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Antonio Accioly

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Energy and momentum for the electromagnetic field described by three outstanding electrodynamics

Antonio Accioly[a)]
Instituto de Física Teórica, Universidade Estadual Paulista, Rua Pamplona 145, 01405-900 São Paulo, SP, Brazil

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A prescription for computing the symmetric energy–momentum tensor from the field equations is presented. The method is then used to obtain the total energy and momentum for the electromagnetic field described by Maxwell electrodynamics, Born–Infeld nonlinear electrodynamics, and Podolsky generalized electrodynamics, respectively.

I. INTRODUCTION

There are two ways of obtaining the expression for the total energy and momentum related to a given relativistic field such as the Maxwell field. The first, which incidentally is described in most college or university textbooks on physics, is based on mechanics and employs concepts such as force, work, and so on. The second way is to make use of the so-called symmetric energy–momentum tensor. Unfortunately, the usual prescription for calculating this tensor is rather cumbersome for those who are not facile with field theory. As a consequence, the study of this powerful and elegant method of easily calculating the total energy and momentum for localized fields is, in general, relegated to graduate courses. Our aim here is to present an elementary algorithm for computing the symmetric energy–momentum tensor from the field equations and apply it to obtain the total energy and momentum for the electromagnetic field related to the three most famous known electrodynamics, i.e., the electrodynamics developed by Maxwell,[1] Born–Infeld,[2] and Podolsky,[3] respectively. By elementary we mean that no effort will be expended on field theoretic technicalities and that correspondingly no formal sophistication will be demanded of the reader.

In Sec. II we outline the algorithm in hand and then apply it to obtain the total energy and momentum for the sourceless Maxwell field. Section III is devoted to the electrodynamics of Born–Infeld and Podolsky, in this order. We present our conclusions in Sec. IV.

In what follows we will work in natural units where $\hbar = c = 1$ and use the Heaviside–Lorentz units with c replaced by unit for the electromagnetic theories. Our conventions for relativity follow nearly all recent field theory texts. We use the metric tensor

$$\eta_{\mu\nu} = \eta^{\mu\nu} = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \\ 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & -1 \end{pmatrix},$$

with Greek indices running over 0, 1, 2, 3. Roman indices—i,j, etc.—denote only the three spatial components. Repeated indices are summed in all cases.

II. THE ALGORITHM

Suppose one wants to find the symmetric energy–momentum tensor for some free relativistic field. We take for granted that this tensor, which from now on will be denoted by $T^{\mu\nu}$ ($T^{\mu\nu}=T^{\nu\mu}$), obeys the differential conservation law:[1]

$$\partial_\nu T^{\mu\nu}=0. \tag{1}$$

The continuity equation (1) yields the conservation of total energy and momentum upon integration over all of three-space at fixed time. Indeed,

$$0=\int d^3\mathbf{x}\partial_\nu T^{\mu\nu}=\frac{d}{dt}\int d^3\mathbf{x}T^{\mu 0}+\int d^3\mathbf{x}\partial_i T^{\mu i}.$$

For localized fields, the second integral—a divergence—gives no contribution, and taking into account that[1]

$$P^\mu=\int d^3\mathbf{x}T^{0\mu}, \tag{2a}$$

where

$$P^\mu=(\epsilon,\mathbf{P}), \tag{2b}$$

is the four-momentum of the field, and ϵ and $\mathbf{P}$ are, respectively, the total energy and three-momentum of the field, we find

$$\frac{dP^\mu}{dt}=0,$$

or, equivalently,

$$\frac{d\epsilon}{dt}=0, \quad \frac{d\mathbf{P}}{dt}=0.$$

If we ''switch on'' a variable external ''current'' which acts as a source for the field, the field equation will be modified and the energy–momentum tensor will no longer satisfy (1), as in the case of the free field. Instead, it will now obey the equation:

$$\partial_\nu T^{\mu\nu}=f^\mu, \tag{3}$$

where the contravariant vector f^μ, which has the dimension of force density, describes the interaction of the field with the current. From (3) we promptly obtain

$$\frac{dP^\mu}{dt}=\int d^3\mathbf{x}f^\mu.$$

This equation tells us that the rate of change of the four-momentum of the field is equal to the ''total force'' $\int d^3\mathbf{x}f^\mu$, which is the analogue of the Minkowski equation for a single particle.[1] Introducing an energy–momentum tensor $T_M^{\mu\nu}$ for the current, i.e., for the matter, by the definition

$$\partial_\nu T_M^{\mu\nu}=-f^\mu,$$

we can rewrite (3) in the form of a continuity equation:

$$\partial_\nu(T^{\mu\nu}+T_M^{\mu\nu})=0.$$

Thus the total energy and momentum of the field and the matter are conserved.

