

IFT - UNESP
INSTITUTO DE FÍSICA TEÓRICA

TESE DE DOUTORAMENTO

IFT-T.004/24

Dark Energy and Neutrinos in Cosmology

Diogo Henrique Francis de Souza

Advisor: *Rogério Rosenfeld*

Co-advisor: *Vivian Miranda*

August, 2024

S729d	Souza, Diogo Henrique Francis de Dark energy and neutrinos in cosmology / Diogo Henrique Francis de Souza. – São Paulo, 2024 129 f.: il. color. Tese (doutorado) – Universidade Estadual Paulista (Unesp), Instituto de Física Teórica (IFT), São Paulo Orientador: Rogério Rosenfeld Coorientador: Vivian Miranda 1. Cosmologia. 2. Energia escura - Astronomia. 3. Neutrinos. I. Título
-------	--

Sistema de geração automática de fichas catalográficas da Unesp. Biblioteca do Instituto de Física Teórica (IFT), São Paulo. Dados fornecidos pelo autor(a).

Dark Energy and Neutrinos in Cosmology

Tese de doutorado apresentada ao Instituto de Física Teórica do Câmpus de São Paulo, da Universidade Estadual Paulista “Júlio de Mesquita Filho”, como parte dos requisitos para obtenção do título Doutor em Física , especialidade Física Teórica.

Comissão Examinadora:

Prof. Dr. ROGÉRIO ROSENFELD (Orientador)

Instituto de Física Teórica/UNESP

Prof. Dr. MIGUEL BOAVISTA QUARTIN

Universidade Federal do Rio de Janeiro

Prof. Dr. FELIPE TOVAR FALCIANO

Centro Brasileiro de Pesquisas Físicas

Profa. Dra. MARIANA PENNA LIMA VITENTI

Universidade de Brasília

Prof. Dr. FELIPE ANDRADE OLIVEIRA

Physik-Institute - University of Zurich

Conceito: Aprovado

São Paulo, 09 de Agosto de 2024.

Em memória de Maria Francisca e José Souza

Agradecimentos

A realização deste trabalho de doutorado foi possível devido às várias pessoas que me apoiaram ao longo da minha trajetória por estes quatro anos. Neste espaço limitado de linhas, expresso aqui os meus agradecimentos a cada uma destas pessoas.

Ao Rogério Rosenfeld, agradeço pela orientação e apoio em todos os momentos durante o doutorado. Sua assistência e dedicação comigo foram essenciais para tornar este trabalho possível. Também agradeço imensamente seu suporte para que eu fosse realizar o doutorado sanduíche. O estímulo do Rogério foi parte fundamental para que o meu intercâmbio se tornasse possível.

Agradeço a Vivian Miranda por ter feito parte do meu doutorado desde o início, e por também ter me recebido na Stony Brook University. Também agradeço o seu suporte fundamental para que o meu intercâmbio de fato pudesse ter se tornado realidade. Foi uma experiência incrível e de valor inestimável. Assim como o Rogério, Vivian foi muito importante para a minha formação.

Agradecer imensamente aos meus familiares que sempre estiveram comigo em todos os momentos; Adelina, Elizângela, Felipe e Sirlene, Mônica, Euvério e Fabíola.

Este trabalho definitivamente não teria sido possível sem um nome em especial; Letícia. Agradeço imensamente por sempre ter estado comigo nos momentos felizes, e também desafiadores, que apenas superei graças ao seu amor, carinho e suporte.

Expresso minhas lembranças para meu pai e meus irmãos; Vinícius e Miguel.

Agradeço João Rebouças, meu colega de doutorado no IFT e parceiro de pesquisa desde o início desta jornada. Sou muito grato por ter compartilhado estes quatro anos com João, com quem pude aprender muito e tive ótimas conversas.

Agradeço também ao Hugo Camacho, Felipe Oliveira e Lucas Faga que sempre de alguma forma estiveram envolvidos durante o meu processo e que portanto foram muito importantes para mim. Também agradeço Raphael e a Renata pelo companheirismo e ótimos momentos durante meu período em Stony Brook.

I want to thank my colleagues at Stony Brook University: Jonathan, Evan, Josh, Mark, and Yijie. I'm very grateful to Jonathan for his assistance with my transition from Brazil to New York and for his help during my early days at Stony Brook.

Agradeço a todos os funcionários do IFT que pude conhecer; ao pessoal da parte técnica e de administração; Rosane, Edna, Ana, Marcus, Rafael, Walter e Heitor; a dona Jô, e ao pessoal da limpeza; Rita, Mayara e Sandra. Às recepcionistas; Edilene, Lauane e Geane. Aos porteiros Adailton, Valdemiro e Genildo. Obrigado a todos vocês.

Expresso os meus agradecimentos a todos os professores de todas as escolas públicas por onde estudei; à professora Cláudia, por suas aulas maravilhosas, à bibliotecária Riva

pela companhia. Ao professor de Física Luslan que teve um papel muito importante na minha trajetória! E claro, agradeço à UFMG e à UFRGS, instituições com professores que me inspiraram com suas aulas incríveis. Sou uma grata consequência de cada um destes lugares que passei e pessoas que conheci.

Por fim, expresso meu agradecimento à CAPES pela bolsa de doutorado, e ao programa CAPES-Print que possibilitou a realização do meu doutorado sanduíche.

Resumo

O paradigma cosmológico predominante é o modelo Λ CDM com curvatura espacialmente plana e com condições iniciais adiabáticas e gaussianas. A letra grega “ Λ ” designa a constante cosmológica, enquanto a sigla “CDM” significa a Matéria Escura Fria e sem pressão. Até hoje, tal modelo tem sido a descrição mais econômica do Universo que é consistente com grande variedades de dados observacionais. Apesar do seu sucesso, o modelo cosmológico padrão não está isento de barreiras. O problema do horizonte e a planicidade são bem conhecidos com possíveis soluções quando trazemos à cena a teoria da Inflação Cósmica. Além disso, os levantamentos cosmológicos alcançaram uma melhorias significativas na precisão das suas medidas de sinais, o que nos permitiu impor restrições mais rigorosas aos parâmetros do modelo Λ CDM. Quando diferentes conjuntos de dados são combinados, os principais problemas que surgem são as chamadas tensões Hubble e S_8 .

Juntamente com a matéria escura, a energia escura e os neutrinos são dois componentes que merecem atenção especial pelo seu papel importante na física de partículas e na cosmologia. Este trabalho está separado em duas partes. Na primeira parte, forneço uma breve introdução à cosmologia, fornecendo referências para maiores detalhes ao leitor interessado. Inicio com as equações de background e de perturbação derivadas da teoria geral da relatividade de Einstein, e termino com as condições iniciais necessárias para resolver essas equações. Na segunda parte desta Tese, me concentro mais especificamente nos neutrinos e no seu efeito na formação de estruturas do universo, e na energia escura como campo escalar. Os dois últimos capítulos tratam de um artigo publicado e de um trabalho em progresso iniciado na Universidade de Stony Brook, respectivamente.

Palavras chaves: cosmologia; neutrinos massivos; energia escura primordial; energy escura em tempos tardios.

Field: física; cosmologia.

Abstract

The prevailing cosmological paradigm is the spatially flat Λ CDM model with adiabatic and Gaussian initial conditions. The Greek letter “ Λ ” designates the cosmological constant while the acronym “CDM” means the pressureless Cold Dark Matter. Until today, this has been the more economical description of the Universe consistent with an assortment of observational probes. Although its success, the benchmark cosmological model is not absent of issues. The horizon and flatness are well-known long-term problems with possible solutions when we bring the Cosmic Inflation theory. Furthermore, cosmological surveys have attained significant precision improvement in their signal measurements, which have enabled us to impose more stringent constraints on the Λ CDM parameters. When different probes are combined, the main challenges are the so-called Hubble and S_8 tensions.

Alongside dark matter, dark energy and neutrinos are two components that deserve special attention for their important role in particle physics and cosmology. This work is separated into two parts. In the first one, I provide a brief introduction to cosmology, providing references for further details to the interested reader. This starts with the background and perturbation equations derived from Einstein’s general theory of relativity and ends with the initial condition required to solve those equations. In the second part of this Thesis, I focus more specifically on neutrinos and their effect on structure formation and dark energy as a scalar field. The last two chapters are about a published paper and a work in progress started at Stony Brook University, respectively.

Key words: cosmology; massive neutrinos; early dark energy; late-time dynamical dark energy.

Field: physics; cosmology.

Contents

I	The Basis of Cosmology	1
1	Background and perturbed universe	2
1.1	Robertson-Walker metric	2
1.2	Cosmological distances	6
1.2.1	Luminosity and angular diameter distance	6
1.3	Friedmann equations	8
1.3.1	Equation of state and background evolution	13
1.4	Linear perturbation theory	18
1.4.1	From smooth to perturbed metric	18
1.4.2	Gauge transformations for metric tensor	19
1.4.3	From smooth to perturbed energy-momentum tensor	21
1.4.4	Perturbed equations of motions in Newtonian gauge	22
1.4.5	Proof of some equations	25
1.4.6	Evolution of the gravitational potential	29
1.5	Cosmic Microwave Background	34
1.5.1	Primordial spectrum and CMB	35
2	Gauge invariant perturbations	42
2.1	Introduction	42
2.2	Setting equations	42
2.3	Building gauge invariant perturbations	44
2.4	Applying the method	45
2.4.1	Bardeen's potentials	45
2.4.2	Comoving curvature perturbation	48
2.4.3	Lukash variable	49
2.4.4	Non-adiabatic pressure perturbation	49
2.4.5	Comoving gauge density perturbation	50
3	Initial conditions	52
3.1	Introduction	52

3.1.1	Cold Dark Matter	53
3.1.2	Neutrinos	53
3.1.3	Photons and baryons	53
3.1.4	The Einstein-Field equations	54
3.2	Early-times and super-horizon initial conditions	56
3.2.1	Zeroth order and higher order correction	60
3.2.2	First order correction	62
3.2.3	Adiabatic initial conditions	63
II	Dark Energy and Neutrinos in Cosmology	64
4	Mass Varying Neutrinos and the Hubble tension	65
4.1	Introduction	65
4.2	Neutrino-assisted EDE	68
4.3	A background analysis of the neutrino-assisted EDE	71
4.4	Initial value of the EDE field	73
4.4.1	Field initially frozen	74
4.4.2	Field initially dynamical	75
4.5	Field displacement	81
4.6	Minimum of the effective potential	82
4.7	Conclusion	84
5	Dark energy parametrizations	86
5.0.1	Constraining EDE with late DE marginalization	87
5.1	General polynomial dark energy equation of state	89
5.1.1	Constant $w(z)$	91
5.1.2	Constant $w(z)$ in bins of redshift	91
5.1.3	Linear $w(z)$ in bins of redshift	92
5.1.4	Quadratic $w(z)$ in bins of redshift	93
5.1.5	Cubic $w(z)$ in bins of redshift	96
5.2	CAMB for Late Dark Energy:	102
5.2.1	Late-Time Dark Energy and Massive Neutrinos with DESI 2024 BAO data	105
	Conclusion	115
	Bibliography	117

Part I

The Basis of Cosmology

Chapter 1

Background and perturbed universe

1.1 Robertson-Walker metric

Cosmology is the study of the universe as a whole, this means that we should be able to describe how the universe evolves, predict its expansion rate through cosmic time, estimate positions and distances, as well as velocities of astrophysical objects (e.g. galaxies) with respect to an observer (e.g. us), etc. This can be understood as the “kinematics” study of the universe. Naturally, the next step is to ask who are the agents (e.g. photons, neutrinos, baryons, dark matter, and dark energy) and forces (e.g. gravity and the repulsive nature of dark energy) responsible for making the universe evolve as it does. In this case, it is essential to know the equations that describe the abundances, evolution, and interactions (that depend on the epochs of the universe) between the various components of the cosmos. Similarly to the previous kinematic case, this can be seen as the “dynamical” study of the universe. This chapter doesn’t intend to exhaust the content of each mentioned topic. Still, it provides the basic elements for an introduction to cosmology and further references where details are far beyond the scope of this thesis. This chapter closely follows the textbook [1] by Daniel Baumann.

We start defining basic quantities used through this text: a coordinate system is denoted by $x^\mu = (x^0, x^i) = (t, \mathbf{x})$. While Greek indices (μ, ν , etc) can assume integer values between 0 and 3, Latin indices (i, j , etc) run over the values 1, 2, 3. Still about the coordinate system, the first component of x^μ , i.e. x^0 , represents time (there are many ways that we can measure time in cosmology, some of the most common are: cosmic time denoted by t , conformal time τ , scale factor a , redshift z and the cosmic microwave background temperature T . When necessary, we will see how these quantities relate to each other). The three-component x^i describes the usual three-dimensional space part of the spacetime coordinate system. An infinitesimal shift in the components of x^μ , represented by dx^μ , can be translated into the invariant line element, ds^2 , through the (spacetime)

metric tensor $g_{\mu\nu}$

$$ds^2 = \sum_{\mu,\nu=0}^3 g_{\mu\nu}(x^\alpha) dx^\mu dx^\nu \equiv g_{\mu\nu}(x^\alpha) dx^\mu dx^\nu, \quad (1.1)$$

where we assume the signature $(-, +, +, +)$ for the metric and Einstein's notation for the sum over repeated indices as indicated in Eq. (1.1). In addition, note that the metric is, in general, a function of the coordinate system $x^\alpha = (t, \mathbf{x})$.

We assume the Cosmological Principle (CP), i.e. that the space is homogeneous and isotropic, which means that the three-space is maximally symmetric (the conversely also is true, i.e., a metric space that is maximally symmetric implies homogeneity and isotropy). By definition, the space is said *homogeneous* if there exist infinitesimal isometries that carry any given point x^μ into any other point in its immediate neighborhood. The space is said *isotropic* about a given point x^μ if there exist infinitesimal isometries that do not change the point x^μ and for which the first derivative exists (see Weinberg's textbook for details on this matter [2]).

The previous definition guarantees the *existence* of maximally symmetric three-space under homogeneity and isotropic conditions. Ultimately, the CP simplifies the four-dimensional line element, Eq. (1.1), for

$$ds^2 = -dt^2 + a^2(t) dl^2, \quad (1.2)$$

$$dl^2 \equiv \gamma_{ij}(x^k) dx^i dx^j, \quad (1.3)$$

where Eq. (1.3) defines the line element of the three-dimensional space and $a(t)$ is the *scale factor* that parameterize the expansion or contraction of the universe. CP also implies in the *uniqueness* of this property, i.e., that exists a coordinate transformation that carries two maximally symmetric metric spaces into the other if the two maximally symmetric metrics have the same *constant intrinsic curvature* k for the three-dimensional space and the same numbers of eigenvalues of each sign [2]. We can deplete the possibilities with just three options: zero curvature or flat space ($k = 0$, e.g. the 3-dimensional Euclidean space $\mathbb{E}^3$), positive curvature ($k = +1$, e.g. the 3-sphere $\mathbb{S}^3$), or negative curvature ($k = -1$, e.g. the 3-hyperboloid $\mathbb{H}^3$). The line element for the spatial part of the metric in Eq. (1.3), dl^2 , for each

case of the constant intrinsic curvature k is given by [1]

$$\text{zero curvature: } dl^2 = d\mathbf{x}^2 = \delta_{ij}dx^i dx^j, \quad (1.4)$$

$$\text{positive curvature: } dl^2 = d\mathbf{x}^2 + du^2, \quad u^2 + \mathbf{x}^2 = R_0^2, \quad udu = -\mathbf{x} \cdot d\mathbf{x}, \quad (1.5)$$

$$\text{negative curvature: } dl^2 = d\mathbf{x}^2 - du^2, \quad u^2 - \mathbf{x}^2 = R_0^2, \quad udu = +\mathbf{x} \cdot d\mathbf{x}, \quad (1.6)$$

where u is an auxiliary coordinate that allows us to embed a three-sphere in a four-dimensional Euclidean space in the case of Eq. (1.5) with R_0 being the radius of the sphere. Switching the sign in front of u we also switch the geometry for the case of a three-hyperboloid embedded in a four-dimensional Lorentzian space represented by Eq. (1.6). We can turn the three Eqs. (1.4) - (1.6) into just one at the price of having explicitly the intrinsic constant curvature k [1]

$$dl^2 = d\mathbf{x}^2 + k \frac{(\mathbf{x} \cdot d\mathbf{x})^2}{R_0^2 - k\mathbf{x}^2} = \gamma_{ij}dx^i dx^j, \\ \gamma_{ij} \equiv \delta_{ij} + k \frac{x_i x_j}{R_0^2 - k(x_m x^m)}, \quad k = 0, +1, -1. \quad (1.7)$$

The observer usually sets a sphere around the Earth when looking into the sky. Therefore, it is useful to write Eq. (1.7) in terms of the spherical coordinate system. In this case, we have $d\mathbf{x}^2 = dr^2 + r^2 d\Omega^2$, $\mathbf{x} \cdot d\mathbf{x} = r dr$, $\mathbf{x}^2 = r^2$ and the line element on a unit sphere is $d\Omega^2 \equiv d\theta^2 + \sin^2\theta d\phi^2$. Simplifying the resulting expression, we obtain

$$dl^2 = \frac{dr^2}{1 - k(r/R_0)^2} + r^2 d\Omega^2. \quad (1.8)$$

The spacetime is not maximally symmetric, but it can be spanned into a family (i.e. sub-spaces) of maximally symmetric metric spaces. The way to do this is to foliate the spacetime into three-dimensional space of constant time, all sewn together by the parameterization function called scale factor $a(t)$ defined in Eq. (1.2) [1,2]. This procedure leads to the Robertson-Walker (RW) metric

$$ds^2 = -dt^2 + a^2(t) \left[\frac{dr^2}{1 - kr^2/R_0^2} + r^2 d\Omega^2 \right]. \quad (1.9)$$

In Eq. (1.9) we have to determine the form of the scale factor (i.e. the why the universe expands or contracts), whether the three-space is flat ($k = 0$), positively ($k = +1$) or negatively ($k = -1$) curved and if not flat, we also have to determine

the intensity of the curvature given by the curvature scale R_0 . In other words, we no longer need to determine the ten components of the symmetric metric tensor $g_{\mu\nu}$, just a time-dependent function, $a(t)$, and two numbers, k and R_0 (if $k \neq 0$).

The RW metric in Eq. (1.9) is left invariant under rescaling symmetry: $a \rightarrow \lambda a$, $r \rightarrow r/\lambda$, $R_0 \rightarrow R_0/\lambda$, or $a \rightarrow a/\lambda$, $r \rightarrow \lambda r$, $R_0 \rightarrow \lambda R_0$. A position given by $\mathbf{r} = (r, \theta, \phi)$ is called a *comoving coordinate system* with *comoving distance* r and the combination $\mathbf{r}_{\text{ph}} = a(t)\mathbf{r}$ is called *physical coordinate system* with *physical distance* $r_{\text{ph}} = a(t)r$ because under the rescaling ($a \rightarrow \lambda a$, $r \rightarrow r/\lambda$ or $a \rightarrow a/\lambda$, $r \rightarrow \lambda r$) r_{ph} is left unchanged. If we take the time derivative of the physical coordinate we have $\mathbf{v}_{\text{ph}} \equiv \dot{\mathbf{r}}_{\text{ph}} = \dot{a}(t)\mathbf{r} + a(t)\dot{\mathbf{r}} = H(t)\mathbf{r}_{\text{ph}} + \mathbf{v}_{\text{pec}}$. Where we have defined the **Hubble parameter** by

$$H \equiv \frac{\dot{a}}{a}, \quad (1.10)$$

where overdot is the derivative with respect to time t . The physical velocity have the contribution of the **Hubble flow**, $H\mathbf{r}_{\text{ph}}$, and the **peculiar velocity** $\mathbf{v}_{\text{pec}} \equiv a\dot{\mathbf{r}}$.

We can rewrite the RW metric in Eq. (1.9) in a more economical way by redefining the metric component g_{rr} as follows $d\chi \equiv dr / \sqrt{1 - kr^2/R_0^2}$. To solve this integral we do the change of variable $\sqrt{k}(r/R_0) = \sin(z) \Rightarrow \sqrt{k}(dr/R_0) = \cos(z)dz$, therefore we have

$$d\chi = \frac{R_0}{\sqrt{k}} \frac{\cos(z)dz}{\sqrt{1 - \sin^2(z)}} = \frac{R_0}{\sqrt{k}} dz \Rightarrow \chi = \frac{R_0}{\sqrt{k}} z. \quad (1.11)$$

Going back to the comoving coordinate r we have

$$\chi = \frac{R_0}{\sqrt{k}} \sin^{-1} \left(\sqrt{k} \frac{r}{R_0} \right), \quad (1.12)$$

$$r = \frac{R_0}{\sqrt{k}} \sin \left(\sqrt{k} \frac{\chi}{R_0} \right). \quad (1.13)$$

We can split Eq. (1.13) into three cases according to the value of k . If $k = +1$ then $r = R_0 \sin(\chi/R_0)$. For $k = 0$ we have $\lim_{k \rightarrow 0} [(R_0/\sqrt{k}) \sin(\sqrt{k}\chi/R_0)] = \chi$. Finally, for $k = -1$ we have $r = (R_0/i) \sin(i\chi/R_0)$ where $i = \sqrt{-1}$. Using that $\sin(i\theta) = i \sinh(\theta)$ we can write the negative curvature case as $r = R_0 \sinh(\chi/R_0)$.

In summary:

$$S_k(\chi) \equiv r = R_0 \times \begin{cases} \sin(\chi/R_0) & \text{if } k = +1 \\ \chi/R_0 & \text{if } k = 0 \\ \sinh(\chi/R_0) & \text{if } k = -1 \end{cases} . \quad (1.14)$$

In this way, the RW metric Eq. (1.9) can compactly be written as follows

$$ds^2 = a^2(\tau) \left[-d\tau^2 + \left(d\chi^2 + S_k^2(\chi) d\Omega^2 \right) \right], \quad (1.15)$$

where we have defined the **conformal time**

$$d\tau \equiv \frac{dt}{a(t)}. \quad (1.16)$$

1.2 Cosmological distances

Determining how far and faster are objects moving with respect to us is not an easy task and may lead to some misconceptions [3]. Despite its complexity, defining and estimating distances it is an important and essential source of cosmological information. For example, evidence that the universe is not static and could actually be expanding, came through Hubble's paper [4] in 1929 that showed a linear relation between velocity and distance of extra-galactic nebulae (Cepheids). The confirmation of the accelerated expansion of the universe using type Ia supernovae led the 2011 Nobel Prize shared by Saul Perlmutter et al. [5], A. D. Riess et al. [6] and Brian P. Schmidt et al. [7]. In all these works, cosmological distances played an important role. Due to its importance, in the following, we discuss the definition of **luminosity distance** and **angular diameter distance**.

1.2.1 Luminosity and angular diameter distance

The luminosity distance is defined using the known energy emitted per unit time or simply luminosity L (hence the name) of an astronomical object. Then we measure the observed flux F . The relation between L and F is mediated by distance, however, the flux F has a unit of energy per unit of time per area, therefore we should be concerned about how these three quantities (area, time, and energy) change in an expanding universe.

In a Euclidean space, the comoving distance r is simplified to χ , then a sphere has an area equal to $4\pi\chi^2$, however, according to Eq. (1.14) the comoving distance depends on the spatial curvature of the universe. Therefore we should generalize to include the possibility of curved space, in this case, we have $A_k \equiv 4\pi S_k^2(\chi)$. There are other two corrections that we should be concerned about. In addition, due to the universe's expansion and assuming a fixed time interval, photons are able to travel smaller distances at late times on the comoving coordinate system than at early times because the associated physical distance at late times is bigger. Therefore the rate at which photons arrive us (or equivalently the number of photons crossing a shell) is smaller than at early times (at the emission epoch) by a factor of $1/(1+z)$. Lastly, we should remember that the photon wavelength is related to its energy by the equation $E = h/\lambda$ where h is Planck's constant. In an expanding universe the wavelength is stretched out (or redshifted) then the energy today is smaller than at emitted time by a factor of $1/(1+z)$. These two last effects produce $A_k \rightarrow A_k(1+z)^2$ then we have the right relation between the flux and the luminosity

$$F = \frac{L}{4\pi S_k^2(\chi)(1+z)^2} \equiv \frac{L}{4\pi d_L^2}, \quad (1.17)$$

where we define the luminosity distance

$$d_L = (1+z)S_k(\chi). \quad (1.18)$$

Note that d_L depends on $S_k(\chi)$, hence is a function of the intrinsic constant spatial curvature k . The value of k is determined by the total amount of energy density. Therefore, ultimately the luminosity distance is a function of the cosmological parameters.

The small angle $\delta\theta \ll 1$ subtended by an object of physical size l (thus of comoving size $l/a = l(1+z)$) where a is the scale factor when the light is emitted is related to its comoving distance by the following

$$\delta\theta = \frac{l(1+z)}{S_k(\chi)} \equiv \frac{l}{d_A}, \quad (1.19)$$

where we define the angular diameter distance by

$$d_A = \frac{S_k(\chi)}{1+z}. \quad (1.20)$$

1.3 Friedmann equations

The Einstein field equation gives the dynamics of the universe. It describes how the Einstein tensor, $G_{\mu\nu}$, responds to the presence of matter and energy distribution $T_{\mu\nu}$

$$G_{\mu\nu} \equiv R_{\mu\nu} - \frac{1}{2}Rg_{\mu\nu} = 8\pi GT_{\mu\nu}, \quad (1.21)$$

where $g_{\mu\nu}$ is the metric tensor that translates the coordinates system, $x^\mu = (t, x^i)$, into an invariant line element, ds^2 (see Eq. (1.9)). $R_{\mu\nu}$ and R are the Ricci tensor and the Ricci scalar, respectively

$$R_{\mu\nu} = \partial_\lambda \Gamma_{\mu\nu}^\lambda - \partial_\nu \Gamma_{\mu\lambda}^\lambda + \Gamma_{\lambda\rho}^\lambda \Gamma_{\mu\nu}^\rho - \Gamma_{\mu\lambda}^\rho \Gamma_{\nu\rho}^\lambda, \quad (1.22)$$

$$R = R^\mu_\mu = g^{\mu\nu} R_{\mu\nu}. \quad (1.23)$$

The relation between the Christoffel symbols and the metric tensor is

$$\Gamma_{\alpha\beta}^\mu = \frac{1}{2}g^{\mu\lambda} (\partial_\alpha g_{\beta\lambda} + \partial_\beta g_{\alpha\lambda} - \partial_\lambda g_{\alpha\beta}). \quad (1.24)$$

Once equipped with the metric tensor, $g_{\mu\nu}$, it is possible to compute the left-hand side (l.h.s) of Eq. (1.21) through the set of Eqs. (1.22)-(1.24). However, we still need to specify the right-hand side (r.h.s) of Eq. (1.21), i.e. the stress energy-momentum tensor $T_{\mu\nu}$. Provided this source term for the Einstein equation, we ended with a differential equation that (given appropriate initial conditions) can be solved to describe the dynamics of the universe. Under the assumption of the CP, it is possible to show [2] that a general expression for the energy-momentum tensor is highly constrained, reducing to the form of a perfect fluid

$$T_{\mu\nu} = (\rho + p)U_\mu U_\nu - Pg_{\mu\nu}, \quad (1.25)$$

where ρ and P are the energy density and the pressure of the fluid, respectively. U_μ is the 4-velocity. Now we have all the ingredients to compute the *Friedmann equations*, which describes the universe's dynamics. In order to do this, we start computing the l.h.s of Eq.(1.21).

Considering a flat universe ($k = 0$) we can evaluate Eq. (1.24) for the RW metric.

In this scenario, Eq. (1.9) becomes

$$\text{Null curvature: } ds^2 = dt^2 - a^2 d\mathbf{x}^2, \quad (1.26)$$

$$\text{Cartesian coordinates: } d\mathbf{x}^2 = d\mathbf{x} \cdot d\mathbf{x} = dx^2 + dy^2 + dz^2. \quad (1.27)$$

Therefore, the components of the RW metric tensor are

$$g_{00} = 1, \quad g_{ij} = -a^2 \delta_{ij}, \quad g^{\mu\lambda} g_{\lambda\nu} = \delta^\mu_\nu, \quad (1.28)$$

where δ_{ij} e δ^μ_ν are the Kronecker delta, and $g^{\mu\lambda}$ is the inverse of the $g_{\lambda\nu}$.

In total, we have $4 \times 4 \times 4 = 64$ Christoffel symbols. Therefore, it is convenient to set one of the indices to zero and compute the spatial part for the others. Using Eq. (1.24), it follows that

$$\begin{aligned} \Gamma_{ij}^0 &= \frac{1}{2} g^{0\lambda} (\partial_i g_{j\lambda} + \partial_j g_{i\lambda} - \partial_\lambda g_{ij}) = -\frac{1}{2} g^{0\lambda} \partial_\lambda g_{ij} = -\frac{1}{2} g^{00} \partial_0 (-a^2 \delta_{ij}) \\ &= a \dot{a} \delta_{ij} \stackrel{\text{or}}{=} a^2 H \delta_{ij}, \end{aligned} \quad (1.29)$$

where we define the *Hubble parameter*

$$H \equiv \frac{1}{a} \frac{da}{dt} = \frac{\dot{a}}{a}. \quad (1.30)$$

Conversely, from Eq. (1.24), we have

$$\begin{aligned} \Gamma_{0j}^i &= \frac{1}{2} g^{i\lambda} (\partial_0 g_{j\lambda} + \partial_j g_{0\lambda} - \partial_\lambda g_{0j}) = \frac{1}{2} g^{i\lambda} \partial_0 g_{j\lambda} = \frac{1}{2} g^{ij} \partial_0 g_{jj} \\ &= \frac{1}{2} \left(-\frac{1}{a^2} \delta^{ij} \right) \partial_0 (-a^2 \delta_{jj}) = \frac{2a\dot{a}}{2a^2} \delta^{ij} \delta_{jj} = \frac{\dot{a}}{a} \delta_j^i \stackrel{\text{or}}{=} H \delta_j^i. \end{aligned} \quad (1.31)$$

Allowing the three indices to be only spatial, we obtain

$$\Gamma_{jk}^i = \frac{1}{2} g^{i\lambda} (\partial_j g_{k\lambda} + \partial_k g_{j\lambda} - \partial_\lambda g_{jk}) = -\frac{1}{2} g^{i\lambda} \partial_\lambda g_{jk} = -\frac{1}{2} g^{i0} \partial_0 g_{jk} = 0 \quad \forall i, j, k. \quad (1.32)$$

Setting simultaneously $\alpha = \beta = 0$ we have

$$\Gamma_{00}^\mu = \frac{1}{2} g^{\mu\lambda} (2\partial_0 g_{0\lambda} - \partial_\lambda g_{00}) = 0 \quad \forall \mu. \quad (1.33)$$

Analogously

$$\Gamma_{0\beta}^0 = \frac{1}{2}g^{0\lambda} (\partial_0 g_{\beta\lambda} + \partial_\beta g_{0\lambda} - \partial_\lambda g_{0\beta}) = 0 \quad \forall \beta. \quad (1.34)$$

Finally, if we switch $\alpha \rightarrow \beta$ and $\beta \rightarrow \alpha$ and using $g_{\alpha\beta} = g_{\beta\alpha}$, from Eq. (1.24) it follows that

$$\Gamma_{\alpha\beta}^\mu = \Gamma_{\beta\alpha}^\mu. \quad (1.35)$$

Therefore, we have for a flat universe with a RW metric the following Christoffel symbols

$$\bullet \quad \Gamma_{ij}^0 = a^2 H \delta_{ij}, \quad (1.36)$$

$$\bullet \quad \Gamma_{0j}^i = H \delta_j^i, \quad (1.37)$$

$$\bullet \quad \Gamma_{jk}^i = \Gamma_{00}^\mu = \Gamma_{0\beta}^0 = 0. \quad (1.38)$$

We illustrate those 64 Christoffel symbols distributed in Eqs. (1.36), (1.37), and (1.38) in Fig. 1.1. As indicated by the figure, most part of the symbols are nulls (represented by the light gray circles).

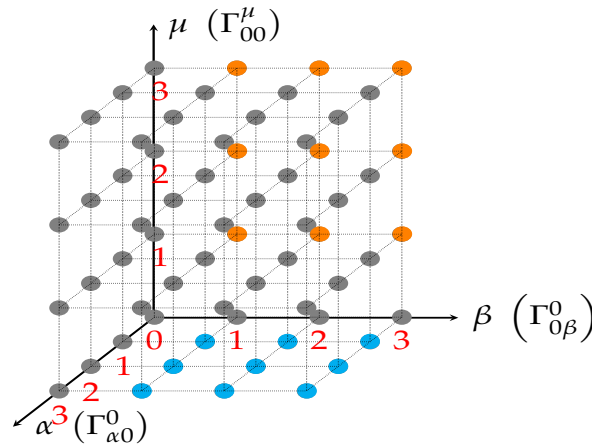


Figure 1.1: Christoffel symbols visualization for the RW metric and flat universe.

We proceed to compute the Ricci tensor and the Ricci scalar for the RW metric. Firstly, consider the following decomposition for the Ricci tensor $R_{\mu\nu} \equiv \{R_{00}, R_{ij}, R_{0i}\}$ and let's compute each of these terms separately. Using the Christoffel symbols displayed in Eqs. (1.36)-(1.38) and the relation between the Ricci tensor

and the Christoffel symbols by Eq. (1.22) we have

$$\begin{aligned}
 R_{00} &= \partial_\lambda \Gamma_{00}^\lambda - \partial_0 \Gamma_{0\lambda}^\lambda + \Gamma_{\lambda\rho}^\lambda \Gamma_{00}^\rho - \Gamma_{0\lambda}^\rho \Gamma_{0\rho}^\lambda = -\partial_0 \Gamma_{0i}^i - \Gamma_{0k}^l \Gamma_{0l}^k \\
 &= -\partial_0 \left(\frac{\dot{a}}{a} \delta_i^i \right) - \left(\frac{\dot{a}}{a} \delta_k^l \frac{\dot{a}}{a} \delta_l^k \right) = - \left(\frac{\ddot{a}a - \dot{a}^2}{a^2} \right) \delta_i^i - \frac{\dot{a}^2}{a^2} \delta_l^l = -3 \left(\frac{\ddot{a}}{a} - \frac{\dot{a}^2}{a^2} + \frac{\dot{a}^2}{a^2} \right) \\
 R_{00} &= -3 \frac{\ddot{a}}{a}.
 \end{aligned} \tag{1.39}$$

The second term in our list is R_{ij} , then by Eq. (1.22) we have

$$\begin{aligned}
 R_{ij} &= \partial_\lambda \Gamma_{ij}^\lambda - \partial_j \Gamma_{i\lambda}^\lambda + \Gamma_{\lambda\rho}^\lambda \Gamma_{ij}^\rho - \Gamma_{i\lambda}^\rho \Gamma_{j\rho}^\lambda = \partial_0 \Gamma_{ij}^0 + \Gamma_{l0}^l \Gamma_{ij}^0 - \left(\Gamma_{im}^0 \Gamma_{j0}^m + \Gamma_{i0}^k \Gamma_{jk}^0 \right) \\
 &= \partial_0 (a\dot{a}\delta_{ij}) + \frac{\dot{a}}{a} \delta_l^l a\dot{a}\delta_{ij} - \left(a\dot{a}\delta_{im} \frac{\dot{a}}{a} \delta_j^m + \frac{\dot{a}}{a} \delta_i^k a\dot{a}\delta_{jk} \right) \\
 &= a\ddot{a}\delta_{ij} + \dot{a}^2 \left(\delta_{ij} + \delta_l^l \delta_{ij} - \delta_{im} \delta_j^m - \delta_i^k \delta_{jk} \right) = a\ddot{a}\delta_{ij} + \dot{a}^2 (\delta_{ij} + 3\delta_{ij} - \delta_{ij} - \delta_{ij}) \\
 &= \left(a\ddot{a} + 2\dot{a}^2 \right) \delta_{ij} = \left(\frac{\ddot{a}}{a} + 2\frac{\dot{a}^2}{a^2} \right) a^2 \delta_{ij} \\
 R_{ij} &= - \left(\frac{\ddot{a}}{a} + 2H^2 \right) g_{ij}, \text{ where } H = \dot{a}/a \text{ and } g_{ij} = -a^2 \delta_{ij}.
 \end{aligned} \tag{1.40}$$

Finally, from the RW metric we have that $g_{0i} = 0$ and $T_{0i} = 0$ for $i = 1, 2, 3$. Therefore, by the Einstein equation we obtain that R_{0i} is zero, as shown bellow

$$G_{0i} = R_{0i} - \frac{1}{2} R g_{0i} = R_{0i} = 8\pi G T_{0i} = 0. \tag{1.41}$$

We advance to compute the Ricci scalar given by Eq. (1.23)

$$\begin{aligned}
 R &= R^\mu_\mu = g^{\mu\nu} R_{\mu\nu} = g^{00} R_{00} + g^{ij} R_{ij} = 1 \cdot \left(-3 \frac{\ddot{a}}{a} \right) - \frac{1}{a^2} \delta^{ij} \left(a\ddot{a} + 2\dot{a}^2 \right) \delta_{ij} \\
 &= -3 \frac{\ddot{a}}{a} - \left(\frac{\ddot{a}}{a} + 2\frac{\dot{a}^2}{a^2} \right) 3\delta^{ij} \delta_{ij} = -6 \left(\frac{\ddot{a}}{a} + \frac{\dot{a}^2}{a^2} \right)
 \end{aligned} \tag{1.42}$$

$$R = -6 \left(\frac{\ddot{a}}{a} + H^2 \right). \tag{1.43}$$

Armed with the Ricci tensor (Eqs. (1.39)-(1.41)) and the Ricci scalar (Eq. (1.43)) for the RW metric and the energy momentum-tensor for the perfect fluid (Eq. (1.25)), we are able to compute the time-time component (i.e. $\mu = \nu = 0$) of the Einstein

equation (1.21)

$$G_{00} = R_{00} - \frac{1}{2}Rg_{00} = 8\pi GT_{00}. \quad (1.44)$$

Using Eq. (1.39) for R_{00} , Eq. (1.43) for R and $T_{00} = \rho$, Eq. (1.44) becomes

$$G_{00} = -3\frac{\ddot{a}}{a} - \frac{1}{2} \left[-6 \left(\frac{\ddot{a}}{a} + \frac{\dot{a}^2}{a^2} \right) \right] \cdot 1 = 8\pi G\rho \Rightarrow \frac{\dot{a}^2}{a^2} = \frac{8\pi G}{3}\rho$$

which can be rewritten as follows

$$\boxed{H^2 = \frac{8\pi G}{3}\rho}. \quad (1.45)$$

Equation (1.45) is known as the *First Friedmann equation* for the dynamics of the universe. It says that an infinitesimal change (e.g. an expansion or contraction) in the scale factor, da , in an infinitesimal time interval, dt , is proportional to the square root of the product between the scale factor by the energy density $\sqrt{a\rho}$ at the current time t .

Analogously, we do the same above procedure for the space-space component (i.e. $\mu = \nu = i$) of the Einstein equation

$$G_{ii} = R_{ii} - \frac{1}{2}Rg_{ii} = 8\pi GT_{ii}. \quad (1.46)$$

Plugging the RW metric, Eq. (1.9), into the energy-momentum tensor, Eq. (1.25), we have $T_{ii} = -Pg_{ii} = Pa^2\delta_{ii}$. In addition, using the Ricci tensor in Eq. (1.40) and again the Ricci scalar in Eq. (1.43) we have

$$G_{ii} = \left(a\ddot{a} + 2\dot{a}^2 \right) \delta_{ii} - \frac{1}{2} \left[-6 \left(\frac{\ddot{a}}{a} + \frac{\dot{a}^2}{a^2} \right) \right] \left(-a^2\delta_{ii} \right) = 8\pi G \left(Pa^2\delta_{ii} \right). \quad (1.47)$$

Simplifying the above expression, we have $2\ddot{a}/a + \dot{a}^2/a^2 = -8\pi GP$ which is the *Second Friedmann equation*

$$\boxed{\frac{\ddot{a}}{a} = -\frac{4\pi G}{3}(\rho + 3P)}. \quad (1.48)$$

1.3.1 Equation of state and background evolution

Inspecting the Friedmann (1.45) we should conclude that to find how scale factor evolves with time, it is necessary to know the energy density ρ , i.e. the expansion history of the universe depends on its energetic content. We can find ρ as follows: in general relativity, the energy-momentum tensor for a perfect fluid, expressed by Eq. (1.25), satisfies the covariant conservation equation

$$\nabla_\mu T^\mu_\nu = \partial_\mu T^\mu_\nu + \Gamma^\mu_{\mu\lambda} T^\lambda_\nu - \Gamma^\lambda_{\mu\nu} T^\mu_\lambda = 0. \quad (1.49)$$

If we set $\nu = 0$ in Eq. (1.49) and use the energy-momentum tensor in Eq. (1.25), we ended with a differential equation for the energy density known as *continuity equation*

$$\dot{\rho} + 3\frac{\dot{a}}{a}(\rho + P) = 0 \Leftrightarrow \dot{\rho} + 3\frac{\dot{a}}{a}\rho(1 + w) = 0, \quad (1.50)$$

where in Eq. (1.50) we have defined an important parameter called *equation of state* (EoS)

$$w \equiv \frac{P}{\rho}. \quad (1.51)$$

The main elements of the universe that contribute to the energy momentum-tensor (thus being important to the universe's expansion history through the Friedmann equations) are *photons*, *neutrinos*, *cold dark matter* (CDM), *baryons* and *dark energy*. If the pressure, P , and the energy density, ρ , are known for some of these listed components, then it should be possible to access its equation of state, w , by Eq. (1.51).

