

Review Article

Applications of Subleading-Color Amplitudes in $\mathcal{N} = 4$ SYM Theory

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A number of features and applications of subleading-color amplitudes of $\mathcal{N} = 4$ SYM theory are reviewed. Particular attention is given to the IR divergences of the subleading-color amplitudes, the relationships of $\mathcal{N} = 4$ SYM theory to $\mathcal{N} = 8$ supergravity, and to geometric interpretations of one-loop subleading-color and N^k MHV amplitudes of $\mathcal{N} = 4$ SYM theory.

1. Introduction

Planar amplitudes of $\mathcal{N} = 4$ SYM theory have been extensively studied by a variety of methods, see, for example, [1–23]. For a recent overview, see [24] and the special issue of *Journal of Physics A*, devoted to “Scattering amplitudes in gauge theories.” Subleading-color (i.e., nonplanar) amplitudes, however, usually receive less attention [25–33]. Nevertheless interesting insights are available from various applications of subleading color amplitudes. One case in point is a possible weak/weak duality between $\mathcal{N} = 4$ SYM theory and $\mathcal{N} = 8$ supergravity [15, 34–46]. Since nonplanar graphs appear on an equal footing with planar graphs in $\mathcal{N} = 8$ supergravity, one needs to understand the nonplanar graphs in $\mathcal{N} = 4$ SYM if the weak/weak duality is to be explored.

This paper will cover three significant topics. Section 2 discusses the IR divergences of various subleading-color amplitudes. In Section 3 the interplay between subleading-color amplitudes of $\mathcal{N} = 4$ SYM theory and amplitudes of $\mathcal{N} = 8$ supergravity will be

considered. Section 4 presents various geometric interpretations of one-loop subleading-color amplitudes, primarily using the tools of momentum twistors and the accompanying polytope interpretation.

In the remainder of this section, we define the notation for the color decomposition, the loop expansion, and the $1/N$ expansion.

At tree level, we can decompose the amplitudes $\mathcal{A}_n$ of $\mathcal{N} = 4$ SYM into *color-ordered* tree amplitudes A_n

$$\begin{aligned}\mathcal{A}_n^{\text{tree}}(12\dots n) &= g^{n-2} \sum_{\sigma \in S_n/Z_n} \text{Tr}(T^{a_{\sigma(1)}} \dots T^{a_{\sigma(n)}}) A_n^{\text{tree}}(\sigma(1) \dots \sigma(n)) \\ &= g^{n-2} \sum_{P(23\dots n)} \text{Tr}(T^{a_1} T^{a_{P(2)}} \dots T^{a_{P(n)}}) A_n^{\text{tree}}(1P(2) \dots P(n)),\end{aligned}\tag{1.1}$$

where in the second line, 1 is fixed and $P(23\dots n)$ is a permutation of $2, 3, \dots, n$ and T^a are $SU(N)$ generators in the fundamental representation, normalized according to $\text{Tr}(T^a T^b) = \delta^{ab}$. The color-ordered amplitudes A_n depend on the momenta and polarizations of the external particles.

The color-ordered amplitudes are not independent. For n -point amplitudes, there is a basis of $(n-2)!$ amplitudes out of the total $n!$, called the Kleiss-Kuijf (KK) basis [47], and we can find the others easily in terms of it [40]. It is based on the existence of the Kleiss-Kuijf relations [47]

$$A_n(1, \{\alpha\}, n, \{\beta\}) = (-1)^{n_\beta} \sum_{\{\sigma\}_i \in \text{OP}(\{\alpha\}, \{\beta^T\})} A_n(1, \{\sigma\}_i, n),\tag{1.2}$$

where σ_i are ordered permutations, that is, ones that keep the order of $\{\alpha\}$ and of $\{\beta^T\}$ inside σ_i . Thus the KK basis is $A_n(1, \mathcal{P}(2, \dots, n-1), n)$, where $\mathcal{P}$ are arbitrary permutations. All the other A_n 's can be recovered from it by the use of the KK relations and cyclicity and reflection invariance

$$A_n(12\dots n) = (-1)^n A_n(n\dots 21).\tag{1.3}$$

At one loop, we can write a similar expansion in color-ordered amplitudes

$$\begin{aligned}\mathcal{A}_n^{1\text{-loop}}(12\dots n) &= g^n \sum_{j=1}^{[n/2]+1} \sum_{\sigma \in S_n/S_{n_j}} Gr_{n;j}(\sigma) A_{n;j}(\sigma(1) \dots \sigma(n)), \\ Gr_{n;1}(1) &= N \text{Tr}(T^{a_1} \dots T^{a_n}), \\ Gr_{n;j}(1) &= \text{Tr}(T^{a_1} \dots T^{a_{j-1}}) \text{Tr}(T^{a_j} \dots T^{a_n}).\end{aligned}\tag{1.4}$$

However, the subleading piece in the $1/N$ expansion can be obtained from the leading piece by

$$A_{n;j}(12\dots, j-1, j, j+1, \dots n) = (-1)^{j-1} \sum_{\sigma \in \text{COP}\{\alpha\}, \{\beta\}} A_{n;1}(\sigma),\tag{1.5}$$

where COP are cyclically ordered permutations, again keeping the order of $\{\alpha\}$ and $\{\beta\}$ fixed up to cyclic permutations.

At arbitrary loops, the decomposition of the four-gluon amplitude takes a form with only single and double traces

$$\begin{aligned} \mathcal{A}_4(1234) = & g^2 \sum_{\sigma \in S_4/\mathbb{Z}_4} \text{Tr}(T^{a_{\sigma(1)}} T^{a_{\sigma(2)}} T^{a_{\sigma(3)}} T^{a_{\sigma(4)}}) N A_{4;1}(\sigma(1)\sigma(2)\sigma(3)\sigma(4)) \\ & + g^2 \sum_{\sigma \in S_4/\mathbb{Z}_2^3} \text{Tr}(T^{a_{\sigma(1)}} T^{a_{\sigma(2)}}) \text{Tr}(T^{a_{\sigma(3)}} T^{a_{\sigma(4)}}) A_{4;3}(\sigma(1)\sigma(2)\sigma(3)\sigma(4)). \end{aligned} \quad (1.6)$$

We also define an explicit basis [48] of single and double traces:

$$\begin{aligned} \mathcal{C}_{[1]} &= \text{Tr}(T^{a_1} T^{a_2} T^{a_3} T^{a_4}), & \mathcal{C}_{[4]} &= \text{Tr}(T^{a_1} T^{a_3} T^{a_2} T^{a_4}), & \mathcal{C}_{[7]} &= \text{Tr}(T^{a_1} T^{a_2}) \text{Tr}(T^{a_3} T^{a_4}), \\ \mathcal{C}_{[2]} &= \text{Tr}(T^{a_1} T^{a_2} T^{a_4} T^{a_3}), & \mathcal{C}_{[5]} &= \text{Tr}(T^{a_1} T^{a_3} T^{a_4} T^{a_2}), & \mathcal{C}_{[8]} &= \text{Tr}(T^{a_1} T^{a_3}) \text{Tr}(T^{a_2} T^{a_4}), \\ \mathcal{C}_{[3]} &= \text{Tr}(T^{a_1} T^{a_4} T^{a_2} T^{a_3}), & \mathcal{C}_{[6]} &= \text{Tr}(T^{a_1} T^{a_4} T^{a_3} T^{a_2}), & \mathcal{C}_{[9]} &= \text{Tr}(T^{a_1} T^{a_4}) \text{Tr}(T^{a_2} T^{a_3}), \end{aligned} \quad (1.7)$$

in terms of which the four-gluon amplitude can be expanded as

$$\mathcal{A}_4(1234) = g^2 \sum_{i=1}^9 A_{[i]} \mathcal{C}_{[i]}. \quad (1.8)$$

The loop expansion of color-ordered amplitudes

$$A_{[i]} = \sum_{L=0}^{\infty} a^L A_{[i]}^{(L)}, \quad N A_{4;1} = \sum_{L=0}^{\infty} a^L A_{4;1}^{(L)}, \quad A_{4;3} = \sum_{L=0}^{\infty} a^L A_{4;3}^{(L)} \quad (1.9)$$

is in terms of the natural 't Hooft loop expansion parameter [7]

$$a \equiv \frac{g^2 N}{8\pi^2} (4\pi e^{-\gamma})^\epsilon, \quad (1.10)$$

where γ is Euler's constant and $\epsilon = (4 - D)/2$. Note that at L loops, the amplitude is at most of order N^L , which means that $A_{[i]}^{(L)}$ starts at $\mathcal{O}(N^0)$.

For a general n -point amplitude, we will have an expansion in an arbitrary number of multitrace color-ordered amplitudes $A_{n;j_1,j_2,\dots,j_k}$.

Besides the loop expansion in the 't Hooft parameter a , we still have a $1/N$ expansion of the amplitudes, which can be understood in 't Hooft's double line notation as an expansion in the topology of the diagrams. For $\mathcal{A}_4$, the expansion in single-trace $A_{4;1}$ and double-trace $A_{4;3}$ amplitudes corresponds to the topology of the outside lines, forming boundaries of the diagrams. For example, at one-loop, the contribution in $A_{4;1}$ to the amplitude is leading, that is, of order N (thus $A_{4;1}$ of order 1), coming from a diagram with the topology of 4 external

lines and a boundary, whereas the contribution of $A_{4;3}$ is subleading, that is, of order N^0 , and comes from a nonplanar diagram with 4 external lines, but arranged on two boundaries. It can be obtained by taking two twists of the 't Hooft double lines on opposite sides of the box, or twists on all 4 sides. Thus the multitrace expansion comes as an expansion in the topology associated with the external lines (number of boundaries for them) and is an expansion in integer powers of $1/N$, corresponding to the number of boundaries of the diagram.

On top of that, we also have an expansion in integer powers of $1/N^2$, independently for $A_{4;1}$ and $A_{4;3}$, corresponding to nonplanar diagrams with handles (a handle gives a factor of $1/N^2$). The expansion terminates at order $\mathcal{O}(N^0)$ for the amplitude, since in the amplitude the powers of N can only be positive. Thus at L -loops, we have $A_{4;1}^{(L)} = \mathcal{O}(1)$ to $\mathcal{O}(1/N^L)$ and $A_{4;3}^{(L)} = \mathcal{O}(1/N)$ to $\mathcal{O}(1/N^L)$. Taken together, we will say that the gluon amplitudes have a $1/N$ expansion.

2. IR Divergences for Subleading $\mathcal{N} = 4$ Four-Gluon Amplitudes

2.1. General Formalism

$\mathcal{N} = 4$ SYM is a UV-finite theory, but IR divergences arise due to the exchange of soft and collinear gluons. These divergences can be regulated using dimensional regularization in $D = 4 - 2\epsilon$ dimensions, in which they appear as poles in a Laurent expansion in ϵ .

In gluon-gluon scattering in $\mathcal{N} = 4$ SYM, IR divergences arise both from soft gluons and from collinear gluons, each of which gives rise to an $\mathcal{O}(1/\epsilon)$ pole at one loop, leading to an $\mathcal{O}(1/\epsilon^2)$ pole at that order. At L loops, the leading IR divergence of the scattering amplitude is therefore $\mathcal{O}(1/\epsilon^{2L})$, arising from multiple soft gluon exchanges.

Subleading-color amplitudes $A^{(L,k)}$, that is, those suppressed by $1/N^k$ relative to the leading-color amplitude at L loops, have less severe IR divergences, being only of $\mathcal{O}(1/\epsilon^{2L-k})$ at L -loops.

In this section, we review the derivation of a compact all-loop-order expression for the IR-divergent part of the $\mathcal{N} = 4$ SYM four-gluon amplitude given in [41, 49]. This result is expressed in terms of the soft (cusp) anomalous dimension $\gamma(a)$, the collinear anomalous dimension $\mathcal{G}_0(a)$, and the soft anomalous dimension matrices $\Gamma^{(\ell)}$ and relies on the assumption that the soft anomalous dimension matrices are mutually commuting, which follows if they are all proportional to $\Gamma^{(1)}$, as has been conjectured in [30, 31, 33, 50]. This compact expression is then used to obtain the coefficient of the leading IR pole (and some subleading poles) of all the subleading-color amplitudes. Explicit values for the anomalous dimensions can be obtained by comparison with various exact results.

We organize the 4-point color-ordered amplitudes $A_{[i]}$ defined in (1.8) into a vector in color space [25, 26]

$$|A\rangle = (A_{[1]}, A_{[2]}, A_{[3]}, A_{[4]}, A_{[5]}, A_{[6]}, A_{[7]}, A_{[8]}, A_{[9]})^T, \quad (2.1)$$

where $(\dots)^T$ denotes the transposed vector. The vector of color-ordered amplitudes factorizes into [27, 29]

$$\left| A\left(\frac{s_{ij}}{\mu^2}, a, \epsilon\right) \right\rangle = J\left(\frac{Q^2}{\mu^2}, a, \epsilon\right) \mathbf{S}\left(\frac{s_{ij}}{Q^2}, \frac{Q^2}{\mu^2}, a, \epsilon\right) \left| H\left(\frac{s_{ij}}{Q^2}, \frac{Q^2}{\mu^2}, a, \epsilon\right) \right\rangle, \quad (2.2)$$

where $|H\rangle$, which is IR-finite as $\epsilon \rightarrow 0$, characterizes the short-distance behavior of the amplitude and where the prefactors J and $\mathbf{S}$ encapsulate the long-distance IR-divergent behavior. The soft function $\mathbf{S}$ is written in boldface to denote that it is a matrix acting on the vector $|H\rangle$. Also s_{ij} is the kinematic invariant $(k_i + k_j)^2$, μ is a renormalization scale, and Q is an arbitrary factorization scale which serves to separate the long- and short-distance behavior.