We return now to Eq. (3). Let us then postulate that f^μ is the simplest contravariant vector constructed with a given current and a suitable derivative of the field. We qualify this statement by means of some examples.

(i) The real Klein–Gordon field. In a sense the simplest field theory is that of a real field $\phi(x)$ that transforms as a Lorentz scalar. This field, which corresponds to electrically neutral particles, is known to satisfy the Klein–Gordon equation[4]

$$(\Box+m^2)\phi(x)=0,$$

where $\Box=\eta^{\mu\nu}\partial_\mu\partial_\nu=\partial_\mu\partial^\mu$. If we introduce a ''source'' $j(x)$ for the field, the field equation turns out to be

$$(\Box+m^2)\phi(x)=j(x).$$

The simplest expression for the ''force density'' f^μ built with a given scalar current $j(x)$ and a derivative of the scalar field is of the form

$$f^\mu=j\partial^\mu\phi=(\partial^\mu\phi)(\Box+m^2)\phi.$$

Note that for some j there is one and only one derivative of the field that couples in the simplest way to j so that the resulting expression is a contravariant vector.

(ii) The Maxwell field. In the absence of sources, Maxwell equations are[5,6]

$$\partial_\nu F^{\mu\nu}=0, \tag{4}$$

$$\partial_\alpha F_{\mu\nu}+\partial_\nu F_{\alpha\mu}+\partial_\mu F_{\nu\alpha}=0, \tag{5}$$

where $F^{\mu\nu}$ denotes the antisymmetric electromagnetic field tensor ($F^{\mu\nu}=-F^{\nu\mu}$). Introducing a source j^μ for the electromagnetic field, the field equations take on the form

$$\partial_\nu F^{\mu\nu}=j^\mu, \tag{6}$$

$$\partial_\alpha F_{\mu\nu}+\partial_\nu F_{\alpha\mu}+\partial_\mu F_{\nu\alpha}=0. \tag{7}$$

The force density is given in this case by

$$f_\alpha=F_{\alpha\mu}j^\mu=F_{\alpha\mu}\partial_\nu F^{\mu\nu}. \tag{8}$$

(iii) The complex Klein–Gordon field. This field is known to obey the equations[4]

$$(\Box+m^2)\phi(x)=0, \quad (\Box+m^2)\phi^*(x)=0,$$

where the ϕ^* field is the complex conjugate of the ϕ field. The complex Klein–Gordon field corresponds to electrically charged particles. Let us then introduce a ''source'' $j^*(x)$ for the ϕ^* field. As a consequence,

$$(\Box+m^2)\phi^*=j^*, \quad (\Box+m^2)\phi=j.$$

The ''force density,'' which of course is a real quantity, has the form

$$\begin{aligned} f^\mu &= j^*\partial^\mu\phi+j\partial^\mu\phi^* \\ &= \partial^\mu\phi(\Box+m^2)\phi^*+\partial^\mu\phi^*(\Box+m^2)\phi. \end{aligned}$$

(iv) The Dirac field. The corresponding field equations are[4]

$$i\gamma^\mu\partial_\mu\psi(x)-m\psi(x)=0,$$

$$\partial_\mu\overline{\psi(x)}i\gamma^\mu+m\overline{\psi(x)}=0,$$

where $\overline{\psi}\equiv\psi^\dagger\gamma^0$ is the adjoint spinor to ψ. The indices labeling spinor components and matrix elements were suppressed for the sake of simplicity. We assume that $\overline{j}=j^\dagger\gamma^0$ is the ''source'' for the ψ field. As a result,

$$i\partial_\mu\overline{\psi}\gamma^\mu+m\overline{\psi}=\overline{j}, \quad i\gamma^\mu\partial_\mu\psi-m\psi=-j.$$

The ''force density,'' which incidentally is an Hermitian quantity, can be written as

$$f_\alpha = \overline{j}\partial_\alpha\psi + \partial_\alpha\overline{\psi}j$$
$$= (i\partial_\mu\overline{\psi}\gamma^\mu + m\overline{\psi})\partial_\alpha\psi - \partial_\alpha\overline{\psi}(i\gamma^\mu\partial_\mu\psi - m\psi).$$

We hope the previous examples have clarified what ''the simplest contravariant vector constructed with a given current and a suitable derivative of the field'' means. Note that there is nothing ambiguous in the prescription for building the force density out of the current and the derivative of the field.

