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The high-eccentricity asymmetric expansion of the disturbing function for non-planar resonant problems

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Abstract. In this paper we present an extension to the non-planar case of the asymmetric expansion of the averaged resonant disturbing function of Ferraz-Mello & Sato (1989, A&A 225, 541-547). Comparions with the exact averaged disturbing function are also presented. The expansion gives a good approximation of the exact function in a wide region around the center of expansion.

Key words: celestial mechanics – solar system: general – methods: analytical

1. Introduction

An important challenge in dynamical astronomy is to find a suitable expansion of the disturbing function R for a given problem. As far as planetary theory is concerned, the Laplace's series are widely used. They basically represent an expansion around coplanar circular orbits. However, these series have generally a poor convergence for high eccentricities and inclinations (see Ferraz-Mello, 1994 for a discussion).

A typical example where the Laplacian expansion can not be used because of its lack of convergence is that of the first-order resonances in the asteroidal main belt (Murray, 1986). In the case of the 2/1 and 3/2 resonances, the radius of convergence in eccentricity of the Laplace's expansion is 0.2 and 0.09, respectively. Recall that both the Griquas (at 2/1) and the Hildas (at 3/2) have eccentricities higher than these values.

It was Woltjer (1928) who first proposed to use a Taylor expansion of the disturbing function around a point other than the origin. There is no reason "a priori" to suppose that such an expansion will not have the same convergence problems of the Laplacian expansion when we are far away from the center of expansion. On the contrary, the larger the eccentricity of the center, the smaller the radius of convergence (see for example Wintner, 1941, §285-§299). However, it is clear that it will be good enough to describe the dynamics of libration as far as the real motion remains close to the center of expansion.

The *asymmetric expansion* of Woltjer allowed him to make a good prediction of the libration period of the 3/4 resonance between Titan and Hyperion. Later on, Ferraz-Mello (1987) reformulated Woltjer's idea in the form of an analytic expansion in the framework of the restricted planar three-body problem for a first-order resonance. This work was extended for resonances of any order by Ferraz-Mello & Sato (1989): We will refer hereafter to their article as FMS89.

The planar asymmetric expansion led to the high-eccentricity libration theory of the Hildas' motion (Ferraz-Mello, 1988; Gallardo & Ferraz-Mello, 1995). It was also used to study the corotation solutions of first-order resonances (Ferraz-Mello, 1989; Ferraz-Mello et al., 1993) and, more recently, to formulate a formal analytic theory for the high-eccentricity small-amplitude librations at first-order resonances (Ferraz-Mello et al., 1997). Numerical applications of the asymmetric expansion also allowed Ferraz-Mello & Klafke (1991) to discover the dynamics of very-high eccentricity libration in the asteroidal 3/1 resonance, and Ferraz-Mello (1997), to construct a symplectic mapping for the 2/1 and 3/2 resonances.

However, it becomes more and more evident that planar models are not sufficient to explain all the properties observed in some resonant systems. The articles of Wisdom (1987) and Henrard et al. (1995) are good examples of this fact. Thus, in many cases, the inclusion of the third dimension in the models is necessary.

An important advance in this direction was already made by Moons (1993, 1994), who extended the basic equations of FMS89, and presented a method to calculate numerically the resonant averaged disturbing function at each point of the phase space for the non-planar model. This method allows us to have an exact representation of the disturbing function. Despite of the problem of the high computational cost of a mean at each point of the phase space, the method was successfully applied in many works (see for example Morbidelli & Moons, 1993; Moons, 1994; Henrard et al., 1995).

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FMS89's equations were also extended by Tsuchida (see Oliveira, 1995) who derived an extension of the asymmetric expansion to the non-planar case in which the plane of motion of the disturbing body is kept fixed. The aim of this paper is to extend the equations of FMS89 to the construction of the spatial asymmetric expansion in terms of the inclinations of both the disturbed and the disturbing bodies with respect to a fixed reference plane. Since this expansion is a polynomial, and its coefficients are computed only once, it is much faster than the numerical calculation of the exact disturbing function. However, it approximates well the disturbing function only in a neighborhood of the center of expansion, and is only useful in the case of moderate amplitude librations. In the case of large amplitude librations, it should be better to use the expansion of Yokoyama (1994), which is an expansion in powers of $(e - e_0)$ combined with a Fourier expansion in the resonant angle σ. However, that expansion was only calculated in the frame of the planar problem.

In Sects. 2 to 6 we present the basic calculations to obtain the asymmetric expansion of the disturbing function. These calculations are extended in Sect. 7 to the case of an oblate potential of the primary body. Finally, in Sect. 8, we present a comparison with the exact averaged disturbing function calculated numerically. The last section is devoted to the conclusions.

