^{[4]}(\zeta_2 \cdot k_1)(\zeta_3 \cdot k_2) \\ & + (4\alpha') I_2^{[4]}(\zeta_2 \cdot k_3)(\zeta_3 \cdot k_1)] \\ & + (\zeta_2 \cdot \zeta_3) [- (4\alpha') I_3^{[4]}(\zeta_1 \cdot k_2)(\zeta_4 \cdot k_1) - (4\alpha') I_3^{[4]}(\zeta_1 \cdot k_2)(\zeta_4 \cdot k_2) \\ & - (4\alpha') I_2^{[4]}(\zeta_1 \cdot k_3)(\zeta_4 \cdot k_2)] \\ & + (\zeta_2 \cdot \zeta_4) [(4\alpha') I_1^{[4]}(\zeta_1 \cdot k_2)(\zeta_3 \cdot k_1) + (4\alpha') I_3^{[4]}(\zeta_1 \cdot k_2)(\zeta_3 \cdot k_2) \\ & + (4\alpha') I_2^{[4]}(\zeta_1 \cdot k_3)(\zeta_3 \cdot k_2)] \\ & + (\zeta_3 \cdot \zeta_4) [- (4\alpha') I_3^{[4]}(\zeta_1 \cdot k_2)(\zeta_2 \cdot k_3) + (4\alpha') I_1^{[4]}(\zeta_1 \cdot k_3)(\zeta_2 \cdot k_1) \\ & - (4\alpha') I_2^{[4]}(\zeta_1 \cdot k_3)(\zeta_2 \cdot k_3)] \} \\ & = 2g^2 F^{[2]}(\alpha') \left\{ (\zeta_1 \cdot \zeta_2) \left[-\frac{4}{s}(\zeta_3 \cdot k_1)(\zeta_4 \cdot k_2) + \left(\frac{4}{s} + \frac{4}{t} \right) (\zeta_3 \cdot k_2)(\zeta_4 \cdot k_1) \right] \right. \\ & + (\zeta_1 \cdot \zeta_3) \left[\frac{4}{s}(\zeta_2 \cdot k_1)(\zeta_4 \cdot k_1) + \frac{4}{s}(\zeta_2 \cdot k_1)(\zeta_4 \cdot k_2) - \frac{4}{t}(\zeta_2 \cdot k_3)(\zeta_4 \cdot k_1) \right] \\ & + (\zeta_1 \cdot \zeta_4) \left[-\frac{4}{s}(\zeta_2 \cdot k_1)(\zeta_3 \cdot k_1) - \left(\frac{4}{s} + \frac{4}{t} \right) (\zeta_2 \cdot k_1)(\zeta_3 \cdot k_2) + \frac{4}{t}(\zeta_2 \cdot k_3)(\zeta_3 \cdot k_1) \right] \\ & + (\zeta_2 \cdot \zeta_3) \left[-\left(\frac{4}{s} + \frac{4}{t} \right) (\zeta_1 \cdot k_2)(\zeta_4 \cdot k_1) - \left(\frac{4}{s} + \frac{4}{t} \right) (\zeta_1 \cdot k_2)(\zeta_4 \cdot k_2) \right. \\ & \left. \left. - \frac{4}{t}(\zeta_1 \cdot k_3)(\zeta_4 \cdot k_2) \right] \right\} \end{aligned}$$

$$\begin{aligned}
& + (\zeta_2 \cdot \zeta_4) \left[\frac{4}{s} (\zeta_1 \cdot k_2) (\zeta_3 \cdot k_1) + \left(\frac{4}{s} + \frac{4}{t} \right) (\zeta_1 \cdot k_2) (\zeta_3 \cdot k_2) + \frac{4}{t} (\zeta_1 \cdot k_3) (\zeta_3 \cdot k_2) \right] \\
& + (\zeta_3 \cdot \zeta_4) \left[- \left(\frac{4}{s} + \frac{4}{t} \right) (\zeta_1 \cdot k_2) (\zeta_2 \cdot k_3) + \frac{4}{s} (\zeta_1 \cdot k_3) (\zeta_2 \cdot k_1) - \frac{4}{t} (\zeta_1 \cdot k_3) (\zeta_2 \cdot k_3) \right] \Big\},
\end{aligned} \tag{35}$$

where⁴²

$$\begin{aligned}
I_1^{[4]} &= \int_0^1 dx_2 x_2^{\alpha' s - 1} (1 - x_2)^{\alpha' t}, & I_2^{[4]} &= \int_0^1 dx_2 x_2^{\alpha' s} (1 - x_2)^{\alpha' t - 1}, \\
I_3^{[4]} &= \int_0^1 dx_2 x_2^{\alpha' s - 1} (1 - x_2)^{\alpha' t - 1}.
\end{aligned} \tag{36}$$

In (35), when we substituted $A_{YM}(1, 2, 3, 4)$ from Eq. (25), we have used that $u = -s - t$.

Comparing the coefficient of the $(\zeta_1 \cdot \zeta_2)(\zeta_3 \cdot k_1)(\zeta_4 \cdot k_2)$ term in both sides of (35) we find that

$$F^{[2]}(\alpha') = \alpha' s I_1^{[4]}, \tag{37}$$

or equivalently,

$$F^{[2]}(\alpha') = 2\alpha' k_2 \cdot k_1 \int_0^1 dx_2 x_2^{2\alpha' k_2 \cdot k_1 - 1} (1 - x_2)^{2\alpha' k_3 \cdot k_2} \tag{38}$$

$$= \int_0^1 dx_2 \left(\prod_{i>j \geq 1}^3 (x_i - x_j)^{2\alpha' k_i \cdot k_j} \right) \frac{2\alpha' k_2 \cdot k_1}{x_2 - x_1}, \tag{39}$$

where $x_1 = 0$ and $x_3 = 1$.

Formula (39) corresponds precisely to formula (32) for the case of $N = 4$, as we had anticipated.

Notice that comparing other $(\zeta \cdot \zeta)^1 (\zeta \cdot k)^2$ terms in Eq. (35) we may arrive to different expressions for $F^{[2]}(\alpha')$. For example, if we compare the coefficient of $(\zeta_1 \cdot \zeta_3)(\zeta_2 \cdot k_3)(\zeta_4 \cdot k_1)$ and the coefficient of $(\zeta_2 \cdot \zeta_3)(\zeta_1 \cdot k_2)(\zeta_4 \cdot k_1)$, in both sides of (35), we find that, respectively,

$$F^{[2]}(\alpha') = \alpha' t I_2^{[4]}, \quad F^{[2]}(\alpha') = \alpha' \frac{st}{s+t} I_3^{[4]}. \tag{40}$$

Integrating by parts in the definition of $I_2^{[4]}$ (and considering the analytic continuation in $\alpha' s$ and $\alpha' t$ of the three integrals in Eq. (36)), we have that

$$I_2^{[4]} = \frac{s}{t} I_1^{[4]}. \tag{41}$$

⁴² In (36) we have introduced the superscript '[4]' as a remainder that the integrals that have appeared are the ones for the 4-point case.

Also, multiplying the identity $1/\{x_2(1-x_2)\} = 1/x_2 + 1/(1-x_2)$, by the term $x_2^{2\alpha's}(1-x_2)^{2\alpha't}$ and integrating in the interval $[0, 1]$, this identity becomes

$$I_3^{[4]} = I_1^{[4]} + I_2^{[4]}. \quad (42)$$

Using relations (41) and (42) it is easy to see that the two alternative expressions for $F^{[2]}(\alpha')$, given in Eq. (40), are equivalent to the one in (37) (and, therefore, equivalent to the one in (39)).

One might even ask what would have happened if we had included the three $(\zeta \cdot \zeta)^2$ terms in both sides of (35) and compared their corresponding coefficients. The answer is that we would have found two of the three integral representations that we already have for $F^{[2]}(\alpha')$ plus a new one:

$$F^{[2]}(\alpha') = \frac{s(\alpha's - 1)}{s + t} \int_0^1 dx_2 x_2^{\alpha's-2} (1-x_2)^{\alpha't}. \quad (43)$$

It can be easily verified that this expression can be obtained using integration by parts in $I_3^{[4]}$, in the second equation in (40).

4.1.2. Case of $N = 5$

In this case the relation (27) guarantees that choosing $T(1, 2, 3, 4, 5) = A_b(1, 2, 3, 4, 5)$ it is possible to write down

$$A_b(1, 2, 3, 4, 5) = F^{[23]}(\alpha') A_{YM}(1, 2, 3, 4, 5) + F^{[32]}(\alpha') A_{YM}(1, 3, 2, 4, 5). \quad (44)$$

Following the procedure proposed in Subsection 3.3, in Eq. (44) we consider only the $(\zeta \cdot \zeta)^1 (\zeta \cdot k)^3$ terms,

$$A_b(1, 2, 3, 4, 5) \big|_{(\zeta \cdot \zeta)^1 (\zeta \cdot k)^3} = F^{[23]}(\alpha') A_{YM}(1, 2, 3, 4, 5) \big|_{(\zeta \cdot \zeta)^1 (\zeta \cdot k)^3} + F^{[32]}(\alpha') A_{YM}(1, 3, 2, 4, 5) \big|_{(\zeta \cdot \zeta)^1 (\zeta \cdot k)^3}, \quad (45)$$

where we are supposed to write all $(\zeta \cdot k)$ terms in the basis given in (14) for $N = 5$.

On one side, in Section 5 of Ref. [23] it has been explained in detail how the $x_5 \rightarrow +\infty$ limit is taken and how the Grassmann integrations are done in Eq. (31), in the case of $N = 5$,⁴³ so all the terms of the left-hand side of Eq. (45) are known.

On the other side, we give the expression for all the $(\zeta \cdot \zeta)^1 (\zeta \cdot k)^3$ terms of $A_{YM}(1, 2, 3, 4, 5)$ (and, therefore, also the corresponding terms of $A_{YM}(1, 3, 2, 4, 5)$) in Eq. (D.7), so the right-hand side of Eq. (45) is also completely known, except for $F^{[23]}(\alpha')$ and $F^{[32]}(\alpha')$.

If we examine the coefficient of $(\zeta_1 \cdot \zeta_2)(\zeta_3 \cdot k_4)(\zeta_4 \cdot k_1)(\zeta_5 \cdot k_2)$ in both sides of Eq. (45) we find that⁴⁴

$$\begin{aligned} \frac{8}{s_1 s_3} F^{[23]}(\alpha') &= 8\alpha'^2 \int_0^1 dx_3 \int_0^{x_3} dx_2 x_2^{\alpha's_1-1} x_3^{\alpha'(s_4-s_1-s_2)} (x_3-x_2)^{\alpha's_2} \\ &\quad \times (1-x_2)^{\alpha'(s_5-s_2-s_3)} (1-x_3)^{\alpha's_3-1}. \end{aligned} \quad (46)$$

⁴³ There are three θ_i 's and five ϕ_j 's in this case.

⁴⁴ In our $N = 5$ calculations we have substituted the $k_i \cdot k_j$ invariants in terms of the independent Mandelstam variables that we have chosen. See Eqs. (D.1) and (D.4).

This equation leads directly to an expression for $F^{[23]}(\alpha')$, which might be written as⁴⁵

$$F^{[23]}(\alpha') = \int_0^1 dx_3 \int_0^{x_3} dx_2 \left(\prod_{i>j \geq 1}^4 (x_i - x_j)^{2\alpha' k_i \cdot k_j} \right) \frac{2\alpha' k_2 \cdot k_1}{x_2 - x_1} \cdot \frac{2\alpha' k_4 \cdot k_3}{x_4 - x_3}, \quad (47)$$

where $x_1 = 0$ and $x_4 = 1$.

The fact that in Eq. (46) $F^{[32]}(\alpha')$ is not present means that the $(\zeta_1 \cdot \zeta_2)(\zeta_3 \cdot k_4)(\zeta_4 \cdot k_1)(\zeta_5 \cdot k_2)$ term is present in $A_b(1, 2, 3, 4, 5)$ and $A_{YM}(1, 2, 3, 4, 5)$, but it is not present in $A_{YM}(1, 3, 2, 4, 5)$.

Similarly, if we examine the coefficient of $(\zeta_1 \cdot \zeta_3)(\zeta_2 \cdot k_4)(\zeta_4 \cdot k_1)(\zeta_5 \cdot k_3)$ in both sides of Eq. (45) we find that

$$\begin{aligned} & -\frac{8}{(s_1 + s_2 - s_4)(s_5 - s_2 - s_3)} F^{[32]}(\alpha') \\ & = 8\alpha'^2 \int_0^1 dx_3 \int_0^{x_3} dx_2 x_2^{\alpha' s_1} x_3^{\alpha'(s_4 - s_1 - s_2) - 1} (x_3 - x_2)^{\alpha' s_2} \\ & \quad \times (1 - x_2)^{\alpha'(s_5 - s_2 - s_3) - 1} (1 - x_3)^{\alpha' s_3}, \end{aligned} \quad (48)$$

from which we can arrive at

$$F^{[32]}(\alpha') = \int_0^1 dx_3 \int_0^{x_3} dx_2 \left(\prod_{i>j \geq 1}^4 (x_i - x_j)^{2\alpha' k_i \cdot k_j} \right) \frac{2\alpha' k_3 \cdot k_1}{x_3 - x_1} \cdot \frac{2\alpha' k_4 \cdot k_2}{x_4 - x_2}. \quad (49)$$

Formulas (47) and (49) correspond to the expressions of the two momentum factors that appear in the case of $N = 5$. These two formulas agree completely with the expected expression for $F^{[23]}(\alpha')$ and $F^{[32]}(\alpha')$, according to formula (33).

The two specific kinematical structures that allowed us to arrive directly to the known expressions of $F^{[23]}(\alpha')$ and $F^{[32]}(\alpha')$ are not the only ones that lead to those expressions. For example, we have checked that examining the coefficient of $(\zeta_1 \cdot \zeta_4)(\zeta_2 \cdot k_1)(\zeta_3 \cdot k_4)(\zeta_5 \cdot k_1)$ and $(\zeta_1 \cdot \zeta_4)(\zeta_2 \cdot k_1)(\zeta_3 \cdot k_4)(\zeta_5 \cdot k_2)$, in both sides of (45), also leads to Eq. (46). We have also checked that examining the coefficient of $(\zeta_1 \cdot \zeta_4)(\zeta_2 \cdot k_4)(\zeta_3 \cdot k_1)(\zeta_5 \cdot k_1)$ and $(\zeta_1 \cdot \zeta_4)(\zeta_2 \cdot k_4)(\zeta_3 \cdot k_1)(\zeta_5 \cdot k_3)$, in both sides of (45), also leads to Eq. (48).

For the sake of completeness we will examine in Eq. (45) two other kinematical structures, namely, $(\zeta_1 \cdot \zeta_2)(\zeta_3 \cdot k_1)(\zeta_4 \cdot k_2)(\zeta_5 \cdot k_1)$ and $(\zeta_1 \cdot \zeta_2)(\zeta_3 \cdot k_4)(\zeta_4 \cdot k_2)(\zeta_5 \cdot k_1)$. This leads, respectively, to the following two relations that $F^{[23]}(\alpha')$ and $F^{[32]}(\alpha')$ should satisfy:

$$\begin{aligned} & \frac{2}{s_1} F^{[23]}(\alpha') + \frac{2(s_2 + s_3 - s_4 - s_5)}{(s_5 - s_2 - s_3)(s_1 + s_2 - s_4)} F^{[32]}(\alpha') = 2\alpha'^2 s_4 I_1^{[5]}, \\ & -\frac{2(s_1 + s_5)}{s_1 s_3} F^{[23]}(\alpha') + \frac{2}{(s_2 + s_3 - s_5)} F^{[32]}(\alpha') = -2\alpha'^2 s_5 I_2^{[5]}, \end{aligned} \quad (50)$$

where

⁴⁵ In (47) we have substituted back the explicit expressions for the $N = 5$ Mandelstam variables.

$$I_1^{[5]} = \int_0^1 dx_3 \int_0^{x_3} dx_2 x_2^{\alpha' s_1 - 1} x_3^{\alpha' (s_4 - s_1 - s_2)} (x_3 - x_2)^{\alpha' s_2} \\ \times (1 - x_2)^{\alpha' (s_5 - s_2 - s_3) - 1} (1 - x_3)^{\alpha' s_3 - 1}, \quad (51)$$

$$I_2^{[5]} = \int_0^1 dx_3 \int_0^{x_3} dx_2 x_2^{\alpha' s_1 - 1} x_3^{\alpha' (s_4 - s_1 - s_2) - 1} (x_3 - x_2)^{\alpha' s_2} \\ \times (1 - x_2)^{\alpha' (s_5 - s_2 - s_3) - 1} (1 - x_3)^{\alpha' s_3}. \quad (52)$$

Solving the linear system in (50) leads to integral expressions for $F^{[23]}(\alpha')$ and $F^{[32]}(\alpha')$ which apparently differ from the ones found in (46) and (48), respectively. But this is exactly the same thing that we analyzed in Eqs. (40) and (43), for the case of $N = 4$: there are integration by parts and partial fraction relations that allow us to write a same momentum factor in apparently different integral representations. Since the expressions found for the momentum factors this time are double integrals, there is a much rich variety of relations that can be found for them and it is not always an immediate thing to prove the equivalence between this different integral representations. In Refs. [48] and [18] it was explained how using these relations, $A_b(1, 2, 3, 4, 5)$ for the first time could be written in a simple form, as a sum of two contributions (in direct analogy to Eq. (44)).

4.1.3. Case of $N = 6$ and $N = 7$

In the case of $N = 6$ and $N = 7$ we will not refer any longer to the alternative integral expressions that show up for the momentum factors (and which can always be proved by combining integration by parts and partial fraction techniques). These expressions will simply have to be equivalent to the ones we are looking for, because of the consistency of our method.

In this case relations (28) and (29) guarantee that choosing $T(1, 2, 3, 4, 5, 6) = A_b(1, 2, 3, 4, 5, 6)$ and $T(1, 2, 3, 4, 5, 6, 7) = A_b(1, 2, 3, 4, 5, 6, 7)$, respectively, it is possible to write down similar relations to the ones in those equations, but relabeling the momentum factors as

$$\lambda^{\{\sigma_6\}} \rightarrow F^{\{\sigma_6\}}(\alpha'), \quad \lambda^{\{\sigma_7\}} \rightarrow F^{\{\sigma_7\}}(\alpha'), \quad (53)$$

where σ_6 and σ_7 are all possible S_{N-3} permutations (for $N = 6$ and $N = 7$) of indices $\{2, 3, 4\}$ and $\{2, 3, 4, 5\}$, respectively.

The final result is that we succeed in arriving at the six and the twenty four momentum factors of formula (33) for $N = 6$ and $N = 7$, respectively, that is:

$$F^{[234]}(\alpha') = \int_0^1 dx_4 \int_0^{x_4} dx_3 \int_0^{x_3} dx_2 \left(\prod_{i>j \geq 1}^5 (x_i - x_j)^{2\alpha' k_i \cdot k_j} \right) \\ \times \left\{ \frac{2\alpha' k_2 \cdot k_1}{x_2 - x_1} \cdot \frac{2\alpha' k_5 \cdot k_4}{x_5 - x_4} \left(\frac{2\alpha' k_3 \cdot k_1}{x_3 - x_1} + \frac{2\alpha' k_3 \cdot k_2}{x_3 - x_2} \right) \right\} \quad (54)$$

and

$$F^{(2345)}(\alpha') = \int_0^1 dx_5 \int_0^{x_5} dx_4 \int_0^{x_4} dx_3 \int_0^{x_3} dx_2 \left(\prod_{i>j \geq 1}^6 (x_i - x_j)^{2\alpha' k_i \cdot k_j} \right) \left\{ \frac{2\alpha' k_2 \cdot k_1}{x_2 - x_1} \frac{2\alpha' k_6 \cdot k_5}{x_6 - x_5} \cdot \left(\frac{2\alpha' k_3 \cdot k_1}{x_3 - x_1} + \frac{2\alpha' k_3 \cdot k_2}{x_3 - x_2} \right) \left(\frac{2\alpha' k_5 \cdot k_4}{x_5 - x_4} + \frac{2\alpha' k_6 \cdot k_4}{x_6 - x_4} \right) \right\}. \quad (55)$$

In Eq. (54) it is understood that $\{x_1 = 0, x_5 = 1\}$ while in Eq. (55) it is understood that $\{x_1 = 0, x_6 = 1\}$.

As we mentioned after Eq. (32), the MSS prescription for finding the remaining momentum factors consists in simply doing a permutation of indices *inside* the curly brackets of (54) and (55), according to the σ_N permutation that is being considered (where $N = 6$ and $N = 7$, respectively). For example, doing $2 \leftrightarrow 3$ in (54) and (55) we arrive, respectively, at

$$F^{(324)}(\alpha') = \int_0^1 dx_4 \int_0^{x_4} dx_3 \int_0^{x_3} dx_2 \left(\prod_{i>j \geq 1}^5 (x_i - x_j)^{2\alpha' k_i \cdot k_j} \right) \times \left\{ \frac{2\alpha' k_3 \cdot k_1}{x_3 - x_1} \cdot \frac{2\alpha' k_5 \cdot k_4}{x_5 - x_4} \left(\frac{2\alpha' k_2 \cdot k_1}{x_2 - x_1} + \frac{2\alpha' k_2 \cdot k_3}{x_2 - x_3} \right) \right\} \quad (56)$$

and

$$F^{(3245)}(\alpha') = \int_0^1 dx_5 \int_0^{x_5} dx_4 \int_0^{x_4} dx_3 \int_0^{x_3} dx_2 \left(\prod_{i>j \geq 1}^6 (x_i - x_j)^{2\alpha' k_i \cdot k_j} \right) \left\{ \frac{2\alpha' k_3 \cdot k_1}{x_3 - x_1} \frac{2\alpha' k_6 \cdot k_5}{x_6 - x_5} \cdot \left(\frac{2\alpha' k_2 \cdot k_1}{x_2 - x_1} + \frac{2\alpha' k_2 \cdot k_3}{x_2 - x_3} \right) \left(\frac{2\alpha' k_5 \cdot k_4}{x_5 - x_4} + \frac{2\alpha' k_6 \cdot k_4}{x_6 - x_4} \right) \right\}. \quad (57)$$

In Appendix G we specify the kinematical terms that allow us to write a linear system of equations which has a *unique* solution for the momentum factors and which is precisely given by the ones in Eqs. (54) and (55) (and for the remaining momentum factors, by means of a σ_N permutation of the corresponding indices). We also specify, in that appendix, the corresponding linear system of equations for the $F^{(\sigma_6)}(\alpha')$'s and for the $F^{(\sigma_7)}(\alpha')$'s.

4.2. For gauge bosons and massless fermions

In this section we briefly explain how to find the amplitudes involving fermions directly from the corresponding amplitude involving only gauge bosons. From the point of view of vertex operators, in the RNS formalism, this is a non-trivial thing to do since it is known that there arise difficulties when calculating amplitudes of n gauge bosons and $2m$ massless fermions in the case of $m \geq 3$ ⁴⁶ (where $n = 1, 2, \dots$) [32].

The shortcut to find the amplitudes involving (any number of even) fermions is that we do not need to compute them from the beginning, by considering correlation functions of vertex operators: the closed formula for the N -point amplitude of gauge bosons and $D = 10$ supersymmetry

⁴⁶ The difficulties have to do with the fact that the total super ghost charge of the involved vertex operators must be -2 and fermion vertex operators are generally known in the $+1/2$ and $-1/2$ picture (therefore, four fermions gives $4 \times (-1/2) = -2$ and this case is fine). We thank N. Berkovits and E. Hatefi for clarifying this issue to us. In spite of this difficulty in Ref. [49] the *pure* fermion 6-point amplitude was obtained using the RNS formalism.

are enough ingredients to find the amplitudes involving fermions. We already saw this in Section 2.3 of Ref. [50], in the case of the open superstring 5-point amplitude, where the ansatz that we proposed for the fermion amplitudes consisted in simply changing the original gauge boson subamplitudes of the 2-dimensional basis ($A_{YM}(1, 2, 3, 4, 5)$ and $A_{F^4}(1, 2, 3, 4, 5)$, in the case of Ref. [50]) by the corresponding expression of their superpartner subamplitudes.

In the N -point case we do exactly the same thing: we change the gauge boson subamplitudes of the basis (in this case given by the Yang–Mills subamplitudes in (5)) by their superpartner subamplitudes. The resulting formula is precisely the one obtained by Mafra, Schlotterer and Stieberger [21] (see Eq. (1)).

Besides the expression in (1) there is no other possibility for the scattering amplitudes involving fermions since, as analyzed in [46], the global supersymmetry requirement is sufficient to determine the structure of the boson/fermion interactions *uniquely*.⁴⁷ In formula (1) this means that the summed variation of all possible boson/fermion subamplitudes, under the supersymmetry transformations,

$$\begin{aligned}\delta\zeta_j^\mu &= \frac{i}{2}(\bar{\epsilon}\gamma^\mu u_j), & \delta u_j &= \frac{i}{2}(\gamma_{\mu\nu}\epsilon)\zeta_j^\mu k_j^\nu, \\ \delta\bar{u}_j &= \frac{i}{2}(\bar{\epsilon}\gamma_{\mu\nu})\zeta_j^\mu k_j^\nu \quad (j = 1, \dots, N)\end{aligned}\quad (58)$$

is zero, after using the on-shell and the physical state conditions, together with momentum conservation.⁴⁸

5. Finding the α' expansion of the momentum factors

It was mentioned in Ref. [35] that the factorization properties of open superstring subamplitudes could be used as a complementary approach to determine the α' expansion of the momentum factors $F^{\{\sigma_N\}}(\alpha')$.⁴⁹ Also, in Ref. [15] the cyclic property of these subamplitudes was used as another complementary approach for that purpose (in the case of $N = 5$, $N = 6$ and $N = 7$).

We agree completely with the observation in [35] and, indeed, together with cyclicity, we will use these sort of factorization properties in this section as part of the tools to find the α' expansion of the momentum factors (see Eqs. (62) and (67)). Our remark at this point is that there is a more general form for the factorization (or tree-level unitarity) relations than the ones considered in Ref. [35], which includes as a particular case the collinear limit considered there (see Eq. (67)). This form, together with the cyclic property of the subamplitude, allows us to obtain all the α' terms of the momentum factors, up to quite high α' order, by just using the very well known 4-point (Veneziano-type) momentum factor.

This is a very non-trivial result that will allow us to bypass α' expansions of worldsheet integrals for $N = 5$, $N = 6$ and $N = 7$, at least up to α'^6 order in the first two cases and α'^4 order in the last one,⁵⁰ in the same spirit that the revisited S-matrix method succeeds in finding

⁴⁷ This analysis can be found, for example, in Sections 5.3.1 and 7.4.1 of Ref. [46]. We thank N. Berkovits for suggesting part of the specific sections of this book to us.

⁴⁸ The formulas in (58) are nothing else than the momentum space version of the supersymmetric transformations of the fields A_μ^a and ψ^a of $D = 10$ super Yang–Mills theory.

⁴⁹ See, for example, Appendix C of this reference.

⁵⁰ Our limitation to go to higher orders in α' , by just considering the expansion of the 4-point gamma factor, is just a computational one since the expressions become extremely huge as the α' order grows. As we explained in

the corresponding α' terms of the OSLEEL which would commonly be determined by open superstring N -point amplitudes (where $N > 4$) [7].

5.1. Tree level unitarity of the amplitudes

In the case of gauge bosons, the tree level unitarity relations state that the N -point subamplitude obeys⁵¹ [47]

$$\mathcal{A}(\zeta_1, k_1; \dots; \zeta_N, k_N) \sim i \int \frac{d^D k}{(2\pi)^D} \frac{\mathcal{A}^\mu(\zeta_1, k_1; \dots; \zeta_{m-1}, k_{m-1}; k) \mathcal{A}_\mu(-k; \zeta_m, k_m; \dots; \zeta_N, k_N)}{-k^2} \quad (59)$$

where $\mathcal{A}^\mu$ and $\mathcal{A}_\mu$ are, respectively, m and $(N + 2 - m)$ -point subamplitudes and where the pole that $\mathcal{A}(\zeta_1, k_1; \dots; \zeta_N, k_N)$ has in the Mandelstam variable $(k_1 + k_2 + \dots + k_{m-1})^2$ is being taken to zero:

$$(k_1 + k_2 + \dots + k_{m-1})^2 \rightarrow 0. \quad (60)$$

All $\mathcal{A}$ subamplitudes in (59) include the ‘ $i\delta$ ’ factor that takes into account momentum conservation (see Eq. (2)) and D ($= 10$) is the spacetime dimension.

Eq. (59) is applicable for $N \geq 4$ and $m = 3, \dots, N - 1$, so there are $(N - 3)$ unitarity relations for Mandelstam variables like $(k_1 + k_2 + \dots + k_{m-1})^2$.

The remaining unitarity relations arise when the other Mandelstam variables,

$$(k_2 + k_3 + \dots + k_{m-1})^2, (k_3 + k_4 + \dots + k_{m-1})^2, \dots, (k_{m-2} + k_{m-1})^2, \quad (61)$$

are individually taken to zero. There are $N(N - 3)/2$ independent Mandelstam variables and that is the total number of independent unitarity relations.

Substituting the explicit dependence of each $\mathcal{A}$ subamplitude in the ‘ $i\delta$ ’ factor in (59) and calculating the D -dimensional integral leads to

$$A(\zeta_1, k_1; \dots; \zeta_N, k_N) \sim \frac{A^\mu(\zeta_1, k_1; \dots; \zeta_{m-1}, k_{m-1}; k) A_\mu(-k; \zeta_m, k_m; \dots; \zeta_N, k_N)}{(k_1 + k_2 + \dots + k_{m-1})^2}, \quad (62)$$

where, now, the A , A^μ and A_μ subamplitudes no longer include the δ dependence and the momentum conservation on each of them must be implicitly assumed. These subamplitudes are the ones that we have been dealing with throughout this work.

A^μ and A_μ are subamplitudes in which the corresponding polarization vector has been taken away:

$$A^\mu(\zeta_1, k_1; \dots; \zeta_{m-1}, k_{m-1}; k) = \frac{\partial}{\partial \zeta_\mu} A(\zeta_1, k_1; \dots; \zeta_{m-1}, k_{m-1}; \zeta, k), \quad (63)$$

$$A_\mu(-k; \zeta_m, k_m; \dots; \zeta_N, k_N) = \frac{\partial}{\partial \zeta^\mu} A(\zeta, -k; \zeta_m, k_m; \dots; \zeta_N, k_N), \quad (64)$$

Subsection 2.2, we believe that using the 4-point amplitude α' expansion, we are able to obtain completely the α' expansion of any N -point momentum factor up to α'^7 order.

⁵¹ In Eq. (59) and the remaining ones that contain expressions involving the subamplitudes, in this subsection, we will not use the index ‘ b ’ on them because their dependence in the polarization vectors ζ_i has been made explicit and, therefore, it is clear that they are gauge boson subamplitudes. As a general rule, we will only use the ‘ b ’ index in the subamplitude variable A when it has not been made explicit its dependence in the polarizations.

where

$$k^\mu = -(k_1^\mu + \dots + k_{m-1}^\mu) = k_m^\mu + \dots + k_N^\mu \quad (65)$$

and where the asymptotic mass-shell condition (60) is being taken into account.

Formula (62) states that the residue that the N -point subamplitude has in $(k_1 + k_2 + \dots + k_{m-1})^2 = 0$ is given by the product of two lower-point subamplitudes (an m -point and a $(N + 2 - m)$ -point one). It is valid at any α' order. In particular, if $\alpha' \rightarrow 0$ it means that Yang–Mills subamplitudes also satisfy it.

When Eq. (62) is considered for the particular case of $m = 3$, it becomes

$$A(\zeta_1, k_1; \dots; \zeta_N, k_N) \sim \frac{A^\mu(\zeta_1, k_1; \zeta_2, k_2; k) A_\mu(-k; \zeta_3, k_3; \dots; \zeta_N, k_N)}{(k_1 + k_2)^2}, \quad (66)$$

or equivalently

$$A(\zeta_1, k_1; \dots; \zeta_N, k_N) \sim \frac{1}{2(k_1 \cdot k_2)} V_{(12)}^\mu \frac{\partial}{\partial \zeta^\mu} A(\zeta, k_1 + k_2; \zeta_3, k_3; \dots; \zeta_N, k_N), \quad (67)$$

where

$$V_{(12)}^\mu = -g[(\zeta_1 \cdot \zeta_2)(k_1 - k_2)^\mu - 2(\zeta_2 \cdot k_1)\zeta_1^\mu + 2(\zeta_1 \cdot k_2)\zeta_2^\mu] \quad (68)$$

is the (contracted) Yang–Mills vertex.

For $m = 3$, condition (60) implies that $(k_1 + k_2)^2 = 2k_1 \cdot k_2$ goes to zero:

$$k_1 \cdot k_2 \rightarrow 0. \quad (69)$$

If k_1 and k_2 are non-zero light-like vectors, then condition (69) implies that these momenta are parallel ($k_1 \parallel k_2$) and, therefore, the unitarity relation (67) is nothing else than the collinear version of the factorization property of gauge boson subamplitudes [51].

In the case of arbitrary m ($= 3, \dots, N - 1$), if $\{k_1, \dots, k_{m-1}\}$ are non-zero physical (Minkowski) momenta, then condition (60) implies that *all* of them are parallel ($k_1 \parallel k_2 \parallel \dots \parallel k_{m-1}$).⁵² Subsequently, all Mandelstam variables should tend to zero (because if the momenta tend to be parallel then $k_i \cdot k_j \rightarrow 0$). This is what happens for the Mandelstam variables in the physical region.

So, the word of caution when considering Eq. (62) subject to condition (60), is that we are considering there the subamplitude expressions where the Mandelstam variables have been analytically continued in the complex plane. Then, the Mandelstam variables are generally out of the physical region and, therefore, condition (60) does not have any implication for any of the other Mandelstam variables: they remain being independent in spite of one of them (namely, $(k_1 + k_2 + \dots + k_{m-1})^2$) tending to zero. This fact will be implicit in the calculations that we will do in Subsections 5.3, 5.4 and 5.5. In these subsections we will use formula (67) and its generalization, Eq. (62), to find α' terms of the N -point amplitude for $N = 5, 6, 7$, by just using the well known open superstring 4-point α' expansion (see Eqs. (F.2) and (F.3)).

⁵² Due to global momentum conservation, it turns out that *all* k_i become parallel, for $i = 1, \dots, N$, not only the first m momenta.

5.2. Analyticity of the momentum factors

Before going to the details of the α' calculations in the N -point amplitudes (where $N = 5, 6, 7$) an important remark proceeds.

From the analysis that we did in Section 4 we have that the N -point subamplitude of gauge bosons in open superstring theory is given by

$$A_b(1, \dots, N) = \sum_{\sigma_N \in S_{N-3}} F^{\{\sigma_N\}}(\alpha') A_{YM}(1, \{2_\sigma, 3_\sigma, \dots, (N-2)_\sigma\}, N-1, N), \quad (70)$$

where the $F^{\{\sigma_N\}}(\alpha')$'s are given by the integral expression (32) (or equivalently, if $N \geq 5$, by Eq. (33)).

In any of the two expressions of the momentum factors we see that the α' dependence of them is completely enclosed in the dimensionless Mandelstam variables, $\alpha' s_i$ and $\alpha' t_j$. So, on one side, if these variables are analytically continued in the complex plane, then the $F^{\{\sigma_N\}}(\alpha')$'s will have a well defined Laurent expansion with respect to $\alpha' s_i = \alpha' t_j = 0$.

On the other side, it is well known that the low energy limit of the gauge boson (tree level) subamplitudes in open superstring theory gives the corresponding Yang–Mills subamplitudes:

$$\lim_{\alpha' \rightarrow 0} A_b(1, \dots, N) = A_{YM}(1, \dots, N). \quad (71)$$

In Eq. (70) this last condition implies that

$$\lim_{\alpha' \rightarrow 0} F^{\{\sigma_N\}}(\alpha') = \begin{cases} 1 & \text{if } \sigma_N = \{2, 3, \dots, N-2\} \\ 0 & \text{if } \sigma_N \neq \{2, 3, \dots, N-2\} \end{cases} \quad (72)$$

where $\{2, 3, \dots, N-2\}$ is the identity permutation of those indices.

The result in (72), together with the behavior of the $F^{\{\sigma_N\}}(\alpha')$'s under an analytical continuation of the dimensionless Mandelstam variables, mentioned before, implies that the momentum factors are analytic functions at the origin of the complex plane ($\alpha' s_i = \alpha' t_j = 0$). This implies that the $F^{\{\sigma_N\}}(\alpha')$'s have a well defined α' expansion as a (Maclaurin) power series.

This is a quite non-trivial result since it is well known, by explicit computations, that the α' expansion of many of the individual multiple integrals which arise in the expanded version of $A_b(1, \dots, N)$ (see Eq. (31)) do indeed have some negative powers of α' .⁵³ In fact, the explicit expression for the $F^{\{\sigma_N\}}(\alpha')$'s, either in (32) or in (33), is given by a $(N-3)$ multiple integral multiplied by α'^{N-3} , where $(N-3)$ is a positive integer number. So, at least the first of those multiple integrals (the one corresponding to $\sigma_N = \{2, 3, \dots, N-2\}$) does have an α' expansion which begins as $\alpha'^{-(N-3)}$.

The analytic behavior of the $F^{\{\sigma_N\}}(\alpha')$'s, when $\alpha' \rightarrow 0$, will be an important fact which is behind the α' expansions which we will obtain in the next three subsections.

5.3. Case of the 5-point momentum factors

In the $N = 5$ case, we will give the details of how using tree level unitarity of the subamplitudes we can arrive at the α'^2 order terms of $F^{\{23\}}(\alpha')$ and $F^{\{32\}}(\alpha')$. The higher order terms of these two momentum factors will be afterwards written as a consequence of repeating the procedure for the corresponding α' order.

⁵³ See, for example, Appendix A.3 of [23] for the case of $N = 5$, and Refs. [19,24] for the case of $N = 6$.