Based on the previous considerations, we now find $T^{\mu\nu}$ for the free Maxwell field. We do that in two steps.

(i) We ''turn on'' a current j^μ: From (3) and (8), we have

$$\partial^\mu T_{\alpha\mu} = f_\alpha = F_{\alpha\mu}j^\mu = F_{\alpha\mu}\partial_\nu F^{\mu\nu}. \tag{9}$$

The right-hand side of this equation can be expanded to

$$F_{\alpha\mu}\partial_\nu F^{\mu\nu} = \partial_\nu(F_{\alpha\mu}F^{\mu\nu}) - (\partial_\nu F_{\alpha\mu})F^{\mu\nu},$$

and, exchanging the dummy indices μ and ν on the second term, we get

$$F_{\alpha\mu}\partial_\nu F^{\mu\nu} = \partial_\nu(F_{\alpha\mu}F^{\mu\nu}) - \tfrac{1}{2}(\partial_\nu F_{\alpha\mu} + \partial_\mu F_{\nu\alpha})F^{\mu\nu}.$$

Using (7), we can replace the term in parentheses with $-\partial_\alpha F_{\mu\nu}$, leading to

$$F_{\alpha\mu}\partial_\nu F^{\mu\nu} = \partial_\nu(F_{\alpha\mu}F^{\mu\nu}) + \tfrac{1}{2}\partial_\alpha F_{\mu\nu}F^{\mu\nu}$$
$$= \partial_\nu(F_{\alpha\mu}F^{\mu\nu}) + \tfrac{1}{4}\partial_\alpha(F_{\mu\nu}F^{\mu\nu}).$$

But,

$$\partial_\alpha = \eta_{\alpha\beta}\partial^\beta = \eta_{\alpha\beta}\eta^{\beta\nu}\partial_\nu = \delta^\nu_\alpha\partial_\nu. \tag{10}$$

Thus

$$F_{\alpha\mu}\partial_\nu F^{\mu\nu} = \partial_\nu[F_{\alpha\mu}F^{\mu\nu} + \tfrac{1}{4}\delta^\nu_\alpha F_{\rho\theta}F^{\rho\theta}]. \tag{11}$$

From (9) and (10) it follows that

$$\partial^\nu T_{\alpha\nu} = \partial_\nu[F_{\alpha\nu}F^{\mu\nu} + \tfrac{1}{4}\delta^\nu_\alpha F_{\rho\theta}F^{\rho\theta}].$$

If we now multiply both sides of the above equation by $\eta^{\alpha\beta}$, we immediately obtain

$$\partial^\nu T^\beta_\nu = \partial_\nu[F^\beta_\mu F^{\mu\nu} + \tfrac{1}{4}\eta^{\beta\nu}F_{\rho\theta}F^{\rho\theta}]. \tag{12}$$

Note that

$$\eta^{\alpha\beta}F_{\alpha\mu} = F^\beta_\mu, \qquad \delta^\nu_\alpha\eta^{\alpha\beta} = \eta^{\nu\beta}.$$

Equation (11) can be rewritten as

$$\partial_\nu T^{\beta\nu} = \partial_\nu[F^\beta_\mu F^{\mu\nu} + \tfrac{1}{4}\eta^{\beta\nu}F_{\rho\theta}F^{\rho\theta}].$$

(ii) We ''turn off'' the current: As a result,

$$f^\beta = \partial_\nu T^{\beta\nu} = \partial_\nu[F^\beta_\mu F^{\mu\nu} + \tfrac{1}{4}\eta^{\beta\nu}F_{\rho\theta}F^{\rho\theta}] = 0.$$

So, the fully contravariant form of the symmetric energy–momentum tensor for the Maxwell field is

$$T^{\beta\nu} = F^\beta_\mu F^{\mu\nu} + \tfrac{1}{4}\eta^{\beta\nu}F_{\rho\theta}F^{\rho\theta}. \tag{13}$$

We may now present a simple algorithm for computing the symmetric energy–momentum tensor from the field equation. Multiply the equation for the field in hand by a suitable derivative of this field so that the resulting expression contains only one free spacetime index and then rewrite it as a four-derivative. Of course, this algorithm is nothing but an ''operational'' summary of the main points we have just discussed. As such, its main role is to work as a mnemonic device for the reader.