2. The disturbing function

Let a, e, I, ω, Ω be the orbital elements in their usual notation, and v, u, ℓ the true, eccentric and mean anomalies, respectively. We will use primed variables for the disturbing body and unprimed ones for the disturbed body. We define the resonant angular variables for a generic resonance $(p+q):p$ in the form

$$\begin{aligned}
\sigma &= \frac{p+q}{q}\lambda' - \frac{p}{q}\lambda - \varpi \\
\sigma' &= \frac{p+q}{q}\lambda' - \frac{p}{q}\lambda - \varpi' \\
\sigma_z &= \frac{p+q}{q}\lambda' - \frac{p}{q}\lambda - \Omega \\
\sigma'_z &= \frac{p+q}{q}\lambda' - \frac{p}{q}\lambda - \Omega' \\
Q &= \frac{\lambda - \lambda'}{q}
\end{aligned} \tag{1}$$

where $\varpi = \omega + \Omega$ and $\lambda = \ell + \varpi$ (also $\varpi' = \omega' + \Omega'$ and $\lambda' = \ell' + \varpi'$). Recall that Q is the so called synodic angle, which is a fast variable in comparison to the others.

Now, we write the disturbing function of the restricted 3-body problem as

$$R = \frac{\mathrm{G}m'}{a'}\left[f + f^1\right] \tag{2}$$

where G is the gravitational constant, m' and a' are the perturber's mass and semimajor axis, and f and f^1 are the direct and indirect parts, respectively, which we write as

$$f = a'(r^2 + r'^2 - 2rr'\cos S)^{-1/2}$$

$$f^1 = -a'\frac{r}{r'^2}\cos S \tag{3}$$

where S is the angle between the position vectors $\mathbf{r}$ and $\mathbf{r}'$. This disturbing function depends on the inclinations only through $\cos S$. Introducing the notation

$$\eta = \sin\frac{I}{2}, \qquad \eta' = \sin\frac{I'}{2} \tag{4}$$

we can write

$$\begin{aligned}
\cos S = &\left(1 - \eta^2 - \eta'^2 + \eta^2\eta'^2\right) \\
&\times \cos\left((v-\sigma) - (v'-\sigma')\right) \\
&+ 2\eta\eta'\sqrt{1-\eta^2}\sqrt{1-\eta'^2} \\
&\times \left[\cos\left((v-\sigma) - (v'-\sigma') + \sigma_z - \sigma'_z\right)\right. \\
&\left. - \cos\left((v-\sigma) + (v'-\sigma') + \sigma_z + \sigma'_z\right)\right] \\
&+ \left(1-\eta^2\right)\eta'^2 \cos\left((v-\sigma) + (v'-\sigma') + 2\sigma'_z\right) \\
&+ \left(1-\eta'^2\right)\eta^2 \cos\left((v-\sigma) + (v'-\sigma') + 2\sigma_z\right) \\
&+ \eta^2\eta'^2 \cos\left((v-\sigma) - (v'-\sigma') + 2\sigma_z - 2\sigma'_z\right).
\end{aligned} \tag{5}$$

On the other hand, R depends on the eccentricities through r, r' and v, v' which involves the solution of Kepler's equation. If we assume that the eccentricity of the perturber is small enough we can use the known elliptic expansions (Brouwer & Clemence, 1961) and write

$$\begin{aligned}
r' &= a'\left[1 - e'\cos\ell' + \frac{1}{2}e'^2(1 - \cos 2\ell') + \mathcal{O}(e'^3)\right] \\
v' &= \ell' + 2e'\sin\ell' + \frac{5}{4}e'^2\sin 2\ell' + \mathcal{O}(e'^3)
\end{aligned} \tag{6}$$

where, in terms of the variables of Eqs. (1), we have

$$\ell' = \sigma' + pQ. \tag{7}$$

A similar assumption is not done in the case of the disturbed body and the dependence of r, v on e will be considered later (see Sect. 5).

Now, using Eqs. (5) and (6) we can write R as a function of the form

$$R = R(r, v, \sigma, \eta, \sigma_z, r'(a', e', \ell'), v'(e', \ell'), \sigma', \eta', \sigma'_z) \tag{8}$$

and expand it in Taylor series. We will do this in two steps, respectively discused in the next two sections.

3. Expansion in e', η' and η

Since e' is a small quantity we start with an expansion around $e' = 0$ up to $\mathcal{O}(e'^2)$ (this part of the expansion will lead to the same expressions that appear in FMS89). We stress the fact that such expansion is sufficient if we are looking for a first-order averaged expansion of R, on the hypothesis that $e' < \mathcal{O}(\sqrt{m'})$. We will also assume that $\eta' \simeq \mathcal{O}(e')$. Note that this assumption is valid for the inclinations of almost all the planets with respect to the invariant plane of the outer Solar System or with respect to the ecliptic. On the other hand, many asteroids have inclinations less than 30° and we will assume $\eta \simeq \mathscr{O}(\sqrt{e'})$. Thus, we will expand R around $\eta' = 0$ and $\eta = 0$ up to orders consistent with $\mathscr{O}(e'^2)$.