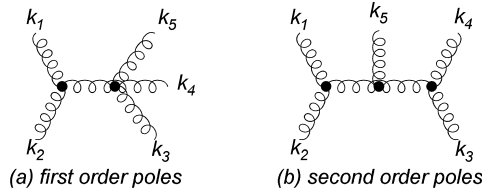


Fig. 1. Two type of Feynman diagrams contribute to the YM 5-point amplitude. Permutations of the legs of these diagrams should also be considered in order to account for the full amplitude.

Table 3
Structures of poles contained in each YM 5-point amplitude.

	First order	Second order
$A_{YM}(1, 2, 3, 4, 5)$	$\alpha_{12}, \alpha_{23},$ $\alpha_{34}, \alpha_{45}, \alpha_{15}$	$\alpha_{12}\alpha_{34}, \alpha_{23}\alpha_{45},$ $\alpha_{34}\alpha_{15}, \alpha_{45}\alpha_{12}, \alpha_{15}\alpha_{23}$
$A_{YM}(1, 3, 2, 4, 5)$	$\alpha_{13}, \alpha_{23},$ $\alpha_{24}, \alpha_{45}, \alpha_{15}$	$\alpha_{13}\alpha_{24}, \alpha_{23}\alpha_{45},$ $\alpha_{24}\alpha_{15}, \alpha_{45}\alpha_{13}, \alpha_{15}\alpha_{23}$

It will turn out to be convenient to introduce here the notation

$$\alpha_{ij} = 2k_i \cdot k_j, \quad (73)$$

where k_i and k_j are, respectively, the i -th and the j -th external leg momentum (and $i < j$). This notation will also be useful in the case of $N = 6$ (Subsection 5.4) and the case of $N = 7$ (Subsection 5.5).

Now, in order to find the α' expansion of $F^{[23]}(\alpha')$ and $F^{[32]}(\alpha')$ within our approach, it will be convenient to write, once again, Eq. (44) in the present subsection:

$$A_b(1, 2, 3, 4, 5) = F^{[23]}(\alpha') A_{YM}(1, 2, 3, 4, 5) + F^{[32]}(\alpha') A_{YM}(1, 3, 2, 4, 5). \quad (74)$$

We begin by looking for the poles of all three subamplitudes involved in Eq. (74). In Fig. 1 we have drawn the two type of Feynman diagrams that contribute to the Yang–Mills 5-point amplitude. Examining all possible diagrams of this type we can infer that $A_{YM}(1, 2, 3, 4, 5)$ and $A_{YM}(1, 3, 2, 4, 5)$ have first and second order poles in the α_{ij} variables (one and two propagators, respectively) that we list in Table 3.

On the other hand, in the left side of Eq. (74), the 5-point superstring subamplitude, $A_b(1, 2, 3, 4, 5)$, presents only first order poles in the same variables that $A_{YM}(1, 2, 3, 4, 5)$ does, namely, $\{\alpha_{12}, \alpha_{23}, \alpha_{34}, \alpha_{45}, \alpha_{15}\}$.⁵⁴ We will refer to these poles as the *physical* poles.

For reasons that will become clear in the next lines, it will be convenient to write $F^{[23]}(\alpha')$ and $F^{[32]}(\alpha')$ in terms of the simple poles that $A_{YM}(1, 2, 3, 4, 5)$ and $A_{YM}(1, 3, 2, 4, 5)$ have, respectively:

$$F^{[23]}(\alpha') = F^{[23]}[\alpha_{12}, \alpha_{23}, \alpha_{34}, \alpha_{45}, \alpha_{15}; \alpha'], \quad (75)$$

$$F^{[32]}(\alpha') = F^{[32]}[\alpha_{13}, \alpha_{23}, \alpha_{24}, \alpha_{45}, \alpha_{15}; \alpha']. \quad (76)$$

⁵⁴ Since $A_b(1, 2, 3, 4, 5)$ agrees with $A_{YM}(1, 2, 3, 4, 5)$ when $\alpha' \rightarrow 0$, and $A_{YM}(1, 2, 3, 4, 5)$ does have second order poles, the statement that $A_b(1, 2, 3, 4, 5)$ has only first order poles is only valid perturbatively, for any *non-zero* order in α' .

Now, since the momentum factors have a well defined α' power series (as it was seen in the previous subsection), we may write that⁵⁵

$$F^{(23)}(\alpha') = 1 + (a_1\alpha_{12} + a_2\alpha_{23} + a_3\alpha_{34} + a_4\alpha_{45} + a_5\alpha_{15})\alpha' + (b_1\alpha_{12}\alpha_{23} + b_2\alpha_{12}\alpha_{34} + b_3\alpha_{12}\alpha_{45} + b_4\alpha_{12}\alpha_{15} + b_5\alpha_{23}\alpha_{34} + b_6\alpha_{23}\alpha_{45} + b_7\alpha_{23}\alpha_{15} + b_8\alpha_{34}\alpha_{45} + b_9\alpha_{34}\alpha_{15} + b_{10}\alpha_{45}\alpha_{15} + b_{11}\alpha_{12}^2 + b_{12}\alpha_{23}^2 + b_{13}\alpha_{34}^2 + b_{14}\alpha_{45}^2 + b_{15}\alpha_{15}^2)\alpha'^2 + \mathcal{O}(\alpha'^3), \quad (77)$$

$$F^{(32)}(\alpha') = (c_1\alpha_{13} + c_2\alpha_{23} + c_3\alpha_{24} + c_4\alpha_{45} + c_5\alpha_{15})\alpha' + (d_1\alpha_{13}\alpha_{23} + d_2\alpha_{13}\alpha_{24} + d_3\alpha_{13}\alpha_{45} + d_4\alpha_{13}\alpha_{15} + d_5\alpha_{23}\alpha_{24} + d_6\alpha_{23}\alpha_{45} + d_7\alpha_{23}\alpha_{15} + d_8\alpha_{24}\alpha_{45} + d_9\alpha_{24}\alpha_{15} + d_{10}\alpha_{45}\alpha_{15} + d_{11}\alpha_{13}^2 + d_{12}\alpha_{23}^2 + d_{13}\alpha_{24}^2 + d_{14}\alpha_{45}^2 + d_{15}\alpha_{15}^2)\alpha'^2 + \mathcal{O}(\alpha'^3). \quad (78)$$

We will find the value of all the a_i 's, b_j 's, c_k 's and d_l 's that participate in these two expansions. We will do this by demanding three requirements:

1. Absence of *unphysical* poles.
 2. Tree level unitarity.
 3. Cyclic invariance.
- (79)

All these requirements will be demanded at every *non-zero* α' order, in $A_b(1, 2, 3, 4, 5)$.

Step 1. Absence of unphysical poles in $A_b(1, 2, 3, 4, 5)$

According to Eq. (74), the only unphysical poles that could arise in $A_b(1, 2, 3, 4, 5)$ are the ones that come from $A_{YM}(1, 3, 2, 4, 5)$, namely, α_{13} and α_{24} . These poles appear always as *simple* poles in $A_{YM}(1, 3, 2, 4, 5)$, in spite of they participating, also, in the second order poles of this subamplitude (see Table 3).

The only possible way to cancel, in the right-hand side of Eq. (74), the independent simple poles that $A_{YM}(1, 3, 2, 4, 5)$ has at $\alpha_{13} = 0$ and $\alpha_{24} = 0$, is demanding that $F^{(32)}(\alpha')$ should be possible to be written with a common $\alpha_{13}\alpha_{24}$ factor:

$$F^{(32)}(\alpha') = \alpha_{13}\alpha_{24}\alpha'^2 f^{(32)}[\alpha_{13}, \alpha_{23}, \alpha_{24}, \alpha_{45}, \alpha_{15}; \alpha'], \quad (80)$$

where $f^{(32)}[\alpha_{13}, \alpha_{23}, \alpha_{24}, \alpha_{45}, \alpha_{15}; \alpha']$ is analytic in all its five dimensionless $\alpha'\alpha_{ij}$ variables.

Comparing Eqs. (80) and (78) we see that the only possibility is that

$$c_i = 0 \quad (\text{for } i = 1, \dots, 5) \quad \text{and} \quad d_j = 0 \quad (\text{for } j = 1, \dots, 15; \text{ except for } j = 2), \quad (81)$$

so, in (78) this automatically leads us to

$$F^{(32)}(\alpha') = d_2\alpha_{13}\alpha_{24}\alpha'^2 + \mathcal{O}(\alpha'^3). \quad (82)$$

Step 2. Unitarity relation for $A_b(1, 2, 3, 4, 5)$

In Appendix F.3.1 we prove that demanding unitarity of $A_b(1, 2, 3, 4, 5)$ with respect to its α_{12} pole implies that the 5-point momentum factor, $F^{(23)}(\alpha')$, is related to the 4-point momentum factor, $F^{(2)}(\alpha')$, by the following relation:

⁵⁵ In Eqs. (77) and (78) we are already using the fact the $F^{(\sigma s)}(\alpha')$'s satisfy low energy requirement in Eq. (72).

$$F^{\{23\}}[0, \alpha_{23}, \alpha_{34}, \alpha_{45}, \alpha_{15}; \alpha'] = F^{\{2\}}[\alpha_{45}, \alpha_{34}; \alpha'], \quad (83)$$

where we have introduced the notation

$$F^{\{2\}}(\alpha') = F^{\{2\}}[\alpha_{12}, \alpha_{23}; \alpha'], \quad (84)$$

$$F^{\{23\}}(\alpha') = F^{\{23\}}[\alpha_{12}, \alpha_{23}, \alpha_{34}, \alpha_{45}, \alpha_{15}; \alpha']. \quad (85)$$

Notice that the two Mandelstam variables in which $F^{\{2\}}$ is being evaluated in (83) are not the same two ones that are implicit in the notation (84).⁵⁶

Relation (83) is an extremely strong constraint since in the left-hand side $F^{\{23\}}$ is being evaluated at arguments which do not appear in the right-hand side of this relation (namely, α_{23} and α_{15}). Here, we will just look for the implications that relation (83) has in the determination of the coefficients in the α' expansion of $F^{\{23\}}(\alpha')$ only up to α'^2 order. In fact, after substituting (77) and (F.3) in (83), we can conclude that

$$\begin{aligned} a_i &= 0 \quad (i = 2, 3, 4, 5), & b_5 &= b_6 = b_7 = 0, & b_8 &= -\frac{\pi^2}{6} = -\zeta(2), \\ b_j &= 0 \quad (j = 9, \dots, 15) \end{aligned} \quad (86)$$

and, therefore, the α' expansion of $F^{\{23\}}(\alpha')$ begins as

$$\begin{aligned} F^{\{23\}}(\alpha') &= 1 + (a_1 \alpha_{12}) \alpha' + (b_1 \alpha_{12} \alpha_{23} + b_2 \alpha_{12} \alpha_{34} + b_3 \alpha_{12} \alpha_{45} + b_4 \alpha_{12} \alpha_{15} \\ &\quad - \zeta(2) \alpha_{34} \alpha_{45}) \alpha'^2 + \mathcal{O}(\alpha'^3). \end{aligned} \quad (87)$$

Step 3. Cyclic invariance of $A_b(1, 2, 3, 4, 5)$

In Appendix F.4.1 we prove that demanding cyclic invariance of $A_b(1, 2, 3, 4, 5)$ in (74) implies that

$$F^{\{23\}}(\alpha') = F_{cycl}^{\{23\}}(\alpha') + \frac{\alpha_{13} + \alpha_{23}}{\alpha_{35}} F_{cycl}^{\{32\}}(\alpha'), \quad (88)$$

$$F^{\{32\}}(\alpha') = \frac{\alpha_{13}}{\alpha_{35}} F_{cycl}^{\{32\}}(\alpha'), \quad (89)$$

where $F_{cycl}^{\{23\}}(\alpha')$ and $F_{cycl}^{\{32\}}(\alpha')$ denote doing $\{k_1 \rightarrow k_2, k_2 \rightarrow k_3, \dots, k_5 \rightarrow k_1\}$ in $F^{\{23\}}(\alpha')$ and $F^{\{32\}}(\alpha')$, respectively.

The $N = 5$ BCJ relations (which we have completely found in Appendix E.2) have played an important role in the intermediate steps to arrive to Eqs. (88) and (89) (see Appendix F.4.1 for more details).

Relations (88) and (89) are to be compared with the corresponding one found in Ref. [15] (namely, Eq. (6.12)) by demanding cyclic symmetry.

In light of the result in (87), after doing the calculations (88) implies that⁵⁷

$$a_1 = 0, \quad b_1 = b_3 = 0, \quad b_2 = -b_4 = d_2 = \zeta(2), \quad (90)$$

so (87) and (82) finally become, respectively,

⁵⁶ The notation introduced in Eq. (84) is not casual: α_{12} and α_{23} are the two Mandelstam variables in which the 4-point amplitudes in Eq. (34) have poles.

⁵⁷ At α'^2 order, relation (89) does not give any new information: it is simply a consistency condition.

$$F^{\{23\}}(\alpha') = 1 + (\alpha_{12}\alpha_{34} - \alpha_{12}\alpha_{15} - \alpha_{34}\alpha_{45})\zeta(2)\alpha'^2 + \mathcal{O}(\alpha'^3), \quad (91)$$

$$F^{\{32\}}(\alpha') = \zeta(2)\alpha_{13}\alpha_{24}\alpha'^2 + \mathcal{O}(\alpha'^3). \quad (92)$$

We have successfully executed steps 1, 2 and 3, together with the corresponding $N = 6$ calculations (see Subsection 5.4), up to α'^6 order, finding *all* the coefficients.⁵⁸ The explicit result up to α'^4 terms is the following:

$$\begin{aligned} F^{\{23\}}(\alpha') = & 1 + \alpha'^2 \zeta(2)(\alpha_{12}\alpha_{34} - \alpha_{34}\alpha_{45} - \alpha_{12}\alpha_{15}) \\ & + \alpha'^3 \zeta(3)(-\alpha_{12}^2\alpha_{34} - 2\alpha_{12}\alpha_{23}\alpha_{34} - \alpha_{12}\alpha_{34}^2 + \alpha_{34}^2\alpha_{45} + \alpha_{34}\alpha_{45}^2 \\ & + \alpha_{12}^2\alpha_{15} + \alpha_{12}\alpha_{15}^2) \\ & + \frac{\alpha'^4}{10} \zeta^2(2)(4\alpha_{12}^3\alpha_{34} + 5\alpha_{12}^2\alpha_{23}\alpha_{34} + 5\alpha_{12}\alpha_{23}^2\alpha_{34} + 4\alpha_{12}^2\alpha_{34}^2 + 5\alpha_{12}\alpha_{23}\alpha_{34}^2 \\ & + 4\alpha_{12}\alpha_{34}^3 + 7\alpha_{12}\alpha_{23}\alpha_{34}\alpha_{45} - 3\alpha_{12}\alpha_{34}^2\alpha_{45} - 4\alpha_{34}^3\alpha_{45} - \alpha_{34}^2\alpha_{45}^2 - 4\alpha_{34}\alpha_{45}^3 \\ & - 4\alpha_{12}^3\alpha_{15} - 3\alpha_{12}^2\alpha_{34}\alpha_{15} + 7\alpha_{12}\alpha_{23}\alpha_{34}\alpha_{15} + 3\alpha_{12}\alpha_{34}\alpha_{45}\alpha_{15} - \alpha_{12}^2\alpha_{15}^2 \\ & - 4\alpha_{12}\alpha_{15}^3) + \mathcal{O}(\alpha'^5), \end{aligned} \quad (93)$$

$$\begin{aligned} F^{\{32\}}(\alpha') = & \alpha'^2 \zeta(2)\alpha_{13}\alpha_{24} \\ & + \alpha'^3 \zeta(3)(\alpha_{13}^2\alpha_{24} + \alpha_{13}\alpha_{23}\alpha_{24} + \alpha_{13}\alpha_{24}^2 - 2\alpha_{13}\alpha_{24}\alpha_{45} - 2\alpha_{13}\alpha_{24}\alpha_{15}) \\ & + \frac{\alpha'^4}{10} \zeta^2(2)(4\alpha_{13}^3\alpha_{24} + 11\alpha_{13}^2\alpha_{23}\alpha_{24} + 14\alpha_{13}\alpha_{23}^2\alpha_{24} + 4\alpha_{13}^2\alpha_{24}^2 \\ & + 11\alpha_{13}\alpha_{23}\alpha_{24}^2 + 4\alpha_{13}\alpha_{24}^3 - 12\alpha_{13}^2\alpha_{24}\alpha_{45} - 12\alpha_{13}\alpha_{23}\alpha_{24}\alpha_{45} - 5\alpha_{13}\alpha_{24}^2\alpha_{45} \\ & + 12\alpha_{13}\alpha_{24}\alpha_{45}^2 - 5\alpha_{13}^2\alpha_{24}\alpha_{15} - 12\alpha_{13}\alpha_{23}\alpha_{24}\alpha_{15} - 12\alpha_{13}\alpha_{24}^2\alpha_{15} \\ & + 7\alpha_{13}\alpha_{24}\alpha_{45}\alpha_{15} + 12\alpha_{13}\alpha_{24}\alpha_{15}^2) + \mathcal{O}(\alpha'^5) \end{aligned} \quad (94)$$

The corresponding full expressions up to α'^6 terms are given in the text files that we have submitted, attached to this work, to the hep-th arXiv preprint basis.

We have confirmed that our results are in perfect agreement with the ones found previously in [18,28,52].

5.4. Case of the 6-point momentum factors

Besides the notation introduced in Eq. (73), here it will be convenient to introduce the notation

$$\beta_{ijk} = \alpha_{ij} + \alpha_{ik} + \alpha_{jk}, \quad (95)$$

where $i < j < k$.

The notation in Eqs. (73) and (95) will also be valid for the $N = 7$ case (Subsection 5.5).

So, we will repeat the procedure that we did in Subsection 5.3 for $N = 5$. For a general N the procedure consists in the following four stages:

⁵⁸ We have also computed α'^7 and higher order calculations, but some undetermined coefficients arise. This is not in conflict with the revisited S-matrix method since this method only guarantees that, at a given α'^p order calculation, the $(\zeta \cdot k)^{p+2}$ terms should be absent in the $(p+2)$ -point amplitude. For $p = 7$ this means to have demanded a 9-point calculation of the terms of the OSLEEL, which we have not done in this work.

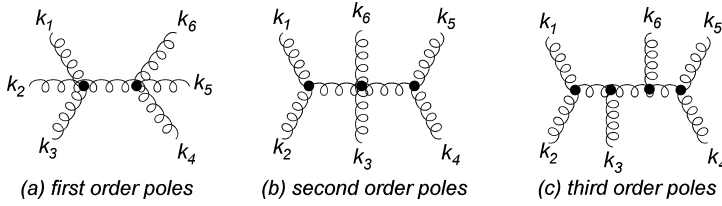


Fig. 2. Three type of Feynman diagrams contribute to the YM 6-point amplitude. Permutations of the legs of these diagrams should also be considered in order to account for the full amplitude.

Table 4

Structures of poles contained in YM 6-point amplitude.

	First order	Second order	Third order
$A_{YM}(1, 2, 3, 4, 5, 6)$	β_{123} β_{234} β_{345}	$\alpha_{12}\alpha_{45}, \alpha_{23}\alpha_{56}, \alpha_{34}\alpha_{16},$ $\alpha_{12}\beta_{345}, \alpha_{23}\beta_{123}, \alpha_{34}\beta_{234},$ $\alpha_{45}\beta_{345}, \alpha_{56}\beta_{123}, \alpha_{16}\beta_{234},$ $\alpha_{12}\beta_{123}, \alpha_{23}\beta_{234}, \alpha_{34}\beta_{345},$ $\alpha_{45}\beta_{123}, \alpha_{56}\beta_{234}, \alpha_{16}\beta_{345}$	$\alpha_{12}\beta_{123}\alpha_{45}, \alpha_{23}\beta_{234}\alpha_{56},$ $\alpha_{12}\beta_{345}\alpha_{45}, \alpha_{23}\beta_{123}\alpha_{56},$ $\alpha_{12}\beta_{345}\alpha_{34}, \alpha_{23}\beta_{123}\alpha_{45},$ $\alpha_{12}\alpha_{34}\alpha_{56}, \alpha_{23}\alpha_{45}\alpha_{16},$ $\alpha_{34}\beta_{345}\alpha_{16}, \alpha_{34}\beta_{234}\alpha_{16},$ $\alpha_{34}\beta_{234}\alpha_{56}$

- (i) Identify the poles of each of the $(N - 3)!$ Yang–Mills subamplitudes,

$$A_{YM}(1, \{2_\sigma, 3_\sigma, \dots, (N - 2)_\sigma\}, N - 1, N), \quad \text{where } \sigma_N \in S_{N-3}.$$

- (ii) Define each $F^{(\sigma_N)}(\alpha')$ momentum factor as a function of the $N(N - 3)/2$ Mandelstam variables which were found in stage i).
- (iii) Write the α' expansion of each $F^{(\sigma_N)}(\alpha')$ up to the desired α' order, in terms of unknown coefficients.
- (iv) Determine the coefficients of the previous α' expansions by following the three steps specified in Subsection 5.3, namely, demand absence of *unphysical* poles, tree level unitarity and cyclic invariance.

Let us do this procedure for the case of $N = 6$.

Stage (i)

In this case there are first, second and third order poles, as we can see in Fig. 2.

From the Feynman diagrams in Fig. 2, and considering cyclic permutations of the external momenta, we obtain the poles of $A_{YM}(1, 2, 3, 4, 5, 6)$, shown in Table 4, where we are using the notation of Eqs. (73) and (95).

The poles for the remaining five Yang–Mills subamplitudes can be obtained by permutations of the indexes $\{2, 3, 4\}$.

Stage (ii)

So, we will work with the six momentum factors as functions of nine independent Mandelstam variables, according to

$$F^{(234)}(\alpha') = F^{(234)}[\alpha_{12}, \alpha_{23}, \alpha_{34}, \alpha_{45}, \alpha_{56}, \alpha_{16}; \beta_{123}, \beta_{234}, \beta_{345}; \alpha'], \quad (96)$$

$$F^{(324)}(\alpha') = F^{(324)}[\alpha_{13}, \alpha_{23}, \alpha_{24}, \alpha_{45}, \alpha_{56}, \alpha_{16}; \beta_{123}, \beta_{234}, \beta_{245}; \alpha'], \quad (97)$$

$$F^{(243)}(\alpha') = F^{(243)}[\alpha_{12}, \alpha_{24}, \alpha_{34}, \alpha_{35}, \alpha_{56}, \alpha_{16}; \beta_{124}, \beta_{234}, \beta_{345}; \alpha'], \quad (98)$$

$$F^{(342)}(\alpha') = F^{(342)}[\alpha_{13}, \alpha_{34}, \alpha_{23}, \alpha_{25}, \alpha_{56}, \alpha_{16}; \beta_{134}, \beta_{234}, \beta_{245}; \alpha'], \quad (99)$$

$$F^{[423]}(\alpha') = F^{[423]}[\alpha_{14}, \alpha_{24}, \alpha_{23}, \alpha_{35}, \alpha_{56}, \alpha_{16}; \beta_{124}, \beta_{234}, \beta_{235}; \alpha'], \quad (100)$$

$$F^{[432]}(\alpha') = F^{[432]}[\alpha_{14}, \alpha_{34}, \alpha_{23}, \alpha_{25}, \alpha_{56}, \alpha_{16}; \beta_{134}, \beta_{234}, \beta_{235}; \alpha']. \quad (101)$$

These are the $N = 6$ generalization of formulas (75) and (76) (seen for the case of $N = 5$).

Stage (iii)

The general form of the α' power series for the momentum factors is the following:

$$F^{[234]}(\alpha') = 1 + \sum_{N=1}^{\infty} \alpha'^N \sum_{\substack{n_1, n_2, n_3, n_4, \\ n_5, n_6, n_7, n_8, n_9=0}}^N A_{n_1, n_2, \dots, n_9}^{(N)} \alpha_{12}^{n_1} \alpha_{23}^{n_2} \alpha_{34}^{n_3} \alpha_{45}^{n_4} \alpha_{56}^{n_5} \alpha_{16}^{n_6} \beta_{123}^{n_7} \beta_{234}^{n_8} \beta_{345}^{n_9}, \quad (102)$$

$$F^{[324]}(\alpha') = \sum_{N=1}^{\infty} \alpha'^N \sum_{\substack{n_1, n_2, n_3, n_4, \\ n_5, n_6, n_7, n_8, n_9=0}}^N B_{n_1, n_2, \dots, n_9}^{(N)} \alpha_{13}^{n_1} \alpha_{23}^{n_2} \alpha_{24}^{n_3} \alpha_{45}^{n_4} \alpha_{56}^{n_5} \alpha_{16}^{n_6} \beta_{123}^{n_7} \beta_{234}^{n_8} \beta_{245}^{n_9}, \quad (103)$$

$$F^{[243]}(\alpha') = \sum_{N=1}^{\infty} \alpha'^N \sum_{\substack{n_1, n_2, n_3, n_4, \\ n_5, n_6, n_7, n_8, n_9=0}}^N C_{n_1, n_2, \dots, n_9}^{(N)} \alpha_{12}^{n_1} \alpha_{24}^{n_2} \alpha_{34}^{n_3} \alpha_{35}^{n_4} \alpha_{56}^{n_5} \alpha_{16}^{n_6} \beta_{124}^{n_7} \beta_{234}^{n_8} \beta_{345}^{n_9}, \quad (104)$$

$$F^{[342]}(\alpha') = \sum_{N=1}^{\infty} \alpha'^N \sum_{\substack{n_1, n_2, n_3, n_4, \\ n_5, n_6, n_7, n_8, n_9=0}}^N D_{n_1, n_2, \dots, n_9}^{(N)} \alpha_{13}^{n_1} \alpha_{34}^{n_2} \alpha_{23}^{n_3} \alpha_{25}^{n_4} \alpha_{56}^{n_5} \alpha_{16}^{n_6} \beta_{134}^{n_7} \beta_{234}^{n_8} \beta_{245}^{n_9}, \quad (105)$$

$$F^{[423]}(\alpha') = \sum_{N=1}^{\infty} \alpha'^N \sum_{\substack{n_1, n_2, n_3, n_4, \\ n_5, n_6, n_7, n_8, n_9=0}}^N E_{n_1, n_2, \dots, n_9}^{(N)} \alpha_{14}^{n_1} \alpha_{24}^{n_2} \alpha_{23}^{n_3} \alpha_{35}^{n_4} \alpha_{56}^{n_5} \alpha_{16}^{n_6} \beta_{124}^{n_7} \beta_{234}^{n_8} \beta_{235}^{n_9}, \quad (106)$$

$$F^{[432]}(\alpha') = \sum_{N=1}^{\infty} \alpha'^N \sum_{\substack{n_1, n_2, n_3, n_4, \\ n_5, n_6, n_7, n_8, n_9=0}}^N F_{n_1, n_2, \dots, n_9}^{(N)} \alpha_{14}^{n_1} \alpha_{34}^{n_2} \alpha_{23}^{n_3} \alpha_{25}^{n_4} \alpha_{56}^{n_5} \alpha_{16}^{n_6} \beta_{134}^{n_7} \beta_{234}^{n_8} \beta_{235}^{n_9}. \quad (107)$$

In all expansions, from (102) to (107), the prime on the internal sum means that the set of indices $\{n_1, n_2, n_3, n_4, n_5, n_6, n_7, n_8, n_9\}$ obeys the constraint

$$n_1 + n_2 + n_3 + n_4 + n_5 + n_6 + n_7 + n_8 + n_9 = N. \quad (108)$$

In these expansions we have already used the condition of low energy behavior, Eq. (72), for the momentum factors: that is why the α' series of $F^{[234]}(\alpha')$ begins with '1' and the five others begin with $\mathcal{O}(\alpha'^1)$ order contributions.⁵⁹

⁵⁹ Once the unitarity and the cyclicity relations have been demanded it will naturally happen that the order '1' coefficients of all expansions will be zero, but as a matter of principles here we are just proposing their general α' expansions in such a way that they respect the low energy requirement in (72) and the analyticity of the momentum factors.

Stage (iv)

Here we apply the three steps that will allow us to obtain the coefficients of the α' expansions in Eqs. (102)–(107). We will do our calculations up to α'^6 order terms.

Step 1. Absence of unphysical poles in $A_b(1, 2, 3, 4, 5, 6)$

Each of the α' expansions initially has 9 coefficients at α'^1 order, 45 coefficients at α'^2 order, 165 coefficients at α'^3 order, 495 coefficients at α'^4 order, 1287 coefficients at α'^5 order and 3003 coefficients at α'^6 order.⁶⁰

Demanding absence of unphysical poles in $A_b(1, 2, 3, 4, 5, 6)$ implies that there should be cancellations between the unphysical simple poles which come in the five Yang–Mills subamplitudes which are different of $A_{YM}(1, 2, 3, 4, 5, 6)$ (which has only physical poles). After this procedure has been done, the number of independent coefficients reduces considerably according to Table 5.

There is also the subtlety that $F^{(234)}(\alpha')$ and the remaining five momentum factors should be such that they cancel the third order poles that come in $A_{YM}(1, 2, 3, 4, 5, 6)$. These poles are not present in $A_b(1, 2, 3, 4, 5, 6)$ (at non-zero α' order).

In the case of the α'^1 order terms, the requirement of absence of unphysical poles is strong enough to forbid them: the coefficients of those terms are all zero and that is the reason of why the first line in Table 5 contains only 0's (zeroes).

We will not write down here the partial result that we have obtained for the expansions in Eqs. (102)–(107), as we did at the end of Step 1 in Subsection 5.3. Instead, for each of the six $F^{(\sigma_6)}(\alpha')$'s we have presented two data at a given order in α' , in Table 5. The first data is the number of undetermined coefficients after the actual step (Step 1) and the second data is the number of undetermined coefficients after Step 2. We see there, that before demanding cyclic symmetry (in Step 3) there is still a big number of undetermined coefficients.

Step 2. Unitarity relations for $A_b(1, 2, 3, 4, 5, 6)$

Since in the $N = 6$ case there are two type of simple poles (α_{ij} and β_{ijk} , see Table 4), demanding unitarity of $A_b(1, 2, 3, 4, 5, 6)$ with respect to each of them will lead to independent unitarity relations. The remaining unitarity relations will be a consequence of the previous ones once cyclic symmetry has been taken into account (in Step 3).

In Appendix F.3.2 we prove that demanding unitarity of $A_b(1, 2, 3, 4, 5, 6)$ with respect to its α_{12} pole implies that the 6-point momentum factors, $F^{(234)}(\alpha')$ and $F^{(243)}(\alpha')$, are related to the 5-point momentum factors, $F^{(23)}(\alpha')$ and $F^{(32)}(\alpha')$, by means of the following two relations:

$$\begin{aligned} F^{(234)}[0, \alpha_{23}, \alpha_{34}, \alpha_{45}, \alpha_{56}, \alpha_{16}, \beta_{123}, \beta_{234}, \beta_{345}; \alpha'] \\ = F^{(23)}[\beta_{123}, \alpha_{34}, \alpha_{45}, \alpha_{56}, \beta_{345}; \alpha'], \end{aligned} \quad (109)$$

$$\begin{aligned} F^{(243)}[0, \alpha_{24}, \alpha_{34}, \alpha_{35}, \alpha_{56}, \alpha_{16}; \beta_{124}, \beta_{234}, \beta_{345}; \alpha'] \\ = F^{(32)}[\beta_{124}, \alpha_{34}, \alpha_{35}, \alpha_{56}, \beta_{345}; \alpha']. \end{aligned} \quad (110)$$

In analogy to relation (83), found after demanding unitarity for the 5-point amplitude, relations (109) and (110) are strong constraints for the coefficients of the α' expansions of $F^{(234)}$ and $F^{(243)}$ since there are four arguments (like α_{23} , α_{16} , β_{123} and β_{234} , in the first case) which are not present in the right-hand side of the corresponding relation.

⁶⁰ In this case, the general formula for the number of coefficients at α'^N order is $(N + 8)/(N! 8!)$.

Table 5

Comparing the initial number of undetermined coefficients with the number of coefficients which remain undetermined after Step 2.

Order	$F^{[234]}(\alpha')$		$F^{[324]}(\alpha')$		$F^{[243]}(\alpha')$		$F^{[342]}(\alpha')$		$F^{[423]}(\alpha')$		$F^{[432]}(\alpha')$	
α'^1	0	0	0	0	0	0	0	0	0	0	0	0
α'^2	44	1	3	0	3	0	3	1	3	0	3	0
α'^3	164	9	25	1	25	2	25	12	25	5	25	0
α'^4	494	45	177	9	177	17	177	70	177	40	177	3
α'^5	1286	167	405	47	405	287	405	405	405	405	405	405
α'^6	3002	1039	1155	615	1155	842	1155	1155	1155	1155	1155	1155

Also, demanding unitarity of $A_b(1, 2, 3, 4, 5, 6)$ with respect to its β_{123} pole implies that the 6-point momentum factors, $F^{[234]}(\alpha')$ and $F^{[324]}(\alpha')$, are related to the 4-point momentum factor, $F^{[2]}(\alpha')$, by means of the relation:

$$\begin{aligned}
 &F^{[234]}[\alpha_{12}, \alpha_{23}, \alpha_{34}, \alpha_{45}, \alpha_{56}, \alpha_{16}, 0, \beta_{234}, \beta_{345}; \alpha'] \\
 &\quad - \frac{\alpha_{12}}{\alpha_{12} + \alpha_{23}} F^{[324]}[-\alpha_{12} - \alpha_{23}, \alpha_{23}, \beta_{234} - \alpha_{23} - \alpha_{34}, \alpha_{45}, \alpha_{56}, \alpha_{16}, 0, \beta_{234}, \\
 &\quad -\alpha_{23} + \alpha_{45} + \alpha_{16} - \beta_{345}; \alpha'] = F^{[2]}[\alpha_{12}, \alpha_{23}; \alpha'] F^{[2]}[\alpha_{56}, \alpha_{45}; \alpha']. \quad (111)
 \end{aligned}$$

In this case there are even stronger constraints⁶¹ because, besides the fact that there are four arguments which are not present in the right-hand side of (111), in the left-hand side a miraculous cancellation of the $(\alpha_{12} + \alpha_{23})$ denominator should happen, since in the right-hand side there is only a product of two α' power series (which involve no denominators at all).

The curious non-zero arguments in which $F^{[324]}$ is being evaluated, in the left-hand side of (111), are simply the original ones (see Eq. (97)), but written in terms of the basis of Mandelstam variables used for $F^{[234]}(\alpha')$ (see Eq. (96)), with $\beta_{123} = 0$.

The three relations that we have written in Eqs. (109), (110) and (111), are conditions for only three of the six momentum factors ($F^{[234]}$, $F^{[243]}$ and $F^{[324]}$). For these momentum factors the number of its undetermined coefficients has been reduced as a consequence of Step 2. This is illustrated in Table 5, in the second data which we have presented for each $F^{[234]}(\alpha')$ at a given order in α' .⁶²

For the remaining momentum factors, the number of undetermined coefficients has not changed from Step 1 to Step 2.

After demanding cyclic invariance the coefficients of all six momentum factors will be related and it will be possible to find them all, at least up to α'^6 order.

Step 3. Cyclic invariance of $A_b(1, 2, 3, 4, 5, 6)$

In Appendix F.4.2 we prove that demanding cyclic invariance for $A_b(1, 2, 3, 4, 5, 6)$, in the closed formula that we have for it,

$$\begin{aligned}
 A_b(1, 2, 3, 4, 5, 6) &= F^{[234]}(\alpha') A_{YM}(1, 2, 3, 4, 5, 6) + F^{[324]}(\alpha') A_{YM}(1, 3, 2, 4, 5, 6) \\
 &\quad + F^{[243]}(\alpha') A_{YM}(1, 2, 4, 3, 5, 6) + F^{[342]}(\alpha') A_{YM}(1, 3, 4, 2, 5, 6) \\
 &\quad + F^{[423]}(\alpha') A_{YM}(1, 4, 2, 3, 5, 6) + F^{[432]}(\alpha') A_{YM}(1, 4, 3, 2, 5, 6), \quad (112)
 \end{aligned}$$

⁶¹ But, in contrast to (109) and (110), there is only one unitarity relation now.