Note that, contrary to what happens in usual field theory, here we do not have to postulate the equation of motion of the material bodies under the influence of the interaction; i.e., we do not have to introduce a formula for the force density representing the action of the field on each body. In fact, the simplicity criterion we have previously introduced determines automatically the force density. Consider, in this vein, the Maxwell theory. From (9), the force density is given by

$$f^\alpha = F^{\alpha\beta}j_\beta,$$

whereupon

$$j^\mu = (\rho, \mathbf{j}),$$

where ρ is the charge density. On the other hand, the components $E^i(B^i)$ of the electric field (magnetic field) can be related to the components of the electromagnetic field tensor as follows:[5,6]

$$E^i = F^{0i}, \tag{14}$$

$$B^i = \tfrac{1}{2}\epsilon^{ilm}F_{lm}, \tag{15}$$

where ϵ^{ilm} is the Levi–Civita density which equals $+1$ (-1) if $i,\ l,\ m$ is an even (odd) permutation from 1, 2, 3, and vanishes if two indices are equal. Therefore,

$$f^0 = F^{0i}j_i = \mathbf{E}\cdot\mathbf{j},$$

and

$$f^k = F^{k\beta}j_\beta = F^{k0}j_0 + F^{ki}j_i = F^{k0}j_0 + \epsilon^{kil}j_iB_l$$
$$= (\rho\mathbf{E})^k + (\mathbf{j}\times\mathbf{B})^k.$$

So,

$$f^\alpha = (\mathbf{E}\cdot\mathbf{j}, \rho\mathbf{E} + \mathbf{j}\times\mathbf{B}),$$

which is nothing but the well-known Lorentz force density.

Let us then find the total energy and momentum for the Maxwell field with zero source. Using (2a), (2b), (13), (14), and (15), yields

$$\epsilon_{\text{field}} = \int d^3\mathbf{x}\left[\frac{1}{2}\,F^{0i}F_{i0} + \frac{1}{4}\,F^{ij}F_{ij}\right]$$
$$= \int d^3\mathbf{x}\,\frac{1}{2}\,[\mathbf{E}^2 + \mathbf{B}^2], \tag{16}$$

$$P^i_{\text{field}} = \int d^3\mathbf{x}F_{0j}F^{ji} = \int d^3\mathbf{x}(\mathbf{E}\times\mathbf{B})^i. \tag{17}$$

Equation (17) can be rewritten as

$$\mathbf{P}_{\text{field}} = \int d^3\mathbf{x}(\mathbf{E}\times\mathbf{B}),$$

from which we trivially obtain the vector **S** representing energy flow, i.e., the Poynting vector,

$$\mathbf{S} = \mathbf{E}\times\mathbf{B}. \tag{18}$$

Note that $S^i = T^{0i}$.

III. THE NONLINEAR ELECTRODYNAMICS OF BORN–INFIELD AND THE HIGHER ORDER ELECTRODYNAMICS OF PODOLSKY

We shall now analyze two interesting attempts to modify Maxwell electrodynamics in order to get rid of the infinities that usually plague that theory. The first was made by Born[7] in the early 30s. He proposed that the field equations concerning Maxwell theory, Eqs. (4) and (5), should be replaced by

$$\partial_\nu\left[\frac{F^{\mu\nu}}{\sqrt{1+F/a^2}}\right]=0, \tag{19}$$

$$\partial_\alpha F_{\mu\nu}+\partial_\nu F_{\alpha\mu}+\partial_\mu F_{\nu\alpha}=0, \tag{20}$$

where $F\equiv\frac{1}{2}F_{\mu\nu}F^{\mu\nu}$ ($F^{\mu\nu}$ is the antisymmetric electromagnetic field tensor) and a is a constant with dimension of (length)^2. Note that Eq. (19) is highly nonlinear. Let us then find the symmetric energy–momentum tensor for this theory. For this purpose, multiply Eq. (19) by $F_{\alpha\mu}$:

$$F_{\alpha\mu}\partial_\nu\left[\frac{F^{\mu\nu}}{\sqrt{1+F/a^2}}\right]=0.$$

Using (20) we obtain

$$F_{\alpha\mu}\partial_\nu\left(\frac{F^{\mu\nu}}{\sqrt{1+F/a^2}}\right)=\partial_\nu\left[\frac{F_{\alpha\mu}F^{\mu\nu}}{\sqrt{1+F/a^2}}\right]+\frac{F^{\mu\nu}\partial_\alpha F_{\mu\nu}}{2\sqrt{1+F/a^2}}.$$