Because $\mathcal{N} = 4$ SYM theory is conformally invariant, the product of jet functions J may be explicitly evaluated as [7]

$$J\left(\frac{Q^2}{\mu^2}, a, \epsilon\right) = \exp\left[-\frac{1}{2}\sum_{\ell=1}^{\infty} a^{\ell} \left(\frac{\mu^2}{Q^2}\right)^{\ell\epsilon} \left(\frac{\gamma^{(\ell)}}{(\ell\epsilon)^2} + \frac{2\mathcal{G}_0^{(\ell)}}{\ell\epsilon}\right)\right], \quad (2.3)$$

where $\gamma^{(\ell)}$ and $\mathcal{G}_0^{(\ell)}$ are the coefficients of the soft (or the Wilson line cusp) and collinear anomalous dimensions of the gluon, respectively. The explicit values for these anomalous dimensions may be obtained from the exact expressions for the planar four-gluon amplitude [7]:

$$\begin{aligned} \gamma(a) &= \sum_{\ell=1}^{\infty} a^{\ell} \gamma^{(\ell)} = 4a - 4\zeta_2 a^2 + 22\zeta_4 a^3 + \dots, \\ \mathcal{G}_0(a) &= \sum_{\ell=1}^{\infty} a^{\ell} \mathcal{G}_0^{(\ell)} = -\zeta_3 a^2 + \left(4\zeta_5 + \frac{10}{3}\zeta_2\zeta_3\right)a^3 + \dots. \end{aligned} \quad (2.4)$$

The soft function $\mathbf{S}$ is given by [27, 29]

$$\mathbf{S}\left(\frac{s_{ij}}{Q^2}, \frac{Q^2}{\mu^2}, a, \epsilon\right) = \text{Pexp}\left[-\frac{1}{2}\int_0^{Q^2} \frac{d\tilde{\mu}^2}{\tilde{\mu}^2} \mathbf{\Gamma}\left(\frac{s_{ij}}{Q^2}, \bar{a}\left(\frac{\mu^2}{\tilde{\mu}^2}, a, \epsilon\right)\right)\right], \quad (2.5)$$

where

$$\mathbf{\Gamma}\left(\frac{s_{ij}}{Q^2}, a\right) = \sum_{\ell=1}^{\infty} a^{\ell} \mathbf{\Gamma}^{(\ell)}, \quad \bar{a}\left(\frac{\mu^2}{\tilde{\mu}^2}, a, \epsilon\right) = \left(\frac{\mu^2}{\tilde{\mu}^2}\right)^{\epsilon} a, \quad (2.6)$$

suppressing the explicit dependence of $\mathbf{\Gamma}^{(\ell)}$ on s_{ij}/Q^2 to lighten the notation.

At this point, we make the assumption that the soft anomalous dimension matrices $\mathbf{\Gamma}^{(\ell)}$ all commute with one another. (This assumption was also used to simplify the IR divergences of QCD in [33]. The assumption is certainly valid through two loops, since $\mathbf{\Gamma}^{(2)} = (1/4)\gamma^{(2)}\mathbf{\Gamma}^{(1)}$, as shown in [28, 29]. In [32], it was established that $\mathbf{\Gamma}^{(3)} = (1/4)\gamma^{(3)}\mathbf{\Gamma}^{(1)}$ for the nonpure gluon contributions. Further, $\mathbf{\Gamma}^{(L)} = (1/4)\gamma^{(L)}\mathbf{\Gamma}^{(1)}$ has been conjectured to hold to all orders in [30, 31, 33, 50]. Difficulties may arise at four loops, however, due to the possibility of quartic

Casimir's terms [31, 32, 51, 52].) Therefore, the path ordering in (2.5) becomes irrelevant, allowing us to explicitly integrate it, obtaining

$$\mathbf{S}\left(\frac{s_{ij}}{Q^2}, \frac{Q^2}{\mu^2}, a, \epsilon\right) = \exp\left[\frac{1}{2}\sum_{\ell=1}^{\infty} a^{\ell} \left(\frac{\mu^2}{Q^2}\right)^{\ell\epsilon} \frac{\Gamma^{(\ell)}}{\ell\epsilon}\right]. \quad (2.7)$$

Combining the exponents of the jet and soft functions into [27, 41]

$$\mathbf{G}^{(\ell)}(\epsilon) = \frac{N^{\ell}}{2} \left(\frac{\mu^2}{Q^2}\right)^{\epsilon} \left[-\left(\frac{\gamma^{(\ell)}}{\epsilon^2} + \frac{2G_0^{(\ell)}}{\epsilon}\right)\mathbb{1} + \frac{1}{\epsilon}\Gamma^{(\ell)}\right], \quad (2.8)$$

we may express the four-gluon amplitude in the compact form

$$|A(\epsilon)\rangle = \exp\left[\sum_{\ell=1}^{\infty} \frac{a^{\ell}}{N^{\ell}} \mathbf{G}^{(\ell)}(\ell\epsilon)\right] |H(\epsilon)\rangle, \quad (2.9)$$

or equivalently

$$|H(\epsilon)\rangle = \sum_{L=0}^{\infty} a^L |H^{(Lf)}(\epsilon)\rangle = \left(\mathbb{1} - \sum_{\ell=1}^{\infty} \frac{a^{\ell}}{N^{\ell}} \mathbf{F}^{(\ell)}(\epsilon)\right) |A(\epsilon)\rangle. \quad (2.10)$$

where the matrices $\mathbf{F}^{(\ell)}(\epsilon)$ will be defined below. (Henceforth we suppress s_{ij} , Q , μ , and a in the arguments of the amplitudes.) Expanding (2.10) through three loops, we obtain the expressions given in [27, 41]

$$\begin{aligned} |A^{(1)}(\epsilon)\rangle &= \frac{1}{N} \mathbf{F}^{(1)}(\epsilon) |A^{(0)}\rangle + |H^{(1f)}(\epsilon)\rangle, \\ |A^{(2)}(\epsilon)\rangle &= \frac{1}{N^2} \mathbf{F}^{(2)}(\epsilon) |A^{(0)}\rangle + \frac{1}{N} \mathbf{F}^{(1)}(\epsilon) |A^{(1)}(\epsilon)\rangle + |H^{(2f)}(\epsilon)\rangle, \\ |A^{(3)}(\epsilon)\rangle &= \frac{1}{N^3} \mathbf{F}^{(3)}(\epsilon) |A^{(0)}\rangle + \frac{1}{N^2} \mathbf{F}^{(2)}(\epsilon) |A^{(1)}\rangle + \frac{1}{N} \mathbf{F}^{(1)}(\epsilon) |A^{(2)}(\epsilon)\rangle + |H^{(3f)}(\epsilon)\rangle, \end{aligned} \quad (2.11)$$

which will be useful in extracting the IR-divergent terms of leading- and subleading-color amplitudes in the following section. (Note that, because of the presence of poles in $\mathbf{F}(\epsilon)$, we will need to know positive powers of ϵ in the expansion of lower loop amplitudes to obtain all the IR-divergent contributions to the L -loop amplitude $A^{(\ell)}$.)

The equivalence of (2.9) and (2.10) follows if the matrices $\mathbf{F}^{(\ell)}(\epsilon)$ are defined through the equation

$$\left(\mathbb{1} - \sum_{\ell=1}^{\infty} \frac{a^{\ell}}{N^{\ell}} \mathbf{F}^{(\ell)}(\epsilon)\right) \exp\left[\sum_{\ell=1}^{\infty} \frac{a^{\ell}}{N^{\ell}} \mathbf{G}^{(\ell)}(\ell\epsilon)\right] = \mathbb{1}. \quad (2.12)$$

First define the functional $X[M]$ via [7]

$$1 + \sum_{\ell=1}^{\infty} a^{\ell} M^{(\ell)} \equiv \exp \left[\sum_{\ell=1}^{\infty} a^{\ell} \left(M^{(\ell)} - X^{(\ell)}[M] \right) \right] \quad (2.13)$$

so that $X^{(1)}[M] = 0$, $X^{(2)}[M] = (1/2)[M^{(1)}]^2$, $X^{(3)}[M] = -(1/3)[M^{(1)}]^3 + M^{(1)}M^{(2)}$, and so forth. This functional was defined for scalar functions $M^{(\ell)}$, but we can also use it for commuting matrices. We have assumed that $\Gamma^{(\ell)}$ and therefore $\mathbf{G}^{(\ell)}$ are mutually commuting, and thus so are $\mathbf{F}^{(\ell)}$, as a result of (2.12). Thus

$$\left(\mathbb{1} - \sum_{\ell=1}^{\infty} \frac{a^{\ell}}{N^{\ell}} \mathbf{F}^{(\ell)}(\epsilon) \right) = \exp \left[\sum_{\ell=1}^{\infty} \frac{a^{\ell}}{N^{\ell}} \left(-\mathbf{F}^{(\ell)}(\epsilon) - X^{(\ell)}[-\mathbf{F}] \right) \right], \quad (2.14)$$

and so (2.12) is equivalent to

$$\mathbf{F}^{(\ell)}(\epsilon) = -X^{(\ell)}[-\mathbf{F}] + \mathbf{G}^{(\ell)}(\ell\epsilon) \quad (2.15)$$

which defines $\mathbf{F}^{(\ell)}$ recursively in terms of $\mathbf{G}^{(\ell')}$ and $\mathbf{F}^{(\ell')}$ with $\ell' < \ell$. The explicit expressions up through three loops

$$\begin{aligned} \mathbf{F}^{(1)}(\epsilon) &= \mathbf{G}^{(1)}(\epsilon), \\ \mathbf{F}^{(2)}(\epsilon) &= -\frac{1}{2} \left[\mathbf{F}^{(1)}(\epsilon) \right]^2 + \mathbf{G}^{(2)}(2\epsilon), \\ \mathbf{F}^{(3)}(\epsilon) &= -\frac{1}{3} \left[\mathbf{F}^{(1)}(\epsilon) \right]^3 - \mathbf{F}^{(1)}(\epsilon) \mathbf{F}^{(2)}(\epsilon) + \mathbf{G}^{(3)}(3\epsilon). \end{aligned} \quad (2.16)$$

agree (up to rescaling by a factor of N^L) with the expressions given in [27] when specialized to the case of $gg \rightarrow gg$ in $\mathcal{N} = 4$ SYM theory.

2.2. $1/N$ Expansion of IR Divergences

In this subsection, we will use the results of the previous subsection to expand the IR-divergent contributions of the four-gluon amplitude in powers of $1/N$.

The L -loop color-ordered amplitudes may be written in a $1/N$ expansion as

$$\left| A^{(L)}(\epsilon) \right\rangle = \sum_{k=0}^L \frac{1}{N^k} \left| A^{(L,k)}(\epsilon) \right\rangle, \quad (2.17)$$

where $|A^{(L,0)}\rangle$ are the leading-color amplitudes, arising from planar diagrams and $|A^{(L,k)}\rangle$, $1 \leq k \leq L$, are the subleading-color amplitudes, which include contributions from nonplanar diagrams as well. The single-trace amplitudes ($i = 1, \dots, 6$) only contain even powers of $1/N$ (relative to the leading-color amplitude), while the double-trace amplitudes ($i = 7, \dots, 9$) only contain odd powers of $1/N$.

We begin by expanding (2.9):

$$|A(\epsilon)\rangle = \sum_{L=0}^{\infty} \sum_{k=0}^L \frac{a^L}{N^k} |A^{(L,k)}(\epsilon)\rangle = \prod_{\ell=1}^{\infty} \sum_{\{n_\ell\}} \frac{1}{n_\ell!} \left(a^\ell \frac{\mathbf{G}^{(\ell)}(\ell\epsilon)}{N^\ell} \right)^{n_\ell} \sum_{\ell_0=0}^{\infty} \sum_{k_0=0}^{\ell_0} \frac{a^{\ell_0}}{N^{k_0}} |H^{(\ell_0,k_0)}(\epsilon)\rangle. \quad (2.18)$$

In the derivation of (2.18), we assumed that the soft-anomalous dimension matrices are mutually commuting. We now assume further that the higher-loop soft-anomalous dimension matrices are all proportional to the one-loop soft-anomalous dimension matrix

$$\mathbf{\Gamma}^{(\ell)} = \frac{\gamma^{(\ell)}}{4} \mathbf{\Gamma}^{(1)} \quad (\text{assumption}) \quad (2.19)$$

as has been conjectured (see footnote 1). This allows us to rewrite (2.8) as

$$\frac{\mathbf{G}^{(\ell)}(\ell\epsilon)}{N^\ell} = \frac{1}{2} \left(\frac{\mu^2}{Q^2} \right)^{\ell\epsilon} \left[- \left(\frac{\gamma^{(\ell)}}{(\ell\epsilon)^2} + \frac{2\mathcal{G}_0^{(\ell)}}{\ell\epsilon} \right) \mathbb{1} + \frac{\gamma^{(\ell)}}{4\ell\epsilon} \mathbf{\Gamma}^{(1)} \right]. \quad (2.20)$$

The one-loop soft anomalous dimension matrix can be written [29]:

$$\mathbf{\Gamma}^{(1)} = -\frac{1}{N} \sum_{i=1}^4 \sum_{j \neq i}^4 \mathbf{T}_i \cdot \mathbf{T}_j \log \left(\frac{-s_{ij}}{Q^2} \right), \quad (2.21)$$

where $\mathbf{T}_i \cdot \mathbf{T}_j = T_i^a T_j^a$ with T_i^a the $\text{SU}(N)$ generators in the adjoint representation. On the basis of (1.7), it has the explicit form [48],

$$\mathbf{\Gamma}^{(1)} = 2 \begin{pmatrix} \alpha & 0 \\ 0 & \delta \end{pmatrix} + \frac{2}{N} \begin{pmatrix} 0 & \beta \\ \gamma & 0 \end{pmatrix}, \quad (2.22)$$

where

$$\alpha = \begin{pmatrix} S+\tau & 0 & 0 & 0 & 0 & 0 \\ 0 & S+\mathcal{U} & 0 & 0 & 0 & 0 \\ 0 & 0 & \tau+\mathcal{U} & 0 & 0 & 0 \\ 0 & 0 & 0 & \tau+\mathcal{U} & 0 & 0 \\ 0 & 0 & 0 & 0 & S+\mathcal{U} & 0 \\ 0 & 0 & 0 & 0 & 0 & S+\tau \end{pmatrix}, \quad \beta = \begin{pmatrix} \tau-\mathcal{U} & 0 & S-\mathcal{U} \\ \mathcal{U}-\tau & S-\tau & 0 \\ 0 & \tau-S & \mathcal{U}-S \\ 0 & \tau-S & \mathcal{U}-S \\ \mathcal{U}-\tau & S-\tau & 0 \\ \tau-\mathcal{U} & 0 & S-\mathcal{U} \end{pmatrix},$$

$$\gamma = \begin{pmatrix} S-\mathcal{U} & S-\tau & 0 & 0 & S-\tau & S-\mathcal{U} \\ 0 & \mathcal{U}-\tau & \mathcal{U}-S & \mathcal{U}-S & \mathcal{U}-\tau & 0 \\ \tau-\mathcal{U} & 0 & \tau-S & \tau-S & 0 & \tau-\mathcal{U} \end{pmatrix}, \quad \delta = \begin{pmatrix} 2S & 0 & 0 \\ 0 & 2\mathcal{U} & 0 \\ 0 & 0 & 2\tau \end{pmatrix} \quad (2.23)$$

with

$$\mathcal{S} = \log\left(-\frac{s}{Q^2}\right), \quad \mathcal{T} = \log\left(-\frac{t}{Q^2}\right), \quad \mathcal{U} = \log\left(-\frac{u}{Q^2}\right). \quad (2.24)$$

If the assumption (2.19) is valid, then the $1/N$ expansion of $\mathbf{G}^{(\ell)}(\ell\epsilon)/N^\ell$ terminates after two terms

$$\frac{\mathbf{G}^{(\ell)}(\ell\epsilon)}{N^\ell} = g_\ell + \frac{1}{N}f_\ell, \quad (2.25)$$

where g_ℓ and f_ℓ can be read from (2.20) and (2.22). We rewrite (2.18) as

$$|A(\epsilon)\rangle = \sum_{L=0}^{\infty} \sum_{k=0}^L \frac{a^L}{N^k} |A^{(L,k)}(\epsilon)\rangle = \prod_{\ell=1}^{\infty} \sum_{\{n_\ell\}} \frac{1}{n_\ell!} \left(a^\ell g_\ell + \frac{a^\ell}{N} f_\ell \right)^{n_\ell} \sum_{\ell_0=0}^{\infty} \sum_{k_0=0}^{\ell_0} \frac{a^{\ell_0}}{N^{k_0}} |H^{(\ell_0,k_0)}(\epsilon)\rangle \quad (2.26)$$

making all N dependence explicit.