In the following, we show an example of how to compute the EoS for photons. We start with the number density, n , weighted by the density of states, g/h^3 , where g is the internal degree of freedom (e.g. spin) and $h = 2\pi\hbar$ is the Planck's constant

$$n = \frac{g}{(2\pi\hbar)^3} \int d^3p f(p). \quad (1.52)$$

where $f(p)$ is the Fermi-Dirac or Bose-Einstein distribution function

$$f(p, T) = \frac{1}{e^{E(p)/K_B T} \pm 1}, \quad (1.53)$$

where the $+$, $-$ signs represent fermions and bosons, respectively. Neglecting the interaction energy between particles, the relativistic energy of a particle with mass m and momentum p is

$$E(p) = \sqrt{p^2 c^2 + m^2 c^4}. \quad (1.54)$$

We use the number density to find ρ and P . While the energy density is weighted by $E(p)$, the pressure is weighted by $p^2/(3E(p))$ (for further details see [1]), then we have

$$\rho = \frac{g}{(2\pi\hbar)^3} \int d^3 p E(p) \quad (1.55)$$

$$P = \frac{g}{(2\pi\hbar)^3} \int d^3 p f(p) \frac{p^2}{3E(p)} \quad (1.56)$$

Now we can write the integrals for n and ρ only in terms of the momentum, which will allow us to solve them

$$n = \frac{g}{2\pi^2\hbar^3} \int_0^\infty dp \frac{p^2}{\exp\left[\frac{\sqrt{p^2 c^2 + m^2 c^4}}{k_B T}\right] \pm 1}, \quad (1.57)$$

$$\rho = \frac{g}{2\pi^2\hbar^3} \int_0^\infty dp \frac{p^2 \sqrt{p^2 c^2 + m^2 c^4}}{\exp\left[\frac{\sqrt{p^2 c^2 + m^2 c^4}}{k_B T}\right] \pm 1}. \quad (1.58)$$

Before to compute the pressure, it is useful to define new variables

$$x \equiv \frac{mc^2}{k_B T}, \quad \xi \equiv \frac{pc}{k_B T}, \quad (1.59)$$

Also is convenient to remember the following identity

$$\int_0^\infty d\xi \frac{\xi^n}{e^\xi - 1} = \zeta(n+1)\Gamma(n+1) = n!\zeta(n+1) = \begin{cases} 2\zeta(3) & \text{if } n = 2, \\ \pi^4/15 & \text{if } n = 3, \end{cases} \quad (1.60)$$

where ζ and Γ are the Zeta and the Gamma functions, respectively.

Now we proceed to compute the EoS for relativistic particles, in particular for photons. In this case, we set the particle's mass to zero ($m = 0$) in the pressure's

equation (1.56)

$$P = \frac{g}{(2\pi\hbar)^3} \int d^3p f(p) \frac{p^2}{3E(p)} = \frac{g}{6\pi^2} \frac{k_B^4}{c^5\hbar^3} T^4 \int_0^\infty d\xi \frac{\xi^3}{e^\xi - 1} = \frac{1}{3} \left(\frac{\pi^2}{30} g \frac{k_B^4}{c^5\hbar^3} T^4 \right).$$

It is possible to show that $\rho = \frac{\pi^2}{30} g \frac{k_B^4}{c^5\hbar^3} T^4$, which in natural units ($k_B = c = \hbar \equiv 1$) reduces to the result of [1]. Then we finally have

$$P = \frac{\rho}{3}. \quad (1.61)$$

To estimate the number density and the energy density of photons today we rewrite the equations for n (Eq. (1.57)) and ρ (Eq. (1.58)) in terms of the new variables, x and ξ , defined in Eq. (1.59)

$$n = \frac{g}{2\pi^2} T^3 \left(\frac{k_B}{\hbar c} \right)^3 I_\pm(x), \quad I_\pm(x) \equiv \int_0^\infty d\xi \frac{\xi^2}{\exp \left[\sqrt{\xi^2 + x^2} \right] \pm 1}, \quad (1.62)$$

$$\rho = \frac{g}{2\pi^2} T^4 \left(\frac{k_B}{\hbar c} \right)^3 \frac{k_B}{c^2} J_\pm(x), \quad J_\pm(x) \equiv \int_0^\infty d\xi \frac{\xi^2 \sqrt{\xi^2 + x^2}}{\exp \left[\sqrt{\xi^2 + x^2} \right] \pm 1}. \quad (1.63)$$

Considering the relativistic limit, $x \rightarrow 0$, and using for the photon temperature $T_0 = 2.73$ K, we have

$$n_{\gamma,0} = \frac{2\zeta(3)}{\pi^2} T_0^3 \times (k_B/\hbar c)^3 \approx 412 \text{ photons/cm}^3, \quad (1.64)$$

likewise for the energy density

$$\rho_{\gamma,0} = \frac{\pi^2}{15} T_0^4 \times (k_B^4/\hbar^3 c^5) \approx 4.7 \times 10^{-34} \text{ g/cm}^3. \quad (1.65)$$

Using the *critical energy* density today

$$\rho_{cr,0} = \frac{3H_0^2}{8\pi G} = 1.29 \times 10^{-29} h^2 \text{ g/cm}^3, \quad (1.66)$$

then the *density parameter* for the CMB is

$$\Omega_{\gamma,0} h^2 = \frac{\rho_{\gamma,0}}{\rho_{cr,0}} h^2 \approx 2.46 \times 10^{-5}. \quad (1.67)$$

Equation (1.61) shows the EoS for photons. However, the universe is made of elements other than photons. Bellow we just summarize the EoS for these components: photons (γ), cold dark matter (c), and dark energy as cosmological constant (Λ) (for further details see [1,2,8])

$$w_\gamma = \frac{1}{3}, \quad w_c = 0, \quad w_\Lambda = -1. \quad (1.68)$$

The previous equation, (1.68), shows that w is time independent for photons, CDM and the cosmological constant, but this is not a rule in general. The EoS for neutrinos and dark energy as a scalar field, for example, varies over time.

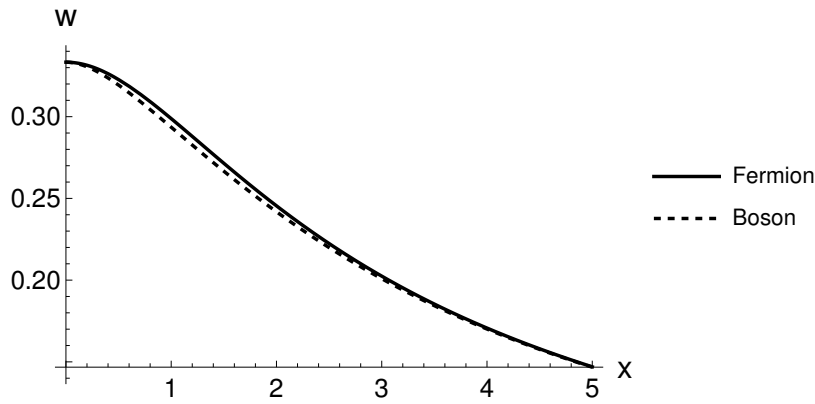


Figure 1.2: Equation of state for fermions and bosons. Note that when the particle's mass is much smaller than the particle's temperature, $m/T \ll 1 \Leftrightarrow x \ll 1$, w approaches to the value of the equation of state for photons, $w \rightarrow 1/3 \approx 0.333$,

Lets look into more details for the case of neutrinos. From the oscillation experiment, we know that neutrinos have mass, then we can not set to zero the mass term in Eq. (1.58), however, we still can study the asymptotic behavior for the EoS for neutrinos (henceforth w_ν) in the very early/late time regime of the universe. The neutrino temperature was bigger (the comparison makes sense in natural units) than its mass, therefore we have the ratio $m \ll T \Leftrightarrow x/\xi \ll 1$. This regime is known as *relativistic* where the neutrino mass can be neglected, and we have $w_\nu \approx 1/3$. On the opposite side, at later times, neutrinos are *non-relativistic*. In the non-relativistic regime, neutrinos behave like CDM, and in this case, we have $w_\nu \approx 0$. Indeed, we can verify this statement numerically by solving the ratio of Eq. (1.56) by Eq. (1.55), the result is shown in figure 1.2. Note that $x \rightarrow 0 \Rightarrow w \rightarrow 1/3$, conversely $x \rightarrow \infty \Rightarrow w \rightarrow 0$.

Once we have the equation of state, we can integrate the continuity equation

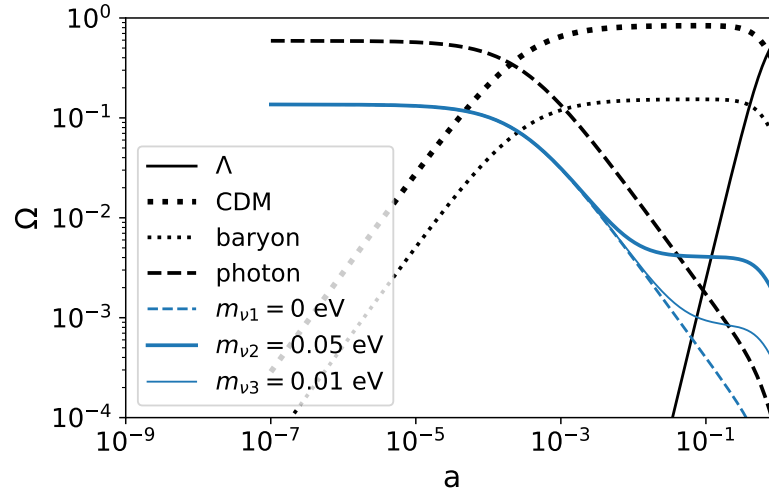


Figure 1.3: The evolution of the density parameter for each component: cosmological constant (Λ), cold dark matter (CDM), baryons, photons, and neutrinos, where for the latter we are assuming one massless neutrino and two massive neutrinos.

(1.50) to obtain the energy density as a function of the scale factor

$$\int_{\rho}^{\rho_0} \frac{d\rho'}{\rho'} = \ln \frac{\rho_0}{\rho} = -3 \int_a^{a_0} \frac{da'}{a'} (1 + w(a')). \quad (1.69)$$

If the equation of state is constant, Eq. (1.69) can be integrated analytically

$$\rho = \rho_0 \left(\frac{a}{a_0} \right)^{-3(1+w)}. \quad (1.70)$$

We can compute Eq. (1.70) for each equation of state in Eq. (1.68)

$$\rho_{\gamma} \propto a^{-4}, \quad \rho_c \propto a^{-3}, \quad \rho_{\Lambda} \propto \text{constant}. \quad (1.71)$$

The energy density for neutrinos, ρ_{ν} , should be computed numerically since its equation of state changes with time. However, notice by Figure. 1.3 that in the limit of high redshift, neutrinos behave like radiation with the equation of state $\approx 1/3$ and in the regime of low redshift, neutrinos behave like dark matter with the equation of state ≈ 0 .

To eliminate the proportionality constant in the previous equations (1.71), we

define the *density parameter*

$$\Omega_{i0} \equiv \frac{\rho_{i0}}{\rho_{cr0}}. \quad (1.72)$$

Using this definition, we have

$$\Omega_\gamma = \Omega_{\gamma0} a^{-4}, \quad \Omega_m = \Omega_{m0} a^{-3}, \quad \Omega_\Lambda = \Omega_{\Lambda0}, \quad \Omega_v = \rho_v / \rho_{cr0}. \quad (1.73)$$

In figure 1.3 we show the evolution for each component.

1.4 Linear perturbation theory

1.4.1 From smooth to perturbed metric

The cosmological principle stated in section 1.1, says that the universe can be represented by spatially flat hyper-surfaces, Σ_τ , of constant time. This translates into the Robertson-Walker metric

$$ds^2 = a^2(\tau) \left[d\tau^2 - \delta_{ij} dx^i dx^j \right]. \quad (1.74)$$

Therefore, homogeneity and isotropy are implicit in the RW metric. However, the universe is not perfectly homogeneous and isotropic, for example: galaxies, voids (regions of space with under-density of matter), clusters, and super-clusters of galaxies form filaments that spread through space. In addition, the temperature fluctuations in the cosmic microwave background are of the order 10^{-5} , which is small but not zero. Therefore, it is important to modify the metric in Eq. (1.74) to take into account those small fluctuations. We can do this by introducing functions that represent small perturbations in the metric (1.74). Then, the modified metric can be written as

$$ds^2 = a^2(\tau) \left[(1 + 2A) d\tau^2 - 2B_i d\tau dx^i - (\delta_{ij} + h_{ij}) dx^i dx^j \right]. \quad (1.75)$$

In this equation, the perturbation functions are: $A(t, \mathbf{x})$, called *temporal lapse function* or the *potential function*, $B_i(t, \mathbf{x})$ named *metric shift* or *vector function*, and $h_{ij}(t, \mathbf{x})$ that introduce perturbations in the *curvature* and the *metric shear*. It is possible to write B_i and h_{ij} into scalar and vector components that satisfy certain properties. Using the Scalar Vector Tensor (SVT) decomposition [1] we can write

the perturbation functions as follows

$$B_i = \partial_i B + \hat{B}_i \quad (\partial^i \hat{B}_i = 0), \quad (1.76)$$

$$h_{ij} = 2C\delta_{ij} + 2\partial_{(i}\partial_{j)}E + 2\partial_{(i}\hat{E}_{j)} + 2\hat{E}_{ij}, \quad (\partial^i \hat{E}_i = \partial^i \hat{E}_{ij} = \delta^{ij} \hat{E}_{ij} = 0), \quad (1.77)$$

where

$$\partial_{(i}\partial_{j)}E \equiv \left(\partial_i \partial_j - \frac{1}{3} \delta_{ij} \nabla^2 \right) E, \quad \partial_{(i}\hat{E}_{j)} \equiv \frac{1}{2} (\partial_i \hat{E}_j + \partial_j \hat{E}_i). \quad (1.78)$$

In table 1.1 we summarize the scalars, vectors and tensors part of the perturbed RW metric, and also the requirements that must be satisfied

Scalars	Vectors	Tensor	Divergenceless	Traceless
A, B, C, E	$\hat{B}_i, \hat{E}_i$	$\hat{E}_{ij}$	$\partial^i \hat{B}_i = \partial^i \hat{E}_i = \partial^i \hat{E}_{ij} = 0$	$\delta^{ij} \hat{E}_{ij} = 0$

Table 1.1: The SVT decomposition and the requirements that must be satisfied.

1.4.2 Gauge transformations for metric tensor

The RW metric expressed by Eq. (1.75) is implicitly written in terms of a specific time slice and, consequently a specific three-dimensional space. To allow the description of these perturbations in other¹ coordinate system we need to perform a gauge transformation. To prevent introducing unphysical perturbations in this process, we can perform the gauge transformation under the requirement that the spacetime interval, ds^2 , remains invariant regardless of the adopted gauge. Such a transformation is performed by the *gauge generator function*: ζ^μ . Starting at the gauge $\mathcal{G}$, we can generate a new one, $\tilde{\mathcal{G}}$, by the following

$$x^\mu \mapsto \tilde{x}^\mu \equiv x^\mu + \zeta^\mu(\tau, x^i), \quad (1.79)$$

$$\zeta^0 \equiv T, \quad (1.80)$$

$$\zeta^i \equiv L^i = \partial^i L + \hat{L}^i, \quad (\partial_i \hat{L}^i = 0), \quad (1.81)$$

where in Eq. (1.81) we have used the SVT decomposition of the spatial part of ζ^μ .

¹The description of these perturbations in other gauges can be more convenient depending on the problem, e.g., in the CMB analysis it is more practical to choose the synchronous gauge, but the spatially-flat gauge is more suitable to study inflation. The Newtonian gauge is more adequate to study the evolution of gravitational potential.

The requirement that the spacetime interval should be invariant under a gauge transformation implies

$$ds^2 = g_{\mu\nu}(x)dx^\mu dx^\nu = \tilde{g}_{\alpha\beta}(\tilde{x})d\tilde{x}^\alpha d\tilde{x}^\beta. \quad (1.82)$$

Using the components of the metric given by Eq. (1.75) into Eq. (1.82) it is possible to show that the perturbations functions transform as

$$\tilde{A} = A - T' - \mathcal{H}T, \quad (1.83)$$

$$\tilde{B}_i = B_i + \partial_i T - L'_i, \quad (1.84)$$

$$\tilde{h}_{ij} = h_{ij} - 2\partial_{(i}L_{j)} - 2\mathcal{H}T\delta_{ij}. \quad (1.85)$$

In terms of the SVT decomposition, these perturbations can be rewritten as follows

• **Scalars**

$$\tilde{A} = A - T' - \mathcal{H}T, \quad (1.86)$$

$$\tilde{B} = B + T - L', \quad (1.87)$$

$$\tilde{C} = C - \mathcal{H}T - \frac{1}{3}\nabla^2 L, \quad (1.88)$$

$$\tilde{E} = E - L \quad (1.89)$$

• **Vectors**

$$\tilde{\hat{B}}_i = \hat{B}_i - \hat{L}'_i, \quad (1.90)$$

$$\tilde{\hat{E}}_i = \hat{E}_i - \hat{L}_i. \quad (1.91)$$

• **Tensor**

$$\tilde{\hat{E}}_{ij} = \hat{E}_{ij} \quad (1.92)$$

We can combine the Eqs. (1.86)-(1.92) in order to eliminate the components of gauge generator functions ζ^μ . This provides us with quantities that are gauge invariant. For example, the tensor component, Eq. (1.92), already is a gauge invariant quantity because the gauge generator is absent.

Following this idea, it is possible to show that the following relations are gauge

invariant

$$\Psi \equiv A + \mathcal{H}(B - E') + (B - E')', \quad (1.93)$$

$$\Phi \equiv -C - \mathcal{H}(B - E') + \frac{1}{3}\nabla^2 E, \quad (1.94)$$

$$\mathcal{R} \equiv C - \frac{1}{3}\nabla^2 E + \mathcal{H}(B + v), \quad (1.95)$$

$$\zeta \equiv C - \frac{1}{3}\nabla^2 E - \frac{\mathcal{H}}{\bar{\rho}'}\delta\rho, \quad (1.96)$$

$$\bar{\rho}'\Delta \equiv \delta\rho + \bar{\rho}'(B + v), \quad (1.97)$$

$$\delta P_{\text{nad}} \equiv \delta P - \frac{\bar{P}'}{\bar{\rho}'}\delta\rho. \quad (1.98)$$

The first two equations, (1.93) and (1.94), are known as the Bardeen's potentials. Eq. (1.95) is known as comoving curvature perturbation and is useful to study inflation.

1.4.3 From smooth to perturbed energy-momentum tensor

In the previous section, we modified the RW metric to take into account small fluctuations in the spacetime, however, the spacetime and the matter-energy distribution are connected by the Einstein equation (1.21), which means that perturbations in the RW metric induce and are induced by perturbations in the matter distribution. Therefore we also need to describe small fluctuations in the energy-momentum tensor [1]

$$T_\nu^\mu = \bar{T}_\nu^\mu + \delta T_\nu^\mu, \quad (1.99)$$

$$\bar{T}_\nu^\mu = (\bar{\rho} + \bar{P})\bar{U}^\mu\bar{U}_\nu - \bar{P}\delta_\nu^\mu, \quad (1.100)$$

$$\delta T_\nu^\mu = (\delta\rho + \delta P)\bar{U}^\mu\bar{U}_\nu + (\bar{\rho} + \bar{P})(\delta u^\mu\bar{U}_\nu + \bar{U}^\mu\delta u_\nu) - \delta P\delta_\nu^\mu - \Pi_\nu^\mu, \quad (1.101)$$

where $\delta(\cdot)$ represents the perturbation. For example, δT_ν^μ is a small deviation from the background energy-momentum tensor $\bar{T}_\nu^\mu$, and so forth. Equation (1.100) is the same as Eq. (1.25) except that we have changed the notation and left the upper bar to denote the unperturbed quantity. Using the perturbed RW metric, Eq. (1.75),

it is possible to obtain the components for δT_ν^μ , these are given by [1]

$$\delta T_0^0 = \delta\rho, \quad (1.102)$$

$$\delta T_0^i = (\bar{\rho} + \bar{P})v^i, \quad (1.103)$$

$$\delta T_j^0 = -(\bar{\rho} + \bar{P})(v_j + B_j), \quad (1.104)$$

$$\delta T_j^i = -\delta P\delta_j^i - \Pi_j^i, \quad (1.105)$$

where $v^i \equiv a\delta U^i$. It is important to note that the perturbations ($\delta\rho$, δP , etc) have the contributions of several components (e.g. photons, neutrinos, baryons, dark matter, dark energy, etc.). This implies that we should sum over all possible components, i.e. $\delta\rho = \sum_I \delta\rho_I$, and the same for δP , v^i and Π^{ij} .

In the same way as the gauge transformations for the metric, we can perform the analogous transformation for the energy-momentum tensor. The coordinate transformation to avoid unphysical perturbations in the energy-momentum tensor is

$$T_\nu^\mu(x) = \frac{\partial x^\mu}{\partial \tilde{x}^\alpha} \frac{\partial \tilde{x}^\beta}{\partial x^\nu} \tilde{T}_\beta^\alpha(\tilde{x}). \quad (1.106)$$

Using the Eqs. (1.102)-(1.105) into Eq. (1.106) we obtain the relations between the perturbations in the gauge $\mathcal{G}$ and $\tilde{\mathcal{G}}$

$$\tilde{\delta\rho} = \delta\rho - T\bar{\rho}', \quad (1.107)$$

$$\tilde{\delta P} = \delta P - T\bar{P}', \quad (1.108)$$

$$\tilde{q}_i = q_i + (\bar{\rho} + \bar{P})L'_i, \quad (1.109)$$

$$\tilde{v}_i = v_i + L'_i, \quad (1.110)$$

$$\tilde{\Pi}_{ij} = \Pi_{ij}, \quad (1.111)$$

where q_i is known as momentum density and is defined by $q_i \equiv (\bar{\rho} + \bar{P})v_i$.

1.4.4 Perturbed equations of motions in Newtonian gauge

Once we have the perturbed metric and the perturbed energy-momentum tensor, in principle should be possible to derive the perturbed equations of motions. We start with perturbations in the Einstein equation, Eq. (1.21), as follows

$$\bar{G}_{\mu\nu} + \delta G_{\mu\nu} = 8\pi G(\bar{T}_{\mu\nu} + \delta T_{\mu\nu}). \quad (1.112)$$

In order to present the main ideas, we use the Newtonian gauge and we adopt the notation of [1] as follows

$$B = E = 0, \quad A \equiv \Psi, \quad C \equiv -\Phi. \quad (1.113)$$

Assuming the Newtonian gauge, we can rewrite the perturbed RW metric in terms of the new variables, Eq. (1.113), which simplifies the metric to

$$ds^2 = a^2(\tau) \left[(1 + 2\Psi) d\tau^2 - \delta_{ij} (1 - 2\Phi) dx^i dx^j \right]. \quad (1.114)$$

Once the gauge and the metric are specified, it is possible to derive all geometric quantities like the Christoffel symbols, the Ricci tensor, and the Ricci Scalar.

Perturbed stress-energy conservation

Using the perturbed energy-momentum tensor into the conservation equation²

$$\nabla_\mu T^\mu_\nu = 0 \quad (1.115)$$

it is possible to derive the perturbed versions of the *continuity* and *Euler equations*.

• Continuity equation

The first one is the continuity equation and is obtained considering $\nu = 0$. The result is

$$\delta' + \left(1 + \frac{\bar{P}}{\bar{\rho}}\right) (\nabla \cdot \mathbf{v} - 3\Phi') + 3\mathcal{H} \left(\frac{\delta P}{\delta \rho} - \frac{\bar{P}}{\bar{\rho}}\right) \delta = 0, \quad (1.116)$$

where $\delta \equiv \delta\rho/\bar{\rho}$ is the dimensionless fractional overdensity³.

• Euler equation

The Euler equation is obtained considering $\nu = i$. The result is

$$\mathbf{v}' + \mathcal{H}\mathbf{v} - 3\mathcal{H}\frac{\bar{P}'}{\bar{\rho}'}\mathbf{v} = -\frac{\nabla\delta P}{\bar{\rho} + \bar{P}} - \nabla\Psi \quad (1.117)$$

Perturbed Einstein equations

• Traceless space-space component

²Note that you will need the perturbed Christoffel symbols.

³Maybe this isn't a good name, because δ represent regions of overdensity but also regions of lack of density (voids). A more appropriate name in common use is *density contrast*.

The trace-free part of the spatial component of the Einstein tensor is given by

$$\partial_{\langle i} \partial_{j \rangle} (\Phi - \Psi) = 0, \quad (1.118)$$

$$\text{If } \Pi_{ij} = 0, \text{ then } \Phi = \Psi. \quad (1.119)$$

where the r.h.s of Eq. (1.118) is zero because we are neglecting the anisotropic stress. If we consider this term, then the r.h.s of Eq. (1.118) is sourced by the term $-8\pi G a^2 \Pi_{ij}$.

• **time-time component**

Considering the 00 component of the Einstein equation, $G_{00} = 8\pi G T_{00}$, we have at zeroth order

$$\mathcal{H}^2 = \frac{8\pi G}{3} a^2 \bar{\rho}, \quad (1.120)$$

and at first order we have

$$\nabla^2 \Phi = 4\pi G a^2 \bar{\rho} \delta + 3\mathcal{H} (\Phi' + \mathcal{H}\Phi). \quad (1.121)$$

• **time-space component**

Considering the 0i component of the Einstein equation, $G_{0i} = 8\pi G T_{0i}$, we are left with

$$\Phi' + \mathcal{H}\Phi = -4\pi G a^2 (\bar{\rho} + \bar{P}) v. \quad (1.122)$$

Combining (1.122) and the (1.121) we have

$$\nabla^2 \Phi = 4\pi G a^2 \bar{\rho} \Delta, \quad (1.123)$$

where $\bar{\rho} \Delta \equiv \bar{\delta} - 3\mathcal{H}(\bar{\rho} + \bar{P})v$.

• **Trace space-space component**

Finally, we consider the trace part of the Einstein equation, that is, $G_i^i = 8\pi G T_i^i$. The result of these computations is, at zeroth order given by

$$2\mathcal{H}' + \mathcal{H}^2 = -8\pi G \bar{P} \quad (1.124)$$

and at first order, we are left with

$$\Phi'' + 3\mathcal{H}\Phi + (2\mathcal{H}' + \mathcal{H}^2)\Phi = 4\pi G a^2 \delta P. \quad (1.125)$$

1.4.5 Proof of some equations

In this section, we want to demonstrate the equations (1.84), (1.85) and prove that the spatial part of the perturbed Christoffel symbol, Γ_{0i}^0 , in equation (1.38) is not zero. In summary, we want to prove the following equations

$$B_i \mapsto \tilde{B}_i = B_i + \partial_i T - L'_i, \quad (1.126)$$

$$h_{ij} \mapsto \tilde{h}_{ij} = h_{ij} - 2\partial_{(i} L_{j)} - 2\mathcal{H}T\delta_{ij}, \quad (1.127)$$

$$\Gamma_{0i}^0 = \partial_i \Psi. \quad (1.128)$$

Proof 1

We start considering the following coordinate transformation

$$X^\mu \mapsto \tilde{X}^\mu \equiv X^\mu + \zeta^\mu(\tau, \mathbf{x}), \quad (1.129)$$

$$X^\mu \rightarrow \{X^0 = \tau, X^i \rightarrow \mathbf{x}\}, \quad (1.130)$$

$$\zeta^0 \equiv T, \quad (1.131)$$

$$\zeta^i \equiv L^i = \partial^i L + \hat{L}^i. \quad (1.132)$$

By requesting the invariance of the spacetime interval we can avoid unphysical perturbations

$$ds^2 = g_{\mu\nu} dX^\mu dX^\nu = \tilde{g}_{\alpha\beta} d\tilde{X}^\alpha d\tilde{X}^\beta. \quad (1.133)$$

The spacetime interval in the coordinate system without/with “tilde” is given by

$$ds^2 = a^2(\tau) \left[(1 + 2A)d\tau^2 - 2B_i dx^i d\tau - (\delta_{ij} + h_{ij}) dx^i dx^j \right] \quad (1.134)$$

$$= a^2(\tilde{\tau}) \left[(1 + 2\tilde{A})d\tilde{\tau}^2 - 2\tilde{B}_i d\tilde{x}^i d\tilde{\tau} - (\delta_{ij} + \tilde{h}_{ij}) d\tilde{x}^i d\tilde{x}^j \right]. \quad (1.135)$$

For clarity, it is very useful to write explicitly each component of the metric tensor. First for frame without “tilde”

$$g_{00} = a^2(\tau)(1 + 2A), \quad (1.136)$$

$$g_{0i} = g_{i0} = -a^2(\tau)B_i, \quad (1.137)$$

$$g_{ij} = g_{ji} = -a^2(\tau)(\delta_{ij} + h_{ij}). \quad (1.138)$$

Analogously for the frame with “tilde”

$$\tilde{g}_{00} = a^2(\tilde{\tau})(1 + 2\tilde{A}), \quad (1.139)$$

$$\tilde{g}_{0i} = \tilde{g}_{i0} = -a^2(\tilde{\tau})\tilde{B}_i, \quad (1.140)$$

$$\tilde{g}_{ij} = \tilde{g}_{ji} = -a^2(\tilde{\tau})(\delta_{ij} + \tilde{h}_{ij}). \quad (1.141)$$

The relation of metric tensor between this coordinates system is given by

$$g_{\mu\nu}(X) = \frac{\partial \tilde{X}^\alpha}{\partial X^\mu} \frac{\partial \tilde{X}^\beta}{\partial X^\nu} \tilde{g}_{\alpha\beta}(\tilde{X}). \quad (1.142)$$

Let us study the components $\mu, \nu \rightarrow 0, i$. It is useful to decompose them α and β as follows: $\alpha, \beta \rightarrow \{0, 0; 0, i; i, 0; i, j\}$.

With Eqs. (1.129)-(1.132) and (1.139)-(1.141), the r.h.s of Eq. (1.142) can be separated in the following terms

$$\frac{\partial \tilde{X}^0}{\partial \tau} \frac{\partial \tilde{X}^0}{\partial X^i} \tilde{g}_{00} = (1 + T')(\partial_i T) a^2(\tilde{\tau})(1 + 2\tilde{A}), \quad (1.143)$$

$$\frac{\partial \tilde{X}^0}{\partial \tau} \frac{\partial \tilde{X}^j}{\partial X^i} \tilde{g}_{0j} = (1 + T')(\delta_j^i + \partial_j \tilde{\zeta}^i)(-a^2(\tilde{\tau})\tilde{B}_j), \quad (1.144)$$

$$\frac{\partial \tilde{X}^j}{\partial \tau} \frac{\partial \tilde{X}^0}{\partial X^i} \tilde{g}_{j0} = (\partial_\tau \tilde{\zeta}^j)(\partial_i T)(-a^2(\tilde{\tau})\tilde{B}_j), \quad (1.145)$$

$$\frac{\partial \tilde{X}^j}{\partial \tau} \frac{\partial \tilde{X}^k}{\partial X^i} \tilde{g}_{jk} = (\partial_\tau \tilde{\zeta}^j)(\delta_i^k + \partial_i \tilde{\zeta}^k)(-a^2(\tilde{\tau})(\delta_{jk} + \tilde{h}_{jk})), \quad (1.146)$$

where $T' := \partial_\tau T$. Now we combining the Eqs. (1.165)-(1.146) in the r.h.s of the (1.142) and eliminating the terms of $\mathcal{O}^2$ in the perturbations terms, we are left with

$$g_{0i} = a^2(\tilde{\tau}) [\partial_i T - \tilde{B}_i - L'_i], \quad (1.147)$$

where $L'_i := \partial_\tau L_i$. The last step is to expand the scale factor $a^2(\tilde{\tau}) = [a(\tau + T)]^2$ where in this equality we have used the 0 component of Eq. (1.129), then we have

$$\begin{aligned} a^2(\tilde{\tau}) &= [a(\tau + T)]^2 = \left[a(\tau) + a'(\tau)T + \frac{1}{2}a''(\tau)T^2 + \dots \right]^2 \\ &= a^2(\tau) \left[1 + \mathcal{H}T + \frac{1}{2} \frac{a''(\tau)}{a'(\tau)} T^2 + \dots \right]^2 \\ &\approx a^2(\tau) [1 + \mathcal{H}T]^2 \quad \text{first approximation} \\ &\approx a^2(\tau) [1 + 2\mathcal{H}T] \quad \text{second approximation.} \end{aligned} \quad (1.148)$$

where $\mathcal{H} = a'/a$ is the comoving Hubble parameter. Substituting Eq. (1.148) in r.h.s of Eq. (1.147), and using (1.137) in the l.h.s of (1.147) we have

$$-a^2(\tau)B_i = a^2(\tau) [1 + 2\mathcal{H}T] [\partial_i T - \tilde{B}_i - L'_i]. \quad (1.149)$$

Expanding and eliminating terms of the order $\mathcal{O}^2$ in the previous equation, we are left with $-B_i = \partial_i T - \tilde{B}_i - L'_i$ which shows that

$$\boxed{\tilde{B}_i = B_i + \partial_i T - L'_i}. \quad (1.150)$$

Proof 2

The demonstration of $\tilde{h}_{ij}$ is analogous to the demonstration for $\tilde{B}_i$. Following the same procedure, we start decomposing the r.h.s of Eq. (1.142), but now for $\mu, \nu \rightarrow i, j$

$$\frac{\partial \tilde{X}^0}{\partial X^i} \frac{\partial \tilde{X}^0}{\partial X^j} \tilde{g}_{00} = (\partial_i T)(\partial_j T)(a^2(\tilde{\tau})(1 + 2\tilde{A})), \quad (1.151)$$

$$\frac{\partial \tilde{X}^0}{\partial X^i} \frac{\partial \tilde{X}^k}{\partial X^j} \tilde{g}_{0k} = (\partial_i T)(\delta_j^k + \partial_j L^k)(-a^2(\tilde{\tau})\tilde{B}_k), \quad (1.152)$$

$$\frac{\partial \tilde{X}^k}{\partial X^i} \frac{\partial \tilde{X}^0}{\partial X^j} \tilde{g}_{k0} = (\delta_i^k + \partial_i L^k)(\partial_j T)(-a^2(\tilde{\tau})\tilde{B}_k), \quad (1.153)$$

$$\frac{\partial \tilde{X}^k}{\partial X^i} \frac{\partial \tilde{X}^l}{\partial X^j} \tilde{g}_{kl} = (\delta_i^k + \partial_i L^k)(\delta_j^l + \partial_j L^l)(a^2(\tilde{\tau})(\delta_{kl} + \tilde{h}_{kl})). \quad (1.154)$$

Combining Eqs. (1.151)-(1.154) into the r.h.s of Eq. (1.142), and eliminating the terms that are of second order in perturbation, $\mathcal{O}^2$, we are left with

$$g_{ij} = a^2(\tilde{\tau}) [\delta_{ij} + \tilde{h}_{ij} + \partial_i L_j + \partial_j L_i]. \quad (1.155)$$

Let's remember the notation for the vector field perturbation:

$$\partial_{(i} L_{j)} \equiv \frac{1}{2} (\partial_i L_j + \partial_j L_i). \quad (1.156)$$

Using Eqs. (1.138), (1.148) and (1.156) in Eq. (1.155) we are left with

$$-a^2(\tau)(\delta_{ij} + h_{ij}) = -a^2(\tau)(1 + 2\mathcal{H}T) [\delta_{ij} + \tilde{h}_{ij} + 2\partial_{(i} L_{j)}]. \quad (1.157)$$

Expanding the r.h.s of the previous equation and eliminating terms of $\mathcal{O}^2$ we get

$h_{ij} = \tilde{h}_{ij} + 2\partial_{(i}L_{j)} + 2\mathcal{H}T\delta_{ij}$. Finally, we have the desired equation

$$\boxed{\tilde{h}_{ij} = h_{ij} - 2\partial_{(i}L_{j)} - 2\mathcal{H}T\delta_{ij}}. \quad (1.158)$$

Proof 3

The final task is to derive the *perturbed connection coefficient* (or the perturbed Christoffel symbol) as shown by Eq.(1.128). The procedure developed here can be easily applied to obtain the other Christoffel symbols. Let's start with the perturbed RW metric tensor

$$g_{\mu\nu} = a^2 \begin{pmatrix} 1 + 2\Psi & 0 \\ 0 & -(1 - 2\Phi)\delta_{ij} \end{pmatrix}. \quad (1.159)$$

The inverse of $g_{\mu\nu}$ is

$$g^{\mu\nu} = \frac{1}{a^2} \begin{pmatrix} 1 - 2\Psi & 0 \\ 0 & -(1 + 2\Phi)\delta^{ij} \end{pmatrix}. \quad (1.160)$$

The relation between the connection coefficients and the metric tensor is

$$\Gamma_{\nu\rho}^{\mu} = \frac{1}{2}g^{\mu\lambda}(\partial_{\nu}g_{\lambda\rho} + \partial_{\rho}g_{\lambda\nu} - \partial_{\lambda}g_{\nu\rho}). \quad (1.161)$$

We want the case $\mu = \nu = 0$ and $\rho = i$, then the relation (1.161) reduces to

$$\Gamma_{0i}^0 = \frac{1}{2}g^{0\lambda}(\partial_0g_{\lambda i} + \partial_i g_{\lambda 0} - \partial_{\lambda}g_{0i}). \quad (1.162)$$

The above equation is non-zero only for $\lambda = 0$. Then, Eq. (1.162) becomes

$$\Gamma_{0i}^0 = \frac{1}{2}g^{00}(\partial_i g_{00}) \quad (1.163)$$

Finally, using $g^{00} = (1 - 2\Psi)/a^2$, $g_{00} = a^2(1 + 2\Psi)$ and eliminating terms second order in Ψ , we obtain the derided expression

$$\boxed{\Gamma_{0i}^0 = \partial_i \Psi}. \quad (1.164)$$

1.4.6 Evolution of the gravitational potential

In this section, we numerically solve a differential equation to obtain the evolution for the gravitational potential Φ . The complete solution is separated into three steps: (i) First we need the general ordinary differential equation governing the evolution of the gravitational potential, (ii) then we need to find the non-constant coefficients of this differential equation. (iii) Finally we need to rewrite this equation in order to be easily numerically integrated.