On the other hand,

$$\frac{F^{\mu\nu}\partial_\alpha F_{\mu\nu}}{2\sqrt{1+F/a^2}}=\frac{1}{4}\,\frac{\partial_\alpha(2F)}{\sqrt{1+F/a^2}}=a^2\partial_\alpha\sqrt{1+F/a^2}$$
$$=a^2\partial_\nu[\,\delta^\nu_\alpha\sqrt{1+F/a^2}]. \tag{21}$$

Thus the symmetric energy–momentum tensor is given by

$$T^{\alpha\nu}=\frac{F^{\alpha\mu}F^{\nu}_{\ \mu}}{\sqrt{1+F/a^2}}+\eta^{\alpha\nu}a^2\sqrt{1+F/a^2}. \tag{22}$$

Substitution of (22) into (2a) leads to the expression for the total energy of the field:

$$\epsilon_{\text{field}}=\int d^3\mathbf{x}\left[\frac{\mathbf{E}^2}{\sqrt{1+\dfrac{\mathbf{B}^2-\mathbf{E}^2}{a^2}}}+a^2\sqrt{1+\frac{\mathbf{B}^2-\mathbf{E}^2}{a^2}}\right].$$

Developing ϵ_{field} in powers of a:

$$\epsilon_{\text{field}}=\int d^3\mathbf{x}\left[a^2+\frac{1}{2}\,(\mathbf{E}^2+\mathbf{B}^2)+\cdots\right],$$

we see that, as a first approximation, the new ϵ_{field} differs from the old (16) by a constant. Yet, this is an infinite constant! Fortunately, only energy differences are observable. Hence, this embarrassing infinite constant is harmless and easily removed by redefining the symmetric energy–momentum tensor to be

$$T^{\mu\nu}=\frac{F^{\alpha\mu}F^{\nu}_{\ \mu}}{\sqrt{1+F/a^2}}+\eta^{\mu\nu}a^2[\,\sqrt{1+F/a^2}-1].$$

The total energy of the field is now given by

$$\epsilon_{\text{field}}=\int d^3\mathbf{x}\left[\frac{\mathbf{E}^2}{\sqrt{1+\dfrac{\mathbf{B}^2-\mathbf{E}^2}{a^2}}}+a^2\left(\sqrt{1+\frac{\mathbf{B}^2-\mathbf{E}^2}{a^2}}-1\right)\right].$$

Expanding the square root and keeping only terms of second order, we recover the expression for the total energy of the Maxwell field; i.e.,

$$\epsilon_{\text{field}}=\int d^3\mathbf{x}\,\frac{1}{2}\,(\mathbf{E}^2+\mathbf{B}^2).$$

The total momentum of the field and the Poynting vector, in turn, are given, respectively, by

$$P^i_{\text{field}}=\int d^3\mathbf{x}\,\frac{(\mathbf{E}\times\mathbf{B})^i}{\sqrt{1+\dfrac{\mathbf{B}^2-\mathbf{E}^2}{a^2}}},\qquad \mathbf{S}=\frac{\mathbf{E}\times\mathbf{B}}{\sqrt{1+\dfrac{\mathbf{B}^2-\mathbf{E}^2}{a^2}}}.$$

Since Born's equations can be derived via a variational principle from a Lagrangian density $\mathscr{L}$, in the limit of small field strengths, $\mathscr{L}$ has to approach $\mathscr{L}^{(0)}=\frac{1}{2}(\mathbf{E}^2-\mathbf{B}^2)$, which is exactly the Lagrangian density leading to Maxwell's equations. The Lagrangian density related to Born's equations is[7]

$$\mathscr{L}=-a^2\sqrt{1+F/a^2}. \tag{23}$$

Expanding the square root, we obtain

$$\mathscr{L}=-a^2+\tfrac{1}{2}(\mathbf{E}^2-\mathbf{B}^2)+\cdots\ .$$

This Lagrangian density, as was expected, differs from the Maxwell one by a constant term. Now, the usual procedure for calculating the symmetric energy–momentum tensor, and consequently the total energy of the field, is entirely based on the Lagrangian density, which incidentally is not a physically measurable quantity in the usual sense but a convenient mathematical tool. As a consequence, the symmetric energy–momentum tensor computed via (23) leads to the same troublesome value for the energy as before. Is it possible to avoid this problem without further redefining the symmetric energy–momentum tensor? Yes, we redefine the Lagrangian density as follows:

$$\mathscr{L}=a^2[1-\sqrt{1+F/a^2}], \tag{24}$$

leading in first approximation to

$$\mathscr{L}=\tfrac{1}{2}(\mathbf{E}^2-\mathbf{B}^2)+\cdots\ .$$

The $T^{\mu\nu}$ computed using (24) gives a finite value for the energy. Note, however, that the original Lagrangian density proposed by Born (23) as well as the Lagrangian density (24) yield the same field equations (19). In a sense this way of sweeping delicate questions under the carpet, despite being mathematically elegant, disguises the underlying physics of the process.