We now determine the power of the leading IR pole of $|A^{(L,k)}(\epsilon)\rangle$. Consider an individual term on the right-hand side of (2.26). By counting powers of a and $1/N$, one sees that this term contributes to $|A^{(L,k)}(\epsilon)\rangle$, with

$$L = \ell_0 + \sum_{\ell=1}^{\infty} \ell n_\ell, \quad k = k_0 + k_1, \quad (2.27)$$

where k_1 is the number of factors f_ℓ present in the term. From (2.20) and (2.22), it is apparent that g_ℓ has a double pole in ϵ , but f_ℓ only has a single pole. The leading IR pole in the term under consideration is therefore $1/\epsilon^p$, where

$$p = 2 \sum_{\ell=1}^{\infty} n_\ell - k_1. \quad (2.28)$$

Combining (2.27) and (2.28), we find

$$p = 2L - k - \left[2 \sum_{\ell=1}^{\infty} (\ell - 1) n_\ell + 2\ell_0 - k_0 \right]. \quad (2.29)$$

Since $k_0 \leq \ell_0$, the term in square brackets is nonnegative, and we conclude that

$$|A^{(L,k)}(\epsilon)\rangle \sim \mathcal{O}\left(\frac{1}{\epsilon^{2L-k}}\right). \quad (2.30)$$

This behavior was previously conjectured in [41] and shown in [49] (subject to the assumptions stated above).

Next we review the derivation [41, 49] of the coefficient of the leading IR pole of $|A^{(L,k)}(\epsilon)\rangle$. Terms in (2.26) contribute to the leading IR pole only when the expression in square brackets in (2.29) vanishes, which occurs when $n_\ell = 0$ for $\ell \geq 2$, and $\ell_0 = k_0 = 0$ (with n_1 unconstrained). In other words, the leading IR divergences are given by [41, 49]

$$|A(\epsilon)\rangle \sim \exp\left[a \frac{\mathbf{G}^{(1)}(\epsilon)}{N}\right] |A^{(0)}\rangle \quad (\text{leading IR divergence}). \quad (2.31)$$

Recalling that

$$\frac{\mathbf{G}^{(1)}(\epsilon)}{N} = \left(\frac{\mu^2}{Q^2}\right)^\epsilon \left[-\frac{2}{\epsilon^2} \mathbb{1} + \frac{1}{\epsilon} \begin{pmatrix} \alpha & 0 \\ 0 & \delta \end{pmatrix} + \frac{1}{N\epsilon} \begin{pmatrix} 0 & \beta \\ \gamma & 0 \end{pmatrix} \right], \quad (2.32)$$

we use (2.31) to obtain the coefficient of the leading IR pole

$$|A^{(L,k)}(\epsilon)\rangle = \frac{(-2)^{L-k}}{k!(L-k)!} \frac{1}{\epsilon^{2L-k}} \begin{pmatrix} 0 & \beta \\ \gamma & 0 \end{pmatrix}^k |A^{(0)}\rangle + \mathcal{O}\left(\frac{1}{\epsilon^{2L-k-1}}\right), \quad (2.33)$$

where the tree-level amplitudes are

$$|H^{(0,0)}\rangle = |A^{(0)}\rangle = -\frac{4iK}{stu} (u, t, s, s, t, u, 0, 0, 0)^T, \quad (2.34)$$

where $s = (k_1 + k_2)^2$, $t = (k_1 + k_4)^2$ and $u = (k_1 + k_3)^2$ are the usual Mandelstam variables, obeying $s + t + u = 0$ for massless external gluons. The factor K , defined in (7.4.42) of [53], depends on the momenta and helicity of the external gluons and is totally symmetric under permutations of the external legs.

The leading IR pole of the planar amplitude is simply

$$|A^{(L,0)}(\epsilon)\rangle = \frac{(-2)^L}{L! \epsilon^{2L}} |A^{(0)}\rangle + \mathcal{O}\left(\frac{1}{\epsilon^{2L-1}}\right). \quad (2.35)$$

The remaining IR divergences, from $\mathcal{O}(1/\epsilon^{2L-1})$ to $\mathcal{O}(1/\epsilon)$, are all proportional to $|A^{(0)}\rangle$ and are given by the (generalized) ABDK equation [7] (see Appendix A of [49]).

We now write an explicit expression for the coefficients of the leading IR poles of *subleading-color* amplitudes. First we use (2.34) and (2.23) to show

$$\gamma \begin{pmatrix} u \\ t \\ s \\ s \\ t \\ u \end{pmatrix} = 2(sY - tX) \begin{pmatrix} 1 \\ 1 \\ 1 \\ 1 \end{pmatrix}, \quad \gamma\beta \begin{pmatrix} 1 \\ 1 \\ 1 \\ 1 \end{pmatrix} = 2(X^2 + Y^2 + Z^2) \begin{pmatrix} 1 \\ 1 \\ 1 \\ 1 \end{pmatrix} \quad (2.36)$$

with

$$X = \log\left(\frac{t}{u}\right), \quad Y = \log\left(\frac{u}{s}\right), \quad Z = \log\left(\frac{s}{t}\right). \quad (2.37)$$

Hence, the leading IR divergence of the subleading-color amplitudes is given by

$$\left| A^{(L,2m+1)}(\epsilon) \right\rangle = \left(\frac{-4iK}{stu} \right) \frac{(-1)^{L-1} 2^{L-m} (X^2 + Y^2 + Z^2)^m (sY - tX)}{(2m+1)!(L-2m-1)! \epsilon^{2L-2m-1}} \begin{pmatrix} 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 1 \\ 1 \\ 1 \end{pmatrix} + \mathcal{O}\left(\frac{1}{\epsilon^{2L-2m-2}}\right), \quad (2.38)$$

$$\left| A^{(L,2m+2)}(\epsilon) \right\rangle = \left(\frac{-4iK}{stu} \right) \frac{(-1)^L 2^{L-m-1} (X^2 + Y^2 + Z^2)^m (sY - tX)}{(2m+2)!(L-2m-2)! \epsilon^{2L-2m-2}} \begin{pmatrix} X - Y \\ Z - X \\ Y - Z \\ Y - Z \\ Z - X \\ X - Y \\ 0 \\ 0 \\ 0 \end{pmatrix} + \mathcal{O}\left(\frac{1}{\epsilon^{2L-2m-3}}\right). \quad (2.39)$$

The results (2.38) and (2.39) were derived in [49], generalizing expressions derived in [41].

2.3. IR Divergences of $A^{(L,1)}$

In this subsection, we consider the subleading-color amplitude $|A^{(L,1)}\rangle$ and derive the first three terms in the Laurent expansion. (It is straightforward to obtain further terms in the Laurent expansion as needed.) Consider all terms in (2.26) for which the expression in square brackets in (2.29) is ≤ 2 :

$$\left| A^{(L)}(\epsilon) \right\rangle = \frac{1}{L!} \left(g_1 + \frac{1}{N} f_1 \right)^L \left| A^{(0)} \right\rangle + \frac{1}{N(L-1)!} \left(g_1 + \frac{1}{N} f_1 \right)^{L-1} \left| H^{(1,1)}(\epsilon) \right\rangle$$

$$\begin{aligned}
& + \frac{1}{(L-2)!} \left(g_1 + \frac{1}{N} f_1 \right)^{L-2} \left(g_2 + \frac{1}{N} f_2 \right) |A^{(0)}\rangle + \frac{1}{(L-1)!} \left(g_1 + \frac{1}{N} f_1 \right)^{L-1} |H^{(1,0)}(\epsilon)\rangle \\
& + \frac{1}{N^2(L-2)!} \left(g_1 + \frac{1}{N} f_1 \right)^{L-2} |H^{(2,2)}(\epsilon)\rangle + \dots \quad (\text{three leading IR poles}),
\end{aligned} \tag{2.40}$$

where we use (2.20) and (2.22) to write

$$\begin{aligned}
g_1 &= \left(\frac{\mu^2}{Q^2} \right)^\epsilon \left[-\frac{2}{\epsilon^2} \mathbb{1} + \frac{1}{\epsilon} \begin{pmatrix} \alpha & 0 \\ 0 & \delta \end{pmatrix} \right], \quad f_1 = \frac{1}{\epsilon} \left(\frac{\mu^2}{Q^2} \right)^\epsilon \begin{pmatrix} 0 & \beta \\ \gamma & 0 \end{pmatrix}, \\
g_2 &= \left(\frac{\mu^2}{Q^2} \right)^{2\epsilon} \left[-\left(\frac{\gamma^{(2)}}{8\epsilon^2} + \frac{\mathcal{G}_0^{(2)}}{2\epsilon} \right) \mathbb{1} + \frac{\gamma^{(2)}}{8\epsilon} \begin{pmatrix} \alpha & 0 \\ 0 & \delta \end{pmatrix} \right], \quad f_2 = \frac{\gamma^{(2)}}{8\epsilon} \left(\frac{\mu^2}{Q^2} \right)^{2\epsilon} \begin{pmatrix} 0 & \beta \\ \gamma & 0 \end{pmatrix}.
\end{aligned} \tag{2.41}$$

To extract the $|A^{(L,1)}\rangle$ amplitude, we employ the identity

$$\begin{aligned}
& \left(g_1 + \frac{1}{N} f_1 \right)^L \Big|_{1/N \text{ piece}} \\
&= L g_1^{L-1} f_1 + \binom{L}{2} g_1^{L-2} [f_1, g_1] + \binom{L}{3} g_1^{L-3} [[f_1, g_1], g_1] + [\dots [[f_1, g_1], g_1], g_1] \dots],
\end{aligned} \tag{2.42}$$

in which the first term on the right-hand side has an expansion that starts with $1/\epsilon^{2L-1}$, the second term has an expansion that starts with $1/\epsilon^{2L-2}$, and so forth. Thus, keeping only the terms proportional to $1/N$ in (2.40), and only the first three terms in the Laurent expansion, we obtain

$$\begin{aligned}
|A^{(L,1)}\rangle &= \frac{1}{(L-1)!} g_1^{L-1} f_1 |A^{(0)}\rangle + \frac{1}{2(L-2)!} g_1^{L-2} [f_1, g_1] |A^{(0)}\rangle + \frac{1}{(L-1)!} g_1^{L-1} |H^{(1,1)}(\epsilon)\rangle \\
&+ \frac{1}{6(L-3)!} g_1^{L-3} [[f_1, g_1], g_1] |A^{(0)}\rangle + \frac{1}{(L-2)!} g_1^{L-2} f_2 |A^{(0)}\rangle + \frac{1}{(L-3)!} g_1^{L-3} f_1 g_2 |A^{(0)}\rangle \\
&+ \frac{1}{(L-2)!} g_1^{L-2} f_1 |H^{(1,0)}(\epsilon)\rangle + \mathcal{O}\left(\frac{1}{\epsilon^{2L-4}}\right),
\end{aligned} \tag{2.43}$$

as obtained in [49].

2.4. IR Divergences of $A^{(L,L)}$

In this subsection, we derive an expression for the coefficient of the IR divergences of the first two terms in the Laurent expansion of the most subleading-color amplitude $|A^{(L,L)}\rangle$.

The only terms in (2.26) that contribute to $|A^{(L,L)}\rangle$ are those with as many factors of $1/N$ as of a . Thus, only f_1 and $|H^{(\ell_0, \ell_0)}\rangle$ can contribute, giving

$$|A^{(L,L)}(\epsilon)\rangle = \sum_{\ell_0=0}^L \frac{1}{(L-\ell_0)!} f_1^{L-\ell_0} |H^{(\ell_0, \ell_0)}(\epsilon)\rangle, \quad \text{where } f_1 = \frac{1}{\epsilon} \left(\frac{\mu^2}{Q^2} \right)^\epsilon \begin{pmatrix} 0 & \beta \\ \gamma & 0 \end{pmatrix} \quad (2.44)$$

exact to all orders in the ϵ expansion. Keeping just the first two terms in the Laurent expansion, we find

$$\begin{aligned} |A^{(L,L)}(\epsilon)\rangle &= \frac{1}{(L-1)!} f_1^{L-1} \left[\frac{1}{L} f_1 |A^{(0)}\rangle + |H^{(1,1)}(\epsilon)\rangle \right] + \mathcal{O}\left(\frac{1}{\epsilon^{L-2}}\right) \\ &= \frac{1}{(L-1)!} \frac{1}{\epsilon^{L-1}} \begin{pmatrix} 0 & \beta \\ \gamma & 0 \end{pmatrix}^{L-1} |A^{(1,1)}(L\epsilon)\rangle + \mathcal{O}\left(\frac{1}{\epsilon^{L-2}}\right). \end{aligned} \quad (2.45)$$

This was derived in [49] and confirms the conjecture made in (4.45) and (4.46) of [41].

2.5. Exact Expressions at One and Two Loops

$\mathcal{N} = 4$ SYM scattering amplitudes may be expressed in terms of planar and nonplanar scalar loop integrals. The two-loop four-gluon scattering amplitude was first computed by Bern et al. [54] (see also [36]). Explicit expressions for these IR-divergent integrals as the Laurent expansions in ϵ were later obtained by Smirnov in the planar case [55] and by Tausk in the nonplanar case [56]. In this subsection, we review these results and some formulas for the $1/N$ expansion of these divergences.

Recall from (2.17) that $A_{[i]}^{(L,k)}$ denotes the L -loop color-ordered amplitude which is subleading by a factor of $1/N^k$ in the $1/N$ expansion. Single-trace amplitudes are denoted by $i = 1, \dots, 6$ and double-trace amplitudes by $i = 7, \dots, 9$ (see (1.7)).