⁶² This data corresponds to the number of coefficients which are still undetermined after Step 2, at a given α' order.

implies the following six relations for the momentum factors:

$$\begin{aligned}
 F^{\{234\}}(\alpha') &= F_{cycl}^{\{234\}}(\alpha') + \frac{\alpha_{56} - \beta_{123}}{\beta_{123} - \alpha_{45} - \alpha_{56}} F_{cycl}^{\{243\}}(\alpha') \\
 &+ \frac{\alpha_{12} - \beta_{123}}{\beta_{123} + \beta_{345} - \alpha_{12} - \alpha_{45}} F_{cycl}^{\{342\}}(\alpha') \\
 &+ \frac{F_{cycl}^{\{432\}}(\alpha')}{(\beta_{123} + \beta_{345} - \alpha_{12} - \alpha_{45})(-\beta_{345} + \alpha_{12} + \alpha_{34} - \alpha_{56})} \\
 &\times [\alpha_{12}(\beta_{123} + \beta_{345} - \alpha_{12} - \alpha_{34} + \alpha_{56} - \alpha_{45}) \\
 &+ \beta_{123}(\alpha_{34} + \alpha_{45} - \beta_{345} - \alpha_{56})] \\
 &+ \frac{F_{cycl}^{\{423\}}(\alpha')}{(\beta_{345} - \alpha_{12} - \alpha_{34} + \alpha_{56})(-\beta_{123} + \alpha_{45} + \alpha_{56})} \alpha_{45}(\alpha_{12} - \beta_{123}), \quad (113)
 \end{aligned}$$

$$\begin{aligned}
 F^{\{324\}}(\alpha') &= \frac{F_{cycl}^{\{342\}}(\alpha')}{\beta_{123} + \beta_{345} - \alpha_{12} - \alpha_{45}} (\alpha_{12} + \alpha_{23} - \beta_{123}) \\
 &+ \frac{F_{cycl}^{\{432\}}(\alpha')}{(\beta_{123} + \beta_{345} - \alpha_{12} - \alpha_{45})(-\beta_{345} + \alpha_{12} + \alpha_{34} - \alpha_{56})} \\
 &\times [\alpha_{12}(\beta_{123} + \beta_{345} - \alpha_{12} - \alpha_{23} - \alpha_{34} + \alpha_{56} - \alpha_{45}) \\
 &+ \beta_{123}(\alpha_{34} + \alpha_{45} - \beta_{345} - \alpha_{56}) + \alpha_{23}(\beta_{345} - \alpha_{34} + \alpha_{56} - \alpha_{45})] \\
 &+ \frac{F_{cycl}^{\{423\}}(\alpha')}{(\beta_{345} - \alpha_{12} - \alpha_{34} + \alpha_{56})(-\beta_{123} + \alpha_{45} + \alpha_{56})} \\
 &\times \alpha_{45}(\alpha_{12} + \alpha_{23} - \beta_{123}), \quad (114)
 \end{aligned}$$

$$\begin{aligned}
 F^{\{243\}}(\alpha') &= F_{cycl}^{\{324\}}(\alpha') + \frac{F_{cycl}^{\{243\}}(\alpha')}{\beta_{123} - \alpha_{45} - \alpha_{56}} (\alpha_{56} - \alpha_{34} - \beta_{123}) \\
 &\times \frac{F_{cycl}^{\{342\}}(\alpha')}{\beta_{123} + \beta_{345} - \alpha_{12} - \alpha_{45}} (\alpha_{12} - \alpha_{34} - \beta_{123}) \\
 &+ \frac{F_{cycl}^{\{432\}}(\alpha')}{(\beta_{123} + \beta_{345} - \alpha_{12} - \alpha_{45})(\beta_{345} - \alpha_{12} - \alpha_{34} + \alpha_{56})} \\
 &\times [\alpha_{34}(-\beta_{123} + \beta_{345} - \alpha_{34} - \alpha_{45} + \alpha_{56}) + (\beta_{123} - \alpha_{56})(\beta_{345} - \alpha_{45})] \\
 &+ \frac{F_{cycl}^{\{423\}}(\alpha')}{(-\beta_{345} + \alpha_{12} + \alpha_{34} - \alpha_{56})(-\beta_{123} + \alpha_{45} + \alpha_{56})} \\
 &\times [\alpha_{56}(\beta_{123} + \alpha_{12} + \alpha_{34} - \alpha_{45} - \alpha_{56}) + (\beta_{123} + \alpha_{34})(\alpha_{45} - \alpha_{12})], \quad (115)
 \end{aligned}$$

$$\begin{aligned}
 F^{\{342\}}(\alpha') &= \frac{F_{cycl}^{\{432\}}(\alpha')}{(\beta_{123} + \beta_{345} - \alpha_{12} - \alpha_{45})(-\beta_{345} + \alpha_{12} + \alpha_{34} - \alpha_{56})} \\
 &\times [\alpha_{12}(\beta_{123} - \beta_{234} + \beta_{345} - \alpha_{12} - \alpha_{45} + \alpha_{56}) \\
 &+ \beta_{123}(\beta_{234} - \beta_{345} - \alpha_{23} + \alpha_{45} - \alpha_{56}) + \alpha_{23}(\beta_{345} - \beta_{234} \\
 &+ \alpha_{23} - \alpha_{45} + \alpha_{56})]
 \end{aligned}$$

$$\begin{aligned}
& + \frac{F_{cycl}^{(423)}(\alpha')}{(\beta_{345} - \alpha_{12} - \alpha_{34} + \alpha_{56})(-\beta_{123} + \alpha_{45} + \alpha_{56})} \\
& \times [\alpha_{23}(\beta_{123} + \beta_{234} - \alpha_{12} - \alpha_{23} - \alpha_{34} + \alpha_{45}) \\
& + (\beta_{123} - \alpha_{12})(\alpha_{34} - \beta_{234} - \alpha_{45})], \\
F^{(423)}(\alpha') & = \frac{F_{cycl}^{(243)}(\alpha')}{\beta_{123} - \alpha_{45} - \alpha_{56}}(\alpha_{23} + \alpha_{56} - \beta_{123} - \beta_{234})
\end{aligned} \tag{116}$$

$$\begin{aligned}
& + \frac{F_{cycl}^{(432)}(\alpha')}{(\beta_{123} + \beta_{345} - \alpha_{12} - \alpha_{45})(\beta_{345} - \alpha_{12} - \alpha_{34} + \alpha_{56})} \\
& \times [\beta_{345}(\beta_{123} + \beta_{234} - \alpha_{23} - \alpha_{56}) + (\alpha_{34} + \alpha_{45})(\alpha_{23} + \alpha_{56} - \beta_{123} - \beta_{234})] \\
& + \frac{F_{cycl}^{(423)}(\alpha')}{(\beta_{345} - \alpha_{12} - \alpha_{34} + \alpha_{56})(-\beta_{123} + \alpha_{45} + \alpha_{56})} \\
& \times [\alpha_{56}(\alpha_{23} + \alpha_{45} - \beta_{123} - \beta_{234} - \alpha_{12} + \alpha_{56}) \\
& + (\alpha_{12} - \alpha_{45})(\beta_{123} + \beta_{234} - \alpha_{23})], \\
F^{(432)}(\alpha') & = \frac{F_{cycl}^{(432)}(\alpha')}{(\beta_{123} + \beta_{345} - \alpha_{12} - \alpha_{45})(\beta_{345} - \alpha_{12} - \alpha_{34} + \alpha_{56})}
\end{aligned} \tag{117}$$

$$\begin{aligned}
& \times [\alpha_{23}(\beta_{123} + \beta_{234} - \beta_{345} - \alpha_{23} + \alpha_{34} + \alpha_{45} - \alpha_{56}) \\
& + (\alpha_{56} - \beta_{234} - \beta_{123})(\alpha_{34} - \beta_{345} + \alpha_{45})] \\
& + \frac{F_{cycl}^{(423)}(\alpha')}{(\beta_{123} - \alpha_{45} - \alpha_{56})(\beta_{345} - \alpha_{12} - \alpha_{34} + \alpha_{56})} \\
& \times [\alpha_{23}(\alpha_{12} - \beta_{123} - \beta_{234} + \alpha_{23} - \alpha_{45}) \\
& + \alpha_{56}(\alpha_{12} + \beta_{123} + \beta_{234} - \alpha_{45} - \alpha_{56}) + (\beta_{123} + \beta_{234})(\alpha_{45} - \alpha_{12})], \\
F^{(432)}(\alpha') & = \frac{F_{cycl}^{(432)}(\alpha')}{(\beta_{123} + \beta_{345} - \alpha_{12} - \alpha_{45})(\beta_{345} - \alpha_{12} - \alpha_{34} + \alpha_{56})}
\end{aligned} \tag{118}$$

where each $F_{cycl}^{(\sigma_6)}(\alpha')$ denotes doing $\{k_1 \rightarrow k_2, k_2 \rightarrow k_3, \dots, k_6 \rightarrow k_1\}$ in the corresponding $F^{(\sigma_6)}(\alpha')$ momentum factor.

The $N = 6$ BCJ relations (which we have completely found in Appendix E.3) have played an important role in the intermediate steps to arrive to Eqs. (113)–(118) (see Appendix F.4.2 for more details).

So, summarizing, we have successfully executed Steps 1, 2 and 3, together with the corresponding $N = 5$ calculations (see Subsection 5.3), up to α'^6 order, finding *all* the coefficients. The explicit result up to α'^3 terms is the following:

$$\begin{aligned}
F^{(234)}(\alpha') & = 1 + \alpha'^2 \zeta(2)(-\beta_{123}\beta_{345} + \beta_{345}\alpha_{12} + \beta_{123}\alpha_{45} - \alpha_{45}\alpha_{56} - \alpha_{12}\alpha_{16}) \\
& + \alpha'^3 \zeta(3)(\beta_{123}^2\beta_{345} + \beta_{123}\beta_{345}^2 - \beta_{345}^2\alpha_{12} - \beta_{345}\alpha_{12}^2 - 2\beta_{345}\alpha_{12}\alpha_{23} \\
& - \beta_{123}^2\alpha_{45} - 2\beta_{234}\alpha_{12}\alpha_{45} + 2\alpha_{12}\alpha_{23}\alpha_{45} - 2\beta_{123}\alpha_{34}\alpha_{45} + 2\alpha_{12}\alpha_{34}\alpha_{45} \\
& - \beta_{123}\alpha_{45}^2 + \alpha_{45}^2\alpha_{56} + \alpha_{45}\alpha_{56}^2 + \alpha_{12}^2\alpha_{16} + \alpha_{12}\alpha_{16}^2) + \mathcal{O}(\alpha'^4), \\
F^{(324)}(\alpha') & = \alpha'^2 \zeta(2)(\beta_{245}\alpha_{13} - \alpha_{13}\alpha_{45})
\end{aligned} \tag{119}$$

$$\begin{aligned}
& + \alpha'^3 \zeta(3)\alpha_{13}[\beta_{245}^2 - 2\beta_{123}(\beta_{245} - \alpha_{45}) \\
& + \alpha_{45}(-\alpha_{13} - \alpha_{23} + 2\alpha_{24} + \alpha_{45} + 2\alpha_{16}) \\
& + \beta_{245}(\alpha_{13} + \alpha_{23} - 2(\alpha_{45} + \alpha_{16}))] + \mathcal{O}(\alpha'^4), \\
F^{(324)}(\alpha') & = \alpha'^2 \zeta(2)(\beta_{245}\alpha_{13} - \alpha_{13}\alpha_{45})
\end{aligned} \tag{120}$$

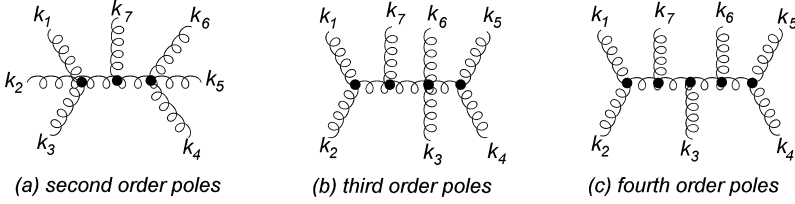


Fig. 3. Tree kind of Feynman diagrams used to calculate the YM subamplitude $A_{YM}(1, 2, 3, 4, 5, 6, 7)$.

$$\begin{aligned}
 F^{[243]}(\alpha') &= \alpha'^2 \zeta(2) (\beta_{124} \alpha_{35} - \alpha_{12} \alpha_{35}) \\
 &\quad + \alpha'^3 \zeta(3) \alpha_{35} [\beta_{124}^2 + \beta_{124} (-2\beta_{345} - 2\alpha_{12} + \alpha_{34} + \alpha_{35} - 2\alpha_{56}) \\
 &\quad + \alpha_{12} (2\beta_{345} + \alpha_{12} + 2\alpha_{24} - \alpha_{34} - \alpha_{35} + 2\alpha_{56})] + \mathcal{O}(\alpha'^4), \quad (121)
 \end{aligned}$$

$$\begin{aligned}
 F^{[342]}(\alpha') &= \alpha'^2 \zeta(2) \alpha_{13} \alpha_{25} \\
 &\quad - \alpha'^3 \zeta(3) \alpha_{13} \alpha_{25} (-2\beta_{134} - \beta_{234} + \alpha_{13} - \alpha_{25} + \alpha_{34} + 2\alpha_{56} + 2\alpha_{16}) \\
 &\quad + \mathcal{O}(\alpha'^4), \quad (122)
 \end{aligned}$$

$$\begin{aligned}
 F^{[423]}(\alpha') &= \alpha'^2 \zeta(2) \alpha_{14} \alpha_{35} \\
 &\quad + \alpha'^3 \zeta(3) \alpha_{14} \alpha_{35} (\beta_{234} + 2\beta_{235} + \alpha_{14} - \alpha_{23} - \alpha_{35} - 2\alpha_{56} - 2\alpha_{16}) \\
 &\quad + \mathcal{O}(\alpha'^4), \quad (123)
 \end{aligned}$$

$$\begin{aligned}
 F^{[432]}(\alpha') &= -\alpha'^2 \zeta(2) \alpha_{14} \alpha_{25} - \alpha'^3 \zeta(3) \alpha_{14} \alpha_{25} (\beta_{234} + \alpha_{14} + \alpha_{25} - 2\alpha_{56} - 2\alpha_{16}) \\
 &\quad + \mathcal{O}(\alpha'^4). \quad (124)
 \end{aligned}$$

The corresponding full expressions up to α'^6 terms are given in the text files that we have submitted, attached to this work, to the hep-th arXiv preprint basis.

We have confirmed that our results are in perfect agreement with the ones found previously in [35,52].

5.5. Case of the 7-point momentum factors

Doing the procedure for the $N = 7$ case is straight forward (although very tedious) once it has been done for $N = 5$ and $N = 6$. We will only mention here a few details.

First, with respect to the poles of $A_{YM}(1, 2, 3, 4, 5, 6, 7)$ (and of the other twenty three subamplitudes of the 7-point basis), it has second, third and fourth order ones. They come from the Feynman diagrams in Fig. 3. The poles can happen in any of the fourteen Mandelstam variables that we have specified in Eq. (D.3): seven α_{ij} 's and also seven β_{ijk} 's.

We have (shortly) presented the poles in Table 6: So, $F^{[2345]}(\alpha')$ will be naturally defined in terms of the $N = 7$ Mandelstam variables that we have defined in (D.3):

$$F^{[2345]}(\alpha') = F^{[2345]} \left[\begin{matrix} \alpha_{12}, & \alpha_{23}, & \alpha_{34}, & \alpha_{56}, & \alpha_{56}, & \alpha_{67}, & \alpha_{17} \\ \beta_{123}, & \beta_{234}, & \beta_{345}, & \beta_{456}, & \beta_{567}, & \beta_{167}, & \beta_{127} \end{matrix}; \alpha' \right]. \quad (125)$$

The Mandelstam variables for the remaining twenty three $F^{[\sigma_7]}(\alpha')$ momentum factors are obtained from the ones that appear in Eq. (125), by considering in them exactly the σ_7 permutation of indices $\{2, 3, 4, 5\}$.

Table 6

Structures of poles contained in YM 7-point amplitude.

	Second order (28 terms)	Third order (84 terms)	Fourth order (42 terms)
$A_{YM}(1, 2, 3, 4, 5, 6, 7)$	$\beta_{167}\beta_{234},$ $\beta_{345}\alpha_{12},$ $\vdots$ $\beta_{567}\alpha_{34}$	$\beta_{127}\beta_{345}\alpha_{12},$ $\beta_{167}\beta_{234}\alpha_{17},$ $\vdots$ $\beta_{234}\beta_{567}\alpha_{23}$	$\beta_{167}\beta_{234}\alpha_{17}\alpha_{23},$ $\beta_{167}\beta_{234}\alpha_{17}\alpha_{34},$ $\vdots$ $\beta_{167}\beta_{345}\alpha_{17}\alpha_{34}$

A computational detail that could be of interest to the reader is that finding the coefficients of the α' expansions, along the same lines presented in Subsection 5.4 (in four stages, in particular, demanding cyclic symmetry after unitarity has been demanded in one physical pole), becomes a heavy task if compared with solving the coefficients from purely demanding unitarity in all poles. These two methods happen to agree at low orders in α' (but not necessarily at high orders⁶³). The advantage of the second method lies in the fact that each of its equations contain a few momentum factors (containing, therefore, a low number of unknowns) while in the first method each of the equations contains many more momentum factors.

In Appendix F.3.3 we have written the momentum factor relations that arise when unitarity of the amplitudes is demanded with respect to their α_{12} and their β_{123} poles. Our complete result is that, when any $\alpha_{ij} \rightarrow 0$ there arise six relations and when any $\beta_{ijk} \rightarrow 0$ there arise only two relations. The first type of relations involves 7-point with 6-point momentum factors while the second type involves 7-point with 4-point and 5-point momentum factors (see Appendix F.3.3 for the detailed relations).

We have found the α' expansions up to α'^4 order.⁶⁴ In the following we have listed these expansions only up to α'^3 order.⁶⁵

$$\begin{aligned}
 F^{\{2345\}}(\alpha') = & 1 - \alpha'^2 \zeta(2) (\beta_{123}(\beta_{127} - \beta_{456}) + \beta_{456}\beta_{567} - \beta_{127}\alpha_{12} + \alpha_{12}\alpha_{17} \\
 & - \beta_{567}\alpha_{56} + \alpha_{56}\alpha_{67}) \\
 & + \alpha'^3 \zeta(3) [\beta_{123}^2(\beta_{127} - \beta_{456}) + \beta_{456}^2\beta_{567} - \beta_{127}^2\alpha_{12} - \beta_{127}\alpha_{12}^2 \\
 & + \alpha_{12}^2\alpha_{17} + \alpha_{12}\alpha_{17}^2 - 2\beta_{127}\alpha_{12}\alpha_{23} \\
 & + \beta_{456}(\beta_{567}^2 + 2\alpha_{12}(-\beta_{234} + \alpha_{23} + \alpha_{34})) - \beta_{567}^2\alpha_{56} - 2\beta_{167}\alpha_{12}\alpha_{56} \\
 & + 2\beta_{234}\alpha_{12}\alpha_{56} + 2\beta_{345}\alpha_{12}\alpha_{56} - 2\alpha_{12}\alpha_{34}\alpha_{56} - 2\beta_{567}\alpha_{45}\alpha_{56} - \beta_{567}\alpha_{56}^2 \\
 & + \beta_{123}(\beta_{127}^2 - \beta_{456}^2 - 2\beta_{456}\alpha_{34} + 2(-\beta_{345} + \alpha_{34} + \alpha_{45})\alpha_{56}) + \alpha_{56}^2\alpha_{67} \\
 & + \alpha_{56}\alpha_{67}^2] + \mathcal{O}(\alpha'^4), \tag{126}
 \end{aligned}$$

⁶³ We have verified this computationally.

⁶⁴ As mentioned before, the reason for not going to higher orders is a purely computational one: we have had memory difficulties to deal with $N = 7$ α'^5 calculations on the computers. In spite of this complication, as we mentioned in Subsection 2.2, we believe that only using the 4-point amplitude information our method is, in principle, able to achieve α'^7 order calculations, for any number of legs.

⁶⁵ There are specific cases of momentum factors which, as remarked in [35], have an α' expansion which begins at α'^4 order. For these cases we have written the α'^4 terms.

$$\begin{aligned}
F^{\{2354\}}(\alpha') &= \alpha'^2 \zeta(2)(-\beta_{123} + \beta_{467})\alpha_{46} \\
&\quad - \alpha'^3 \zeta(3)\alpha_{46}[-\beta_{123}^2 + 2\beta_{456}\beta_{467} - \beta_{467}^2 - 2\beta_{235}\alpha_{12} + 2\alpha_{12}\alpha_{23} \\
&\quad + 2\alpha_{12}\alpha_{35} - \beta_{467}\alpha_{45} - \beta_{467}\alpha_{46} + \beta_{123}(-2\beta_{456} + 2\beta_{467} - 2\alpha_{35} + \alpha_{45} \\
&\quad + \alpha_{46} - 2\alpha_{67}) + 2\beta_{467}\alpha_{67}] + \mathcal{O}(\alpha'^4), \tag{127}
\end{aligned}$$

$$\begin{aligned}
F^{\{2435\}}(\alpha') &= \alpha'^2 \zeta(2)(\beta_{124} - \alpha_{12})(\beta_{356} - \alpha_{56}) \\
&\quad + \alpha'^3 \zeta(3)[\beta_{124}^2(\beta_{356} - \alpha_{56}) \\
&\quad + \beta_{124}(\beta_{356}^2 + \beta_{356}(-2\beta_{567} - 2\alpha_{12} + \alpha_{34} - 2\alpha_{56}) \\
&\quad - 2\beta_{127}(\beta_{356} - \alpha_{56}) + \alpha_{56}(2\beta_{567} + 2\alpha_{12} - \alpha_{34} + 2\alpha_{35} + \alpha_{56})) \\
&\quad + \alpha_{12}(-\beta_{356}^2 + 2\beta_{127}(\beta_{356} - \alpha_{56}) \\
&\quad - \alpha_{56}(2\beta_{567} + \alpha_{12} + 2\alpha_{24} - \alpha_{34} + 2\alpha_{35} + \alpha_{56})) \\
&\quad + \alpha_{12}\beta_{356}(2\beta_{567} + \alpha_{12} + 2\alpha_{24} - \alpha_{34} + 2\alpha_{56}) + \mathcal{O}(\alpha'^4), \tag{128}
\end{aligned}$$

$$\begin{aligned}
F^{\{2453\}}(\alpha') &= \alpha'^2 \zeta(2)(\beta_{346} - \alpha_{34} - \alpha_{46})(-\beta_{345} - \beta_{467} + \alpha_{35} + \alpha_{67}) \\
&\quad - \alpha'^3 \zeta(3)[2\alpha_{12}(\beta_{346} - \alpha_{34} - \alpha_{46})(-\beta_{167} + \beta_{235} - \beta_{467} + \alpha_{67}) \\
&\quad + (\beta_{346} - \alpha_{34} - \alpha_{46})(\beta_{345} + \beta_{467} - \alpha_{35} - \alpha_{67}) \\
&\quad \times (2\beta_{125} - 2\beta_{127} + \beta_{345} + \beta_{346} - \beta_{467} - 2\alpha_{34} - \alpha_{46} - \alpha_{67})] \\
&\quad + \mathcal{O}(\alpha'^4), \tag{129}
\end{aligned}$$

$$\begin{aligned}
F^{\{2534\}}(\alpha') &= -\alpha'^2 \zeta(2)(\beta_{127} - \beta_{345} - \beta_{356} + \alpha_{35})(\beta_{124} - \beta_{367} + \alpha_{45}) \\
&\quad + \alpha'^3 \zeta(3)(\beta_{127} - \beta_{345} - \beta_{356} + \alpha_{35})[\beta_{124}^2 + \beta_{345}\beta_{367} - \beta_{356}\beta_{367} + \beta_{367}^2 \\
&\quad + 2\beta_{245}\alpha_{12} - 2\alpha_{12}\alpha_{24} + 2\beta_{367}\alpha_{36} - \beta_{345}\alpha_{45} + \beta_{356}\alpha_{45} - 3\beta_{367}\alpha_{45} \\
&\quad - 2\alpha_{12}\alpha_{45} - 2\alpha_{36}\alpha_{45} + 2\alpha_{45}^2 + \beta_{127}(-\beta_{367} + \alpha_{45}) - 2\beta_{367}\alpha_{67} + 2\alpha_{45}\alpha_{67} \\
&\quad + \beta_{124}(\beta_{127} - \beta_{345} + \beta_{356} - 2\beta_{367} - 2\alpha_{36} + 3\alpha_{45} + 2\alpha_{67})] \\
&\quad + \mathcal{O}(\alpha'^4), \tag{130}
\end{aligned}$$

$$\begin{aligned}
F^{\{2543\}}(\alpha') &= -\alpha'^2 \zeta(2)(\beta_{125} - \alpha_{12})\alpha_{36} \\
&\quad - \alpha'^3 \zeta(3)\alpha_{36}[\beta_{125}^2 + \beta_{125}(-2\beta_{127} + \beta_{345} - 2\alpha_{12} + \alpha_{36} - 2\alpha_{67}) \\
&\quad + \alpha_{12}(2\beta_{127} - \beta_{345} + \alpha_{12} + 2\alpha_{25} - \alpha_{36} + 2\alpha_{67})] + \mathcal{O}(\alpha'^4), \tag{131}
\end{aligned}$$

$$\begin{aligned}
F^{\{3245\}}(\alpha') &= \alpha'^2 \zeta(2)(\beta_{137} - \beta_{456})\alpha_{13} \\
&\quad + \alpha'^3 \zeta(3)\alpha_{13}[\beta_{137}^2 - 2\beta_{123}(\beta_{137} - \beta_{456}) + \beta_{456}^2 - \beta_{456}\alpha_{13} + 2\beta_{456}\alpha_{17} \\
&\quad - \beta_{456}\alpha_{23} + \beta_{137}(-2\beta_{456} + \alpha_{13} - 2\alpha_{17} + \alpha_{23}) + 2\beta_{456}\alpha_{24} + 2\beta_{245}\alpha_{56} \\
&\quad - 2\alpha_{24}\alpha_{56} - 2\alpha_{45}\alpha_{56}] + \mathcal{O}(\alpha'^4), \tag{132}
\end{aligned}$$

$$F^{\{3254\}}(\alpha') = -2\alpha'^3 \zeta(3)\alpha_{13}\alpha_{25}\alpha_{46} + \mathcal{O}(\alpha'^4), \tag{133}$$

$$\begin{aligned}
F^{(3425)}(\alpha') &= \alpha'^2 \zeta(2)(-\beta_{137} - \beta_{234} + \alpha_{17} + \alpha_{24})(\beta_{134} - \alpha_{13} - \alpha_{34}) \\
&\quad - \alpha'^3 \zeta(3)(\beta_{134} - \alpha_{13} - \alpha_{34})[-\beta_{137}^2 + \beta_{234}^2 + 2\beta_{234}\beta_{256} - 2\beta_{234}\beta_{567} \\
&\quad - \beta_{234}\alpha_{13} - 2\beta_{234}\alpha_{17} - 2\beta_{256}\alpha_{17} + 2\beta_{567}\alpha_{17} + \alpha_{13}\alpha_{17} + \alpha_{17}^2 \\
&\quad + \beta_{134}(\beta_{137} + \beta_{234} - \alpha_{17} - \alpha_{24}) \\
&\quad - \beta_{234}\alpha_{24} - 2\beta_{256}\alpha_{24} + 2\beta_{567}\alpha_{24} + \alpha_{13}\alpha_{24} + \alpha_{17}\alpha_{24} - 2\beta_{234}\alpha_{34} \\
&\quad + 2\alpha_{17}\alpha_{34} + 2\alpha_{24}\alpha_{34} + \beta_{137}(2\beta_{256} - 2\beta_{567} - \alpha_{13} + \alpha_{24} - 2\alpha_{34} - 2\alpha_{56}) \\
&\quad - 2\beta_{167}\alpha_{56} + 2\beta_{245}\alpha_{56} + 2\alpha_{17}\alpha_{56}] + \mathcal{O}(\alpha'^4), \tag{134}
\end{aligned}$$

$$\begin{aligned}
F^{(3452)}(\alpha') &= \alpha'^2 \zeta(2)(\beta_{137} - \beta_{245} - \beta_{256} + \alpha_{25})(-\beta_{134} + \beta_{267} - \beta_{345} + \alpha_{34}) \\
&\quad + \alpha'^3 \zeta(3)(\beta_{137} - \beta_{245} - \beta_{256} + \alpha_{25})(\beta_{134} - \beta_{267} + \beta_{345} - \alpha_{34}) \\
&\quad \times (\beta_{134} + \beta_{137} - \beta_{245} + \beta_{256} - \beta_{267} + 2\beta_{345} - 2\alpha_{26} - 2\alpha_{34} + 2\alpha_{67}) \\
&\quad + \mathcal{O}(\alpha'^4), \tag{135}
\end{aligned}$$

$$\begin{aligned}
F^{(3524)}(\alpha') &= \alpha'^2 \zeta(2)\alpha_{14}\alpha_{36} \\
&\quad - \alpha'^3 \zeta(3)\alpha_{14}\alpha_{36}(-2\beta_{145} - \beta_{167} - 2\beta_{236} + \alpha_{14} + 2\alpha_{17} + \alpha_{23} + \alpha_{36} \\
&\quad + \alpha_{45} + 2\alpha_{67}) + \mathcal{O}(\alpha'^4), \tag{136}
\end{aligned}$$

$$\begin{aligned}
F^{(3542)}(\alpha') &= \alpha'^2 \zeta(2)(\beta_{137} + \beta_{167} - \beta_{245} - \alpha_{17})(\beta_{135} - \alpha_{13} - \alpha_{35}) \\
&\quad - \alpha'^3 \zeta(3)(\beta_{137} + \beta_{167} - \beta_{245} - \alpha_{17})(\beta_{135} - \alpha_{13} - \alpha_{35}) \\
&\quad \times (-\beta_{135} + \beta_{137} - \beta_{167} + \beta_{345} + \alpha_{13} + \alpha_{17} - \alpha_{24} - 2\alpha_{26} + \alpha_{35} \\
&\quad - \alpha_{45} + 2\alpha_{67}) + \mathcal{O}(\alpha'^4), \tag{137}
\end{aligned}$$

$$\begin{aligned}
F^{(4235)}(\alpha') &= \alpha'^2 \zeta(2)(\beta_{147} - \beta_{356} - \alpha_{23})(-\beta_{124} - \beta_{234} + \beta_{567} + \alpha_{24}) \\
&\quad + \alpha'^3 \zeta(3)(\beta_{124} + \beta_{234} - \beta_{567} - \alpha_{24})[-\beta_{147}^2 + \beta_{234}\beta_{356} - \beta_{356}^2 \\
&\quad - \beta_{356}\beta_{567} + 2\beta_{356}\alpha_{14} - 2\beta_{356}\alpha_{17} + \beta_{124}(\beta_{147} - \beta_{356} - \alpha_{23}) + \beta_{234}\alpha_{23} \\
&\quad - 3\beta_{356}\alpha_{23} - \beta_{567}\alpha_{23} + 2\alpha_{14}\alpha_{23} - 2\alpha_{17}\alpha_{23} - 2\alpha_{23}^2 - 2\beta_{235}\alpha_{56} + 2\alpha_{23}\alpha_{56} \\
&\quad + 2\alpha_{35}\alpha_{56} + \beta_{147}(-\beta_{234} + 2\beta_{356} + \beta_{567} - 2\alpha_{14} + 2\alpha_{17} + 3\alpha_{23})] \\
&\quad + \mathcal{O}(\alpha'^4), \tag{138}
\end{aligned}$$

$$\begin{aligned}
F^{(4253)}(\alpha') &= -\frac{1}{10}\alpha'^4 \zeta^2(2)(\beta_{157} - \beta_{234} - \beta_{346} + \alpha_{34})(-\beta_{167} + \beta_{245} - \beta_{367} + \alpha_{67}) \\
&\quad \times [7\alpha_{15}\alpha_{24} + 17\alpha_{24}(\beta_{167} - \beta_{234} - \beta_{245} + \alpha_{24}) - 3\alpha_{15}(\beta_{346} - \alpha_{34} - \alpha_{36}) \\
&\quad - 10(\beta_{167} - \beta_{234} - \beta_{245} + \alpha_{24})(\beta_{346} - \alpha_{34} - \alpha_{36})] + \mathcal{O}(\alpha'^5), \tag{139}
\end{aligned}$$

$$\begin{aligned}
F^{(4325)}(\alpha') &= -\alpha'^2 \zeta(2)\alpha_{14}(\beta_{256} - \alpha_{56}) \\
&\quad - \alpha'^3 \zeta(3)\alpha_{14}[\beta_{256}^2 + \beta_{234}(\beta_{256} - \alpha_{56}) \\
&\quad + \alpha_{56}(2\beta_{567} - \alpha_{14} + 2\alpha_{17} + 2\alpha_{25} + \alpha_{56}) \\
&\quad + \beta_{256}(-2\beta_{567} + \alpha_{14} - 2(\alpha_{17} + \alpha_{56}))] + \mathcal{O}(\alpha'^4), \tag{140}
\end{aligned}$$

$$\begin{aligned}
 F^{(4352)}(\alpha') &= \frac{1}{10} \alpha'^4 \zeta^2(2) (-\beta_{147} - \beta_{167} + \beta_{235} + \alpha_{17}) (-\beta_{167} + \beta_{245} - \beta_{367} + \alpha_{67}) \\
 &\quad \times [10\alpha_{24}(-\beta_{124} - \beta_{245} + \beta_{367} + \alpha_{24}) + 27\alpha_{24}\alpha_{35} \\
 &\quad + 3(-\beta_{124} - \beta_{245} + \beta_{367} + \alpha_{24})(\beta_{147} - \beta_{235} - \beta_{356} + \alpha_{35}) \\
 &\quad + 10\alpha_{35}(\beta_{147} - \beta_{235} - \beta_{356} + \alpha_{35})] + \mathcal{O}(\alpha'^5), \tag{141}
 \end{aligned}$$

$$\begin{aligned}
 F^{(4523)}(\alpha') &= \alpha'^2 \zeta(2) (\beta_{137} + \beta_{167} - \beta_{245} - \alpha_{17}) (\beta_{167} - \beta_{235} + \beta_{467} - \alpha_{67}) \\
 &\quad - \alpha'^3 \zeta(3) (\beta_{137} + \beta_{167} - \beta_{245} - \alpha_{17}) (\beta_{167} - \beta_{235} + \beta_{467} - \alpha_{67}) \\
 &\quad \times [-2\beta_{135} + \beta_{137} + \beta_{167} - 2\beta_{235} - 2\beta_{245} - 2\beta_{246} + \beta_{467} \\
 &\quad + 2\alpha_{13} + \alpha_{17} + 3\alpha_{24} - \alpha_{25} + 3\alpha_{35} + 2\alpha_{46} + \alpha_{67}] + \mathcal{O}(\alpha'^4), \tag{142}
 \end{aligned}$$

$$\begin{aligned}
 F^{(4532)}(\alpha') &= -\alpha'^2 \zeta(2) (\beta_{236} - \alpha_{23} - \alpha_{26}) (\beta_{145} - \alpha_{14} - \alpha_{45}) \\
 &\quad - \alpha'^3 \zeta(3) (\beta_{236} - \alpha_{23} - \alpha_{26}) (\beta_{145} - \alpha_{14} - \alpha_{45}) \\
 &\quad \times (\beta_{145} + \beta_{167} + \beta_{236} - \alpha_{14} - 2\alpha_{17} + \alpha_{26} - \alpha_{45} - 2\alpha_{67}) \\
 &\quad + \mathcal{O}(\alpha'^4), \tag{143}
 \end{aligned}$$

$$\begin{aligned}
 F^{(5234)}(\alpha') &= \alpha'^2 \zeta(2) (-\beta_{125} - \beta_{235} + \beta_{467} + \alpha_{25}) (\beta_{157} - \beta_{234} - \beta_{346} + \alpha_{34}) \\
 &\quad + \alpha'^3 \zeta(3) (\beta_{125} + \beta_{235} - \beta_{467} - \alpha_{25}) (\beta_{157} - \beta_{234} - \beta_{346} + \alpha_{34}) \\
 &\quad \times (\beta_{125} - \beta_{157} + 2\beta_{234} - \beta_{235} + \beta_{346} + \beta_{467} - 2\alpha_{15} + 2\alpha_{17} - 2\alpha_{34}) \\
 &\quad + \mathcal{O}(\alpha'^4), \tag{144}
 \end{aligned}$$

$$\begin{aligned}
 F^{(5243)}(\alpha') &= \frac{1}{10} \alpha'^4 \zeta^2(2) (-\beta_{147} - \beta_{167} + \beta_{235} + \alpha_{17}) (-\beta_{134} + \beta_{267} - \beta_{345} + \alpha_{34}) \\
 &\quad \times [3\alpha_{26}(\beta_{134} - \alpha_{14} - \alpha_{34}) - 7\alpha_{26}\alpha_{35} \\
 &\quad + 10(\beta_{134} - \alpha_{14} - \alpha_{34})(\beta_{167} - \beta_{235} - \beta_{345} + \alpha_{35}) \\
 &\quad - 17\alpha_{35}(\beta_{167} - \beta_{235} - \beta_{345} + \alpha_{35})] + \mathcal{O}(\alpha'^5), \tag{145}
 \end{aligned}$$

$$\begin{aligned}
 F^{(5324)}(\alpha') &= \alpha'^2 \zeta(2) (\beta_{246} - \alpha_{24} - \alpha_{46}) (\beta_{167} - \beta_{235} + \beta_{467} - \alpha_{67}) \\
 &\quad + \alpha'^3 \zeta(3) (\beta_{167} - \beta_{234} + \beta_{246} - \beta_{467} + 2\alpha_{15} - 2\alpha_{17} + \alpha_{23} - \alpha_{24} + \alpha_{35} \\
 &\quad - \alpha_{46} - \alpha_{67}) (\beta_{246} - \alpha_{24} - \alpha_{46}) (\beta_{167} - \beta_{235} + \beta_{467} - \alpha_{67}) \\
 &\quad + \mathcal{O}(\alpha'^4), \tag{146}
 \end{aligned}$$

$$\begin{aligned}
 F^{(5342)}(\alpha') &= \frac{1}{10} \alpha'^4 \zeta^2(2) \alpha_{15} \alpha_{26} [-3\beta_{246} \alpha_{15} + 10\alpha_{15} \alpha_{24} + \beta_{135} (3\beta_{246} - 10\alpha_{24} - 3\alpha_{26}) \\
 &\quad + 3\alpha_{15} \alpha_{26} - 10\beta_{246} \alpha_{35} + 27\alpha_{24} \alpha_{35} + 10\alpha_{26} \alpha_{35}] + \mathcal{O}(\alpha'^5), \tag{147}
 \end{aligned}$$

$$\begin{aligned}
 F^{(5423)}(\alpha') &= -\alpha'^2 \zeta(2) (\beta_{236} - \alpha_{23} - \alpha_{36}) (\beta_{145} - \alpha_{15} - \alpha_{45}) \\
 &\quad - \alpha'^3 \zeta(3) (\beta_{236} - \alpha_{23} - \alpha_{36}) (\beta_{145} - \alpha_{15} - \alpha_{45}) \\
 &\quad \times (\beta_{145} + \beta_{167} + \beta_{236} + \alpha_{15} - 2\alpha_{17} - \alpha_{23} - \alpha_{36} - 2\alpha_{67}) \\
 &\quad + \mathcal{O}(\alpha'^4), \tag{148}
 \end{aligned}$$

$$\begin{aligned}
 F^{(5432)}(\alpha') &= \alpha'^2 \zeta(2) \alpha_{15} \alpha_{26} + \alpha'^3 \zeta(3) \alpha_{15} \alpha_{26} (\beta_{167} + \alpha_{15} - 2\alpha_{17} + \alpha_{26} - 2\alpha_{67}) \\
 &\quad + \mathcal{O}(\alpha'^4). \tag{149}
 \end{aligned}$$

The corresponding expressions of the α'^4 terms for the remaining terms are included in the text files that we have submitted, attached to this work, to the hep-th arXiv preprint basis.