Denoting $\mathscr{R} = f + f^1$ we write

$$\begin{aligned} R = \frac{\mathrm{G}m'}{a'} \Bigg[& \mathscr{R}_0 \\ & + \frac{\partial \mathscr{R}}{\partial e'} e' + \frac{1}{2}\frac{\partial^2 \mathscr{R}}{\partial e'^2} e'^2 + \frac{\partial \mathscr{R}}{\partial \eta'} \eta' + \frac{1}{2}\frac{\partial^2 \mathscr{R}}{\partial \eta'^2} \eta'^2 \\ & + \frac{\partial \mathscr{R}}{\partial \eta} \eta + \frac{1}{2}\frac{\partial^2 \mathscr{R}}{\partial \eta^2} \eta^2 + \frac{1}{6}\frac{\partial^3 \mathscr{R}}{\partial \eta^3} \eta^3 + \frac{1}{24}\frac{\partial^4 \mathscr{R}}{\partial \eta^4} \eta^4 \\ & + \frac{\partial^2 \mathscr{R}}{\partial e' \partial \eta} e'\eta + \frac{1}{2}\frac{\partial^3 \mathscr{R}}{\partial e' \partial \eta^2} e'\eta^2 + \frac{\partial^2 \mathscr{R}}{\partial e' \partial \eta'} e'\eta' \\ & + \frac{\partial^2 \mathscr{R}}{\partial \eta' \partial \eta} \eta'\eta + \frac{1}{2}\frac{\partial^3 \mathscr{R}}{\partial \eta' \partial \eta^2} \eta'\eta^2 + \mathscr{O}(e'^3) \Bigg] \end{aligned} \quad (9)$$

and hereafter we assume that all the derivatives are evaluated at $e' = \eta' = \eta = 0$. Since the expansion is symmetric with respect to the inclinations, the D'Alembert rule holds and, so, just the terms in the even powers of the inclinations should survive. Hence, we must have

$$\frac{\partial \mathscr{R}}{\partial \eta'} = \frac{\partial \mathscr{R}}{\partial \eta} = \frac{\partial^3 \mathscr{R}}{\partial \eta^3} = \frac{\partial^2 \mathscr{R}}{\partial e' \partial \eta} = \frac{\partial^2 \mathscr{R}}{\partial e' \partial \eta'} = \frac{\partial^3 \mathscr{R}}{\partial \eta' \partial \eta^2} = 0. \quad (10)$$

The remaining derivatives are not null, and in order to calculate them we divide $\mathscr{R}$ in its direct and indirect parts.

First, we introduce the notation $c = \cos S$ and

$$b = \frac{r^2 + r'^2 - 2rr'c}{a'^2}. \quad (11)$$

Thus for the direct part we can write

$$\begin{aligned} f_0 &= b_0^{-1/2} \\ \frac{\partial f}{\partial e'} &= -b_0^{-3/2} \left[\frac{1}{a'}\frac{\partial r'}{\partial e'} \left(1 - \frac{r}{a'}c_0\right) - \frac{r}{a'}\frac{\partial c}{\partial v'}\frac{\partial v'}{de'} \right] \\ \frac{\partial^2 f}{\partial e'^2} &= -b_0^{-3/2} \Bigg[\left(\frac{1}{a'}\frac{\partial r'}{\partial e'}\right)^2 + \frac{1}{a'}\frac{\partial^2 r'}{\partial e'^2}\left(1 - \frac{r}{a'}c_0\right) \\ &\qquad - \frac{r}{a'}\frac{\partial c}{\partial v'}\frac{\partial^2 v'}{\partial e'^2} - 2\frac{r}{a'}\frac{\partial c}{\partial v'}\frac{\partial v'}{\partial e'}\left(\frac{1}{a'}\frac{\partial r'}{\partial e'}\right) \\ &\qquad - \frac{r}{a'}\frac{\partial^2 c}{\partial v'^2}\left(\frac{\partial v'}{\partial e'}\right)^2 \Bigg] + 3b_0^{1/2}\left(\frac{\partial f}{\partial e'}\right)^2 \\ \frac{\partial^2 f}{\partial \eta'^2} &= b_0^{-3/2}\frac{r}{a'}\frac{\partial^2 c}{\partial \eta'^2} \\ \frac{\partial^2 f}{\partial \eta^2} &= b_0^{-3/2}\frac{r}{a'}\frac{\partial^2 c}{\partial \eta^2} \\ \frac{\partial^4 f}{\partial \eta^4} &= 9b_0^{1/2}\left(\frac{\partial^2 f}{\partial \eta^2}\right)^2 \\ \frac{\partial^3 f}{\partial e' \partial \eta^2} &= b_0^{-3/2}\left[\frac{r}{a'}\frac{\partial^2 c}{\partial \eta^2}\left(\frac{1}{a'}\frac{\partial r'}{\partial e'}\right) + \frac{r}{a'}\frac{\partial^3 c}{\partial v' \partial \eta^2}\frac{\partial v'}{\partial e'} \right] \\ &\quad + 3b_0^{1/2}\frac{\partial f}{\partial e'}\frac{\partial^2 f}{\partial \eta^2} \\ \frac{\partial^2 f}{\partial \eta' \partial \eta} &= b_0^{-3/2}\frac{r}{a'}\frac{\partial^2 c}{\partial \eta' \partial \eta} \end{aligned} \quad (12)$$