The fact that Lagrangian density (23) does not reproduce Maxwell Lagrangian density in the limit of small field strengths, among other things, led Born and Infeld to reformulate Born's original theory. Since the Lagrangian density $\mathscr{L}$ must be a Lorentz scalar and the electromagnetic field has only two gauge invariant Lorentz scalars, namely,

$$F=\tfrac{1}{2}F_{\mu\nu}F^{\mu\nu}=\mathbf{B}^2-\mathbf{E}^2,$$

$$G^2=(\tfrac{1}{4}F^*_{\mu\nu}F^{\mu\nu})^2=(\mathbf{E}\cdot\mathbf{B})^2,$$

where ${}^*F^{\mu\nu}=\frac{1}{2}\epsilon^{\mu\nu\rho\sigma}F_{\rho\sigma}$ is the dual field strength tensor, $\mathscr{L}$ can be a function of F and G^2 only. Born and Infield opted for the following function of the invariants, as their Lagrangian density[2]

$$\mathscr{L}=a^2\left[1-\sqrt{1+\frac{F}{a^2}-\frac{G^2}{a^4}}\right],$$

where a has the meaning of a maximum field strength. So, for fields much weaker than a we recover Maxwell Lagrangian density by expanding the square root and keeping only terms of second order. They chose this Lagrangian density in analogy to the relativistic Lagrangian for a free particle, $L=m[1-\sqrt{1-v^2}]$, which in the nonrelativistic limit $(v\ll 1)$ reduces to the well-known expression $L=mv^2/2$. Note that just as the limiting velocity $v=1$ is of no importance in classical mechanics, the maximum field strength a is irrelevant for classical electrodynamics. The field equations for Born–Infeld theory are

$$\partial_\nu\left[\frac{F^{\mu\nu}-{}^*F^{\mu\nu}G/a^2}{\sqrt{1+\frac{F}{a^2}-\frac{G^2}{a^4}}}\right]=0,$$

$$\partial_\alpha F_{\mu\nu}+\partial_\nu F_{\alpha\mu}+\partial_\mu F_{\nu\alpha}=0.$$

Using our algorithm we find that the symmetric energy–momentum tensor for that theory is given by

$$T^{\mu\nu}=\frac{F^{\mu\rho}F_\rho^{\ \nu}+G^2\eta^{\mu\nu}/a^2}{\sqrt{1+\frac{F}{a^2}-\frac{G^2}{a^4}}}+\eta^{\mu\nu}a^2\sqrt{1+\frac{F}{a^2}-\frac{G^2}{a^4}}.$$

In deriving this result, use has been made of the identity

$${}^*F^{\mu\nu}F_{\nu\rho}=-G\,\delta^\mu_\rho.$$

Just as we have done before, we redefine the symmetric energy–momentum tensor to avoid troublesome constants:

$$T^{\mu\nu}=\frac{F^{\mu\rho}F_\rho^{\ \nu}+G^2\eta^{\mu\nu}/a^2}{\sqrt{1+\frac{F}{a^2}-\frac{G^2}{a^4}}}+\eta^{\mu\nu}a^2\left[\sqrt{1+\frac{F}{a^2}-\frac{G^2}{a^4}}-1\right].$$