We have confirmed that our results are in perfect agreement with the ones found previously in [52].

5.6. Higher order α' expansion of the momentum factors

5.6.1. Subtleties in the revisited S-matrix method

In this subsection we begin recalling that the general form of the α'^p bosonic terms of the OSLEEL in (3) (where $p \geq 2$) is given by

$$\mathcal{L}_{\text{eff}}^{(p)} = \frac{1}{g^2} (2\alpha')^p [F^{p+2} + D^2 F^{p+1} + D^4 F^p + \dots + D^{2p-4} F^4]. \quad (150)$$

There are two subtleties in the first step (out of two) of the revisited S-matrix method, when it is used to find all the α'^p terms in (150):

1. It requires to demand the absence of $(\zeta \cdot k)^N$ terms in the N -point amplitude, from $N = 4$ up to $N = p + 2$.

This means that the method requires to compute up to a $(p + 2)$ -point amplitude from the lagrangian terms in (150) and from the lower order α' ones.⁶⁶

So, for high values of p ($p \geq 5$, for example) the computations become quite heavy, although they will always be very much simpler than the corresponding ones in the traditional S-matrix method.

2. When gradually imposing the constraint *absence of $(\zeta \cdot k)^N$ terms in the N -point amplitude*, starting from $N = 4$ and ending with $N = p + 2$, at each stage the coefficients of the $D^{2(p+2-N)} F^N$ terms do not necessarily become completely constrained: some further restrictions for them could still come when demanding *absence of $(\zeta \cdot k)^N$ terms* for a higher value of N .

This means that, in order to guarantee that the number of independent coefficients of the constrained basis (see Table 1) has been achieved, it is strictly necessary to have demanded the absence of $(\zeta \cdot k)^N$ terms in the N -point amplitude for all $N = 4, \dots, (p + 2)$.

An example of this subtlety first happens at α'^4 order. In the determination of them that we did in [7], when we demanded *absence of $(\zeta \cdot k)^4$ terms in the 4-point amplitude*, this left only two undetermined coefficients in the $D^4 F^4$ terms, say, c_1 and c_2 . Afterwards, when we demanded *absence of $(\zeta \cdot k)^5$ terms in the 5-point amplitude*, this restriction fixed many (but not all) of the coefficients of the $D^2 F^5$ terms in terms of c_1 and c_2 : there still remained *other* undetermined coefficients. Finally, when we demanded *absence of $(\zeta \cdot k)^6$ terms in the 6-point amplitude*, all the coefficients (the F^6 , the $D^2 F^5$ and the $D^4 F^4$ ones⁶⁷) became completely determined, even without using information from the $F^{[2]}(\alpha')$ momentum factor: that is why in Table 1 we wrote that the dimension of the constrained basis at α'^4 order is *zero*.

The two subtleties that we have mentioned above for the first step of the revisited S-matrix method in the determination of the $\mathcal{L}_{\text{eff}}^{(p)}$ terms of the OSLEEL, have a direct counterpart in the

⁶⁶ It is not required to know the full contribution of these amplitudes but only their $(\zeta \cdot k)^N$ terms.

⁶⁷ The $D^4 F^4$ ones were what we referred to as c_1 and c_2 .

determination of the α'^p terms of the momentum factors of the N -point amplitudes (from $N = 4$ up to $N = p + 2$).

This means that, once we have obtained the N -point momentum factors up to a given α' order, say α'^{p-2} (where $p \geq 4$), if we want to determine the α'^p terms of these N -point momentum factors (where $N \geq 4$ ⁶⁸) we have to demand the three requirements in (79) for them, from $N = 4$ up to $N = p + 2$,⁶⁹ and only then the revisited S-matrix method proposes, in the second step, that all their coefficients will be found using a lower n -point momentum factor at α'^p order.

For example, besides the case that we mentioned about the determination of the α'^4 terms of the OSLEEL (which for the scattering amplitudes corresponds to determining the 4, 5 and 6-point momentum factors at that α' order), in the calculations that we did in the present work to find the α'^6 terms for 5 and 6-point momentum factors (see Subsections 5.3 and 5.4), using the three requirements in (79) for $N = 5$ is not enough information to determine *all* the 5-point coefficients, unless we also demand those three requirements for the 6-point momentum factors as well: this completely determines the coefficients of the 5 and the 6-point momentum factors (once we use the known α' expansion of the 4-point momentum factor, $F^{(2)}(\alpha')$).

5.6.2. A conjecture for the higher order α' terms of the momentum factors

So far, we have exposed that the revisited S-matrix method imposes many strong constraints to the coefficients of the α'^p terms of the OSLEEL (or equivalently, to the N -point momentum factors at that α' order, from $N = 4$ to $N = p + 2$).

From the scattering amplitudes point of view, it is clear that if the method succeeds finding the N -point momentum factors at α'^p order, from $N = 4$ to $N = p + 2$, then it is guaranteed that for $N > p + 2$ the N -point momentum factors will also be determined at that α' order. The reason for this is simple: the α'^p terms of the OSLEEL (and the lower order ones) would be all known and, therefore, using this lagrangian *any* N -point amplitude can be computed up to that order in α' .

What still remains to be further examined is the freedom that remains in the coefficients of the OSLEEL after the first step of the revisited S-matrix approach has been applied at α'^p order. In Subsection 2.2 we have proposed that freedom can be completely fixed from the knowledge of the α' expansion of the 5-point amplitude. The only solid reason that we have for this, until this moment, has to do with the fact that the α' expansion of the 5-point amplitude (presumably) contains all the MZV's that appear at higher N -point amplitudes. As mentioned in Subsection 2.2, this is something that can also be seen in the recent discovery that there is a unique and same generator for the MZV's which allows to obtain all the momentum factors, at *any* α' order [16].

So, in this work, we will leave the statement that the revisited S-matrix method plus the 5-point amplitude α' expansion are enough ingredients to find the *full* bosonic terms of the OSLEEL (or equivalently, they are enough ingredients to find *any* N -point amplitude, at *any* α' order) as a conjecture.

⁶⁸ Although in Subsections 5.3, 5.4 and 5.5 we have already used the known α' expansion of the 4-point momentum factor, $F^{(2)}(\alpha')$, this information is formally part of the second (and last) step of the revisited S-matrix method, so we are ignoring it at this moment.

⁶⁹ In the case of $N = 4$ the only requirement that is needed is cyclic invariance.

6. Summary and conclusions

We have successfully derived a closed formula for the tree level N -point amplitude of open massless superstrings (for $3 \leq N \leq 7$), considering bosons and fermions, using the RNS formalism.⁷⁰ This is the same formula first found by Mafra, Schlotterer and Stieberger in [21], using the pure spinor formalism [30].

Spacetime supersymmetry has been present throughout our approach, but in a *non-manifestly* manner. First, it has been present implicitly in the computation of the N -point gauge boson amplitude in a closed form (see Eq. (70)) because, in order to arrive to it, the condition of absence of $(\zeta \cdot k)^N$ terms⁷¹ has been used. And second, it has been used to find *uniquely* the amplitudes involving fermions, once the N -point formula for gauge bosons, Eq. (70), has been found.

We believe that a deeper understanding of our procedure (like working with an induction hypothesis in N , for example) can eventually arrive to the MSS formula in Eq. (1), for arbitrary N .

The kinematic analysis that we have required to arrive to MSS formula, naturally leads us to a space of N -point gauge boson subamplitudes which is $(N - 3)!$ -dimensional (at least for $3 \leq N \leq 7$).⁷² At this point is where the basis of Yang–Mills subamplitudes first proposed by Bern, Carrasco and Johansson [37], plays an important role in our procedure. Once this basis has been identified as a basis for $\mathcal{V}_N$, then the MSS formula and the explicit BCJ relations themselves become linear algebra problems in which the components of a certain vector, with respect to a given basis, are desired to be found. We have done these calculations in Section 4 and Appendix E, respectively.

Following the same spirit of the revisited S-matrix approach [7], we have found α' correction terms to the open superstring N -point amplitudes (where $N = 5, 6, 7$) by only using the well known 4-point amplitude α' expansion (see Eqs. (F.2) and (F.3)) and demanding cyclic symmetry and tree level unitarity for the scattering amplitudes. We have done these calculations to at most α'^6 order. It is quite remarkable that we have not needed to compute, for the calculations that we have proposed to do in this work, any coefficient as a numerical series or any integral involving polylogarithms.

We expect that, within our approach, only for α'^8 order onwards⁷³ the α' expansion of the 5-point amplitude (which is nowadays explicitly known up to α'^{22} order [28,52]) will be required to obtain the remaining coefficients of the series.⁷⁴ We will leave the calculations of higher α' terms of the N -point amplitude, where $N \geq 6$ and where *non-trivial* MZV's show up, checking that the 5-point amplitude information is enough, to be done somewhere else.

In Table 7 we have summarized the existing parallel of the revisited S-matrix method in the determination of the OSLEEL and the corresponding scattering amplitude α' calculations.

In a forthcoming work we will use the revisited S-matrix method to compute the α'^5 order terms of the OSLEEL (in analogy to the calculations that we did in [7]) and also to compute those terms in the $N = 7$ scattering amplitude [53].

⁷⁰ An important advantage of our procedure consists in the fact that we have not needed to work with fermion vertex operators at any moment, because it is known that scattering of n gauge bosons with $2m$ massless fermions in open superstring theory, along the RNS formalism, is a complicated issue for $m \geq 3$.

⁷¹ This is the main observation of our revisited S-matrix method [7].

⁷² In the main body of this work we have referred to this space by $\mathcal{V}_N$. See Section 3 for more details.

⁷³ With the remarkable exception of the α'^9 order terms, as mentioned in Subsection 2.2.

⁷⁴ These coefficients would be the ones which have dependence in the non-trivial MZV's that we referred to in Subsection 2.2.

Table 7
Comparison between the revisited S-matrix method in the determination of the OSLEEL and the corresponding scattering amplitude α' calculations.

Step of the revisited S-matrix method at α'^p order	Open superstring low energy effective lagrangian at α'^p order	Open superstring scattering amplitude at α'^p order
<i>Step I:</i>		
Requirement of absence of $(\xi \cdot k)^N$ terms in the N -point amplitude ($N = 4, \dots, p + 2$)	Reduces the general basis to a constrained basis of terms	Finds that the N -point amplitude can be written to a constrained basis of terms in terms of the $\mathcal{B}_N$ basis (see Eq. (5))
<i>Step II:</i>		
Use of n -point amplitudes information ($n \ll p + 2$) (presumably, only $n = 4$ and $n = 5$)	Determines <i>all</i> the coefficients of the constrained basis	Determines <i>all</i> the momentum factors

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Appendix A. Conventions

We use the following convention for the Minkowski metric:

$$\eta_{\mu\nu} = \text{diag}(-, +, \dots, +). \tag{A.1}$$

Gauge fields are matrices in the adjoint representation of the Lie group internal space, so that $A_\mu = A_\mu^a \lambda^a$, where

$$(\lambda^a)^{bc} = -if^{abc}. \tag{A.2}$$

In (A.2) the f^{abc} 's are the Lie group structure constants and the λ^a 's obey the normalization condition

$$\text{tr}(\lambda^a \lambda^b) = \delta^{ab}. \tag{A.3}$$

The field strength is defined by

$$F_{\mu\nu} = \partial_\mu A_\nu - \partial_\nu A_\mu - i[A_\mu, A_\nu]. \tag{A.4}$$

Appendix B. Known relations for Yang–Mills subamplitudes before BCJ

Before the BCJ relations were found, based on the conjecture between color and kinematics [37], other relations were very well known for the Yang–Mills (tree level) subamplitudes [51]:

- 1. On-shell gauge invariance (item 3 in Eq. (9)).

2. Cyclicity:

$$A(1, 2, \dots, N-1, N) = A(2, 3, \dots, N, 1) = \dots = A(N, 1, \dots, N-2, N-1). \quad (\text{B.1})$$

3. Reflection:

$$A(1, 2, \dots, N-1, N) = (-1)^N A(N-1, N-2, \dots, 1, N). \quad (\text{B.2})$$

4. Dual Ward identity⁷⁵:

$$A(1, 2, 3, \dots, N) + A(2, 1, 3, \dots, N) + A(2, 3, 1, \dots, N) + \dots + A(2, 3, \dots, 1, N) = 0. \quad (\text{B.3})$$

5. Kleiss–Kuijf relations [55]⁷⁶:

$$A(1, \{\alpha\}, N, \{\beta\}) = (-1)^{n_\beta} \sum_{\{\sigma\}_i \in \mathcal{OP}(\{\alpha\}, \{\beta\})} A(1, \{\sigma\}_i, N). \quad (\text{B.4})$$

It is known that the first three of these relations are obeyed, also, by gauge boson subamplitudes in open string theory [33] and that there is a string theory version of the Kleiss–Kuijf relations in (B.4) [38,8].

The Kleiss–Kuijf relations in (B.4) are extremely important since using them it was possible to find an $(N-2)!$ -dimensional set of ‘independent’ Yang–Mills subamplitudes from which all the remaining ones could be obtained [55].⁷⁷

Appendix C. Calculations that lead to one 4-point kinematical expression

In this appendix we give the details of the results mentioned in Subsection 3.2.2, namely, we give the details of how starting with the general expression in (22) for $T(1, 2, 3, 4)$ we arrive to (26), after demanding on-shell gauge invariance.

On-shell gauge invariance of $T(1, 2, 3, 4)$ means that it becomes zero (after demanding momentum conservation, the physical state condition (8) and the mass-shell condition (7)) whenever any of the polarizations vectors ζ_i becomes k_i ($i = 1, 2, 3, 4$). Let us see this:

1. Case of $\zeta_1 \rightarrow k_1$

In this case, using that $k_1 \cdot k_2 = s/2$ and $k_1 \cdot k_3 = -(s+t)/2$, where s and t are the Mandelstam variables defined in Eq. (24), we can arrive to

$$\begin{aligned} T(1, 2, 3, 4)|_{\zeta_1=k_1} &= [(\lambda_1 + \lambda_5 + \lambda_9)(\zeta_2 \cdot k_1)(\zeta_3 \cdot k_1)(\zeta_4 \cdot k_1) + (\lambda_2 + \lambda_6)(\zeta_2 \cdot k_1)(\zeta_3 \cdot k_1)(\zeta_4 \cdot k_2) \\ &\quad + (\lambda_3 + \lambda_{10})(\zeta_2 \cdot k_1)(\zeta_3 \cdot k_2)(\zeta_4 \cdot k_1) + \lambda_4(\zeta_2 \cdot k_1)(\zeta_3 \cdot k_2)(\zeta_4 \cdot k_2) \\ &\quad + (\lambda_7 + \lambda_{11})(\zeta_2 \cdot k_3)(\zeta_3 \cdot k_1)(\zeta_4 \cdot k_1) + \lambda_8(\zeta_2 \cdot k_3)(\zeta_3 \cdot k_1)(\zeta_4 \cdot k_2) \\ &\quad + \lambda_{12}(\zeta_2 \cdot k_3)(\zeta_3 \cdot k_2)(\zeta_4 \cdot k_1)] \end{aligned}$$

⁷⁵ This identity is also known as ‘subcyclic property’ or ‘photon decoupling identity’.

⁷⁶ See Section II of Ref. [37], for example, for the details about the intrinsic notation used on these relations.

⁷⁷ After the discovery of the BCJ relations it became understood that these $(N-2)!$ were not really all ‘independent’: only a subset of $(N-3)!$ of them is.

$$\begin{aligned}
& + \left[\left\{ -\frac{1}{2}(s+t)\lambda_{15} + \frac{1}{2}\lambda_{13}s + \rho_3 \right\} (\zeta_2 \cdot \zeta_3)(\zeta_4 \cdot k_1) \right. \\
& + \left\{ -\frac{1}{2}(s+t)\lambda_{16} + \frac{1}{2}\lambda_{14}s \right\} (\zeta_2 \cdot \zeta_3)(\zeta_4 \cdot k_2) \\
& + \left\{ -\frac{1}{2}(s+t)\lambda_{19} + \frac{1}{2}\lambda_{17}s + \rho_2 \right\} (\zeta_2 \cdot \zeta_4)(\zeta_3 \cdot k_1) \\
& + \left\{ -\frac{1}{2}(s+t)\lambda_{20} + \frac{1}{2}\lambda_{18}s \right\} (\zeta_2 \cdot \zeta_4)(\zeta_3 \cdot k_2) \\
& + \left\{ -\frac{1}{2}(s+t)\lambda_{23} + \frac{1}{2}\lambda_{21}s + \rho_1 \right\} (\zeta_3 \cdot \zeta_4)(\zeta_2 \cdot k_1) \\
& \left. + \left\{ -\frac{1}{2}(s+t)\lambda_{24} + \frac{1}{2}\lambda_{22}s \right\} (\zeta_3 \cdot \zeta_4)(\zeta_2 \cdot k_3) \right]. \quad (\text{C.1})
\end{aligned}$$

In (C.1) the first square bracket contains seven $(\zeta \cdot k)^3$ terms while the second square bracket contains six $(\zeta \cdot \zeta)^1(\zeta \cdot k)^1$ ones. Since the complete expression in (C.1) should be zero and all its kinematic terms are linearly independent then the coefficient of each of them should be zero. This leads precisely to the linear system of equations of Eq. (23) in the main text.

2. Case of $\zeta_2 \rightarrow k_2$

In this case, using that $k_1 \cdot k_2 = s/2$ and $k_2 \cdot k_3 = t/2$, we can arrive to

$$\begin{aligned}
& T(1, 2, 3, 4)|_{\zeta_2=k_2} \\
& = [\lambda_1(\zeta_1 \cdot k_2)(\zeta_3 \cdot k_1)(\zeta_4 \cdot k_1) + (\lambda_2 + \lambda_{17})(\zeta_1 \cdot k_2)(\zeta_3 \cdot k_1)(\zeta_4 \cdot k_2) \\
& + (\lambda_3 + \lambda_{13})(\zeta_1 \cdot k_2)(\zeta_3 \cdot k_2)(\zeta_4 \cdot k_1) + (\lambda_4 + \lambda_{14} + \lambda_{18})(\zeta_1 \cdot k_2)(\zeta_3 \cdot k_2)(\zeta_4 \cdot k_2) \\
& + \lambda_{19}(\zeta_1 \cdot k_3)(\zeta_3 \cdot k_1)(\zeta_4 \cdot k_2) + \lambda_{15}(\zeta_1 \cdot k_3)(\zeta_3 \cdot k_2)(\zeta_4 \cdot k_1) \\
& + (\lambda_{16} + \lambda_{20})(\zeta_1 \cdot k_3)(\zeta_3 \cdot k_2)(\zeta_4 \cdot k_2)] \\
& + \left[\frac{1}{2}\{\lambda_5s + \lambda_7t\}(\zeta_1 \cdot \zeta_3)(\zeta_4 \cdot k_1) + \left\{ \frac{1}{2}(\lambda_6s + \lambda_8t) + \rho_2 \right\} (\zeta_1 \cdot \zeta_3)(\zeta_4 \cdot k_2) \right. \\
& + \frac{1}{2}\{\lambda_9s + \lambda_{11}t\}(\zeta_1 \cdot \zeta_4)(\zeta_3 \cdot k_1) + \left\{ \frac{1}{2}(\lambda_{10}s + \lambda_{12}t) + \rho_3 \right\} (\zeta_1 \cdot \zeta_4)(\zeta_3 \cdot k_2) \\
& \left. + \left\{ \frac{1}{2}(\lambda_{21}s + \lambda_{22}t) + \rho_1 \right\} (\zeta_3 \cdot \zeta_4)(\zeta_1 \cdot k_2) + \frac{1}{2}\{\lambda_{23}s + \lambda_{24}t\}(\zeta_3 \cdot \zeta_4)(\zeta_1 \cdot k_3) \right]. \quad (\text{C.2})
\end{aligned}$$

In a similar way to what was argued in the previous case, the expression in (C.2) should be zero and this leads to the following set of thirteen equations (which happens to be linearly independent):

$$\begin{aligned}
& \lambda_1 = 0, \quad \lambda_{15} = 0, \quad \lambda_{19} = 0, \quad \lambda_2 + \lambda_{17} = 0, \\
& \lambda_3 + \lambda_{13} = 0, \quad \lambda_4 + \lambda_{14} + \lambda_{18} = 0, \quad \lambda_{16} + \lambda_{20} = 0, \\
& 2\rho_1 + \lambda_{21}s + \lambda_{22}t = 0, \quad 2\rho_2 + \lambda_6s + \lambda_8t = 0, \quad 2\rho_3 + \lambda_{10}s + \lambda_{12}t = 0, \\
& \lambda_5s + \lambda_7t = 0, \quad \lambda_9s + \lambda_{11}t = 0, \quad \lambda_{23}s + \lambda_{24}t = 0. \quad (\text{C.3})
\end{aligned}$$

3. Case of $\zeta_3 \rightarrow k_3$

In this case, besides using the expressions for products of momenta in terms of Mandelstam variables ($k_1 \cdot k_3 = -(s+t)/2$ and $k_2 \cdot k_3 = t/2$), we need to use the condition

$$(\zeta_4 \cdot k_3) = -(\zeta_4 \cdot k_1) - (\zeta_4 \cdot k_2), \quad (\text{C.4})$$

which comes from the gauge (or the physical state) condition in (8) and momentum conservation. Doing these substitutions, after $\zeta_3 \rightarrow k_3$, we can arrive to

$$\begin{aligned} T(1, 2, 3, 4)|_{\zeta_3=k_3} &= [-\lambda_{21}(\zeta_1 \cdot k_2)(\zeta_2 \cdot k_1)(\zeta_4 \cdot k_1) - \lambda_{21}(\zeta_1 \cdot k_2)(\zeta_2 \cdot k_1)(\zeta_4 \cdot k_2) \\ &\quad + (\lambda_{13} - \lambda_{22})(\zeta_1 \cdot k_2)(\zeta_2 \cdot k_3)(\zeta_4 \cdot k_1) + (\lambda_{14} - \lambda_{22})(\zeta_1 \cdot k_2)(\zeta_2 \cdot k_3)(\zeta_4 \cdot k_2) \\ &\quad + (\lambda_5 - \lambda_{23})(\zeta_1 \cdot k_3)(\zeta_2 \cdot k_1)(\zeta_4 \cdot k_1) + (\lambda_6 - \lambda_{23})(\zeta_1 \cdot k_3)(\zeta_2 \cdot k_1)(\zeta_4 \cdot k_2) \\ &\quad + (\lambda_7 + \lambda_{15} - \lambda_{24})(\zeta_1 \cdot k_3)(\zeta_2 \cdot k_3)(\zeta_4 \cdot k_1) \\ &\quad + (\lambda_8 + \lambda_{16} - \lambda_{24})(\zeta_1 \cdot k_3)(\zeta_2 \cdot k_3)(\zeta_4 \cdot k_2)] \\ &\quad + \left[\left\{ -\frac{1}{2}(s+t)\lambda_1 + \frac{1}{2}\lambda_3 t - \rho_1 \right\} (\zeta_1 \cdot \zeta_2)(\zeta_4 \cdot k_1) \right. \\ &\quad + \left\{ -\frac{1}{2}(s+t)\lambda_2 + \frac{1}{2}\lambda_4 t - \rho_1 \right\} (\zeta_1 \cdot \zeta_2)(\zeta_4 \cdot k_2) \\ &\quad + \left\{ -\frac{1}{2}(s+t)\lambda_9 + \frac{1}{2}\lambda_{10} t \right\} (\zeta_1 \cdot \zeta_4)(\zeta_2 \cdot k_1) \\ &\quad + \left\{ -\frac{1}{2}(s+t)\lambda_{11} + \frac{1}{2}\lambda_{12} t + \rho_3 \right\} (\zeta_1 \cdot \zeta_4)(\zeta_2 \cdot k_3) \\ &\quad + \left\{ -\frac{1}{2}(s+t)\lambda_{17} + \frac{1}{2}\lambda_{18} t \right\} (\zeta_2 \cdot \zeta_4)(\zeta_1 \cdot k_2) \\ &\quad \left. + \left\{ -\frac{1}{2}(s+t)\lambda_{19} + \frac{1}{2}\lambda_{20} t + \rho_2 \right\} (\zeta_2 \cdot \zeta_4)(\zeta_1 \cdot k_3) \right]. \quad (\text{C.5}) \end{aligned}$$

This time demanding the previous expression to vanish leads to fourteen equations, but clearly, two of them are identical ($-\lambda_{21} = 0$), so we just have a linear system of thirteen equations (which, once more, happens to be linearly independent) and which is equivalent to the following one:

$$\begin{aligned} \lambda_{21} &= 0, & \lambda_{23} - \lambda_6 &= 0, & \lambda_{22} - \lambda_{14} &= 0, & \lambda_{24} - \lambda_8 - \lambda_{16} &= 0, \\ \lambda_7 - \lambda_8 + \lambda_{15} - \lambda_{16} &= 0, & \lambda_{13} - \lambda_{14} &= 0, & \lambda_5 - \lambda_6 &= 0, \\ 2\rho_1 + \lambda_1(s+t) - \lambda_3 t &= 0, & 2\rho_2 - \lambda_{19}(s+t) + \lambda_{20} t &= 0, \\ 2\rho_3 - \lambda_{11}(s+t) + \lambda_{12} t &= 0, & \lambda_4 t - \lambda_3 t + \lambda_1(s+t) - \lambda_2(s+t) &= 0, \\ \lambda_{10} t - \lambda_9(s+t) &= 0, & \lambda_{18} t - \lambda_{17}(s+t) &= 0. \end{aligned} \quad (\text{C.6})$$

4. Case of $\zeta_4 \rightarrow k_4$

Similarly to the previous case, besides using the expressions for products of momenta in terms of Mandelstam variables ($k_1 \cdot k_4 = t/2$ and $k_2 \cdot k_4 = -(s+t)/2$), we need to use the conditions

$$\begin{aligned}
 (\zeta_1 \cdot k_4) &= -(\zeta_1 \cdot k_2) - (\zeta_1 \cdot k_3), & (\zeta_2 \cdot k_4) &= -(\zeta_2 \cdot k_1) - (\zeta_2 \cdot k_3), \\
 (\zeta_3 \cdot k_4) &= -(\zeta_3 \cdot k_1) - (\zeta_3 \cdot k_2), & &
 \end{aligned}
 \tag{C.7}$$

which come from the gauge (or the physical state) condition in (8) and momentum conservation. Doing these substitutions, after $\zeta_4 \rightarrow k_4$, we can arrive to

$$\begin{aligned}
 T(1, 2, 3, 4)|_{\zeta_4=k_4} &= -\left[(\lambda_9 + \lambda_{17} + \lambda_{21})(\zeta_1 \cdot k_2)(\zeta_2 \cdot k_1)(\zeta_3 \cdot k_1) \right. \\
 &\quad + (\lambda_{10} + \lambda_{18} + \lambda_{21})(\zeta_1 \cdot k_2)(\zeta_2 \cdot k_1)(\zeta_3 \cdot k_2) \\
 &\quad + (\lambda_{11} + \lambda_{17} + \lambda_{22})(\zeta_1 \cdot k_2)(\zeta_2 \cdot k_3)(\zeta_3 \cdot k_1) \\
 &\quad + (\lambda_{12} + \lambda_{18} + \lambda_{22})(\zeta_1 \cdot k_2)(\zeta_2 \cdot k_3)(\zeta_3 \cdot k_2) \\
 &\quad + (\lambda_9 + \lambda_{19} + \lambda_{23})(\zeta_1 \cdot k_3)(\zeta_2 \cdot k_1)(\zeta_3 \cdot k_1) \\
 &\quad + (\lambda_{10} + \lambda_{20} + \lambda_{23})(\zeta_1 \cdot k_3)(\zeta_2 \cdot k_1)(\zeta_3 \cdot k_2) \\
 &\quad + (\lambda_{11} + \lambda_{19} + \lambda_{24})(\zeta_1 \cdot k_3)(\zeta_2 \cdot k_3)(\zeta_3 \cdot k_1) \\
 &\quad \left. + (\lambda_{12} + \lambda_{20} + \lambda_{24})(\zeta_1 \cdot k_3)(\zeta_2 \cdot k_3)(\zeta_3 \cdot k_2) \right] \\
 &\quad + \left[\left\{ -\frac{1}{2}(s+t)\lambda_2 + \frac{1}{2}\lambda_1 t - \rho_1 \right\} (\zeta_1 \cdot \zeta_2)(\zeta_3 \cdot k_1) \right. \\
 &\quad + \left\{ -\frac{1}{2}(s+t)\lambda_4 + \frac{1}{2}\lambda_3 t - \rho_1 \right\} (\zeta_1 \cdot \zeta_2)(\zeta_3 \cdot k_2) \\
 &\quad + \left\{ -\frac{1}{2}(s+t)\lambda_6 + \frac{1}{2}\lambda_5 t - \rho_2 \right\} (\zeta_1 \cdot \zeta_3)(\zeta_2 \cdot k_1) \\
 &\quad + \left\{ -\frac{1}{2}(s+t)\lambda_8 + \frac{1}{2}\lambda_7 t - \rho_2 \right\} (\zeta_1 \cdot \zeta_3)(\zeta_2 \cdot k_3) \\
 &\quad + \left\{ -\frac{1}{2}(s+t)\lambda_{14} + \frac{1}{2}\lambda_{13} t - \rho_3 \right\} (\zeta_2 \cdot \zeta_3)(\zeta_1 \cdot k_2) \\
 &\quad \left. + \left\{ -\frac{1}{2}(s+t)\lambda_{16} + \frac{1}{2}\lambda_{15} t - \rho_3 \right\} (\zeta_2 \cdot \zeta_3)(\zeta_1 \cdot k_3) \right].
 \end{aligned}
 \tag{C.8}$$

As it happened in the case of $\zeta_3 \rightarrow k_3$, demanding the previous expression to vanish leads to fourteen equations, where only thirteen of them are linearly independent. The set of equations is equivalent to the following thirteen ones:

$$\begin{aligned}
 \lambda_{11} - \lambda_{12} + \lambda_{19} - \lambda_{20} &= 0, & \lambda_{10} + \lambda_{11} - \lambda_{12} + \lambda_{17} + \lambda_{21} &= 0, \\
 \lambda_9 - \lambda_{10} - \lambda_{11} + \lambda_{12} &= 0, & \lambda_{11} - \lambda_{12} + \lambda_{17} - \lambda_{18} &= 0, & \lambda_{10} + \lambda_{20} + \lambda_{23} &= 0, \\
 \lambda_{12} + \lambda_{20} + \lambda_{24} &= 0, & \lambda_{11} + \lambda_{17} + \lambda_{22} &= 0, \\
 2\rho_1 - \lambda_1 t + \lambda_2(s+t) &= 0, & 2\rho_2 - \lambda_5 t + \lambda_6(s+t) &= 0, \\
 2\rho_3 - \lambda_{15} t + \lambda_{16}(s+t) &= 0, \\
 \lambda_3 t + (\lambda_2 - \lambda_4)(s+t) - \lambda_1 t &= 0, & \lambda_7 t + (\lambda_6 - \lambda_8)(s+t) - \lambda_5 t &= 0, \\
 \lambda_{13} t + (\lambda_{16} - \lambda_{14})(s+t) - \lambda_{15} t &= 0.
 \end{aligned}
 \tag{C.9}$$

Now, the solution of the linear system formed by Eqs. (23), (C.3), (C.6) and (C.9), is given by

$$\begin{aligned}
\lambda_1 &= 0, & \lambda_4 &= 0, & \lambda_8 &= 0, \\
\lambda_{12} &= 0, & \lambda_{15} &= 0, & \lambda_{19} &= 0, & \lambda_{21} &= 0, \\
\lambda_7 &= -\lambda_{11} = \lambda_{16} = -\lambda_{20} = \lambda_{24}, \\
\lambda_2 &= -\lambda_5 = -\lambda_6 = \lambda_9 = -\lambda_{17} = -\lambda_{23} = \lambda_{24} \frac{t}{s}, \\
-\lambda_3 &= \lambda_{10} = \lambda_{13} = \lambda_{14} = -\lambda_{18} = \lambda_{22} = \lambda_{24} \frac{s+t}{s},
\end{aligned} \tag{C.10}$$

where λ_{24} is completely arbitrary.

In this way, substituting the final solution found in Eq. (C.10), in the original expression for $T(1, 2, 3, 4)$, given in Eq. (22), this subamplitude becomes a kinematical expression which contains as a global factor λ_{24} . Due to the arbitrariness of λ_{24} , for convenience we may rewrite it as

$$\lambda_{24} = 4g^2\lambda/t. \tag{C.11}$$

Afterwards, using appropriately the relations in (C.7) plus the relations

$$u = -s - t, \tag{C.12}$$

$$(\zeta_4 \cdot k_3) = -(\zeta_4 \cdot k_1) - (\zeta_4 \cdot k_2), \tag{C.13}$$

we can finally arrive to the symmetric formula in (25).

In (25) the expression inside the curly brackets is precisely the well known 4-point kinematic factor [33].

Appendix D. Finding an N -point basis for the scattering amplitude ($N = 5, 6, 7$)

In this appendix we give the details of the calculations that support the main claim of Section 3, namely, that the $\mathcal{B}_N$ set of subamplitudes, given in Eq. (5), is a basis for the space of N -point amplitudes $\mathcal{V}_N$, defined by the requirements in Eq. (9) (where the polarizations and the momenta obey relations (6), (7) and (8)).

The importance of this claim is that the N -point tree level scattering subamplitudes of gauge bosons in open superstring theory (and also in Yang–Mills theory) can all be written as a linear combination of the subamplitudes in $(\zeta \cdot \zeta)^1 (\zeta \cdot k)^{N-2}$ terms of $A_{YM}(1, \dots, N)$. At least we have succeeded in doing so for $3 \leq N \leq 7$.

We begin by defining, in Appendix D.1, the independent Mandelstam variables that appear in an N -point scattering process of massless particles (for $N = 5, 6, 7$). We also write there the expression for the remaining scalar invariants, $k_i \cdot k_j$, in terms of the corresponding Mandelstam variables. Then, in Appendix D.2 we present, in an abbreviated fashion, the expressions of the $(\zeta \cdot \zeta)^1 (\zeta \cdot k)^{N-2}$ terms of $A_{YM}(1, \dots, N)$. As argued at the end of Subsection 3.3, the explicit expression of these terms will allow us to find the momentum factors in the case of open superstring subamplitudes (Section 4) and also in the case of Yang–Mills subamplitudes (Appendix E).

In Appendix D.3 we use the $(\zeta \cdot \zeta)^1 (\zeta \cdot k)^{N-2}$ terms of $A_{YM}(1, \dots, N)$, written in Appendix D.2, to prove that the $\mathcal{B}_N$ set is linearly independent in $\mathcal{V}_N$.

We end by checking, in Appendix D.4, that $\mathcal{V}_N$, as defined by the requirements in Eq. (9), is an $(N-3)!$ -dimensional space (at least in the case of $N = 5, 6, 7$) and, therefore, $\mathcal{B}_N$ is indeed a basis for $\mathcal{V}_N$.

D.1. Mandelstam variables

In general, in an N -point scattering process of massless particles there are $N(N-3)/2$ independent Mandelstam variables. In the cases of $N = 5, 6, 7$ these variables can be respectively chosen as⁷⁸

$$N = 5: \quad s_i = (k_i + k_{i+1})^2 = 2k_i \cdot k_{i+1} \quad (i = 1, \dots, 5). \quad (\text{D.1})$$

$$\begin{aligned} N = 6: \quad s_i &= (k_i + k_{i+1})^2 = 2k_i \cdot k_{i+1} \quad (i = 1, \dots, 6), \\ t_j &= (k_j + k_{j+1} + k_{j+2})^2 = 2(k_j \cdot k_{j+1} + k_j \cdot k_{j+2} + k_{j+1} \cdot k_{j+2}) \\ &\quad (j = 1, 2, 3). \end{aligned} \quad (\text{D.2})$$

$$\begin{aligned} N = 7: \quad s_i &= (k_i + k_{i+1})^2 = 2k_i \cdot k_{i+1} \quad (i = 1, \dots, 7), \\ t_j &= (k_j + k_{j+1} + k_{j+2})^2 = 2(k_j \cdot k_{j+1} + k_j \cdot k_{j+2} + k_{j+1} \cdot k_{j+2}) \\ &\quad (j = 1, \dots, 7). \end{aligned} \quad (\text{D.3})$$

In Eqs. (D.1), (D.2) and (D.3) it is implicit the identification of indexes ($6 \leftrightarrow 1$), ($7 \leftrightarrow 1$) and ($8 \leftrightarrow 1$, $9 \leftrightarrow 2$), respectively. For example, according to this identification of indexes, in (D.3) we have that $t_7 = (k_7 + k_8 + k_9)^2 = (k_7 + k_1 + k_2)^2 = 2(k_7 \cdot k_1 + k_7 \cdot k_2 + k_1 \cdot k_2)$.