(i) General equation: the gravitational potential: The first-order in perturbation theory for the components $0,0$ and i,i in the Einstein equations are, respectively, given by

$$\nabla^2 \Phi = 4\pi G a^2 \delta\rho + 3\mathcal{H}(\Phi' + \mathcal{H}\Phi), \quad (1.165)$$

$$\Phi'' + 3\mathcal{H}\Phi' + (2\mathcal{H}' + \mathcal{H}^2)\Phi = 4\pi G a^2 \delta P. \quad (1.166)$$

Consider the following: $\nabla^2 \rightarrow -k^2$ and $\text{Eq.}(1.165) \times \left(-\frac{\bar{P}}{\bar{\rho}}\right) + \text{Eq.}(1.166)$, then we have

$$\Phi'' + 3\mathcal{H} \left(1 + \frac{\bar{P}}{\bar{\rho}}\right) \Phi' + \left[2\mathcal{H}' + \mathcal{H}^2 + 3\mathcal{H}^2 \frac{\bar{P}}{\bar{\rho}}\right] \Phi + \frac{\bar{P}}{\bar{\rho}} k^2 \Phi \quad (1.167)$$

$$= 4\pi G a^2 \delta\rho \left(\frac{\delta P}{\delta\rho} - \frac{\bar{P}}{\bar{\rho}}\right) \approx 0. \quad (1.168)$$

Analogously, we also consider the zeroth order in the Einstein equations of the components $0,0$ and i,i . These correspond to the first and second Friedmann equations, respectively

$$\mathcal{H}^2 = \frac{8\pi G}{3} a^2 \bar{\rho}, \quad (1.169)$$

$$2\mathcal{H}' + \mathcal{H}^2 = -8\pi G a^2 \bar{P}. \quad (1.170)$$

Performing the following operation $\text{Eq.}(1.169) \times \left(3\frac{\delta P}{\delta\rho}\right) + \text{Eq.}(1.170)$ we have

$$2\mathcal{H}' + \mathcal{H}^2 + 3\mathcal{H}^2 \frac{\delta P}{\delta\rho} = 2\mathcal{H}' + \mathcal{H}^2 + 3\mathcal{H}^2 \frac{\bar{P}}{\bar{\rho}} = 8\pi G a^2 \bar{P} \left(\frac{\delta P}{\delta\rho} - \frac{\bar{P}}{\bar{\rho}}\right) \approx 0. \quad (1.171)$$

Substituting Eq. (1.171) back to Eq. (1.168), we have

$$\boxed{\Phi''(\tau) + 3\mathcal{H}(1 + w_{\text{eff}})\Phi'(\tau) + \Phi + k^2 w_{\text{eff}}\Phi(\tau) = 0}. \quad (1.172)$$

Equation (1.172) is an ordinary differential equation with non-constant coefficients. Therefore, to solve this equation we need to find these terms. Firstly, we define the *effective equation of state* as

$$w_{\text{eff}} \equiv \frac{\bar{P}}{\bar{\rho}}. \quad (1.173)$$

(ii) Early times: radiation + pressureless matter: for this first solution, we simplify the problem and solve the differential equation (1.172) in the regime of radiation and cold dark matter (pressureless matter) domination. Therefore, the energy density and pressure can be written as follows

$$\bar{\rho} = \bar{\rho}_r + \bar{\rho}_m = \frac{\bar{\rho}_{r,0}}{a^4} + \frac{\bar{\rho}_{m,0}}{a^3} \quad (1.174)$$

$$\bar{P} = \bar{P}_r + \bar{P}_m \approx \bar{P}_r = w_r \rho_r = \frac{1}{3} \rho_r \quad (1.175)$$

$$\Omega_{I,0} \equiv \frac{\bar{\rho}_{I,0}}{\rho_{\text{crit},0}}, \quad I = r \text{ or } m. \quad (1.176)$$

Using Eqs. (1.174)-(1.176), the effective equation of state, Eq. (1.173), can be rewritten as follows

$$w_{\text{eff}} = \frac{1}{3 \left(1 + \frac{\Omega_{m,0}}{\Omega_{r,0}} a \right)}. \quad (1.177)$$

For convenience, we define a new variable y as follows

$$a_{\text{eq}} = \frac{\Omega_{r,0}}{\Omega_{m,0}}, \quad y \equiv \frac{a}{a_{\text{eq}}}. \quad (1.178)$$

Using these changes of variable, the effective equation of state can be written as

$$\boxed{w_{\text{eff}} = \frac{1}{3(1+y)}}. \quad (1.179)$$

The final task is to find the form of $\mathcal{H}$ as a function of y in these simplified radiation + CDM universe models. The relation between the conformal Hubble parameter, $\mathcal{H}$, and Hubble parameter, H , is

$$\mathcal{H}^2 = a^2 H^2. \quad (1.180)$$

Considering only radiation and pressureless matter, the Hubble parameter is

$$H^2 = H_0^2 \left[\frac{\Omega_{r,0}}{a^4} + \frac{\Omega_{m,0}}{a^3} \right] \Rightarrow \mathcal{H}^2 = H_0^2 \left[\frac{\Omega_{r,0}}{a^2} + \frac{\Omega_{m,0}}{a} \right]. \quad (1.181)$$

Using the change of variable is not difficult to show that the conformal Hubble parameter is given by

$$\mathcal{H}^2 = \frac{H_0^2 \Omega_{m,0}}{a_{\text{eq}}} \left[\frac{1}{y} + \frac{1}{y^2} \right], \text{ or } \boxed{\mathcal{H} = \frac{H_0 \sqrt{\Omega_{m,0}}}{\sqrt{a_{\text{eq}}}} \frac{\sqrt{1+y}}{y}}. \quad (1.182)$$

Taking the derivative of the conformal Hubble parameter with respect y we have

$$\boxed{\dot{\mathcal{H}} = -\frac{H_0 \sqrt{\Omega_{m,0}}}{\sqrt{a_{\text{eq}}}} \frac{2+y}{2y^2 \sqrt{1+y}}}. \quad (1.183)$$

(iii) Gravitational potential: change of variable: finally, we need to rewrite the Eq. (1.172) in terms of the new variable y . The first step is to express $d\tau$ in terms of dy , we have

$$\frac{d}{d\tau} = \frac{a}{a} \frac{da}{d\tau} \frac{d}{da} = \mathcal{H} a \frac{d}{da} = \mathcal{H} (y a_{\text{eq}}) \frac{d}{d(y a_{\text{eq}})}, \quad (1.184)$$

then we have

$$\boxed{\frac{d}{d\tau} = \mathcal{H} y \frac{d}{dy}}. \quad (1.185)$$

It is going to be useful for the second derivative of Eq. (1.185)

$$\frac{d^2}{d\tau^2} = \frac{d}{d\tau} \frac{d}{d\tau} = \mathcal{H} y \frac{d}{dy} \left(\mathcal{H} y \frac{d}{dy} \right), \text{ then} \quad (1.186)$$

$$\boxed{\frac{d^2}{d\tau^2} = \mathcal{H} y \left[\dot{\mathcal{H}} y \frac{d}{dy} + \mathcal{H} \frac{d}{dy} + \mathcal{H} y \frac{d^2}{dy^2} \right]}. \quad (1.187)$$

Just to recap, we are using the following notation

$$\frac{d}{d\tau} \equiv ()', \quad \frac{d}{dy} \equiv (\dot{ }). \quad (1.188)$$

Applying the Eqs. (1.185) and (1.188) into Eq. (1.172), after some algebraic manip-

ulations, we are left with

$$\ddot{\Phi}(y) + \frac{1}{\mathcal{H}y} [\dot{\mathcal{H}}y + \mathcal{H}(4 + 3w_{\text{eff}})] \dot{\Phi}(y) + \frac{k^2 w_{\text{eff}}}{\mathcal{H}^2 y^2} \Phi(y) = 0. \quad (1.189)$$

Finally, using Eqs. (1.179), (1.182) and (1.183) into Eq. (1.189) we have the desired equation of the potential in terms of the new variable y

$$\boxed{\ddot{\Phi}_k(y) + \left[\frac{8 + 7y}{2y(1 + y)} \right] \dot{\Phi}_k(y) + \left[\frac{a_{\text{eq}} k^2}{3H_0^2 \Omega_{m,0}(1 + y)^2} \right] \Phi_k(y) = 0}. \quad (1.190)$$

We use the new notation $\Phi(y) \rightarrow \Phi_k(y)$ to remember that this equation depends on the Fourier mode k .

(iv) Numerical results: To compute equation (1.190) numerically, we can use any programming language that allows us to solve ordinary differential equations (ODE), for example, `Python`. ODEs in `Python` can be solved with the function called `odeint` from the package `Scipy`. The function `odeint` only solves first-order ODEs, or a system of *coupled* ODEs. For this reason, we need to transform the second order Eq. (1.190) into a system of two coupled ODEs.⁴ To break the ODE into two other ones, we pay the price of gaining a new variable, here defined by v , so Eq. (1.190) becomes

$$v_k \equiv \dot{\Phi}_k, \quad (1.191)$$

$$\dot{v}_k(y) = -A(y)v_k(y) - B_k(y)\Phi_k(y). \quad (1.192)$$

The coefficients are defined by

$$A(y) \equiv \frac{8 + 7y}{2y(1 + y)}, \quad B_k(y) \equiv \frac{a_{\text{eq}} k^2}{3H_0^2 \Omega_{m,0}(1 + y)^2}. \quad (1.193)$$

We have assumed natural units where the speed of light in a vacuum is 1, but here we need to reintroduce this constant for a correct numerical computation. By dimensional analysis we see that $[H] = \text{km/s/Mpc}$ and $[k] = 1/\text{Mpc}$, therefore we need multiply k by c , i.e. we re-scale the value of k by $k \rightarrow kc$. `Python` also return to us the *rate* of the gravitational potential with respect to the variable y . The result is shown in figures 1.4 and 1.5.

⁴`Mathematica` is more suitable for this task, but here we use an open source software for illustrative purposes only.

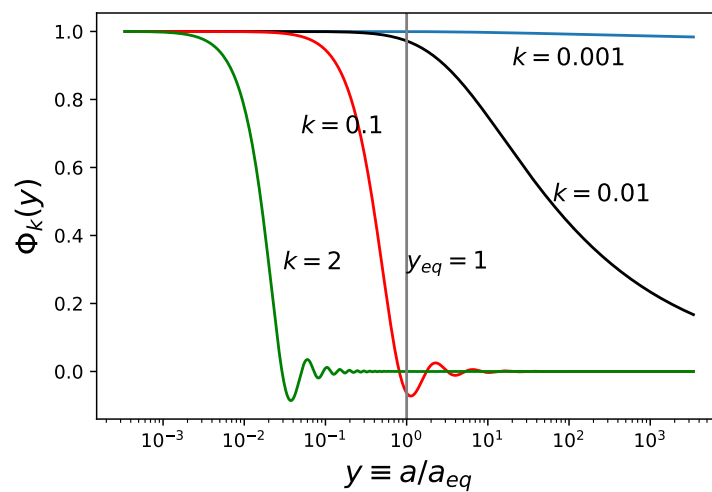


Figure 1.4: Evolution of perturbed gravitational potential

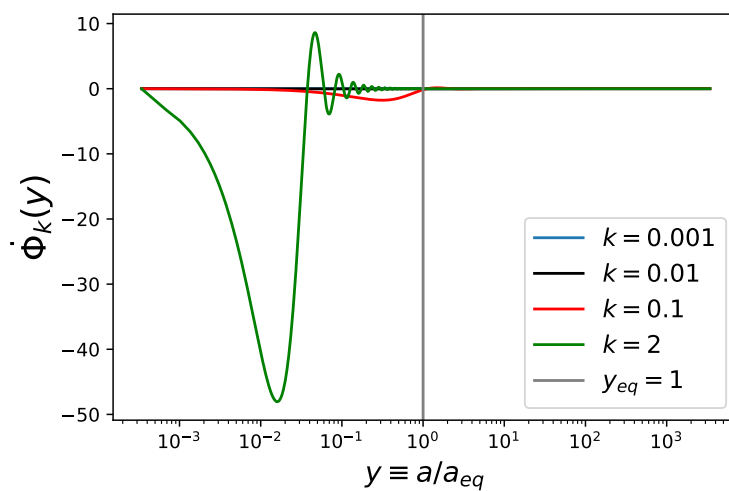


Figure 1.5: Evolution of the rate of the perturbed gravitational potential.

1.5 Cosmic Microwave Background

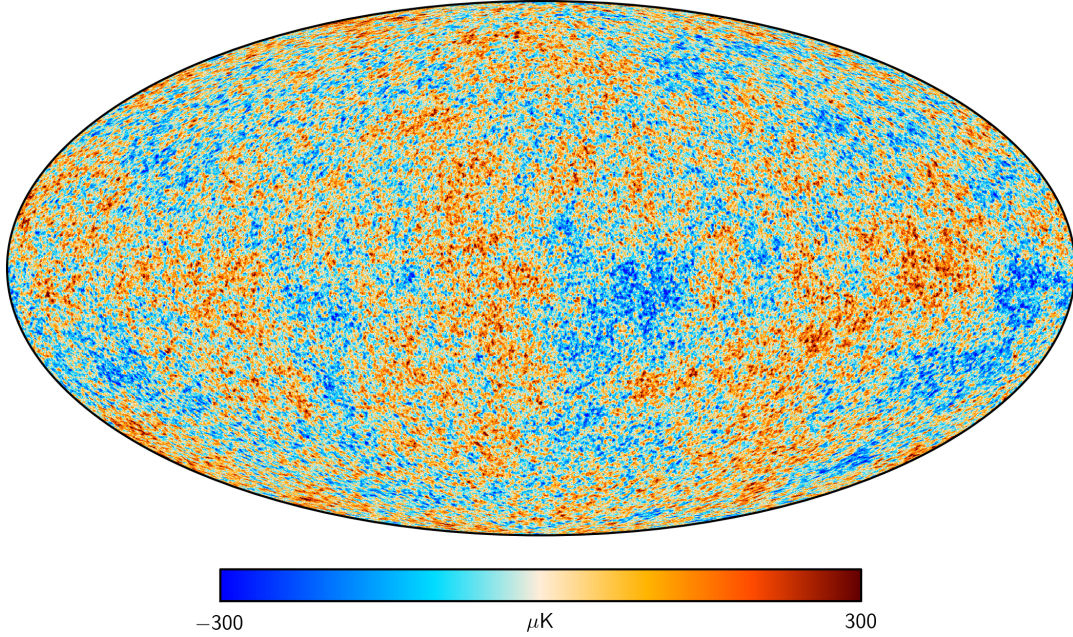


Figure 1.6: Figure from [Planck image gallery](#) [9]

The light that permeates throughout the universe is known as Cosmic Microwave Background (CMB) formed 370000 years after the Big Bang temporal-singularity. This radiation was first discovered by Penzias and Wilson [10] allowing them to share the Physics Nobel Prize in 1978. In Figure 1.6 we show a map of the CMB temperature anisotropy measured by the Planck satellite [9]. We can write a general expression for the temperature as a function of the sky direction:

$$T(\mathbf{n}) \equiv \bar{T}_0 [1 + \Theta(\mathbf{n})], \quad (1.194)$$

where $\bar{T}_0$ is the average CMB temperature and $\Theta(\mathbf{n}) \equiv \delta T(\mathbf{n})/\bar{T}_0$ is the fractional temperature fluctuation. In this part, we closely follow the [1] notation. As we will see, it is possible to write a functional form for $\Theta(\mathbf{n})$ (thus for $T(\mathbf{n})$ if $\bar{T}_0$ is known), however, observationally we should turn our finds into a statistical equation known as two-point correlation function:

$$C(\theta) \equiv \langle \Theta(\mathbf{n})\Theta(\mathbf{n}') \rangle, \quad (1.195)$$

where θ is the angular separation in the sky between the unitary vectors $\mathbf{n}$ and $\mathbf{n}'$ and defined through $\cos \theta = \mathbf{n} \cdot \mathbf{n}'$. This is because a broad variety of theoretical

models of the early universe predicts that $\Theta(\mathbf{n})$ should satisfy a Gaussian distribution [11]. If this actually is the case, then we should be able to capture all features of the CMB temperature anisotropies, $\theta(\mathbf{n})$, with a single function of the angular separation $C(\theta)$ given by Eq. (1.195).

The same cosmological information in $C(\theta)$ is contained in the angular power spectra denoted by C_ℓ . To see this, we first expand the CMB fractional temperature anisotropy in terms of spherical harmonics

$$\Theta(\mathbf{n}) = \sum_{l=2}^{\infty} \sum_{m=-l}^l a_{lm} Y_{lm}(\mathbf{n}). \quad (1.196)$$

Substituting Eq. (1.196) into Eq. (1.195) we obtain

$$C(\theta) = \sum_{\ell} \frac{2\ell + 1}{4\pi} C_{\ell} P_{\ell}(\cos \theta) \quad (1.197)$$

where P_{ℓ} are the Legendre polynomials, and we have used the assumption of Gaussian statistics (predicted by early universe models) for the fractional temperature anisotropy:

$$\delta_{mm'} \delta_{\ell\ell'} C_{\ell} = \langle a_{\ell m} a_{\ell' m'}^* \rangle, \quad (1.198)$$

Since the P_{ℓ} 's are known, if we determine C_{ℓ} we automatically find $C(\theta)$ by means of Eq. (1.197). In Eq. (1.196) note that m varies in the range $\pm\ell$, thus for a fixed ℓ there are $2\ell + 1$ possible values for m . Therefore, summing over all possible values for m , we can obtain an estimator for C_{ℓ}

$$\hat{C}_{\ell} = \frac{1}{(2\ell + 1)} \sum_{m=-\ell}^{\ell} |a_{\ell m}|^2. \quad (1.199)$$

Note that the estimator is unbiased, $\langle \hat{C}_{\ell} \rangle - C_{\ell} = 0$. However, there is an intrinsic error due to the cosmic variance $[2/(2\ell + 1)]^{1/2} C_{\ell}$ [12] which is dominated on larger angular scales.

1.5.1 Primordial spectrum and CMB

What was the mechanism (inflation, topological defects, etc) behind the generation of the primordial spectrum density perturbation that 13.8 billion years after the Big Bang led the universe that we see today? In order to use CMB to

find an answer to this question, we first need a connection between the angular power spectra, C_ℓ 's, and the primordial spectrum of density perturbations, $\Delta_{\mathcal{R}}^2(k)$. The first step in this direction is to know how the photons (i.e. a property of the photons: energy E , wavelength λ , etc) evolve after decoupling in a perturbed universe. The geodesic equation governing the free photon propagation is

$$\frac{dP^\mu}{d\eta} = -\Gamma_{\nu\rho}^\mu \frac{P^\nu P^\rho}{P^0}. \quad (1.200)$$

Using the Christoffel symbols derived in proof 3, section 1.4.5, we can rewrite Eq. (1.200) for $\mu = 0$ (because it is related with the photon's energy) as follow

$$\frac{dP^0}{d\eta} = -(\mathcal{H} + \Psi') P^0 - 2\partial_i \Psi P^i - [\mathcal{H} - \Phi' - 2\mathcal{H}(\Phi + \Psi)] \delta_{ij} \frac{P^i P^j}{P^0} \quad (1.201)$$

Given a particle with mass m , physical momentum $p = (g_{ij} P^i P^j)^{1/2}$, energy $E = (p^2 + m^2)^{1/2}$, the mass-shell constraint is given by (see e.g. [13]) $g_{\mu\nu} P^\mu P^\nu = (1 + 2\Psi)(P^0)^2 + p^2 = m^2$. For the photon, $m = 0$, thus we obtain

$$\begin{aligned} P^0 &= \frac{E}{a}(1 - \Psi), \\ P^i &= \frac{E}{a}(1 + \Phi)p^i \end{aligned} \quad (1.202)$$

substituting Eq. (1.202) into (1.201) and solving for the energy at first order in perturbation, we get

$$\frac{d \ln E}{d\eta} = -\mathcal{H} + \Phi' - p^i \partial_i \Psi. \quad (1.203)$$

The equation (1.203) tells how the photon energy varies in an expanding (due the Hubble term) and inhomogeneous universe (due the metric perturbations). At this point, we are interested solely on the effect of metric perturbations, thus we have to discount the Hubble flow from the right hand side of Eq. (1.203), which leads

$$\frac{d \ln \epsilon}{d\eta} = \Phi' + \Psi' - \frac{d\Psi}{d\eta} \quad (1.204)$$

where $\epsilon \equiv aE$. The Eq. (1.204) tells us how the photon's energy (corrected by the redshifting due to the Hubble flow) is affected by local perturbations of the Hubble flow (Φ') which means that some parts of the universe expand slightly faster, or

lower than others one plus potential wells (the whole term $\Psi' - d\Phi/d\eta = p^i \partial_i \Phi$) caused by the intervening matter of the large scale structure.

To obtain how much was the photon's energy deviated from its initial value at the decoupling time (η_*) to its value seen today (η_0) by our telescopes, we should solve Eq. (1.204) for the energy, a result known as *line-of-sight solution* [1] or *line-of-sight integral* [14]. A proper way to perform this calculation needs to take into account several physical processes beyond the gravitation redshift, such as the adiabatic growth of perturbations, baryon velocity-induced fluctuations, and photon diffusion. Those effects are then combined and solved with the Euler, Continuity, and Boltzmann equation with the introduction of the *visibility function*, which is the probability that a photon last scattered within a time interval of $[\eta, \eta + d\eta]$ (see [15] and references therein). For illustrative purposes, we assume an instantaneous decoupling at η_* (see chapter 7 of [1] or Appendix B for detailed computation), and the solution for Eq. (1.204) becomes

$$\frac{\delta T}{\bar{T}} \Big|_0 = \frac{\delta T}{\bar{T}} \Big|_* + \Psi_* - \Psi_0 + \int_{\eta_0}^{\eta_*} d\eta \left(\frac{\partial \Phi}{\partial \eta} + \frac{\partial \Psi}{\partial \eta} \right). \quad (1.205)$$

To obtain Eq. (1.205) we have used that after decoupling the photon's distribution function have its shape unmodified once they trajectories are along geodesics, and since the Bose-Einstein distribution is a function of E/T , then the proportionality relation holds, $E \propto T \Rightarrow E \propto \bar{T}(1 + \delta T/\bar{T})$. The first term in Eq. (1.205) has the contribution from two sources: the temperature fluctuation induced by photon density contrast, $\delta_\gamma/4$, and a Doppler shift due to the scattering between photons and electrons at the last scattering surface, which is proportional to the projection of the baryon velocity onto the line-of-sight direction, $-\mathbf{n} \cdot \mathbf{v}_b$, where the minus sight means that the photon momentum vector has a direction opposite to the unit vector $\mathbf{n}$. The term $\Psi_* - \Psi_0$ in Eq.(1.204) simplifies to Ψ_* because the local gravitational potential, Ψ_0 , modify the CMB power spectra as a whole, thus this term is unobserved [16] and can be ignored in the equation. Rearranging the terms we have $\delta T/\bar{T}|_* + \Psi_* - \Psi_0 = (\delta_{\gamma*}/4 + \Psi_*) - \mathbf{n} \cdot \mathbf{v}_{b*}$ where the first term inside the parenthesis is recognized as the *Sachs-Wolfe effect* due to Sachs and Wolfe (1967) [17]. The second term, $-\mathbf{n} \cdot \mathbf{v}_{b*}$, is the Doppler shift. Finally, the integral in Eq.(1.204) is called the *integrated Sachs-Wolfe effect* that accounts for temperature fluctuation induced by the intervening distribution of matter through

the universe [1, 16]. In this way, Eq. (1.205) can be written as

$$\Theta(\mathbf{n}) \equiv \frac{\delta T}{\bar{T}}(\mathbf{n}) = \frac{1}{4}\delta\gamma_* + \Psi_* - \mathbf{n} \cdot \mathbf{v}_{b*} + \int_{\eta_*}^{\eta_0} (\Phi' + \Psi') d\eta. \quad (1.206)$$

In general, it is easier to work in Fourier space, where plane waves can be decomposed in Fourier modes and evolve independently and in the end, we just need to perform a superposition of these modes. This constitutes the so-called *spatial to angular projection* approach [1]. Let us define new functions $F_*(k)\mathcal{R}_i(\mathbf{k}) \equiv \delta\gamma_*/4 + \Psi_*$ and $G_*(k)\mathcal{R}_i(\mathbf{k}) \equiv |\mathbf{v}_{b*}| = v_{b*}$ where $\mathcal{R}_i$ is the initial curvature perturbation that takes into account the linear evolution of fluctuations. Therefore, Eq. (1.206) becomes

$$\Theta(\mathbf{n}) = \int \frac{d^3k}{(2\pi)^3} e^{i\mathbf{k} \cdot \mathbf{x}_*} \Theta_\ell(k) R_i. \quad (1.207)$$

where we have defined the *transfer function* $\Theta_\ell(k) \equiv F_* - i(\hat{\mathbf{k}} \cdot \hat{\mathbf{n}})G_*$. Note that in this equation we neglected the ISW effect, which is small in comparison with the other ones, with the exception at $\ell < 10$ (see e.g. figure 5.2 in [14]) which is a little bit larger than the Doppler effect, but that cause no much prejudice for our example. If we take the Legendre expansion of the exponential, $e^{i(k\chi_*)\hat{\mathbf{k}} \cdot \hat{\mathbf{n}}} = \sum_\ell i^\ell (2\ell + 1) j_\ell(k\chi_*) P_\ell(\hat{\mathbf{k}} \cdot \hat{\mathbf{n}})$, and its derivative with respect to the module of its argument, we can rewrite the equation (1.207) as follows

$$\Theta(\mathbf{n}) = \sum_\ell i^\ell (2\ell + 1) \int \frac{d^3k}{(2\pi)^3} \Theta_\ell(k) R_i(\mathbf{k}) P_\ell(\hat{\mathbf{k}} \cdot \hat{\mathbf{n}}). \quad (1.208)$$

Finally, substituting Eq.(1.208) into the two-point correlation function (1.195) and comparing the result with the CMB temperature power spectra in Eq. (1.197) we have

$$C_\ell = 4\pi \int d(\ln k) \Theta_\ell^2(k) \Delta_{\mathcal{R}}^2(k), \quad (1.209)$$

where the primordial curvature perturbation, $\Delta_{\mathcal{R}}(k)$, is defined via the two-point correlation function of the initial curvature perturbation, $\mathcal{R}_i(\mathbf{k})$, in the following way: $\langle \mathcal{R}_i(\mathbf{n}) \mathcal{R}_i(\mathbf{n}') \rangle = 2\pi^2/k^3 \Delta_{\mathcal{R}}^2(k) (2\pi)^3 \delta(\mathbf{k} + \mathbf{k}')$.

As we have mentioned, the proper way to derive the CMB temperature anisotropy power spectra is via Boltzmann's kinetic theory (coupled with the

Einstein equations) that takes into account deviations of the particle distribution function from thermal equilibrium due to the universe expansion, particle scatterings, annihilations and a sharply peaked, but non-instantaneous, photon decoupling as we have assumed in our derivation of equation (1.209). Although the purpose here is not to provide all the details on CMB calculations, we provide a scheme on how real CMB power spectra are estimated (see [14] for details or appendix B of [1]). In figure 1.7 the horizontal line represents the last scattering surface and a photon that starts its journey through the space until be captured by our telescopes today. This photon carries much information about the physical process on the LSS, and in figure 1.7 we show the essential equations that allow us to relate the CMB photon with the universe's characteristics such as mass-energy distribution and the physics of recombination.

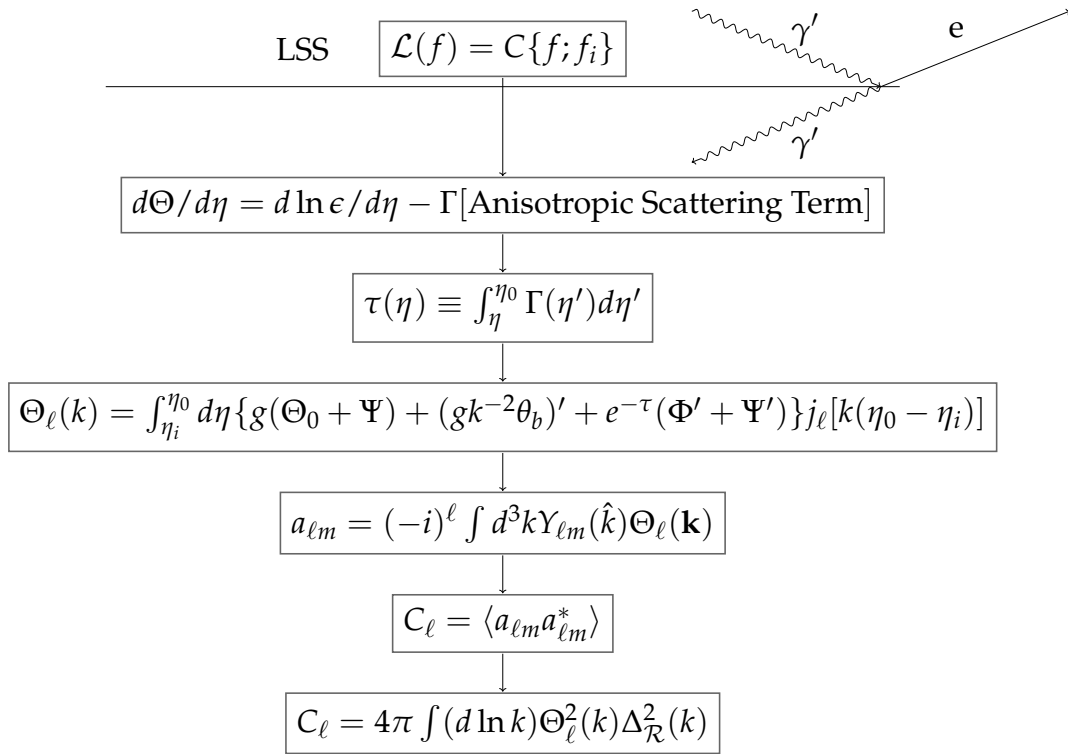


Figure 1.7: A schematic representation of how we convert the early universe's information carried by the CMB photons when it reaches us today.

We can interpret figure 1.7 as follows. Initially, photons were tightly coupled with baryons, exchanging energy and momentum in a scattering process with a rate higher than the universe expansion. From the photon's phase-space point of

view, this implies in an evolution of the number of photons at a given point (P_μ, x^i) and time $t = x^0$. This is characterized by the distribution function $f(P_\mu, x^\nu)$ with a dynamics described by the Boltzmann equation

$$\mathcal{L}(f) \equiv \frac{df}{d\eta}(\eta, \mathbf{x}, \epsilon, \hat{\mathbf{p}}) = \frac{\partial f}{\partial \eta} + \frac{\partial f}{\partial \mathbf{x}} \cdot \frac{d\mathbf{x}}{d\eta} + \frac{\partial f}{\partial \epsilon} \frac{d\epsilon}{d\eta} + \text{CMB Lensing} = C\{f; f_i\} \quad (1.210)$$

where the left side in (1.210) is the Liouville operator which simply tells us how the distribution function varies, with respect to η , in response to external interactions with other particles given by each distribution function, f_i , where i can be electrons, neutrinos, etc. The “CMB lensing” term describes deviation in the CMB photon direction and is given by $(\partial f / \partial \hat{\mathbf{p}}) \cdot (\partial \hat{\mathbf{p}} / \partial \eta)$. This term is second order in perturbation (which is called “secondary” CMB effect) so we drop off its contribution.

The distribution function in terms of the fractional temperature fluctuation, Θ , is given by

$$f(\eta, \mathbf{x}, \epsilon, \hat{\mathbf{p}}) = \frac{1}{\exp \left[\frac{\epsilon}{a\bar{T}(\eta)(1+\Theta(\eta, \mathbf{x}, \hat{\mathbf{p}}))} \right] - 1}. \quad (1.211)$$

Combining Eq. (1.211) with (1.210) and taking into account the Isotropic Thompson Scattering, $\text{ITS} \equiv \Theta - \Theta_0 - \hat{\mathbf{p}} \cdot \mathbf{v}_b$, and the Anisotropic Thompson Scattering, $\text{ATS} \equiv -3/(16\pi) \int d\hat{\mathbf{p}}_i \Theta(\hat{\mathbf{p}}_i) [(\hat{\mathbf{p}}_i \cdot \hat{\mathbf{p}})^2 - 1/3]$, where Θ_0 is the monopole and $\hat{\mathbf{p}}_i$ is the incoming photon’s momentum in the scattering process with the target electron. This produces the second equation from top to bottom in figure 1.7. The solution of this differential equation was mentioned before, it is known as the “line-of-sight solution”, which is obtained by introducing the optical depth, $\tau(\eta)$, that describes the opacity of the universe; higher the opacity is higher is the probability of photon be scattered between η and η_0 . The mathematical advantage of this quantity is that it helps to integrate the Boltzmann equation, which turns on into the fourth equation shown in figure 1.7.

The fractional temperature fluctuation was defined in equation (1.196) in terms of spherical harmonics $Y_{\ell m}$ with weights $a_{\ell m}$. These quantities can be computed just by inverting Eq. (1.196), in terms of Θ_ℓ , as shown in the fifth equation, from top to bottom, of figure 1.7. We take the average of these weights which finally produce the CMB power spectra in terms of the transfer function, $\Theta_\ell(k)$, and the

primordial power spectra.

Chapter 2

Gauge invariant perturbations

2.1 Introduction

In chapter 1.1 we saw expressions that do not change under a gauge transformation, for example, the Bardeen variables and the comoving curvature perturbation. How can we obtain these quantities? This chapter explains the procedure and performs detailed computations to obtain these and other gauge-invariant perturbations.

The universe is described by the RW metric $g_{\mu\nu}$. This metric is the contribution of two terms: $\bar{g}_{\mu\nu}$ which denotes the metric at background level (or at zeroth order in perturbation theory) and $\delta g_{\mu\nu}$ that represents small perturbation in the metric (or the first order in perturbation theory). Through this chapter we use the notation of [1]. As we saw in chapter 1.1, the *perturbation functions* (PF) are A , B_i , and h_{ij} . In order to avoid the *gauge problem*, i.e. the generation of unphysical perturbations, we need to perform the gauge transformation requesting the invariance of the spacetime interval, i.e. $d\tilde{s}^2 = ds^2$. This “jump” from the gauge $\mathcal{G}$ to the other gauge $\tilde{\mathcal{G}}$ is performed by the *gauge generator functions* given by ζ^μ (introduced in chapter 1.1), such that $|\zeta^\mu| \ll 1$ in order to guarantee that the perturbations can be linearized.

2.2 Setting equations

The perturbed RW metric is given by

$$g_{\mu\nu} = a^2(\tau) \begin{Bmatrix} [1 + 2A(\tau, \mathbf{x})] & -B_i(\tau, \mathbf{x}) \\ -B_i(\tau, \mathbf{x}) & -[\delta_{ij} + h_{ij}(\tau, \mathbf{x})] \end{Bmatrix}. \quad (2.1)$$

Let's remember the SVT decomposition for the perturbation functions. These are given by

$$B_i = \partial_i B + \tilde{B}_i \quad (\partial^i B_i = 0), \quad (2.2)$$

$$h_{ij} = 2C\delta_{ij} + 2\partial_{\langle i}\partial_{j\rangle}E + 2\partial_{(i}\tilde{E}_{j)} + 2\tilde{E}_{ij} \quad (\tilde{E}_i^i = \partial^i \tilde{E}_i = \partial^i \tilde{E}_{ij} = 0), \quad (2.3)$$

$$\partial_{\langle i}\partial_{j\rangle}E = \left(\partial_i\partial_j - \frac{1}{3}\delta_{ij}\nabla^2 \right) E, \quad (2.4)$$

$$\partial_{(i}\tilde{E}_{j)} = \frac{1}{2} (\partial_i\tilde{E}_j + \partial_j\tilde{E}_i). \quad (2.5)$$

For convenience, we also repeat the gauge generator function

$$\tilde{x}^\mu = x^\mu + \zeta^\mu(\tau, \mathbf{x}), \quad (2.6)$$

$$\zeta^0 \equiv T, \quad (2.7)$$

$$\zeta^i \equiv L_i = \partial^i L + \tilde{L}^i \quad (\partial^i \tilde{L}_i = 0). \quad (2.8)$$

From the invariance of the spacetime interval, $d\tilde{s}^2 = ds^2$, it is possible to show that the components of the metric in the gauge $\mathcal{G}$, $g_{\mu\nu}$, relates with the components of the metric in the gauge $\tilde{\mathcal{G}}$, $\tilde{g}_{\mu\nu}$, by the following expression

$$g_{\mu\nu}(\tau, \mathbf{x}) = \tilde{g}_{\mu\nu}(\tau, \mathbf{x}) + \partial_\alpha g_{\mu\nu}(\tau, \mathbf{x})\zeta^\alpha + \partial_\mu \tilde{\zeta}^\rho g_{\rho\nu}(\tau, \mathbf{x}) + \partial_\nu \tilde{\zeta}^\rho g_{\rho\mu}(\tau, \mathbf{x}). \quad (2.9)$$

To demonstrate the Eq. (2.9), we start from the invariance of the line elements between the gauges, i.e.

$$ds^2 = d\tilde{s}^2 \Rightarrow g_{\mu\nu}(x)dx^\mu dx^\nu = \tilde{g}_{\alpha\beta}(\tilde{x})d\tilde{x}^\alpha d\tilde{x}^\beta. \quad (2.10)$$

The coordinate transformation is given by $d\tilde{x}^\alpha = (\partial\tilde{x}^\alpha/\partial x^\mu)dx^\mu$ and $d\tilde{x}^\beta = (\partial\tilde{x}^\beta/\partial x^\nu)dx^\nu$ we have

$$g_{\mu\nu}(x) = \frac{\partial\tilde{x}^\alpha}{\partial x^\mu} \frac{\partial\tilde{x}^\beta}{\partial x^\nu} \tilde{g}_{\alpha\beta}(\tilde{x}) \quad (2.11)$$

Using the gauge transformation given by Eq. (2.6) into Eq. (2.11) we have

$$g_{\mu\nu}(x) = \left(\delta_\mu^\alpha + \partial_\mu \tilde{\zeta}^\alpha \right) \left(\delta_\nu^\beta + \partial_\nu \tilde{\zeta}^\beta \right) \tilde{g}_{\alpha\beta}(\tilde{x}). \quad (2.12)$$

Expand Eq. (2.12) and neglect quadratic terms in the gauge generator $\tilde{\zeta}^\alpha$. In addition, it is useful to consider the expansion of the $\tilde{g}_{\alpha\beta}(\tilde{x})$ around the neighborhood

of x , this is given by

$$\tilde{g}_{\alpha\beta}(\tilde{x}) = \tilde{g}_{\alpha\beta}(\tilde{x}^\mu) = \tilde{g}_{\alpha\beta}(x^\mu + \zeta^\mu) \approx \tilde{g}_{\alpha\beta}(x^\mu) + \partial_\mu \tilde{g}_{\alpha\beta}(x^\mu) \zeta^\mu(x), \quad \text{or} \quad (2.13)$$

$$\tilde{g}_{\alpha\beta}(\tilde{x}) \approx \tilde{g}_{\alpha\beta}(x) + \partial_\mu \tilde{g}_{\alpha\beta}(x) \zeta^\mu(x) \quad (2.14)$$

Substituting Eq. (2.14) into Eq. (2.12) and expanding the expression up to first order in perturbation, we are left with

$$\begin{aligned} g_{\mu\nu}(x) &= \left(\delta_\mu^\alpha + \partial_\mu \zeta^\alpha \right) \left(\delta_\nu^\beta + \partial_\nu \zeta^\beta \right) (\tilde{g}_{\alpha\beta} + \partial_\sigma \tilde{g}_{\alpha\beta} \zeta^\sigma) \\ &= \left(\delta_\mu^\alpha \delta_\nu^\beta + \delta_\mu^\alpha \partial_\nu \zeta^\beta + \delta_\nu^\beta \partial_\mu \zeta^\alpha \right) (\tilde{g}_{\alpha\beta} + \partial_\sigma \tilde{g}_{\alpha\beta} \zeta^\sigma) \\ &= \delta_\mu^\alpha \delta_\nu^\beta \tilde{g}_{\alpha\beta} + \delta_\mu^\alpha \delta_\nu^\beta (\partial_\sigma \tilde{g}_{\alpha\beta}) \zeta^\sigma + \delta_\mu^\alpha (\partial_\nu \zeta^\beta) \tilde{g}_{\alpha\beta} + \delta_\nu^\beta (\partial_\mu \zeta^\alpha) \tilde{g}_{\alpha\beta} \\ &= \tilde{g}_{\mu\nu}(x) + \partial_\sigma \tilde{g}_{\mu\nu}(x) \zeta^\sigma + \partial_\mu \zeta^\alpha \tilde{g}_{\nu\alpha}(x) + \partial_\nu \zeta^\beta \tilde{g}_{\mu\beta}(x), \end{aligned} \quad (2.15)$$

where Eq. (2.15) is the same as Eq. (2.9), what was to be shown. Applying Eq. (2.9) to each metric component, we obtain the relations satisfied the gauge generator between the gauges $\mathcal{G}$ and $\tilde{\mathcal{G}}$. However, this computation was already performed in section (1.4.5) for B_i and h_{ij} , the result is

$$\tilde{A} = A - T' - \mathcal{H}T, \quad (2.16)$$

$$\tilde{B}_i = B + \partial_i T - L'_i, \quad (2.17)$$

$$\tilde{h}_{ij} = h_{ij} - 2\partial_{(i} L_{j)} - 2\mathcal{H}T\delta_{ij}. \quad (2.18)$$

2.3 Building gauge invariant perturbations

In terms of the SVT decomposition, we can separate the perturbations contributions, Eqs. (2.16)-(2.18), into scalars, vectors, and tensors part. Let's concentrate on the scalar perturbation part. Then we have

$$\tilde{A} = A - T' - \mathcal{H}T, \quad (2.19)$$

$$\tilde{B} = B + T - L', \quad (2.20)$$

$$\tilde{C} = C - \mathcal{H}T - \frac{1}{3}\nabla^2 L, \quad (2.21)$$

$$\tilde{E} = E - L. \quad (2.22)$$

In the same sense, we can decompose the components of the energy-momentum

tensor (again, taking just the scalar part)

$$\tilde{\delta\rho} = \delta\rho - \bar{\rho}'T, \quad (2.23)$$

$$\tilde{\delta P} = \delta P - \bar{P}'T, \quad (2.24)$$

$$\tilde{v} = v + L', \quad (2.25)$$

$$\tilde{q} = q + (\bar{\rho} + \bar{P}) L'. \quad (2.26)$$

Now we have everything that we need in order to describe the general method to find gauge invariant combinations. Firstly, note that the Eqs. (2.19)-(2.26) can be expressed in a most general way as follows $\tilde{A} = \tilde{A}(A, T, L)$, $\tilde{B} = \tilde{B}(B, T, L)$, ..., $\tilde{q} = \tilde{q}(q, T, L)$. Generically, this can be written as $\tilde{Q} = \tilde{Q}(Q, T, L)$. These quantities are dependent of the gauge generator function T and L , this means that the perturbed quantity, $\tilde{Q}$, also is gauge dependent. Then, the conclusion is:

If we combine the equations from (2.19) to (2.26) in order to eliminate the gauge generator functions, T and L , then the result is a gauge invariant quantity.

2.4 Applying the method

We proceed to apply the previous statement in order to find useful combinations that are gauge-independent.

2.4.1 Bardeen's potentials

The first gauge invariant quantity widely used in cosmology is the Bardeen's potentials (or variables), which we define below.

First Bardeen's potential

Consider the following equations

$$\tilde{A} = A - T' - \mathcal{H}T, \quad (2.27)$$

$$\tilde{B} = B + T - L', \quad (2.28)$$

$$\tilde{E} = E - L. \quad (2.29)$$

In order to eliminate the gauge generator functions T and L (as well T' and L'), we perform the following manipulations

$$-\text{Eq. (2.29)}' \Rightarrow -\tilde{E}' = -E' + L', \quad (2.30)$$

$$\Downarrow$$

$$\text{Eq. (2.28)} + \text{Eq. (2.30)} \Rightarrow \tilde{B} - \tilde{E}' = B - E' + T. \quad (2.31)$$

Consider now the following manipulations with Eq. (2.31)

$$\mathcal{H} \times \text{Eq. (2.31)} \Rightarrow \mathcal{H}(\tilde{B} - \tilde{E}') = \mathcal{H}(B - E') + \mathcal{H}T, \quad (2.32)$$

$$\text{Eq. (2.31)}' \Rightarrow (\tilde{B} - \tilde{E}')' = (B - E')' + T'. \quad (2.33)$$

Finally, if we do: Eq. (2.27)+Eq. (2.32)+Eq. (2.33), then we obtain the following

$$\begin{aligned} \tilde{A} + \mathcal{H}(\tilde{B} - \tilde{E}') + (\tilde{B} - \tilde{E}')' &= A - \mathcal{T}' - \mathcal{H}\mathcal{T}' + \mathcal{H}(B - E') + \mathcal{H}\mathcal{T}' + (B - E')' + \mathcal{T}' \\ \tilde{A} + \mathcal{H}(\tilde{B} - \tilde{E}') + (\tilde{B} - \tilde{E}')' &= A + \mathcal{H}(B - E') + (B - E')'. \end{aligned} \quad (2.34)$$

The l.h.s of Eq. (2.34) is expressed in the gauge $\tilde{\mathcal{G}}$, and the r.h.s is expressed in the gauge $\mathcal{G}$, however, how both sides of Eq. (2.34) are equal, then we conclude that the quantity defined by

$$\boxed{\Psi \equiv A + \mathcal{H}(B - E') + (B - E')'} \quad (2.35)$$

is a gauge invariant quantity. The Eq. (2.35) is known as the *first Bardeen's potential*.