At one loop, the single-trace amplitudes are given by [34]

$$A_{[1]}^{(1,0)} = M^{(1)}(s, t) A_{[1]}^{(0)} = 2iK I_4^{(1)}(s, t) \quad (2.46)$$

with the other single-trace amplitudes $A_{[2]}^{(1,0)}$ and $A_{[3]}^{(1,0)}$ obtained by letting $t \leftrightarrow u$, and $s \leftrightarrow u$, respectively. The identities $A_{[1]}^{(L)} = A_{[6]}^{(L)}$, $A_{[2]}^{(L)} = A_{[5]}^{(L)}$, and $A_{[3]}^{(L)} = A_{[4]}^{(L)}$ are satisfied at all loop orders. In (2.46), $I_4^{(1)}(s, t)$ denotes the scalar box integral

$$\begin{aligned} M^{(1)}(s, t) &= -\frac{1}{2} s t I_4^{(1)}(s, t), \\ I_4^{(1)}(s, t) &= I_4^{(1)}(t, s) = -i\mu^{2\epsilon} e^{\epsilon\gamma} \pi^{-D/2} \int \frac{d^D p}{p^2 (p-k_1)^2 (p-k_1-k_2)^2 (p+k_4)^2}, \end{aligned} \quad (2.47)$$

an explicit expression for which is given, for example, in [7].

The one-loop double-trace amplitudes are given by [34]

$$A_{[7]}^{(1,1)} = A_{[8]}^{(1,1)} = A_{[9]}^{(1,1)} = 2 \left(A_{[1]}^{(1,0)} + A_{[2]}^{(1,0)} + A_{[3]}^{(1,0)} \right) \quad (2.48)$$

$$= 4iK \left[I_4^{(1)}(s, t) + I_4^{(1)}(t, u) + I_4^{(1)}(u, s) \right]. \quad (2.49)$$

Relation (2.48) follows from the one-loop U(1) decoupling identity [57].

At two loops, the leading-color single-trace amplitude is given by [54]

$$A_{[1]}^{(2,0)} = M^{(2)}(s, t) A_{[1]}^{(0)} = -iK \left[s I_4^{(2)P}(s, t) + t I_4^{(2)P}(t, s) \right], \quad (2.50)$$

where $I_4^{(2)P}(s, t)$ denotes the scalar double-box (planar) integral

$$M^{(2)}(s, t) = \frac{1}{4} s t \left[s I_4^{(2)P}(s, t) + t I_4^{(2)P}(t, s) \right],$$

$$I_4^{(2)P}(s, t) = \left(-i\mu^{2\epsilon} e^{\epsilon\gamma} \pi^{-D/2} \right)^2 \int \frac{d^D p d^D q}{p^2 (p+q)^2 q^2 (p-k_1)^2 (p-k_1-k_2)^2 (q-k_4)^2 (q-k_3-k_4)^2}, \quad (2.51)$$

an explicit expression for which is given, for example, in [7]. The double-trace amplitude is [54]

$$A_{[7]}^{(2,1)} = -2iK \left[s \left(3I_4^{(2)P}(s, t) + 2I_4^{(2)NP}(s, t) + 3I_4^{(2)P}(s, u) + 2I_4^{(2)NP}(s, u) \right) \right. \\ \left. - t \left(I_4^{(2)NP}(t, s) + I_4^{(2)NP}(t, u) \right) - u \left(I_4^{(2)NP}(u, s) + I_4^{(2)NP}(u, t) \right) \right], \quad (2.52)$$

and the subleading-color single-trace amplitude is [54]

$$A_{[1]}^{(2,2)} = -2iK \left[s \left(I_4^{(2)P}(s, t) + I_4^{(2)NP}(s, t) + I_4^{(2)P}(s, u) + I_4^{(2)NP}(s, u) \right) \right. \\ \left. + t \left(I_4^{(2)P}(t, s) + I_4^{(2)NP}(t, s) + I_4^{(2)P}(t, u) + I_4^{(2)NP}(t, u) \right) \right. \\ \left. - 2u \left(I_4^{(2)P}(u, s) + I_4^{(2)NP}(u, s) + I_4^{(2)P}(u, t) + I_4^{(2)NP}(u, t) \right) \right], \quad (2.53)$$

where $I_4^{(2)NP}(s, t)$ denotes the two-loop nonplanar integral

$$I_4^{(2)NP}(s, t) = \left(-i\mu^{2\epsilon} e^{\epsilon\gamma} \pi^{-D/2} \right)^2 \int \frac{d^D p d^D q}{p^2 (p+q)^2 q^2 (p-k_2)^2 (p+q+k_1)^2 (q-k_3)^2 (q-k_3-k_4)^2}, \quad (2.54)$$

an explicit expression for which is given in [56]. All the other single- and double-trace amplitudes $A_{[i]}^{(2,k)}$ are obtained by making the appropriate permutations of s, t , and u in these expressions.

It is well known [7] that planar amplitudes have the property of uniform transcendentality. It is less obvious but nevertheless true [41] that subleading-color $\mathcal{N} = 4$ amplitudes at one and two loops (and presumably beyond) also have uniform transcendentality. What makes this surprising is that the nonplanar integral $I_4^{(2)NP}(s, t)$ that contributes to $A^{(2,1)}$ and $A^{(2,2)}$ does not have uniform transcendentality [39, 58]. The subleading transcendentality parts, however, cancel out in the expressions (2.52) and (2.53). (The same thing happens for the two-loop four-point amplitude of $\mathcal{N} = 8$ supergravity [39, 58].)

The two-loop amplitudes obey the following group theory relations [59]:

$$\begin{aligned} A_{[7]}^{(2,1)} &= 2\left(A_{[1]}^{(2,0)} + A_{[2]}^{(2,0)} + A_{[3]}^{(2,0)}\right) - A_{[3]}^{(2,2)}, \\ A_{[8]}^{(2,1)} &= 2\left(A_{[1]}^{(2,0)} + A_{[2]}^{(2,0)} + A_{[3]}^{(2,0)}\right) - A_{[1]}^{(2,2)}, \\ A_{[9]}^{(2,1)} &= 2\left(A_{[1]}^{(2,0)} + A_{[2]}^{(2,0)} + A_{[3]}^{(2,0)}\right) - A_{[2]}^{(2,2)} \end{aligned} \quad (2.55)$$

and may be easily verified using (2.50), (2.52), and (2.53). In addition, we have

$$A_{[1]}^{(2,2)} + A_{[2]}^{(2,2)} + A_{[3]}^{(2,2)} = 0, \quad (2.56)$$

also easily verified using (2.53). Together these equations imply

$$6 \sum_{i=1}^3 A_{[i]}^{(2,0)} - \sum_{i=7}^9 A_{[i]}^{(2,1)} = 0 \quad (2.57)$$

which is the two-loop generalization of the U(1) decoupling relation (2.48). Both (2.56) and (2.57) are encapsulated in the equation

$$6 \sum_{i=1}^3 A_{[i]}^{(L)} - N \sum_{i=7}^9 A_{[i]}^{(L)} = 0, \quad L \leq 2, \quad (2.58)$$

which is valid through two loops.

At one loop, we also saw that one can relate all the subleading-color amplitudes $A_{n;j}$ to the leading amplitude $A_{n;1}$ via the group theory relation (1.5).

We now list some explicit formulas for the IR-divergent pieces of one- and two-loop amplitudes that will be of use in the following section. At one loop, the leading 4-point amplitude is given by (2.46) with

$$M^{(1)}(s, t) = -\frac{1}{\epsilon^2} \left(\frac{\mu^2}{-s} \right)^\epsilon - \frac{1}{\epsilon^2} \left(\frac{\mu^2}{-t} \right)^\epsilon + \frac{1}{2} \log^2 \left(\frac{s}{t} \right) + \frac{2\pi^2}{3} + \mathcal{O}(\epsilon), \quad (2.59)$$

while the exact relation (2.48) can be used to write both the IR-divergent and IR-finite contributions to the double-trace subleading-color amplitude

$$\left| A^{(1,1)}(\epsilon) \right\rangle = \left(\frac{-8iK}{stu} \right) \left[\left(\frac{\mu^2}{-u} \right)^\epsilon \frac{(sY - tX)}{\epsilon} - (s + t)XY \right] \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix} + \mathcal{O}(\epsilon), \quad (2.60)$$

where we have only included the [7–9] components of $A_{[ij]}^{(1,1)}$ as the others vanish.

At two loops, the planar amplitude is given by (2.50) with [60]

$$M^{(2)}(\epsilon) = \frac{1}{2} \left[M^{(1)}(\epsilon) \right]^2 - \left(\zeta_2 + \epsilon \zeta_3 + \epsilon^2 \zeta_4 \right) M^{(1)}(2\epsilon) - \frac{\pi^4}{72} + \mathcal{O}(\epsilon). \quad (2.61)$$

The two-loop double trace amplitude has an IR divergence given by the general formula (2.38), which yields

$$\left| A^{(2,1)}(\epsilon) \right\rangle = \left(\frac{-8iK}{stu} \right) \frac{(-2)(sY - tX)}{\epsilon^3} \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix} + \mathcal{O}\left(\frac{1}{\epsilon^2}\right). \quad (2.62)$$

Finally, the subleading-color single-trace amplitude is given by (2.45) which in this case yields

$$\left| A^{(2,2)}(\epsilon) \right\rangle = \frac{1}{\epsilon} \begin{pmatrix} X - Y \\ Z - X \\ Y - Z \\ Y - Z \\ Z - X \\ X - Y \end{pmatrix} A_{[7]}^{(1,1)}(2\epsilon) + \mathcal{O}(\epsilon^0). \quad (2.63)$$

Only the [1] through [6] components are listed, as the [7] through [9] components vanish.

3. Subleading-Color Amplitudes of $\mathcal{N} = 4$ SYM and Amplitudes of $\mathcal{N} = 8$ Supergravity

The $\text{AdS}_5/\text{CFT}_4$ correspondence provides a strong/weak duality between $\mathcal{N} = 4$ SYM and $\mathcal{N} = 8$ supergravity. These relationships have been extensively explored and exploited. There are also numerous indications of a weak/weak duality between the two theories, although this latter possibility is much less developed. Nevertheless this may be a very fruitful approach in attempts to understand relationships between the two theories. A lot of work has been done to relate the perturbation expansions of these two theories [15, 34–38, 41–46, 61, 62]. Part of this program is the extension of the tree-level KLT theories, but

many relations have been found at loop level as well. Since this work is extensive, we will not attempt to review it all here. Since nonplanar graphs appear on an equal footing with planar graphs in $\mathcal{N} = 8$ supergravity, it seems important to understand nonplanar graphs in $\mathcal{N} = 4$ SYM if a weak-weak duality is to be explored. This is the focus of this section.

We will review the known exact relations between the 4-point functions of subleading $\mathcal{N} = 4$ SYM and those of $\mathcal{N} = 8$ supergravity, at one and two loops. For more than two loops, the known relation for $n = 4$ is for the leading IR singularity only. One application of these ideas for $n = 5$ at one loop is a new form of (tree level) KLT relations. Others are possible relations between $\mathcal{N} = 4$ subleading-color amplitudes and $\mathcal{N} = 8$ sugra for $n \geq 5$.

3.1. One and Two-Loop Relations

In this subsection, we demonstrate the existence of some exact relations between $\mathcal{N} = 4$ SYM amplitudes and $\mathcal{N} = 8$ supergravity amplitudes at the one- and two-loop levels. The L -loop N -independent SYM amplitude $A^{(L,L)}$ may be expected to be related to the L -loop supergravity amplitude, as both have $\mathcal{O}(1/\epsilon^L)$ leading IR divergences. Other subleading-color SYM amplitudes $A^{(L,k)}$ have $\mathcal{O}(1/\epsilon^{2L-k})$ leading IR divergences and consequently satisfy relations involving lower-loop supergravity amplitudes. The normalization of $A_{\text{SYM}}^{(L,2m+1)}(s,t)$ is arbitrary. We have chosen one that is most natural in the context of the SYM/supergravity relations presented in this subsection.

In this section we use the notation

$$A_{\text{SYM}}^{(L,2m)}(s,t) = a^L A_{[1]}^{(L,2m)}, \quad A_{\text{SYM}}^{(L,2m+1)}(s,t) = -\frac{a^L}{\sqrt{2}} A_{[8]}^{(L,2m+1)}, \quad (3.1)$$

noting that the other components $A_{[i]}^{(L,k)}$ are obtained by permutations of s, t , and u . However, we omit the argument (s,t) for functions that are completely symmetric under permutations of s, t , and u .

Factor out the tree amplitude to define

$$M_{\text{SYM}}^{(L,k)}(s,t) = \frac{A_{\text{SYM}}^{(L,k)}(s,t)}{A_{\text{SYM}}^{(0)}(s,t)}, \quad (3.2)$$

so that the coupling constant a^L is now included in the definition of $M_{\text{SYM}}^{(L,k)}(s,t)$, and where.

$$A_{\text{SYM}}^{(0)}(s,t) = -\frac{4iK}{st}. \quad (3.3)$$

In what follows we denote $A_{\text{SYM}}^{\text{tree}}(ij \dots k) = A(ij \dots k)$ (see also (2.34)). Recall that the one-loop N -independent SYM four-gluon amplitude is given by (2.47), obtaining

$$A_{\text{SYM}}^{(1,1)} = -2\sqrt{2}iK \left[\frac{g^2 N}{8\pi^2} (4\pi e^{-\gamma})^\epsilon \right] \left[I_4^{(1)}(s,t) + I_4^{(1)}(t,u) + I_4^{(1)}(u,s) \right]. \quad (3.4)$$

The one-loop supergravity four-graviton amplitude may be expressed as [34, 36]

$$A_{\text{SG}}^{(1)} = 8iK^2 \left[\frac{(\kappa/2)^2}{8\pi^2} (4\pi e^{-\gamma})^\epsilon \right] \left[I_4^{(1)}(s, t) + I_4^{(1)}(t, u) + I_4^{(1)}(u, s) \right]. \quad (3.5)$$

after stripping off a factor of $(\kappa/2)^2$ for a four-point amplitude. The supergravity amplitude is proportional to K^2 rather than K due to the KLT relations [63] (a manifestation of the relation “closed string = (open string)²”). This factor is also present in the tree-level supergravity amplitude, so we can factor it out as follows:

$$A_{\text{SG}}^{(1)} = A_{\text{SG}}^{(0)} M_{\text{SG}}^{(1)} = \left(\frac{16iK^2}{stu} \right) M_{\text{SG}}^{(1)}. \quad (3.6)$$

Defining $\lambda_{\text{SYM}} = g^2 N$ and $\lambda_{\text{SG}} = (\kappa/2)^2$, one observes that the one-loop SYM and supergravity amplitudes are related by

$$M_{\text{SYM}}^{(1,1)}(s, t) = \sqrt{2} \frac{\lambda_{\text{SYM}}}{\lambda_{\text{SG}} u} M_{\text{SG}}^{(1)}. \quad (3.7)$$

In other words, the ratio of the one-loop subleading-color SYM and the one-loop supergravity amplitudes (after factoring out the tree amplitudes) is simply proportional to the ratio of coupling constants, where we encounter the effective dimensionless coupling $\lambda_{\text{SG}} u$ for supergravity because λ_{SG} is dimensionful.