The remaining scalar products of the momenta that appear in the corresponding N -point scattering process of massless particles can all be written in terms of the corresponding Mandelstam variables, in the following way⁷⁹:

$$\begin{aligned} N = 5: \quad 2k_1 \cdot k_3 &= s_4 - s_1 - s_2, \quad 2k_1 \cdot k_4 = s_2 - s_5 - s_4, \quad 2k_2 \cdot k_4 = s_5 - s_2 - s_3, \\ 2k_2 \cdot k_5 &= s_3 - s_1 - s_5, \quad 2k_3 \cdot k_5 = s_1 - s_4 - s_3, \end{aligned} \quad (\text{D.4})$$

$$\begin{aligned} N = 6: \quad 2k_1 \cdot k_3 &= t_1 - s_1 - s_2, \quad 2k_1 \cdot k_4 = s_2 + s_5 - t_1 - t_2, \quad 2k_1 \cdot k_5 = t_1 - s_5 - s_6, \\ 2k_2 \cdot k_4 &= t_2 - s_2 - s_3, \quad 2k_2 \cdot k_5 = s_3 + s_6 - t_2 - t_3, \quad 2k_2 \cdot k_6 = t_3 - s_1 - s_6, \\ 2k_3 \cdot k_5 &= t_3 - s_3 - s_4, \quad 2k_3 \cdot k_6 = s_1 + s_4 - t_1 - t_3, \quad 2k_4 \cdot k_6 = t_1 - s_4 - s_5, \end{aligned} \quad (\text{D.5})$$

$$\begin{aligned} N = 7: \quad 2k_1 \cdot k_3 &= t_1 - s_1 - s_2, \quad 2k_1 \cdot k_4 = s_2 + t_5 - t_1 - t_2, \\ 2k_1 \cdot k_5 &= s_6 + t_2 - t_5 - t_6, \quad 2k_1 \cdot k_6 = t_6 - s_6 - s_7, \\ 2k_2 \cdot k_4 &= t_2 - s_2 - s_3, \quad 2k_2 \cdot k_5 = s_3 + t_6 - t_2 - t_3, \\ 2k_2 \cdot k_6 &= s_7 + t_3 - t_6 - t_7, \quad 2k_2 \cdot k_7 = t_7 - s_1 - s_7, \\ 2k_3 \cdot k_5 &= t_3 - s_3 - s_4, \quad 2k_3 \cdot k_6 = s_4 + t_7 - t_3 - t_4, \\ 2k_3 \cdot k_7 &= s_1 + t_4 - t_1 - t_7, \quad 2k_4 \cdot k_6 = t_4 - s_4 - s_5, \\ 2k_4 \cdot k_7 &= t_1 + s_5 - t_4 - t_5, \quad 2k_5 \cdot k_7 = t_5 - s_5 - s_6. \end{aligned} \quad (\text{D.6})$$

D.2. $(\zeta \cdot \zeta)^1 (\zeta \cdot k)^{N-2}$ terms of the Yang–Mills N -point subamplitude

As mentioned at the end of Subsection 3.3, our proposal is that it will be enough to consider the $(\zeta \cdot \zeta)^1 (\zeta \cdot k)^{N-2}$ terms of the Yang–Mills N -point subamplitude, in order to find the momentum

⁷⁸ The s_i and t_j variables that appear in Eqs. (D.1), (D.2) and (D.3), are the ones that naturally appear as poles of $A_{YM}(1, \dots, N)$, for $N = 5, 6, 7$, respectively.

⁷⁹ This comes from demanding momentum conservation and the mass-shell condition for the external massless states.

factors that appear in the BCJ relations (and also the momentum factors that appear in the closed expression of the open superstring scattering subamplitudes). So, in the next subsection of this appendix we will give the explicit expression of those terms in the case of $N = 5$. We have obtained it by calculating the 5-point subamplitude in Yang–Mills theory (in the Lorentz gauge) by using Feynman rules. We have also written the $(\zeta \cdot k)$ terms of them in the basis mentioned in Eq. (14). For the Mandelstam variables we are using the conventions of Eq. (D.1).

In the case of $N = 6$ and $N = 7$ we have written in Appendix D.2.2 only the general structure of the $(\zeta \cdot \zeta)^1(\zeta \cdot k)^4$ and the $(\zeta \cdot \zeta)^1(\zeta \cdot k)^5$ terms, respectively, using the convention for the Mandelstam variables in (D.2) and (D.3). Although we have used the complete expression of them in order to achieve important results like the ones in Eqs. (28) and (29), it is not instructive to write down those huge expressions here.⁸⁰ We have verified that those expressions satisfy basic properties of the Yang–Mills subamplitudes, like cyclicity and reflection. In any case, the interested reader can request the authors for the complete explicit expressions of the 6- and 7-point subamplitudes.

D.2.1. Case of $N = 5$

$$\begin{aligned}
 & A_{YM}(1, 2, 3, 4, 5)|_{(\zeta \cdot \zeta)^1(\zeta \cdot k)^3} \\
 &= 8g^3 \left[(\zeta_2 \cdot \zeta_3) \left\{ - \left(\frac{1}{s_1 s_4} + \frac{1}{s_2 s_4} \right) (\zeta_1 \cdot k_2)(\zeta_4 \cdot k_1)(\zeta_5 \cdot k_3) \right. \right. \\
 &\quad - \left(\frac{1}{s_1 s_4} + \frac{1}{s_2 s_4} + \frac{1}{s_2 s_5} \right) (\zeta_1 \cdot k_2)(\zeta_4 \cdot k_2)(\zeta_5 \cdot k_3) \\
 &\quad + \left(\frac{1}{s_1 s_3} + \frac{1}{s_2 s_4} + \frac{1}{s_3 s_5} + \frac{1}{s_4 s_1} + \frac{1}{s_5 s_2} \right) (\zeta_1 \cdot k_2)(\zeta_4 \cdot k_3)(\zeta_5 \cdot k_1) \\
 &\quad + \left(\frac{1}{s_1 s_3} + \frac{1}{s_2 s_4} + \frac{1}{s_3 s_5} + \frac{1}{s_4 s_1} + \frac{1}{s_5 s_2} \right) (\zeta_1 \cdot k_2)(\zeta_4 \cdot k_3)(\zeta_5 \cdot k_2) \\
 &\quad + \frac{1}{s_2 s_4} (\zeta_1 \cdot k_3)(\zeta_4 \cdot k_1)(\zeta_5 \cdot k_2) - \left(\frac{1}{s_2 s_4} + \frac{1}{s_2 s_5} \right) (\zeta_1 \cdot k_3)(\zeta_4 \cdot k_2)(\zeta_5 \cdot k_1) \\
 &\quad - \left(\frac{1}{s_2 s_4} + \frac{1}{s_2 s_5} \right) (\zeta_1 \cdot k_3)(\zeta_4 \cdot k_2)(\zeta_5 \cdot k_3) \\
 &\quad + \left(\frac{1}{s_2 s_4} + \frac{1}{s_3 s_5} + \frac{1}{s_5 s_2} \right) (\zeta_1 \cdot k_3)(\zeta_4 \cdot k_3)(\zeta_5 \cdot k_2) - \frac{1}{s_2 s_5} (\zeta_1 \cdot k_4)(\zeta_4 \cdot k_2)(\zeta_5 \cdot k_3) \\
 &\quad \left. + \left(\frac{1}{s_2 s_5} + \frac{1}{s_3 s_5} \right) (\zeta_1 \cdot k_4)(\zeta_4 \cdot k_3)(\zeta_5 \cdot k_2) \right\} \\
 &\quad + (\zeta_1 \cdot \zeta_4) \left\{ - \frac{1}{s_1 s_4} (\zeta_2 \cdot k_1)(\zeta_3 \cdot k_1)(\zeta_5 \cdot k_1) - \frac{1}{s_1 s_4} (\zeta_2 \cdot k_1)(\zeta_3 \cdot k_1)(\zeta_5 \cdot k_2) \right. \\
 &\quad - \frac{1}{s_1 s_4} (\zeta_2 \cdot k_1)(\zeta_3 \cdot k_1)(\zeta_5 \cdot k_3) - \left(\frac{1}{s_1 s_4} + \frac{1}{s_2 s_4} \right) (\zeta_2 \cdot k_1)(\zeta_3 \cdot k_2)(\zeta_5 \cdot k_1) \\
 &\quad \left. - \left(\frac{1}{s_1 s_4} + \frac{1}{s_2 s_4} \right) (\zeta_2 \cdot k_1)(\zeta_3 \cdot k_2)(\zeta_5 \cdot k_2) - \left(\frac{1}{s_1 s_4} + \frac{1}{s_2 s_4} \right) (\zeta_2 \cdot k_1)(\zeta_3 \cdot k_2)(\zeta_5 \cdot k_3) \right\}
 \end{aligned}$$

⁸⁰ We have specified on these expressions the number of terms that they have. See Eqs. (D.8) and (D.9).

$$\begin{aligned}
& + \frac{1}{s_1 s_3} (\zeta_2 \cdot k_1) (\zeta_3 \cdot k_4) (\zeta_5 \cdot k_1) + \frac{1}{s_1 s_3} (\zeta_2 \cdot k_1) (\zeta_3 \cdot k_4) (\zeta_5 \cdot k_2) \\
& + \frac{1}{s_2 s_4} (\zeta_2 \cdot k_3) (\zeta_3 \cdot k_1) (\zeta_5 \cdot k_1) + \frac{1}{s_2 s_4} (\zeta_2 \cdot k_3) (\zeta_3 \cdot k_1) (\zeta_5 \cdot k_2) \\
& + \frac{1}{s_2 s_4} (\zeta_2 \cdot k_3) (\zeta_3 \cdot k_1) (\zeta_5 \cdot k_3) - \left(\frac{1}{s_2 s_5} + \frac{1}{s_5 s_3} \right) (\zeta_2 \cdot k_3) (\zeta_3 \cdot k_4) (\zeta_5 \cdot k_1) \\
& + \frac{1}{s_2 s_5} (\zeta_2 \cdot k_4) (\zeta_3 \cdot k_2) (\zeta_5 \cdot k_1) - \frac{1}{s_3 s_5} (\zeta_2 \cdot k_4) (\zeta_3 \cdot k_4) (\zeta_5 \cdot k_1) \Bigg] \\
& + (\text{cyclic permutations})
\end{aligned} \tag{D.7}$$

D.2.2. Case of $N = 6$ and $N = 7$

The $(\zeta \cdot \zeta)^1 (\zeta \cdot k)^4$ and the $(\zeta \cdot \zeta)^1 (\zeta \cdot k)^5$ terms of the 6 and the 7-point Yang–Mills subamplitudes have the following structure⁸¹:

$$\begin{aligned}
& A_{YM}(1, 2, 3, 4, 5, 6) |_{(\zeta \cdot \zeta)^1 (\zeta \cdot k)^4} \\
& = 8g^4 \left[2(\zeta_1 \cdot \zeta_2) \left\{ \frac{1}{s_1 s_5 t_1} (\zeta_3 \cdot k_1) (\zeta_4 \cdot k_1) (\zeta_5 \cdot k_1) (\zeta_6 \cdot k_2) + (67 \text{ terms}) \right\} \right. \\
& \quad + 2(\zeta_1 \cdot \zeta_3) \left\{ \frac{1}{s_1 s_5 t_1} (\zeta_2 \cdot k_1) (\zeta_4 \cdot k_1) (\zeta_5 \cdot k_1) (\zeta_6 \cdot k_3) + (57 \text{ terms}) \right\} \\
& \quad + (\zeta_1 \cdot \zeta_4) \left\{ \frac{1}{s_1 s_5 t_1} (\zeta_2 \cdot k_1) (\zeta_3 \cdot k_1) (\zeta_5 \cdot k_1) (\zeta_6 \cdot k_4) + (51 \text{ terms}) \right\} \Bigg] \\
& \quad + (\text{cyclic permutations})
\end{aligned} \tag{D.8}$$

and

$$\begin{aligned}
& A_{YM}(1, 2, 3, 4, 5, 6, 7) |_{(\zeta \cdot \zeta)^1 (\zeta \cdot k)^5} \\
& = 32g^5 \left[(\zeta_1 \cdot \zeta_2) \left\{ \frac{1}{s_1 s_6 t_1 t_5} (\zeta_3 \cdot k_1) (\zeta_4 \cdot k_1) (\zeta_5 \cdot k_1) (\zeta_6 \cdot k_1) (\zeta_7 \cdot k_2) + (565 \text{ terms}) \right\} \right. \\
& \quad + (\zeta_1 \cdot \zeta_3) \left\{ \frac{1}{s_1 s_6 t_1 t_5} (\zeta_2 \cdot k_1) (\zeta_4 \cdot k_1) (\zeta_5 \cdot k_1) (\zeta_6 \cdot k_1) (\zeta_7 \cdot k_3) + (466 \text{ terms}) \right\} \\
& \quad + (\zeta_1 \cdot \zeta_4) \left\{ \frac{1}{s_1 s_6 t_1 t_5} (\zeta_2 \cdot k_1) (\zeta_3 \cdot k_1) (\zeta_5 \cdot k_1) (\zeta_6 \cdot k_1) (\zeta_7 \cdot k_4) + (408 \text{ terms}) \right\} \Bigg] \\
& \quad + (\text{cyclic permutations}).
\end{aligned} \tag{D.9}$$

D.3. Linear independence of the set $\mathcal{B}_N$

The color/kinematic duality in Yang–Mills theory [37] and the monodromy relations found for the open string subamplitudes [38,8] succeed in finding all N -point subamplitudes of Yang–Mills theory in terms of a linear combinations of a set of $(N - 3)!$ subamplitudes (which can be

⁸¹ In the 6-point case, in Eq. (D.8), there is no mistake in the fact that the $(\zeta_1 \cdot \zeta_4)$ term does not carry a factor ‘2’, as every other terms does: this factor will arise when summing the cyclic permutations.

chosen to be the ones in the $\mathcal{B}_N$ set of Eq. (5)).⁸² The procedure of each of these approaches is self consistent in the sense that it is found a linear system for the subamplitudes which is over-determined; then, it is found that this linear system has a *unique* solution (which consists, precisely, in the BCJ relations for the N -point subamplitudes). But this result still does not guarantee that this set $\mathcal{B}_N$ is indeed a basis of $\mathcal{V}_N$: nothing forbids that there could still be additional restrictions that relate the Yang–Mills subamplitudes in $\mathcal{B}_N$.

In fact, it is known that in $D = 4$, the $N = 5$ BCJ relations can all be found in terms of *only one* Yang–Mills subamplitude, instead of two (see Section 5.2 of Ref. [45]). This does not invalidate the BCJ relations that have been found for $N = 5$: it simply states that, at least in $D = 4$, the set $\mathcal{B}_5$ is not linearly independent and the BCJ relations can have an even simpler form.

In the way that the BCJ relations have been found [37,38,8], their validity is for *any* spacetime dimension D . So, what may happen (and the mentioned result of Ref. [45] is an explicit evidence for it) is that if N is sufficiently high (as compared to D), there may still be further relations among the Yang–Mills subamplitudes of the $\mathcal{B}_N$ set in Eq. (5).

The same as in [37] and [38,8], our approach throughout this work has only considered the general case, that is, the one in which the BCJ relations are written being valid for *any* spacetime dimension D (or equivalently, we will assume that N is sufficiently small, as compared to D , such that no new extra relations between the subamplitudes arise).

Having stated this subtlety about the dependence of the basis of $\mathcal{V}_N$ in the spacetime dimension D , we will proceed with our analysis only using arguments that are valid for any spacetime dimension.

In this section of Appendix D we will see that in fact, as mentioned at the end of Subsection 3.3, considering only the $(\zeta \cdot \zeta)^1 (\zeta \cdot k)^{N-2}$ terms of the Yang–Mills N -point subamplitudes, there is enough information (which is even redundant when we consider *all* those terms) to prove the linear independence of the set of subamplitudes $\mathcal{B}_N$, given in Eq. (5).

D.3.1. Case of $N = 5$

In order to prove that $\mathcal{B}_5 = \{A_{YM}(1, 2, 3, 4, 5), A_{YM}(1, 3, 2, 4, 5)\}$ is a linearly independent set in $\mathcal{V}_5$, we need to consider the null linear combination of $A_{YM}(1, 2, 3, 4, 5)$ and $A_{YM}(1, 3, 2, 4, 5)$:

$$\lambda^{\{23\}} A_{YM}(1, 2, 3, 4, 5) + \lambda^{\{32\}} A_{YM}(1, 3, 2, 4, 5) = 0. \quad (\text{D.10})$$

Here we want to prove that (D.10) happens if and only if $\lambda^{\{23\}} = \lambda^{\{32\}} = 0$.

Considering only the $(\zeta \cdot \zeta)^1 (\zeta \cdot k)^3$ terms in (D.10), we have that

$$\lambda^{\{23\}} A_{YM}(1, 2, 3, 4, 5) \big|_{(\zeta \cdot \zeta)^1 (\zeta \cdot k)^3} + \lambda^{\{32\}} A_{YM}(1, 3, 2, 4, 5) \big|_{(\zeta \cdot \zeta)^1 (\zeta \cdot k)^3} = 0, \quad (\text{D.11})$$

and using the explicit expression for the $(\zeta \cdot \zeta)^1 (\zeta \cdot k)^3$ terms of the 5-point subamplitude, given in (D.7), we have

⁸² In the case of the color/kinematic duality approach, the authors of [37] have conjectured the explicit form of BCJ relations for arbitrary N , and they have found evidence for their conjecture up to $N = 8$, but the proof for arbitrary N has not been given yet.

$$\begin{aligned}
& 8g^3 \left[\left\{ -\lambda^{\{23\}} \left(\frac{1}{s_1 s_4} + \frac{1}{s_2 s_4} \right) + \lambda^{\{32\}} \frac{1}{s_2 s_4} \right\} (\zeta_2 \cdot \zeta_3)(\zeta_1 \cdot k_2)(\zeta_4 \cdot k_1)(\zeta_5 \cdot k_3) \right. \\
& + \left\{ -\lambda^{\{23\}} \frac{1}{s_1 s_4} - \lambda^{\{32\}} \frac{1}{(s_4 - s_1 - s_2)s_4} \right\} (\zeta_1 \cdot \zeta_4)(\zeta_2 \cdot k_1)(\zeta_3 \cdot k_1)(\zeta_5 \cdot k_1) \\
& \left. + (140 \text{ terms}) \right] = 0.
\end{aligned} \tag{D.12}$$

Since the $(\zeta \cdot \zeta)^1(\zeta \cdot k)^3$ terms in (D.12) are linearly independent, then this equation is valid if and only if the coefficient of each of these terms is zero:

$$\begin{aligned}
& -\lambda^{\{23\}} \left(\frac{1}{s_1 s_4} + \frac{1}{s_2 s_4} \right) + \lambda^{\{32\}} \frac{1}{s_2 s_4} = 0, \\
& -\lambda^{\{23\}} \frac{1}{s_1 s_4} - \lambda^{\{32\}} \frac{1}{(s_4 - s_1 - s_2)s_4} = 0, \quad \dots
\end{aligned} \tag{D.13}$$

(D.13) is a linear homogeneous system of 142 equations for $\lambda_1^{(5)}$ and $\lambda_2^{(5)}$. From the first two equations, explicitly written in (D.13), it is easy to see that the *unique* solution of this system is the trivial one:

$$\lambda^{\{23\}} = \lambda^{\{32\}} = 0. \tag{D.14}$$

D.3.2. Case of $N = 6$ and $N = 7$

Doing exactly the same procedure of the previous subsection, but for $N = 6$ and $N = 7$, using the corresponding expressions of the $(\zeta \cdot \zeta)^1(\zeta \cdot k)^4$ and the $(\zeta \cdot \zeta)^1(\zeta \cdot k)^5$ terms, mentioned in Eqs. (D.8) and (D.9), respectively, we arrive to the same conclusion, namely, that the six $\lambda^{\{\sigma_6\}}$'s and the twenty four $\lambda^{\{\sigma_7\}}$'s are *all* zero, implying that the $\mathcal{B}_6$ and $\mathcal{B}_7$ sets are linearly independent in their corresponding spaces, $\mathcal{V}_6$ and $\mathcal{V}_7$.

An interesting remark is that we have not needed to examine *all* the $(\zeta \cdot \zeta)^1(\zeta \cdot k)^{N-2}$ terms of the Yang–Mills N -point subamplitudes (for $N = 5, 6, 7$), in order to arrive at the conclusion that the $\lambda^{\{\sigma_N\}}$'s are all zero: it has been enough to just analyze the $(\zeta_1 \cdot \zeta_2)(\zeta \cdot k)^{N-2}$ kinematical structures (which are just one of the many possible structures of the $(\zeta \cdot \zeta)^1(\zeta \cdot k)^{N-2}$ terms).

D.4. Dimension of $\mathcal{V}_N$ and a basis for it

In the case of $N = 3$ and $N = 4$, in Subsections 3.2.1 and 3.2.2, respectively, it was checked that the dimension of $\mathcal{V}_N$ is, in fact,

$$\dim(\mathcal{V}_N) = (N - 3)!, \tag{D.15}$$

and that the set of subamplitudes $\mathcal{B}_N$, given in Eq. (5), is indeed a basis of this space.

In this subsection we will argue that the validity of the formula (D.15) and of the set $\mathcal{B}_N$ proposed as a basis for $\mathcal{V}_N$, given in Eq. (5), indeed holds for $N = 5, 6, 7$.

We will argue this in Appendix D.4.1 for the $N = 5$ case, with some detail, and in Appendix D.4.2 for the $N = 6$ and the $N = 7$ case. In these last two cases we will just mention the final results, because the intermediate formulas are simply too big and they do not give any additional idea.

Before going into the details of Appendices D.4.1 and D.4.2 we warn the reader that our calculations have not lead directly to the set $\mathcal{B}_N$ as a basis for $\mathcal{V}_N$ (as made explicit in Eq. (5)): what we have found for each $N (= 5, 6, 7)$ is another set of $(N - 3)!$ independent kinematical

expressions⁸³ which become a basis for the corresponding $\mathcal{V}_N$ space. Within our approach, it would simply be too tricky to arrive, for any $N \geq 5$, directly to the set $\mathcal{B}_N$.

So, at this point we must admit that, without knowing the BCJ result for the basis of Yang–Mills subamplitudes, given in Eq. (5), demanding on-shell gauge invariance would have led us to the validity of Eq. (D.15), but we would have hardly arrived to a basis in which *all* the elements can be directly associated to Yang–Mills theory.

D.4.1. Case of $N = 5$

Let $T(1, 2, 3, 4, 5) \in \mathcal{V}_5$. According to Table 2, this element of $\mathcal{V}_5$ can be constructed from 270 $(\zeta \cdot \zeta)^1(\zeta \cdot k)^3$ and from 45 $(\zeta \cdot \zeta)^2(\zeta \cdot k)^1$ terms, which, after being written using the $(\zeta \cdot k)$ terms of Eq. (14), are all linearly independent. So, initially, it contains 315 kinematical terms.

Then, we impose the third requirement that the list in Eq. (9) demands for the elements of $\mathcal{V}_5$, namely, $T(1, 2, 3, 4, 5)$ should satisfy on-shell gauge invariance:

$$T(1, 2, 3, 4, 5)|_{\zeta_i=k_i} = 0 \quad (i = 1, 2, \dots, 5). \quad (\text{D.16})$$

Doing exactly the same procedure that we explained in Subsection 3.2.2 and in Appendix C for the $N = 4$ case, in this case we have arrived to the conclusion that $T(1, 2, 3, 4, 5)$ can be written in terms of two known (and independent) kinematic expressions:

$$T(1, 2, 3, 4, 5) = \rho_1^{(5)} K_1^{(5)}(\zeta, k) + \rho_2^{(5)} K_2^{(5)}(\zeta, k), \quad (\text{D.17})$$

where $K_1^{(5)}(\zeta, k)$ and $K_2^{(5)}(\zeta, k)$ are the known (but big) kinematic expressions (without poles) and $\{\rho_1^{(5)}, \rho_2^{(5)}\}$ are the arbitrary momentum factors. So, by construction $K_1^{(5)}(\zeta, k)$ and $K_2^{(5)}(\zeta, k)$ are (on-shell) gauge invariant kinematical expressions.

$K_1^{(5)}(\zeta, k)$ consists in 133 $(\zeta \cdot k)^3(\zeta \cdot \zeta)^1$ and 33 $(\zeta \cdot k)^1(\zeta \cdot \zeta)^2$ independent terms while $K_2^{(5)}(\zeta, k)$ consists in 134 $(\zeta \cdot k)^3(\zeta \cdot \zeta)^1$ and 31 $(\zeta \cdot k)^1(\zeta \cdot \zeta)^2$ independent ones. These kinematical expressions have the following form⁸⁴:

$$\begin{aligned} K_1^{(5)}(\zeta, k) = & g^3 \left[\left\{ (s_2 + s_3 - s_5)(s_1 s_2 + s_2 s_5 + s_3 s_4 + s_3 s_5 - s_2 s_3)(\zeta_1 \cdot k_2)(\zeta_2 \cdot k_3)(\zeta_3 \cdot k_4) \right. \right. \\ & \times (\zeta_4 \cdot \zeta_5) + 132(\zeta \cdot k)^3(\zeta \cdot \zeta)^1 \text{ terms} \Big\} \\ & + \left\{ s_2 s_3(s_1 s_2 + s_2 s_5 + s_3 s_4 - s_2 s_3 - s_4 s_5)(\zeta_1 \cdot k_4)(\zeta_2 \cdot \zeta_4)(\zeta_3 \cdot \zeta_5) \right. \\ & \left. \left. + 32(\zeta \cdot k)^1(\zeta \cdot \zeta)^2 \text{ terms} \right\} \right], \quad (\text{D.18}) \end{aligned}$$

$$\begin{aligned} K_2^{(5)}(\zeta, k) = & g^3 \left[\left\{ -(s_1 s_2 s_4 + s_1 s_3 s_4 + s_1 s_3 s_5 + s_2 s_3 s_5 + s_2 s_4 s_5)(\zeta_1 \cdot k_2)(\zeta_2 \cdot k_3)(\zeta_3 \cdot k_4) \right. \right. \\ & \times (\zeta_4 \cdot \zeta_5) + 133(\zeta \cdot k)^3(\zeta \cdot \zeta)^1 \text{ terms} \Big\} \\ & + \left\{ s_2(s_1 s_2 s_4 + s_1 s_3 s_4 + s_1 s_3 s_5 + s_2 s_3 s_5 + s_2 s_4 s_5)(\zeta_1 \cdot k_4)(\zeta_2 \cdot \zeta_4)(\zeta_3 \cdot \zeta_5) \right. \\ & \left. \left. + 30(\zeta \cdot k)^1(\zeta \cdot \zeta)^2 \text{ terms} \right\} \right]. \quad (\text{D.19}) \end{aligned}$$

In (D.18) and (D.19) $\{s_1, s_2, s_3, s_4, s_5\}$ are the five independent Mandelstam variables that appear in massless 5-point scattering, and that were defined in Eq. (D.1).

So, our result in (D.17) indeed verifies that $\dim(\mathcal{V}_5) = 2$, in agreement with (D.15).

⁸³ These kinematical expressions are on-shell gauge invariant.

⁸⁴ If it has any usefulness to the reader, the explicit expressions of $K_1^{(5)}(\zeta, k)$ and $K_2^{(5)}(\zeta, k)$ can be requested to the authors.

Since in Appendix D.3.1 we proved that the two-element set $\mathcal{B}_5$ is linearly independent in $\mathcal{V}_5$, it is clear that a change of basis can be done, namely,

$$\{K_1^{(5)}(\zeta, k), K_2^{(5)}(\zeta, k)\} \rightarrow \{A_{YM}(1, 2, 3, 4, 5), A_{YM}(1, 3, 2, 4, 5)\}, \quad (\text{D.20})$$

and, therefore, $\mathcal{B}_5$ is a possible basis of $\mathcal{V}_5$.

So, at the end, instead of (D.17) it is possible to write

$$T(1, 2, 3, 4, 5) = \lambda^{[23]} \cdot A_{YM}(1, 2, 3, 4, 5) + \lambda^{[32]} \cdot A_{YM}(1, 3, 2, 4, 5), \quad (\text{D.21})$$

as claimed in Eq. (27) of the main text of this work.

D.4.2. Case of $N = 6$ and $N = 7$

Let $T(1, 2, 3, 4, 5, 6) \in \mathcal{V}_6$ and $T(1, 2, 3, 4, 5, 6, 7) \in \mathcal{V}_7$. Although the computational effort becomes greater as N grows, we have succeeded in finding that, repeating exactly the same procedure that we have described in Subsection 3.2.2 and in Appendix C for the $N = 4$ case, and in Appendix D.4.1 for the $N = 5$ case, we arrive that these two elements can be respectively written as

$$\begin{aligned} T(1, 2, 3, 4, 5, 6) &= \rho_1^{(6)} K_1^{(6)}(\zeta, k) + \rho_2^{(6)} K_2^{(6)}(\zeta, k) + \rho_3^{(6)} K_3^{(6)}(\zeta, k) + \rho_4^{(6)} K_4^{(6)}(\zeta, k) \\ &\quad + \rho_5^{(6)} K_5^{(6)}(\zeta, k) + \rho_6^{(6)} K_6^{(6)}(\zeta, k) \end{aligned} \quad (\text{D.22})$$

and

$$\begin{aligned} T(1, 2, 3, 4, 5, 6, 7) &= \rho_1^{(7)} K_1^{(7)}(\zeta, k) + \rho_2^{(7)} K_2^{(7)}(\zeta, k) + \rho_3^{(7)} K_3^{(7)}(\zeta, k) + \rho_4^{(7)} K_4^{(7)}(\zeta, k) \\ &\quad + \rho_5^{(7)} K_5^{(7)}(\zeta, k) + \rho_6^{(7)} K_6^{(7)}(\zeta, k) + \rho_7^{(7)} K_7^{(7)}(\zeta, k) \\ &\quad + \rho_8^{(7)} K_8^{(7)}(\zeta, k) + \rho_9^{(7)} K_9^{(7)}(\zeta, k) + \rho_{10}^{(7)} K_{10}^{(7)}(\zeta, k) \\ &\quad + \rho_{11}^{(7)} K_{11}^{(7)}(\zeta, k) + \rho_{12}^{(7)} K_{12}^{(7)}(\zeta, k) + \rho_{13}^{(7)} K_{13}^{(7)}(\zeta, k) \\ &\quad + \rho_{14}^{(7)} K_{14}^{(7)}(\zeta, k) + \rho_{15}^{(7)} K_{15}^{(7)}(\zeta, k) + \rho_{16}^{(7)} K_{16}^{(7)}(\zeta, k) \\ &\quad + \rho_{17}^{(7)} K_{17}^{(7)}(\zeta, k) + \rho_{18}^{(7)} K_{18}^{(7)}(\zeta, k) + \rho_{19}^{(7)} K_{19}^{(7)}(\zeta, k) \\ &\quad + \rho_{20}^{(7)} K_{20}^{(7)}(\zeta, k) + \rho_{21}^{(7)} K_{21}^{(7)}(\zeta, k) + \rho_{22}^{(7)} K_{22}^{(7)}(\zeta, k) \\ &\quad + \rho_{23}^{(7)} K_{23}^{(7)}(\zeta, k) + \rho_{24}^{(7)} K_{24}^{(7)}(\zeta, k), \end{aligned} \quad (\text{D.23})$$

where the $K_i^{(N)}(\zeta, k)$'s are known kinematic expressions (without poles).

In the case of $N = 6$ we have found our result in (D.22) analytically, in the same way as we did in (D.17).

In the case of $N = 7$ we have obtained our result numerically: we have found the explicit form of the $K_i^{(7)}(\zeta, k)$'s only for fixed values of the Mandelstam variables.

The important result is that, independently of being an analytic or a numerical result, we can conclude that

$$\dim(\mathcal{V}_6) = 6 \quad \text{and} \quad \dim(\mathcal{V}_7) = 24, \quad (\text{D.24})$$

in agreement with formula (D.15).

Since in Appendix D.3.2 we proved that the six-element set $\mathcal{B}_6$ and the twenty four-element set $\mathcal{B}_7$ are linearly independent in $\mathcal{V}_6$ and $\mathcal{V}_7$, respectively, due to the matching with the dimensions found in (D.24), these sets can be chosen as basis of the corresponding spaces.

Appendix E. Kinematical derivation of BCJ relations

BCJ relations were discovered by Bern, Carrasco and Johansson in 2008 by means of a conjectured duality between color and kinematics in Yang–Mills theories [37]. These relations state that the $(N - 1)!$ Yang–Mills subamplitudes that appear in the N -point formula (see Eq. (2)⁸⁵) can be written as linear combinations of a subset which contains only $(N - 3)!$ of them. Although not exactly the same, from the work in [37] it can be easily seen that the set of subamplitudes that we propose in Eq. (5) can be chosen as a basis for the Yang–Mills subamplitudes in the BCJ relations.⁸⁶

A rigorous proof of the BCJ relations was found first using string theory methods, independently, by the authors of [38] and [8]. Later, a pure field theory proof of these relations was also found in [54] (at least for $N = 4, 5, 6$).

In this appendix we present an alternative derivation of BCJ relations for $N = 4, 5, 6, 7$. We will not deal with color/kinematic duality of Yang–Mills subamplitudes at any moment. We will not need to consider Kleiss–Kuijf relations [55] either. From our perspective the result is that BCJ relations arise as a natural consequence of the basis found for $\mathcal{V}_N$, considered in Section 3 of this work. The only unknowns in the BCJ relations will then be the momentum factors that participate in them as the components of a given Yang–Mills subamplitude with respect to the $\mathcal{B}_N$ basis.

As mentioned at the end of Subsection 3.3, our proposal is that it is enough to consider the explicit expression of the $(\zeta \cdot \zeta)^1 (\zeta \cdot k)^{N-2}$ terms of the N -point Yang–Mills subamplitudes, in order to obtain the momentum factors that appear in the BCJ relations. These subamplitudes can be obtained using Feynman rules from the Yang–Mills lagrangian (in the Lorentz gauge) or by selecting them in the low energy limit of the corresponding subamplitude of gauge bosons in string theory.

We will find the BCJ relations for a set of $(N - 1)!/2 - (N - 3)!$ Yang–Mills subamplitudes. The first of these numbers, $(N - 1)!/2$, comes from considering the number of independent subamplitudes, from the point of view of cyclic and reflection symmetries (see Eqs. (B.1) and (B.2), respectively). The second one, $(N - 3)!$, is simply the number of Yang–Mills subamplitudes of the $\mathcal{B}_N$ basis, which are taken from this set.

Here, we will give the details of our derivation of the BCJ relations for the case of $N = 4$ and $N = 5$. The procedure for $N = 6$ and $N = 7$ is exactly the same one, but the intermediate formulas become too big, so we will just present some examples of these relations.

E.1. Case of $N = 4$

In this case there are $(N - 1)!/2 = 3$ subamplitudes which are not related by cyclic neither reflection symmetry. We will choose them to be $A_{YM}(1, 2, 3, 4)$, $A_{YM}(1, 3, 4, 2)$ and $A_{YM}(1, 4, 2, 3)$.⁸⁷

We will now apply Eq. (26) for $T(1, 2, 3, 4) = A_{YM}(1, 3, 4, 2)$, namely:

$$A_{YM}(1, 3, 4, 2) = \lambda^{[2]} \cdot A_{YM}(1, 2, 3, 4). \quad (\text{E.1})$$

where $\lambda^{[2]}$ is the momentum factor that we want to determine.

⁸⁵ Formula (2) was written in the Introduction as valid for gauge bosons in tree level open superstring theory, but it is well known that it is also valid for tree level Yang–Mills N -point amplitudes [51].

⁸⁶ The reason of our choice of basis is, of course, that we want to arrive to the MSS formula in Eq. (1), which is written in this basis.

⁸⁷ Notice that the first subamplitude is $A_{YM}(1, 2, 3, 4)$, that is, the one that has been chosen as a basis for $\mathcal{V}_4$ in Eq. (26).