Second Bardeen's potential

Consider the following equations

$$\tilde{B} = B + T - L', \quad (2.36)$$

$$\tilde{C} = C - \mathcal{H}T - \frac{1}{3}\nabla^2 L, \quad (2.37)$$

$$\tilde{E} = E - L. \quad (2.38)$$

Performing the following manipulations

$$-\text{Eq. (2.37)} \Rightarrow -\tilde{C} = -C + \mathcal{H}T + \frac{1}{3}\nabla^2 L, \quad (2.39)$$

$$\frac{1}{3}\nabla^2 \text{Eq. (2.38)} \Rightarrow \frac{1}{3}\nabla^2 \tilde{E} = \frac{1}{3}\nabla^2 E - \frac{1}{3}\nabla^2 L, \quad (2.40)$$

$\Downarrow$

$$\text{Eq. (2.39)} + \text{Eq. (2.40)} \Rightarrow -\tilde{C} + \frac{1}{3}\nabla^2 \tilde{E} = -C + \frac{1}{3}\nabla^2 E + \mathcal{H}T. \quad (2.41)$$

In addition, consider the following computations

$$-\text{Eq. (2.38)'} \Rightarrow -\tilde{E}' = -E' + L', \quad (2.42)$$

$$\text{Eq. (2.36)} + \text{Eq. (2.42)} \Rightarrow \tilde{B} - \tilde{E}' = B - E' + T, \quad (2.43)$$

$\Downarrow$

$$-\mathcal{H} \times \text{Eq. (2.43)} \Rightarrow -\mathcal{H}(\tilde{B} - \tilde{E}') = -\mathcal{H}(B - E') - \mathcal{H}T \quad (2.44)$$

Finally, summing the equations Eq. (2.41)+Eq. (2.44), we have

$$\begin{aligned} -\tilde{C} + \frac{1}{3}\nabla^2 \tilde{E} - \mathcal{H}(\tilde{B} - \tilde{E}') &= -C + \frac{1}{3}\nabla^2 E + \mathcal{H}T - \mathcal{H}(B - E') - \mathcal{H}T, \\ -\tilde{C} - \mathcal{H}(\tilde{B} - \tilde{E}') + \frac{1}{3}\nabla^2 \tilde{E} &= -C - \mathcal{H}(B - E') + \frac{1}{3}\nabla^2 E. \end{aligned} \quad (2.45)$$

In the same way of Eq. (2.34), we conclude from the Eq. (2.45) that the quantity defined by

$$\boxed{\Phi \equiv -C - \mathcal{H}(B - E') + \frac{1}{3}\nabla^2 E} \quad (2.46)$$

is a gauge invariant perturbation. The Eq. (2.46) is known as the *second Bardeen's potential*.

2.4.2 Comoving curvature perturbation

Consider the following equations

$$\tilde{B} = B + T - L', \quad (2.47)$$

$$\tilde{C} = C - \mathcal{H}T - \frac{1}{3}\nabla^2 L, \quad (2.48)$$

$$\tilde{E} = E - L, \quad (2.49)$$

$$\tilde{v} = v + L'. \quad (2.50)$$

Performing the following manipulations

$$-\frac{1}{3}\nabla^2 \text{Eq. (2.49)} \Rightarrow -\frac{1}{3}\nabla^2 \tilde{E} = -\frac{1}{3}\nabla^2 E + \frac{1}{3}\nabla^2 L, \quad (2.51)$$

$\Downarrow$

$$\text{Eq. (2.48)} + \text{Eq. (2.51)} \Rightarrow \tilde{C} - \frac{1}{3}\nabla^2 \tilde{E} = C - \frac{1}{3}\nabla^2 E - \mathcal{H}T. \quad (2.52)$$

In addition, consider the following computation

$$\mathcal{H} \times [\text{Eq. (2.47)} + \text{Eq. (2.50)}] \Rightarrow \mathcal{H}(\tilde{B} + \tilde{v}) = \mathcal{H}(B + v) + \mathcal{H}T. \quad (2.53)$$

Finally, doing Eq. (2.52)+Eq. (2.53) we are left with

$$\tilde{C} - \frac{1}{3}\nabla^2 \tilde{E} + \mathcal{H}(\tilde{B} + \tilde{v}) = C - \frac{1}{3}\nabla^2 E + \mathcal{H}(B + v). \quad (2.54)$$

From equation Eq. (2.54), we conclude that the quantity defined by

$$\boxed{\mathcal{R} \equiv C - \frac{1}{3}\nabla^2 E + \mathcal{H}(B + v)} \quad (2.55)$$

is a gauge invariant perturbation. Eq. (2.55) is known as the *comoving curvature perturbation*.

2.4.3 Lukash variable

Let's consider the following equations

$$\tilde{C} = C - \mathcal{H}T - \frac{1}{3}\nabla^2 L, \quad (2.56)$$

$$\tilde{E} = E - L, \quad (2.57)$$

$$\tilde{\delta\rho} = \delta\rho - \bar{\rho}'T. \quad (2.58)$$

Performs the following manipulations

$$-\frac{1}{3}\nabla^2 \text{Eq. (2.57)} \Rightarrow -\frac{1}{3}\nabla^2 \tilde{E} = -\frac{1}{3}\nabla^2 E + \frac{1}{3}\nabla^2 L, \quad (2.59)$$

$\Downarrow$

$$\text{Eq. (2.56)} + \text{Eq. (2.59)} \Rightarrow \tilde{C} - \frac{1}{3}\nabla^2 \tilde{E} = C - \frac{1}{3}\nabla^2 E - \mathcal{H}T. \quad (2.60)$$

In addition, consider the following computation

$$-\frac{\mathcal{H}}{\bar{\rho}'} \times \text{Eq. (2.58)} \Rightarrow -\frac{\mathcal{H}}{\bar{\rho}'}\tilde{\delta\rho} = -\frac{\mathcal{H}}{\bar{\rho}'}\delta\rho + \mathcal{H}T. \quad (2.61)$$

Finally, combining Eq. (2.60)+Eq. (2.61) we are left with

$$\tilde{C} - \frac{1}{3}\nabla^2 \tilde{E} - \frac{\mathcal{H}}{\bar{\rho}'}\tilde{\delta\rho} = C - \frac{1}{3}\nabla^2 E - \frac{\mathcal{H}}{\bar{\rho}'}\delta\rho. \quad (2.62)$$

Therefore, the quantity defined by

$$\boxed{\zeta \equiv C - \frac{1}{3}\nabla^2 E - \frac{\mathcal{H}}{\bar{\rho}'}\delta\rho} \quad (2.63)$$

is a gauge invariant perturbation known as *Lukash variable*.

2.4.4 Non-adiabatic pressure perturbation

Consider the following equations

$$\tilde{\delta\rho} = \delta\rho - \bar{\rho}'T, \quad (2.64)$$

$$\tilde{\delta P} = \delta P - \bar{P}'T. \quad (2.65)$$

Performing the following manipulation

$$-\frac{\bar{P}'}{\bar{\rho}'} \text{Eq. (2.64)} \Rightarrow -\frac{\bar{P}'}{\bar{\rho}'} \tilde{\delta\rho} = -\frac{\bar{P}'}{\bar{\rho}'} \delta\rho + \bar{P}' T. \quad (2.66)$$

Finally, consider the following manipulation: Eq. (2.65)+Eq. (2.66), then we are left with

$$\tilde{\delta P} - \frac{\bar{P}'}{\bar{\rho}'} \tilde{\delta\rho} = \delta P - \frac{\bar{P}'}{\bar{\rho}'} \delta\rho. \quad (2.67)$$

Finally, the quantity defined by

$$\boxed{\delta P_{\text{nad}} \equiv \delta P - \frac{\bar{P}'}{\bar{\rho}'} \delta\rho} \quad (2.68)$$

is a gauge invariant perturbation known as *non-adiabatic pressure perturbation*.

2.4.5 Comoving gauge density perturbation

This chapter ends with the comoving gauge density perturbation. Consider the following equations

$$\tilde{B} = B + T - L', \quad (2.69)$$

$$\tilde{v} = v + L', \quad (2.70)$$

$$\tilde{\delta\rho} = \delta\rho - \bar{\rho}' T. \quad (2.71)$$

Performs the following manipulations

$$\text{Eq. (2.69)} + \text{Eq. (2.70)} \Rightarrow \tilde{B} + \tilde{v} = B + v + T, \quad (2.72)$$

$\Downarrow$

$$\bar{\rho}' \times \text{Eq. (2.72)} \Rightarrow \bar{\rho}'(\tilde{B} + \tilde{v}) = \bar{\rho}'(B + v) + \bar{\rho}' T. \quad (2.73)$$

Finally, consider the following operation: Eq. (2.71)+Eq. (2.73), then we are left with

$$\tilde{\delta\rho} + \bar{\rho}'(\tilde{B} + \tilde{v}) = \delta\rho + \bar{\rho}'(B + v). \quad (2.74)$$

Therefore, the quantity defined by

$$\boxed{\bar{\rho}' \Delta \equiv \delta\rho + \bar{\rho}'(B + v)} \quad (2.75)$$

is a gauge invariant quantity known as comoving gauge density perturbation. We summarize all the gauge-invariant perturbations obtained in this chapter

$$\boxed{\Psi \equiv A + \mathcal{H}(B - E') + (B - E')'} \quad (2.76)$$

$$\boxed{\Phi \equiv -C - \mathcal{H}(B - E') + \frac{1}{3}\nabla^2 E} \quad (2.77)$$

$$\boxed{\mathcal{R} \equiv C - \frac{1}{3}\nabla^2 E + \mathcal{H}(B + v)} \quad (2.78)$$

$$\boxed{\zeta \equiv C - \frac{1}{3}\nabla^2 E - \frac{\mathcal{H}}{\bar{\rho}'}\delta\rho} \quad (2.79)$$

$$\boxed{\bar{\rho}' \Delta \equiv \delta\rho + \bar{\rho}'(B + v)} \quad (2.80)$$

$$\boxed{\delta P_{\text{nad}} \equiv \delta P - \frac{\bar{P}'}{\bar{\rho}'}\delta\rho} \quad (2.81)$$

Chapter 3

Initial conditions

3.1 Introduction

In this chapter we describe the initial conditions required to solve the perturbed system of coupled Einstein-Boltzmann equations that describe the evolution of the metric components and for the evolution of the energy density for the components of the universe as photons, cold dark matter, neutrinos, etc. Initial conditions are important and a required step to study the evolution of the perturbed equations of general relativity.

We start with the equations of motion for all components in the synchronous gauge according to [18] and then we perform a change of variable to switch to the notation of [19]. This is useful because [19] build an Ordinary Differential Equation system with all variables of interest to derive the initial conditions - here we follow the same steps as in this reference.

$$\cdot = \frac{d}{d\tau}, \quad ' = \frac{d}{dx} \quad (3.1)$$

$$x = k\tau \quad (3.2)$$

$$\tilde{\delta}_i = \delta_i/x \quad (3.3)$$

$$\tilde{t}_i = t_i/x^2 = \theta_i/(kx^2) \quad (3.4)$$

$$\Theta = h' \quad (3.5)$$

$$\tilde{\sigma}_\nu = \sigma_\nu/x = \Theta_\nu/x \quad (3.6)$$

$$\tilde{F}_{\nu 3} = F_{\nu 3}/x^2. \quad (3.7)$$

In the following subsections the equation on the left-hand side of the arrow “ $\Longrightarrow$ ” is in the notation of [18]. Conversely, the equation in the right-hand side of the arrow “ $\Longrightarrow$ ” is the same notation as in the reference [19].

3.1.1 Cold Dark Matter

Since [19] does not set to zero the cold dark matter velocity $\tilde{t}_c$ in the synchronous gauge, for consistency we also kept the term θ_c in the notation of [18], then we obtain

$$\dot{\delta}_c = -\theta_c - \frac{\dot{h}}{2} \implies \frac{d\delta_c}{d\ln x} = -\tilde{\delta}_c - x^2\tilde{t}_c - \frac{1}{2}\Theta, \quad (3.8)$$

$$\dot{\theta}_c = -\frac{\dot{a}}{a}\theta_c \implies t'_c = -2\tilde{t}_c - \frac{x}{k}\frac{\dot{a}}{a}\tilde{t}_c. \quad (3.9)$$

3.1.2 Neutrinos

The neutrino equations of motion are

$$\dot{\delta}_\nu = -\frac{4}{3}\theta_\nu - \frac{2}{3}\dot{h} \implies \frac{d\tilde{\delta}_\nu}{d\ln x} = -\tilde{\delta}_\nu - \frac{4}{3}\tilde{t}_\nu x^2 - \frac{2}{3}\Theta, \quad (3.10)$$

$$\dot{\theta}_\nu = k^2 \left(\frac{1}{4}\delta_\nu - \Theta_\nu \right) \implies \frac{d\tilde{t}_\nu}{d\ln x} = -2\tilde{t}_\nu + \frac{1}{4}\tilde{\delta}_\nu - \tilde{\sigma}_\nu, \quad (3.11)$$

$$\dot{\Theta}_\nu = \frac{4\theta_\nu}{15} - \frac{3k}{10}\frac{F_{\nu 3}}{7} + \frac{2\dot{h}}{15} + \frac{4\dot{\eta}}{5} \implies \frac{d\tilde{\sigma}_\nu}{d\ln x} = -\tilde{\sigma}_\nu + \frac{4x^2\tilde{t}_\nu}{15} - \frac{3x^2\tilde{F}_{\nu 3}}{10} + \frac{2\Theta}{15} + \frac{4\eta'}{5}, \quad (3.12)$$

$$\dot{F}_{\nu 3} = \frac{1}{7}\frac{3k}{5}F_{\nu 2} = \frac{6k}{7}\Theta_\nu \implies \frac{d\tilde{F}_{\nu 3}}{d\ln x} = -2\tilde{F}_{\nu 3} + \frac{6\tilde{\sigma}_\nu}{7}. \quad (3.13)$$

3.1.3 Photons and baryons

The density contrast for photons and baryons is given respectively by

$$\dot{\delta}_\gamma = -\frac{4}{3}\theta_\gamma - \frac{2}{3}\dot{h} \implies \frac{d\tilde{\delta}_\gamma}{d\ln x} = -\tilde{\delta}_\gamma - \frac{4}{3}\tilde{t}_\gamma x^2 - \frac{2}{3}\Theta, \quad (3.14)$$

$$\dot{\delta}_b = -\theta_b - \frac{1}{2}\dot{h} \implies \frac{d\tilde{\delta}_b}{d\ln x} = -\tilde{\delta}_b - x^2\tilde{t}_b - \frac{1}{2}\Theta. \quad (3.15)$$

Note that the reference [19] uses the tight-coupling approximation once it defines a common value for the divergence of the fluid velocity $\tilde{t}_{\gamma b}$ instead of two separated values, $\tilde{t}_\gamma$ and $\tilde{t}_b$ as in Eqs. (3.14) and (3.15), respectively. The equations for $\dot{\theta}_\gamma$ and

$\dot{\theta}_b$ are given by

$$\dot{\theta}_\gamma = k^2 \left(\frac{1}{4} \delta_\gamma - \Theta_\gamma \right) + a n_e \sigma_T (\theta_b - \theta_\gamma), \quad (3.16)$$

$$\dot{\theta}_b = -\frac{\dot{a}}{a} \theta_b + c_s^2 k^2 \delta_b + \frac{4\bar{\rho}_\gamma}{3\bar{\rho}_b} a n_e \sigma_T (\theta_\gamma - \theta_b), \quad (3.17)$$

where $\Theta_\gamma = \sigma_\gamma$ is the photon anisotropic stress, n_e is the electron density, σ_T is the Thomson scattering cross-section and $\bar{\rho}_\gamma$ and $\bar{\rho}_b$ are the unperturbed energy densities for photons and baryons. To obtain a single equation for the plasma photon-baryon, we should linearly combine the equations (3.16) and (3.17) in order to eliminate the Thomson scattering cross section and then we set the common value $\theta_\gamma = \theta_b = \theta_{\gamma b}$. This algebra produces the following result

$$\left(1 + \frac{4\bar{\rho}_\gamma}{3\bar{\rho}_b} \right) \dot{\theta}_{\gamma b} = \frac{k^2 \delta_\gamma \bar{\rho}_\gamma}{3\bar{\rho}_b} - \frac{\dot{a}}{a} \theta_{\gamma b}, \quad (3.18)$$

and performing the change of variables according to Eqs. (3.1) - (3.7) we have

$$\frac{d\tilde{t}_{\gamma b}}{d \ln x} = -2\tilde{t}_{\gamma b} + \frac{\tilde{\delta}_\gamma}{4 \left(\frac{3}{4} \frac{\bar{\rho}_b}{\bar{\rho}_\gamma} + 1 \right)} - \frac{(\dot{a}/a)(x/k)}{\left(1 + \frac{4\bar{\rho}_\gamma}{3\bar{\rho}_b} \right)} \tilde{t}_{\gamma b} \quad (3.19)$$

In the tight-coupling regime, we neglect the photon anisotropic stress Θ_γ ,

3.1.4 The Einstein-Field equations

According to [18] the Einstein equations in the synchronous gauge are

$$k^2 \eta - \frac{1}{2} \frac{\dot{a}}{a} \dot{h} = 4\pi G a^2 \delta T_0^0(\text{syn}) \quad (3.20)$$

$$k^2 \eta = 4\pi G a^2 (\bar{\rho}_i + \bar{P}_i) \theta^i(\text{syn}) \quad (3.21)$$

$$\ddot{h} + 2\frac{\dot{a}}{a} \dot{h} - 2k^2 \eta = -8\pi G a^2 \delta T_i^i(\text{syn}) \quad (3.22)$$

$$\ddot{h} + 6\ddot{\eta} + 2\frac{\dot{a}}{a} (\dot{h} + 6\dot{\eta}) - 2k^2 \eta = -24\pi G a^2 (\bar{\rho} + \bar{P}) \Theta(\text{syn}) \quad (3.23)$$

We will show the details of the notation transformation below

$$k^2 \eta - \frac{1}{2} \frac{\dot{a}}{a} \frac{kdh}{kd\tau} = -\frac{3}{2} \left(\frac{8\pi G a^2 \rho_{\text{tot}}}{3} \right) \sum_{i=\gamma, \nu, c, b} \frac{\delta \rho_i}{\rho_{\text{tot}}} \quad (3.24)$$

using the Friedmann equation and expanding the summation over all fluid components, we have

$$k^2\eta - \frac{k}{2} \left(\frac{\dot{a}}{a} \right) \left(\frac{dh}{dx} \right) = -\frac{3}{2} \left(\frac{\dot{a}}{a} \right)^2 \left\{ \frac{\rho_{\text{rad}}}{\rho_{\text{tot}}} \left[\frac{\delta\rho_\gamma}{\rho_\gamma} \frac{\rho_\gamma}{\rho_{\text{rad}}} + \frac{\delta\rho_\nu}{\rho_\nu} \frac{\rho_\nu}{\rho_{\text{rad}}} \right] + \frac{\rho_{\text{mat}}}{\rho_{\text{tot}}} \left[\frac{\delta\rho_c}{\rho_c} \frac{\rho_c}{\rho_{\text{mat}}} + \frac{\delta\rho_b}{\rho_b} \frac{\rho_b}{\rho_{\text{mat}}} \right] \right\} \quad (3.25)$$

where now we use the definition for the density contrast $\delta_i = \rho_i/\rho_i$, and the definitions for the fractions according to [19]: $R_\gamma = \rho_\gamma/\rho_{\text{rad}} = \Omega_\gamma/\Omega_{\text{rad}}$, and similarly for the other components R_ν , R_c and R_b . Therefore, Eq. (3.25) can be written as

$$k^2\eta - \frac{k}{2} \frac{2\tau+1}{\tau^2+\tau} h' = -\frac{3}{2} \frac{(2\tau+1)^2}{(\tau^2+\tau)^2} \left\{ \frac{1}{(2\tau+1)^2} [\delta_\gamma R_\gamma + \delta_\nu R_\nu] + \frac{4(\tau^2+\tau)}{(2\tau+1)^2} [\delta_c R_c + \delta_b R_b] \right\} \quad (3.26)$$

Switching from τ to $x/k = \tau$, using $h' = \Theta$ and simplifying the ks, we finally have

$$\eta - \frac{2x/k+1}{x^2(x/k+1)} = -\frac{3}{2} \left[\frac{R_\gamma\delta_\gamma + R_\nu\delta_\nu}{x^2(x/k+1)^2} + \frac{4(R_c\delta_c + R_b\delta_b)}{kx(x/k+1)} \right] \quad (3.27)$$

Following an analogous procedure for Eq. (3.21) we have

$$k^2\dot{\eta} = \frac{3}{2} \left(\frac{8\pi G a^2 \rho_{\text{tot}}}{3} \right) \frac{1}{\rho_{\text{tot}}} \left\{ \frac{\rho_{\text{rad}}}{\rho_{\text{tot}}} \left[\frac{\rho_\gamma}{\rho_{\text{rad}}} (1+w_\gamma) \theta_\gamma + \frac{\rho_\nu}{\rho_{\text{rad}}} (1+w_\nu) \theta_\nu \right] + \frac{\rho_{\text{mat}}}{\rho_{\text{tot}}} \left[\frac{\rho_b}{\rho_{\text{mat}}} (1+w_b) \theta_b + \frac{\rho_c}{\rho_{\text{mat}}} (1+w_c) \theta_c \right] \right\} \quad (3.28)$$

and performing the change of variables from τ to x we have

$$\eta' = \frac{2(R_\gamma t_\gamma + R_\nu t_\nu)}{x^2(x/k+1)^2} + \frac{6(R_b t_{\gamma b} + R_c t_c)}{kx(x/k+1)}. \quad (3.29)$$

Following a similar approach we can rewrite the Eq. (3.22) as

$$\Theta' + \frac{2(2x/k + 1)}{x/k + 1} \Theta - 2\eta = \frac{-3(R_\gamma \delta_\gamma + R_\nu \delta_\nu)}{x^2(x/k + 1)}, \quad (3.30)$$

and Eq. (3.23) as

$$\Theta'' + 6\eta'' + \frac{2(2x/k + 1)}{x(x/k + 1)} (\Theta' + 6\eta'') - 2\eta = -\frac{12(R_\gamma \sigma_\gamma + R_\nu \sigma_\nu)}{x^2(x/k + 1)^2} - \frac{36(R_b \sigma_{\gamma b} + R_c \sigma_c)}{kx(x/k + 1)}. \quad (3.31)$$

3.2 Early-times and super-horizon initial conditions

The evolution in time for the perturbed components can be obtained once we provide initial perturbations. The formalism adopted here for the procedure of obtaining these initial conditions (IC) is close to [19, 20]. Firstly, we organize the variables to be integrated into a unified vector notation as follows

$$\mathbf{u}^T = [\tilde{\delta}_\gamma, \tilde{\delta}_\nu, \tilde{\delta}_c, \tilde{\delta}_b, \tilde{t}_{\gamma b}, \tilde{t}_\nu, \tilde{t}_c, \tilde{\sigma}_\nu, \tilde{F}_\nu^{(3)}, \Theta, \eta], \quad (3.32)$$

where “T” means transpose. These functions can be simplified if we Taylor-expand all coefficients around $x/k = \tau = 0$. By “coefficient” we mean any quantity that multiply $\tilde{\delta}_\gamma, \tilde{\delta}_\nu, \dots, \eta$ in Eq. (3.32). These procedures allow us to separate terms proportional to x^0 (constant), x , x^2 , and so forth. In summary, we ended with an ODE system that can be written in the matrix form, as shown by the following equation

$$\frac{d\mathbf{u}}{d \ln x} = [A_0 + A_1 x + A_2 x^2 + \dots] \mathbf{u}, \quad (3.33)$$

where $A_0, A_1, A_2, \dots$ are matrices that contain coefficients of terms of different orders in x [19]. We numerically computed these matrices up to the fourth order and the results are shown in Eqs. (3.34) - (3.38)

(3.34)

(3.35)

(3.36)

$$(3.37)$$

$$A_4^T = \begin{pmatrix} 0 & 0 & 0 & 0 & \frac{9R_b^2(-9R_bR_\gamma + 9R_b^2 + R_\gamma^2)}{4k^4R_\gamma^4} & 0 & 0 & 0 & 0 & \frac{-30R_\gamma}{k^4} & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & \frac{-30R_\nu}{k^4} & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & \frac{12R_c}{k^4} & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & \frac{12R_b}{k^4} & 0 \\ 0 & 0 & 0 & 0 & \frac{9R_b^2(-12R_bR_\gamma + 9R_b^2 + 2R_\gamma^2)}{k^4R_\gamma^4} & 0 & 0 & \frac{-8(3R_b - 5R_\gamma)}{5k^4} & 0 & 0 & \frac{6R_b - 8R_\gamma}{k^3} \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & \frac{8R_\nu}{k^4} & 0 & 0 & \frac{-8R_\nu}{k^3} \\ 0 & 0 & 0 & 0 & 0 & 0 & \frac{1}{k^4} & \frac{-24R_c}{5k^4} & 0 & 0 & \frac{6R_c}{k^3} \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & \frac{1}{k^4} & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \end{pmatrix}. \quad (3.38)$$

An approximate solution can be obtained considering the lowest order in the Taylor-expansion (i.e. neglecting all terms except A_0) for the system Eq. (3.33). In this case, we have

$$\mathbf{u}_0(x) = \sum_{\lambda} c_{\lambda} x^{\lambda} \mathbf{u}_0^{(\lambda)} \quad (3.39)$$

where c_{λ} are weights that indicate the contribution of each normal mode, while $\mathbf{u}_{\lambda}$ are eigenvectors with eigenvalues λ for the matrix A_0 . Let's show that Eq. (3.39) is a solution for Eq. (3.33) at lowest order. Taking the derivative of Eq. (3.39) with respect to $\ln x$ we have

$$\frac{d\mathbf{u}_0}{d \ln x} = \sum_{\lambda} \lambda c_{\lambda} x^{\lambda} \mathbf{u}_0^{(\lambda)}. \quad (3.40)$$

Applying A_0 to Eq. (3.39) we have

$$A_0 \mathbf{u}_0 = A_0 \left(\sum_{\lambda} c_{\lambda} x^{\lambda} \mathbf{u}_0^{(\lambda)} \right) = \sum_{\lambda} c_{\lambda} x^{\lambda} A_0 \mathbf{u}_0^{(\lambda)} = \sum_{\lambda} c_{\lambda} x^{\lambda} \lambda \mathbf{u}_0^{(\lambda)} \quad (3.41)$$

where we can identify the right hand side of Eq. (3.41) as being the *rhs* of Eq. (3.40) which proves that Eq. (3.39) is the lowest order solution for the equation (3.33). Using Eq. (3.34) we can find the eigenvectors and eigenvalues for A_0 . The results

are shown in the equations (3.42) and (3.43), respectively

$$\begin{pmatrix} 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & \frac{R_\nu}{R_\nu-1} & 1 & 0 & 0 & 0 & 0 & 0 \\ \frac{4R_\nu}{R_\nu-1} & 4 & 0 & 0 & \frac{R_\nu}{R_\nu-1} & 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 0 & \frac{-7}{6} & 0 & \frac{7(\sqrt{5}\sqrt{5-32R_\nu}+5)}{60} & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & \frac{-7}{6} & 0 & \frac{-7(\sqrt{5}\sqrt{5-32R_\nu}-5)}{60} & 1 & 0 & 0 \\ \frac{-1}{3} & \frac{-1}{3} & \frac{-1}{4} & \frac{-1}{4} & \frac{-1}{36} & \frac{-(4R_\nu+23)}{144R_\nu+540} & 0 & \frac{2}{12R_\nu+45} & \frac{4}{84R_\nu+315} & 1 & 0 \\ \frac{1}{3} & \frac{1}{3} & \frac{1}{4} & \frac{1}{4} & \frac{1}{12} & \frac{-1}{12} & 0 & 0 & 0 & 1 & 0 \end{pmatrix}, \quad (3.42)$$

$$\left[-3, -2, -2, -1, -1, -1, 0, \frac{-\sqrt{1-\frac{32R_\nu}{5}}-3}{2}, \frac{\sqrt{5}\sqrt{5-32R_\nu}-15}{10}, -3, 1 \right]. \quad (3.43)$$

3.2.1 Zeroth order and higher order correction

The solution here is written as $\mathbf{u}_i^{(\lambda)}$ where λ is the eigenvalue and i is the order correction. At zeroth order the second subscript is $i = 0$, at first order is $i = 1$, and so forth for higher order corrections.

Adiabatic mode (AD) $\lambda = 1$

$$\mathbf{u}_0^{\text{AD}} = \left[\frac{-1}{3}, \frac{-1}{3}, \frac{-1}{4}, \frac{-1}{4}, \frac{-1}{36}, \frac{-(23+4R_\nu)}{36(15+4R_\nu)}, 0, \frac{2}{3(15+4R_\nu)}, \frac{4}{21(15+4R_\nu)}, 1, 0 \right]. \quad (3.44)$$

Baryon isocurvature (BI) mode: $\lambda = -1$

$$\mathbf{u}_0^{\text{BI}} = [0, 0, 0, 1, 0, 0, 0, 0, 0, 0, 0]. \quad (3.45)$$

Cold dark matter isocurvature (CI) mode: $\lambda = -1$

$$\mathbf{u}_0^{\text{CI}} = [0, 0, 1, 0, 0, 0, 0, 0, 0, 0, 0]. \quad (3.46)$$

Neutrino density isocurvature (NDI) mode: $\lambda = -1$

$$\mathbf{u}_0^{\text{NDI}} = \left[\frac{4R_\nu}{R_\nu - 1}, 4, 0, 0, \frac{R_\nu}{R_\nu - 1}, 1, 0, 0, 0, 0, 0 \right]. \quad (3.47)$$

Neutrino velocity isocurvature (NVI) mode: $\lambda = -2$

$$\mathbf{u}_0^{\text{NVI}} = \left[0, 0, 0, 0, \frac{R_\nu}{R_\nu - 1}, 1, 0, 0, 0, 0, 0 \right]. \quad (3.48)$$

We can use each previous normal mode (i.e. with fixed λ) to extend the solution to high-order corrections

$$\mathbf{u}^{(\lambda)} = \left(\mathbf{u}_0^{(\lambda)} + \mathbf{u}_1^{(\lambda)} x^1 + \mathbf{u}_2^{(\lambda)} x^2 + \dots \right) x^\lambda, \quad (3.49)$$

and we obtain the following hierarchy

$$[(\lambda + 1)\mathcal{I} - A_0]\mathbf{u}_1^{(\lambda)} = A_1\mathbf{u}_0^{(\lambda)}, \quad (3.50)$$

$$[(\lambda + 2)\mathcal{I} - A_0]\mathbf{u}_2^{(\lambda)} = A_1\mathbf{u}_1^{(\lambda)} + A_2\mathbf{u}_0^{(\lambda)}, \quad (3.51)$$

$$[(\lambda + 3)\mathcal{I} - A_0]\mathbf{u}_3^{(\lambda)} = A_1\mathbf{u}_2^{(\lambda)} + A_2\mathbf{u}_1^{(\lambda)} + A_3\mathbf{u}_0^{(\lambda)}, \quad (3.52)$$

$$[(\lambda + 4)\mathcal{I} - A_0]\mathbf{u}_4^{(\lambda)} = A_1\mathbf{u}_3^{(\lambda)} + A_2\mathbf{u}_2^{(\lambda)} + A_3\mathbf{u}_1^{(\lambda)} + A_4\mathbf{u}_0^{(\lambda)}, \quad (3.53)$$

where $\mathcal{I}$ denotes the 11×11 dimensional identity matrix. Let's show Eq. (3.50) and the others follow a similar approach. The solution and the matrix up to first-order correction are

$$\mathbf{u}^{(\lambda)} = \left(\mathbf{u}_0^{(\lambda)} + x\mathbf{u}_1^{(\lambda)} \right) x^\lambda \quad (3.54)$$

$$A = A_0 + xA_1 \quad (3.55)$$

Taking the derivative of Eq. (3.54) with respect to $\ln x$ we have

$$\frac{d\mathbf{u}^{(\lambda)}}{d \ln x} = \left[\lambda \mathbf{u}_0^{(\lambda)} + (\lambda + 1)x\mathbf{u}_1^{(\lambda)} \right] x^\lambda. \quad (3.56)$$

Operating Eq. (3.55) into Eq. (3.54) and neglecting terms higher than first order in x , we have

$$A\mathbf{u}^{(\lambda)} = \left[\lambda \mathbf{u}_0^{(\lambda)} + xA_0\mathbf{u}_1^{(\lambda)} + xA_1\mathbf{u}_0^{(\lambda)} \right] x^\lambda. \quad (3.57)$$

We must have the equality

$$\frac{d\mathbf{u}^{(\lambda)}}{d \ln x} = A\mathbf{u}^{(\lambda)}, \quad (3.58)$$

then from the *rhs* of Eq. (3.56) and Eq. (3.57) we have

$$\begin{aligned} \lambda \mathbf{u}_0^{(\lambda)} + (\lambda + 1)x\mathbf{u}_1^{(\lambda)} &= \lambda \mathbf{u}_0^{(\lambda)} + xA_0\mathbf{u}_1^{(\lambda)} + xA_1\mathbf{u}_0^{(\lambda)} \\ x(\lambda + 1 - A_0)\mathbf{u}_1^{(\lambda)} &= xA_1\mathbf{u}_0^{(\lambda)} \\ [(\lambda + 1)\mathcal{I} - A_0]\mathbf{u}_1^{(\lambda)} &= A_1\mathbf{u}_0^{(\lambda)}, \end{aligned} \quad (3.59)$$

where Eq. (3.59) is exactly Eq. (3.50), which proves the relation. A similar procedure can be applied to obtain the other hierarchy relations in Eqs. (3.51), (3.52) and (3.53).

Applying the Eq. (3.50) for λ corresponding to the adiabatic mode, $1, -1, -1, -1$ and -2 we obtain:

3.2.2 First order correction

Adiabatic mode ($\lambda = 1$):

$$\begin{aligned} \mathbf{u}_1^{\text{AD}} = & \left[\frac{4}{15k'}, \frac{4}{15k'}, \frac{1}{5k'}, \frac{1}{5k'}, \frac{5R_b - R_\nu + 1}{60k - 60kR_\nu}, \frac{8R_\nu^2 + 50R_\nu + 275}{60k(8R_\nu^2 + 90R_\nu + 225)}, 0, \right. \\ & \left. \frac{2(4R_\nu - 5)}{3k(8R_\nu^2 + 90R_\nu + 225)}, \frac{4R_\nu - 5}{7k(8R_\nu^2 + 90R_\nu + 225)}, \frac{-6}{5k'}, \frac{-(5 + 4R_\nu)}{12(15 + 4R_\nu)} \right] \end{aligned} \quad (3.60)$$

Baryon isocurvature mode ($\lambda = -1$):

$$\mathbf{u}_1^{\text{BI}} = \left[\frac{-8R_b}{3k}, \frac{-8R_b}{3k}, \frac{-2R_b}{k}, \frac{-2R_b}{k}, \frac{-R_b}{3k}, \frac{-R_b}{3k}, 0, 0, 0, \frac{4R_b}{k}, 0 \right] \quad (3.61)$$

CDM isocurvature mode ($\lambda = -1$).

$$\mathbf{u}_1^{\text{CI}} = \left[\frac{-8R_c}{3k}, \frac{-8R_c}{3k}, \frac{-2R_c}{k}, \frac{-2R_c}{k}, \frac{-R_c}{3k}, \frac{-R_c}{3k}, 0, 0, 0, \frac{4R_c}{k}, 0 \right] \quad (3.62)$$

$\mathbf{u}_1^{\text{NDI}}$ = Neutrino isocurvature density mode ($\lambda = -1$):

$$\left[0, 0, 0, 0, \frac{3R_b R_\nu}{k(R_\nu - 1)^2}, 0, 0, 0, 0, 0, 0\right] \quad (3.63)$$

$\mathbf{u}_1^{\text{NVI}}$ = Neutrino isocurvature velocity mode ($\lambda = -2$):

$$\left[0, 0, 0, 0, \frac{3R_b R_\nu}{k(R_\nu - 1)^2}, 0, 0, 0, 0, 0, 0\right] \quad (3.64)$$

3.2.3 Adiabatic initial conditions

From the previous sections, we can combine the zeroth and first order corrections to obtain the adiabatic mode for the initial conditions

$$\delta_\gamma = \delta_\nu = -\frac{1}{3}k^2\tau^2 + \mathcal{O}(\tau^3) \quad (3.65)$$

$$\delta_c = \delta_b = -\frac{1}{4}k^2\tau^2 + \mathcal{O}(\tau^3) \quad (3.66)$$

$$\theta_{\gamma b} = -\frac{1}{36}k^4\tau^3 + \mathcal{O}(\tau^4) \quad (3.67)$$

$$\theta_\nu = -\frac{(23 + 4R_\nu)}{36(15 + 4R_\nu)}k^4\tau^3 + \mathcal{O}(\tau^4) \quad (3.68)$$

$$\theta_c = 0 \quad (3.69)$$

$$\sigma_\nu = \frac{2}{3(15 + 4R_\nu)}k^2\tau^2 + \mathcal{O}(\tau^3) \quad (3.70)$$

$$F_\nu^{(3)} = \frac{4}{21(15 + 4R_\nu)}k^3\tau^3 + \mathcal{O}(\tau^4) \quad (3.71)$$

$$\Theta = h' = k\tau \Rightarrow h = \frac{1}{2}k^2\tau^2 + \mathcal{O}(\tau^3) \quad (3.72)$$

$$\eta = 1 - \frac{5 + 4R_\nu}{12(15 + 4R_\nu)}k^2\tau^2 + \mathcal{O}(\tau^3). \quad (3.73)$$

Note that the results from Eq. (3.65) to Eq. (3.73) agree with results in [21, 22]. To recover the results in [19] we must perform a transformation from the synchronous gauge (used in this chapter) to the Newtonian gauge.

Part II

Dark Energy and Neutrinos in Cosmology

Chapter 4

Mass Varying Neutrinos and the Hubble tension

The idea of neutrino-assisted early dark energy (ν EDE), where a coupling between neutrinos and the scalar field that models early dark energy (EDE) is considered, was introduced with the aim of reducing some of the fine-tuning and coincidence problems that appear in usual EDE models. In order to be relevant in ameliorating the H_0 tension, the contribution of EDE to the total energy density (f_{EDE}) should be around 10% near the redshift of matter-radiation equality. We verify under which conditions ν EDE models can fulfill these requirements for a model with a quartic self-coupling of the EDE field and an exponential coupling to neutrinos. We find that in the situation where the EDE field is frozen initially, the contribution to f_{EDE} can be significant but it is not sensitive to the neutrino-EDE coupling and does not address the EDE coincidence problem. On the other hand, if the EDE field starts already dynamically at the minimum of the effective potential, it tracks this time-dependent minimum that presents a feature triggered by the neutrino transition from relativistic to nonrelativistic particles. This feature generates f_{EDE} in a natural way around this transition epoch, which roughly coincides with the matter-radiation equality redshift. For the set of parameters that we considered we did not find values that satisfy the requirements on the background cosmological evolution to mitigate the Hubble tension in a natural way in this particular ν EDE model.

4.1 Introduction

Despite its great success in being able to describe a vast amount of observables from different epochs of the evolution of the Universe, the standard spatially flat Λ CDM model has recently been under stress mainly from the so-called Hubble tension arising from measurements of the Hubble constant from the cosmic mi-

crowave background (CMB) data in comparison to local measurements (for recent reviews, see [23–25]). One of the most promising solutions to the Hubble tension is a class of models called Early Dark Energy (EDE) [26–31]. The current status of EDE models proposed to resolve the Hubble tension was recently reviewed in [32].

A non-negligible amount of a new dark energy component that becomes dynamic around the matter-radiation equality era and quickly dissipates afterward would decrease the sound horizon that enters the indirect measurement of the Hubble constant from the CMB data, leading to an increase in the estimated value of the Hubble parameter today; hence EDE can ameliorate the Hubble tension [33]. In fact, the existence of EDE has already been proposed in the context of late quintessence models with global attractor solutions, and its effect on the CMB and structure formation was studied in [34] and [35]. In the more recent versions of EDE motivated by the Hubble tension, a scalar field with an axion-like or a pure ϕ^4 potential was proposed to model the EDE component [27, 28, 36]. It has been shown that if EDE contributes with a fraction $f_{\text{EDE}} \approx 10\%$ of the total energy density of the Universe in an epoch with redshift around $\log_{10}(z) \sim 3.5$, its presence is expected to decrease the statistical significance of the Hubble tension (see, e.g., [29]).