Finally, rewrite (3.7) in the manifestly permutation-symmetric form

$$\frac{1}{3} \left[(\lambda_{\text{SG}} u) M_{\text{SYM}}^{(1,1)}(s, t) + \text{c.p.} \right] = \sqrt{2} \lambda_{\text{SYM}} M_{\text{SG}}^{(1)}, \quad (3.8)$$

(where c.p. denotes cyclic permutations of s , t , and u) even though $u M_{\text{SYM}}^{(1,1)}(s, t)$ is already symmetric under permutations. A similar symmetrized relation can be written for the one-loop leading-color amplitude

$$(\lambda_{\text{SG}} u) M_{\text{SYM}}^{(1,0)}(s, t) + \text{c.p.} = -\lambda_{\text{SYM}} M_{\text{SG}}^{(1)} \quad (3.9)$$

obtained from the one-loop decoupling relation (2.48) together with (3.7).

Now turn to *two loops*. There are some relations between SYM and supergravity amplitudes that hold only for the IR-divergent terms. The easiest case to analyze is the two-loop N -independent SYM amplitude $A_{\text{SYM}}^{(2,2)}(s, t)$, since, from (2.63),

$$A_{\text{SYM}}^{(2,2)}(s, t) = -\sqrt{2} a \frac{X - Y}{\epsilon} A_{\text{SYM}}^{(1,1)}(2\epsilon) + \mathcal{O}(\epsilon^0). \quad (3.10)$$

This can be rewritten as

$$M_{\text{SYM}}^{(2,2)}(s, t) = -2a \frac{\lambda_{\text{SYM}}}{\lambda_{\text{SG}} u} \left(\frac{X - Y}{\epsilon} \right) M_{\text{SG}}^{(1)}(2\epsilon) + \mathcal{O}(\epsilon^0), \quad (3.11)$$

where $X = \log(t/u)$, $Y = \log(u/s)$, $Z = \log(s/t)$, as in (2.37), thus obtaining a relation to the one-loop supergravity amplitude.

Using the relation $M_{\text{SG}}^{(2)}(\epsilon) = (1/2)[M_{\text{SG}}^{(1)}(\epsilon)]^2 + \mathcal{O}(\epsilon^0)$ between the one- and two-loop supergravity amplitudes [41, 58, 64, 65], we can write this as

$$\frac{1}{3} \left[(\lambda_{\text{SG}} u)^2 M_{\text{SYM}}^{(2,2)}(s, t) + \text{c.p.} \right] = 2\lambda_{\text{SYM}}^2 M_{\text{SG}}^{(2)}, \quad (3.12)$$

where this relation is exact (!), as may be easily verified by using the exact expression for the N -independent SYM amplitude [54] and from (2.53)

$$\begin{aligned} M_{\text{SYM}}^{(2,2)}(s, t) = \frac{a^2 s t}{2} & \left[s \left(I_4^{(2)P}(s, t) + I_4^{(2)NP}(s, t) + I_4^{(2)P}(s, u) + I_4^{(2)NP}(s, u) \right) \right. \\ & + t \left(I_4^{(2)P}(t, s) + I_4^{(2)NP}(t, s) + I_4^{(2)P}(t, u) + I_4^{(2)NP}(t, u) \right) \\ & \left. - 2u \left(I_4^{(2)P}(u, s) + I_4^{(2)NP}(u, s) + I_4^{(2)P}(u, t) + I_4^{(2)NP}(u, t) \right) \right] \end{aligned} \quad (3.13)$$

and that for the two-loop supergravity amplitude [36]

$$M_{\text{SG}}^{(2)} = -\frac{s^3 t u}{4} \left[\frac{(\kappa/2)^2}{8\pi^2} (4\pi e^{-\gamma})^\epsilon \right]^2 \left[I_4^{(2)P}(s, t) + I_4^{(2)NP}(s, t) + I_4^{(2)P}(s, u) + I_4^{(2)NP}(s, u) \right] + \text{c.p.}, \quad (3.14)$$

where $I_4^{(2)P}$ and $I_4^{(2)NP}$ are the two-loop planar and nonplanar 4-point functions.

Now consider the two-loop subleading-color amplitude $M_{\text{SYM}}^{(2,1)}$. The two-loop decoupling relation (2.57) can be rewritten as

$$-\sqrt{2} \left[u M_{\text{SYM}}^{(2,1)}(s, t) + \text{c.p.} \right] = 6 \left[u M_{\text{SYM}}^{(2,0)}(s, t) + \text{c.p.} \right]. \quad (3.15)$$

Using the ABDK relation [60]

$$M_{\text{SYM}}^{(2,0)}(\epsilon) = \frac{1}{2} \left[M_{\text{SYM}}^{(1,0)}(\epsilon) \right]^2 + a f^{(2)}(\epsilon) M_{\text{SYM}}^{(1,0)}(2\epsilon) + \mathcal{O}(\epsilon), \quad f^{(2)}(\epsilon) = -\left(\zeta_2 + \epsilon \zeta_3 + \epsilon^2 \zeta_4 \right), \quad (3.16)$$

together with (3.9), we can rewrite (3.15) as

$$\begin{aligned} & \frac{1}{3} \left[(\lambda_{\text{SG}} u) M_{\text{SYM}}^{(2,1)}(s, t) + \text{c.p.} \right] + \frac{1}{\sqrt{2}} \left\{ (\lambda_{\text{SG}} u) \left[M_{\text{SYM}}^{(1,0)}(s, t) \right]^2 + \text{c.p.} \right\} \\ &= \sqrt{2} \frac{\lambda_{\text{SYM}}^2}{8\pi^2} (4\pi e^{-r})^\epsilon f^{(2)}(\epsilon) M_{\text{SG}}^{(1)}(2\epsilon) + \mathcal{O}(\epsilon). \end{aligned} \quad (3.17)$$

Unlike (3.12), however, (3.17) only holds through $\mathcal{O}(\epsilon^0)$, which relates to the one-loop supergravity amplitude rather than the two-loop one.

Note that (3.8) and (3.12) can be written as

$$\frac{1}{3} \left[(\lambda_{\text{SG}} u)^L M_{\text{SYM}}^{(L,L)}(s, t) + \text{c.p.} \right] = \left(\sqrt{2} \lambda_{\text{SYM}} \right)^L M_{\text{SG}}^{(L)} \quad (3.18)$$

for $L = 0, 1$, and 2 . Can this relation be valid at higher loops? It turns out not to be the case, but we can still find some relations valid for $L \geq 3$.

3.2. Three or More Loops

On the supergravity side, there is an exact exponentiation formula [64, 65], which implies

$$M_{\text{SG}}^{(L)} = \frac{1}{L!} \left[M_{\text{SG}}^{(1)} \right]^L + \mathcal{O}\left(\frac{1}{\epsilon^{L-2}}\right) = \frac{1}{L!} \left[\frac{-\lambda_{\text{SG}}(sY - tX)}{8\pi^2 \epsilon} \right]^L + \mathcal{O}\left(\frac{1}{\epsilon^{L-1}}\right). \quad (3.19)$$

Since the leading IR divergences of $A^{(L,L)}$ is $\mathcal{O}(1/\epsilon^L)$, one can show that the following relations hold:

$$\begin{aligned} & \left[\lambda_{\text{SG}}^2 \frac{s^2 + t^2 + u^2}{3} \right]^k \frac{1}{3} \left[(\lambda_{\text{SG}} u) M_{\text{SYM}}^{(2k+1, 2k+1)}(s, t; \epsilon) + \text{c.p.} \right] \\ &= \lambda_{\text{SYM}}^{2k+1} \frac{2^{2k+1/2}}{(2k+1)!} \left[M_{\text{SG}}^{(2)}(\epsilon) + \frac{1}{6} \left(\frac{\lambda_{\text{SG}}}{8\pi^2} \right)^2 \left(\frac{sX + tY + uZ}{\epsilon} \right)^2 \right]^k M_{\text{SG}}^{(1)}(\epsilon) + \mathcal{O}\left(\frac{1}{\epsilon^{2k}}\right), \end{aligned} \quad (3.20)$$

for $L = 2k + 1$

$$\begin{aligned} & \left[\lambda_{\text{SG}}^2 \frac{s^2 + t^2 + u^2}{3} \right]^k \frac{1}{3} \left[(\lambda_{\text{SG}} u)^2 M_{\text{SYM}}^{(2k+2, 2k+2)}(s, t; \epsilon) + \text{c.p.} \right] \\ &= \lambda_{\text{SYM}}^{2k+2} \frac{2^{2k+2}}{(2k+2)!} \left[M_{\text{SG}}^{(2)}(\epsilon) + \frac{1}{6} \left(\frac{\lambda_{\text{SG}}}{8\pi^2} \right)^2 \left(\frac{sX + tY + uZ}{\epsilon} \right)^2 \right]^k M_{\text{SG}}^{(2)}(\epsilon) + \mathcal{O}\left(\frac{1}{\epsilon^{2k+1}}\right) \end{aligned} \quad (3.21)$$

for $L = 2k + 2$ (where $k = 0, 1, 2, \dots$).

That is, we have an exact relation at L -loops for the leading IR divergence $\sim \mathcal{O}(1/\epsilon^L)$, with an untested relation for the subleading divergence of $\mathcal{O}(1/\epsilon^{L-1})$; see also (2.45).

An interesting fact is that either (3.18) or (3.20) and (3.21) without the extra term, and also the relation (3.17), have a possible interpretation in terms of the 't Hooft string picture of the $1/N$ expansion. Thus at least in the case of $L = 1, 2$, (3.18) and (3.17) still do, so one can hope that there is a correct relation at higher L yet to be determined.

3.3. New KLT Relations

One of the pioneering connections between SYM and supergravity theories are the KLT relations [63], originally proved using string theory methods [35, 63]. More recently, alternate versions of KLT relations have been presented based on field theoretic techniques at the tree level [44, 45]. One form of these new relations has manifest $(n - 3)!$ permutation symmetry for the n -point functions, and another has $(n - 2)!$ symmetry, but requires regularization as a consequence of singularities. They are part of a flurry of recent activity relating $\mathcal{N} = 4$ SYM and $\mathcal{N} = 8$ supergravity, including [40, 42, 43, 46, 61, 66–68] (among older works see also [37, 69, 70]). Recent work applying the KLT relations includes [71–74]. In our quest for SYM-supergravity relations, we first review previous KLT relations; we then note that $A_{5,3}$ and the 1-loop supergravity amplitude both have $1/\epsilon$ IR divergences. We present here a tree-level KLT relation for the $n = 5$ -point amplitudes derived in [75], using information from one-loop SYM and supergravity amplitudes and their IR divergences. This results in a KLT relation for 5-point functions with $2(n - 2)!$ manifest symmetry, without the need for regularization. These KLT relations are proved explicitly using the helicity spinor formalism and the Parke-Taylor formula. In analogy with Section 3.1 on 4-point functions of $\mathcal{N} = 8$ supergravity and subleading-color $\mathcal{N} = 4$ SYM theories, both with the $1/\epsilon$ IR divergence, we explore the possibility that the 1-loop 5-point supergravity amplitude can be expressed as a linear combination of the $A_{5,3}$ SYM amplitudes. In particular a linear relation is proposed among the $1/\epsilon$ IR divergences of the two theories.

At tree level, the KLT relations are quadratic relations between the n -point amplitudes of $\mathcal{N} = 4$ SYM and those of $\mathcal{N} = 8$ supergravity. In these relations, the helicity information is all contained within the amplitudes, and the coefficients are all function of the kinematic invariants s_{ij} only.

These relations relate graviton tree amplitudes with sums of squares (products) of gauge tree amplitudes. The original KLT relations were derived from string theory in the $\alpha' \rightarrow 0$ limit [35, 63] and can be expressed as (we use the notation of [37])

$$\begin{aligned}
 A_{n,\text{sugra}}^{\text{tree}}(12 \dots n) &= (-1)^{n+1} \left[A_n(12 \dots n) \sum_{\text{perms}} f(i_1 \dots i_j) \bar{f}(l_1 \dots l_{j'}) \right. \\
 &\quad \left. \times A_n(i_1, \dots, i_j, 1, n-1, l_1, \dots, l_{j'}, n) \right] + \mathcal{P}(2, \dots, n-2) \\
 f(i_1, \dots, i_j) &= s(1, i_j) \prod_{m=1}^{j-1} \left(s(1, i_m) + \sum_{k=m+1}^j g(i_m, i_k) \right), \\
 \bar{f}(l_1, \dots, l_{j'}) &= s(l_1, n-1) \prod_{m=2}^{j'} \left(s(l_m, n-1) + \sum_{k=1}^{m-1} g(l_k, l_m) \right),
 \end{aligned} \tag{3.22}$$

where “perms” are $(i_1, \dots, i_j) \in \mathcal{P}(2, \dots, n/2)$, $(l_1, \dots, l_{j'}) \in \mathcal{P}(n/2 + 1, \dots, n - 2)$, $j = n/2 - 1$, $j' = n/2 - 2$, and $g_{ij} = s_{ij}$ if $i > j$ and zero otherwise.

In [44, 45], new forms of the KLT relations for any n -point function were found. They are both written in terms of the functions:

$$\begin{aligned} \mathcal{S}[i_1 \dots i_k \mid j_1 \dots j_k] &= \prod_{t=1}^k \left(s_{i_t 1} + \sum_{q>t}^k \theta(i_t, i_q) s_{i_t i_q} \right), \\ \tilde{\mathcal{S}}[i_1 \dots i_k \mid j_1 \dots j_k] &= \prod_{t=1}^k \left(s_{j_t n} + \sum_{q<t}^k \theta(j_q, j_t) s_{j_q j_t} \right), \end{aligned} \quad (3.23)$$

where $\theta(i_t, i_q)$ is zero if (i_t, i_q) has the same order in both sets $\mathcal{O} = \{i_1, \dots, i_k\}$ and $\mathcal{Q} = \{j_1, \dots, j_k\}$ and is 1 otherwise, and similarly for $\theta(j_q, j_t)$.