Using the explicit expression for $A_{YM}(1, 2, 3, 4)$ given in Eq. (25) (and the corresponding one for $A_{YM}(1, 3, 4, 2)$), and afterwards only selecting the $(\zeta \cdot \zeta)^1(\zeta \cdot k)^2$ terms on them, Eq. (E.1) becomes

$$\begin{aligned}
 & -8g^2 \frac{1}{us} \left\{ \frac{1}{2} u \left[(\zeta_1 \cdot k_2)(\zeta_4 \cdot k_3)(\zeta_3 \cdot \zeta_2) + (\zeta_3 \cdot k_4)(\zeta_2 \cdot k_1)(\zeta_1 \cdot \zeta_4) \right. \right. \\
 & \quad + (\zeta_1 \cdot k_4)(\zeta_2 \cdot k_3)(\zeta_3 \cdot \zeta_4) + (\zeta_3 \cdot k_2)(\zeta_4 \cdot k_1)(\zeta_1 \cdot \zeta_3) \left. \right] \\
 & \quad + \frac{1}{2} s \left[(\zeta_3 \cdot k_1)(\zeta_2 \cdot k_4)(\zeta_4 \cdot \zeta_1) + (\zeta_4 \cdot k_2)(\zeta_1 \cdot k_3)(\zeta_3 \cdot \zeta_2) \right. \\
 & \quad + (\zeta_3 \cdot k_2)(\zeta_1 \cdot k_4)(\zeta_4 \cdot \zeta_2) + (\zeta_4 \cdot k_1)(\zeta_2 \cdot k_3)(\zeta_3 \cdot \zeta_1) \left. \right] \\
 & \quad + \frac{1}{2} t \left[(\zeta_1 \cdot k_3)(\zeta_2 \cdot k_4)(\zeta_4 \cdot \zeta_3) + (\zeta_4 \cdot k_2)(\zeta_3 \cdot k_1)(\zeta_1 \cdot \zeta_2) + (\zeta_1 \cdot k_2)(\zeta_3 \cdot k_4)(\zeta_4 \cdot \zeta_2) \right. \\
 & \quad \left. \left. + (\zeta_4 \cdot k_3)(\zeta_2 \cdot k_1)(\zeta_1 \cdot \zeta_3) \right] \right\} \\
 & = -\lambda^{[2]} 8g^2 \frac{1}{st} \left\{ \frac{1}{2} s \left[(\zeta_1 \cdot k_4)(\zeta_3 \cdot k_2)(\zeta_2 \cdot \zeta_4) + (\zeta_2 \cdot k_3)(\zeta_4 \cdot k_1)(\zeta_1 \cdot \zeta_3) \right. \right. \\
 & \quad + (\zeta_1 \cdot k_3)(\zeta_4 \cdot k_2)(\zeta_2 \cdot \zeta_3) + (\zeta_2 \cdot k_4)(\zeta_3 \cdot k_1)(\zeta_1 \cdot \zeta_4) \left. \right] \\
 & \quad + \frac{1}{2} t \left[(\zeta_2 \cdot k_1)(\zeta_4 \cdot k_3)(\zeta_3 \cdot \zeta_1) + (\zeta_3 \cdot k_4)(\zeta_1 \cdot k_2)(\zeta_2 \cdot \zeta_4) \right. \\
 & \quad + (\zeta_2 \cdot k_4)(\zeta_1 \cdot k_3)(\zeta_3 \cdot \zeta_4) + (\zeta_3 \cdot k_1)(\zeta_4 \cdot k_2)(\zeta_2 \cdot \zeta_1) \left. \right] \\
 & \quad + \frac{1}{2} u \left[(\zeta_1 \cdot k_2)(\zeta_4 \cdot k_3)(\zeta_3 \cdot \zeta_2) + (\zeta_3 \cdot k_4)(\zeta_2 \cdot k_1)(\zeta_1 \cdot \zeta_4) + (\zeta_1 \cdot k_4)(\zeta_2 \cdot k_3)(\zeta_3 \cdot \zeta_4) \right. \\
 & \quad \left. \left. + (\zeta_3 \cdot k_2)(\zeta_4 \cdot k_1)(\zeta_1 \cdot \zeta_2) \right] \right\}. \tag{E.2}
 \end{aligned}$$

Next, we write all $(\zeta \cdot k)$ terms in both sides of (E.2) in the basis mentioned in (14), by using the relations (12) for this case, namely,

$$(\zeta_4 \cdot k_3) = -(\zeta_4 \cdot k_1) - (\zeta_4 \cdot k_2), \tag{E.3}$$

$$(\zeta_1 \cdot k_4) = -(\zeta_1 \cdot k_2) - (\zeta_1 \cdot k_3), \tag{E.4}$$

$$(\zeta_2 \cdot k_4) = -(\zeta_2 \cdot k_1) - (\zeta_2 \cdot k_3), \tag{E.5}$$

$$(\zeta_3 \cdot k_4) = -(\zeta_3 \cdot k_1) - (\zeta_3 \cdot k_2). \tag{E.6}$$

Once both sides of (E.2) have been written in terms of the same basis for the $(\zeta \cdot \zeta)^1(\zeta \cdot k)^2$ terms, it is possible to compare their coefficients, arriving at a set of eight linearly dependent equations for $\lambda^{[2]}$. After using that $u = -s - t$ it can be seen that there is really only one linearly independent equation, namely, $\lambda^{[2]} = t/u$, from which it comes that

$$A_{YM}(1, 3, 4, 2) = \frac{t}{u} A_{YM}(1, 2, 3, 4). \tag{E.7}$$

Doing exactly the same procedure to find $A_{YM}(1, 4, 2, 3)$ in terms of $A_{YM}(1, 2, 3, 4)$ we find that

$$A_{YM}(1, 4, 2, 3) = \frac{s}{u} A_{YM}(1, 2, 3, 4). \tag{E.8}$$

Eqs. (E.7) and (E.8) are the independent BCJ relations for $N = 4$. Any other of the remaining twenty one 4-point subamplitudes can be written in terms of $A_{YM}(1, 2, 3, 4)$ by means of these two relations and using cyclic or/and reflection symmetries.

It is clear that in this case ($N = 4$) it would have been very much simple to just compare the coefficients of the $(\zeta \cdot \zeta)^2$ kinematical terms since there are only three of them in the whole subamplitude. The result would have been just the same one: $\lambda^{[2]} = t/u$. But, as indicated in the final paragraph of Subsection 3.3, our proposal for the N -point subamplitudes, for any value of N , is that it will always be enough to consider the $(\zeta \cdot \zeta)^1(\zeta \cdot k)^{N-2}$ terms and this is what we have confirmed here.

E.2. Case of $N = 5$

In this case there are $(N - 1)!/2 = 12$ subamplitudes which are not related by cyclic neither reflection symmetry. We will choose them to be $A_{YM}(1, 2, 3, 4, 5)$, $A_{YM}(1, 2, 4, 3, 5)$, $A_{YM}(1, 4, 2, 3, 5)$, $A_{YM}(2, 1, 3, 4, 5)$, $A_{YM}(2, 3, 1, 4, 5)$, $A_{YM}(2, 1, 4, 3, 5)$ and six additional 5-point subamplitudes obtained from the previous ones by exchanging labels 2 and 3.⁸⁸

In this subsection we will present, in some detail, the calculations that lead to the BCJ relation for the subamplitude $A_{YM}(2, 1, 3, 4, 5)$, that is, we will find the momentum factors, $\lambda^{[23]}$ and $\lambda^{[32]}$, that allow to write $A_{YM}(2, 1, 3, 4, 5)$ in terms of $A_{YM}(1, 2, 3, 4, 5)$ and $A_{YM}(1, 3, 2, 4, 5)$, according to Eq. (27):

$$A_{YM}(2, 1, 3, 4, 5) = \lambda^{[23]} A_{YM}(1, 2, 3, 4, 5) + \lambda^{[32]} A_{YM}(1, 3, 2, 4, 5). \quad (\text{E.9})$$

Using the explicit expression for the $(\zeta \cdot \zeta)^1(\zeta \cdot k)^3$ terms of $A_{YM}(1, 2, 3, 4, 5)$, given in Eq. (D.7), the corresponding terms of $A_{YM}(1, 3, 2, 4, 5)$ and $A_{YM}(2, 1, 3, 4, 5)$ can be obtained. Substituting all these $(\zeta \cdot \zeta)^1(\zeta \cdot k)^3$ terms in (E.9) we arrive to an equation which has the following form⁸⁹:

$$\begin{aligned} & 8g^3 \left[(\zeta_1 \cdot \zeta_3) \left\{ - \left(\frac{1}{s_1 s_4} + \frac{1}{(s_4 - s_1 - s_2) s_4} \right) (\zeta_2 \cdot k_1)(\zeta_4 \cdot k_2)(\zeta_5 \cdot k_3) + (9 \text{ terms}) \right\} \right. \\ & \quad + (\zeta_3 \cdot \zeta_4) \left\{ + \frac{1}{s_1 s_3} (\zeta_1 \cdot k_2)(\zeta_2 \cdot k_4)(\zeta_5 \cdot k_3) + (15 \text{ terms}) \right\} \\ & \quad \left. + (\text{cyclic permutations of indexes } (2, 1, 3, 4, 5)) \right] \\ & = \lambda^{[23]} 8g^3 \left[(\zeta_2 \cdot \zeta_3) \left\{ - \left(\frac{1}{s_1 s_4} + \frac{1}{s_2 s_4} \right) (\zeta_1 \cdot k_2)(\zeta_4 \cdot k_1)(\zeta_5 \cdot k_3) + (9 \text{ terms}) \right\} \right. \\ & \quad + (\zeta_1 \cdot \zeta_4) \left\{ - \frac{1}{s_1 s_4} (\zeta_2 \cdot k_1)(\zeta_3 \cdot k_1)(\zeta_5 \cdot k_1) + (13 \text{ terms}) \right\} \\ & \quad \left. + (\text{cyclic permutations of indexes } (1, 2, 3, 4, 5)) \right] \\ & \quad + \lambda^{[32]} 8g^3 \left[(\zeta_3 \cdot \zeta_2) \left\{ - \left(\frac{1}{(s_4 - s_1 - s_2) s_4} + \frac{1}{s_2 s_4} \right) (\zeta_1 \cdot k_3)(\zeta_4 \cdot k_1)(\zeta_5 \cdot k_2) \right. \right. \\ & \quad \left. \left. + (9 \text{ terms}) \right\} \right] \end{aligned}$$

⁸⁸ We have chosen the same set of twelve 5-point subamplitudes of Ref. [38].

⁸⁹ See the convention for the Mandelstam variables s_i in Eq. (D.1) of Appendix D.

$$\begin{aligned}
& + (\zeta_1 \cdot \zeta_4) \left\{ -\frac{1}{(s_4 - s_1 - s_2)s_4} (\zeta_3 \cdot k_1)(\zeta_2 \cdot k_1)(\zeta_5 \cdot k_1) + (13 \text{ terms}) \right\} \\
& + (\text{cyclic permutations of indexes } (1, 3, 2, 4, 5)) \Big]. \tag{E.10}
\end{aligned}$$

In this equation we have preferred to present explicitly only some of the $(\zeta \cdot \zeta)^1(\zeta \cdot k)^3$ terms in order to save some space on it.⁹⁰

The peculiar factor $(s_4 - s_1 - s_2)$ in some of the denominators of Eq. (E.10) simply corresponds to $k_1 \cdot k_3$, as given in Eq. (D.1).

Writing all $(\zeta \cdot k)$ terms in both sides of (E.10) in the basis mentioned in (14), once again, by appropriately using the relations (12), we arrive to a set of 142 linearly dependent equations,⁹¹ finding $\lambda^{[23]}(\alpha') = (s_5 - s_3)/(s_3 - s_1 - s_5)$ and $\lambda^{[32]}(\alpha') = (s_5 - s_2 - s_3)/(s_3 - s_1 - s_5)$ as a consistent unique solution of it (as expected). Therefore, Eq. (E.9) becomes

$$\begin{aligned}
A_{YM}(2, 1, 3, 4, 5) &= \frac{s_5 - s_3}{s_3 - s_1 - s_5} A_{YM}(1, 2, 3, 4, 5) \\
&+ \frac{s_5 - s_2 - s_3}{s_3 - s_1 - s_5} A_{YM}(1, 3, 2, 4, 5). \tag{E.11}
\end{aligned}$$

Doing this same procedure we arrive to the following additional relations:

$$\begin{aligned}
A_{YM}(1, 2, 4, 3, 5) &= \frac{s_4 - s_1}{s_1 - s_3 - s_4} A_{YM}(1, 2, 3, 4, 5) \\
&+ \frac{s_4 - s_1 - s_2}{s_1 - s_3 - s_4} A_{YM}(1, 3, 2, 4, 5), \tag{E.12}
\end{aligned}$$

$$\begin{aligned}
A_{YM}(2, 3, 1, 4, 5) &= -\frac{s_1 s_3}{(s_3 - s_1 - s_5)(s_2 - s_4 - s_5)} A_{YM}(1, 2, 3, 4, 5) \\
&+ \frac{(s_1 + s_5)(s_5 - s_2 - s_3)}{(s_3 - s_1 - s_5)(s_2 - s_4 - s_5)} A_{YM}(1, 3, 2, 4, 5), \tag{E.13}
\end{aligned}$$

$$\begin{aligned}
A_{YM}(1, 4, 2, 3, 5) &= -\frac{s_1 s_3}{(s_1 - s_3 - s_4)(s_2 - s_4 - s_5)} A_{YM}(1, 2, 3, 4, 5) \\
&+ \frac{(s_3 + s_4)(s_4 - s_1 - s_2)}{(s_1 - s_3 - s_4)(s_2 - s_4 - s_5)} A_{YM}(1, 3, 2, 4, 5), \tag{E.14}
\end{aligned}$$

$$\begin{aligned}
A_{YM}(2, 1, 4, 3, 5) &= \frac{p(s_1, s_2, s_3, s_4, s_5)}{(s_1 - s_3 - s_4)(s_2 - s_4 - s_5)(s_3 - s_1 - s_5)} A_{YM}(1, 2, 3, 4, 5) \\
&- \frac{(s_4 + s_5)(s_4 - s_1 - s_2)(s_5 - s_2 - s_3)}{(s_1 - s_3 - s_4)(s_2 - s_4 - s_5)(s_3 - s_1 - s_5)} A_{YM}(1, 3, 2, 4, 5), \tag{E.15}
\end{aligned}$$

where the polynomial appearing in (E.15) is given by

$$\begin{aligned}
p(s_1, s_2, s_3, s_4, s_5) &= s_3 s_4^2 - s_3 s_2 s_4 + s_3 s_5 s_4 - s_3 s_5 s_1 - s_3 s_4 s_1 + s_1 s_5 s_4 - s_1 s_5 s_2 \\
&+ s_5 s_2 s_4 - s_5^2 s_4 + s_5^2 s_1 - s_5 s_4^2. \tag{E.16}
\end{aligned}$$

⁹⁰ It is clear, from the expression of the $(\zeta \cdot \zeta)^1(\zeta \cdot k)^3$ terms of $A_{YM}(1, 2, 3, 4, 5)$, given in Eq. (D.7), that we could have written explicitly *all* those kinematic terms in Eq. (E.10).

⁹¹ This set comes from demanding that the same 142 $(\zeta \cdot \zeta)^1(\zeta \cdot k)^3$ considered in Appendix D, in Eq. (D.12), are linearly independent.

After using the expressions for the momentum products $k_i \cdot k_j$, given in Eqs. (D.1) and (D.4) of Appendix D, it can be verified that all five relations agree with the BCJ ones found in Eq. (13) of Ref. [38].⁹²

The BCJ relations for the remaining subamplitudes can be obtained from Eqs. (E.11)–(E.15) by just exchanging labels 2 and 3.⁹³

This achieves writing ten of the twelve subamplitudes, mentioned at the beginning of this section of Appendix E, in terms of the remaining two ones.

E.3. Case of $N = 6$

In this case there are $(N - 1)!/2 = 60$ subamplitudes which are not related by cyclic neither reflection symmetry. The outcome of our procedure is that we have succeeded in finding an expression for 54 of them in terms of the six ones of $\mathcal{B}_6$, as predicted in Eq. (28).

We have found that these 54 relations can be classified into two types. In the first type, 42 subamplitudes are in fact given as a linear combination of the six ones of $\mathcal{B}_6$, like for example⁹⁴:

$$\begin{aligned}
 & A_{YM}(1, 4, 5, 3, 2, 6) \\
 &= -\frac{s_1(-t_1 + s_2 + t_3 - s_6)}{(-s_1 + t_3 - s_6)(-t_1 + s_2 - s_6 + s_4)} A_{YM}(1, 2, 3, 4, 5, 6) \\
 &+ \frac{(t_1 - s_1 - s_2)(s_2 + t_3 - s_6)}{(-s_1 + t_3 - s_6)(-t_1 + s_2 - s_6 + s_4)} A_{YM}(1, 3, 2, 4, 5, 6) \\
 &+ \frac{s_1(t_1 - s_2 + s_3 - t_3 + s_6)}{(-s_1 + t_3 - s_6)(-t_1 + s_2 - s_6 + s_4)} A_{YM}(1, 2, 4, 3, 5, 6) \\
 &+ \frac{(s_2 - t_2 + s_3 - s_1)(t_3 - s_6 - t_1 + s_2 - s_3)}{(-s_1 + t_3 - s_6)(-t_1 + s_2 - s_6 + s_4)} A_{YM}(1, 3, 4, 2, 5, 6) \\
 &- \frac{(-t_3 - t_2 + s_6 + s_3)(t_1 - s_1 - s_2)}{(-s_1 + t_3 - s_6)(-t_1 + s_2 - s_6 + s_4)} A_{YM}(1, 4, 2, 3, 5, 6) \\
 &- \frac{(-t_3 - t_2 + s_6 + s_3)(t_1 - s_1 - s_2 + s_3)}{(-s_1 + t_3 - s_6)(-t_1 + s_2 - s_6 + s_4)} A_{YM}(1, 4, 3, 2, 5, 6), \tag{E.17} \\
 &A_{YM}(1, 5, 4, 3, 2, 6) \\
 &= \frac{s_1(s_2 + t_3 - s_6 - t_1)(s_5 + s_6 - s_4 - t_2)}{(s_1 + s_5 - t_2)(s_1 + s_6 - t_3)(s_6 + t_1 - s_2 - s_4)} A_{YM}(1, 2, 3, 4, 5, 6) \\
 &+ \frac{(s_1 + s_2 - t_1)(s_6 - s_2 - t_3)(s_5 + s_6 - s_4 - t_2)}{(s_1 + s_5 - t_2)(s_1 + s_6 - t_3)(s_6 + t_1 - s_2 - s_4)} A_{YM}(1, 3, 2, 4, 5, 6) \\
 &+ \frac{s_1(s_3 + s_4 - t_3)(t_1 + t_2 - s_2 - s_5)}{(s_1 + s_5 - t_2)(s_1 + s_6 - t_3)(s_6 + t_1 - s_2 - s_4)} A_{YM}(1, 2, 4, 3, 5, 6)
 \end{aligned}$$

⁹² Since in Ref. [38] those authors work with open string theory subamplitudes, in order to see the matching of our results with theirs it should be taken the low energy limit of their relations.

⁹³ In order for these relations to have the same format as the first ones, the s_i Mandelstam variables should first be rewritten in terms of momentum products $k_i \cdot k_j$, then exchange labels 2 and 3 and then write the final expression in terms of the Mandelstam variables again.

⁹⁴ See Eq. (D.2) for the definition of the nine independent Mandelstam variables that appear in the 6-point scattering process: there are six s_i 's and three t_j 's.

$$\begin{aligned}
& + \frac{(s_1 + s_2 - t_1)(s_4 + t_2 - s_5 - s_6)(t_2 + t_3 - s_3 - s_6)}{(s_1 + s_5 - t_2)(s_1 + s_6 - t_3)(s_6 + t_1 - s_2 - s_4)} A_{YM}(1, 3, 4, 2, 5, 6) \\
& + \frac{(s_3 + s_4 - t_3)(s_2 + s_5 - t_1 - t_2)(s_1 + t_2 - s_2 - s_3)}{(s_1 + s_5 - t_2)(s_1 + s_6 - t_3)(s_6 + t_1 - s_2 - s_4)} A_{YM}(1, 4, 2, 3, 5, 6) \\
& + \frac{(s_6 + s_1 - s_3 - s_4)(t_1 + t_2 - s_2 - s_5)(t_2 + t_3 - s_3 - s_6)}{(s_1 + s_5 - t_2)(s_1 + s_6 - t_3)(s_6 + t_1 - s_2 - s_4)} A_{YM}(1, 4, 3, 2, 5, 6).
\end{aligned} \tag{E.18}$$

In the second type, 12 subamplitudes are found to be given only in terms of three subamplitudes of $\mathcal{B}_6$, like for example⁹⁵:

$$\begin{aligned}
& A_{YM}(1, 2, 3, 5, 4, 6) \\
& = \frac{t_1 - s_5}{t_1 - s_5 - s_4} A_{YM}(1, 2, 3, 4, 5, 6) - \frac{t_1 - s_5 + s_3}{t_1 - s_5 - s_4} A_{YM}(1, 2, 4, 3, 5, 6) \\
& + \frac{s_5 - t_1 - t_2 + s_2}{-s_5 - s_4 + t_1} A(1, 3, 4, 2, 5, 6),
\end{aligned} \tag{E.19}$$

$$\begin{aligned}
& A_{YM}(1, 2, 4, 5, 3, 6) \\
& = \frac{t_1 - s_1}{s_1 - t_3 + s_4 - t_1} A_{YM}(1, 2, 3, 4, 5, 6) + \frac{t_1 - s_1 - s_2}{s_1 - t_3 + s_4 - t_1} A_{YM}(1, 3, 2, 4, 5, 6) \\
& + \frac{s_3 + t_1 - s_1}{s_1 - t_3 + s_4 - t_1} A_{YM}(1, 2, 4, 3, 5, 6),
\end{aligned} \tag{E.20}$$

$$\begin{aligned}
& A_{YM}(1, 4, 3, 5, 2, 6) \\
& = -\frac{s_1}{s_1 - t_3 + s_6} A_{YM}(1, 2, 4, 3, 5, 6) - \frac{s_1 - s_3 + t_2}{s_1 - t_3 + s_6} A_{YM}(1, 4, 3, 2, 5, 6) \\
& + \frac{s_2 + s_3 - s_1 - t_2}{s_1 - t_3 + s_6} A(1, 3, 4, 2, 5, 6).
\end{aligned} \tag{E.21}$$

We have checked that all our $N = 6$ results agree with the ones in Ref. [56] (which were obtained in detail using the BCJ color/kinematic duality).

E.4. Case of $N = 7$

In this case there are $(N - 1)!/2 = 360$ subamplitudes which are not related by cyclic neither reflection symmetry. This time we have found 336 of them in terms of the twenty four ones of $\mathcal{B}_7$, as predicted in Eq. (29).

The 336 BCJ relations that we have found can be classified into three types. In the first type, 216 subamplitudes are in fact given as a linear combination of the twenty four ones of $\mathcal{B}_7$. In the second type, 72 subamplitudes are given only in terms of twelve of the ones of $\mathcal{B}_7$. And in the third type, 48 subamplitudes are given only in terms of four of the ones of $\mathcal{B}_7$.⁹⁶

A few examples of the third type of BCJ relations are the following⁹⁷:

⁹⁵ Notice, in Eqs. (E.19), (E.20) and (E.21), that the three subamplitudes of $\mathcal{B}_6$ that appear in them are not necessarily all the same.

⁹⁶ Because of computer limitations, we have arrived to these results numerically, for the first two types of BCJ relations, and analytically for the last ones. See Eqs. (E.22), (E.23) and (E.24).

⁹⁷ See Eq. (D.3) for the definition of the fourteen independent Mandelstam variables that appear in the 7-point scattering process: there are seven s_i 's and seven t_j 's.

$$\begin{aligned}
& A_{YM}(1, 2, 3, 4, 6, 5, 7) \\
&= \frac{t_5 - s_6}{s_5 + s_6 - t_5} A_{YM}(1, 2, 3, 4, 5, 6, 7) + \frac{s_4 + t_5 - s_6}{s_5 + s_6 - t_5} A_{YM}(1, 2, 3, 5, 4, 6, 7) \\
&+ \frac{t_3 + t_5 - s_3 - s_6}{s_5 + s_6 - t_5} A_{YM}(1, 2, 4, 5, 3, 6, 7) + \frac{t_5 + t_6 - s_6 - t_2}{s_5 + s_6 - t_5} A_{YM}(1, 3, 4, 5, 2, 6, 7),
\end{aligned} \tag{E.22}$$

$$\begin{aligned}
& A_{YM}(1, 2, 3, 5, 6, 4, 7) \\
&= \frac{t_5 - t_1}{s_5 + t_1 - t_4 - t_5} A_{YM}(1, 2, 3, 4, 5, 6, 7) + \frac{s_4 + t_5 - t_1}{s_5 + t_1 - t_4 - t_5} A_{YM}(1, 2, 3, 5, 4, 6, 7) \\
&+ \frac{t_5 - s_3 - t_1}{s_5 + t_1 - t_4 - t_5} A_{YM}(1, 2, 4, 3, 5, 6, 7) + \frac{s_2 + t_5 - t_1 - t_2}{s_5 + t_1 - t_4 - t_5} A_{YM}(1, 3, 4, 2, 5, 6, 7),
\end{aligned} \tag{E.23}$$

$$\begin{aligned}
& A_{YM}(1, 2, 4, 3, 6, 5, 7) \\
&= \frac{t_5 - s_6}{s_5 + s_6 - t_5} A_{YM}(1, 2, 4, 3, 5, 6, 7) + \frac{t_3 + t_5 - s_3 - s_4 - s_6}{s_5 + s_6 - t_5} A_{YM}(1, 2, 5, 3, 4, 6, 7) \\
&+ \frac{t_3 + t_5 - s_3 - s_6}{s_5 + s_6 - t_5} A_{YM}(1, 2, 5, 4, 3, 6, 7) + \frac{t_5 + t_6 - s_6 - t_2}{s_5 + s_6 - t_5} A_{YM}(1, 3, 5, 4, 2, 6, 7).
\end{aligned} \tag{E.24}$$

Our confidence in the correctness of these (and the remaining 45) relations is the fact that the linear system that we have found for the momentum factors in each of them, following the procedure of Subsection 3.3, is overdetermined, and we have found a unique and consistent solution for it.

Appendix F. Momentum factors in the open superstring formula

In this appendix we will consider the α' expansion of the momentum factors that appear in the N -point open superstring formula. These are the $F^{(\sigma_N)}(\alpha')$'s that appear in Eq. (1).

In Appendix F.1 we will consider the momentum factor that appears in the case of $N = 4$ and in Appendix F.2 we will consider the case of $N \geq 5$.

F.1. Gamma factor

As mentioned in [57], using the Taylor expansion for $\ln \Gamma(1 + z)$,⁹⁸

$$\ln \Gamma(1 + z) = -\gamma z + \sum_{k=2}^{\infty} (-1)^k \frac{\zeta(k)}{k} z^k \quad (-1 < z \leq 1), \tag{F.1}$$

it may be proved that the explicit α' expansion for the gamma factor is given by

$$F^{(2)}(\alpha') = \alpha'^2 s t \frac{\Gamma(\alpha' s) \Gamma(\alpha' t)}{\Gamma(1 + \alpha' s + \alpha' t)} = \exp \left\{ \sum_{k=2}^{\infty} \frac{\zeta(k)}{k} \alpha'^k ((s + t)^k - s^k - t^k) \right\}. \tag{F.2}$$

⁹⁸ See formula (10.44c) of [58], for example.

So, up to $\mathcal{O}(\alpha'^6)$ terms we have that

$$\begin{aligned}
 F^{(2)}(\alpha') = & 1 - \frac{\pi^2}{6} st \alpha'^2 + \zeta(3)(s+t)st \alpha'^3 - \frac{\pi^4}{360}(4s^2 + st + 4t^2)st \alpha'^4 \\
 & - \left[\frac{\pi^2}{6} \zeta(3)s^2 t^2 (s+t) - \zeta(5)(s^3 + 2s^2 t + 2st^2 + t^3)st \right] \alpha'^5 \\
 & + \left[\frac{1}{2} \zeta(3)^2 s^2 t^2 (s+t)^2 \right. \\
 & \left. - \frac{\pi^6}{15 \cdot 120} (16s^4 + 12s^3 t + 23s^2 t^2 + 12st^3 + 16t^4)st \right] \alpha'^6 \\
 & + \mathcal{O}(\alpha'^7).
 \end{aligned} \tag{F.3}$$

F.2. $N = 5$ and higher N -point momentum factors

In Appendix F.2.1 we make a short review about MZV 's. We present enough material for the reader to become aware of the *non-trivial* MZV 's that we mentioned in Section 2 of the main body of this work ($\zeta(3, 5)$, $\zeta(3, 7)$, $\zeta(3, 3, 5)$, etc.).

Then, in Appendix F.2.2 we briefly review the structure of the α' expansion of the N -point momentum factors (for $N \geq 5$).⁹⁹ Here it will be seen that the *non-trivial* MZV 's only show up for the first time at α'^8 order.

F.2.1. Multiple zeta values (MZV 's)

MZV 's are defined as

$$\zeta(n_1, \dots, n_r) := \sum_{0 < k_1 < \dots < k_r} \prod_{l=1}^r \frac{1}{k_l^{n_l}}, \tag{F.4}$$

where all the n_l 's are positive integers and the last one of the list, n_r , should be greater than 1.

In (F.4),

$$r \quad \text{and} \quad w = \sum_{l=1}^r n_l \tag{F.5}$$

are called, respectively, the ‘depth’ and the ‘weight’ (or the transcendentality) of $\zeta(n_1, \dots, n_r)$.

In the case of $r = 1$, (F.4) becomes a single zeta value, that is, Riemann's zeta function evaluated at an integer $n \geq 2$, $\zeta(n)$. In this case the weight is $w = n$.

The following are a few examples of (F.4) for $r = 2$ and weights 3, 4 and 5, respectively:

$$\zeta(1, 2) := \sum_{k_2=2}^{\infty} \sum_{k_1=1}^{k_2-1} \frac{1}{k_1^1 k_2^2} = \zeta(3), \tag{F.6}$$

$$\zeta(2, 2) := \sum_{k_2=2}^{\infty} \sum_{k_1=1}^{k_2-1} \frac{1}{k_1^2 k_2^2} = \frac{3}{10} \zeta(2)^2, \tag{F.7}$$

⁹⁹ This subsection has been taken literally (except for some differences in the notation) from part of Section 3.2 of Ref. [14].

$$\zeta(1, 4) := \sum_{k_2=2}^{\infty} \sum_{k_1=1}^{k_2-1} \frac{1}{k_1^1 k_2^4} = 2\zeta(5) - \zeta(2)\zeta(3). \quad (\text{F.8})$$

In order to compute these double zeta values and arrive to the expressions that we have written in these equations, in terms of single zeta ones, harmonic sums and its algebraic property [25] can be used, for example.

An extremely important data mine of proven results for MZV's is Ref. [42]. Higher order coefficients of momentum factor α' expansions in string theory, for $N \geq 5$, are based on the MZV's basis of this reference (see [14] and [28], for example).

For $w \leq 7$ it is always possible to write any MZV as a rational linear combination of products of $\zeta(n)$'s, where $n \leq w$ (as in (F.6), (F.7) and (F.8), for example). But already at $w = 8$ there is a first MZV (which is present in the MZV basis of Ref. [42]), namely $\zeta(3, 5)$, which is believed to not admit such sort of expression [41].

Other MZV's, for higher weights, which are also believed that cannot be written as a rational linear combinations of products of $\zeta(n)$'s, and which are present in the MZV basis of Ref. [42], are $\zeta(3, 7)$, $\zeta(3, 3, 5)$, $\zeta(3, 9)$ and $\zeta(1, 1, 4, 6)$. Section 2 of Ref. [14] contains tables in which these type of MZV's can be found up to $w = 16$. We recall that in Section 2 of the main body of the present work we referred to these peculiar MZV's as “non-trivial MZV's”.

F.2.2. Structure of the α' expansion of the N -point momentum factors ($N \geq 5$)

Following [14], we will first consider the case $N = 5$.

From Eq. (44) we can write that

$$A_b(1, 2, 3, 4, 5) = F^{\{23\}}(\alpha') A_{YM}(1, 2, 3, 4, 5) + F^{\{32\}}(\alpha') A_{YM}(1, 3, 2, 4, 5), \quad (\text{F.9})$$

$$A_b(1, 3, 2, 4, 5) = \tilde{F}^{\{23\}}(\alpha') A_{YM}(1, 3, 2, 4, 5) + \tilde{F}^{\{32\}}(\alpha') A_{YM}(1, 2, 3, 4, 5), \quad (\text{F.10})$$

where Eq. (F.9) is literally the same one written in (44) and Eq. (F.10) has been obtained from the first one by interchanging indexes 2 and 3, therefore

$$\tilde{F}^{\{23\}}(\alpha') = F^{\{23\}}(\alpha')|_{2 \leftrightarrow 3}, \quad \tilde{F}^{\{32\}}(\alpha') = F^{\{32\}}(\alpha')|_{2 \leftrightarrow 3}. \quad (\text{F.11})$$

In matrix notation, Eqs. (F.9) and (F.10) can be written as

$$\vec{A} = \mathbb{F} \vec{A}_{YM}, \quad (\text{F.12})$$

where $\vec{A}$ and $\vec{A}_{YM}$ are the 2-component vectors given by

$$\vec{A} = \begin{pmatrix} A_b(1, 2, 3, 4, 5) \\ A_b(1, 3, 2, 4, 5) \end{pmatrix}, \quad \vec{A}_{YM} = \begin{pmatrix} A_{YM}(1, 2, 3, 4, 5) \\ A_{YM}(1, 3, 2, 4, 5) \end{pmatrix}, \quad (\text{F.13})$$

and where $\mathbb{F}$ is a 2×2 matrix given by

$$\mathbb{F} = \begin{pmatrix} F^{\{23\}}(\alpha') & F^{\{32\}}(\alpha') \\ \tilde{F}^{\{32\}}(\alpha') & \tilde{F}^{\{23\}}(\alpha') \end{pmatrix}. \quad (\text{F.14})$$

In Ref. [14] the authors propose that the α' expansion of $\mathbb{F}$ can be decomposed in the following way¹⁰⁰:

¹⁰⁰ We will not explain here what is the prescription for the ordering colons, ‘: ... :’, in Eq. (F.15). The interested reader can find this detail in Eq. (3.18) of Ref. [14].

$$\mathbb{F} = \left\{ \mathbb{I} + \sum_{n=1}^{\infty} \mathbb{P}_{2n} \zeta(2)^n \alpha'^{2n} \right\} \mathbb{Q} : \exp \left\{ \sum_{n=1}^{\infty} \mathbb{M}_{2n+1} \zeta(2n+1) \alpha'^{2n+1} \right\}, \quad (\text{F.15})$$

where the $\mathbb{P}_{2n}$'s, $\mathbb{Q}$ and the $\mathbb{M}_{2n+1}$'s are also 2×2 matrices,¹⁰¹ given by

$$\mathbb{P}_{2n} = \mathbb{F}|_{\zeta(2n)\alpha'^{2n}}, \quad (\text{F.16})$$

$$\mathbb{Q} = \mathbb{I} + \sum_{n=8}^{\infty} \mathbb{Q}_n \alpha'^n, \quad (\text{F.17})$$

$$\mathbb{M}_{2n+1} = \mathbb{F}|_{\zeta(2n+1)\alpha'^{2n+1}}. \quad (\text{F.18})$$

So, from Eqs. (F.16) and (F.18) we see that the $\mathbb{P}_{2n}$'s and the $\mathbb{M}_{2n+1}$'s matrices contain the dependence of $\mathbb{F}$ in the 5-point Mandelstam variables that we have specified in Eq. (D.1).

In (F.17) the $\mathbb{Q}_n$'s are 2×2 matrices given in terms of $w = n$ MZV's and commutators of the $\mathbb{M}_{2r+1}$ matrices:

$$\begin{aligned} \mathbb{Q}_8 &= \frac{1}{5} \zeta(3, 5) [\mathbb{M}_5, \mathbb{M}_3], & \mathbb{Q}_9 &= \mathbb{O}, \\ \mathbb{Q}_{10} &= \left\{ \frac{3}{14} \zeta(5)^2 + \frac{1}{14} \zeta(3, 7) \right\} [\mathbb{M}_7, \mathbb{M}_3], & \text{etc.} \end{aligned} \quad (\text{F.19})$$

In Section 3.2 of [14] the authors give explicit expressions for $\mathbb{Q}_{11}, \mathbb{Q}_{12}, \dots, \mathbb{Q}_{16}$. Higher weight *non-trivial* MZV's (like $\zeta(3, 3, 5)$, $\zeta(3, 9)$, $\zeta(1, 1, 4, 6)$, and many others) show up on these $\mathbb{Q}_n$'s.

So we see that, in the conjectured relation in (F.15), the *non-trivial* MZV's are *all* contained in matrix $\mathbb{Q}$ and that, due to expression (F.17), these *non-trivial* MZV's arise, for the first time, only at weight $w = 8$.

The authors of [14] have tested their conjecture in (F.15) up to α'^{16} order. The authors of [15] have gone further, having checked this conjecture up to α'^{21} order.

Now, for the case of $N > 5$, the authors of [14] conjectured that the $(N-3)! \times (N-3)!$ generalization of matrix $\mathbb{F}^{102}$ in (F.14), satisfies exactly *the same* conjecture in Eq. (F.15), but for the corresponding $\mathbb{P}_{2n}$, $\mathbb{Q}$ and $\mathbb{M}_{2n+1}$ matrices. The definitions in (F.16), (F.17) and (F.18) will now lead to $(N-3)! \times (N-3)!$ matrix expressions for them, and these expressions now depend on the N -point Mandelstam variables (because $\mathbb{F}$ does so).

The authors of [15] have tested this last conjecture up to α'^9 order for $N = 6$ and up to α'^7 order for $N = 7$.

F.3. Unitarity relations for the momentum factors

F.3.1. Case of $N = 5$

Here we give the details of the proof of the relation between the 5- and the 4-point momentum factors, $F^{\{23\}}(\alpha')$ and $F^{\{2\}}(\alpha')$, respectively (see Eq. (83)).

In the $N = 5$ case there are five independent Mandelstam variables (which are identified with the poles of the amplitude) and, therefore, there are five unitarity relations like the one in Eq. (67). We will give the details of the unitarity relation of $A_b(1, 2, 3, 4, 5)$ with respect to its pole α_{12} .

¹⁰¹ In Eqs. (F.15) and (F.17) $\mathbb{I}$ denotes the 2×2 identity matrix.

¹⁰² It is clear from MSS formula in Eq. (1), that the matrix generalization of (F.12) will contain an $(N-3)! \times (N-3)!$ matrix $\mathbb{F}$ and $(N-3)!$ -component vectors $\vec{A}$ and $\vec{A}_{YM}$.

According to Eq. (67), when $\alpha_{12} \rightarrow 0$, in this case we have that

$$A(\zeta_1, k_1; \zeta_2, k_2; \dots; \zeta_5, k_5) \sim \frac{1}{\alpha_{12}} V_{(12)}^\mu \frac{\partial}{\partial \zeta^\mu} A(\zeta, k_1 + k_2; \zeta_3, k_3; \zeta_4, k_4; \zeta_5, k_5), \quad (\text{F.20})$$

$$A_{YM}(\zeta_1, k_1; \zeta_2, k_2; \dots; \zeta_5, k_5) \sim \frac{1}{\alpha_{12}} V_{(12)}^\mu \frac{\partial}{\partial \zeta^\mu} A_{YM}(\zeta, k_1 + k_2; \zeta_3, k_3; \zeta_4, k_4; \zeta_5, k_5), \quad (\text{F.21})$$

where the expression for $V_{(12)}^\mu$ is given in Eq. (68). We have made explicit the case of Yang–Mills subamplitudes in (F.21) because we will soon need this relation.

It is understood that the limit $\alpha_{12} \rightarrow 0$ also is being taken in the 4-point subamplitudes on the right-hand side of Eqs. (F.20) and (F.21).