However, in the standard EDE scenario, some level of fine-tuning is necessary in order to solve the Hubble tension: the scalar field mass and its initial conditions need to have specific values in order for the EDE model to contribute the required amount of energy density at around the matter-radiation equality epoch. This is sometimes referred to as the EDE coincidence problem.

Some models were proposed to address the EDE coincidence problem by introducing a coupling of the EDE field to matter fields. For instance, a coupling of EDE to dark matter as a trigger for its dynamics was recently studied in [37, 38].

Another similar mechanism was proposed in [39, 40], where a coupling of the EDE field to neutrinos was introduced. There are two distinct consequences of introducing such a coupling: (i) a modification in the original EDE field potential leading to a new effective potential and (ii) a dynamical effect of a “kick” to the EDE field. This kick is associated with the transition of neutrinos from being relativistic to nonrelativistic particles when the temperature of the universe falls below $T = \mathcal{O}(m_\nu)$, which causes a jump in the trace of the neutrino energy-momentum tensor. This jump can displace the value of the EDE field and then start its oscillations and dissipation. This mechanism can naturally trigger the

EDE field at around the matter-radiation equality since the temperature at this epoch is approximately at the scale of neutrino masses.

In this work, we explore this neutrino-assisted EDE model (ν EDE) in more detail. In order to do this, we developed an implementation of the ν EDE model in CAMB¹ at the background level to analyze the evolution of the EDE scalar field in the time-dependent effective potential given some initial conditions and model parameters and the computations are consistently performed in the Einstein frame. We study two cases for the initial conditions: the field starting either at some arbitrary value or starting at the minimum of the effective potential (see, e.g., Brax et al. [41]). In the first case, we consider the field initially frozen at an arbitrary value by Hubble friction ($|V''_{\text{eff}}/H^2| < 1$). In the second case, we chose parameters of the model such that the field has already relaxed to the minimum of the effective potential.

We show that in the first case, the model can result in a cosmologically significant amount of f_{EDE} at the redshift required to ameliorate the Hubble tension. However, using the approach proposed in [38], we point out that this result is not due to the neutrino-EDE coupling, and it does not address the EDE coincidence problem. In the second case, the model provides a natural value for the initial condition of the EDE field given by the minimum of the effective potential [39, 40, 42]. Nevertheless, the contribution to f_{EDE} is too small to address the Hubble tension.

An important feature of the ν EDE model is the neutrino kick [39, 40] which may occur by a similar mechanism previously studied in [41, 43–45]. The latter are chameleon models addressing the late dark energy with a different class of self-interaction potentials. In the model studied here we do not see the effects of the neutrino kick on the EDE field with a displacement from the minimum of the effective potential by solving its field evolution equation. Rather, we show that the jump in the trace of the neutrino energy-momentum tensor is reflected in a change in the minimum of the effective potential that the EDE is following adiabatically. It is this change that leads to an increase in f_{EDE} .

This work is organized as follows. In Section 4.2, we present a summary of the model and proceed to Section 4.3 for a description of the methods used in the background analysis. Our results are presented in Section 4.4 where we differentiate the discussion in the two regimes described above: in Section 4.4.1 the field is initially frozen by Hubble drag while in Section 4.4.2 the field is initially dynamical. In both scenarios we compute the fraction of EDE and assess the

¹<https://github.com/cmbant/CAMB>

importance of the neutrino-EDE coupling. We present our conclusions in Section 4.7.

4.2 Neutrino-assisted EDE

The idea of neutrinos coupled to a dark energy field was introduced in the so-called “Mass-Variable Neutrinos” (MaVaN) models, since this coupling induces a variation of the neutrino mass due to the cosmic evolution of the background dark energy (also known as quintessence) field [46,47]. The effects of mass-varying neutrinos on the CMB anisotropies and large-scale structures (LSS), both at the background and perturbative levels, were studied in [48–50]. The issue of fine-tuning was discussed in the context of quintessence models in [51] and more recently in MaVaN models in [52]. We now present a brief summary of the model.

Consider a general action of the following form:

$$\begin{aligned}
 S = \int d^4x \sqrt{-g} \left[\frac{1}{2} M_{pl}^2 R - \frac{1}{2} \nabla_\mu \phi \nabla^\mu \phi - V(\phi) \right] \\
 + S_{dm}[\Psi_{dm}, \tilde{g}_{\mu\nu}^{dm}] + S_b[\Psi_b, \tilde{g}_{\mu\nu}^b] \\
 + S_\gamma[\Psi_\gamma, \tilde{g}_{\mu\nu}^\gamma] + S_\nu[\Psi_\nu, \tilde{g}_{\mu\nu}^\nu],
 \end{aligned} \tag{4.1}$$

where Ψ_{dm} , Ψ_b , Ψ_γ and Ψ_ν are the dark matter, baryons, photon, and neutrino fields, respectively, $M_{pl} \equiv (8\pi G)^{-1/2}$ is the reduced Planck mass and g is the determinant of the metric $g_{\mu\nu}$ in the Einstein-frame. The Jordan-frame metric, $\tilde{g}_{\mu\nu}$, for each species are related with $g_{\mu\nu}$ by $\tilde{g}_{\mu\nu}^i = A_i^2(\phi) g_{\mu\nu}$ where i represents all the components cited above. It is interesting to note that in the case $A_{dm} = A_b = A_\gamma = A_\nu = 1$ and $V(\phi) = \mu^4 [1 - \cos(\phi/f)]^n$, where f is the energy scale where a shift symmetry is spontaneously broken and μ is a small explicit shift symmetry-breaking energy scale that generates an axion-like potential, one obtains the canonical uncoupled axion-like EDE model [26,27]. When $A_b = A_\gamma = A_\nu = 1$, $A_{dm} \equiv A = \exp(\beta\phi/M_{pl})$ and $V(\phi) = \lambda\phi^4$ we obtain the chameleon EDE model, recently studied in [37]. Finally, when $A_{dm} = A_b = A_\gamma = 1$, $A_\nu \equiv A = \exp(\beta\phi/M_{pl})$ and $V(\phi) = \lambda\phi^4/4$ we obtain the neutrino-assisted EDE (ν EDE) model studied in [39,40]. This is the model considered in the following. Notice that ν EDE has the same framework and is described by the same equations as for the MaVaNs model, the main difference being that for the latter case, the quintessence potential is $V(\phi) = V_0 \exp(-\sigma\phi)$ which is required to achieve the late time cosmic

acceleration.

The coupling function that changes the metric between the Einstein and Jordan frames can be related to the time-varying neutrino mass as $m_\nu(\phi) = m_{\nu 0}A(\phi)$ where $m_{\nu 0}$ is the neutrino mass today. The average neutrino energy density (ρ_ν) and pressure (p_ν) are given in terms of an integral over the comoving momentum (q) of the usual unperturbed Fermi-Dirac distribution $f_0(q)$:

$$\rho_\nu(\phi) = \frac{1}{a^4} \int q^2 dq d\Omega \epsilon f_0(q), \quad (4.2)$$

$$p_\nu(\phi) = \frac{1}{3a^4} \int q^2 dq d\Omega \frac{q^2}{\epsilon} f_0(q), \quad (4.3)$$

$$f_0(q) = \frac{g_\nu}{h_P^3} \frac{1}{e^{q/k_B T_{\nu 0}} + 1}, \quad (4.4)$$

$$\epsilon^2 = q^2 + a^2 m_\nu^2(\phi), \quad (4.5)$$

where g_ν , $T_{\nu 0}$, h_P , k_B are the number of spin degrees of freedom, the neutrino temperature today, the Planck and Boltzmann constant, respectively. In equation (4.4) we have assumed that neutrinos decoupled from the primordial plasma when they are still relativistic, so the phase-space distribution preserves the Fermi-Dirac form with a dependence only on the comoving momentum.

Taking the time derivative of Eq. (4.2) and using that the energy conservation is determined by the coupled system of neutrinos and the scalar field, we obtain the modified Klein-Gordon equation

$$\ddot{\phi} + 3H\dot{\phi} + \frac{dV(\phi)}{d\phi} = -\frac{d \ln m_\nu(\phi)}{d\phi} (\rho_\nu - 3p_\nu), \quad (4.6)$$

where the dot is the derivative with respect to time. The self-interaction EDE potential assumed in this work is

$$V(\phi) = \lambda \phi^4 / 4. \quad (4.7)$$

This is the potential used in the so-called “rock ‘n’ roll” models of EDE [53].

A useful quantity that we will use is the neutrino trace. This can be obtained

combining Eq. (4.2) with Eq. (4.3) as follow

$$\rho_\nu(\phi) - 3p_\nu(\phi) = \frac{1}{a^4} \int q^2 dq d\Omega \epsilon(\phi) f_0(q) - \frac{1}{a^4} \int q^2 dq d\Omega \frac{q^2}{\epsilon(\phi)} f_0(q) \quad (4.8)$$

$$= \frac{1}{a^4} \int q^2 dq d\Omega \left(\epsilon(\phi) - \frac{q^2}{\epsilon(\phi)} \right) f_0(q) \quad (4.9)$$

$$= \frac{1}{a^4} \int q^2 dq d\Omega \left(\frac{\epsilon^2(\phi) - q^2}{\epsilon(\phi)} \right) f_0(q), \text{ using Eq. (4.5)} \quad (4.10)$$

$$= \frac{1}{a^4} \int q^2 dq d\Omega \left(\frac{q^2 + a^2 m_\nu^2 - q^2}{\epsilon(\phi)} \right) f_0(q), \text{ simplifying} \quad (4.11)$$

$$\rho_\nu(\phi) - 3p_\nu(\phi) = \frac{a^2 m_\nu^2(\phi)}{a^4} \int q^2 dq d\Omega \frac{1}{\epsilon(\phi)} f_0(q). \quad (4.12)$$

Now let's take the derivative of Eq. (4.2) with respect to the scalar field

$$\frac{d\rho_\nu(\phi)}{d\phi} = \frac{1}{a^4} \int q^2 dq d\Omega \frac{d\epsilon(\phi)}{d\phi} f_0(q), \text{ using Eq. (4.5)} \quad (4.13)$$

$$= \frac{1}{a^4} \int q^2 dq d\Omega \left[\frac{1}{2} (q^2 + a^2 m_\nu^2)^{-1/2} 2a^2 m_\nu \frac{dm_\nu(\phi)}{d\phi} \right] f_0(q), \text{ simplifying} \quad (4.14)$$

$$= a^2 m_\nu(\phi) \frac{dm_\nu(\phi)}{d\phi} \frac{1}{a^4} \int q^2 dq d\Omega \frac{1}{\epsilon(\phi)} f_0(q), \text{ multiplying by } \frac{m_\nu(\phi)}{m_\nu(\phi)} \quad (4.15)$$

$$= \frac{1}{m_\nu(\phi)} \frac{dm_\nu(\phi)}{d\phi} \frac{a^2 m_\nu^2(\phi)}{a^4} \int q^2 dq d\Omega \frac{1}{\epsilon(\phi)} f_0(q). \quad (4.16)$$

We note that the product of the last two terms on the right-hand side of Eq. (4.16) is exactly the neutrino trace given by Eq. (4.12). Finally, using the identity $(1/y)(dy/dx) = d(\ln y)/dx$ we can rewrite Eq. (4.16) in the compact form:

$$\frac{d\rho_\nu(\phi)}{d\phi} = \frac{d \ln m_\nu(\phi)}{d\phi} (\rho_\nu(\phi) - 3p_\nu(\phi)). \quad (4.17)$$

For definiteness, we are going to adopt a coupling of the form:

$$A(\phi) = \exp(\beta\phi/M_{pl}), \quad (4.18)$$

where β is a dimensionless constant factor giving the coupling strength between neutrinos and the EDE field. The fact that in EDE models the energy density must dissipate quickly after recombination ensures that $m_\nu(\phi) \rightarrow m_{\nu 0}$ today. With this

choice of interaction, the Klein-Gordon equation takes the familiar form:

$$\ddot{\phi} + 3H\dot{\phi} + \frac{dV(\phi)}{d\phi} = -\frac{\beta}{M_{pl}}\Theta_\nu, \quad (4.19)$$

where $\Theta_\nu \equiv -\rho_\nu + 3p_\nu$ is the trace of the energy-momentum tensor for the neutrinos. Therefore, one can define an effective potential where

$$V'_{\text{eff}} = V'(\phi) - \frac{\beta}{M_{pl}}\Theta_\nu. \quad (4.20)$$

and using equation (4.17) we write the redshift-dependent effective potential as

$$V_{\text{eff}}(\phi) = V(\phi) + \rho_\nu(\phi, z). \quad (4.21)$$

An important quantity throughout this work is the redshift-dependent minimum of the effective potential given by [40]²:

$$\phi_{\text{min}} = - \left(\frac{\beta\Theta_\nu}{\lambda M_{Pl}} \right)^{1/3}, \quad (4.22)$$

The free parameters of the model are the scalar field self-coupling constant λ , a parameter β that characterizes the scalar field coupling to neutrinos, and the initial conditions for the field ϕ_i and $\dot{\phi}_i$. In the following we will adopt three degenerate neutrinos with $\sum m_{\nu 0} = 0.3$ eV which is below the upper bound found in [54] with *Planck* data only.

4.3 A background analysis of the neutrino-assisted EDE

As previously stated, the free parameters of the model are λ , β , and ϕ_i (we will assume $\dot{\phi}_i = 0$, i.e. the field is frozen initially). In order to gain some intuition on the parameter λ , notice that in the axion-like EDE potential with $V(\phi) = \mu^4[1 - \cos(\phi/f)]^n$, an expansion around the minimum for $n = 1$ gives the axion mass $m_a = \mu^2/f$ and the quartic coupling $\lambda = m_a^2/f^2$. In scalar field models, it is well-known that the field becomes dynamical (i.e. it “thaws”) when

²Notice that since we are working in the Einstein frame there is an implicit dependence of Θ_ν on the field and the coupling constant β . This dependence can be neglected only if $|\beta\phi/M_{Pl}| \ll 1$. We take this dependence into account in our numerical code.

its mass is of the order of the Hubble parameter. Therefore, if we want the EDE field to become dynamical before recombination ($H \approx (1 \text{ eV})^2 / M_{Pl} \approx \mathcal{O}(10^{-27})$ eV) and assuming a typical symmetry breaking scale of $f = 10^{15}$ GeV one expects very small values of the self-coupling of $\lambda = \mathcal{O}(10^{-100})$. Even in anharmonic potentials with $n > 1$, this behavior is present [27], and in our case the value of λ controls the epoch when EDE becomes dynamical, with a redshift usually denoted by z_c . More generally, the field becomes dynamical when [41]

$$\frac{V''_{\text{eff}}}{H^2} = \mathcal{O}(1). \quad (4.23)$$

Another important quantity in EDE models is the maximum contribution of the EDE component to the total energy budget of the universe at z_c , denoted by f_{EDE} :

$$f_{\text{EDE}}(z_c) = \frac{\rho_{\text{EDE}}(z_c)}{\rho_{\text{tot}}(z_c)}, \quad (4.24)$$

and it is determined by the model parameters. In fact, in an EDE model, there is a simple relation between λ , ϕ_i , and z_c :

$$\lambda \approx \frac{T_0^4}{9M_{Pl}^2 \phi_i^2} (1 + z_c)^4, \quad (4.25)$$

where T_0 is the CMB temperature today.

The solution to the Hubble tension in EDE models should fulfill the basic requirements [31, 38]

$$f_{\text{EDE}} \sim 0.1 \text{ and } \log_{10} z_c \sim 3.5 \quad (4.26)$$

and that the EDE field is dynamic just prior to matter-radiation equality. However, usual EDE models suffer from a coincidence problem due to the arbitrary initial conditions. Extensions of EDE models including interactions with other components such as dark matter [37] or neutrinos [39, 40] may address the EDE coincidence problem via some triggering mechanism [38]. In order to investigate the impact of the new coupling β on the fraction of EDE produced in the extended model we introduce the quantity

$$\Delta \equiv \frac{f_{\text{EDE}}}{f_{\text{EDE}}^{\beta=0}}, \quad (4.27)$$

which is defined by the ratio between the fraction of EDE for the coupled ($\beta \neq 0$) and uncoupled ($\beta = 0$) scenarios. In the limit that $\Delta \rightarrow 1$ the f_{EDE} is mainly sourced by the bare potential [38].

In the top panel of Fig. 4.1 we show an example of the background evolution for cold dark matter (CDM), photons (γ), neutrinos ν and the EDE scalar field plus a cosmological constant ($\phi + \Lambda$) where the conditions in Eq. (4.26) are satisfied. In the bottom panel, one can see that in this case, the quantity Δ is close to 1 at the EDE peak ($z \sim 2 \times 10^3$), indicating that the coupling to neutrinos is not relevant to generating f_{EDE} . We will show below cases with the opposite situation.

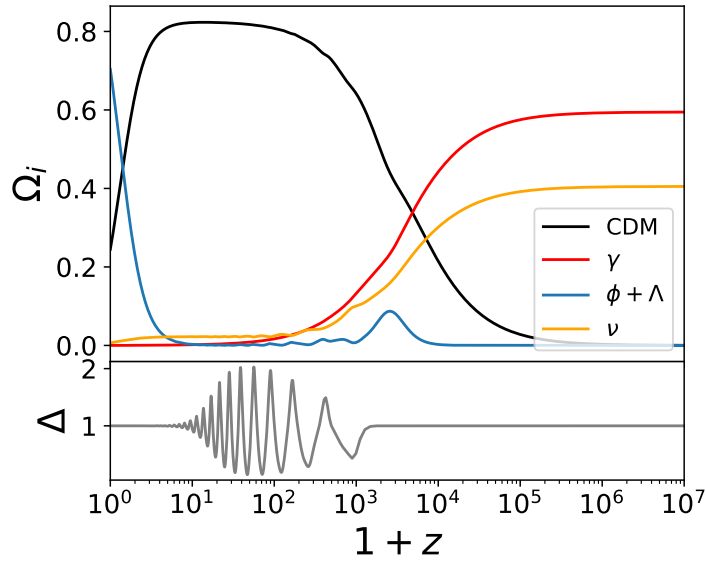


Figure 4.1: The background evolution for cold dark matter (CDM), photons (γ), neutrinos ν , and the scalar field plus a cosmological constant ($\phi + \Lambda$). The parameters used are $\beta = 10$, $\phi_i = -0.6 M_{Pl}$ and $\log_{10} \lambda = -109$. The surge of EDE is around at $z \sim 2 \times 10^3$. The bottom panel shows the quantity defined in Eq. (4.27) which is the ratio between the coupled and uncoupled cases.

4.4 Initial value of the EDE field

Previously we assumed an arbitrary initial value for the EDE field. We now want to address this issue in more detail. We identify two situations: a) the field is still frozen initially and starts to follow the minimum of the effective potential after its thawing and b) the field is already dynamical initially and therefore it follows the minimum of the effective potential. This separation is somewhat

arbitrary since at large enough redshifts the field is frozen. However, in practical terms, we need to choose an initial redshift from which we start evolving the field numerically. The separation between these two regimes at a given redshift z_i can be estimated by the condition

$$\frac{V''_{\text{eff}}(\phi_{\text{min}})}{H^2} = 3\lambda^{1/3}\beta^{2/3} \left(\frac{\Theta_\nu}{H^3 M_{\text{Pl}}} \right)^{2/3} \Big|_{z_i} = \mathcal{O}(1). \quad (4.28)$$

In Fig. 4.2 we show the contours in the parameter space (λ, β) where the field is already dynamic at some given redshifts. The figure provides estimates of the values of these coupling separating the two cases.

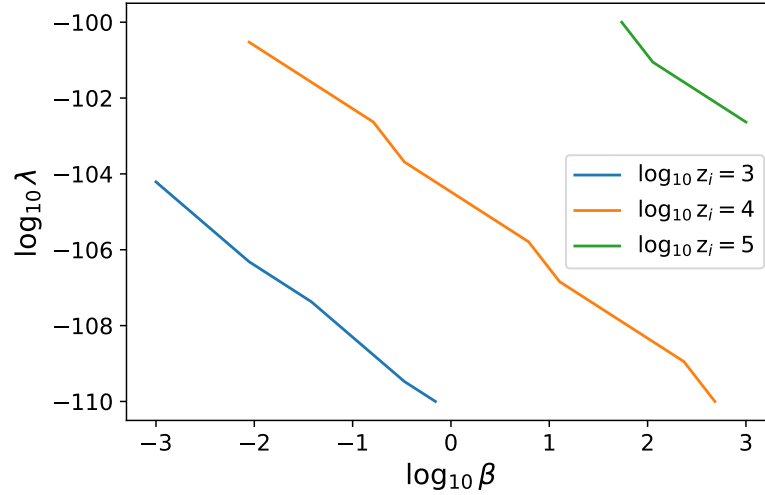


Figure 4.2: Values of λ and β for which $|V''_{\text{eff}}/H^2 - 1| = 0.1$ for $z_i = 10^3, 10^4$ and 10^5 . For regions above the curves, the field is dynamical at the corresponding redshift.

The motivation for studying these two cases separately below is that in the first case, one could argue that the initial value of the frozen field does not have a natural value whereas in the second case the value of the field is naturally close to the minimum of the effective potential. Indeed we find different dynamical behaviours in these two cases.

4.4.1 Field initially frozen

In Figure 4.3 we plot the effective potential given in Eq. (4.21) as a function of the field for different values of the redshift starting at $z_i = 10^5$ for $\beta = 10$ and

$\lambda = 10^{-109}$, where $\rho_\nu(\phi, z)$ is given by Eq. (4.2). At high redshifts, the neutrino contribution dominates and it is a smooth function of ϕ . At lower redshifts, the contribution from $V(\phi)$ to the effective potential starts to become more relevant. For these parameters, we plot the evolution of the field for three different values of ϕ_i : one starting at the minimum of the effective potential and the other two are chosen arbitrarily for comparison. We also show the minimum of the effective potential given by Eq. (4.22) that, due to the coupling, evolves with redshift.

In all the cases the field is frozen at z_i , corresponding to the first line in the plot. We find numerically that for the initial values of the field that we chose it remains frozen until approximately the redshift that satisfies Eq. (4.25) is reached.

Notice that as the neutrino contribution becomes small at lower redshifts, the minimum of the effective potential approaches the minimum of the bare potential $V(\phi)$ at $\phi = 0$.

After thawing the field starts to roll in the effective potential tracking and oscillates around its minimum. This behavior of the field is shown both by the inset zoom in Figure 4.3 and at the top panel of Figure 4.4, where one can see the field relaxing to $\phi = 0$, dissipating the energy density contained in it.

In the bottom panel of Figure 4.4 we show the contribution to f_{EDE} in this example. One can see that there could be a significant contribution depending on the initial value of the field. However, the coupling to neutrinos does not seem to play a large role in this case. We investigate this issue by computing the evolution of the field with the same initial conditions but without coupling to neutrinos ($\beta = 0$). This is shown as the dashed lines in the bottom panel of Figure 4.4, where one can see that although the coupling introduces changes in the behavior of the field the modification to the maximum of f_{EDE} is negligible. In the case of the largest contribution, we find $|f_{\text{EDE}}/f_{\text{EDE}}^{\beta=0} - 1| = 5 \times 10^{-4}$. Therefore, in this case, one needs the usual fine-tuning of initial conditions found in regular EDE models in order to have a significant contribution to f_{EDE} .

4.4.2 Field initially dynamical

In this subsection, we are interested in studying the case where the field is already dynamical at the initial redshift considered. The motivation is that in this case, the field is already tracking the minimum of effective potential and hence its value is not arbitrary. We will show below that in order for the field to become dynamic at an early redshift, a larger value of the self-coupling constant is

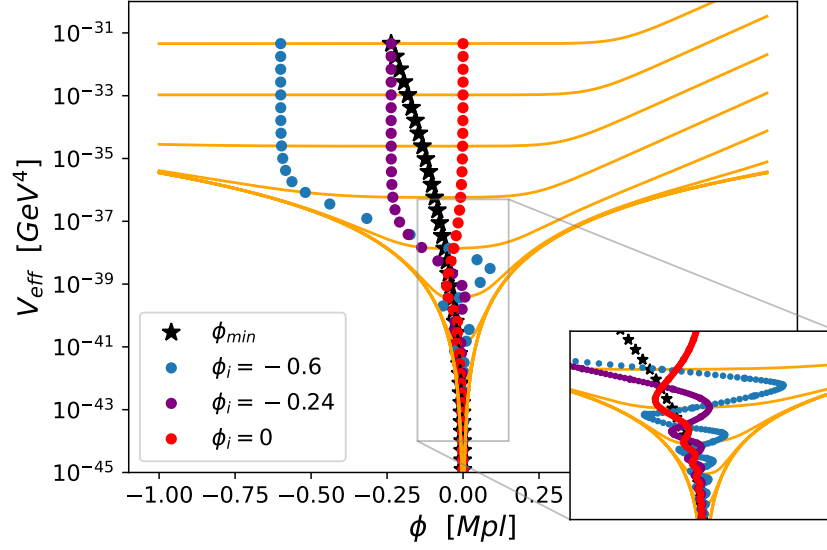


Figure 4.3: The orange lines are snapshots for the effective potential for decreasing values of redshift (the first line from top to bottom starts at $z_i = 10^5$ and the redshift interval between two subsequent lines is $\log_{10}(z_n/z_{n+1}) \approx 0.41$). The black stars indicate the minimum of the effective potential and the blue, purple, and red dots are the scalar field evolution starting at different initial conditions as indicated in the label. Notice that in the case represented by the purple dots, the field starts at the minimum of the effective potential but does not follow it until thawing. Also, a zoom of the region where the field is dynamical and oscillates around the minimum of the effective potential. In this figure we used $\beta = 10$ and $\lambda = 10^{-109}$.

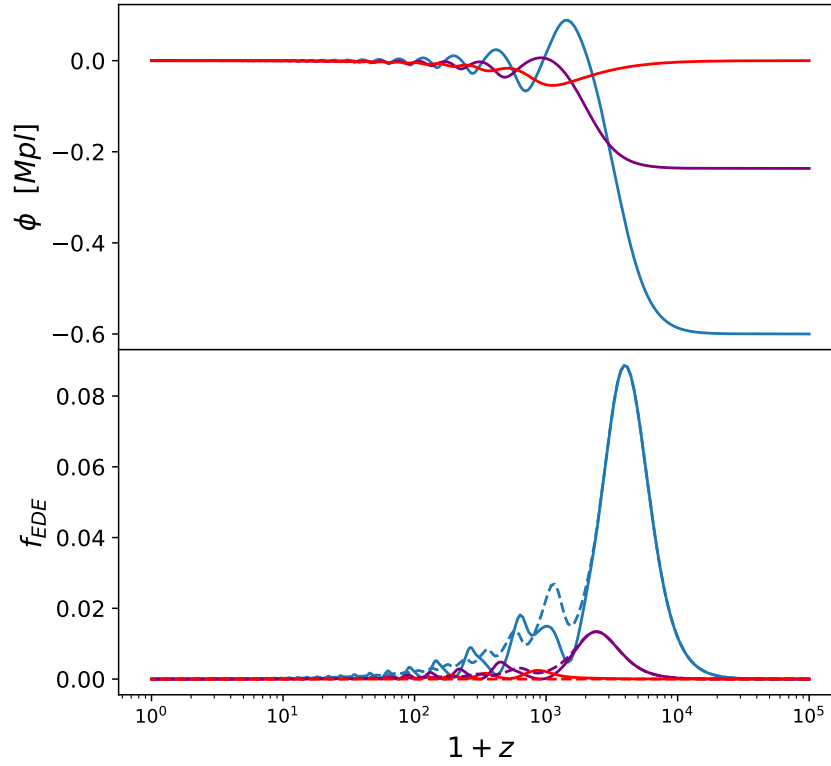


Figure 4.4: The scalar field evolution and the fraction of energy density for the same three initial conditions as in Fig. 4.3: $\phi_i/M_{Pl} = -0.6, -0.24, 0$ for the blue, purple and red lines, respectively. The dashed lines in the bottom panel show the uncoupled case with all other parameters kept fixed at the same values.

necessary but this leads to a small fraction of EDE.

We begin by showing some examples of this behavior in Figure 4.5. We choose parameters in such a way that the field becomes dynamical and starts to oscillate around the minimum of the effective potential at some $z = \mathcal{O}(10^5)$. More specifically, we start the field evolution at a high redshift $z = 10^7$ at $\phi_i/M_{Pl} = -10^{-4}$, where it is still frozen and show how it approaches the minimum of the effective potential at $z = \mathcal{O}(10^5)$ for different values of β and λ . We also show that the difference is minimal when compared to the evolution starting at $z = 10^5$ with the field at the minimum of the effective potential.

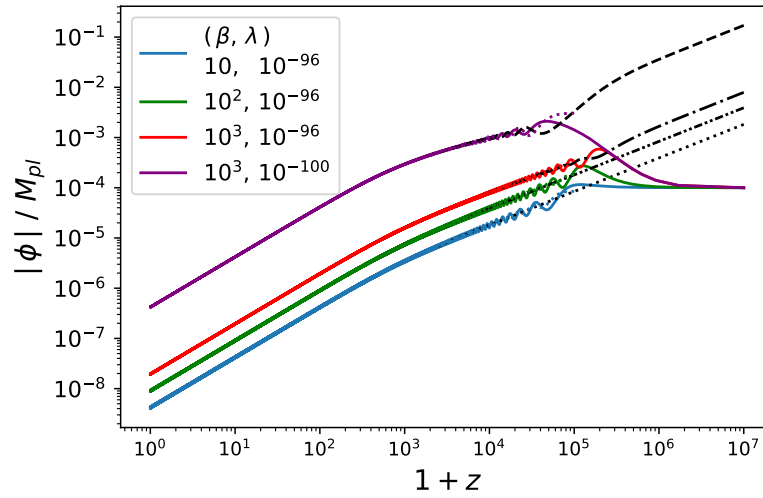


Figure 4.5: The full colored lines are the field evolution for $\phi_i = -10^{-4}M_{Pl}$ at $z_i = 10^7$ and considering different values of the parameters β and λ as indicated in the label. Each field track its minimum as described by Eq. (4.22) and it is indicated by the different stylized black lines. The dotted colored lines are the field evolution starting at $z_i = 10^5$ in their respective minimum for the same set of parameters β and λ .

We now turn to the contribution of the EDE field ϕ to the energy density of the Universe, showing in Figure 4.6 the behavior of f_{EDE} for the cases described above. We compare the numerical computation with the analytical estimate obtained by assuming that the EDE field is at the minimum of the effective potential (Eq. (4.22)) and that the dominant contribution comes from the bare potential $V(\phi)$ [40]:

$$f_{\text{EDE}} = \frac{1}{12} \frac{\beta^{4/3}}{\lambda^{1/3}} \left(\frac{\Theta_\nu(\phi)}{H^{3/2} M_{Pl}^{5/2}} \right)^{4/3}, \quad (4.29)$$

computed assuming that $\beta\phi/M_{Pl} < 1$ in order to use the approximate expression

Eq. (4.22). However, in the equation above we use the value of ϕ from CAMB. One can see that the analytical estimate is a reasonable approximation of the average of the oscillations in f_{EDE} . We also notice that even for large values of the coupling ($\beta = 1000$) a cosmologically significant level of f_{EDE} is not obtained³. The same behavior is also seen if one decreases the field self-coupling λ , as expected from Eq. (4.29) shown in the bottom panel of Fig. 4.6: an increase in f_{EDE} is obtained for smaller values of λ at the cost of a decrease in the relevant redshift z_c . A more thorough investigation including a comparison with data using a Bayesian analysis is necessary to reach a more definitive conclusion of whether this case can help to ease the H_0 tension.

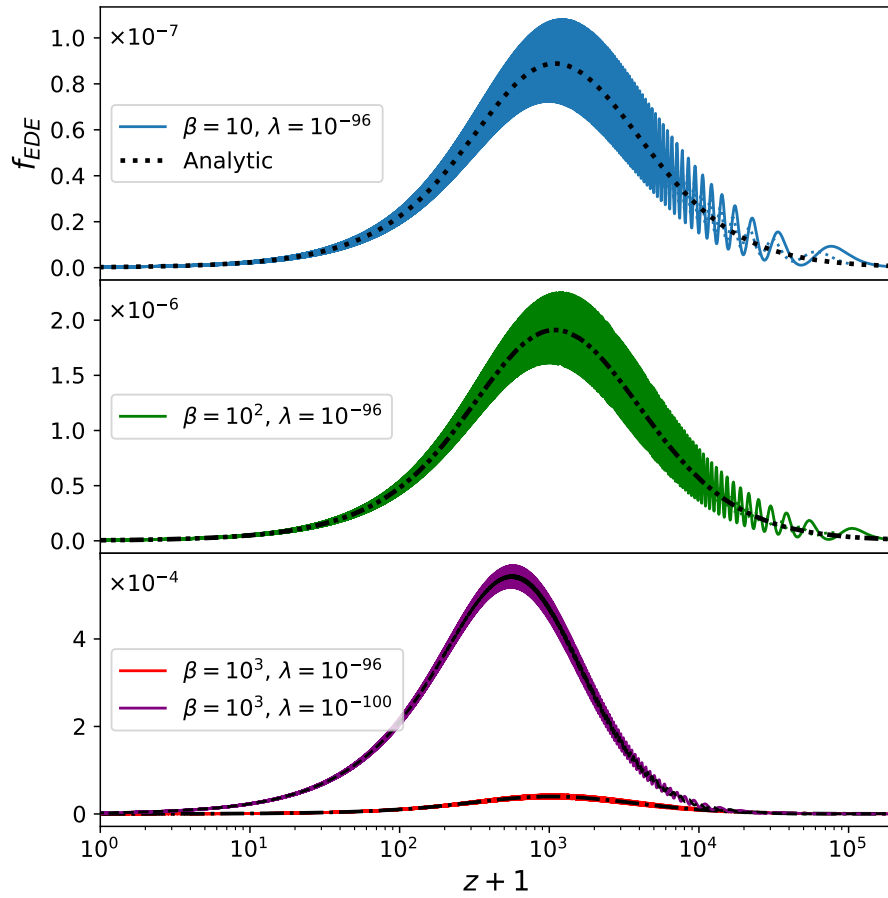


Figure 4.6: Fraction of EDE for the same set of parameters as in Fig. 4.5. The different stylized black lines are the analytic prediction for the f_{EDE} according to Eq. (4.29).

³In principle it is possible to obtain a significant amount of EDE for still larger values of β . Indeed, we find that $\beta \geq \mathcal{O}(10^6)$ could in principle lead to $f_{\text{EDE}} = \mathcal{O}(10\%)$ but these large values of β can not be considered natural.

A prominent feature of ν EDE models is the kick in the EDE field provided by the coupling with neutrinos that occurs when neutrinos transition from the relativistic to the nonrelativistic regimes. An intriguing possibility is that this mechanism could explain the EDE coincidence problem since the neutrino mass energy scale roughly coincides with the matter-radiation equality epoch. We now want to assess the impact of the neutrino coupling in the contribution of the EDE field to the total energy density.

In order to evaluate the impact of the coupling to neutrinos we will use the criteria in Eq. (4.27) and compare the two cases, with and without coupling. In both cases, we start the field evolution at the minimum of the effective potential at $z = 10^5$ in the situation where the field is already dynamic but in the first case, we study the full model whereas in the second case, we set the neutrino coupling β to zero, keeping all other parameters equal. In Figure 4.7 we show a comparison in f_{EDE} for both cases, and one can see that the effect of the neutrino-EDE coupling is prominent. Although the contribution is the same initially, in the uncoupled case the field tends to zero (the minimum of the bare potential) while the coupled case follows Eq. (4.22) which for larger β can significantly differ from zero. This results in a large difference between the two cases, which impacts f_{EDE} by several orders of magnitude.

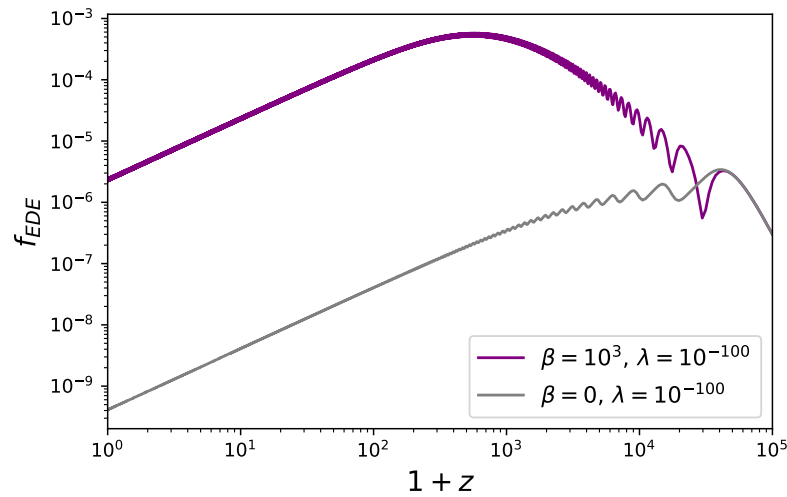


Figure 4.7: A comparison of f_{EDE} between the coupled and uncoupled ($\beta = 0$) scenarios with the field starting at the minimum of the effective potential at $z = 10^5$ while keeping fixed all the other cosmological and model parameters. The purple line is the case with the largest f_{EDE} shown in the bottom of Fig. 4.6.

4.5 Field displacement

In this section, we derive an analytical estimate for the EDE field displacement induced by the neutrino kick. We begin with the Klein-Gordon equation of the EDE field coupled to neutrinos given by Eq. (4.19), assuming that the EDE field is close to the minimum of the bare potential $V(\phi)$:

$$\ddot{\phi} + 3H\dot{\phi} = \frac{-\beta}{M_{pl}}(\rho_\nu - 3p_\nu). \quad (4.30)$$

Following [48, 49] we introduce the ϕ -independent quantities

$$\tilde{\rho}_\nu \equiv \rho_\nu e^{-\beta\phi/M_{pl}}; \quad \tilde{p}_\nu \equiv p_\nu e^{-\beta\phi/M_{pl}} \quad (4.31)$$

such that

$$\frac{d}{d\phi}(\rho_\nu - 3p_\nu) = \frac{d}{d\phi}[(\tilde{\rho}_\nu - 3\tilde{p}_\nu)e^{\beta\phi/M_{pl}}] = \frac{\beta}{M_{pl}}(\rho_\nu - 3p_\nu). \quad (4.32)$$

Therefore, Eq. (4.30) becomes

$$\ddot{\phi} + 3H\dot{\phi} = \frac{\beta}{M_{pl}}\tilde{\Theta}^{(\nu)}e^{\beta\phi/M_{pl}}, \quad (4.33)$$

where $\tilde{\Theta}_\nu \equiv -\tilde{\rho}_\nu + 3\tilde{p}_\nu$. This function is given by

$$\tilde{\Theta}_\nu = -\frac{g_\nu T_\nu^4}{2\pi^2}\tau(m_\nu/T_\nu), \quad (4.34)$$

$$\tau(x) = x^2 \int_x^\infty du \frac{\sqrt{u^2 - x^2}}{e^u + 1}, \quad (4.35)$$

where $x \equiv m_\nu/T_\nu$. Assuming that the kick occurs within the radiation-dominated epoch,

$$\ddot{\phi} + 3H\dot{\phi} \approx -\frac{45}{\pi^4} \left(\frac{4}{11}\right)^{4/3} \frac{\beta g_\nu}{g_\star(T_\gamma)} H^2 M_{pl} \tau(x) e^{\beta\phi/M_{pl}}, \quad (4.36)$$

where $g_\star(T)$ is the effective number of relativistic species at temperature T .

In the case of an instantaneous transition (see e.g. [41]) and using $g_\nu = 4$ and $g_\star = 3.38$ one obtains the approximate equation:

$$\Delta\phi/M_{pl} \approx -0.03\beta e^{\beta\Delta\phi/M_{pl}}, \quad (4.37)$$

the solution of which can be written as:

$$\Delta\phi/M_{Pl} = -\frac{W_0(0.03\beta^2)}{\beta}, \quad (4.38)$$

where W_0 is the principal branch of the Lambert function.

Eq. (4.37) can be written in the form

$$\begin{aligned} ye^y &= x, \\ y &\equiv -\beta\Delta\phi/M_{Pl}, \\ x &\equiv 0.03\beta^2. \end{aligned} \quad (4.39)$$

The solution of Eq. (4.39), i.e., $y(x)$, is given by the principal branch of the Lambert W -function. Then we finally recover Eq. (4.38).

4.6 Minimum of the effective potential

The derivative of the effective potential is given by Eq. (4.20)

$$V'_{\text{eff}}(\phi) = V'(\phi) - \frac{\beta}{M_{Pl}} \tilde{\Theta}_\nu e^{\beta\phi/M_{Pl}}, \quad (4.40)$$

where in the last term was used the transformation (4.31). The minimum of the effective potential is the solution of

$$\lambda\phi_{\min}^3 = \frac{\beta}{M_{Pl}} \tilde{\Theta}_\nu e^{\beta\phi_{\min}/M_{Pl}}, \quad (4.41)$$

which is given by

$$\phi_{\min} = -\frac{3}{\beta} W_0 \left[\left(\frac{\beta^4 \tilde{\Theta}_\nu}{27\lambda M_{Pl}^4} \right)^{1/3} \right] M_{Pl}. \quad (4.42)$$

Note that an expansion of Eq. (4.42) reduces to the solution

$$\phi_{\min} = - \left(\frac{\beta \tilde{\Theta}_\nu}{\lambda M_{Pl}} \right)^{1/3}, \quad (4.43)$$

assuming $|\beta\phi|/M_{Pl} \ll 1$ [40]. We compare the two solutions for the minimum evolution in Fig. 4.8.