A form of KLT relations was found in [44], but needs to be regularized, due to a singular denominator

$$\begin{aligned} A_{n,\text{sugra}}^{\text{tree}} &= (-1)^n \sum_{\gamma, \beta} \frac{\tilde{A}_n(n, \gamma_{2,n-1}, 1) \mathcal{S}[\gamma_{2,n-1}, \beta_{2,n-1}]_{p_1} A_n(1, \beta_{2,n-1}, n)}{s_{12\dots n-1}}, \\ A_{n,\text{sugra}}^{\text{tree}} &= (-1)^n \sum_{\beta, \gamma} \frac{A_n(n, \beta_{2,n-1}, 1) \tilde{\mathcal{S}}[\beta_{2,n-1}, \gamma_{2,n-1}]_{p_n} \tilde{A}_n(1, \gamma_{2,n-1}, n)}{s_{23\dots n}}. \end{aligned} \quad (3.24)$$

However they have a large $(n-2)!$ manifest symmetry. Another set was proven in [45] which is nonsingular

$$\begin{aligned} A_{n,\text{sugra}}^{\text{tree}} &= (-1)^{n+1} \sum_{\sigma \in S_{n-3}} \sum_{\alpha \in S_{j-1}} \sum_{\beta \in S_{n-j-2}} A_n(1, \sigma_{2,j}, \sigma_{j+1,n-2}, n-1, n) \mathcal{S}[\alpha_{\sigma(2),\sigma(j)} \mid \sigma_{2,j}]_{p_1} \\ &\quad \times \tilde{\mathcal{S}}[\sigma_{j+1,n-2} \mid \beta_{\sigma(j+1),\sigma(n-2)}, n]_{p_n} \tilde{A}_n(\alpha_{\sigma(2),\sigma(j)}, 1, n-1, \beta_{\sigma(j+1),\sigma(n-1)}, n) \end{aligned} \quad (3.25)$$

but with only $(n-3)!$ manifest symmetry.

The original KLT relation for the 5-point function is

$$A_{5,\text{sugra}}^{\text{tree}} = s_{12}s_{34}A(12345)\tilde{A}(21435) + s_{13}s_{24}A(13245)\tilde{A}(31425) \quad (3.26)$$

and has $(n-3)! = 2!$ symmetry, whereas the KLT relations (3.25) become, explicitly,

$$\begin{aligned}
A_{5,\text{sugra}}^{\text{tree}} &= \sum_{\sigma, \tilde{\sigma} \in S_2} \tilde{A}(45, \tilde{\sigma}_{23}, 1) A(1, \sigma_{23}, 45) S[\tilde{\sigma}_{2,3} \mid \sigma_{2,3}]_{p_1} \\
&= s_{12}s_{13}(A(45231)A(12345) + A(45321)A(13245)) + s_{13}(s_{12} + s_{23})A(45231)A(13245) \\
&\quad + s_{12}(s_{13} + s_{23})A(45321)A(12345), \\
A_{5,\text{sugra}}^{\text{tree}} &= \sum_{\sigma, \tilde{\sigma} \in S_2} \tilde{A}(14, \tilde{\sigma}_{23}, 5) A(1, \sigma_{23}, 45) \tilde{S}[\sigma_{2,3} \mid \tilde{\sigma}_{2,3}]_{p_4} \\
&= s_{24}s_{34}[A(12345)A(14235) + A(13245)A(14325)] + s_{34}(s_{24} + s_{23})A(12345)A(14325) \\
&\quad + s_{24}(s_{34} + s_{23})A(13245)A(14235)
\end{aligned} \tag{3.27}$$

and have $(n-3)! = 2!$ symmetry.

We now derive another KLT relation for 5-point amplitudes using information about subleading one-loop amplitudes.

As we saw in (1.5), the $A_{n;j}$ are related to the $A_{n;1}$ via group theory. In particular, for 5-point amplitudes, one has a single-trace amplitude $A_{5;1}$ and a double-trace amplitude $A_{5;3}$ related by [76]

$$A_{5;3}(45123) = \sum_{\sigma \in \text{COPI}_4^{123}} A_{5;1}(\sigma(1), \dots, \sigma(4), 5). \tag{3.28}$$

The single-trace amplitude is given by

$$A_5^{(1,0)}(12345) \equiv A_{5;1}(12345) = -\frac{1}{4} A(12345) \sum_{\text{cyclic}} F^{(1)}(s, t, m^2), \tag{3.29}$$

where

$$F^{(1)}(s, t, m^2) = st I_5^{(1)}(s, t, m^2) \tag{3.30}$$

is the dimensionless one-mass box, and $I^{(1)}(s, t, m^2)$ is the 1-loop scalar box integral (2.47) with momenta 3, 4 in the same corner and $m^2 = P^2 = (p_3 + p_4)^2$.

Substituting in (3.28), we find

$$A_{5;3}(fg; hij) = \sum_{abcde \in 30 \text{ fixed terms}} F(cde; ab) [s_{abcde;+;fghij} A(abcde) + s_{abcde;-;fghij} A(abedc)]. \tag{3.31}$$

Here $s_{abcde;\pm;fghij}$ are signs, defined as follows. The relative sign is plus if ab belong to hij , and minus otherwise, and the overall sign is plus if the permutation of hij inside $abcde$ is even, and minus if it is odd.

The 1-loop $\mathcal{N} = 8$ supergravity amplitude is [62], written in terms of the scalar 1 m box $I(123, 45)$ (with momenta 4,5 on the same corner of the box), and the dimensionless box $F(123; 45)$ is

$$A_{5,\text{sugra}}^{1\text{-loop}}(12q_3q_2q_1) = -\frac{1}{2} \sum_{30 \text{ perms}} s_{q_2q_1} s_{12} s_{2q_3} A(12q_3q_2q_1) A(12q_3q_1q_2) F(12q_3; q_2q_1) \quad (3.32)$$

or

$$A_{5,\text{sugra}}^{1\text{-loop}}(12345) = -\frac{1}{2} \sum_{30 \text{ perms}} F(cde; ab) s_{cd} s_{de} s_{ab} A(cdeab) A(cdeba). \quad (3.33)$$

The IR behavior of the 1-loop 1 m scalar box is

$$\begin{aligned} I_{4,1m}(s, t, m^2) &= \frac{r_\Gamma}{s_{12}s_{23}} \left\{ \frac{2}{\epsilon^2} [(-s_{12})^{-\epsilon} + (-s_{23})^{-\epsilon} - (-s_{45})^{-\epsilon}] + \text{finite} \right\} \Rightarrow \\ F(cde; ab) &\simeq \frac{r_\Gamma}{\epsilon^2} [(-s_{cd})^{-\epsilon} + (-s_{de})^{-\epsilon} - (-s_{ab})^{-\epsilon}] + \text{finite} \\ r_\Gamma &= \frac{\Gamma(1+\epsilon)\Gamma^2(1-\epsilon)}{\Gamma(1-2\epsilon)}, \end{aligned} \quad (3.34)$$

where $D = 4 - 2\epsilon$.

3.3.1. IR Behavior of the Double-Trace 1-Loop SYM Amplitude $A_{5,3}$

Using (3.34), we find

$$\begin{aligned} A_{5,3}(fg; hij) &= \sum_{abcd \in 30 \text{ terms}} F(cde; ab) [s_{abcde; +; fghij} A(abcde) + s_{abcde; -; fghij} A(abedc)] \\ &\sim \frac{r_\Gamma}{\epsilon^2} \sum_{abcd \in 30 \text{ terms}} [s_{cd}^{-\epsilon} + s_{de}^{-\epsilon} - s_{ab}^{-\epsilon}] [s_{abcde; +; fghij} A(abcde) + s_{abcde; -; fghij} A(abedc)]. \end{aligned} \quad (3.35)$$

Organizing the coefficients of each divergence, we find

$$A_{5,3}(fg; lmn) \simeq \frac{r_\Gamma}{\epsilon^2} \sum_{i < j} (-s_{ij})^{-\epsilon} \sum_{abc \neq i, j} \epsilon_{lmn} [A(ijabc)], \quad (3.36)$$

where $\epsilon_{lmn}[A(ijabc)]$ means $A(ijabc)$ is multiplied by the sign of the permutation of l, m, n inside i, j, a, b, c , and the sum over a, b, c contains all the 6 terms of the arbitrary permutation of the $a, b, c \neq i, j$.

The leading $(1/\epsilon^2)$ divergence of $A_{5,3}(45; 123)$, given by

$$\sum_{i < j} \sum_{abc \neq i, j} \epsilon_{123} [A(ijabc)], \quad (3.37)$$

vanishes by explicit computation, so that the leading IR divergence of $A_{5,3}$ is $1/\epsilon$, as expected from a generalization of the subleading-color amplitude of the 4-gluon amplitude [39, 49]. (The vanishing of the $1/\epsilon^2$ IR divergence of (3.37) is also a consequence of (4.7).)

3.3.2. IR Behavior of $\mathcal{N} = 8$ Supergravity One-Loop Amplitudes and KLT Relations

Using (3.32), we obtain

$$\begin{aligned} A_{5,\text{sugra}}^{1\text{-loop}}(12345) &= -\frac{1}{2} \sum_{30 \text{ perms}} F(cde; ab) s_{cd} s_{de} s_{ab} A(cdeab) A(cdeba) \\ &\simeq -\frac{1}{2\epsilon^2} \sum_{30 \text{ perms}} [s_{cd}^{-\epsilon} + s_{de}^{-\epsilon} - s_{ab}^{-\epsilon}] s_{cd} s_{de} s_{ab} A(cdeab) A(cdeba). \end{aligned} \quad (3.38)$$

Organizing the terms by IR divergences, we obtain

$$\begin{aligned} A_{5,\text{sugra}}^{1\text{-loop}} &\simeq \frac{1}{\epsilon^2} \sum_{i < j} s_{ij}^{1-\epsilon} \times \left[\sum_d s_{cd} s_{de} A(ijcde) A(ijecd) + \sum_c s_{ic} s_{ab} A(ijabc) A(ijbac) \right. \\ &\quad \left. + \sum_c s_{jc} s_{ab} A(ijcba) A(ijcab) \right]. \end{aligned} \quad (3.39)$$

On the other hand, we know that the IR behavior of the 1-loop n -point supergravity amplitude is [77]

$$A_{n,\text{sugra}}^{1\text{-loop}}(1 \dots n) \simeq \frac{1}{\epsilon^2} A_{n,\text{sugra}}^{\text{tree}}(1 \dots n) \sum_{i < j} s_{ij}^{1-\epsilon} \quad (3.40)$$

which means that we must have the KLT relation

$$\begin{aligned} A_{5,\text{sugra}}^{\text{tree}}(12345) &= \sum_d s_{cd} s_{de} A(ijcde) A(ijecd) + \sum_c s_{ic} s_{ab} A(ijabc) A(ijbac) \\ &\quad + \sum_c s_{jc} s_{ab} A(ijcba) A(ijcab), \quad \forall (ij). \end{aligned} \quad (3.41)$$

Note that it has the larger manifest symmetry of $2 \times (n-2)! = 2 \times 3!$ and has no need to be regularized.

The tree-level KLT formula (3.41) has been derived using the IR behavior of 1-loop computations. However it can be proved explicitly. To do so, use the helicity spinor formalism and the Parke-Taylor formula [78], which states that

$$A_{n,\text{SYM}}^{\text{tree}}(1^+ 2^+ \dots i^- \dots j^- \dots n^+) = \frac{\langle ij \rangle^4}{\langle 12 \rangle \langle 23 \rangle \dots \langle n1 \rangle}, \quad (3.42)$$

or for our case, for instance choosing 1^-2^- ,

$$A(1^-2^-3^+4^+5^+) = \frac{\langle 12 \rangle^4}{\langle 12 \rangle \langle 23 \rangle \langle 34 \rangle \langle 45 \rangle \langle 51 \rangle}. \quad (3.43)$$

A similar formula exists for the supergravity amplitude [37]

$$A_{5,\text{sugra}}^{\text{tree}}(1^-2^-3^+4^+5^+) = \frac{\langle 12 \rangle^8 \epsilon(1234)}{N(5)}, \quad (3.44)$$

where

$$\begin{aligned} \epsilon(ijkl) &= 4i\epsilon_{\mu\nu\rho\sigma} k_i^\mu k_j^\nu k_k^\rho k_l^\sigma, \\ N(5) &= \prod_{i=1}^4 \prod_{j=i+1}^5 \langle ij \rangle. \end{aligned} \quad (3.45)$$

A specific case of (3.41) is proved, namely,

$$\begin{aligned} A_{5,\text{sugra}}^{\text{tree}}(12345) &= s_{34}s_{45}A(12345)A(12543) + s_{53}s_{34}A(12534)A(12435) + s_{45}s_{53}A(12453)A(12354) \\ &+ s_{23}s_{45}A(12345)A(12354) + s_{24}s_{35}A(12435)A(12453) + s_{25}s_{34}A(12534)A(12543) \\ &+ s_{13}s_{45}A(21345)A(21354) + s_{14}s_{35}A(21435)A(21453) + s_{15}s_{34}A(21534)A(21543). \end{aligned} \quad (3.46)$$

The others follow from permutations and symmetry.

One makes use of helicity spinor identities to verify that the right-hand side of (3.46) is equal to (3.44), proving the KLT relation.

3.3.3. Relation between $A_{5,\text{sugra}}^{\text{one-loop}}$ and $A_{5;3}$

Motivated by the fact that the leading IR divergence of the $n = 5$ -point supergravity amplitude and that of $A_{5;3}$ are both of order $1/\epsilon$ at 1 loop, one investigates whether $A_{5,\text{sugra}}^{\text{1-loop}}$ can be expressed as a linear combination of $A_{5;3}$ amplitudes. One uses information from (3.34) to (3.38) and finds a relation valid for IR divergences, and then one conjectures how one possibly could extend to a relation for the full amplitudes.

Based on what happened at 4 points at 1 and 2 loops, as discussed in Section 3.1, we want to find $A_{5,\text{sugra}}^{\text{1-loop}}$ as a linear combination of the $A_{5;3}$ amplitudes.

In analogy with the 4-point function, we would like to find a relation of the type

$$\begin{aligned} A_{5,\text{sugra}}^{1\text{-loop}}(12345) &= \sum_{i \in fghij} \beta_i A_{5,3}(i) \\ &= \sum_{abcde \in 30 \text{ fixed terms}} F(cde; ab) \sum_{i \in fghij} \beta_i [s_{abcde;+i} A(abcde) + s_{abcde;-i} A(abedc)]. \end{aligned} \quad (3.47)$$

On the other hand,

$$A_{5,\text{sugra}}^{1\text{-loop}}(12345) = \sum_{abcde \in 30 \text{ fixed terms}} F(cde; ab) \alpha_{abcde}, \quad (3.48)$$

where from (3.33),

$$\alpha_{abcde} = -\frac{1}{2} s_{cd} s_{de} s_{ab} A(cdeab) A(cdeba) \quad (3.49)$$

which means that we need

$$\alpha_{abcde} = \sum_{i \in fghij} \beta_i [s_{abcde;+i} A(abcde) + s_{abcde;-i} A(abedc)] \quad (3.50)$$

to be satisfied, which are 30 equations for 10 unknowns (β_i), so (3.50) is not guaranteed to have solutions.