The proof of (83) goes as follows. Using Eq. (34), we may write the 4-point amplitude of the right-hand side of (F.20) as

$$\begin{aligned} & A(\zeta, k_1 + k_2; \zeta_3, k_3; \zeta_4, k_4; \zeta_5, k_5) \\ &= F^{[2]}[2(k_1 + k_2) \cdot k_3, 2k_3 \cdot k_4; \alpha'] A_{YM}(\zeta, k_1 + k_2; \zeta_3, k_3; \zeta_4, k_4; \zeta_5, k_5), \end{aligned} \quad (\text{F.22})$$

where, after using Eqs. (D.1) and (D.4), we may prove that $2(k_1 + k_2) \cdot k_3 = 2(k_4 \cdot k_5 - k_1 \cdot k_2)$, and since we are considering the limit $\alpha_{12} \rightarrow 0$, we have that

$$2(k_1 + k_2) \cdot k_3 \rightarrow \alpha_{45} \quad (\text{F.23})$$

and in (F.22) we are left with

$$\begin{aligned} & A(\zeta, k_1 + k_2; \zeta_3, k_3; \zeta_4, k_4; \zeta_5, k_5) \\ &= F^{[2]}[\alpha_{45}, \alpha_{34}; \alpha'] A_{YM}(\zeta, k_1 + k_2; \zeta_3, k_3; \zeta_4, k_4; \zeta_5, k_5). \end{aligned} \quad (\text{F.24})$$

Substituting (F.24) in (F.20) we have that

$$\begin{aligned} & A(\zeta_1, k_1; \zeta_2, k_2; \dots; \zeta_5, k_5) \\ & \sim F^{[2]}[\alpha_{45}, \alpha_{34}; \alpha'] \left\{ \frac{1}{\alpha_{12}} V_{(12)}^\mu \frac{\partial}{\partial \zeta^\mu} A_{YM}(\zeta, k_1 + k_2; \zeta_3, k_3; \zeta_4, k_4; \zeta_5, k_5) \right\}, \end{aligned} \quad (\text{F.25})$$

and identifying the term in the curly brackets (in the $\alpha_{12} \rightarrow 0$ limit, which is being taken) as the Yang–Mills 5-point subamplitude (see Eq. (F.21)) we finally arrive at

$$A(\zeta_1, k_1; \zeta_2, k_2; \dots; \zeta_5, k_5) \sim F^{[2]}[\alpha_{45}, \alpha_{34}; \alpha'] A_{YM}(\zeta_1, k_1; \zeta_2, k_2; \dots; \zeta_5, k_5), \quad (\text{F.26})$$

or even using the simpler notation,

$$A_b(1, 2, 3, 4, 5) \sim F^{[2]}[\alpha_{45}, \alpha_{34}; \alpha'] A_{YM}(1, 2, 3, 4, 5). \quad (\text{F.27})$$

This relation is to be compared with the one in (74). Taking there the $\alpha_{12} \rightarrow 0$ limit, the leading divergent term in the right-hand side comes from $F^{[23]}(\alpha') A_{YM}(1, 2, 3, 4, 5)$.¹⁰³ Comparing this leading term with the one in (F.27) we finally arrive at Eq. (83).

¹⁰³ The other term, $F^{[32]}(\alpha') A_{YM}(1, 3, 2, 4, 5)$, remains finite when this limit is taken because it does not have any pole at $\alpha_{12} = 0$.

F.3.2. Case of $N = 6$

Here we give the details of the proof of the relation between two of the 6 and the two 5-point momentum factors, $\{F^{[234]}(\alpha'), F^{[234]}(\alpha')\}$ and $\{F^{[23]}(\alpha'), F^{[32]}(\alpha')\}$, respectively (see Eqs. (109) and (110)). We also give here the details of the proof of the relation between two of the 6 and the 4-point momentum factors, $\{F^{[234]}(\alpha'), F^{[324]}(\alpha')\}$ and $F^{[2]}(\alpha')$, respectively (see Eq. (111)).

In the $N = 6$ case there are nine independent Mandelstam variables (which are identified with the poles of the amplitude). Six of these variables (the α_{ij} 's) have to do with the poles coming from two adjacent legs and the other three Mandelstam variables (the β_{ijk} 's) have to do with the poles that come from three adjacent legs. Therefore, the first six unitarity relations are like the one in Eq. (67) and the remaining three ones are like the one in Eq. (62), with $m = 4$.

Here we will give the details of the unitarity relations of $A_b(1, 2, 3, 4, 5, 6)$ with respect to its α_{12} pole and of the one with respect to its β_{123} pole.

(i) Case of $\alpha_{12} \rightarrow 0$

According to Eq. (67), when $\alpha_{12} \rightarrow 0$, in this case we have that¹⁰⁴

$$A_b(1, 2, 3, 4, 5, 6) \sim \frac{1}{\alpha_{12}} V_{(12)}^\mu \frac{\partial}{\partial \zeta^\mu} A(\zeta, k_1 + k_2; \zeta_3, k_3; \zeta_4, k_4; \zeta_5, k_5; \zeta_6, k_6), \quad (\text{F.28})$$

$$A_{YM}(1, 2, 3, 4, 5, 6) \sim \frac{1}{\alpha_{12}} V_{(12)}^\mu \frac{\partial}{\partial \zeta^\mu} A_{YM}(\zeta, k_1 + k_2; \zeta_3, k_3; \zeta_4, k_4; \zeta_5, k_5; \zeta_6, k_6), \quad (\text{F.29})$$

$$A_{YM}(1, 2, 4, 3, 5, 6) \sim \frac{1}{\alpha_{12}} V_{(12)}^\mu \frac{\partial}{\partial \zeta^\mu} A_{YM}(\zeta, k_1 + k_2; \zeta_4, k_4; \zeta_3, k_3; \zeta_5, k_5; \zeta_6, k_6), \quad (\text{F.30})$$

where the expression for $V_{(12)}^\mu$ is given in Eq. (68). We have made explicit the case of Yang–Mills subamplitudes, in (F.29) and (F.30), because we will soon need these relations.

It is understood that the limit $\alpha_{12} \rightarrow 0$ also is being taken in the 5-point subamplitudes on the right-hand side of Eqs. (F.28), (F.29) and (F.30).

Using Eq. (44), we may write the 5-point amplitude of the right-hand side of (F.28) as

$$\begin{aligned} & A(\zeta, k_1 + k_2; \zeta_3, k_3; \dots; \zeta_6, k_6) \\ &= F^{[23]} [2(k_1 + k_2) \cdot k_3, 2k_3 \cdot k_4, 2k_4 \cdot k_5, 2k_5 \cdot k_6, 2k_6 \cdot (k_1 + k_2); \alpha'] \\ & \quad \times A_{YM}(\zeta, k_1 + k_2; \zeta_3, k_3; \zeta_4, k_4; \zeta_5, k_5; \zeta_6, k_6) \\ & \quad + F^{[32]} [2(k_1 + k_2) \cdot k_4, 2k_3 \cdot k_4, 2k_3 \cdot k_5, 2k_5 \cdot k_6, 2k_6 \cdot (k_1 + k_2); \alpha'] \\ & \quad \times A_{YM}(\zeta, k_1 + k_2; \zeta_4, k_4; \zeta_3, k_3; \zeta_5, k_5; \zeta_6, k_6), \end{aligned} \quad (\text{F.31})$$

where, after using Eqs. (D.2) and (D.5), we may prove that $2(k_1 + k_2) \cdot k_3 = \beta_{123} - \alpha_{12}$ and also $2k_6 \cdot (k_1 + k_2) = \beta_{345} - \alpha_{12}$, and since we are considering the limit $\alpha_{12} \rightarrow 0$, we have that

$$2(k_1 + k_2) \cdot k_3 \rightarrow \beta_{123}, \quad 2k_6 \cdot (k_1 + k_2) \rightarrow \beta_{345}. \quad (\text{F.32})$$

Also using Eqs. (D.2) and (D.5), we may prove that $2(k_1 + k_2) \cdot k_4 = \beta_{124} - \alpha_{12}$ and, therefore, in the limit $\alpha_{12} \rightarrow 0$,

$$2(k_1 + k_2) \cdot k_4 \rightarrow \beta_{124}. \quad (\text{F.33})$$

¹⁰⁴ For reasons of space in the writing, we are using two different notations in both sides Eqs. (F.28), (F.29) and (F.30): in the left-hand side we are using $A(1, 2, \dots, N)$ to denote $A(\zeta_1, k_1; \zeta_2, k_2; \dots; \zeta_N, k_N)$.

So, substituting (F.32) and (F.33) in (F.31) we are left with¹⁰⁵

$$\begin{aligned} A(\zeta, k_1 + k_2; \zeta_3, k_3; \dots; \zeta_6, k_6) \\ = F^{(23)}[\beta_{123}, \alpha_{34}, \alpha_{45}, \alpha_{56}, \beta_{345}; \alpha'] A_{YM}(\zeta, k_1 + k_2; \zeta_3, k_3; \zeta_4, k_4; \zeta_5, k_5; \zeta_6, k_6) \\ + F^{(32)}[\beta_{124}, \alpha_{34}, \alpha_{35}, \alpha_{56}, \beta_{345}; \alpha'] A_{YM}(\zeta, k_1 + k_2; \zeta_4, k_4; \zeta_3, k_3; \zeta_5, k_5; \zeta_6, k_6). \end{aligned} \quad (\text{F.34})$$

Substituting (F.34) in (F.28) we have that

$$\begin{aligned} A(1, 2, 3, 4, 5, 6) \sim F^{(23)}[\beta_{123}, \alpha_{34}, \alpha_{45}, \alpha_{56}, \beta_{345}; \alpha'] \\ \times \left\{ \frac{1}{\alpha_{12}} V_{(12)}^\mu \frac{\partial}{\partial \zeta^\mu} A_{YM}(\zeta, k_1 + k_2; \zeta_3, k_3; \zeta_4, k_4; \zeta_5, k_5; \zeta_6, k_6) \right\} \\ + F^{(32)}[\beta_{124}, \alpha_{34}, \alpha_{35}, \alpha_{56}, \beta_{345}; \alpha'] \\ \times \left\{ \frac{1}{\alpha_{12}} V_{(12)}^\mu \frac{\partial}{\partial \zeta^\mu} A_{YM}(\zeta, k_1 + k_2; \zeta_4, k_4; \zeta_3, k_3; \zeta_5, k_5; \zeta_6, k_6) \right\}, \end{aligned} \quad (\text{F.35})$$

and identifying the terms in the curly brackets (in the $\alpha_{12} \rightarrow 0$ limit, which is being taken) as the Yang–Mills 6-point subamplitudes (see Eqs. (F.29) and (F.30)) we finally arrive at

$$\begin{aligned} A(1, 2, 3, 4, 5, 6) \sim F^{(23)}[\beta_{123}, \alpha_{34}, \alpha_{45}, \alpha_{56}, \beta_{345}; \alpha'] \\ \times A_{YM}(\zeta_1, k_1; \zeta_2, k_2; \zeta_3, k_3; \zeta_4, k_4; \zeta_5, k_5; \zeta_6, k_6) \\ + F^{(32)}[\beta_{124}, \alpha_{34}, \alpha_{35}, \alpha_{56}, \beta_{345}; \alpha'] \\ \times A_{YM}(\zeta_1, k_1; \zeta_2, k_2; \zeta_4, k_4; \zeta_3, k_3; \zeta_5, k_5; \zeta_6, k_6), \end{aligned} \quad (\text{F.36})$$

or even using the simpler notation,

$$\begin{aligned} A(1, 2, 3, 4, 5, 6) \sim F^{(23)}[\beta_{123}, \alpha_{34}, \alpha_{45}, \alpha_{56}, \beta_{345}; \alpha'] A_{YM}(1, 2, 3, 4, 5, 6) \\ + F^{(32)}[\beta_{124}, \alpha_{34}, \alpha_{35}, \alpha_{56}, \beta_{345}; \alpha'] A_{YM}(1, 2, 4, 3, 5, 6). \end{aligned} \quad (\text{F.37})$$

This relation is to be compared with the one in (112). Taking there the $\alpha_{12} \rightarrow 0$ limit, there are two leading divergent terms in the right-hand side: they come from $F^{(234)}(\alpha') A_{YM}(1, 2, 3, 4, 5, 6)$ and $F^{(243)}(\alpha') A_{YM}(1, 2, 4, 3, 5, 6)$.¹⁰⁶ Comparing these leading terms with the ones in (F.37) we finally arrive at Eqs. (109) and (110).

(ii) *Case of $\beta_{123} \rightarrow 0$*

According to Eq. (62) for $m = 4$, when $\beta_{123} \rightarrow 0$ in the $N = 6$ case we have that¹⁰⁷

¹⁰⁵ In passing from Eq. (F.31) to Eq. (F.34) we have gone back to the notations (73) and (95) for the momentum invariants.

¹⁰⁶ The other four terms remain finite when this limit is taken because they do not have any pole at $\alpha_{12} = 0$.

¹⁰⁷ For reasons of space in the writing, we are using two different notations in both sides Eqs. (F.28), (F.29) and (F.30): in the left-hand side we are using $A(1, 2, \dots, N)$ to denote $A(\zeta_1, k_1; \zeta_2, k_2; \dots; \zeta_N, k_N)$.

$$\begin{aligned}
A_b(1, 2, 3, 4, 5, 6) &\sim \frac{1}{\beta_{123}} \frac{\partial}{\partial \zeta_\mu} A(\zeta_1, k_1; \zeta_2, k_2; \zeta_3, k_3; \zeta, k) \\
&\times \frac{\partial}{\partial \zeta^\mu} A(\zeta, -k; \zeta_4, k_4; \zeta_5, k_5; \zeta_6, k_6),
\end{aligned} \tag{F.38}$$

$$\begin{aligned}
A_{YM}(1, 2, 3, 4, 5, 6) &\sim \frac{1}{\beta_{123}} \frac{\partial}{\partial \zeta_\mu} A_{YM}(\zeta_1, k_1; \zeta_2, k_2; \zeta_3, k_3; \zeta, k) \\
&\times \frac{\partial}{\partial \zeta^\mu} A_{YM}(\zeta, -k; \zeta_4, k_4; \zeta_5, k_5; \zeta_6, k_6),
\end{aligned} \tag{F.39}$$

where we have made explicit the case of the Yang–Mills subamplitude in (F.39), because we will soon need it.

It is understood that the limit $\beta_{123} \rightarrow 0$ also is being taken in the 4-point subamplitudes on the right-hand side of Eqs. (F.38) and (F.39).

In relations (F.38) and (F.39) the momentum k^μ is given by

$$k^\mu = -(k_1 + k_2 + k_3)^\mu = (k_4 + k_5 + k_6)^\mu. \tag{F.40}$$

Using Eq. (34), we may write the 4-point subamplitudes of the right-hand side of (F.38) as

$$\begin{aligned}
&A(\zeta_1, k_1; \zeta_2, k_2; \zeta_3, k_3; \zeta, k) \\
&= F^{[2]}[\alpha_{12}, \alpha_{23}; \alpha'] A_{YM}(\zeta_1, k_1; \zeta_2, k_2; \zeta_3, k_3; \zeta, k),
\end{aligned} \tag{F.41}$$

$$\begin{aligned}
&A(\zeta, -k; \zeta_4, k_4; \zeta_5, k_5; \zeta_6, k_6) \\
&= F^{[2]}[-2k \cdot k_4, \alpha_{45}; \alpha'] A_{YM}(\zeta, -k; \zeta_4, k_4; \zeta_5, k_5; \zeta_6, k_6),
\end{aligned} \tag{F.42}$$

where k^μ is given by Eq. (F.40).

It may be proved, using Eqs. (D.2) and (D.5), that $-2k \cdot k_4 = \alpha_{56} - \beta_{123}$, and since we are considering the limit $\beta_{123} \rightarrow 0$, we have that

$$-2k \cdot k_4 \rightarrow \alpha_{56}. \tag{F.43}$$

So, in (F.42) we have that

$$\begin{aligned}
&A(\zeta, -k; \zeta_4, k_4; \zeta_5, k_5; \zeta_6, k_6) \\
&= F^{[2]}[\alpha_{56}, \alpha_{45}; \alpha'] A_{YM}(\zeta, -k; \zeta_4, k_4; \zeta_5, k_5; \zeta_6, k_6).
\end{aligned} \tag{F.44}$$

Substituting (F.41) and (F.44) in (F.38), it becomes

$$\begin{aligned}
A_b(1, 2, 3, 4, 5, 6) &\sim F^{[2]}[\alpha_{12}, \alpha_{23}; \alpha'] F^{[2]}[\alpha_{56}, \alpha_{45}; \alpha'] \\
&\times \left\{ \frac{1}{\beta_{123}} \frac{\partial}{\partial \zeta_\mu} A_{YM}(\zeta_1, k_1; \zeta_2, k_2; \zeta_3, k_3; \zeta, k) \right. \\
&\times \left. \frac{\partial}{\partial \zeta^\mu} A_{YM}(\zeta, -k; \zeta_4, k_4; \zeta_5, k_5; \zeta_6, k_6) \right\},
\end{aligned} \tag{F.45}$$

and identifying the terms in the curly brackets (in the $\beta_{123} \rightarrow 0$ limit, which is being taken) as the Yang–Mills 6-point subamplitude (see Eq. (F.39)) we arrive at

$$A_b(1, 2, 3, 4, 5, 6) \sim F^{[2]}[\alpha_{12}, \alpha_{23}; \alpha'] F^{[2]}[\alpha_{56}, \alpha_{45}; \alpha'] A_{YM}(1, 2, 3, 4, 5, 6). \tag{F.46}$$

On the other side, taking the $\beta_{123} \rightarrow 0$ limit in (112), we have that

$$A_b(1, 2, 3, 4, 5, 6) \sim F^{[234]}(\alpha') A_{YM}(1, 2, 3, 4, 5, 6) + F^{[324]}(\alpha') A_{YM}(1, 3, 2, 4, 5, 6), \quad (\text{F.47})$$

because, according to the unitarity relation, Eq. (F.39), when $\beta_{123} \rightarrow 0$ $A_{YM}(1, 2, 3, 4, 5, 6)$ and $A_{YM}(1, 3, 2, 4, 5, 6)$ are the only divergent terms in (112). In fact, according to (62) for $m = 4$, the unitarity relation for $A_{YM}(1, 3, 2, 4, 5, 6)$, when $\beta_{123} \rightarrow 0$, is given by

$$A_{YM}(1, 3, 2, 4, 5, 6) \sim \frac{1}{\beta_{123}} \frac{\partial}{\partial \zeta_\mu} A_{YM}(\zeta_1, k_1; \zeta_3, k_3; \zeta_2, k_2; \zeta, k) \times \frac{\partial}{\partial \zeta^\mu} A_{YM}(\zeta, -k; \zeta_4, k_4; \zeta_5, k_5; \zeta_6, k_6), \quad (\text{F.48})$$

where k^μ is given by Eq. (F.40).

Now, from the $N = 4$ BCJ relations we have that

$$A_{YM}(\zeta_1, k_1; \zeta_3, k_3; \zeta_2, k_2; \zeta, k) = -\frac{\alpha_{12}}{\alpha_{12} + \alpha_{23}} A_{YM}(\zeta_1, k_1; \zeta_2, k_2; \zeta_3, k_3; \zeta, k), \quad (\text{F.49})$$

so, in (F.48), it becomes

$$A_{YM}(1, 3, 2, 4, 5, 6) \sim -\frac{\alpha_{12}}{\alpha_{12} + \alpha_{23}} \left\{ \frac{1}{\beta_{123}} \frac{\partial}{\partial \zeta_\mu} A_{YM}(\zeta_1, k_1; \zeta_2, k_2; \zeta_3, k_3; \zeta, k) \times \frac{\partial}{\partial \zeta^\mu} A_{YM}(\zeta, -k; \zeta_4, k_4; \zeta_5, k_5; \zeta_6, k_6) \right\}. \quad (\text{F.50})$$

Identifying the expression in the curly brackets with the leading behavior of $A_{YM}(1, 3, 2, 4, 5, 6)$ (see Eq. (F.48)), we have that

$$A_{YM}(1, 3, 2, 4, 5, 6) \sim -\frac{\alpha_{12}}{\alpha_{12} + \alpha_{23}} A_{YM}(1, 2, 3, 4, 5, 6). \quad (\text{F.51})$$

Substituting (F.51) in (F.47) leads us to

$$A_b(1, 2, 3, 4, 5, 6) \sim \left[F^{[234]}(\alpha') - \frac{\alpha_{12}}{\alpha_{12} + \alpha_{23}} F^{[324]}(\alpha') \right] A_{YM}(1, 2, 3, 4, 5, 6). \quad (\text{F.52})$$

Finally, comparing expressions (F.46) and (F.52) we conclude, precisely, the desired relation between $F^{[234]}(\alpha')$, $F^{[324]}(\alpha')$ and $F^{[2]}(\alpha')$ (see Eq. (111)).

In order to derive conclusions for the α' series of the momentum factors which are involved in relation (111), one has to write the arguments which in one of them are different (say, $\{\alpha_{13}, \alpha_{24}, \beta_{245}\}$, in the case of $F^{[324]}(\alpha')$) in terms of the nine Mandelstam arguments of the other ($F^{[234]}(\alpha')$), taking into account the condition $\beta_{123} \rightarrow 0$. Using relations (D.2) and (D.5) it can easily be seen that the expressions for α_{13} , α_{24} and β_{245} are indeed the ones that we have written in Eq. (111).

F.3.3. Case of $N = 7$

(i) Case of $\alpha_{12} \rightarrow 0$

It may be proved that demanding the unitarity relation (67) to be obeyed by $A_b(1, 2, 3, 4, 5, 6, 7)$ implies the following six relations for the 7-point momentum factors, $F^{[\sigma_7]}(\alpha')$, in which the $\sigma_7 = \{2, \sigma'_6\}$, where σ'_6 is a permutation of indices $\{3, 4, 5\}$:

$$\begin{aligned}
 & F^{(2345)} \left[\begin{array}{cccccc} 0, & \alpha_{23}, & \alpha_{34}, & \alpha_{56}, & \alpha_{56}, & \alpha_{17}, & \alpha_{17} \\ \beta_{123}, & \beta_{234}, & \beta_{345}, & \beta_{456}, & \beta_{567}, & \beta_{167}, & \beta_{127} \end{array}; \alpha' \right] \\
 &= F^{(234)} [\beta_{123}, \alpha_{34}, \alpha_{45}, \alpha_{56}, \alpha_{67}, \beta_{127}, \beta_{567}, \beta_{345}, \beta_{456}; \alpha'], \tag{F.53}
 \end{aligned}$$

$$\begin{aligned}
 & F^{(2354)} \left[\begin{array}{cccccc} 0, & \alpha_{23}, & \alpha_{35}, & \alpha_{45}, & \alpha_{46}, & \alpha_{67}, & \alpha_{17} \\ \beta_{123}, & \beta_{235}, & \beta_{345}, & \beta_{456}, & \beta_{467}, & \beta_{167}, & \beta_{127} \end{array}; \alpha' \right] \\
 &= F^{(243)} [\beta_{123}, \alpha_{35}, \alpha_{45}, \alpha_{46}, \alpha_{67}, \beta_{127}, \beta_{467}, \beta_{345}, \beta_{456}; \alpha'], \tag{F.54}
 \end{aligned}$$

$$\begin{aligned}
 & F^{(2435)} \left[\begin{array}{cccccc} 0, & \alpha_{24}, & \alpha_{34}, & \alpha_{35}, & \alpha_{56}, & \alpha_{67}, & \alpha_{17} \\ \beta_{124}, & \beta_{234}, & \beta_{345}, & \beta_{356}, & \beta_{567}, & \beta_{167}, & \beta_{127} \end{array}; \alpha' \right] \\
 &= F^{(324)} [\beta_{124}, \alpha_{34}, \alpha_{35}, \alpha_{56}, \alpha_{67}, \beta_{127}, \beta_{567}, \beta_{345}, \beta_{356}; \alpha'], \tag{F.55}
 \end{aligned}$$

$$\begin{aligned}
 & F^{(2534)} \left[\begin{array}{cccccc} 0, & \alpha_{25}, & \alpha_{35}, & \alpha_{34}, & \alpha_{46}, & \alpha_{67}, & \alpha_{17} \\ \beta_{125}, & \beta_{235}, & \beta_{345}, & \beta_{346}, & \beta_{467}, & \beta_{167}, & \beta_{127} \end{array}; \alpha' \right] \\
 &= F^{(423)} [\beta_{125}, \alpha_{35}, \alpha_{34}, \alpha_{46}, \alpha_{67}, \beta_{127}, \beta_{467}, \beta_{345}, \beta_{346}; \alpha'], \tag{F.56}
 \end{aligned}$$

$$\begin{aligned}
 & F^{(2453)} \left[\begin{array}{cccccc} 0, & \alpha_{24}, & \alpha_{45}, & \alpha_{35}, & \alpha_{36}, & \alpha_{67}, & \alpha_{17} \\ \beta_{124}, & \beta_{245}, & \beta_{345}, & \beta_{356}, & \beta_{367}, & \beta_{167}, & \beta_{127} \end{array}; \alpha' \right] \\
 &= F^{(342)} [\beta_{124}, \alpha_{45}, \alpha_{35}, \alpha_{36}, \alpha_{67}, \beta_{127}, \beta_{367}, \beta_{345}, \beta_{356}; \alpha'], \tag{F.57}
 \end{aligned}$$

$$\begin{aligned}
 & F^{(2543)} \left[\begin{array}{cccccc} 0, & \alpha_{25}, & \alpha_{45}, & \alpha_{34}, & \alpha_{36}, & \alpha_{67}, & \alpha_{17} \\ \beta_{123}, & \beta_{245}, & \beta_{345}, & \beta_{346}, & \beta_{367}, & \beta_{167}, & \beta_{127} \end{array}; \alpha' \right] \\
 &= F^{(432)} [\beta_{123}, \alpha_{45}, \alpha_{34}, \alpha_{36}, \alpha_{67}, \beta_{127}, \beta_{367}, \beta_{345}, \beta_{346}; \alpha']. \tag{F.58}
 \end{aligned}$$

(ii) Case of $\beta_{123} \rightarrow 0$

It may be proved that demanding the unitarity relation (92) to be obeyed by $A_b(1, 2, 3, 4, 5, 6, 7)$ implies the following two relations for the 7-point momentum factors, $\{F^{(2345)}(\alpha'), F^{(3245)}(\alpha'), F^{(2354)}(\alpha'), F^{(3254)}(\alpha')\}$, and the 4-point and 5-point momentum factors, $\{F^{(2)}(\alpha'), F^{(23)}(\alpha'), F^{(32)}(\alpha')\}$:

$$\begin{aligned}
 & F^{(2345)} \left[\begin{array}{cccccc} \alpha_{12}, & \alpha_{23}, & \alpha_{34}, & \alpha_{45}, & \alpha_{56}, & \alpha_{67}, & \alpha_{17} \\ 0, & \beta_{234}, & \beta_{345}, & \beta_{456}, & \beta_{567}, & \beta_{167}, & \beta_{127} \end{array}; \alpha' \right] \\
 & - \frac{\alpha_{12}}{\alpha_{12} + \alpha_{23}} F^{(3245)} \left[\begin{array}{cccccc} \alpha_{13}, & \alpha_{23}, & \alpha_{24}, & \alpha_{45}, & \alpha_{56}, & \alpha_{67}, & \alpha_{17} \\ 0, & \beta_{234}, & \beta_{245}, & \beta_{456}, & \beta_{567}, & \beta_{167}, & \beta_{137} \end{array}; \alpha' \right] \\
 &= F^{(2)} [\alpha_{12}, \alpha_{23}; \alpha'] F^{(23)} [\beta_{567}, \alpha_{45}, \alpha_{56}, \alpha_{67}, \beta_{127}; \alpha'], \tag{F.59}
 \end{aligned}$$

$$\begin{aligned}
 & F^{(2354)} \left[\begin{array}{cccccc} \alpha_{12}, & \alpha_{23}, & \alpha_{35}, & \alpha_{45}, & \alpha_{46}, & \alpha_{67}, & \alpha_{17} \\ 0, & \beta_{235}, & \beta_{345}, & \beta_{456}, & \beta_{467}, & \beta_{167}, & \beta_{127} \end{array}; \alpha' \right] \\
 & - \frac{\alpha_{12}}{\alpha_{12} + \alpha_{23}} F^{(3254)} \left[\begin{array}{cccccc} \alpha_{13}, & \alpha_{23}, & \alpha_{25}, & \alpha_{45}, & \alpha_{46}, & \alpha_{67}, & \alpha_{17} \\ 0, & \beta_{235}, & \beta_{245}, & \beta_{456}, & \beta_{467}, & \beta_{167}, & \beta_{137} \end{array}; \alpha' \right] \\
 &= F^{(2)} [\alpha_{12}, \alpha_{23}; \alpha'] F^{(32)} [\beta_{467}, \alpha_{45}, \alpha_{46}, \alpha_{67}, \beta_{456}; \alpha']. \tag{F.60}
 \end{aligned}$$

Care must be taken when using relations (F.53)–(F.58) and/or relations (F.59)–(F.60) to compute the coefficients of the α' series of the momentum factors. It is understood implicitly in them that all Mandelstam variables are written in terms of the ones for $F^{(2345)}(\alpha')$, namely, $\{\alpha_{12}, \alpha_{23}, \alpha_{34}, \alpha_{45}, \alpha_{56}, \alpha_{67}, \alpha_{17}, \beta_{123}, \beta_{234}, \beta_{345}, \beta_{456}, \beta_{567}, \beta_{167}, \beta_{127}\}$ where $\alpha_{12} = 0$ or $\beta_{123} = 0$, depending on the relation that is being used.

F.4. Cyclic transformation of the momentum factors

F.4.1. Case of $N = 5$

Here we will prove that relations (88) and (89) arise when we demand cyclic symmetry to be obeyed by $A_b(1, 2, 3, 4, 5)$.

Cyclic invariance of $A_b(1, 2, 3, 4, 5)$ means that $A_b(1, 2, 3, 4, 5) = A_b(2, 3, 4, 5, 1)$, so in (74) cyclic invariance implies that

$$\begin{aligned} A_b(1, 2, 3, 4, 5) &= F_{cycl}^{[23]}(\alpha') A_{YM}(2, 3, 4, 5, 1) + F_{cycl}^{[32]}(\alpha') A_{YM}(2, 4, 3, 5, 1) \\ &= F_{cycl}^{[23]}(\alpha') A_{YM}(1, 2, 3, 4, 5) + F_{cycl}^{[32]}(\alpha') A_{YM}(1, 2, 4, 3, 5), \end{aligned} \quad (\text{F.61})$$

where in (F.61) we have used the cyclic property of Yang–Mills subamplitudes.

$F_{cycl}^{[23]}(\alpha')$ and $F_{cycl}^{[32]}(\alpha')$ denote doing $\{k_1 \rightarrow k_2, k_2 \rightarrow k_3, \dots, k_5 \rightarrow k_1\}$ in $F^{[23]}(\alpha')$ and $F^{[32]}(\alpha')$, respectively.

Now, the BCJ relation for $A_{YM}(1, 2, 4, 3, 5)$ (see Eq. (E.12)) says that

$$A_{YM}(1, 2, 4, 3, 5) = \frac{\alpha_{13} + \alpha_{23}}{\alpha_{35}} A_{YM}(1, 2, 3, 4, 5) + \frac{\alpha_{13}}{\alpha_{35}} A_{YM}(1, 3, 2, 4, 5), \quad (\text{F.62})$$

where $\alpha_{13} = k_1 \cdot k_3$ and $\alpha_{35} = k_3 \cdot k_5$ are given in terms of the Mandelstam variables in Eq. (D.4).

So, substituting (F.62) in (F.61) we have that

$$\begin{aligned} A_b(1, 2, 3, 4, 5) &= \left\{ F_{cycl}^{[23]}(\alpha') + \frac{\alpha_{13} + \alpha_{23}}{\alpha_{35}} F_{cycl}^{[32]}(\alpha') \right\} A_{YM}(1, 2, 3, 4, 5) \\ &\quad + \left\{ \frac{\alpha_{13}}{\alpha_{35}} F_{cycl}^{[32]}(\alpha') \right\} A_{YM}(1, 2, 3, 4, 5) \end{aligned} \quad (\text{F.63})$$

and comparison of this last relation with (74) leads precisely to relations (88) and (89).

F.4.2. Case of $N = 6$

Cyclic invariance of $A_b(1, 2, 3, 4, 5, 6)$ means that $A_b(1, 2, 3, 4, 5, 6) = A_b(2, 3, 4, 5, 6, 1)$, so in (112) cyclic invariance implies that

$$\begin{aligned} A_b(1, 2, 3, 4, 5, 6) &= F_{cycl}^{[234]}(\alpha') A_{YM}(1, 2, 3, 4, 5, 6) + F_{cycl}^{[324]}(\alpha') A_{YM}(2, 4, 3, 5, 6, 1) \\ &\quad + F_{cycl}^{[243]}(\alpha') A_{YM}(2, 3, 5, 4, 6, 1) + F_{cycl}^{[342]}(\alpha') A_{YM}(2, 4, 5, 3, 6, 1) \\ &\quad + F_{cycl}^{[423]}(\alpha') A_{YM}(2, 5, 3, 4, 6, 1) + F_{cycl}^{[432]}(\alpha') A_{YM}(2, 5, 4, 3, 6, 1) \\ &= F_{cycl}^{[234]}(\alpha') A_{YM}(1, 2, 3, 4, 5, 6) + F_{cycl}^{[324]}(\alpha') A_{YM}(1, 2, 4, 3, 5, 6) \\ &\quad + F_{cycl}^{[243]}(\alpha') A_{YM}(1, 2, 3, 5, 4, 6) + F_{cycl}^{[342]}(\alpha') A_{YM}(1, 2, 4, 5, 3, 6) \\ &\quad + F_{cycl}^{[423]}(\alpha') A_{YM}(1, 2, 5, 3, 4, 6) + F_{cycl}^{[432]}(\alpha') A_{YM}(1, 2, 5, 4, 3, 6), \end{aligned} \quad (\text{F.64})$$

where in the second equality we have used the cyclic property of Yang–Mills subamplitudes.

$F_{cycl}^{[\sigma_6]}(\alpha')$ denotes doing $\{k_1 \rightarrow k_2, k_2 \rightarrow k_3, \dots, k_6 \rightarrow k_1\}$ in $F^{[\sigma_6]}(\alpha')$, for all six permutations σ_6 .

Now, in order to write the right-hand side of (F.64) in terms of the original Yang–Mills basis, we use the $N = 6$ BCJ relations. In Appendix E.3 we have found them but, in order to save

space, there we have made explicit only a few examples of them. In particular in Eqs. (E.19) and (E.20) we have the BCJ relations for $A_{YM}(1, 2, 3, 5, 4, 6)$ and $A_{YM}(1, 2, 4, 5, 3, 6)$, respectively. Writing these equations in the conventions of the α_{ij} 's and the β_{ijk} 's, they are given by

$$\begin{aligned} A_{YM}(1, 2, 3, 5, 4, 6) &= \frac{\beta_{123} - \alpha_{56}}{\beta_{123} - \alpha_{56} - \alpha_{45}} A_{YM}(1, 2, 3, 4, 5, 6) \\ &\quad - \frac{\beta_{123} - \alpha_{56} + \alpha_{34}}{\beta_{123} - \alpha_{56} - \alpha_{45}} A_{YM}(1, 2, 4, 3, 5, 6) \\ &\quad + \frac{\alpha_{56} - \beta_{123} - \beta_{234} + \alpha_{23}}{-\alpha_{56} - \alpha_{45} + \beta_{123}} A(1, 3, 4, 2, 5, 6), \end{aligned} \quad (\text{F.65})$$

$$\begin{aligned} A_{YM}(1, 2, 4, 5, 3, 6) &= \frac{\beta_{123} - \alpha_{12}}{\alpha_{12} - \beta_{345} + \alpha_{45} - \beta_{123}} A_{YM}(1, 2, 3, 4, 5, 6) \\ &\quad + \frac{\beta_{123} - \alpha_{12} - \alpha_{23}}{\alpha_{12} - \beta_{345} + \alpha_{45} - \beta_{123}} A_{YM}(1, 3, 2, 4, 5, 6) \\ &\quad + \frac{\alpha_{34} + \beta_{123} - \alpha_{12}}{\alpha_{12} - \beta_{345} + \alpha_{45} - \beta_{123}} A_{YM}(1, 2, 4, 3, 5, 6). \end{aligned} \quad (\text{F.66})$$

Notice that there is one subamplitude in (F.65) which is not present in (F.66), and vice versa.

Substituting the expressions (F.65), (F.66) and the BCJ relations for $A_{YM}(1, 2, 4, 3, 5, 6)$,

$A_{YM}(1, 2, 5, 3, 4, 6)$ and $A_{YM}(1, 2, 5, 4, 3, 6)$ (which we have not made explicit here) in the second equality in (F.64), and then comparing the coefficient of each subamplitude, $A_{YM}(1, \{2_\sigma, 3_\sigma, 4_\sigma\}, 5, 6)$, with the corresponding one in Eq. (112), we arrive to the six relations for the $F^{(\sigma_6)}(\alpha')$'s in terms of the $F^{(\sigma_6)}_{\text{cycl}}(\alpha')$'s, given in Eqs. (113)–(118).

Appendix G. Linear system for the $F^{(\sigma_6)}(\alpha')$ and the $F^{(\sigma_7)}(\alpha')$ momentum factors

We have implemented the N -point formula in Eq. (31) computationally, for $4 \leq N \leq 7$, in such a way that we have done on it all the Grassmann integrations (for the $(N - 2)$ θ_i 's and for the N ϕ_j 's). So we have an explicit expression for the $A_b(1, \dots, N)$ in terms of momentum factors ($= (N - 3)$ -dimensional integrals) and the kinematical structures that we have listed in Eq. (10).

In this appendix we give the details of how we have found the $F^{(\sigma_6)}(\alpha')$ and the $F^{(\sigma_7)}(\alpha')$ momentum factors of Subsection 4.1.3.