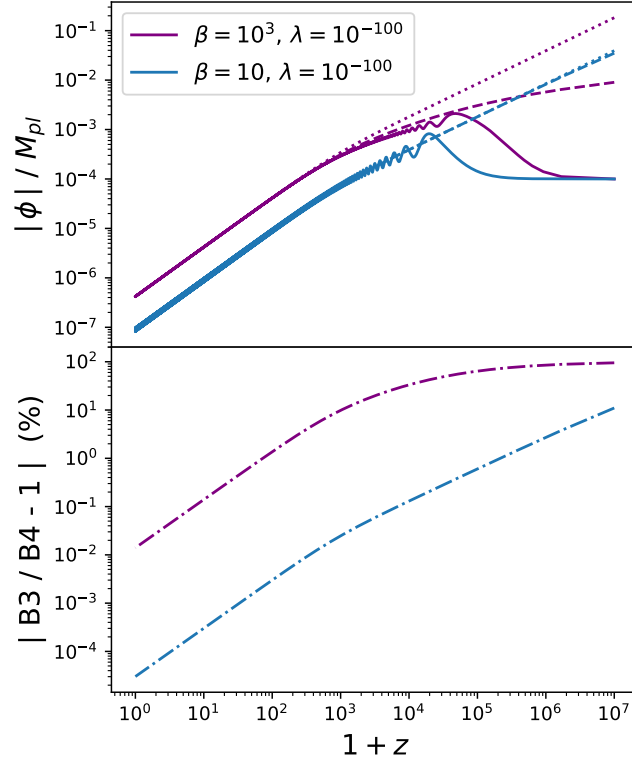


Figure 4.8: In the top panel, the full blue and purple lines show the field evolution for the parameters $\beta = 10, 1000$ and the same $\lambda = 10^{-100}$ and $\phi_i = -10^{-4}M_{Pl}$. The dashed and dotted lines are the minimum evolution given by Eq. (4.42) and Eq. (4.43), respectively. For a small value of β the two solutions are more close to each other, this is why is difficult to see the dotted blue line in the top panel. In the bottom panel is shown the difference between the two analytic solutions.

4.7 Conclusion

Neutrino-assisted early dark energy models were introduced with the aim of reducing some of the fine-tuning required in usual EDE models with respect to the parameters characterizing the scalar field self-coupling and its initial displacement.

The idea that a “neutrino kick” occurring around the matter-radiation equality epoch could trigger the EDE field is interesting for addressing the EDE coincidence problem.

In this Chapter, we show results from an implementation of the ν EDE model in CAMB at the background level to study f_{EDE} and Δ (which are background quantities), quantifying the EDE scalar field contribution to the total energy density and the impact of the neutrino-EDE coupling, respectively.

We have performed this analysis considering two cases depending on whether or not the EDE field is frozen at the initial redshift of its evolution. We find that if the field is initially frozen, its value can be arbitrary and the model can help in ameliorating the Hubble tension. However, since $\Delta \sim 1$ the neutrino-EDE coupling is not relevant in this case and the EDE coincidence problem is not addressed.

In contrast, as pointed out in [40, 42], the opposite scenario is attractive since there is a natural initial value for the field: the minimum of the effective potential, ϕ_{min} , which is time-dependent. We show that the field follows ϕ_{min} without experiencing a significant dynamical kick from its evolution equation, ie without a displacement from the minimum. However, ϕ_{min} is affected by the change in the trace of the neutrino energy-momentum tensor, which generates a bump in f_{EDE} since the field follows the minimum.

We find that on average f_{EDE} is well reproduced by the analytical prediction in Eq. (4.29). Furthermore, from Fig. 4.7 one can see that in this regime the neutrino-EDE coupling plays a non-negligible role, leading to large modifications in f_{EDE} . However, the attained value of f_{EDE} does not satisfy the criteria in Eq. (4.26) for easing the Hubble tension. One might think that decreasing the value of the self-coupling λ would increase the value of f_{EDE} , as follows from Eq. (4.29). This is correct but one should notice that this would also imply in a field that is frozen until later times (since it results in a smaller $V''(\phi)$), which changes this model to the first case discussed above. In addition, we verified that couplings as large as $\beta = 1000$ do not produce a significant amount of EDE.

We conclude that ν EDE models, with a self-interaction potential and a coupling

of the type $V(\phi) \propto \phi^4$ and $A(\phi) = \exp(\beta\phi/M_{Pl})$, respectively, do not seem to naturally provide a significant amount of early dark energy which is one of the criteria usually required for the model to ease the Hubble tension. However, a more thorough investigation is certainly needed. Different potentials and couplings should be analyzed, such as the ones proposed in Ref. [38] where a model with EDE interacting with dark matter is studied using a monodromy-type potential and a ϕ^2 coupling. Additionally, a successful model must not only be able to alleviate the Hubble tension at the background level but also produce perturbations that agree with the current CMB and LSS data. Hence, a comparison with data using Bayesian statistics at the perturbative level (as implemented in [48–50] for neutrinos coupled to a quintessence field) is necessary in order to reach a more definitive conclusion of whether this class of models is capable of alleviating the H_0 tension and possibly the S_8 tension as well. In particular, the integrated contribution of ν EDE in addition to its peak value contribution should be carefully analyzed.

Once the perturbations equations for the ν EDE model are implemented in a Boltzmann solver, it may be interesting to obtain constraints on these model parameters (β and λ) using data from large scale structure, cosmic microwave background, baryon acoustic oscillation, and supernovae.

Chapter 5

Late-Time Dark Energy and Massive Neutrinos with DESI Y1 BAO data

The prevailing cosmological paradigm is the spatially flat Λ CDM model with adiabatic and Gaussian initial conditions. The Greek letter “ Λ ” designates the cosmological constant while the acronym “CDM” means the pressureless Cold Dark Matter. Until today this has been the more economical description of the Universe that is consistent with an assortment of observational probes [5, 6, 55–62]. Although its success, the benchmark cosmological model is not absent of issues. The horizon and flatness are well-known long-term problems [2, 63, 64] with possible solutions when we bring to the scene the Cosmic Inflation theory [65]. On the other hand, over the past few decades, cosmological surveys have attained significant improvement in precision in their signal measurements, thus allowing tighter constraints for the Λ CDM model. When different probes are combined for the parameter estimation, the main challenges that arise are the so-called Hubble (or H_0) and S_8 (or σ_8) tensions.

The Hubble constant, H_0 , quantifies the expansion rate of the universe at the present time. The Hubble tension arises from the disagreement between two types of measurements of H_0 , classified as (i) “*early*” (and model dependent), which yields a lower value for H_0 compared to (ii) “*late*” (and model-independent) measurements of H_0 . Typical probes of type (i) are Cosmic Microwave Background (CMB), from Planck, ACT and SPT-3G [9, 55, 66], and combination between Baryon Acoustic Oscillation (BAO) from BOSS and eBOSS DR12 with prior on the baryon density from Big Bang Nucleosynthesis (BBN) [56, 67–70]. Conversely, assessments of the Hubble constant by means of probes of kind (ii), are the ones based on the distance-redshift relation employed in observations of Cepheid variable and Type Ia Supernovae (SNIa) [71–77] (see [78, 79] for an extended list of estimation of H_0). To pose the Hubble tension in numbers, we chose one representative value for each group, *i* and *ii*. Using high multipole (small angular scales) temperature

and polarization TT, TE, and EE with large-scale polarization (lowE), the Planck Collaboration [55] report an estimation of $H_0 = 67.27 \pm 0.60$ [km/s/Mpc] at 68% CL, which is 5σ smaller than direct measurements, e.g. from SH0ES collaboration [80] using Cepheid-SNIa sample yielding a baseline result of $H_0 = 73.04 \pm 1.04$ at the same confidence level.

A second cosmological tension arises with measurements of matter fluctuations amplitude (or the standard deviation $\sigma_R = \langle \delta_R^2 \rangle^{0.5}$) within a sphere of radius equal to $R = 8h^{-1}\text{Mpc}$, denoted by σ_8 or usually as $S_8 \equiv \sigma_8(\Omega_m/0.3)^{0.5}$ where Ω_m is the total amount of non-relativistic matter today. The S_8 (or σ_8) tension. Conversely to what happens with the Hubble tension, in this case, are CMB primary anisotropies estimates that constraints the S_8 parameter in a higher value compared to low-redshift probes such as weak gravitational lensing (WL) and galaxy clustering [62, 81–92]. Using the same dataset to report the Hubble tension, the Planck collaboration reports $S_8 = 0.834 \pm 0.016$ at 68% confidence level [55] while the DES-Y3 cosmic shear measurements from [86, 87] presents $S_8 = 0.759^{+0.025}_{-0.023}$ and joint cosmological analysis of multi-probe WL and spectroscopic galaxy clustering [89] finds $S_8 = 0.766^{+0.020}_{-0.014}$. A complementary list and discussion on S_8 tension can be found in [79] and references therein.

5.0.1 Constraining EDE with late DE marginalization

We investigate the Hubble and S_8 tensions in the paper [93]¹, where we applied the piecewise (see section 5.1.2 for more details) equation of state to model the late-time dark energy and performing a marginalization over this late-time expansion, it is obtained constraints on the early dark energy parameters. The main driving question of this work was whether the addition of a late-time smooth dark energy component could improve the fraction of early dark energy back when adding large-scale structure data to, e.g., cosmic microwave background, type Ia supernovae, and baryon acoustic oscillation.

In this work [93], we considered the equation of state for late dark energy as being constant in 3, 5, or 10 redshift bins, whilst the EDE is modeled by an effective fluid description [27]. Although one could use a microphysics-inspired description for EDE, such as axion-like scalar field mostly used in this kind of analysis [26, 94–97], this goes into numerical issues arising from the code attempts to integrate the fast field oscillation when it approaches the minimum of the

¹Check the public available code [CAMB-MultiFluid](#).

potential. The parametrization via the effective fluid description allows us to overcome this numerical difficulty.

To analyze this early plus late dark energy phenomenology, we considered in [93] different data set combinations that we shorten for “DC i ” and $i=1-5$ as shown in table 5.1. In this table, the CMB means ACT DR4 [98] CMB TT ($\ell \in [600, 4500]$), TE and EE ($\ell \in [350, 4500]$) supplemented with Planck 2018 data for $\ell < 650$. Type Ia supernovae from Pantheon [99], and BAO from three different sources; 6dFGS [100], SDSS DR7 [101], and SDSS BOSS DR12 [102]. CMBL stands for CMB lensing from Planck 2018, and finally, strong lensing from H0LiCOW [103] is shortened by SL.

DC i	Dataset Combination
DC1	CMB + BAO + SNe + CMBL
DC2	CMB + BAO + SNe
DC3	CMB + BAO + SNe + CMBL + SL
DC4	CMB + SNe + CMBL
DC5	CMB + SNe + SL

Table 5.1: Different dataset combination considered in the analysis of [93].

On top of, e.g., DC1, are added LSS measurements from Dark Energy Survey (DES) [104] from 3×2 pt but also only with cosmic shear since the most constraining power on S_8 and Ω_m comes from this probe. For example, Fig. 1 of [93] displays 2D and 1D constraints on H_0 , Ω_m , S_8 and f_{EDE} and in Eq. 4.2 of [93] it is shown the constraint on the fraction of EDE with late-time expansion marginalization $f_{\text{EDE}} = 0.087^{+0.028}_{-0.035}$ for DC1 and adding LSS measurements, it is found $f_{\text{EDE}} = 0.067^{+0.019}_{-0.025}$ and $f_{\text{EDE}} = 0.063^{+0.019}_{-0.024}$ for DES-Y3 cosmic shear and DES-Y3 3×2 pt, respectively. From these results, we conclude that DES data decreases the value of f_{EDE} , which worsens the Hubble tension. This work is important because it shows that in order to solve simultaneously the Hubble and S_8 tensions, one needs to go beyond a smooth late dark energy.

As previously stated, the work [93] uses a piecewise constant equation of state for late-time expansion. The following sections describe this and other parameterizations for $w(z)$ that are being investigated in light of DESI Y1 data.

5.1 General polynomial dark energy equation of state

Let's consider a general polynomial function for the dark energy equation of state at the i th redshift bin

$$w^{(i+1) \text{ th bin}} = w(z_i < z < z_{i+1}) = \sum_{n=0}^N w_{an,i} (z - z_i)^n, \quad (5.1)$$

where $w_i \equiv w_{a0,i}$, $w_{a,i} \equiv w_{a1,i}$, $w_{aa,i} \equiv w_{a2,i}$, $w_{aaa,i} \equiv w_{a3,i}$ and so forth. These parameters have to be determined by imposing some condition, e.g., continuity and smoothness for $w(z)$ at the transition between redshift bins, i.e., the derivative must exist. The energy density is obtained from the continuity equation, and using $w = P/\rho$, we have

$$\dot{\rho} + 3\frac{\dot{a}}{a}(\rho + P) = 0 \Rightarrow \dot{\rho} = -3\frac{\dot{a}}{a}\rho(1 + w(a)) \Rightarrow \frac{d\rho}{\rho} = -3\frac{da}{a}(1 + w(a))$$

$$\boxed{\int_{\rho}^{\rho_0} \frac{d\rho'}{\rho'} = -3 \int_a^{a_0} \frac{da'}{a'} (1 + w(a'))}, \quad a < a_0 = 1, \quad (5.2)$$

or in terms of redshift we have $1 + z = 1/a \Rightarrow dz = da/a = -dz/(1 + z)$ therefore

$$\boxed{\int_{\rho_0}^{\rho} \frac{d\rho'}{\rho'} = 3 \int_{z_0}^z \frac{dz'}{1 + z'} (1 + w(z'))}, \quad z > z_0 = 0. \quad (5.3)$$

From Eq. (5.1), we obtain Eq. (5.3)

$$\int_{\rho_i}^{\rho} \frac{d\rho'}{\rho'} = 3 \int_{z_i}^z \frac{dz'}{1 + z'} + 3 \sum_{n=0}^N w_{an,i} \int_{z_i}^z \frac{dz'}{1 + z'} (z' - z_i)^n, \quad (5.4)$$

where N is the maximum degree of the polynomial. Using the binomial theorem, we can rewrite the polynomial expression $(z' - z_i)^n$ as follows:

$$(z' - z_i)^n = \sum_{k=0}^n \binom{n}{k} (-z_i)^{n-k} z'^k. \quad (5.5)$$

Substituting Eq (5.5) into Eq (5.4) we have

$$\begin{aligned} \int_{\rho_i}^{\rho} \frac{d\rho'}{\rho'} &= 3 \int_{z_i}^z \frac{dz'}{1+z'} + 3 \sum_{n=0}^N w_{an,i} \int_{z_i}^z \frac{dz'}{1+z'} \left[\sum_{k=0}^n \binom{n}{k} (-z_i)^{n-k} z'^k \right] \\ &= 3 \int_{z_i}^z \frac{dz'}{1+z'} + 3 \sum_{n=0}^N w_{an,i} \sum_{k=0}^n \binom{n}{k} (-z_i)^{n-k} \int_{z_i}^z \frac{dz'}{1+z'} z'^k. \end{aligned} \quad (5.6)$$

Let's consider the indefinite integral:

$$\begin{aligned} \int \frac{dz}{1+z} z^k &= \int \frac{dz}{1+z} (z+1-1) z^{k-1} = \int dz z^{k-1} - \int \frac{dz}{1+z} z^{k-1} \\ &= \frac{z^k}{k} - \frac{z^{k-1}}{k-1} + \frac{z^{k-2}}{k-2} - \int dz z^{k-4} + \int \frac{dz}{1+z} (z+1-1) z^{k-5} = \dots \end{aligned} \quad (5.7)$$

if we continue the process as indicated in Eq (5.7), we can show that

$$\begin{aligned} \int \frac{dz}{1+z} z^k &= \ln(1+z), \text{ for } k=0 \\ \int \frac{dz}{1+z} z^k &= (-1)^k \ln(1+z) + \sum_{\ell=1}^k \frac{(-1)^{\ell-1}}{k-\ell+1} z^{k-\ell+1}, \text{ for } k=1,2,3,\dots \end{aligned} \quad (5.8)$$

which can be shown as having a hypergeometric form

$$\int \frac{dz}{1+z} z^k = \frac{z^{k+1} {}_2F_1(1, k+1; k+2; -z)}{k+1} + \text{constant}, \quad (5.9)$$

where ${}_2F_1(a, b; c; z)$ is the hypergeometric function. Indeed, we can test this solution by means of computing the Eq. (5.9) for a few values of k . This is shown in the following table

k	Solution
0	$\ln(1+z)$
1	$z - \ln(1+z)$
2	$[z^2 - 2z + 2\ln(1+z)]/2$
3	$[2z^3 - 3z^2 + 6z - 6\ln(1+z)]/6$

We can easily check that the Eq. (5.8) recovers the results in the table 5.1. Finally, the dark energy density is given by substituting either Eq. (5.8) or Eq. (5.9) into

Eq. (5.6). We opted to use the closed form of the latter one, then we have

$$\rho = \rho_i \left(\frac{1+z}{1+z_i} \right)^3 \exp \left\{ 3 \sum_{n=0}^N w_{an,i} \sum_{k=0}^n \binom{n}{k} \frac{(-z_i)^{n-k} z^{k+1}}{k+1} \left[{}_2F_1(1, k+1; k+2; -z) - {}_2F_1(1, k+1; k+2; -z_i) \right] \right\}. \quad (5.10)$$

Once we have a general expression for the dark energy equation of state and energy density, we proceed to a showcase

5.1.1 Constant $w(z)$

If the EoS is merely a constant, then we can easily integrate the expression Eq. (5.2) as shown in the following

$$\int_{\rho}^{\rho_0} \frac{d\rho'}{\rho'} = -3(1+w) \int_a^{a_0} \frac{da'}{a'} \Rightarrow \boxed{\rho = \rho_0 \left(\frac{a}{a_0} \right)^{-3(1+w)}} \text{ if } w = \text{const.} \quad (5.11)$$

5.1.2 Constant $w(z)$ in bins of redshift

Using Eq. (5.2) we can compute the energy density for each bin of redshift (or scale factor) as we do in the following

$$\text{1st bin: } \rho(a_1 < a < a_0) = \rho_0 \left(\frac{a}{a_0} \right)^{-3(1+w_0)}, \quad (5.12)$$

$$\text{2nd bin: } \rho(a_2 < a < a_1) = \rho_1 \left(\frac{a}{a_1} \right)^{-3(1+w_1)}, \quad (5.13)$$

$$\text{3rd bin: } \rho(a_3 < a < a_2) = \rho_2 \left(\frac{a}{a_2} \right)^{-3(1+w_2)}, \quad (5.14)$$

To guarantee the uniqueness of the solution we must request the continuity of the integral at the bins limit. Let us first request the match between the boundaries of the 1st and 2nd bins, and then between the 2nd and 3rd bins.

Match: 2nd - 1st bins Assuming $a_0=1$ we have

$$\lim_{a \rightarrow a_1^+} \rho^{(2\text{nd bin})} = \lim_{a \rightarrow a_1^-} \rho^{(1\text{st bin})} \Rightarrow \boxed{\rho^{2\text{nd bin}} = \rho_0 a_1^{-3(w_0-w_1)} a^{-3(1+w_1)}} \quad (5.15)$$

Match: 3rd - 2nd bins

$$\lim_{a \rightarrow a_2^+} \rho^{(3\text{rd bin})} = \lim_{a \rightarrow a_2^-} \rho^{(2\text{nd bin})} \Rightarrow \boxed{\rho^{3\text{rd bin}} = \rho_0 a_1^{-3(w_0-w_1)} a_2^{-3(w_1-w_2)} a^{-3(1+w_2)}}. \quad (5.16)$$

Generalizing the match: (n+1)th - nth bins

The energy density at the (n+1)th redshift bin is

$$\rho(a_{n+1} < a < a_n) = \rho_n \left(\frac{a}{a_n} \right)^{-3(1+w_{n+1})}, \quad (5.17)$$

and assuming $a_0 = 1$, and $a_i = (1 + z_i)^{-1}$ we obtain

$$\boxed{\rho(a_{n+1} < a < a_n) = \rho_0 \left[\prod_{i=1}^n (1 + z_i)^{3(w_{i-1}-w_i)} \right] a^{-3(1+w_n)}} \quad (5.18)$$

5.1.3 Linear $w(z)$ in bins of redshift

In this case, we allow the EoS to be a linear function of redshift, then for a given redshift bin, $w(z)$ takes the form:

$$w^{(i+1) \text{ th bin}} = w(z_i < z < z_{i+1}) = w_i + w_{a,i}(z - z_i) = w_i + w_{a,i}\Delta z, \quad (5.19)$$

$$w_{a,i} = \frac{\Delta w}{\Delta z} = \frac{w_{i+1} - w_i}{z_{i+1} - z_i}, \quad (5.20)$$

$$\boxed{w^{(i+1) \text{ th bin}}(z) = w_i + \left(\frac{w_{i+1} - w_i}{z_{i+1} - z_i} \right) (z - z_i) \quad \forall z \in [z_i, z_{i+1}], i = 0, 1, 2, \dots} \quad (5.21)$$

Once we have the EoS, we can obtain the energy density using Eq. (5.3):

$$\int_{\rho_i}^{\rho} \frac{d\rho'}{\rho'} = 3 \int_{z_i}^z \frac{dz'}{1+z'} (1 + w^{(i+1)}(z')). \quad (5.22)$$

Inserting Eq. (5.21) into the Eq. (5.22) and solving the integral, the energy density at the i th bin is:

$$\boxed{\rho^{(i+1) \text{ th bin}}(z) = \rho_i \left(\frac{1+z}{1+z_i} \right)^{3[1+w_i-w_{a,i}(1+z_i)]} e^{3w_{a,i}(z-z_i)}}, \quad (5.23)$$

$\forall z \in [z_i, z_{i+1}]$, $i = 0, 1, 2, \dots$ and $w_{a,i}$ given by Eq. (5.20). The last step is to determine ρ_i in terms of ρ_0 . For this purposes simply note that $\rho^{(i+1)}(z_{i+1}) = \rho^{(i+2)}(z_{i+1})$ because $\rho^{(i+2)}(z)$ is defined in $[z_{i+1}, z_{i+2}]$. If we force this equality, we obtain:

$$\rho_{i+1} = \rho_i \left(\frac{1 + z_{i+1}}{1 + z_i} \right)^{3[1+w_i-w_{a,i}(1+z_{i+1})]} e^{3w_{a,i}(z_{i+1}-z_i)}. \quad (5.24)$$

5.1.4 Quadratic $w(z)$ in bins of redshift

We start writing the equation of the state

$$\begin{aligned} w^{(i+1) \text{ th bin}} = w(z_i < z < z_{i+1}) &= w_i + w_{a,i}\Delta z_i + w_{aa,i}\Delta z_i^2 \\ &= w_i + w_{a,i}(z - z_i) + w_{aa,i}(z - z_i)^2 \end{aligned} \quad (5.25)$$

Once we have the equation of state, we can obtain the energy density using the Eq. (5.3) which can be generalized to

$$\begin{aligned} \int_{\rho_i}^{\rho} \frac{d\rho'}{\rho'} &= 3 \int_{z_i}^z \frac{dz'}{1+z'} (1 + w^{(i+1)}(z')) \text{ Using Eq (5.25)} \\ &= 3 \int_{z_i}^z \frac{dz'}{1+z'} \left[1 + w_i + w_{a,i}(z' - z_i) + w_{aa,i}(z' - z_i)^2 \right] \\ &= 3 \int_{z_i}^z \frac{dz'}{1+z'} \left[1 + w_i + w_{a,i}(z' - z_i) + w_{aa,i}(z'^2 + z_i^2 - 2z'z_i) \right] \\ &= 3(1 + w_i - w_{a,i}z_i + w_{aa,i}z_i^2) \int_{z_i}^z \frac{dz'}{1+z'} + 3(w_{a,i} - 2w_{aa,i}z_i) \\ &\quad \int_{z_i}^z \frac{dz'}{1+z'} z' + 3w_{aa,i} \int_{z_i}^z \frac{dz'}{1+z'} z'^2 \end{aligned} \quad (5.26)$$

For simplicity, let's define the constant terms in Eq (5.26) as follows:

$$A_{0,i} \equiv 3(1 + w_i - w_{a,i}z_i + w_{aa,i}z_i^2), \quad (5.27)$$

$$A_{1,i} \equiv 3(w_{a,i} - 2w_{aa,i}z_i), \quad (5.28)$$

$$A_{2,i} \equiv 3w_{aa,i}, \quad (5.29)$$

with the above definition of A_0 , A_1 and A_2 we can continue the manipulation of Eq (5.26):

$$\begin{aligned}
\int_{\rho_i}^{\rho} \frac{d\rho'}{\rho'} &= A_{0,i} \int_{z_i}^z \frac{dz'}{1+z'} + A_{1,i} \int_{z_i}^z \frac{dz'}{1+z'} z' + A_{2,i} \int_{z_i}^z \frac{dz'}{1+z'} z'^2 \\
&= A_{0,i} \int_{z_i}^z \frac{dz'}{1+z'} + A_{1,i} \int_{z_i}^z \frac{dz'}{1+z'} (z' + 1 - 1) + A_{2,i} \int_{z_i}^z \frac{dz'}{1+z'} (z' + 1 - 1)^2 \\
&= A_{0,i} \int_{z_i}^z \frac{dz'}{1+z'} + A_{1,i} \left[\int_{z_i}^z dz' - \int_{z_i}^z \frac{dz'}{1+z'} \right] + \\
&\quad + A_{2,i} \left[\int_{z_i}^z \frac{dz'}{1+z'} (z' + 1)^2 + \int_{z_i}^z \frac{dz'}{1+z'} 1^2 + \int_{z_i}^z \frac{dz'}{1+z'} 2(z' + 1)(-1) \right] \\
&= A_{0,i} \int_{z_i}^z \frac{dz'}{1+z'} + A_{1,i} \left[\int_{z_i}^z dz' - \int_{z_i}^z \frac{dz'}{1+z'} \right] + \\
&\quad A_{2,i} \left[\int_{z_i}^z dz' z' + \int_{z_i}^z dz' + \int_{z_i}^z \frac{dz'}{1+z'} - 2 \int_{z_i}^z dz' \right] \\
&= (A_{0,i} - A_{1,i} + A_{2,i}) \int_{z_i}^z \frac{dz'}{1+z'} + (A_{1,i} - A_{2,i}) \int_{z_i}^z dz' + A_{2,i} \int_{z_i}^z dz' z' \\
\ln \frac{\rho}{\rho_i} &= (A_{0,i} - A_{1,i} + A_{2,i}) \ln \left(\frac{1+z}{1+z_i} \right) + (A_{1,i} - A_{2,i})(z - z_i) + \frac{1}{2} A_{2,i} (z^2 - z_i^2)
\end{aligned} \tag{5.30}$$

Taking the exponential of the Eq (5.30) we can finally obtain ρ as function of redshift and the constants A_0 , A_1 and A_2 :

$$\rho^{(i+1) \text{ th bin}}(z) = \rho_i \left(\frac{1+z}{1+z_i} \right)^{A_{0,i}-A_{1,i}+A_{2,i}} e^{(A_{1,i}-A_{2,i})(z-z_i)+A_{2,i}(z^2-z_i^2)/2}. \tag{5.31}$$

Quadratic $w(z)$: boundary conditions for 2 bins

Assuming two bins (by constructions, we actually have 3 (2+1) bins because we must specify what happens with $w(z)$ after the 2nd bin):

$$w^{(1)}(z) = w_0 + w_{a0}(z - z_0) + w_{aa0}(z - z_0)^2, \quad z \in [z_0 = 0, z_1], \quad \text{1st bin} \tag{5.32}$$

$$w^{(2)}(z) = w_1 + w_{a1}(z - z_1) + w_{aa1}(z - z_1)^2, \quad z \in [z_1, z_2], \quad \text{2nd bin} \tag{5.33}$$

$$w^{(3)}(z) = w_2 + w_{a2}(z - z_2) + w_{aa2}(z - z_2)^2, \quad z \in [z_2, z_3 = \infty), \quad \text{3rd bin} \tag{5.34}$$

In principle, we have 6 unknowns variables: w_{a0} , w_{aa0} , w_{a1} , w_{aa1} , w_{a2} and w_{aa2} but in practice we have just 4 unknowns variables because we assume that the third bin is a straight, (no curvature: $w_{aa2} = 0$), and horizontal, (no slope: $w_{a2} = 0$)

line passing by $w_2 = -1$ in order to recover the cosmological constant from the standard Λ CDM model. Assuming the continuity of the function, $w(z_{\text{internal knot}})$, and its derivative, $dw(z_{\text{internal knot}})/dz$, at each the internal knot (which guarantees a smooth handoff from one bin to another) we construct the following table:

function continuity	1st & 2nd bins	$w^{(1)}(z_1) = w^{(2)}(z_1) = w_1$
derivative continuity	1st & 2nd bins	$\frac{d}{dz}w^{(1)}(z_1) = \frac{d}{dz}w^{(2)}(z_1) = w_{a1}$
function continuity	2nd & 3rd bins	$w^{(2)}(z_2) = w^{(3)}(z_2) = w_2 \equiv -1$
derivative continuity	2nd & 3rd bins	$\frac{d}{dz}w^{(2)}(z_2) = \frac{d}{dz}w^{(3)}(z_2) = w_{a2}$
no slope - horizontal line	3rd bin	$w_{a2} = 0$
no curvature - straight line	3rd bin	$w_{aa2} = 0$

Table 5.2: This table shows that we have (in principle) 6 equations for 6 variables, which already is fine because the set is closed and thus is solvable. But note that the third bin is always known in order to recover the cosmological constant. So the last two conditions are said to be trivial, and in practice, we just have 4 remaining variables to be found.

Applying the boundaries conditions displayed in table 5.2 to the three equations (5.32), (5.33) and (5.34) we have:

$$w_0 + w_{a0}(z_1 - z_0) + w_{aa0}(z_1 - z_0)^2 = w_1 \iff w_{a0}\Delta z_1 + w_{aa0}\Delta z_1^2 = \Delta w_1 \quad (5.35)$$

$$w_{a0} + 2w_{aa0}(z_1 - z_0) = w_{a1} \iff w_{a0} + 2w_{aa0}\Delta z_1 - w_{a1} = 0 \quad (5.36)$$

$$w_1 + w_{a1}(z_2 - z_1) + w_{aa1}(z_2 - z_1)^2 = -1 \iff w_{a1}\Delta z_2 + w_{aa1}\Delta z_2^2 = \Delta w_2 \quad (5.37)$$

$$w_{a1} + 2w_{aa1}(z_2 - z_1) = 0 \iff w_{a1} + 2w_{aa1}\Delta z_2 = 0 \quad (5.38)$$

where in Eq (5.35) I have defined

$$\begin{aligned} \Delta z_1 &\equiv z_1 - z_0 = z_1 - 0 \quad \text{and} \quad \Delta w_1 \equiv w_1 - w_0, \\ \Delta z_2 &\equiv z_2 - z_1 \quad \text{and} \quad \Delta w_2 \equiv w_2 - w_1 = -1 - w_1. \end{aligned} \quad (5.39)$$

Lets firstly operate with the Eqs (5.37) and (5.38):

$$\begin{aligned} &- \text{Eq (5.38)} \times \Delta z_2 + \text{Eq (5.37)} \\ &\implies \left(-w_{a1}\Delta z_2 - 2w_{aa1}\Delta z_2^2 \right) + \left(w_{a1}\Delta z_2 + w_{aa1}\Delta z_2^2 \right) = \Delta w_2 \\ &\therefore -w_{aa1}\Delta z_2^2 = \Delta w_2 \implies \boxed{w_{aa1} = -\frac{\Delta w_2}{\Delta z_2^2}}. \end{aligned} \quad (5.40)$$

Substituting Eq (5.40) into Eq (5.38) we have

$$w_{a1} + 2 \left(-\frac{\Delta w_2}{\Delta z_2^2} \right) \Delta z_2 = 0 \implies \boxed{w_{a1} = 2 \frac{\Delta w_2}{\Delta z_2}}. \quad (5.41)$$

Finally lets operate with the two remaining equations (5.35) and (5.36):

$$\begin{aligned} & -2 \text{ Eq (5.35) } + \text{ Eq (5.36) } \times \Delta z_1 \\ & \implies \left(-2w_{a0}\Delta z_1 - 2w_{aa0}\Delta z_1^2 \right) + \left(w_{a0}\Delta z_1 + 2w_{aa0}\Delta z_1^2 - w_{a1}\Delta z_1 \right) \\ & = -2\Delta w_1 + 0 \times \Delta z_1 \\ & \therefore -w_{a0}\Delta z_1 - w_{a1}\Delta z_1 = -2\Delta w_1 \implies w_{a0} = 2 \frac{\Delta w_1}{\Delta z_1} - w_{a1} \text{ From Eq (5.41)} \\ & \therefore \boxed{w_{a0} = 2 \left(\frac{\Delta w_1}{\Delta z_1} - \frac{\Delta w_2}{\Delta z_2} \right)}. \end{aligned} \quad (5.42)$$

Substituting Eq (5.42) into Eq (5.35) we have

$$\boxed{w_{aa0} = -\frac{\Delta w_1}{\Delta z_1^2} + 2 \frac{\Delta w_2}{\Delta z_1 \Delta z_2}}. \quad (5.43)$$

5.1.5 Cubic $w(z)$ in bins of redshift

$$\begin{aligned} w^{(i+1) \text{ th bin}} &= w(z_i < z < z_{i+1}) \\ &= w_i + w_{a,i}\Delta z_i + w_{aa,i}\Delta z_i^2 + w_{aaa,i}\Delta z_i^3 \\ &= w_i + w_{a,i}(z - z_i) + w_{aa,i}(z - z_i)^2 + w_{aaa,i}(z - z_i)^3 \end{aligned} \quad (5.44)$$

Once we have the equation of state, we can obtain the energy density using the

Eq. (5.3) which can be generalized to

$$\begin{aligned}
\int_{\rho_i}^{\rho} \frac{d\rho'}{\rho'} &= 3 \int_{z_i}^z \frac{dz'}{1+z'} (1 + w^{(i+1)}(z')) \text{ Using Eq (5.44)} \\
&= 3 \int_{z_i}^z \frac{dz'}{1+z'} \left[1 + w_i + w_{a,i}(z' - z_i) + w_{aa,i}(z' - z_i)^2 + w_{aaa,i}(z' - z_i)^3 \right] \\
&= 3 \int_{z_i}^z \frac{dz'}{1+z'} [1 + w_i + w_{a,i}(z' - z_i) + w_{aa,i}(z'^2 + z_i^2 - 2z'z_i) \\
&\quad + w_{aaa,i}(z'^3 - 3z'^2z_i + 3z'z_i^2 - z_i^3)] \\
&= 3(1 + w_i - w_{a,i}z_i + w_{aa,i}z_i^2 - w_{aaa,i}z_i^3) \int_{z_i}^z \frac{dz'}{1+z'} \\
&\quad + 3(w_{a,i} - 2w_{aa,i}z_i + 3w_{aaa,i}z_i^2) \int_{z_i}^z \frac{dz'}{1+z'} z' \\
&\quad + 3(w_{aa,i} - 3w_{aaa,i}z_i) \int_{z_i}^z \frac{dz'}{1+z'} z'^2 \\
&\quad + 3w_{aaa,i} \int_{z_i}^z \frac{dz'}{1+z'} z'^3 \\
&= A_{0,i} \int_{z_i}^z \frac{dz'}{1+z'} + A_{1,i} \int_{z_i}^z \frac{dz'}{1+z'} z' + A_{2,i} \int_{z_i}^z \frac{dz'}{1+z'} z'^2 + A_{3,i} \int_{z_i}^z \frac{dz'}{1+z'} z'^3,
\end{aligned} \tag{5.45}$$

where now we have the updated version of the Eqs (5.27), (5.28), (5.29) and a new variable, $A_{3,i}$ that support just the term proportional to the third derivative term:

$$A_{0,i} \equiv 3(1 + w_i - w_{a,i}z_i + w_{aa,i}z_i^2 - w_{aaa,i}z_i^3), \tag{5.46}$$

$$A_{1,i} \equiv 3(w_{a,i} - 2w_{aa,i}z_i + 3w_{aaa,i}z_i^2), \tag{5.47}$$

$$A_{2,i} \equiv 3(w_{aa,i} - 3w_{aaa,i}z_i), \tag{5.48}$$

$$A_{3,i} \equiv 3w_{aaa,i}. \tag{5.49}$$

We don't need to perform the integral of all terms in Eq (5.45) because the Eq (5.30) already provides the results up till the term proportional to $A_{2,i}$. The remaining

integral is the following:

$$\begin{aligned}
 \int_{z_i}^z \frac{dz'}{1+z'} z'^3 &= \int_{z_i}^z \frac{dz'}{1+z'} z' z'^2 = \int_{z_i}^z \frac{dz'}{1+z'} z' (z'^2 - 1^2 + 1) \\
 &= \int_{z_i}^z \frac{dz'}{1+z'} z' [(z' + 1)(z' - 1) + 1] \\
 &= \int_{z_i}^z \frac{dz'}{1+z'} z' (z' + 1)(z' - 1) + \int_{z_i}^z \frac{dz'}{1+z'} (z' + 1 - 1) \\
 &= \int_{z_i}^z dz' (z'^2 - z') + \int_{z_i}^z dz' - \int_{z_i}^z \frac{dz'}{1+z'} \\
 &= \frac{1}{3}(z^3 - z_i^3) - \frac{1}{2}(z^2 - z_i^2) + (z - z_i) - \ln \left(\frac{1+z}{1+z_i} \right). \quad (5.50)
 \end{aligned}$$

Finally, with Eq (5.50) we can write the updated version of Eq (5.30) as follow:

$$\begin{aligned}
 \ln \frac{\rho}{\rho_i} &= (A_{0,i} - A_{1,i} + A_{2,i} - A_{3,i}) \ln \left(\frac{1+z}{1+z_i} \right) \\
 &\quad + (A_{1,i} - A_{2,i} + A_{3,i})(z - z_i) + \frac{1}{2}(A_{2,i} - A_{3,i})(z^2 - z_i^2) + \frac{1}{3}A_{3,i}(z^3 - z_i^3). \quad (5.51)
 \end{aligned}$$

Taking the exponential of Eq (5.51) we finally have the “updated” version of Eq (5.31):

$$\begin{aligned}
 \rho^{(i+1) \text{ th bin}}(z) &= \rho_i \left(\frac{1+z}{1+z_i} \right)^{A_{0,i} - A_{1,i} + A_{2,i} - A_{3,i}} \\
 &\quad \times e^{(A_{1,i} - A_{2,i} + A_{3,i})(z - z_i) + (A_{2,i} - A_{3,i})(z^2 - z_i^2)/2 + A_{3,i}(z^3 - z_i^3)/3}. \quad (5.52)
 \end{aligned}$$

Cubic $w(z)$: boundary conditions for 2 bins

Assuming two bins (by constructions, we actually have 3 (2+1) bins since we need to specify what happens with $w(z)$ after the 2nd bin)

$$w^{(1)}(z) = w_0 + w_{a0}(z - z_0) + w_{aa0}(z - z_0)^2 + w_{aaa0}(z - z_0)^3, \quad (5.53)$$

$$w^{(2)}(z) = w_1 + w_{a1}(z - z_1) + w_{aa1}(z - z_1)^2 + w_{aaa1}(z - z_1)^3, \quad (5.54)$$

$$w^{(3)}(z) = w_2 + w_{a2}(z - z_2) + w_{aa2}(z - z_2)^2 + w_{aaa2}(z - z_2)^3, \quad (5.55)$$

In principle, we have 9 unknowns variables: w_{a0} , w_{aa0} , w_{aaa0} , w_{a1} , w_{aa1} , w_{aaa1} , w_{a2} , w_{aa2} and w_{aaa2} but in practice we have just 6 unknowns variables because we assume that the third bin is a straight line (no curvature: $w_{aa2} = 0$) a hori-

zontal line (no slope: $w_{a2} = 0$) passing by $w_2 = -1$ and also has no $w_{aaa,2}$, i.e., $w_{aaa,2} = 0$. Those requirements for the last bin guarantee that we recover the cosmological constant from the standard Λ CDM model. Assuming the continuity of the function, $w(z_{\text{internal knot}})$, its first and second derivatives, $dw(z_{\text{internal knot}})/dz$, $d^2w(z_{\text{internal knot}})/dz^2$, at each the internal knot (which guarantees a smooth hand-off from one bin to another) we came up with the following table:

function continuity	1st & 2nd bins	$w^{(1)}(z_1) = w_1$
first derivative continuity	1st & 2nd bins	$\frac{d}{dz}w^{(1)}(z_1) = w_{a1}$
second derivative continuity	1st & 2nd bins	$\frac{d^2}{dz^2}w^{(1)}(z_1) = w_{aa1}$
function continuity	2nd & 3rd bins	$w^{(2)}(z_2) = w_2 \equiv -1$
first derivative continuity	2nd & 3rd bins	$\frac{d}{dz}w^{(2)}(z_2) = w_{a2}$
second derivative continuity	2nd & 3rd bins	$\frac{d^2}{dz^2}w^{(2)}(z_2) = w_{aa2}$
no slope - horizontal line	3rd bin	$w_{a2} = 0$
no curvature - straight line	3rd bin	$w_{aa2} = 0$
no jerk	3rd bin	$w_{aaa2} = 0$

Table 5.3: This table shows that we have (in principle) 9 equations for 9 variables, which already is fine because the set is closed and thus is solvable. But note that the third bin is always known in order to recover the cosmological constant. So the last three conditions are said to be trivial, and in practice, we just have 6 remaining variables to be found.