The energy, the momentum, and the Poynting vector, are now given, respectively, by

$$\epsilon_{\text{field}}=\int d^3\mathbf{x}\left[\frac{\mathbf{E}^2+(\mathbf{E}\cdot\mathbf{B}/a)^2}{\sqrt{1+\frac{\mathbf{B}^2-\mathbf{E}^2}{a^2}-\frac{(\mathbf{E}\cdot\mathbf{B})^2}{a^4}}}+a^2\left(\sqrt{1+\frac{\mathbf{B}^2-\mathbf{E}^2}{a^2}-\frac{(\mathbf{E}\cdot\mathbf{B})^2}{a^4}}-1\right)\right],$$

$$\mathbf{P}_{\text{field}}=\int d^3\mathbf{x}\,\frac{\mathbf{E}\times\mathbf{B}}{\sqrt{1+\frac{\mathbf{B}^2-\mathbf{E}^2}{a^2}-\frac{(\mathbf{E}\cdot\mathbf{B})^2}{a^4}}},$$

$$\mathbf{S}=\frac{\mathbf{E}\times\mathbf{B}}{\sqrt{1+\frac{\mathbf{B}^2-\mathbf{E}^2}{a^2}-\frac{(\mathbf{E}\cdot\mathbf{B})^2}{a^4}}}.$$

An attractive feature of this theory is that it can account for a stable electron with a finite size and finite self-energy.

The second notable attempt to modify Maxwell electrodynamics was made by Podolsky[3] in the early 40s. In generalizing the equations of Maxwell electrodynamics, Podolsky made every effort to leave the ordinary assumptions of Maxwell theory as nearly unaltered as possible. His higher-order electrodynamics is described by the equations

$$(1+a^2\Box)\partial_\nu F^{\mu\nu}=0, \tag{25}$$

$$\partial_\alpha F_{\mu\nu}+\partial_\nu F_{\alpha\mu}+\partial_\mu F_{\nu\alpha}=0, \tag{26}$$

where a is a real parameter with dimension of length and $\Box\equiv\partial_\beta\partial^\beta$. It is worth mentioning that the usual prescription for obtaining the symmetric energy–momentum tensor for higher-order field theories is difficult to handle even for those that are familiar with field theory. Fortunately our recipe allows us to carry out this computation in a simple way.

To find $T^{\mu\nu}$ for Podolsky generalized electrodynamics we multiply (25) by $F_{\alpha\mu}$. Using (11) we obtain

$$\partial_\nu[F_{\alpha\mu}F^{\mu\nu}+\tfrac{1}{4}\delta^\nu_\alpha F_{\rho\theta}F^{\rho\theta}]+F_{\alpha\mu}a^2\Box\partial_\nu F^{\mu\nu}=0.$$

The second term can be transformed as follows:

$$a^2F_{\alpha\mu}\Box\partial_\nu F^{\mu\nu}=[\partial_\nu(F_{\alpha\mu}\Box F^{\mu\nu})-\partial_\nu F_{\alpha\mu}\Box F^{\mu\nu}]a^2.$$

From (26) and the above equation we obtain

$$a^2F_{\alpha\mu}\Box\partial_\nu F^{\mu\nu}=a^2[\partial_\nu(F_{\alpha\mu}\Box F^{\mu\nu})+\partial_\alpha(\tfrac{1}{2}F_{\mu\nu}\Box F^{\mu\nu})-\tfrac{1}{2}F^{\mu\nu}\Box\partial_\alpha F^{\mu\nu}].$$

On the other hand,

$$\begin{aligned}\tfrac{1}{2}F^{\mu\nu}\Box\partial_\alpha F_{\mu\nu}&=\tfrac{1}{2}F^{\mu\nu}\Box(\partial_\nu F_{\alpha\mu}+\partial_\mu F_{\nu\alpha})\\&=-F^{\mu\nu}\Box\partial_\nu F_{\alpha\mu}\\&=-\partial_\nu(F^{\mu\nu}\Box F_{\alpha\mu})+\partial_\nu F^{\mu\nu}\Box F_{\alpha\mu}.\end{aligned}$$

Similarly,

$$\begin{aligned}\partial_\nu F^{\mu\nu}\Box F_{\alpha\mu}&=\partial_\nu F^{\mu\nu}\partial^\beta\partial_\beta F_{\alpha\mu}\\&=-\partial_\nu F^{\mu\nu}\partial^\beta(\partial_\mu F_{\beta\nu}+\partial_\alpha F_{\mu\beta})\\&=-\partial_\mu(\partial_\nu F^{\mu\nu}\partial^\beta F_{\beta\alpha})-\partial_\alpha(\tfrac{1}{2}\partial_\nu F^{\mu\nu}\partial^\beta F_{\mu\beta}).\end{aligned}$$