The 30 equations can then be rewritten, using the explicit form of α_{abcde} , and a new notation that will prove useful, as

$$-\frac{1}{2} s_{ab} s_{bc} s_{de} A(abcde) A(abced) = \sum_{fg; hij} \beta_{(fg)} \epsilon_{hij} [A(abcde)] \left(1 - \epsilon_{hij}(de) \frac{A(abced)}{A(abcde)} \right), \quad (3.51)$$

where $\epsilon_{hij}(de)$ is plus if both d, e belong to h, i, j , and minus otherwise.

In order to see if a unique solution for the β_i is possible, one can match the IR behaviors on the two sides of (3.47). Expressing the IR behaviors of the lhs and the rhs,

$$\frac{1}{e^2} A_{5,\text{sugra}}^{\text{tree}}(12345) \sum_{i < j} s_{ij} (-s_{ij})^{-\epsilon} = \frac{r_\Gamma}{e^2} \sum_{k \in fg; lmn} \beta_k \sum_{i < j} (-s_{ij})^{-\epsilon} \sum_{abc \neq i, j} \epsilon_{lmn} [A(ijabc)] \quad (3.52)$$

which means that one requires, using the vanishing of the $1/e^2$ IR divergence,

$$A_{5,\text{sugra}}^{\text{tree}}(12345) s_{ij} = \sum_{k \in fg; lmn} \beta_k \sum_{abc \neq i, j} \epsilon_{lmn} [A(ijabc)]. \quad (3.53)$$

If we denote the $A_{5,\text{sugra}}^{\text{tree}}(12345)$ by just M_5 , then the lhs is a vector column of (ij) , $M_5 s_{ij}$. Also denote $\sum_{abc \neq i,j} \epsilon_{lmn} [A(ijabc)]$ as $N_{(ij),(fg)}$, so that

$$N_{(ij),(fg)} \beta_{(fg)} = M_5 s_{ij} \implies [\beta_{(fg)}] = [N_{(ij),(fg)}]^{-1} M_5 s_{ij}. \quad (3.54)$$

Note that the index (fg) on the matrix N has 10 values, and these values can also be identified by the lmn of $\epsilon_{lmn} [A(ijabc)]$, since it corresponds to the same 10 terms, picking out a group (fg) or (lmn) out of 1, 2, 3, 4, 5.

At this point, however, note that the vanishing of the leading IR divergence in (3.37) means that

$$\sum_{(ij)} N_{(ij),(fg)} = 0, \quad (3.55)$$

that is, that the matrix N has rank 9 instead of 10. One then needs to work with the corresponding 9×9 reduced matrix $N_{\text{red};(ij),(fg)}$ and give the 10th coefficient $\beta_{(fg)}$ an arbitrary value.

Therefore one has found a linear relation, (3.47), with coefficients obtained from (3.54), which is satisfied by the IR divergences, and containing an arbitrary parameter. Of course, it is still not clear that the remaining $\beta_{(fg)}$ are unique. For that, one must calculate the rank of N_{red} . If its rank is less than 9, the solution is parametrized by more than one parameter, since then some of the remaining β 's will be undetermined. As the algebra is quite involved, this is a project for further work.

In order to see if (3.47) is also satisfied for the full amplitude, one must substitute the solution for $\beta_{(fg)}$ back in (3.51) and see if these equations are satisfied, since now one needs to check whether the 30 equations are satisfied by substituting the 10 unknowns $\beta_{(fg)}$ solved as in (3.54). The verification of (3.47) for $n = 5$ is analogous with that for the (successful) relation (3.9) for $n = 4$. Therefore, it would be interesting if (3.47), (3.50) were true. (As (3.18) exemplifies for $L = 1, n = 4$, (3.47) may not be the only equation relating $\mathcal{N} = 8$ supergravity to $\mathcal{N} = 4$ SYM for $L = 1, n = 5$.)

In principle the strategy described above can be applied to higher n -point amplitudes. Namely, one can analyze the IR behavior of the results for $\mathcal{N} = 4$ SYM and $\mathcal{N} = 8$ supergravity at 1 loop and compare these to the known behavior, which would imply a relation among tree amplitudes from SYM and a KLT-type relation from the supergravity. Finally, one can relate the subleading-color SYM and supergravity amplitudes and use the consistency of the IR behavior to fix the proposed relation. For $n = 6, L = 1$, the results of Bjerrum-Bohr et al. [38] are suitable for this purpose.

4. Geometric Interpretations of Subleading-Color Amplitudes

4.1. Polytope Picture

4.1.1. Polytopes for MHV Leading Amplitudes

In [79], a simple picture was found for the 1-loop color-ordered leading amplitudes of $\mathcal{N} = 4$ SYM theory, in terms of the volume of a closed polytope in AdS_5 . In [80], it was generalized to subleading-color amplitudes. The picture for the leading MHV amplitude was

obtained as follows. We start by writing the amplitude in a space dual to momenta, thus trivializing momentum conservation $\sum_i p_i = 0$, by $p_i = x_i - x_{i+1}$. Then, for example, the 1-loop dimensionless massless box function (in 4 dimensions, which is of course IR divergent) $F_{0m}(1234) = -(1/2)stI_4^{(1)}(s, t)$, with $I_4^{(1)}(s, t)$ in (2.47) becomes

$$F_{0m}(1, 2, 3, 4) = i \int \frac{d^4 x_0}{2\pi^2} \frac{(x_1 - x_3)^2 (x_2 - x_4)^2}{(x_0 - x_1)^2 (x_0 - x_2)^2 (x_0 - x_3)^2 (x_0 - x_4)^2}. \quad (4.1)$$

We then construct $x_{a\dot{a}} = x^\mu (\sigma_\mu)_{a\dot{a}}$ and finally map

$$x^{a\dot{a}} \longrightarrow X^{AB} = \begin{pmatrix} -\frac{1}{2}\epsilon^{\alpha\beta}x^2 & ix_{\dot{\beta}}^\alpha \\ -ix_{\dot{\alpha}}^\beta & \epsilon_{\dot{\alpha}\dot{\beta}} \end{pmatrix}. \quad (4.2)$$

Here the X 's, satisfying

$$\begin{aligned} X^2 &\equiv \frac{1}{2}\epsilon_{ABCD}X^{AB}X^{CD} = 0, \\ X_i \cdot X_j &= -(x_i - x_j)^2 \end{aligned} \quad (4.3)$$

are coordinate patches on the quadric $X \cdot X = 0$ in RP^5 , with $X^{AB} \sim \lambda X^{AB}$ being their homogeneous coordinates. These X 's are considered as vertices situated at the boundary of an AdS_5 and are simple bitwistors living in twistor space, that is, there exist twistors A^A and B^B such that $X^{AB} = A^{[A}B^{B]}$ (a twistor A^A is made of $(A^\alpha, A_{\dot{\alpha}})$).

Consider a box function characterized by vertices X_1, X_2, X_3, X_4 . Then the following function of the Feynman parameters $\alpha_i \in (0, 1)$ with $\sum \alpha_i = 1$,

$$X(\alpha) = \alpha_1 X_1 + \alpha_2 X_2 + \alpha_3 X_3 + \alpha_4 X_4, \quad (4.4)$$

is a map to RP^5 , but such that $X(\alpha) \cdot X(\alpha) \neq 0$, and in fact they vary over a tetrahedron in RP^5 . After normalizing by

$$Y(\alpha) = \frac{X(\alpha)}{\sqrt{X(\alpha) \cdot X(\alpha)}}, \quad (4.5)$$

one obtains $Y(\alpha) \cdot Y(\alpha) = 1$, which means that $Y(\alpha)$ lies in the Euclidean AdS_5 . Since straight lines $X(\alpha)$ are mapped to geodesics in AdS_5 , the edges and faces of the tetrahedron in AdS_5 are geodesic, which by definition makes the tetrahedron *ideal*.

The value of the IR-finite 4-mass box matches twice the volume of the tetrahedron in AdS_5 . The IR-divergent lower mass functions need to be regularized, either in dimensional regularization, or using a mass regularization as in [79], modifying $X^2 = 0$ to $X \cdot X = \mu^2(X \cdot I)$, with I being a fixed point (a useful choice of I is $X_i \cdot I = 1$, for all i).

The one-loop MHV n -point amplitudes divided by the tree MHV amplitudes are given by the sum of 1-mass and 2-mass easy box functions with coefficient one, which add up to the volume of a closed 3-dimensional polytope (without a boundary) with n vertices.

Note that by this definition, the volume of a tetrahedron comes with a sign, determined by the order of the dual space vertices x_i in the box function $F(i, j, k, l)$. That also induces an orientation (sign) for the triangular faces of the tetrahedron, determined by whether the missing vertex from $(ijkl)$ is in an even or odd position. Faces with same vertices and different orientation (sign) can be glued together, forming a continuous object.

4.1.2. Polytopes for MHV Subleading-Color Amplitudes

For subleading-color amplitudes, we want to use (1.5) to relate to the leading amplitudes and expand in the KK basis (1.2), where we will obtain a nice geometrical interpretation.

We start with the 5-point amplitude as an example. The ratio of the leading 1-loop MHV to the tree level MHV amplitudes is the volume of a boundary of a 4-simplex,

$$\frac{A_{5,1}^{\text{MHV}}(12345)}{A_5^{\text{MHV}}(12345)} \equiv M_5^{\text{MHV}}(12345) = \sum_{\text{cyclic}} I(x_1, x_2, x_3, x_4, (x_5)) \equiv V(x_1, x_2, x_3, x_4, x_5). \quad (4.6)$$

Here $I(x_1, x_2, x_3, x_4, (x_5))$ is the volume of the tetrahedron with vertices x_1, x_2, x_3, x_4 , equal to $F(1, 2, 3, 4)$, and the missing vertex (x_5) is added in brackets since the cyclicity involves all 5 points; $V(x_1, x_2, x_3, x_4, x_5)$ is the volume of the boundary of the 4-simplex in twistor space, with $(y)_i \rightarrow (Y)_i$; that is, we map the arguments of V into twistor space.

Using (1.5) and writing the tree amplitudes in terms of the KK basis, we obtain

$$\begin{aligned} A_{5,3}(12345) = & A_5(12345)[(M_5(12345) - M_5(41235)) + (M_5(43125) - M_5(31245))] \\ & + A_5(12435)[(M_5(12435) - M_5(31245)) + (M_5(34125) - M_5(41235))] \\ & + A_5(14235)[(M_5(14235) - M_5(31425)) + (M_5(34125) - M_5(41235))] \\ & + A_5(13245)[(M_5(23145) - M_5(42315)) + (M_5(43125) - M_5(31245))] \\ & + A_5(13425)[(M_5(23145) - M_5(31425)) + (M_5(43125) - M_5(24315))] \\ & + A_5(14325)[(M_5(23145) - M_5(31425)) + (M_5(34125) - M_5(23415))]. \end{aligned} \quad (4.7)$$

We see that for each KK basis member we have the sums of two terms which are some simple differences of M_5 's. In fact these differences can be written as the differences of two polytope volumes, which in turn can be written as the volume of a simple polytope. For instance, the coefficient of the KK basis element $A_5(12345)$ in (4.7) is

$$\begin{aligned} & (M_5^{\text{MHV}}(12345) - M_5^{\text{MHV}}(41235)) + (M_5^{\text{MHV}}(43125) - M_5^{\text{MHV}}(31245)) \\ & = [V(x_1, x_2, x_3, x_4, x_5) - V((x_4 - x_5 + x_1), x_1, x_2, x_3, x_4)] \\ & \quad + [V(x_4, (x_1 + x_4 - x_5), x_1, (x_1 - x_3 + x_4), (x_2 - x_3 + x_4)) \\ & \quad - V(x_1, (x_1 - x_3 + x_4), (x_2 - x_3 + x_4), x_4, x_5)]. \end{aligned} \quad (4.8)$$

In the two brackets, the two volumes of opposite sign correspond to polytopes with $n - 1 = 4$ points common out of $n = 5$, and the relative sign is such that we can write this as the volume of the polytope obtained by the taking the union of the two polytopes.

By examining the $n = 6$ case [80] as well, we can understand the general pattern for $A_{n,3}$. The general formula is

$$A_{n,3}^{\text{MHV}}(n-1, n, 1, 2, \dots, n-2) = \sum_{\{\sigma\}_i \in \text{OP}(\{\alpha\}, \{\beta^T\})|_{j_{\max}}} A_n^{\text{MHV}}(1, \{\sigma\}_i, n) \times \sum_{n-1 \in \{\alpha\}, \{\beta\}; j_{\max} \in \{\alpha\}, \{\beta\}} (-)^{n_\beta} M_n^{\text{MHV}}(\{\beta\}, 1, \{\alpha\}, n) \quad (4.9)$$

with M_n^{MHV} being the volume of a closed polytope and pairs of opposite sign M_n 's adding up to another closed polytope. It is obtained as follows. In the above, just from (1.5), we have $M_n(\{\beta\}, 1, \{\alpha\}, n)$, where $n-1$ is either in $\{\alpha\}$ or in $\{\beta\}$, and otherwise $\{\alpha\}$ contains $2, 3, \dots, k$ and $\{\beta\}$ contains $k+1, \dots, n-2$.

For the tree amplitude prefactors, when using the KK relations (1.2), from $\{\alpha\}$ and $\{\beta\}$ we form the permutation $\{\sigma\}_i$ which contains $\{\alpha\}$ and $\{\beta^T\}$, keeping the ordering, that is, in the KK basis amplitude, we have $A(1, \{\sigma\}, n)$. Here if we extract the $n-1$, then $\{\alpha\} = 2, \dots, j_{\max}$ is ordered; that is, it goes from left to right in the permutation, and then $\{\beta\} = j_{\max}+1, \dots, n-2$ is transposed and still ordered; that is, it goes from right to left. The same $j_{\max}$ (extracted from the resulting KK basis member) is obtained from either k or $k+1$. That means that there are exactly 4 terms corresponding to the same KK basis member, corresponding to both $j_{\max}$ and $n-1$ belonging to either $\{\alpha\}$ or $\{\beta\}$.

The sign of the terms is obtained from the sign in the KK relations (1.2), that is, $(-1)^{n_\beta}$, where here $\{\beta\}$ refers to the individual $M_n(\{\beta\}, 1, \{\alpha\}, n)$ term. The fact that $j_{\max}$ belongs to either $\{\alpha\}$ or $\{\beta\}$ means that in $M_n(\{\beta\}, 1, \{\alpha\}, n)$ we have $j_{\max}$ either at the end of $\{\alpha\}$, or at the beginning of $\{\beta\}$; that is, we have a flip of $j_{\max}n$ versus $nj_{\max}$ in between terms with different signs, hence a different n_β (with or without $j_{\max}$). The exception is when actually $(n-1)$ is at the end of $\{\alpha\}$, and not $j_{\max}$, in which case the same flip is now $(n-1)n$ versus $n(n-1)$, and the same relative minus sign applies.