G.1. Case of the $F^{(\sigma_6)}(\alpha')$'s

Choosing $T(1, 2, 3, 4, 5, 6) = A_b(1, 2, 3, 4, 5, 6)$ in Eq. (28) and considering then only the $(\zeta \cdot \zeta)^1 (\zeta \cdot k)^4$ terms, we have that:

$$\begin{aligned} A_b(1, 2, 3, 4, 5, 6)|_{(\zeta \cdot \zeta)^1 (\zeta \cdot k)^4} &= F^{[234]}(\alpha') A_{YM}(1, 2, 3, 4, 5, 6)|_{(\zeta \cdot \zeta)^1 (\zeta \cdot k)^4} \\ &\quad + F^{[324]}(\alpha') A_{YM}(1, 3, 2, 4, 5, 6)|_{(\zeta \cdot \zeta)^1 (\zeta \cdot k)^4} \\ &\quad + F^{[243]}(\alpha') A_{YM}(1, 2, 4, 3, 5, 6)|_{(\zeta \cdot \zeta)^1 (\zeta \cdot k)^4} \\ &\quad + F^{[342]}(\alpha') A_{YM}(1, 3, 4, 2, 5, 6)|_{(\zeta \cdot \zeta)^1 (\zeta \cdot k)^4} \\ &\quad + F^{[423]}(\alpha') A_{YM}(1, 4, 2, 3, 5, 6)|_{(\zeta \cdot \zeta)^1 (\zeta \cdot k)^4} \\ &\quad + F^{[432]}(\alpha') A_{YM}(1, 4, 3, 2, 5, 6)|_{(\zeta \cdot \zeta)^1 (\zeta \cdot k)^4}. \end{aligned} \quad (\text{G.1})$$

Considering the coefficient of the following six kinematical structures,

$$\begin{aligned} & \{(\zeta_1 \cdot \zeta_2)(\zeta_3 \cdot k_1)(\zeta_4 \cdot k_5)(\zeta_5 \cdot k_1)(\zeta_6 \cdot k_2), (\zeta_1 \cdot \zeta_2)(\zeta_3 \cdot k_2)(\zeta_4 \cdot k_5)(\zeta_5 \cdot k_1)(\zeta_6 \cdot k_2), \\ & (\zeta_1 \cdot \zeta_2)(\zeta_3 \cdot k_5)(\zeta_4 \cdot k_1)(\zeta_5 \cdot k_1)(\zeta_6 \cdot k_2), (\zeta_1 \cdot \zeta_2)(\zeta_3 \cdot k_5)(\zeta_4 \cdot k_2)(\zeta_5 \cdot k_1)(\zeta_6 \cdot k_2), \\ & (\zeta_1 \cdot \zeta_3)(\zeta_2 \cdot k_5)(\zeta_4 \cdot k_1)(\zeta_5 \cdot k_1)(\zeta_6 \cdot k_3), (\zeta_1 \cdot \zeta_3)(\zeta_2 \cdot k_5)(\zeta_4 \cdot k_3)(\zeta_5 \cdot k_1)(\zeta_6 \cdot k_3)\}, \end{aligned} \quad (\text{G.2})$$

in both sides of (G.1), we have arrived, respectively, at the following six linearly independent equations for the momentum factors¹⁰⁸:

$$\begin{aligned} -F^{\{324\}}(\alpha')s_1 + F^{\{234\}}(\alpha')(s_1 + s_2 - t_1) &= s_1s_4(s_1 + s_2 - t_1)t_1\alpha'^3 \left[\frac{1}{x_2x_3(1-x_4)} \right], \\ -F^{\{324\}}(\alpha')s_1 + F^{\{234\}}(\alpha')(s_1 + s_2) &= s_1s_2s_4t_1\alpha'^3 \left[\frac{1}{x_2(x_3-x_2)(1-x_4)} \right], \\ -F^{\{423\}}(\alpha')s_1 + F^{\{243\}}(\alpha')(-s_2 - s_5 + t_1 + t_2) \end{aligned} \quad (\text{G.3})$$

$$\begin{aligned} &= s_1(s_1 - s_3 + s_5 - t_1)(s_2 + s_5 - t_1 - t_2)(s_3 + s_4 - t_3)\alpha'^3 \left[\frac{1}{x_2(1-x_3)x_4} \right], \\ F^{\{423\}}(\alpha')s_1 + F^{\{243\}}(\alpha')(-s_1 + s_2 + s_3 - t_2) \\ &= s_1(s_1 - s_3 + s_5 - t_1)(s_2 + s_3 - t_2)(s_3 + s_4 - t_3)\alpha'^3 \left[\frac{1}{x_2(1-x_3)(x_4-x_2)} \right], \\ F^{\{432\}}(\alpha')(s_1 + s_2 - t_1) + F^{\{342\}}(\alpha')(s_1 + s_2 - s_3 - t_1) \end{aligned} \quad (\text{G.4})$$

$$\begin{aligned} &= -s_3(s_1 + s_2 - t_1)(-s_1 + s_3 + s_5 - t_2)(s_3 + s_6 - t_2 - t_3)\alpha'^3 \left[\frac{1}{(1-x_2)x_3(x_4-x_3)} \right], \\ F^{\{432\}}(\alpha')(s_1 + s_2 - t_1) + F^{\{342\}}(\alpha')(-s_2 - s_5 + t_1 + t_2) \\ &= (s_1 + s_2 - t_1)(s_2 + s_5 - t_1 - t_2)(s_1 - s_3 - s_5 + t_2)(-s_3 - s_6 + t_2 + t_3)\alpha'^3 \\ &\quad \times \left[\frac{1}{(1-x_2)x_3x_4} \right] \end{aligned} \quad (\text{G.5})$$

where we are using the notation [24]

$$\begin{aligned} [f(x_2, x_3, x_4)] &= \int_0^1 dx_4 \int_0^{x_4} dx_3 \int_0^{x_3} dx_2 x_2^{2\alpha'k_2 \cdot k_1} x_3^{2\alpha'k_3 \cdot k_1} x_4^{2\alpha'k_4 \cdot k_1} (x_3 - x_2)^{2\alpha'k_3 \cdot k_2} \\ &\quad \times (x_4 - x_2)^{2\alpha'k_4 \cdot k_2} (1 - x_2)^{2\alpha'k_5 \cdot k_2} (x_4 - x_3)^{2\alpha'k_4 \cdot k_3} \\ &\quad \times (1 - x_3)^{2\alpha'k_5 \cdot k_3} (1 - x_4)^{2\alpha'k_5 \cdot k_4} \times f(x_2, x_3, x_4), \end{aligned} \quad (\text{G.6})$$

$$= \int_0^1 dx_4 \int_0^{x_4} dx_3 \int_0^{x_3} dx_2 \left(\prod_{i>j \geq 1}^5 (x_i - x_j)^{2\alpha'k_i \cdot k_j} \right) \times f(x_2, x_3, x_4). \quad (\text{G.7})$$

In Eq. (G.7) we have assumed that $\{x_1 = 0, x_5 = 1\}$.

Notice that the six equations come in three blocks of two equations and two unknowns each.

Solving the first block of two equations, Eq. (G.3), gives

¹⁰⁸ On these equations we introduce the $N = 6$ Mandelstam variables. See Eqs. (D.2) and (D.5).

$$F^{\{234\}}(\alpha') = \alpha'^3 \int_0^1 dx_4 \int_0^{x_4} dx_3 \int_0^{x_3} dx_2 \left(\prod_{i>j\geq 1}^5 (x_i - x_j)^{2\alpha' k_i \cdot k_j} \right) \times \left\{ \frac{s_1}{x_2} \cdot \frac{s_4}{1-x_4} \cdot \left(\frac{t_1 - s_1 - s_2}{x_3} + \frac{s_2}{x_3 - x_2} \right) \right\}, \quad (\text{G.8})$$

$$F^{\{324\}}(\alpha') = \alpha'^3 \int_0^1 dx_4 \int_0^{x_4} dx_3 \int_0^{x_3} dx_2 \left(\prod_{i>j\geq 1}^5 (x_i - x_j)^{2\alpha' k_i \cdot k_j} \right) \times \left\{ \frac{t_1 - s_1 - s_2}{x_3} \cdot \frac{s_4}{1-x_4} \cdot \left(\frac{s_1}{x_2} + \frac{s_2}{x_2 - x_3} \right) \right\}. \quad (\text{G.9})$$

Substituting back the $N = 6$ Mandelstam variables (see Eqs. (D.2) and (D.5)) in (G.8) and (G.9), it is easy to see that the expressions for $F^{\{234\}}(\alpha')$ and $F^{\{324\}}(\alpha')$ correspond exactly to the ones given in Eqs. (54) and (56) of the main body of this work, respectively.

Solving the two blocks of equations in (G.4) and (G.5) it can be verified, also, that the expression for $F^{\{243\}}(\alpha')$, $F^{\{423\}}(\alpha')$, $F^{\{342\}}(\alpha')$ and $F^{\{432\}}(\alpha')$ are in agreement with the one in (54) by doing the corresponding σ_6 permutation of indices $\{2, 3, 4\}$ inside the curly brackets of that equation.

G.2. Case of the $F^{\{\sigma_7\}}(\alpha')$'s

Choosing $T(1, 2, 3, 4, 5, 6, 7) = A_b(1, 2, 3, 4, 5, 6, 7)$ in Eq. (29) and considering then only the $(\zeta \cdot \zeta)^1 (\zeta \cdot k)^5$ terms, we have that:

$$A_b(1, 2, 3, 4, 5, 6, 7)|_{(\zeta \cdot \zeta)^1 (\zeta \cdot k)^5} = F^{\{2345\}}(\alpha') A_{YM}(1, 2, 3, 4, 5, 6, 7)|_{(\zeta \cdot \zeta)^1 (\zeta \cdot k)^5} + F^{\{2354\}}(\alpha') A_{YM}(1, 2, 3, 5, 4, 6, 7)|_{(\zeta \cdot \zeta)^1 (\zeta \cdot k)^5} + F^{\{2435\}}(\alpha') A_{YM}(1, 2, 4, 3, 5, 6, 7)|_{(\zeta \cdot \zeta)^1 (\zeta \cdot k)^5} + \dots + F^{\{5432\}}(\alpha') A_{YM}(1, 5, 4, 3, 2, 6, 7)|_{(\zeta \cdot \zeta)^1 (\zeta \cdot k)^5} \quad (\text{G.10})$$

Considering the coefficient of the following twenty four kinematical structures,

$$\begin{aligned} & \{ (\zeta_1 \cdot \zeta_2)(\zeta_3 \cdot k_2)(\zeta_4 \cdot k_6)(\zeta_5 \cdot k_6)(\zeta_6 \cdot k_1)(\zeta_7 \cdot k_2), \\ & (\zeta_1 \cdot \zeta_2)(\zeta_3 \cdot k_1)(\zeta_4 \cdot k_6)(\zeta_5 \cdot k_6)(\zeta_6 \cdot k_1)(\zeta_7 \cdot k_2), \\ & (\zeta_1 \cdot \zeta_2)(\zeta_3 \cdot k_2)(\zeta_4 \cdot k_5)(\zeta_5 \cdot k_6)(\zeta_6 \cdot k_1)(\zeta_7 \cdot k_3), \\ & (\zeta_1 \cdot \zeta_2)(\zeta_3 \cdot k_1)(\zeta_4 \cdot k_5)(\zeta_5 \cdot k_6)(\zeta_6 \cdot k_1)(\zeta_7 \cdot k_2), \\ & (\zeta_1 \cdot \zeta_2)(\zeta_3 \cdot k_6)(\zeta_4 \cdot k_2)(\zeta_5 \cdot k_3)(\zeta_6 \cdot k_1)(\zeta_7 \cdot k_2), \\ & (\zeta_1 \cdot \zeta_2)(\zeta_3 \cdot k_6)(\zeta_4 \cdot k_1)(\zeta_5 \cdot k_3)(\zeta_6 \cdot k_1)(\zeta_7 \cdot k_2), \\ & (\zeta_1 \cdot \zeta_2)(\zeta_3 \cdot k_2)(\zeta_4 \cdot k_5)(\zeta_5 \cdot k_6)(\zeta_6 \cdot k_1)(\zeta_7 \cdot k_3), \\ & (\zeta_1 \cdot \zeta_2)(\zeta_3 \cdot k_1)(\zeta_4 \cdot k_5)(\zeta_5 \cdot k_6)(\zeta_6 \cdot k_1)(\zeta_7 \cdot k_2), \\ & (\zeta_1 \cdot \zeta_4)(\zeta_2 \cdot k_6)(\zeta_3 \cdot k_1)(\zeta_5 \cdot k_2)(\zeta_6 \cdot k_1)(\zeta_7 \cdot k_4), \\ & (\zeta_1 \cdot \zeta_4)(\zeta_2 \cdot k_6)(\zeta_3 \cdot k_4)(\zeta_5 \cdot k_2)(\zeta_6 \cdot k_1)(\zeta_7 \cdot k_3), \\ & (\zeta_1 \cdot \zeta_4)(\zeta_2 \cdot k_6)(\zeta_3 \cdot k_1)(\zeta_5 \cdot k_6)(\zeta_6 \cdot k_1)(\zeta_7 \cdot k_4), \end{aligned}$$

$$\begin{aligned}
& (\zeta_1 \cdot \zeta_4)(\zeta_2 \cdot k_6)(\zeta_3 \cdot k_4)(\zeta_5 \cdot k_6)(\zeta_6 \cdot k_1)(\zeta_7 \cdot k_3), \\
& (\zeta_1 \cdot \zeta_4)(\zeta_2 \cdot k_6)(\zeta_3 \cdot k_2)(\zeta_5 \cdot k_4)(\zeta_6 \cdot k_1)(\zeta_7 \cdot k_4), \\
& (\zeta_1 \cdot \zeta_4)(\zeta_2 \cdot k_6)(\zeta_3 \cdot k_2)(\zeta_5 \cdot k_4)(\zeta_6 \cdot k_1)(\zeta_7 \cdot k_5), \\
& (\zeta_1 \cdot \zeta_4)(\zeta_2 \cdot k_6)(\zeta_3 \cdot k_6)(\zeta_5 \cdot k_4)(\zeta_6 \cdot k_1)(\zeta_7 \cdot k_4), \\
& (\zeta_1 \cdot \zeta_4)(\zeta_2 \cdot k_3)(\zeta_3 \cdot k_6)(\zeta_5 \cdot k_1)(\zeta_6 \cdot k_1)(\zeta_7 \cdot k_4), \\
& (\zeta_1 \cdot \zeta_3)(\zeta_2 \cdot k_6)(\zeta_4 \cdot k_2)(\zeta_5 \cdot k_3)(\zeta_6 \cdot k_1)(\zeta_7 \cdot k_5), \\
& (\zeta_1 \cdot \zeta_3)(\zeta_2 \cdot k_6)(\zeta_4 \cdot k_6)(\zeta_5 \cdot k_3)(\zeta_6 \cdot k_1)(\zeta_7 \cdot k_3), \\
& (\zeta_1 \cdot \zeta_3)(\zeta_2 \cdot k_4)(\zeta_4 \cdot k_6)(\zeta_5 \cdot k_3)(\zeta_6 \cdot k_1)(\zeta_7 \cdot k_3), \\
& (\zeta_1 \cdot \zeta_3)(\zeta_2 \cdot k_6)(\zeta_4 \cdot k_6)(\zeta_5 \cdot k_1)(\zeta_6 \cdot k_1)(\zeta_7 \cdot k_3), \\
& (\zeta_1 \cdot \zeta_2)(\zeta_3 \cdot k_6)(\zeta_4 \cdot k_3)(\zeta_5 \cdot k_2)(\zeta_6 \cdot k_1)(\zeta_7 \cdot k_5), \\
& (\zeta_1 \cdot \zeta_2)(\zeta_3 \cdot k_6)(\zeta_4 \cdot k_6)(\zeta_5 \cdot k_2)(\zeta_6 \cdot k_1)(\zeta_7 \cdot k_2), \\
& (\zeta_1 \cdot \zeta_2)(\zeta_3 \cdot k_6)(\zeta_4 \cdot k_6)(\zeta_5 \cdot k_1)(\zeta_6 \cdot k_1)(\zeta_7 \cdot k_2), \\
& (\zeta_1 \cdot \zeta_2)(\zeta_3 \cdot k_4)(\zeta_4 \cdot k_6)(\zeta_5 \cdot k_1)(\zeta_6 \cdot k_1)(\zeta_7 \cdot k_2) \}
\end{aligned} \tag{G.11}$$

in both sides of (G.10), we have arrived, respectively, at the following twenty four linearly independent equations for the momentum factors¹⁰⁹:

$$\begin{aligned}
& (-F^{\{3254\}}(\alpha')s_1 + F^{\{2354\}}(\alpha')(s_1 + s_2 - t_1))s_5 + F^{\{3245\}}(\alpha')s_1(s_4 + s_5 - t_4) \\
& - F^{\{2345\}}(\alpha')(s_1 + s_2 - t_1)(s_4 + s_5 - t_4) \\
& = -\alpha'^4 s_1 s_5 (s_1 + s_2 - t_1) t_1 (s_4 + s_5 - t_4) t_4 \left[\frac{1}{x_2 x_3 (1 - x_4)(1 - x_5)} \right], \\
& -s_5 (F^{\{2354\}}(\alpha')(s_1 + s_2 - t_1) + F^{\{3254\}}(\alpha')(-s_1 + t_1)) \\
& - F^{\{3245\}}(\alpha')(s_1 - t_1)(s_4 + s_5 - t_4) + F^{\{2345\}}(\alpha')(s_1 + s_2 - t_1)(s_4 + s_5 - t_4) \\
& = \alpha'^4 (s_2 s_5 (s_1 + s_2 - t_1) t_1 (s_4 + s_5 - t_4) t_4) \left[\frac{1}{x_3 (x_3 - x_2)(1 - x_4)(1 - x_5)} \right], \\
& F^{\{3245\}}(\alpha')s_1(s_4 + s_5) + s_5 (-F^{\{3254\}}(\alpha')s_1 + F^{\{2354\}}(\alpha')(s_1 + s_2 - t_1)) \\
& - F^{\{2345\}}(\alpha')(s_4 + s_5)(s_1 + s_2 - t_1) \\
& = -\alpha'^4 s_1 s_4 s_5 (s_1 + s_2 - t_1) t_1 t_4 \left[\frac{1}{x_2 x_3 (1 - x_5)(x_5 - x_4)} \right], \\
& -F^{\{3245\}}(\alpha')(s_4 + s_5)(s_1 - t_1) + F^{\{2345\}}(\alpha')(s_4 + s_5)(s_1 + s_2 - t_1) \\
& - s_5 (F^{\{2354\}}(\alpha')(s_1 + s_2 - t_1) + F^{\{3254\}}(\alpha')(-s_1 + t_1)) \\
& = \alpha'^4 s_2 s_4 s_5 (s_1 + s_2 - t_1) t_1 t_4 \left[\frac{1}{x_3 (x_3 - x_2)(1 - x_5)(x_5 - x_4)} \right],
\end{aligned} \tag{G.12}$$

$$\left(\begin{array}{l} 5 \text{ blocks of 4 equations each, obtained from (G.12)} \\ \text{by permutations of indices } (2, 3, 4, 5) \end{array} \right), \tag{G.13}$$

where, in this case, we are using the notation

¹⁰⁹ On these equations we have introduced the $N = 7$ Mandelstam variables. See Eqs. (D.3) and (D.6).

$$\begin{aligned}
[g(x_2, x_3, x_4, x_5)] &= \int_0^1 dx_5 \int_0^{x_5} dx_4 \int_0^{x_4} dx_3 \int_0^{x_3} dx_2 x_2^{2\alpha'k_2 \cdot k_1} x_3^{2\alpha'k_3 \cdot k_1} x_4^{2\alpha'k_4 \cdot k_1} x_5^{2\alpha'k_5 \cdot k_1} \\
&\quad \times (x_3 - x_2)^{2\alpha'k_3 \cdot k_2} (x_4 - x_2)^{2\alpha'k_4 \cdot k_2} (x_5 - x_2)^{2\alpha'k_5 \cdot k_2} \\
&\quad \times (1 - x_2)^{2\alpha'k_6 \cdot k_2} (x_4 - x_3)^{2\alpha'k_4 \cdot k_3} (x_5 - x_3)^{2\alpha'k_5 \cdot k_3} \\
&\quad \times (1 - x_3)^{2\alpha'k_6 \cdot k_3} (x_5 - x_4)^{2\alpha'k_5 \cdot k_4} (1 - x_4)^{2\alpha'k_6 \cdot k_4} \\
&\quad \times (1 - x_5)^{2\alpha'k_6 \cdot k_5} \times g(x_2, x_3, x_4, x_5)
\end{aligned} \tag{G.14}$$

$$\begin{aligned}
&= \int_0^1 dx_5 \int_0^{x_5} dx_4 \int_0^{x_4} dx_3 \int_0^{x_3} dx_2 \left(\prod_{i>j \geq 1}^6 (x_i - x_j)^{2\alpha'k_i \cdot k_j} \right) \\
&\quad \times g(x_2, x_3, x_4, x_5)
\end{aligned} \tag{G.15}$$

In Eq. (G.15) we have assumed that $\{x_1 = 0, x_6 = 1\}$.

Notice that the twenty four equations come in six blocks of four equations and four unknowns each.

Solving the first block of four equations, Eq. (G.12), gives

$$\begin{aligned}
F^{[2345]}(\alpha') &= \alpha'^4 \int_0^1 dx_5 \int_0^{x_5} dx_4 \int_0^{x_4} dx_3 \int_0^{x_3} dx_2 \left(\prod_{i>j \geq 1}^6 (x_i - x_j)^{2\alpha'k_i \cdot k_j} \right) \\
&\quad \times \left\{ \frac{s_1}{x_2} \cdot \frac{s_5}{1 - x_5} \cdot \left(\frac{t_1 - s_1 - s_2}{x_3} + \frac{s_2}{x_3 - x_2} \right) \cdot \left(\frac{s_4}{x_5 - x_4} + \frac{t_4 - s_4 - s_5}{1 - x_4} \right) \right\},
\end{aligned} \tag{G.16}$$

$$\begin{aligned}
F^{[2354]}(\alpha') &= \alpha'^4 \int_0^1 dx_5 \int_0^{x_5} dx_4 \int_0^{x_4} dx_3 \int_0^{x_3} dx_2 \left(\prod_{i>j \geq 1}^6 (x_i - x_j)^{2\alpha'k_i \cdot k_j} \right) \\
&\quad \times \left\{ \frac{s_1}{x_2} \cdot \frac{t_4 - s_4 - s_5}{1 - x_4} \cdot \left(\frac{t_1 - s_1 - s_2}{x_3} + \frac{s_2}{x_3 - x_2} \right) \cdot \left(\frac{s_4}{x_4 - x_5} + \frac{s_5}{1 - x_5} \right) \right\},
\end{aligned} \tag{G.17}$$

$$\begin{aligned}
F^{[3245]}(\alpha') &= \alpha'^4 \int_0^1 dx_5 \int_0^{x_5} dx_4 \int_0^{x_4} dx_3 \int_0^{x_3} dx_2 \left(\prod_{i>j \geq 1}^6 (x_i - x_j)^{2\alpha'k_i \cdot k_j} \right) \\
&\quad \times \left\{ \frac{t_1 - s_1 - s_2}{x_3} \cdot \frac{s_5}{1 - x_5} \cdot \left(\frac{s_1}{x_2} + \frac{s_2}{x_2 - x_3} \right) \cdot \left(\frac{s_4}{x_5 - x_4} + \frac{t_4 - s_4 - s_5}{1 - x_4} \right) \right\},
\end{aligned} \tag{G.18}$$

$$\begin{aligned}
F^{[3254]}(\alpha') &= \alpha'^4 \int_0^1 dx_5 \int_0^{x_5} dx_4 \int_0^{x_4} dx_3 \int_0^{x_3} dx_2 \left(\prod_{i>j \geq 1}^6 (x_i - x_j)^{2\alpha'k_i \cdot k_j} \right) \\
&\quad \times \left\{ \frac{t_1 - s_1 - s_2}{x_3} \cdot \frac{t_4 - s_4 - s_5}{1 - x_4} \cdot \left(\frac{s_1}{x_2} + \frac{s_2}{x_2 - x_3} \right) \cdot \left(\frac{s_4}{x_4 - x_5} + \frac{s_5}{1 - x_5} \right) \right\}.
\end{aligned} \tag{G.19}$$

Substituting back the $N = 7$ Mandelstam variables (see Eqs. (D.3) and (D.6)) in (G.16) and (G.18), it is easy to see that the expressions for $F^{\{2345\}}(\alpha')$ and $F^{\{3245\}}(\alpha')$ correspond exactly to the ones given in Eqs. (55) and (57) of the main body of this work, respectively.

Solving the five blocks of equations in (G.13) it can be verified, also, that the expression for the remaining twenty $F^{(\sigma\tau)}(\alpha')$'s are in agreement with the one in (54) by doing the corresponding $\sigma\tau$ permutation of indices $\{2, 3, 4, 5\}$ inside the curly brackets of that equation.

References

- [1] E. Hatefi, More on Ramond–Ramond, SYM, WZ couplings and their corrections in IIA, arXiv:1403.7167;
E. Hatefi, Selection rules and RR couplings on non-BPS branes, J. High Energy Phys. 1311 (2013) 204, arXiv:1307.3520;
E. Hatefi, All order α' higher derivative corrections to non-BPS branes of type IIB superstring theory, J. High Energy Phys. 1307 (2013) 002, arXiv:1304.3711;
E. Hatefi, Closed string Ramond–Ramond proposed higher derivative interactions on fermionic amplitudes in IIB, Nucl. Phys. B 880 (2014) 1, arXiv:1302.5024.
- [2] M.R. Garousi, S-matrix elements from T-duality, Nucl. Phys. B 869 (2013) 216, arXiv:1204.4978;
M.R. Garousi, Tree-level S-matrix elements from S-duality, J. High Energy Phys. 1204 (2012) 140, arXiv:1201.2556.
- [3] E. Hatefi, I.Y. Park, More on closed string induced higher derivative interactions on D-branes, Phys. Rev. D 85 (2012) 125039, arXiv:1203.5553.
- [4] K. Becker, G. Guo, D. Robbins, Four-derivative brane couplings from string amplitudes, J. High Energy Phys. 1112 (2011) 050, arXiv:1110.3831;
K. Becker, G. Guo, D. Robbins, Disc amplitudes, picture changing and space–time actions, J. High Energy Phys. 1201 (2012) 127, arXiv:1106.3307;
K. Becker, G. Guo, D. Robbins, Higher derivative brane couplings from T-duality, J. High Energy Phys. 1009 (2010) 029, arXiv:1007.0441.
- [5] M.R. Garousi, Ricci curvature corrections to type II supergravity, Phys. Rev. D 87 (2013) 025006, arXiv:1210.4379.
- [6] M.R. Garousi, T-duality of the Riemann curvature corrections to supergravity, Phys. Lett. B 718 (2013) 1481, arXiv:1208.4459.
- [7] L.A. Barreiro, R. Medina, Revisiting the S-matrix approach to the open superstring low energy effective lagrangian, J. High Energy Phys. 10 (2012) 108, arXiv:1208.6066.
- [8] S. Stieberger, Open and closed vs. pure open string disk amplitudes, arXiv:0907.2211.
- [9] H. Gomez, C.R. Mafra, The closed-string 3-loop amplitude and S-duality, arXiv:1308.6567.
- [10] M.B. Green, C.R. Mafra, O. Schlotterer, Multiparticle one-loop amplitudes and S-duality in closed superstring theory, arXiv:1307.3534.
- [11] C.R. Mafra, O. Schlotterer, The structure of n -point one-loop open superstring amplitudes, arXiv:1203.6215.
- [12] S. Stieberger, T.R. Taylor, Superstring/supergravity correspondence in Grassmannian formulation, arXiv:1306.1844.
- [13] S. Stieberger, T.R. Taylor, Superstring amplitudes as Mellin transform of supergravity, Nucl. Phys. B 873 (2013) 65, arXiv:1303.1532.
- [14] O. Schlotterer, S. Stieberger, Motivic multiple zeta values and superstring amplitudes, arXiv:1205.1516.
- [15] J. Broedel, O. Schlotterer, S. Stieberger, Polylogarithms, multiple zeta values and superstring amplitudes, arXiv:1304.7267.
- [16] J. Broedel, O. Schlotterer, S. Stieberger, T. Terasoma, All order α' expansion of superstring trees from the Drinfeld associator, arXiv:1304.7304.
- [17] S. Stieberger, Closed superstring amplitudes, single-valued multiple zeta values and Deligne associator, arXiv:1310.3259.
- [18] L.A. Barreiro, R. Medina, 5-field terms in the open superstring effective action, J. High Energy Phys. 03 (2005) 055, <http://xxx.lanl.gov/abs/hep-th/0503182>.
- [19] D. Oprisa, S. Stieberger, Six gluon open superstring disk amplitude, multiple hypergeometric series and Euler–Zagier sums, <http://xxx.lanl.gov/abs/hep-th/0509042>;
S. Stieberger, T.R. Taylor, Amplitude for N-gluon superstring scattering, Phys. Rev. Lett. 97 (2006) 21601, <http://xxx.lanl.gov/abs/hep-th/0607184>;
S. Stieberger, T.R. Taylor, Multi-gluon scattering in open superstring theory, Phys. Rev. D 74 (2006) 126007, <http://xxx.lanl.gov/abs/hep-th/0609175>.

- [20] S. Stieberger, T. Taylor, Supersymmetry relations and MHV amplitudes in superstring theory, *Nucl. Phys. B* 793 (2008) 83, [arXiv:0708.0574](http://arxiv.org/abs/0708.0574).
- [21] C.R. Mafra, O. Schlotterer, S. Stieberger, Complete N -point superstring disk amplitude I. Pure spinor computation, *Nucl. Phys. B* 873 (2013) 419, [arXiv:1106.2645](http://arxiv.org/abs/1106.2645).
- [22] Y. Kitazawa, Effective lagrangian for the open superstring from a 5-point function, *Nucl. Phys. B* 289 (1987) 599.
- [23] F. Brandt, F. Machado, R. Medina, The open superstring 5-point amplitude revisited, *J. High Energy Phys.* 07 (2002) 071, <http://xxx.lanl.gov/abs/hep-th/0208121>.
- [24] H. Gomes, Uso de Somas Harmônicas no cálculo dos coeficientes das expansões de algumas funções não elementares, Undergraduate Research Study, Universidade Federal de Itajubá, Minas Gerais, Brasil, 2010, Portuguese version, unpublished.
- [25] J.A.M. Vermaseren, Harmonic sums, Mellin transforms and integrals, *Int. J. Mod. Phys. A* 14 (1999) 2037, <http://xxx.lanl.gov/abs/hep-ph/9806280>.
- [26] E. Remiddi, J.A.M. Vermaseren, Harmonic polylogarithms, *Int. J. Mod. Phys. A* 15 (2000) 725, <http://xxx.lanl.gov/abs/hep-ph/9905237>.
- [27] C.V. Paiva, Uso de polilogaritmos harmônicos na determinação de algumas integrais impróprias múltiplas, Scientific Initiation work, Universidade Federal de Itajubá, Minas Gerais, Brasil, 2004, Portuguese version, unpublished.
- [28] R.H. Boels, On the field theory expansion of superstring five point amplitudes, *Nucl. Phys. B* 876 (2013) 215, [arXiv:1304.7918](http://arxiv.org/abs/1304.7918).
- [29] J.M. Drummond, E. Ragoucy, Superstring amplitudes and the associator, *J. High Energy Phys.* 1308 (2013) 135, [arXiv:1301.0794](http://arxiv.org/abs/1301.0794).
- [30] N. Berkovits, Super-Poincaré covariant quantization of the superstring, *J. High Energy Phys.* 0004 (2000) 18, <http://xxx.lanl.gov/abs/hep-th/0001035>.
- [31] C.R. Mafra, O. Schlotterer, S. Stieberger, D. Tsimpis, A recursive method for SYM n -point tree amplitudes, *Phys. Rev. D* 83 (2011) 126012, [arXiv:1012.3981](http://arxiv.org/abs/1012.3981).
- [32] D. Friedan, E. Martinec, S. Shenker, Conformal invariance, supersymmetry and string theory, *Nucl. Phys. B* 271 (1986) 93.
- [33] J.H. Schwarz, Superstring theory, *Phys. Rep.* 89 (1982) 233.
- [34] F. Brown, Multiple zeta values and periods of moduli spaces $\mathcal{M}_{0,n}$, *Ann. Sci. Éc. Norm. Supér.* 42 (2009) 371, [arXiv:math/0606419](http://arxiv.org/abs/math/0606419).
- [35] C.R. Mafra, O. Schlotterer, S. Stieberger, Complete N -point superstring disk amplitude II. Amplitude and hypergeometric function structure, *Nucl. Phys. B* 873 (2013) 461, [arXiv:1106.2646](http://arxiv.org/abs/1106.2646).
- [36] H. Kawai, D.C. Lewellen, S.-H.H. Tye, A relation between tree amplitudes of closed and open strings, *Nucl. Phys. B* 269 (1986) 1.
- [37] Z. Bern, J.J.M. Carrasco, H. Johansson, New relations for gauge theory amplitudes, *Phys. Rev. D* 78 (2008) 085011, [arXiv:0805.3993](http://arxiv.org/abs/0805.3993).
- [38] N.E. Bjerrum-Bohr, P.H. Damgaard, P. Vanhove, Minimal basis for gauge theory amplitudes, *Phys. Rev. Lett.* 103 (2009) 161602, [arXiv:0907.1425](http://arxiv.org/abs/0907.1425).
- [39] A.A. Tseytlin, Vector field effective action in the open superstring theory, *Nucl. Phys. B* 276 (1986) 391; A.A. Tseytlin, Vector field effective action in the open superstring theory, *Nucl. Phys. B* 291 (1987) 876, Erratum.
- [40] P. Koerber, A. Sevrin, The non-abelian Born–Infeld action through order α'^3 , *J. High Energy Phys.* 10 (2001) 003, <http://xxx.lanl.gov/abs/hep-th/0108169>;
P. Koerber, A. Sevrin, Testing the α'^3 term in the non-abelian open superstring effective action, *J. High Energy Phys.* 09 (2001) 009, <http://xxx.lanl.gov/abs/hep-th/0109030>;
P. Koerber, A. Sevrin, The non-abelian D-brane effective action through order α'^4 , *J. High Energy Phys.* 10 (2002) 046, <http://xxx.lanl.gov/abs/hep-th/0208044>;
P. Koerber, Abelian and non-abelian D-brane effective actions, *Fortschr. Phys.* 52 (2004) 871, <http://xxx.lanl.gov/abs/hep-th/0405227>.
- [41] F. Brown, On the decomposition of motivic zeta values, [arXiv:1102.1310](http://arxiv.org/abs/1102.1310).
- [42] J. Blümlein, D. Broadhurst, J.A.M. Vermaseren, The multiple zeta value data mine, *Comput. Phys. Commun.* 181 (2010) 582, [arXiv:0907.2557](http://arxiv.org/abs/0907.2557).
- [43] M. De Roo, M.G.C. Eenink, The effective action for the 4-point functions in abelian open superstring theory, *J. High Energy Phys.* 0308 (2003) 036, <http://xxx.lanl.gov/abs/hep-th/0307211>.
- [44] O. Chandia, R. Medina, 4-point effective actions in open and closed superstring theory, *J. High Energy Phys.* 0311 (2003) 003, <http://xxx.lanl.gov/abs/hep-th/0310015>.
- [45] H. Nastase, H.J. Schnitzer, On KLT and SYM-supergravity relations from 5-point 1-loop amplitudes, *J. High Energy Phys.* 1101 (2011) 048, [arXiv:1011.2487](http://arxiv.org/abs/1011.2487).

- [46] M.B. Green, J.H. Schwarz, E. Witten, *Superstring Theory, vol. 1: Introduction*, Cambridge Monographs on Mathematical Physics, Cambridge Univ. Press, Cambridge, UK, 1987.
- [47] J. Polchinski, *String Theory, vol. 1: An Introduction to the Bosonic String*, Cambridge Univ. Press, Cambridge, UK, 1998.
- [48] F. Machado, R. Medina, The open superstring and the non-abelian Born–Infeld theory, *Nucl. Phys. B, Proc. Suppl.* 127 (2004) 166.
- [49] V.A. Kosteletzky, O. Lechtenfeld, D. Sahdev, The six-fermion amplitude in the superstring, *Phys. Lett. B* 183 (1987) 299.
- [50] L.A. Barreiro, R. Medina, Higher N -point amplitudes in open superstring theory, *PoS IC2006* (2006) 038, <http://xxx.lanl.gov/abs/hep-th/0611349>.
- [51] M.L. Mangano, S.J. Parke, Multi-parton amplitudes in gauge theories, *Phys. Rep.* 200 (6) (1996) 301, <http://xxx.lanl.gov/abs/hep-th/0509223>.
- [52] J. Broedel, O. Schlotterer, S. Stieberger, α' -expansion of open superstring amplitudes, wwwth.mpp.mpg.de/members/stieberg/mzv/index.html.
- [53] L.A. Barreiro, R. Medina, work in preparation.
- [54] D. Vaman, Y.-P. Yao, Constraints and generalized gauge transformations on tree-level gluon and graviton amplitudes, *J. High Energy Phys.* 1011 (2010) 028, [arXiv:1007.3475](http://arxiv.org/abs/1007.3475).
- [55] R. Kleiss, H. Kuijf, Multi-gluon cross-sections and five jet production at hadron colliders, *Nucl. Phys. B* 312 (1989) 616.
- [56] H. Gomes, Dualidade cor-cinémática na teoria de Yang–Mills e as relações BCJ, M.Sc. Thesis, Universidade Federal de Itajubá, Minas Gerais, Brasil, 2013, Portuguese version, unpublished.
- [57] A. Collinucci, M. De Roo, M.G.C. Eenink, Supersymmetric Yang–Mills theory at order α'^3 , *J. High Energy Phys.* 06 (2002) 024, <http://xxx.lanl.gov/abs/hep-th/0205150>.
- [58] G. Arfken, *Mathematical Methods for Physicists*, third edition, Academic Press, San Diego, USA, 1985.