Hence,

$$\begin{aligned}a^2F_{\alpha\mu}\Box\partial_\nu F^{\mu\nu}=a^2\partial_\nu[&F_{\alpha\mu}\Box F^{\mu\nu}+F^{\mu\nu}\Box F_{\alpha\mu}\\&-\partial^\beta F_{\alpha\beta}\partial_\mu F^{\nu\mu}+\tfrac{1}{2}\delta^\nu_\alpha(F_{\rho\theta}\Box F^{\rho\theta}\\&+\partial_\beta F^{\rho\theta}\partial^\beta F_{\rho\theta})].\end{aligned}$$

The symmetric energy–momentum tensor can then be expressed as

$$\begin{aligned}T^{\alpha\nu}=F^{\alpha\mu}F_\mu^{\ \nu}+\frac{1}{4}\eta^{\alpha\nu}F_{\rho\theta}F^{\rho\theta}+\frac{a^2}{2}\,\eta^{\alpha\nu}[&F_{\rho\theta}\Box F^{\rho\theta}\\+\partial_\theta F^{\rho\theta}\partial^\beta F_{\rho\beta}]-a^2[&F^{\alpha\mu}\Box F_\mu^{\ \nu}+F^{\nu\mu}\Box F_\mu^{\ \alpha}\\+\partial_\beta F^{\alpha\beta}\partial_\mu F^{\nu\mu}],&\end{aligned}$$

which agrees with the Podolsky result.

In electrostatics this gives for the energy,

$$\mathscr{E}_{\text{field}}=\int d^3\mathbf{x}\frac{1}{2}\{\mathbf{E}^2-a^2[(\nabla\cdot\mathbf{E})^2+2\mathbf{E}\cdot\nabla^2\mathbf{E}]\}. \tag{27}$$

Taking into account that $\nabla\times\mathbf{E}=0$, and assuming that $\mathbf{E}\nabla\cdot\mathbf{E}$ vanishes at infinity faster than $1/r^2$, one finds that Eq. (27) can easily be put in the form

$$\mathscr{E}_{\text{field}}=\frac{1}{2}\int d^3\mathbf{x}[\mathbf{E}^2+a^2(\nabla\cdot\mathbf{E})^2], \tag{28}$$

which is obviously positive. In the static case the scalar potential due to a point charge turns out to be[3]

$$\phi=\frac{Q}{4\pi r}(1-e^{-r/a}),$$

which approaches a finite value $Q/4\pi a$ as r approaches zero. Making use of $\mathbf{E}=-\nabla\phi$, we easily obtain the expression for the electrostatic field due to a point charge:

$$\mathbf{E}(\mathbf{r})=\frac{Q}{4\pi}\left[\frac{1}{r^2}-e^{-r/a}\left(\frac{1}{r^2}+\frac{1}{ar}\right)\right]\frac{\mathbf{r}}{r}. \tag{29}$$

Substituting (29) into (28) and performing the integration, we find that the energy for the fields of a point charge is given by $Q^2/2a$. Thus, unlike Maxwell electrodynamics, Podolsky generalized electrodynamics leads to a finite value for the energy of a point charge in the whole space. Of course, the momentum of the field in this case is equal to zero. The expressions for the total energy and momentum of the field in the general case are not very illuminating, so they will not be displayed here.

It is worth mentioning that our method tells us that the force law for Podolsky generalized electrodynamics is precisely the Lorentz force. If we consult Podolsky's paper we find that this is the very force law he assumed for his theory.

IV. CONCLUSION

We have devised an algorithm for computing the symmetric energy–momentum tensor from the field equations. The prescription can be used for both ordinary and higher-order field theory. It does not matter whether the field in hand is a tensor or a spinor field.

A last remark: The algorithm can be easily extended to curved space.[8,9]

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[a)]Electronic mail: accioly@axp.ift.unesp.br

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HIGH VOLTAGE

''Oh, and about lab rules,'' he added. ''The first rule is 'It's your job and not mine.' The second is 'I don't take any excuses,' and the third is 'Lab hours are 7:30 to 5:30, with half an hour for lunch.' You should be able to get all your lab work done between those hours. Harvard's Physics Department lost a student last year when he fell asleep into his experiment's high-voltage power supply at 2:00 in the morning. He was found dead the next day. We don't want that kind of thing to happen here in the Sloan Lab.''

Me neither. ''Uh huh''.

''You can do computer work at night and read journal articles. And try taking a swim after dinner for half an hour or so. I've found that exercising at that time makes me need less sleep and wakes me up so I can work another four or five hours. Come on downstairs—I'll show you your cell.''

Pepper White, *The Idea Factory—Learning to Think at MIT* (Penguin Books, New York, 1991), p. 79.