Since the pair in the difference in the $()$ bracket multiplying KK basis members has the same $n-2$ permutation, and the remaining two terms are flipped, we have the difference of two n -polytopes with a common $n-1$ -polytope, just as in the 5-point case.

We can generalize to $A_{n,j}$ also, obtaining

$$A_{n,j}^{\text{MHV}}(n-j+2, \dots, n, 1, \dots, n-j+1) = \sum_{\{\sigma\}_i \in \text{OP}(\{\alpha\}, \{\beta^T\})|_{j_{\max} \in \{1, \dots, n-j+1\}, j_{\max} \in \{n-j+2, \dots, n-1\}}} A_n^{\text{MHV}}(1, \{\sigma\}_i, n) \times \sum_{\{n-1, \dots, n-j+2\} \in \{\alpha\}, \{\beta\}; j_{\max} \in \{\alpha\}, \{\beta\}} (-)^{n_\beta+j-1} M_n^{\text{MHV}}(\{\beta\}, 1, \{\alpha\}, n), \quad (4.10)$$

where again we have pairs of M_n^{MHV} 's of different signs and with $n-1$ common vertices adding up to give other closed polytopes (of $n+1$ vertices).

The new features with respect to the $A_{n,3}$ are as follows. The KK basis elements that we get are of a special type: if we take out $n-1, \dots, n-j+2$ from the amplitude, then the situation should be like the one for $n=3$, namely, in the remaining permutation we go from 1 to a $j_{\max}$ towards the right, and then towards the left. But moreover, in $n-1, \dots, n-j+2$, we also have some ordering: some of them are in $\{\alpha\}$, some in $\{\beta^T\}$, which means that $n-1, \dots, l_{\max}+1$ is cyclic (i.e., towards the right), and $n-j+2, \dots, l_{\max}$ is also cyclic (i.e., we change the direction of the cyclicity at $l_{\max}$).

The number of terms multiplying a KK basis member is even, corresponding to having $j_{\max}$ in $\{\alpha\}$ or $\{\beta\}$ and any number of the $j-2$ terms $\{n-1, \dots, n-j+2\}$ in $\{\alpha\}$ and the rest in $\{\beta\}$. They come in pairs, the pairs corresponding to $j_{\max}$ being just before n or just after, or otherwise one of the $\{n-1, \dots, n-j+2\}$ being either just before, or just after n , and the pairs as before having different sign. The sign of the terms is then simply $(-1)^{j-1+n_\beta}$. In terms of polytopes, the two terms of different sign correspond as before to polytopes with only a vertex differing between them, which means they again add up to another polytope with one more vertex.

As a simple application of this analysis, we note that (4.7), (4.9), and (4.10) show that the amplitudes M_n come in alternating pairs. Each of these M_n has leading IR singularity $1/\epsilon^2 + \mathcal{O}(1/\epsilon)$ and therefore at one-loop $A_{n,j}$ has only a $1/\epsilon$ IR singularity.

4.1.3. Polytope Picture for the 6-Point Leading NMHV Amplitude

The leading (planar) gluon amplitudes $A_{6;1}^{\text{NMHV}}$ for the split-helicity configuration are [81]

$$A_{6;1}^{\text{NMHV}}(1^+2^+3^+4^-5^-6^-) = \frac{c_\Gamma}{2} \left(B_1 W_6^{(1)} + B_2 W_6^{(2)} + B_3 W_6^{(3)} \right), \quad (4.11)$$

where $W_6^{(i)}$ are cyclic permutations of $W_6^{(1)}$, and $W_6^{(i+3)} \equiv W_6^{(i)}$, given in terms of box functions by

$$W_6^{(i)} = F_{6;i}^{1m} + F_{6;i+3}^{1m} + F_{6;2;i+1}^{2mh} + F_{6;2;i+4}^{2mh}, \quad (4.12)$$

and the F 's are dimensionless boxes. We can write polytope interpretations for the $W_6^{(i)}$'s based on the fact that the F 's have polytope interpretation. Denoting for instance by (4561(23)) what was previously called $I(x_4, x_5, x_6, x_1(x_2, x_3))$, we write

$$\begin{aligned} W_6^{(1)} &= (4561(23)) + (1234(56)) + (12(3)4(5)6) + (45(6)1(2)3) \\ &\equiv A_1 + A_3 + A_2 + A_4, \\ W_6^{(2)} &= (5612(34)) + (2345(61)) + (23(4)5(6)1) + (56(1)2(3)4) \\ &\equiv A_5 + A_7 + A_6 + A_8, \\ W_6^{(3)} &= (6123(45)) + (3456(12)) + (34(5)6(1)2) + (61(2)3(4)5) \\ &\equiv A_9 + A_{11} + A_{10} + A_{12}, \end{aligned} \quad (4.13)$$

where the A 's are tetrahedra defined in the order they appear in the $W_6^{(i)}$ above, while for example for the 6-point MHV amplitude, we have

$$\begin{aligned}
 A_{6,1}^{\text{MHV}}(123456) &= A(123456)[(12(3)45(6)) + (23(4)56(1)) + (34(5)61(2)) \\
 &\quad + (1234(56)) + (2345(61)) + (3456(12)) \\
 &\quad + (4561(23)) + (5612(34)) + (6123(45))] \\
 &= A(123456)[A_{13} + A_{14} + A_{15} + A_3 + A_7 + A_{11} + A_1 + A_5 + A_9],
 \end{aligned} \tag{4.14}$$

where again the various A 's are defined in the order they appear. However, because of the spin coefficients of $W_6^{(i)}$, we cannot find a simple polytope interpretation for the subleading-color amplitudes.

4.2. Momentum Twistor Representation

4.2.1. Momentum Twistor Representation for Leading N^k MHV Amplitudes

Instead, we can use a momentum twistor [17, 20] representation for the N^k MHV superamplitudes in order to find a simple formula for the subleading N^k MHV amplitudes.

The MHV tree-level color-ordered superamplitudes are given by the Nair formula [82], a supersymmetric generalization of the Parke-Taylor formula [78, 83],

$$\mathcal{A}_{n,2}(12 \dots n) = \frac{\delta^4\left(\sum_{i=1}^n \lambda_i \tilde{\lambda}_i\right) \delta^8\left(\sum_{i=1}^n \lambda_i \tilde{\eta}^i\right)}{\langle 12 \rangle \langle 23 \rangle \dots \langle n-1, n \rangle \langle n, 1 \rangle}, \tag{4.15}$$

where as usual $\langle ij \rangle \equiv \epsilon^{\alpha\beta} \lambda_\alpha^{(i)} \lambda_\beta^{(j)}$, $\tilde{\eta}$ is a spinor with an index $I = 1, \dots, 4$ for supersymmetries suppressed, and the 2 in $A_{n,2}$ refers to R -charge, since the N^k MHV amplitude has $m = k+2$ R -charge.

The leading singularities of an amplitude are the discontinuities of the amplitude over the singularities where we put a maximum number of propagators on-shell, as explained in [18], where a conjecture for these leading singularities was proposed.

In terms of supermomentum twistors Z_i , the leading singularity of the (color-ordered, planar, i.e., leading) N^k MHV superamplitude is [19, 23]

$$\mathcal{L}_{n,m} = \frac{\delta^4\left(\sum \lambda \tilde{\lambda}\right) \delta^8\left(\sum \lambda \tilde{\eta}\right)}{\langle 12 \rangle \langle 23 \rangle \dots \langle n1 \rangle} \int \frac{d^{nk} \mathfrak{D}}{\text{Vol}(Gl(2))} \frac{\prod_{\mu=1}^k \delta^{4|4}\left(\sum_{i=1}^n \mathfrak{D}_{\mu i} Z_i\right)}{(12 \dots k)(23 \dots k+1) \dots (n12k-1)} = \mathcal{L}_{n,2} \times \mathcal{R}_{n,k}, \tag{4.16}$$

where $k = m - 2$. The prefactor $\mathcal{L}_{n,2}$ is the tree MHV amplitude (4.15), and the integral $\mathcal{R}_{n,k} = \mathcal{R}_{n,m-2}$, is Yangian invariant. This object is dual conformal *covariant*, only $\mathcal{R}_{n,k}$ being dual conformal invariant, and the tree amplitude is covariant.

The one-loop amplitudes of $\mathcal{N} = 4$ SYM can be reduced to just boxes via the van Neerven and Vermaseren procedure, with some coefficients. The leading singularities also

coincide with the coefficients of these box functions [18]. For one-loop MHV, the coefficients of the boxes are known to be just the MHV tree amplitudes, agreeing with the result above.

4.2.2. Subleading N^k MHV Amplitudes in Momentum Twistor Space

The planar (leading) color-ordered N^k MHV amplitude is a sum of permutations of boxes with coefficients equal to the leading singularities,

$$A_{n,1}(1 \dots n) = \sum_{\sigma} \mathcal{L}_{n,k}(\sigma) I_{n,4}(\sigma) = \sum A_n^{\text{MHV}}(\sigma) R_{n,k}(\sigma) I_{n,4}(\sigma), \quad (4.17)$$

where $I_{n,4}$ are boxes. At 6 points, the permutations σ combine such that we can organize the sum as a sum over cyclic permutations, with several boxes having the same coefficient [18]. For this coefficient we can factorize the tree MHV amplitude, which is cyclically invariant, so that it appears as a common factor

$$A_{6,1}(1 \dots 6) = A_6^{\text{MHV}}(1 \dots 6) \sum_{\lambda=\text{cyclic}} R_{6,k}(\lambda) \sum_{\sigma/\lambda} I_{6,4}(\sigma). \quad (4.18)$$

At higher n -point, the situation is slightly more complicated. The box diagrams are ordered in groups that can be cyclically permuted, for each group having a given formula for the residue, but unlike 6-point, the residue is not universal for all the groups [18]. However, all the diagrams still have the external legs in the original order, which means, since the MHV tree amplitude is cyclically invariant, that we can again factorize the MHV tree amplitude, obtaining for planar N^k MHV amplitudes

$$\begin{aligned} A_{n,1}(1 \dots n) &= A_n^{\text{MHV}}(1 \dots n) \sum_{\text{groups of diagrams}} \sum_{\lambda=\text{cyclic}} R_{n,k}(\lambda) \sum_{\sigma/\lambda} I_{n,4}(\sigma) \\ &\equiv A_n^{\text{MHV}}(1 \dots n) M_{n,k}(1 \dots n) \end{aligned} \quad (4.19)$$

which implicitly defines $M_{n,k}$.

We now finally note that we have the same formula for $A_{n,1}(1 \dots n)$ in terms of A_n^{MHV} and $M_{n,k}$ from the previous section on polytopes, so we can apply the same calculations we used to obtain the MHV $A_{n,j}$ in terms of $A_{n,1}$ in Section 2. We just have to change the definition of $M_{n,k}$ as in (4.19) and thus also drop the polytope interpretation of $M_{n,k}$. But otherwise the same (4.10) found in the MHV case holds in the general N^k MHV case as well, as can be seen from (4.9).

4.2.3. Application to the 6-Point NMHV Amplitude

For the superamplitude, we use an explicit form of the twistor formula (4.18), doing the twistor space integrals over the 1-loop NMHV contours. The result is [84, 85]

$$\begin{aligned} A_{6,1}^{(1)\text{NMHV}}(123456) &= \frac{a}{2} A_6^{(0)\text{MHV}}(123456) \\ &\quad \times \left[(R_{413} + R_{146})W_6^{(1)} + (R_{524} + R_{251})W_6^{(2)} + (R_{635} + R_{362})W_6^{(3)} \right] \\ &\equiv A_6^{(0)\text{MHV}}(123456)M_6^{(1)\text{NMHV}}(123456). \end{aligned} \quad (4.20)$$

From the $R_{n,k}$ terms in (4.18), one gets the sum of basic dual conformal invariant R -invariants $R_{j,j+3,j+5}$ above. Here the $R_{j,j+3,j+5}$ are given by

$$\begin{aligned} R_{rst} &= -\frac{\langle s-1s \rangle \langle t-1t \rangle \delta^{(4)}(\Xi_{rst})}{x_{st}^2 \langle r|x_{rt}x_{ts}|s-1 \rangle \langle r|x_{rt}x_{ts}|s \rangle \langle r|x_{rs}x_{st}|t-1 \rangle \langle r|x_{rs}x_{st}|t \rangle}, \\ \Xi_{rst} &= \sum_t^{r-1} \eta_i \langle i|x_{ts}x_{sr}|r \rangle + \sum_r^{s-1} \eta_i \langle i|x_{st}x_{tr}|r \rangle, \\ x_{st} &= x_s - x_t = \sum_{i=s}^{t-1} p_i. \end{aligned} \quad (4.21)$$

As explained before, we can then perform the same combinatorics that led us to (4.9), just that now we use $M_6^{(1)\text{NMHV}}(123456)$ instead of the $M_6(123456)$.

5. Summary

We have reviewed a number of features of subleading-color amplitudes of $\mathcal{N} = 4$ SYM theory, a subject considerably less developed than that of the leading (planar) amplitudes. Nevertheless this topic should not be ignored if the structure of perturbative $\mathcal{N} = 8$ supergravity and its relationship to $\mathcal{N} = 4$ SYM theory are to be understood, as nonplanar graphs appear on an equal footing in $\mathcal{N} = 8$ supergravity.

After presenting a detailed description of the IR divergences of $\mathcal{N} = 4$ SYM theory, we obtained the leading (and some subleading) IR divergences of subleading-color amplitudes at L loops and tested these against known exact results for one- and two-loop four-point functions. These ideas applied to the one-loop five-point function led to a new KLT relation, as well as possible new relations between $\mathcal{N} = 4$ SYM and $\mathcal{N} = 8$ supergravity amplitudes. A geometric interpretation of the one-loop subleading and $N^k\text{MHV}$ amplitudes of $\mathcal{N} = 4$ SYM was presented in the last section.

Since reformulations and extensions of known results frequently lead to new insights, we advocate that continued study of subleading-color amplitudes is likely to be fruitful. In particular, it would be important for our understanding of the relation of $\mathcal{N} = 4$ SYM to $\mathcal{N} = 8$ supergravity to extend (3.20) and (3.21) to subleading IR divergences, and to higher n -point functions. An example of the latter is the speculative (3.47) for $L = 1$ and $n = 5$. However,

(3.8) and (3.17) remind us that (3.47) may not be the only way to relate the two theories, so that the subjects discussed in this paper should provide many opportunities for future work.

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