Firstly let's define the following notations:

$$\begin{aligned}
 z_0 &= 0 \\
 w_2 &= -1, w_{a2} = w_{aa2} = w_{aaa2} = 0, \\
 \Delta z_1 &\equiv z_1 - z_0 = z_1 \\
 \Delta w_1 &\equiv w_1 - w_0, \\
 \Delta z_2 &\equiv z_2 - z_1, \\
 \Delta w_2 &\equiv w_2 - w_1 = -1 - w_1.
 \end{aligned} \tag{5.56}$$

Applying the boundaries conditions displayed in table 5.3 to the three equations

(5.53), (5.54) and (5.55) we obtain:

$$\begin{aligned}
w_{a0}\Delta z_1 + w_{aa0}\Delta z_1^2 + w_{aaa0}\Delta z_1^3 &= \Delta w_1 \\
w_{a0} + 2w_{aa0}\Delta z_1 + 3w_{aaa0}\Delta z_1^2 &= w_{a1} \\
w_{aa0} + 3w_{aaa0}\Delta z_1 &= w_{aa1} \\
w_{a1}\Delta z_2 + w_{aa1}\Delta z_2^2 + w_{aaa1}\Delta z_2^3 &= \Delta w_2 = -1 - w_1 \\
2w_{aa1}\Delta z_2 + 3w_{aaa1}\Delta z_2^2 &= -w_{a1} \\
3w_{aaa1}\Delta z_2 &= -w_{aa1}
\end{aligned} \tag{5.57}$$

The notation of the system in Eq (5.57) is overloaded, and in order to manually solve this 6×6 equation, we first need a clean notation to avoid making mistakes, so let's consider the following change of notation:

$$\begin{aligned}
x_0 &\equiv w_{a0}, \quad y_0 \equiv w_{aa0}, \quad u_0 \equiv w_{aaa0}, \\
x_1 &\equiv w_{a1}, \quad y_1 \equiv w_{aa1}, \quad u_1 \equiv w_{aaa1}, \\
A_0 &\equiv \Delta z_1, \quad B_0 \equiv \Delta w_1, \\
A_1 &\equiv \Delta z_2, \quad B_1 \equiv \Delta w_2.
\end{aligned} \tag{5.58}$$

The system in Eq (5.57) expressed in terms of the new notation (5.58) becomes more readable:

$$x_0 A_0 + y_0 A_0^2 + u_0 A_0^3 = B_0 \tag{5.59}$$

$$x_0 + 2y_0 A_0 + 3u_0 A_0^2 = x_1 \tag{5.60}$$

$$y_0 + 3u_0 A_0 = y_1 \tag{5.61}$$

$$x_1 A_1 + y_1 A_1^2 + u_1 A_1^3 = B_1 \tag{5.62}$$

$$x_1 + 2y_1 A_1 + 3u_1 A_1^2 = 0 \tag{5.63}$$

$$y_1 + 3u_1 A_1 = 0. \tag{5.64}$$

Let's solve the system backward. Doing Eq (5.62) $- A_1 \times$ Eq (5.63) we have

$$\begin{aligned}
(x_1 A_1 + y_1 A_1^2 + u_1 A_1^3) - (x_1 A_1 + 2y_1 A_1^2 + 3u_1 A_1^3) &= B_1 - 0 \\
y_1 A_1^2 + u_1 A_1^3 - 2y_1 A_1^2 - 3u_1 A_1^3 &= B_1 \\
-y_1 A_1^2 - 2u_1 A_1^3 &= B_1.
\end{aligned} \tag{5.65}$$

Doing $A_1^2 \times \text{Eq (5.64)} + \text{Eq (5.65)}$ we have:

$$(y_1 A_1^2 + 3u_1 A_1^3) + (-y_1 A_1^2 - 2u_1 A_1^3) = B_1 \Rightarrow u_1 A_1^3 = B_1$$

$$\therefore u_1 = \frac{B_1}{A_1^3} \Rightarrow \boxed{u_1 \equiv w_{aaa1} = \frac{\Delta w_2}{(\Delta z_2)^3}}, \quad (5.66)$$

where in Eq (5.66) I have used the change of variables in Eq (5.58) to go back to the original notation. Substituting the result given by Eq (5.66) into Eq (5.64) we have:

$$y_1 = -3u_1 A_1 = -3 \left(\frac{\Delta w_2}{\Delta z_2^3} \right) \Delta z_2 \Rightarrow \boxed{y_1 \equiv w_{aa1} = -3 \frac{\Delta w_2}{(\Delta z_2)^2}}. \quad (5.67)$$

Inserting Eqs (5.66)-(5.67) into Eq (5.62) we have:

$$x_1 A_1 + y_1 A_1^2 + u_1 A_1^3 = B_1$$

$$w_{a1} \Delta z_2 = - \left(-3 \frac{\Delta w_2}{\Delta z_2^2} \right) \Delta z_2^2 - \left(\frac{\Delta w_2}{\Delta z_2^3} \right) \Delta z_2^3 + \Delta w_2$$

$$w_{a1} \Delta z_2 = 3\Delta w_2 - \Delta w_2 + \Delta w_2$$

$$w_{a1} \Delta z_2 = 3\Delta w_2 \Rightarrow \boxed{x_1 \equiv w_{a1} = 3 \frac{\Delta w_2}{\Delta z_2}}. \quad (5.68)$$

Now we have just three equations to solve. Doing Eq (5.59) $- A_0 \times \text{Eq (5.60)}$

$$(x_0 A_0 + y_0 A_0^2 + u_0 A_0^3) - A_0(x_0 + 2y_0 A_0 + 3u_0 A_0^2) = B_0 - x_1 A_0$$

$$y_0 A_0^2 + u_0 A_0^3 - 2y_0 A_0^2 - 3u_0 A_0^3 = B_0 - x_1 A_0$$

$$-y_0 A_0^2 - 2u_0 A_0^3 = B_0 - x_1 A_0. \quad (5.69)$$

Doing Eq $A_0^2 \times \text{Eq (5.61)} + \text{Eq (5.69)}$ we obtain

$$A_0^2(y_0 + 3u_0 A_0) + (-y_0 A_0^2 - 2u_0 A_0^3) = y_1 A_0^2 + (B_0 - x_1 A_0)$$

$$3u_0 A_0^3 - 2u_0 A_0^3 = y_1 A_0^2 + B_0 - x_1 A_0 \Rightarrow u_0 A_0^3 = -x_1 A_0 + y_1 A_0^2 + B_0$$

$$u_0 = -x_1 \frac{1}{A_0^2} + y_1 \frac{1}{A_0} + B_0 \frac{1}{A_0^3} \quad (5.70)$$

$$\boxed{u_0 \equiv w_{aaa0} = \frac{\Delta w_1}{\Delta z_1^3} - 3 \frac{\Delta w_2}{\Delta z_1 \Delta z_2} \left(\frac{1}{\Delta z_1} + \frac{1}{\Delta z_2} \right)}. \quad (5.71)$$

Inserting Eq (5.71) into Eq (5.61) we have:

$$y_0 + 3u_0A_0 = y_1 \Rightarrow y_0 = y_1 - 3u_0A_0 \quad (5.72)$$

$$\begin{aligned} y_0 &= -3\frac{\Delta w_2}{\Delta z_2^2} - 3\left[\frac{\Delta w_1}{\Delta z_1^3} - 3\frac{\Delta w_2}{\Delta z_1\Delta z_2}\left(\frac{1}{\Delta z_1} + \frac{1}{\Delta z_2}\right)\right]\Delta z_1 \\ y_0 &= -3\frac{\Delta w_2}{\Delta z_2^2} - 3\frac{\Delta w_1}{\Delta z_1^3} + 9\frac{\Delta w_2}{\Delta z_2}\left(\frac{1}{\Delta z_1} + \frac{1}{\Delta z_2}\right) \\ \therefore y_0 \equiv w_{a0} &= -3\frac{\Delta w_1}{\Delta z_1^3} + 3\frac{\Delta w_2}{\Delta z_2}\left(\frac{3}{\Delta z_1} + \frac{2}{\Delta z_2}\right). \end{aligned} \quad (5.73)$$

Substituting the Eqs (5.71) and (5.73) into the Eq (5.59) and solving for x_0 we have:

$$x_0A_0 + y_0A_0^2 + u_0A_0^3 = B_0 \Rightarrow x_0 = w_{a0} = 3\frac{\Delta w_1}{\Delta z_1} - 3\frac{\Delta w_2}{\Delta z_2}\left(2 + \frac{\Delta z_1}{\Delta z_2}\right). \quad (5.74)$$

5.2 CAMB for Late Dark Energy:

The previous sections present many parametrizations for the dark energy equation of state. If we want to go further and probe these parameters, namely $w_0, w_1, w_2, \dots, w_n$ for a given redshift window, with real data, then in practice we have to implement these equations in a Boltzmann code, say CAMB² or CLASS³ and use a code for parameter exploration such as Cobaya⁴ or CosmoSIS⁵, and then a tool for statistical analysis, as for example Getdist⁶.

All these previous parametrizations, and some others, such as the hyperbolic tangent, given by equation (21) in the paper [105], and the user-defined number of redshift bins for the constant binned $w(z)$, are implemented in our modified version of CAMB that we call CAMBLateDE. This code is publicly available at <https://github.com/SBU-COSMOLIKE/CAMBLateDE>. A Jupyter Notebook illustrating these parametrizations can be found at [LateDE_demo.ipynb](#)⁷.

Table xxx show all the current parametrizations implemented in CAMBLateDE at the writing of this Thesis

²<https://camb.readthedocs.io/en/latest/#>

³https://lesgourg.github.io/class_public/class.html

⁴<https://cobaya.readthedocs.io/en/latest/index.html>

⁵<https://cosmosis.readthedocs.io/en/latest/#>

⁶<https://getdist.readthedocs.io/en/latest/#>

⁷If the reader wants to test this CAMBLateDE and faces any problem (installation, compilation, etc), please don't hesitate to contact me at diogo.henrique@unesp.br or open an issue on GitHub.

DEmodel	Description	CAMB parameter
1	$w(z)$ constant	w_0
2	w_0-w_a	w_0, w_1
3	Constant bin $w(z)$ with 2 bins	w_0, w_1, z_1, z_2
4	Constant bin $w(z)$ with 3 bins	$w_0, w_1, w_2, z_1, z_2, z_3$
5	Constant bin $w(z)$ with 5 bins	$w_0, \dots, w_4, z_1, \dots, z_5$
6	Constant bin $w(z)$ with 10 bins	$w_0, \dots, w_9, z_1, \dots, z_{10}$
7	Linear bin $w(z)$ with 2 bins	w_0, w_1, z_1, z_2
8	Linear bin $w(z)$ with 3 bins	$w_0, w_1, w_2, z_1, z_2, z_3$
9	Linear bin $w(z)$ with 5 bins	$w_0, \dots, w_4, z_1, \dots, z_5$
10	Linear bin $w(z)$ with 10 bins	$w_0, \dots, w_9, z_1, \dots, z_{10}$
11	Quadratic bin $w(z)$ with 2 bins	w_0, w_1, z_1, z_2
12	Quadratic bin $w(z)$ with 3 bins	$w_0, w_1, w_2, z_1, z_2, z_3$
13	Quadratic bin $w(z)$ with 5 bins	$w_0, \dots, w_4, z_1, \dots, z_5$
14	Cubic bin $w(z)$ with 2 bins	w_0, w_1, z_1, z_2
15	Cubic bin $w(z)$ with 3 bins	$w_0, w_1, w_2, z_1, z_2, z_3$
16	Cubic bin $w(z)$ with 5 bins	$w_0, \dots, w_4, z_1, \dots, z_5$
17	Hyperbolic tangent	$w_0, w_1, z_1, \text{sigma}$
18	Constant bin $w(z)$	$\text{max_num_of_bins}, w_knot, z_knot$

Table 5.4: All the parametrizations implemented so far in CAMBLateDE.

In table 5.4 the dark energy model is selected in CAMBLateDE via the parameter `DEmodel` which is an integer between 1 and 18 that corresponds to a given equation of state parameterization. The parameters $w_0 \dots w_9, z_1 \dots z_{10}$ and sigma are all floats. Figure 5.1 displays the equation of state and energy density for dark energy. It is assumed two redshift bins $[0, 1.5]$ and $[1.5, 3]$, and beyond the second redshift bin, we recover the usual scenario of the cosmological constant, i.e. $w = -1$. Note the continuity of the functions at each bin's transition.

The `DEmodel=18` is a generalization for the models 3 - 6. The user should provide to CAMBLateDE the number of bins with the parameter `max_num_of_bins` $\equiv N_{\text{bins}}$, and the corresponding equation of state and redshift bins up til N_{bins} . This is passed via the list `w_knot=[w1, ..., wNbins]` and `z_knot=[z1, ..., zNbins]`.

One of the simplest single parameter extensions beyond the Λ CDM model can be obtained by allowing $\sum m_\nu$ to vary. Constraints for this model are frequently done by many survey collaborations. For example, using the likelihood for the power spectra of temperature (TT), temperature-polarization (TE), polarization-polarization (EE) and its associated low- ℓ polarization (lowE) together with CMB lensing and BAO, the *Planck* collaboration [55] found an upper bond of $\sum m_\nu < 0.12$ eV at 95% CL. The same result was obtained using the gravitational lensing mass

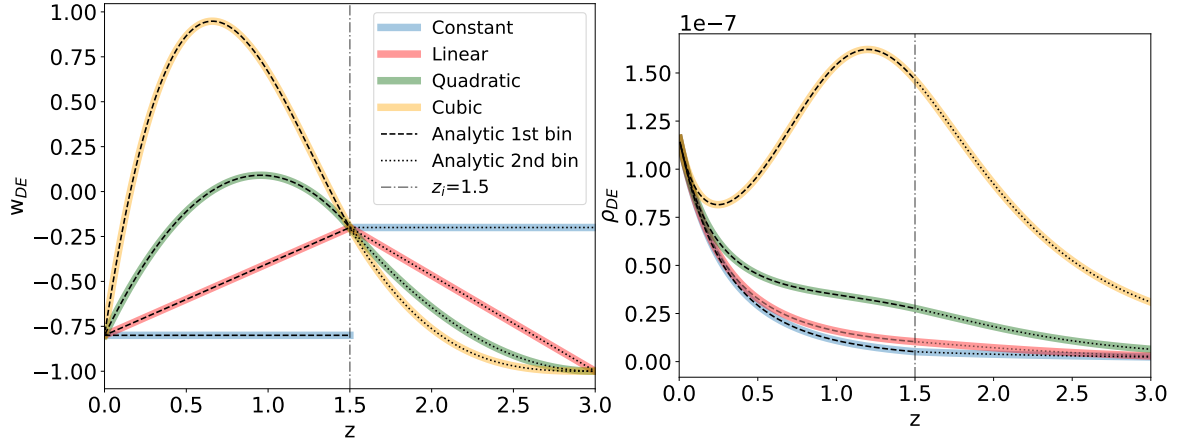


Figure 5.1: Dark energy equation of state on the left and the corresponding energy density on the right. Colored lines are outputs of CAMBLat_{eDE}, and the black dashed and dotted lines are the respective analytic solutions. We adopted two redshift bins from $[0, 1.5]$ and $[1.5, 3]$. The vertical dashed-dotted line is the redshift boundary between these two bins.

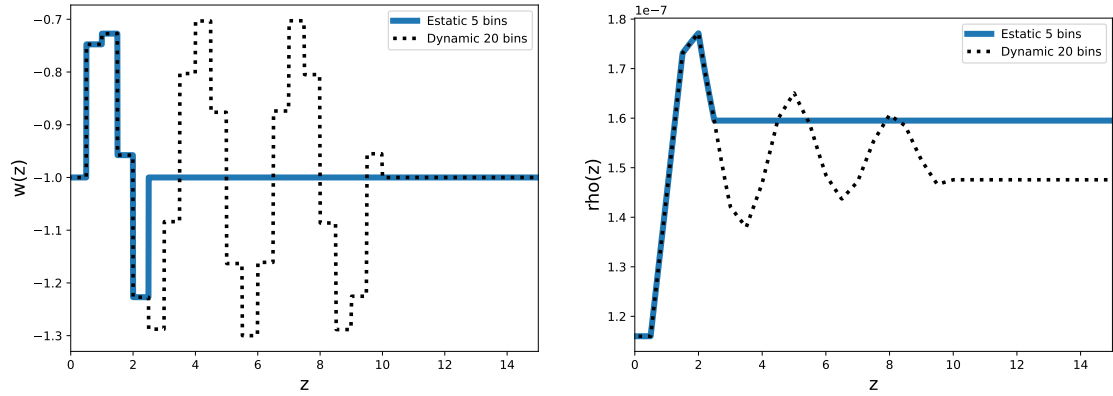


Figure 5.2: Dark energy equation of state on the left and the corresponding energy density on the right. Comparison between $DE_{model}=5$ with fixed 5 bins and $DE_{model}=18$ that allows the user to provide the desired amount of bins, in this case, 20 redshift bins between 0 and 10.

map of the Atacama Cosmology Telescope (ACT) DR6 combined with BAO data and the Planck CMB anisotropies (c.f. Eq. 10 in [106]).

The South Pole Telescope third-generation (SPT-3G) 2018 EE and TT power spectra, combined with *Planck* and BAO, reports an upper bound in the neutrino mass as $\sum m_\nu < 0.13$ eV at 95% CL [66]. However, this constraint by SPT-3G considered a prior on τ (so it is a 5+1 parameter extension) whilst the *Planck* and ACT analysis kept τ as a free parameter. Using the three two-point correlation function (3x2pt) from Dark Energy Survey (DES) in combination with data of Redshift Space Distortion (RSD), BAO, SNIa, *Planck* CMB and CMB lensing, the DES Y3 reports a constraint in the neutrino mass as being $\sum m_\nu < 0.13$ eV at 95% CL [62]. The use of this large external dataset on top of DES data is argued in [62] (see also [61]) due to the weaker power of the 3x2pt analysis in providing enough constraining resolution on neutrino masses, so its marginalized posterior can accommodate large $\sum m_\nu$ which is highly limited by its prior. Last but not least survey we highlight the reported bounds on neutrino masses from the recent BAO, quasar, and Lyman- α forest measurements from the Dark Energy Spectroscopic Instrument (DESI) Y1 [107]. DESI BAO with *Planck* CMB provides stringent constraint; $\sum m_\nu < 0.072$ eV at 95% CL. These upper bound values for $\sum m_\nu$ are very close to the minimum limit of the IH, which is consistent with the mildly preferred NH results from neutrino oscillation experiments such as T2K and NO ν A [108, 109].

I implemented all the parameterizations in table 5.4 in a modified version of the Boltzmann solver CAMB. We then analyzed some of these models (in particular, the constant binned $w(z)$ for five and ten redshift bins and the hyperbolic equation of state) with BAO data combined with CMB, CMB lensing, and supernovae. In the following section, I show the results of the paper [110].

5.2.1 Late-Time Dark Energy and Massive Neutrinos with DESI 2024 BAO data

Dark energy models under analysis

The BAO dataset from the first year of the Dark Energy Survey Spectroscopic Instrument (DESI) [107] combined with the Type Ia supernovae data from the fifth year of the Dark Energy Survey (DESY5) [111] and CMB anisotropies from *Planck* [55] provides a hint for deviation from the flat Λ CDM with a preference of 3.9σ for $w_0 w_a$ CDM. This same analysis was performed using supernovae from two other

different surveys. Using supernovae from Union3 [112] and Pantheon+ [113], it is found a 3.5σ and 2.5σ preference for w_0w_a CDM over the Λ CDM, respectively. The DESI Collaboration used the w_0w_a parametrization for the dark energy equation of state, where $w(a) = w_0 + w_a(1 - a)$. The reconstruction of the $w(z)$ from the DESI analysis shows that the dark energy equation of state possesses a pantheon behavior ($w < -1$) for redshifts $z > 0.5$ and a thawing behavior ($w > -1$) nowadays.

In the work [110], we analyze the dark energy models $\text{DE}_{\text{model}} = 5, 6$, and 17 as shown in table 5.4. The $\text{DE}_{\text{models}} = 5$ and 6 are the constant $w(z)$ in five and ten redshift bins. This parametrization was discussed in section 5.1.2. To perform the analysis of these two models, we decided to use the binned strategy

For five bins: $[0.3, 0.6, 0.9, 1.2, 1.5]$

For ten bins: $[0.1, 0.2, 0.3, 0.4, 0.5, 0.6, 0.7, 0.8, 0.9, 1.0]$

We turn to the hyperbolic model ($\text{DE}_{\text{model}}=17$), which we named by “transition model” in [110] with equation of state given by

$$w(z) = w_0 + \left(\frac{w_1 - w_0}{2} \right) \left[1 + \tanh \left(\frac{z - z_c}{\sigma} \right) \right]. \quad (5.75)$$

This model represents a smooth transition from the equation of state at early times (represented by w_1) to the dark energy equation of state at late times (represented by w_0). These asymptotic regimes are reached for $z - z_c \gg \sigma$ (representing the high redshift limit) and $|z - z_c| \gg \sigma$ with $z < z_c$ (representing the low redshift limit). z_c represents the redshift at which $w(z_c) = (w_0 + w_1)/2$. σ is the parameter that controls the redshift width of the transition. The energy density of this model is given by

$$\rho_{\text{DE}} = \Omega_{\text{DE}} \rho_{\text{cr}} \exp \left(3 \int_0^z \frac{1 + w(z')}{1 + z'} dz' \right), \quad (5.76)$$

and it is solved numerically by CAMBLateDE.

Dataset, dataset combination and analysis methodology

To analyze the different dark energy models, we used data from DESI Y1 BAO, Planck 2018 CMB, and ACT DR4 CMB, both temperature and polarization anisotropies. CMB lensing from ACT DR6. Finally, the Type Ia Supernovae from DESY5, Union3, Pantheon+ or Pantheon. The different dataset combinations are

displayed below:

- Planck 2018 (P18) + DESI-Y1 BAO (DESI) + Pantheon (Pan);
- Planck 2018 (P18) + DESI-Y1 BAO (DESI) + Pantheon + (Pan+);
- Planck 2018 (P18) + DESI-Y1 BAO (DESI) + Union-3 (Uni3);
- Planck 2018 (P18) + DESI-Y1 BAO (DESI) + DES-Y5 (DESY5).

Constraints on the neutrino mass sum are assessed by changing the the CMB dataset from Planck to ACT+Planck ($\ell < 650$):

- ACT DR4 (ACT) + Planck 2018 $\ell < 650$ (P650) + DESI + Pan;
- ACT DR4 (ACT) + Planck 2018 $\ell < 650$ (P650) + DESI + Pan+;
- ACT DR4 (ACT) + Planck 2018 $\ell < 650$ (P650) + DESI + Uni3;
- ACT DR4 (ACT) + Planck 2018 $\ell < 650$ (P650) + DESI + DESY5.

Finally, constraints on the neutrino mass sum also is obtained with the CMB lensing (probe of large scale structure) from ACT DR6:

- ACT + P650 + ACT DR6 Lensing (ACTL) + DESI + Pan;
- ACT + P650 + ACT DR6 Lensing (ACTL) + DESI + Pan+;
- ACT + P650 + ACT DR6 Lensing (ACTL) + DESI + Uni3;
- ACT + P650 + ACT DR6 Lensing (ACTL) + DESI + DESY5.

The analysis of these dark energy models and constraints on neutrino mass sum with these dataset combinations is performed with the Markov Chain Monte Carlo (MCMC), where we obtain the posterior distribution of the model parameters. The priors used are displayed in Table 1 of [110].

Results for the Transition model

Figure 5.3 displays the 2D region preference for the parameters of the transition model (with fixed $\sigma = 0.05$) in addition to the matter energy density parameter Ω_m . Except for the Pantheon, the $\{w_0, w_1\}$ plane for the Pantheon+, Union3, and DESY5 is contained (at 95% confidence level) in the quadrant where $w_1 < -1$ and $w_0 > -1$ which suggests a transition from a phantom limit to a non-phantom regime. The transition redshift z_c is weakly constrained at around the value of $z_c \approx 0.6$.

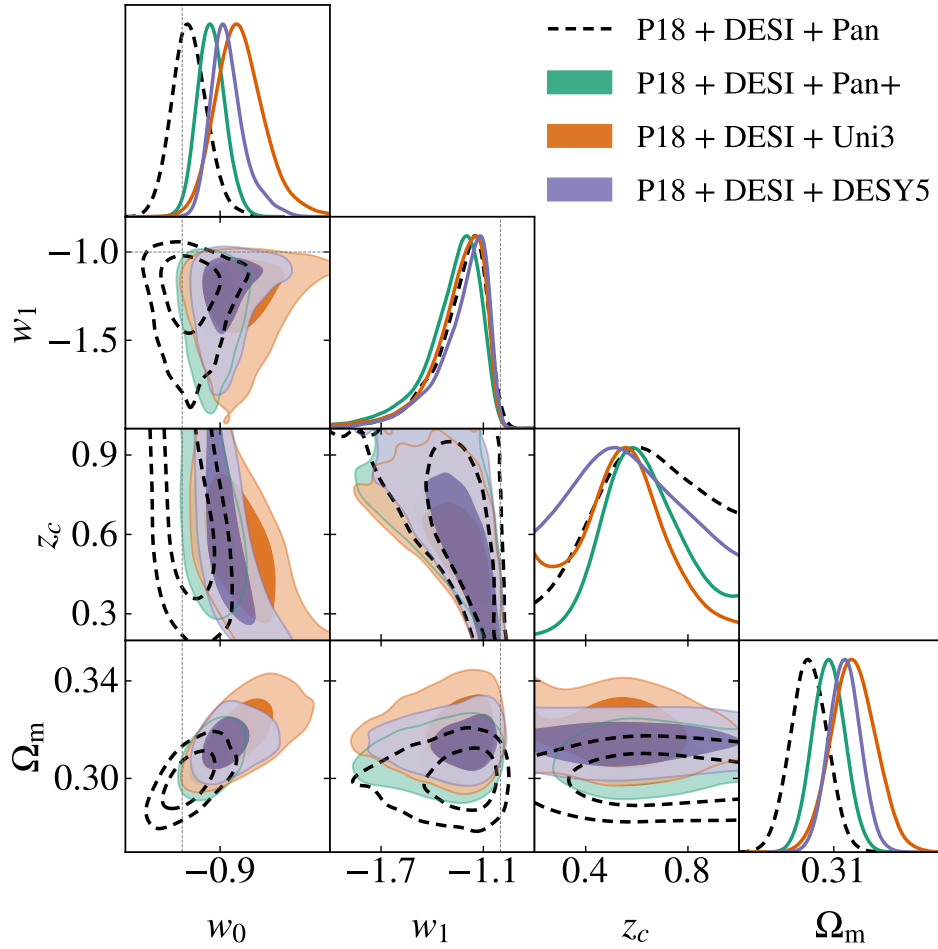


Figure 5.3: Two-dimensional contours region (68% and 95%) for the transition model parameters and the matter energy density parameter.

Results for the binned $w(z)$ model

We analyzed the constant binned $w(z)$ for 5 and 10 redshift bins. In opposition to the w_0w_a and the transition models, the binned $w(z)$ intends to be the most general as possible as we increase the number of bins. Figure 5.4 shows the reconstruction for these two cases. For both scenarios within 1σ or 2σ , the model-agnostic is consistent with the cosmological constant, $w = -1$, and/or with the best-fit of the w_0w_a parametrization. It is challenging to conclude much from this result because this model-agnostic approach has many parameters⁸, and the error bars among the bins are correlated.

⁸In the case studied in our work [110] it has 5 and 10 free parameters, but notice that CAMB_{lateDE} allows an arbitrary number of parameters for the `DEmodel=18`. Of course, one can add as many bins as possible with this feature, but this may come at the cost of computational expenses and low performance.

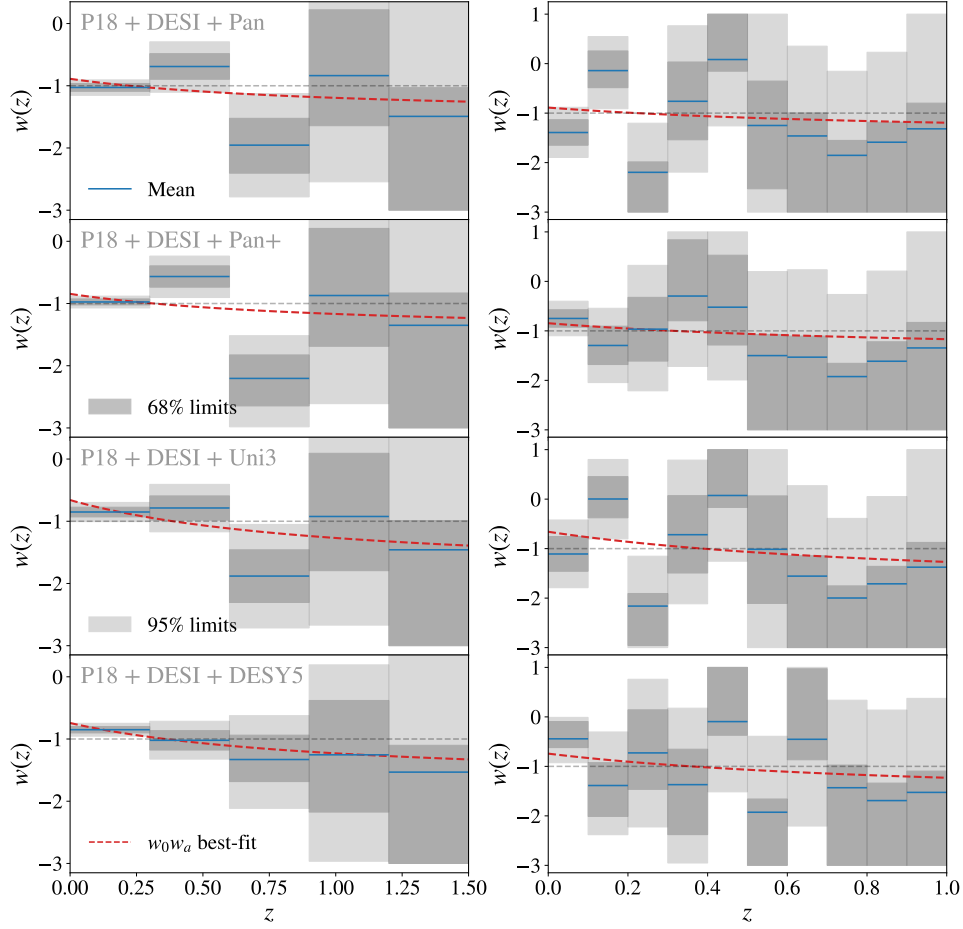


Figure 5.4: Reconstruction of the dark energy equation of state for 5 redshift bins (left) and 10 redshift bins (right), assuming CMB + DESI for the different types of supernovae datasets from Pantheon (Pan), Pantheon+ (Pan+), Union3 (Uni3), and DES year 5 (DESY5). The blue horizontal line is the mean value of w in each redshift bin with its 68% and 95% error bars. For comparison, the red dotted line is the best-fit of w_0w_a parametrization.

The Principal Component Analysis (PCA) method allows us to overcome this difficulty [114–118]. This method consists of extracting the covariance matrix of the w_i parameters. This forms a real and asymmetric matrix, which can be diagonalized, and we can obtain the eigenfunctions, $e_i(z)$, and the eigenvalues, α_i , where the correlation, assuming the new PC basis, between α_i and α_j for $i \neq j$, is zero. Therefore, we can expand the dark energy equation of state into this new PCA basis as follows:

$$w(z) = -1 + \sum_{i=1}^{N_{\text{bins}}} \alpha_i e_i(z), \quad (5.77)$$

where the sum term accounts for possible deviation from the cosmological constant -1 corresponding to $\alpha = (0, \dots, 0)$. Figure 5.5 shows principal components for the cases of constant w in 5 and 10 redshift bins for DESI + Planck 2018 + Pantheon+ dataset.

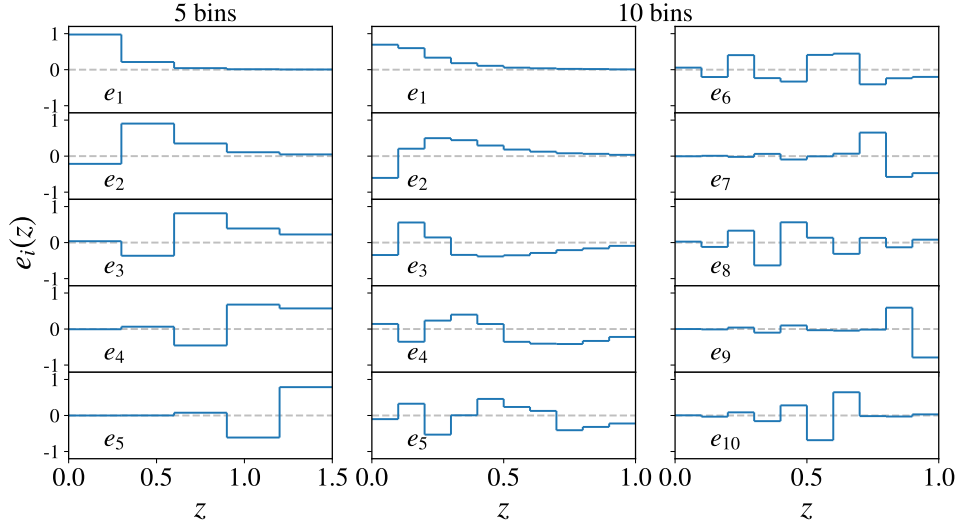


Figure 5.5: Principal components $e_i(z)$ for the constant binned $w(z)$ assuming 5 redshift bins (left panel), and 10 redshift bins (middle and right panels) for the dataset combination DESI + Planck 2018 + Pantheon+.

Figure 5.6 displays the 1-D marginalized amplitude, α_i , of the principal component, $e_i(z)$, for each dataset combination of DESI + Planck 2018 + Supernovae. The top row shows the case of $N_{\text{bins}} = 5$, and the middle and bottom rows display the case of $N_{\text{bins}} = 10$. For the 5 bins, α_1 (except for Pantheon) and α_3 are the amplitudes that exhibit a mild deviation from 0 compared to the remaining other amplitudes $\alpha_{i=2,4,5}$.

Figure 5.7 shows the “filtered” $w(z)$ which is defined through the Eq. (5.77) for the Principal Component $e_i(z)$ with smaller error bars on the Principal Components amplitude α_i , that deviates from zero. In the case of 5 bins, this corresponds to α_1 and α_3 , and in the case of 10 bins this corresponds to α_1 , α_4 and α_5 .

For the case of 5 redshift bins, the Figure 5.7 shows that the binned $w(z)_{\text{filtered}}$ have a different behavior not captured by the $w_0 w_a$ parametrization: $w(z)_{\text{filtered}}$ at high redshift ($1.2 < z < 1.5$) start at a value $w < -1$, goes to lower values in $0.6 < z < 1.2$ (except for P18+DESI+DESY5 which is consistent with -1 at 2σ), then it crosses the phantom barrier ($w = -1$) with $w > -1$ for $0.3 < z < 0.6$ and finally have a thanwing behavior nowadays ($0 < z < 0.3$) with $w_0 > -1$. This result is consistent with previous analysis from the literature using the $w_0 w_a$

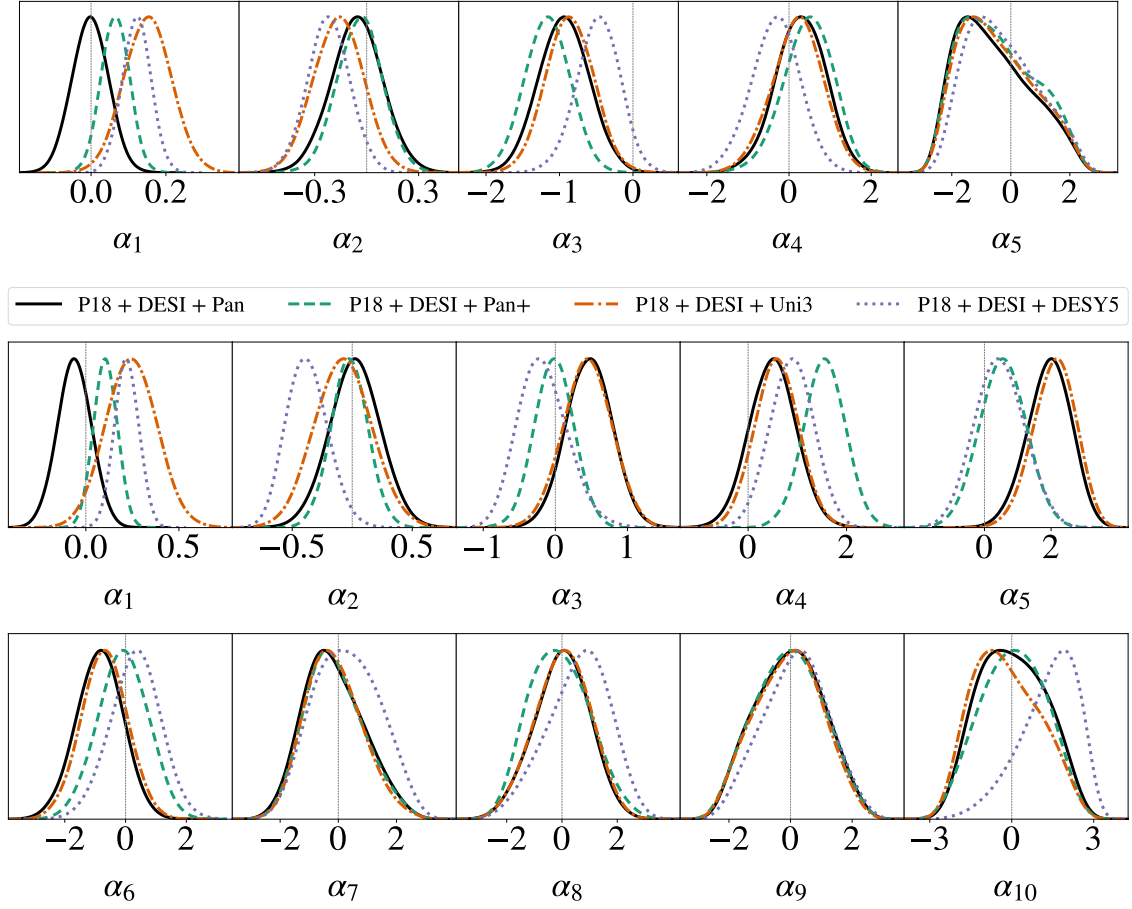


Figure 5.6: The 1-dimensional marginalized constraint on the PC amplitude, α_i , for the constant w in 5 redshift bins (top row), and 10 redshift bins (middle and bottom rows). The cosmological constant corresponds to $\alpha_i = 0$ as indicated by the vertical gray line at each panel.

[107, 111, 112] that shows a thawing behavior for the dark energy EoS today.

Constraints on $\sum m_\nu$

The previous analysis has assumed two massless neutrinos and one massive neutrino with $\sum m_\nu = 0.06$ eV, i.e. the minimal value for the sum mass in the normal hierarchy scenario. In this Section, we consider the neutrino mass sum $\sum m_\nu$ as a free parameter, assuming three massive neutrinos and a degenerate hierarchy with a uniform prior in the neutrino mass sum in the range $[0.01, 0.6]$. The upper limit on $\sum m_\nu$ at 2σ is shown in table 5.5 for different dark energy models. Notice that a factor of 1.5 - 1.9 enlarges the constraints obtained with the ACT+Pl650 dataset compared to P18. However, including ACT DR6 lensing data brings back the constraints on the same level as P18.

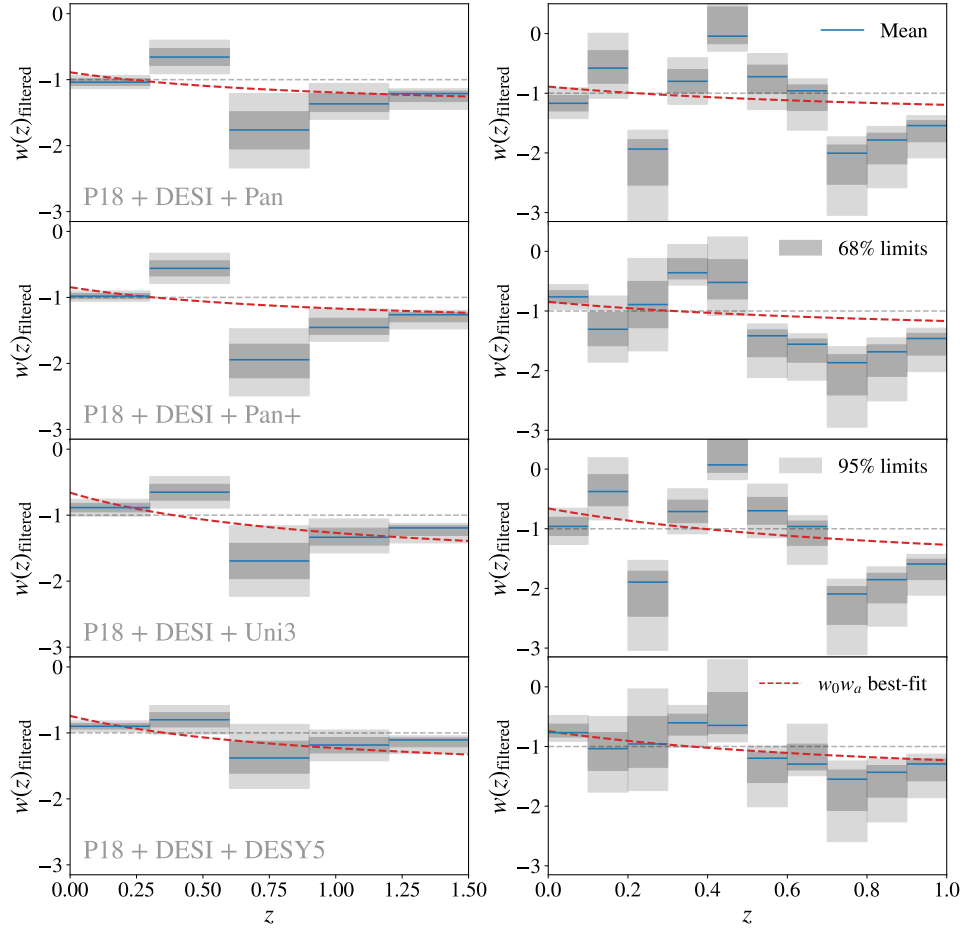


Figure 5.7: Filtered (see the text) reconstruction of the dark energy equation of state for 5 redshift bins (left) and 10 redshift bins (right). The blue line is the mean and gray-shaded regions are 68% and 95% confidence interval. For comparison, the reconstruction for w_0w_a is also displayed in the red dashed line.

The 1D marginalized constraints on $\sum m_\nu$ for different dark energy models and dataset combination is shown in Figure 5.8. The w_0w_a parameterization relaxes the constraint on $\sum m_\nu$ by a factor 1.8 - 2.2 compared to Λ CDM model. The binned $w(z)$ with 5 redshift bins possesses a similar behavior for datasets with Pantheon and Pantheon+. For Union3 and DESY5 with ACT+P650 and ACT+P650+ACTL, the binned $w(z)$ with 10 redshift bins have tighter constraints than w_0w_a ; one would not expect that since the former model has a more significant number of parameters compared to w_0w_a . However, previous work has shown that constraints on $\sum m_\nu$ prefer negative values under the Λ CDM model [119]. This effect is enhanced by the “lensing anomaly” seen in the Planck 2018 temperature and polarization power spectra [55]. Therefore, if negative values of $\sum m_\nu$ are preferred, we have tighter constraints in the Bayesian analysis once we assume physical priors that

Upper limit (95% C. L.) on $\sum m_\nu$ (eV)				
P18 + DESI +	Pan	Pan+	Uni3	DESY5
Λ CDM	0.10	0.11	0.11	0.12
w_0w_a	0.18	0.17	0.21	0.19
5 – bin $w(z)$	0.19	0.19	0.19	0.18
10 – bin $w(z)$	0.16	0.16	0.16	0.15
ACT+P650 + DESI +	Pan	Pan+	Uni3	DESY5
Λ CDM	0.16	0.18	0.18	0.20
w_0w_a	0.31	0.28	0.34	0.32
5 – bin $w(z)$	0.32	0.33	0.32	0.31
10 – bin $w(z)$	0.26	0.26	0.26	0.24
ACT+P650 + ACTL + DESI +	Pan	Pan+	Uni3	DESY5
Λ CDM	0.10	0.11	0.11	0.12
w_0w_a	0.22	0.21	0.24	0.23
5 – bin $w(z)$	0.23	0.23	0.23	0.22
10 – bin $w(z)$	0.17	0.18	0.17	0.17

Table 5.5: Upper limit on the sum of the neutrino mass at 2σ for different dataset combinations and dark energy models.

exclude negative values of neutrino masses. This may be the case of the constant $w(z)$ in 5 and 10 redshift bins, as seen by the figure 5.8, but we can not confirm this hypothesis with our current analysis. This is reserved for future work.

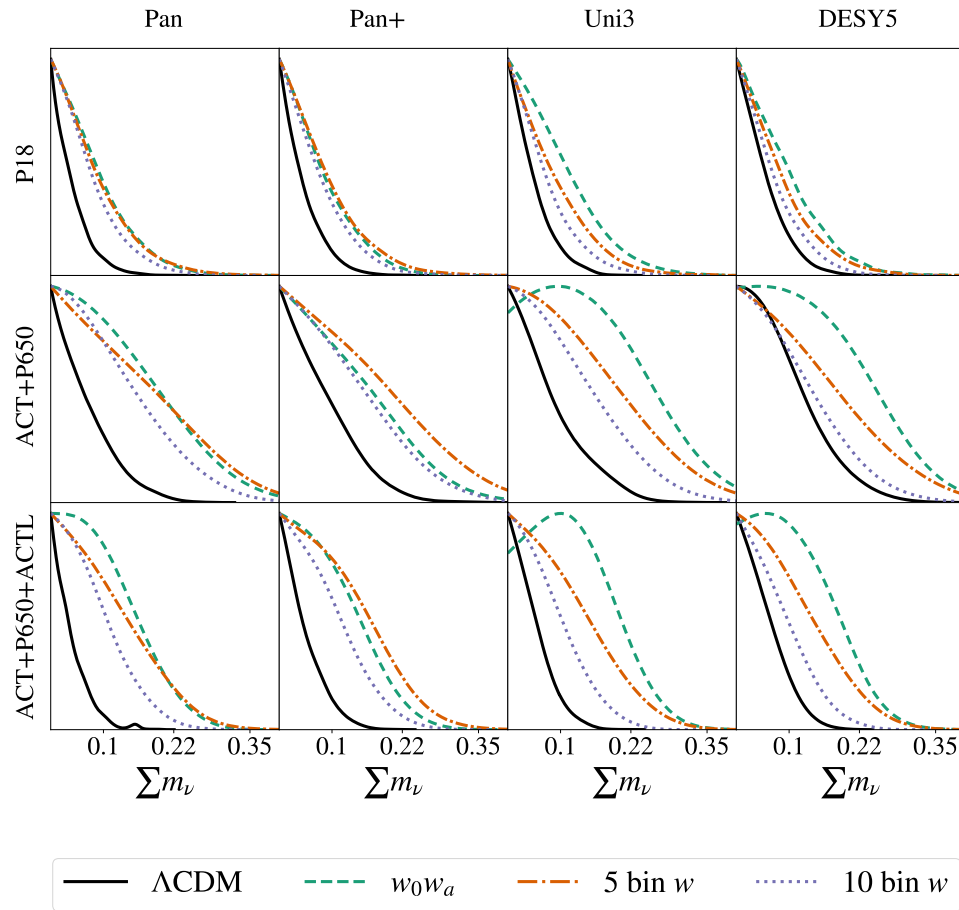


Figure 5.8: 1D marginalized constraint on Σm_ν for different dark energy models and dataset combination.

Conclusion

This Thesis is divided into two parts. The first part presents the basics of cosmology. Chapter 1 presents the background equations, the Robertson-Walker metric, the angular and luminosity distances, the Friedmann equations that describe a homogeneous and isotropic universe at a large scale, and the energy density evolution for each fluid component. Chapter 2 explains the gauge invariant method and how to build gauge-invariant perturbations. In the following, a detailed description and steps are provided for obtaining those gauge-invariant perturbation quantities. Chapter 3 discusses the initial conditions in the early universe required to solve the Boltzmann-Einstein equations for each component: cold dark matter, neutrinos, and the coupled photons-baryons fluid. An approximate solution is also provided within the early-times and super-horizon initial conditions scenario.

The second part of this Thesis has two chapters, 4 and 5, that summarize the research developed during the PhD. Chapter 4 is the result of the implementation at the background level in a Boltzmann-Einstein solver of the neutrino-assisted early dark energy model. This allows us to access the fraction of early-dark energy since it is a background quantity, and a comparison from the literature shows that the numeric implementation and the theoretical prediction agree with each other on average. However, it is challenging to find parameters (coupling strength and self-interaction potential) that push the fraction of early-dark energy close to 10%. Despite these results, we state that a more detailed analysis, which includes the perturbation equations and performs a Bayesian analysis, is required.

Finally, Chapter 5 explains in detail the implementation in the Boltzmann-Einstein solver several extensions for Λ , the cosmological constant, and beyond the $w_0 w_a$ parametrization. Starting from the most straightforward constant equation of state in bins of redshift to more complex parametrization such as linear, quadratic, and cubic. Despite these multiple implementations, we ended with a work that performs the Bayesian inference for the constant dark energy equation-of-state in five and ten redshift bins. The paper goes beyond and includes a more recent implementation of the transition model, given by a hyperbolic tangent function and the monodromic K-essence model. For this analysis, we use a dataset combination that includes BAO from DESI, CMB from Planck 2018, ACT DR4,

CMB lensing from ACT DR6, and supernovae from Pantheon, Pantheon+, Union3 or DES Y5. We also reassess the constraints on the sum of the neutrino masses for the constant binned dark energy equation of state and compare with the Λ CDM model and the w_0w_a parametrization.

The results of this work are consistent with those using the w_0w_a parametrization that presents hints for dynamical dark energy with thawing behavior. However, we highlight that it may be essential to use parametrizations other than the w_0w_a to explore further possible complex behavior of the dark energy. We also underline the importance of reassessing the neutrino mass sum, given that these two components have a degenerate effect on the structure formation.

Both results presented in the second part of this thesis contribute to actual (at the moment of this written) discussions in the literature of interesting problems such as the early-dark energy models, dynamical dark energy models, and massive neutrinos. In all cases, it is essential to reappraise these analyses in light of synergies of datasets from multi-probe surveys coming in the following decades, such as the LSST, Euclid, and the Roman Space Telescope—precise measurements of BAO, 3×2 pt, supernova, and weak gravitation lensing require outstanding control of systematic errors. In parallel, other statistical methods beyond the Bayesian analysis, such as profile likelihood, are being used and improved. All these efforts, in combination, have the potential to shed light on the nature of dark energy.

Bibliography

- [1] D. Baumann, *Cosmology*. Cambridge University Press, 7 2022.
- [2] S. Weinberg, *Gravitation and Cosmology: Principles and Applications of the General Theory of Relativity*. New York: John Wiley and Sons, 1972.
- [3] T. M. Davis and C. H. Lineweaver, “Expanding confusion: common misconceptions of cosmological horizons and the superluminal expansion of the universe,” *Publ. Astron. Soc. Austral.*, vol. 21, p. 97, 2004.
- [4] E. Hubble, “A Relation between Distance and Radial Velocity among Extra-Galactic Nebulae,” *Contributions from the Mount Wilson Observatory*, vol. 3, pp. 23–28, Jan. 1929.
- [5] S. Perlmutter *et al.*, “Measurements of Ω and Λ from 42 high redshift supernovae,” *Astrophys. J.*, vol. 517, pp. 565–586, 1999.
- [6] A. G. Riess *et al.*, “Observational evidence from supernovae for an accelerating universe and a cosmological constant,” *Astron. J.*, vol. 116, pp. 1009–1038, 1998.
- [7] B. P. Schmidt *et al.*, “The High Z supernova search: Measuring cosmic deceleration and global curvature of the universe using type Ia supernovae,” *Astrophys. J.*, vol. 507, pp. 46–63, 1998.
- [8] S. Dodelson, *Modern Cosmology*. Amsterdam: Academic Press, 2003.
- [9] P. A. R. Ade *et al.*, “Planck 2015 results. XIII. Cosmological parameters,” *Astron. Astrophys.*, vol. 594, p. A13, 2016.
- [10] A. A. Penzias and R. W. Wilson, “A Measurement of Excess Antenna Temperature at 4080 Mc/s,” *apj*, vol. 142, pp. 419–421, July 1965.
- [11] “The Scientific programme of Planck,” 4 2006.
- [12] M. Kamionkowski and A. Loeb, “Getting around cosmic variance,” *Phys. Rev. D*, vol. 56, pp. 4511–4513, 1997.

- [13] S. Dodelson and F. Schmidt, *Modern Cosmology*. Academic Press, Elsevier Science, 2 ed., 2020.
- [14] J. Lesgourgues, G. Mangano, G. Miele, and S. Pastor, *Neutrino Cosmology*. Cambridge University Press, 2013.
- [15] W. Hu and N. Sugiyama, “Anisotropies in the cosmic microwave background: An Analytic approach,” *Astrophys. J.*, vol. 444, pp. 489–506, 1995.
- [16] L. Amendola and S. Tsujikawa, *Dark Energy: Theory and Observations*. Cambridge University Press, 2010.
- [17] R. K. Sachs and A. M. Wolfe, “Perturbations of a Cosmological Model and Angular Variations of the Microwave Background,” *apj*, vol. 147, p. 73, Jan. 1967.
- [18] C.-P. Ma and E. Bertschinger, “Cosmological perturbation theory in the synchronous versus conformal Newtonian gauge,” 1 1994.
- [19] J. Chluba and D. Grin, “CMB spectral distortions from small-scale isocurvature fluctuations,” *Mon. Not. Roy. Astron. Soc.*, vol. 434, pp. 1619–1635, 2013.
- [20] M. Doran, C. M. Muller, G. Schafer, and C. Wetterich, “Gauge-invariant initial conditions and early time perturbations in quintessence universes,” *Phys. Rev. D*, vol. 68, p. 063505, 2003.
- [21] J. R. Shaw and A. Lewis, “Massive neutrinos and magnetic fields in the early universe,” *prd*, vol. 81, p. 043517, Feb. 2010.
- [22] M. Bucher, K. Moodley, and N. Turok, “The General primordial cosmic perturbation,” *Phys. Rev. D*, vol. 62, p. 083508, 2000.
- [23] E. Di Valentino, O. Mena, S. Pan, L. Visinelli, W. Yang, A. Melchiorri, D. F. Mota, A. G. Riess, and J. Silk, “In the realm of the hubble tension—a review of solutions,” *Classical and Quantum Gravity*, vol. 38, no. 15, p. 153001, 2021.
- [24] M. Kamionkowski and A. G. Riess, “The hubble tension and early dark energy,” *arXiv preprint arXiv:2211.04492*, 2022.

- [25] N. Schöneberg, G. F. Abellán, A. P. Sánchez, S. J. Witte, V. Poulin, and J. Lesgourgues, “The h_0 olympics: A fair ranking of proposed models,” *arXiv e-prints*, 2021.
- [26] T. Karwal and M. Kamionkowski, “Dark energy at early times, the hubble parameter, and the string axiverse,” *Physical Review D*, vol. 94, nov 2016.
- [27] V. Poulin, T. L. Smith, D. Grin, T. Karwal, and M. Kamionkowski, “Cosmological implications of ultralight axionlike fields,” *Physical Review D*, vol. 98, oct 2018.
- [28] P. Agrawal, F.-Y. Cyr-Racine, D. Pinner, and L. Randall, “Rock ‘n’ Roll Solutions to the Hubble Tension,” *arXiv e-prints*, p. arXiv:1904.01016, Apr. 2019.
- [29] V. Poulin, T. L. Smith, T. Karwal, and M. Kamionkowski, “Early dark energy can resolve the hubble tension,” *Physical Review Letters*, vol. 122, jun 2019.
- [30] T. L. Smith, V. Poulin, and M. A. Amin, “Oscillating scalar fields and the hubble tension: a resolution with novel signatures,” *Physical Review D*, vol. 101, no. 6, p. 063523, 2020.
- [31] L. Herold and E. G. M. Ferreira, “Resolving the Hubble tension with early dark energy,” *Phys. Rev. D*, vol. 108, no. 4, p. 043513, 2023.
- [32] V. Poulin, T. L. Smith, and T. Karwal, “The Ups and Downs of Early Dark Energy solutions to the Hubble tension: A review of models, hints and constraints circa 2023,” *Phys. Dark Univ.*, vol. 42, p. 101348, 2023.
- [33] E. V. Linder and G. Robbers, “Shifting the universe: early dark energy and standard rulers,” *Journal of Cosmology and Astroparticle Physics*, vol. 2008, p. 004, jun 2008.
- [34] R. R. Caldwell, M. Doran, C. M. Müller, G. Schäfer, and C. Wetterich, “Early quintessence in light of the wilkinson microwave anisotropy probe,” *The Astrophysical Journal*, vol. 591, no. 2, p. L75, 2003.
- [35] M. Bartelmann, M. Doran, and C. Wetterich, “Non-linear structure formation in cosmologies with early dark energy,” *Astronomy & Astrophysics*, vol. 454, no. 1, pp. 27–36, 2006.

- [36] T. L. Smith, V. Poulin, and M. A. Amin, "Oscillating scalar fields and the hubble tension: A resolution with novel signatures," *Physical Review D*, vol. 101, Mar 2020.
- [37] T. Karwal, M. Raveri, B. Jain, J. Khoury, and M. Trodden, "Chameleon early dark energy and the hubble tension," *Physical Review D*, vol. 105, no. 6, p. 063535, 2022.
- [38] M.-X. Lin, E. McDonough, J. C. Hill, and W. Hu, "Dark matter trigger for early dark energy coincidence," *Phys. Rev. D*, vol. 107, no. 10, p. 103523, 2023.
- [39] J. Sakstein and M. Trodden, "Early dark energy from massive neutrinos as a natural resolution of the hubble tension.," *Physical review letters*, vol. 124 16, p. 161301, 2019.
- [40] M. C. González, Q. Liang, J. Sakstein, and M. Trodden, "Neutrino-assisted early dark energy: theory and cosmology," *Journal of Cosmology and Astroparticle Physics*, vol. 2021, 2020.
- [41] P. Brax, C. van de Bruck, A.-C. Davis, J. Khoury, and A. Weltman, "Detecting dark energy in orbit: The cosmological chameleon," *Physical Review D*, vol. 70, no. 12, p. 123518, 2004.
- [42] M. Carrillo González, Q. Liang, J. Sakstein, and M. Trodden, "Neutrino-Assisted Early Dark Energy is a Natural Resolution of the Hubble Tension," 2 2023.
- [43] A. Coc, K. A. Olive, J.-P. Uzan, and E. Vangioni, "Big bang nucleosynthesis constraints on scalar-tensor theories of gravity," *Phys. Rev. D*, vol. 73, p. 083525, 2006.
- [44] A. L. Erickcek, N. Barnaby, C. Burrage, and Z. Huang, "Catastrophic Consequences of Kicking the Chameleon," *Phys. Rev. Lett.*, vol. 110, p. 171101, 2013.
- [45] A. L. Erickcek, N. Barnaby, C. Burrage, and Z. Huang, "Chameleons in the Early Universe: Kicks, Rebounds, and Particle Production," *Phys. Rev. D*, vol. 89, no. 8, p. 084074, 2014.
- [46] R. Fardon, A. E. Nelson, and N. Weiner, "Dark energy from mass varying neutrinos," *Journal of Cosmology and Astroparticle Physics*, vol. 2004, no. 10, p. 005, 2004.

- [47] R. D. Peccei, “Neutrino models of dark energy,” *Physical Review D*, vol. 71, no. 2, p. 023527, 2005.
- [48] A. W. Brookfield, C. van de Bruck, D. F. Mota, and D. Tocchini-Valentini, “Cosmology with massive neutrinos coupled to dark energy,” *Phys. Rev. Lett.*, vol. 96, p. 061301, 2006.
- [49] A. W. Brookfield, C. Van de Bruck, D. Mota, and D. Tocchini-Valentini, “Cosmology of mass-varying neutrinos driven by quintessence: theory and observations,” *Physical Review D*, vol. 73, no. 8, p. 083515, 2006.
- [50] K. Ichiki and Y.-Y. Keum, “Primordial neutrinos, cosmological perturbations in interacting dark-energy model: Cmb and lss,” *Journal of Cosmology and Astroparticle Physics*, vol. 2008, no. 06, p. 005, 2008.
- [51] U. França and R. Rosenfeld, “Fine tuning in quintessence models with exponential potentials,” *JHEP*, vol. 10, p. 015, 2002.
- [52] M. Maziashvili and V. Tsintsabadze, “Avoiding parameter fine-tuning in mass varying neutrino models of de?,” *arXiv preprint arXiv:2302.00380*, 2023.
- [53] P. Agrawal, F.-Y. Cyr-Racine, D. Pinner, and L. Randall, “Rock ‘n’ roll solutions to the Hubble tension,” *Phys. Dark Univ.*, vol. 42, p. 101347, 2023.
- [54] E. Di Valentino, A. Melchiorri, and J. Silk, “Cosmological constraints in extended parameter space from the Planck 2018 Legacy release,” *JCAP*, vol. 01, p. 013, 2020.
- [55] N. Aghanim *et al.*, “Planck 2018 results. VI. Cosmological parameters,” *Astron. Astrophys.*, vol. 641, p. A6, 2020. [Erratum: *Astron. Astrophys.* 652, C4 (2021)].
- [56] S. Alam *et al.*, “Completed SDSS-IV extended Baryon Oscillation Spectroscopic Survey: Cosmological implications from two decades of spectroscopic surveys at the Apache Point Observatory,” *Phys. Rev. D*, vol. 103, no. 8, p. 083533, 2021.
- [57] P. de Bernardis *et al.*, “A Flat universe from high resolution maps of the cosmic microwave background radiation,” *Nature*, vol. 404, pp. 955–959, 2000.

- [58] A. Balbi *et al.*, “Constraints on cosmological parameters from MAXIMA-1,” *Astrophys. J. Lett.*, vol. 545, pp. L1–L4, 2000. [Erratum: *Astrophys.J.Lett.* 558, L145–L146 (2001), Erratum: *Astrophys.J.* 558, L145–L146 (2001)].
- [59] A. H. Jaffe *et al.*, “Cosmology from MAXIMA-1, BOOMERANG and COBE / DMR CMB observations,” *Phys. Rev. Lett.*, vol. 86, pp. 3475–3479, 2001.
- [60] C. B. Netterfield *et al.*, “A measurement by Boomerang of multiple peaks in the angular power spectrum of the cosmic microwave background,” *Astrophys. J.*, vol. 571, pp. 604–614, 2002.
- [61] T. M. C. Abbott *et al.*, “Dark Energy Survey year 1 results: Cosmological constraints from galaxy clustering and weak lensing,” *Phys. Rev. D*, vol. 98, no. 4, p. 043526, 2018.
- [62] T. M. C. Abbott *et al.*, “Dark Energy Survey Year 3 results: Cosmological constraints from galaxy clustering and weak lensing,” *Phys. Rev. D*, vol. 105, no. 2, p. 023520, 2022.
- [63] W. Rindler, “Visual horizons in world models,” *mnras*, vol. 116, p. 662, Jan. 1956.
- [64] C. W. Misner, K. S. Thorne, and J. A. Wheeler, *Gravitation*. San Francisco: W. H. Freeman, 1973.
- [65] A. H. Guth, “Inflationary universe: A possible solution to the horizon and flatness problems,” *Phys. Rev. D*, vol. 23, pp. 347–356, Jan 1981.
- [66] L. Balkenhol *et al.*, “Constraints on Λ CDM extensions from the SPT-3G 2018 EE and TE power spectra,” *Phys. Rev. D*, vol. 104, no. 8, p. 083509, 2021.
- [67] T. Colas, G. D’amico, L. Senatore, P. Zhang, and F. Beutler, “Efficient Cosmological Analysis of the SDSS/BOSS data from the Effective Field Theory of Large-Scale Structure,” *JCAP*, vol. 06, p. 001, 2020.
- [68] G. D’Amico, J. Gleyzes, N. Kokron, K. Markovic, L. Senatore, P. Zhang, F. Beutler, and H. Gil-Marín, “The Cosmological Analysis of the SDSS/BOSS data from the Effective Field Theory of Large-Scale Structure,” *JCAP*, vol. 05, p. 005, 2020.

- [69] O. H. E. Philcox, M. M. Ivanov, M. Simonović, and M. Zaldarriaga, “Combining Full-Shape and BAO Analyses of Galaxy Power Spectra: A 1.6\% CMB-independent constraint on H_0 ,” *JCAP*, vol. 05, p. 032, 2020.
- [70] M. M. Ivanov, M. Simonović, and M. Zaldarriaga, “Cosmological Parameters from the BOSS Galaxy Power Spectrum,” *JCAP*, vol. 05, p. 042, 2020.
- [71] A. G. Riess, S. Casertano, W. Yuan, J. B. Bowers, L. Macri, J. C. Zinn, and D. Scolnic, “Cosmic Distances Calibrated to 1% Precision with Gaia EDR3 Parallaxes and Hubble Space Telescope Photometry of 75 Milky Way Cepheids Confirm Tension with Λ CDM,” *Astrophys. J. Lett.*, vol. 908, no. 1, p. L6, 2021.
- [72] A. G. Riess, S. Casertano, W. Yuan, L. M. Macri, and D. Scolnic, “Large Magellanic Cloud Cepheid Standards Provide a 1% Foundation for the Determination of the Hubble Constant and Stronger Evidence for Physics beyond Λ CDM,” *Astrophys. J.*, vol. 876, no. 1, p. 85, 2019.
- [73] A. G. Riess *et al.*, “A 2.4% Determination of the Local Value of the Hubble Constant,” *Astrophys. J.*, vol. 826, no. 1, p. 56, 2016.
- [74] L. Breuval *et al.*, “The Milky Way Cepheid Leavitt law based on Gaia DR2 parallaxes of companion stars and host open cluster populations,” *Astron. Astrophys.*, vol. 643, p. A115, 2020.
- [75] C. R. Burns *et al.*, “The Carnegie Supernova Project: Absolute Calibration and the Hubble Constant,” *Astrophys. J.*, vol. 869, no. 1, p. 56, 2018.
- [76] B. Follin and L. Knox, “Insensitivity of the distance ladder Hubble constant determination to Cepheid calibration modelling choices,” *Mon. Not. Roy. Astron. Soc.*, vol. 477, no. 4, pp. 4534–4542, 2018.
- [77] D. Camarena and V. Marra, “Local determination of the Hubble constant and the deceleration parameter,” *Phys. Rev. Res.*, vol. 2, no. 1, p. 013028, 2020.
- [78] E. Di Valentino, O. Mena, S. Pan, L. Visinelli, W. Yang, A. Melchiorri, D. F. Mota, A. G. Riess, and J. Silk, “In the realm of the Hubble tension—a review of solutions,” *Class. Quant. Grav.*, vol. 38, no. 15, p. 153001, 2021.
- [79] E. Abdalla *et al.*, “Cosmology intertwined: A review of the particle physics, astrophysics, and cosmology associated with the cosmological tensions and anomalies,” *JHEAp*, vol. 34, pp. 49–211, 2022.

- [80] A. G. Riess *et al.*, “A Comprehensive Measurement of the Local Value of the Hubble Constant with $1 \text{ km s}^{-1} \text{ Mpc}^{-1}$ Uncertainty from the Hubble Space Telescope and the SH0ES Team,” *Astrophys. J. Lett.*, vol. 934, no. 1, p. L7, 2022.
- [81] M. Asgari *et al.*, “KiDS-1000 Cosmology: Cosmic shear constraints and comparison between two point statistics,” *Astron. Astrophys.*, vol. 645, p. A104, 2021.
- [82] S. Joudaki *et al.*, “CFHTLenS revisited: assessing concordance with Planck including astrophysical systematics,” *Mon. Not. Roy. Astron. Soc.*, vol. 465, no. 2, pp. 2033–2052, 2017.
- [83] O. H. E. Philcox and M. M. Ivanov, “BOSS DR12 full-shape cosmology: Λ CDM constraints from the large-scale galaxy power spectrum and bispectrum monopole,” *Phys. Rev. D*, vol. 105, no. 4, p. 043517, 2022.
- [84] M. Asgari *et al.*, “KiDS+VIKING-450 and DES-Y1 combined: Mitigating baryon feedback uncertainty with COSEBIs,” *Astron. Astrophys.*, vol. 634, p. A127, 2020.
- [85] S. Joudaki *et al.*, “KiDS+VIKING-450 and DES-Y1 combined: Cosmology with cosmic shear,” *Astron. Astrophys.*, vol. 638, p. L1, 2020.
- [86] A. Amon *et al.*, “Dark Energy Survey Year 3 results: Cosmology from cosmic shear and robustness to data calibration,” *Phys. Rev. D*, vol. 105, no. 2, p. 023514, 2022.
- [87] L. F. Secco *et al.*, “Dark Energy Survey Year 3 results: Cosmology from cosmic shear and robustness to modeling uncertainty,” *Phys. Rev. D*, vol. 105, no. 2, p. 023515, 2022.
- [88] A. Loureiro *et al.*, “KiDS and Euclid: Cosmological implications of a pseudo angular power spectrum analysis of KiDS-1000 cosmic shear tomography,” *Astron. Astrophys.*, vol. 665, p. A56, 2022.
- [89] C. Heymans *et al.*, “KiDS-1000 Cosmology: Multi-probe weak gravitational lensing and spectroscopic galaxy clustering constraints,” *Astron. Astrophys.*, vol. 646, p. A140, 2021.

- [90] H. Hildebrandt *et al.*, “KiDS+VIKING-450: Cosmic shear tomography with optical and infrared data,” *Astron. Astrophys.*, vol. 633, p. A69, 2020.
- [91] T. M. C. Abbott *et al.*, “Dark Energy Survey Year 1 Results: Cosmological constraints from cluster abundances and weak lensing,” *Phys. Rev. D*, vol. 102, no. 2, p. 023509, 2020.
- [92] E. Macaulay, I. K. Wehus, and H. K. Eriksen, “Lower Growth Rate from Recent Redshift Space Distortion Measurements than Expected from Planck,” *Phys. Rev. Lett.*, vol. 111, no. 16, p. 161301, 2013.
- [93] J. Rebouças, J. Gordon, D. H. F. de Souza, K. Zhong, V. Miranda, R. Rosenfeld, T. Eifler, and E. Krause, “Early dark energy constraints with late-time expansion marginalization,” *JCAP*, vol. 02, p. 042, 2024.
- [94] J. C. Hill *et al.*, “Atacama Cosmology Telescope: Constraints on prerecombination early dark energy,” *Phys. Rev. D*, vol. 105, no. 12, p. 123536, 2022.
- [95] V. Poulin, T. L. Smith, and A. Bartlett, “Dark energy at early times and ACT data: A larger Hubble constant without late-time priors,” *Phys. Rev. D*, vol. 104, no. 12, p. 123550, 2021.
- [96] J. C. Hill, E. McDonough, M. W. Toomey, and S. Alexander, “Early dark energy does not restore cosmological concordance,” *Phys. Rev. D*, vol. 102, no. 4, p. 043507, 2020.
- [97] M. M. Ivanov, E. McDonough, J. C. Hill, M. Simonović, M. W. Toomey, S. Alexander, and M. Zaldarriaga, “Constraining Early Dark Energy with Large-Scale Structure,” *Phys. Rev. D*, vol. 102, no. 10, p. 103502, 2020.
- [98] S. Aiola *et al.*, “The Atacama Cosmology Telescope: DR4 Maps and Cosmological Parameters,” *JCAP*, vol. 12, p. 047, 2020.
- [99] D. M. Scolnic *et al.*, “The Complete Light-curve Sample of Spectroscopically Confirmed SNe Ia from Pan-STARRS1 and Cosmological Constraints from the Combined Pantheon Sample,” *Astrophys. J.*, vol. 859, no. 2, p. 101, 2018.
- [100] F. Beutler, C. Blake, M. Colless, D. H. Jones, L. Staveley-Smith, L. Campbell, Q. Parker, W. Saunders, and F. Watson, “The 6dF Galaxy Survey: baryon acoustic oscillations and the local Hubble constant,” *mnras*, vol. 416, pp. 3017–3032, Oct. 2011.

- [101] A. J. Ross, L. Samushia, C. Howlett, W. J. Percival, A. Burden, and M. Manera, “The clustering of the SDSS DR7 main Galaxy sample – I. A 4 per cent distance measure at $z = 0.15$,” *Mon. Not. Roy. Astron. Soc.*, vol. 449, no. 1, pp. 835–847, 2015.
- [102] S. Alam *et al.*, “The clustering of galaxies in the completed SDSS-III Baryon Oscillation Spectroscopic Survey: cosmological analysis of the DR12 galaxy sample,” *Mon. Not. Roy. Astron. Soc.*, vol. 470, no. 3, pp. 2617–2652, 2017.
- [103] S. H. Suyu *et al.*, “H0LiCOW – I. H0 Lenses in COSMOGRAIL’s Wellspring: program overview,” *Mon. Not. Roy. Astron. Soc.*, vol. 468, no. 3, pp. 2590–2604, 2017.
- [104] O. Friedrich *et al.*, “Dark Energy Survey year 3 results: covariance modelling and its impact on parameter estimation and quality of fit,” *Mon. Not. Roy. Astron. Soc.*, vol. 508, no. 3, pp. 3125–3165, 2021.
- [105] P. A. R. Ade *et al.*, “Planck 2015 results. XIV. Dark energy and modified gravity,” *Astron. Astrophys.*, vol. 594, p. A14, 2016.
- [106] M. S. Madhavacheril *et al.*, “The Atacama Cosmology Telescope: DR6 Gravitational Lensing Map and Cosmological Parameters,” *Astrophys. J.*, vol. 962, no. 2, p. 113, 2024.
- [107] DESI Collaboration, A. G. Adame, J. Aguilar, S. Ahlen, S. Alam, D. M. Alexander, M. Alvarez, O. Alves, A. Anand, U. Andrade, E. Armengaud, S. Avila, A. Aviles, H. Awan, B. Bahr-Kalus, S. Bailey, C. Baltay, A. Bault, J. Behera, S. BenZvi, A. Bera, F. Beutler, D. Bianchi, C. Blake, R. Blum, S. Brieden, A. Brodzeller, D. Brooks, E. Buckley-Geer, E. Burtin, R. Calderon, R. Canning, A. Carnero Rosell, R. Cereskaite, J. L. Cervantes-Cota, S. Chabanier, E. Chaussidon, J. Chaves-Montero, S. Chen, X. Chen, T. Claybaugh, S. Cole, A. Cuceu, T. M. Davis, K. Dawson, A. de la Macorra, A. de Mattia, N. Deiosso, A. Dey, B. Dey, Z. Ding, P. Doel, J. Edelstein, S. Eftekharzadeh, D. J. Eisenstein, A. Elliott, P. Fagrelus, K. Fanning, S. Ferraro, J. Ereza, N. Findlay, B. Flaugher, A. Font-Ribera, D. Forero-Sánchez, J. E. Forero-Romero, C. S. Frenk, C. Garcia-Quintero, E. Gaztañaga, H. Gil-Marín, S. G. A. Gontcho, A. X. Gonzalez-Morales, V. Gonzalez-Perez, C. Gordon, D. Green, D. Gruen, R. Gsponer, G. Gutierrez, J. Guy, B. Hadzhiyska, C. Hahn, M. M. S. Hanif, H. K. Herrera-Alcantar, K. Honscheid, C. Howlett, D. Huterer, V. Iršič,

- M. Ishak, S. Juneau, N. G. Karaçaylı, R. Kehoe, S. Kent, D. Kirkby, A. Kremin, A. Krolewski, Y. Lai, T. W. Lan, M. Landriau, D. Lang, J. Lasker, J. M. Le Goff, L. Le Guillou, A. Leauthaud, M. E. Levi, T. S. Li, E. Linder, K. Lodha, C. Magneville, M. Manera, D. Margala, P. Martini, M. Maus, P. McDonald, L. Medina-Varela, A. Meisner, J. Mena-Fernández, R. Miquel, J. Moon, S. Moore, J. Moustakas, N. Mudur, E. Mueller, A. Muñoz-Gutiérrez, A. D. Myers, S. Nadathur, L. Napolitano, R. Neveux, J. A. Newman, N. M. Nguyen, J. Nie, G. Niz, H. E. Noriega, N. Padmanabhan, E. Paillas, N. Palanque-Delabrouille, J. Pan, S. Penmetsa, W. J. Percival, M. M. Pieri, M. Pinon, C. Poppett, A. Porredon, F. Prada, A. Pérez-Fernández, I. Pérez-Ràfols, D. Rabinowitz, A. Raichoor, C. Ramírez-Pérez, S. Ramirez-Solano, C. Ravoux, M. Rashkovetskyi, M. Rezaie, J. Rich, A. Rocher, C. Rockosi, N. A. Roe, A. Rosado-Marin, A. J. Ross, G. Rossi, R. Ruggeri, V. Ruhlmann-Kleider, L. Samushia, E. Sanchez, C. Saulder, E. F. Schlafly, D. Schlegel, M. Schubnell, H. Seo, A. Shafieloo, R. Sharples, J. Silber, A. Slosar, A. Smith, D. Sprayberry, T. Tan, G. Tarlé, P. Taylor, S. Trusov, L. A. Ureña-López, R. Vaisakh, D. Valcin, F. Valdes, M. Vargas-Magaña, L. Verde, M. Walther, B. Wang, M. S. Wang, B. A. Weaver, N. Weaverdyck, R. H. Wechsler, D. H. Weinberg, M. White, J. Yu, Y. Yu, S. Yuan, C. Yèche, E. A. Zaborowski, P. Zarrouk, H. Zhang, C. Zhao, R. Zhao, R. Zhou, T. Zhuang, and H. Zou, “DESI 2024 VI: Cosmological Constraints from the Measurements of Baryon Acoustic Oscillations,” *arXiv e-prints*, p. arXiv:2404.03002, Apr. 2024.
- [108] K. Abe and other, “Measurement of neutrino and antineutrino oscillations by the t2k experiment including a new additional sample of ν_e interactions at the far detector,” *Phys. Rev. D*, vol. 96, p. 092006, Nov 2017.
- [109] M. A. Acero *et al.*, “First Measurement of Neutrino Oscillation Parameters using Neutrinos and Antineutrinos by NOvA,” *Phys. Rev. Lett.*, vol. 123, no. 15, p. 151803, 2019.
- [110] J. Rebouças, D. H. F. de Souza, K. Zhong, V. Miranda, and R. Rosenfeld, “Investigating Late-Time Dark Energy and Massive Neutrinos in Light of DESI Y1 BAO,” 8 2024.
- [111] A. Möller, M. Smith, M. Sako, M. Sullivan, M. Vincenzi, P. Wiseman, P. Armstrong, J. Asorey, D. Brout, D. Carollo, T. M. Davis, C. Frohmaier, L. Galbany, K. Glazebrook, L. Kelsey, R. Kessler, G. F. Lewis, C. Lidman, U. Malik,

- R. C. Nichol, D. Scolnic, B. E. Tucker, T. M. C. Abbott, M. Agüena, S. Allam, J. Annis, E. Bertin, S. Bocquet, D. Brooks, D. L. Burke, A. Carnero Rosell, M. Carrasco Kind, J. Carretero, F. J. Castander, C. Conselice, M. Costanzi, M. Crocce, L. N. da Costa, J. De Vicente, S. Desai, H. T. Diehl, P. Doel, S. Everett, I. Ferrero, D. A. Finley, B. Flaugher, D. Friedel, J. Frieman, J. García-Bellido, D. W. Gerdes, D. Gruen, R. A. Gruendl, J. Gschwend, G. Gutierrez, K. Herner, S. R. Hinton, D. L. Hollowood, K. Honscheid, D. J. James, K. Kuehn, N. Kuropatkin, O. Lahav, M. March, J. L. Marshall, F. Menanteau, R. Miquel, R. Morgan, A. Palmese, F. Paz-Chinchón, A. Pieres, A. A. Plazas Malagón, A. K. Romer, A. Roodman, E. Sanchez, V. Scarpine, M. Schubnell, S. Serrano, I. Sevilla-Noarbe, E. Suchyta, G. Tarle, D. Thomas, C. To, and T. N. Varga, “The dark energy survey 5-yr photometrically identified type Ia supernovae,” *Mon. Not. Roy. Astron. Soc.*, vol. 514, pp. 5159–5177, Aug. 2022.
- [112] D. Rubin, G. Aldering, M. Betoule, A. Fruchter, X. Huang, A. G. Kim, C. Lidman, E. Linder, S. Perlmutter, P. Ruiz-Lapuente, and N. Suzuki, “Union Through UNITY: Cosmology with 2,000 SNe Using a Unified Bayesian Framework,” *arXiv e-prints*, p. arXiv:2311.12098, Nov. 2023.
- [113] D. Scolnic, D. Brout, A. Carr, A. G. Riess, T. M. Davis, A. Dwomoh, D. O. Jones, N. Ali, P. Charvu, R. Chen, E. R. Peterson, B. Popovic, B. M. Rose, C. M. Wood, P. J. Brown, K. Chambers, D. A. Coulter, K. G. Dettman, G. Dimitriadis, A. V. Filippenko, R. J. Foley, S. W. Jha, C. D. Kilpatrick, R. P. Kirshner, Y.-C. Pan, A. Rest, C. Rojas-Bravo, M. R. Siebert, B. E. Stahl, and W. Zheng, “The Pantheon+ Analysis: The Full Data Set and Light-curve Release,” *Astrophys. J.*, vol. 938, p. 113, Oct. 2022.
- [114] D. Huterer and G. Starkman, “Parameterization of dark-energy properties: A Principal-component approach,” *Phys. Rev. Lett.*, vol. 90, p. 031301, 2003.
- [115] D. Huterer and A. Cooray, “Uncorrelated estimates of dark energy evolution,” *Phys. Rev. D*, vol. 71, p. 023506, 2005.
- [116] E. V. Linder and D. Huterer, “How many dark energy parameters?,” *Phys. Rev. D*, vol. 72, p. 043509, 2005.
- [117] R. G. Crittenden, L. Pogosian, and G.-B. Zhao, “Investigating dark energy experiments with principal components,” *JCAP*, vol. 12, p. 025, 2009.

-
- [118] D. Huterer and H. V. Peiris, “Dynamical behavior of generic quintessence potentials: Constraints on key dark energy observables,” *Phys. Rev. D*, vol. 75, p. 083503, 2007.
- [119] D. Green and J. Meyers, “The Cosmological Preference for Negative Neutrino Mass,” *arXiv e-prints*, p. arXiv:2407.07878, July 2024.