where $c_0 = \cos S|_{e'=\eta'=\eta=0}$ and

$$b_0 = 1 + \frac{r^2}{a'^2} - 2\frac{r}{a'}c_0. \quad (13)$$

For the indirect part, after straightforward calculations, we obtain

$$\begin{aligned} f_0^1 &= -\frac{r}{a'}c_0 \\ \frac{\partial f^1}{\partial e'} &= -\frac{r}{a'}\frac{\partial c}{\partial v'}\frac{\partial v'}{\partial e'} + 2\frac{r}{a'}c_0\left(\frac{1}{a'}\frac{\partial r'}{\partial e'}\right) \\ \frac{\partial^2 f^1}{\partial e'^2} &= -6\frac{r}{a'}c_0\left(\frac{1}{a'}\frac{\partial r'}{\partial e'}\right)^2 + 2\frac{r}{a'}c_0\left(\frac{1}{a'}\frac{\partial^2 r'}{\partial e'^2}\right) \\ &\quad + 4\frac{r}{a'}\frac{\partial c}{\partial v'}\frac{\partial v'}{\partial e'}\left(\frac{1}{a'}\frac{\partial r'}{\partial e'}\right) - \frac{r}{a'}\frac{\partial c}{\partial v'}\frac{\partial^2 v'}{\partial e'^2} \\ &\quad - \frac{r}{a'}\frac{\partial^2 c}{\partial v'^2}\left(\frac{\partial v'}{\partial e'}\right)^2 \\ \frac{\partial^2 f^1}{\partial \eta'^2} &= -\frac{r}{a'}\frac{\partial^2 c}{\partial \eta'^2} \\ \frac{\partial^2 f^1}{\partial \eta^2} &= -\frac{r}{a'}\frac{\partial^2 c}{\partial \eta^2} \\ \frac{\partial^4 f^1}{\partial \eta^4} &= 0 \\ \frac{\partial^3 f^1}{\partial e' \partial \eta^2} &= -\frac{r}{a'}\frac{\partial^3 c}{\partial v' \partial \eta^2}\frac{\partial v'}{\partial e'} + 2\frac{r}{a'}\frac{\partial^2 c}{\partial \eta^2}\left(\frac{1}{a'}\frac{\partial r'}{\partial e'}\right) \\ \frac{\partial^2 f^1}{\partial \eta' \partial \eta} &= -\frac{r}{a'}\frac{\partial^2 c}{\partial \eta' \partial \eta}. \end{aligned} \quad (14)$$

Now, we introduce the functions

$$\begin{aligned} W_1 &= \frac{r}{a}\cos(v - \sigma) \\ W_2 &= \frac{r}{a}\sin(v - \sigma) \end{aligned} \quad (15)$$

which are the same as defined in FMS89 (see Eq. 18 in that paper). These definitions together with Eq. (7) lead, after some calculations, to the following expressions for the derivatives at $e' = \eta' = \eta = 0$

$$\begin{aligned} \frac{r^2}{a'^2} &= \alpha^2(W_1^2 + W_2^2) \\ \frac{r}{a'}c_0 &= \alpha(W_1\cos pQ + W_2\sin pQ) \\ \frac{r}{a'}\frac{\partial c}{\partial v'} &= -\alpha(W_1\sin pQ - W_2\cos pQ) \\ \frac{r}{a'}\frac{\partial^2 c}{\partial \eta^2} &= -4\alpha(W_1\sin\sigma_z + W_2\cos\sigma_z) \\ &\quad \times(\sin pQ\cos\sigma_z + \cos pQ\sin\sigma_z) \end{aligned}$$

$$
\begin{aligned}
\frac{r}{a'}\frac{\partial^2 c}{\partial\eta'\partial\eta} &= 4\alpha(W_1\sin\sigma_z + W_2\cos\sigma_z)\\
&\quad\times(\sin pQ\cos\sigma'_z + \cos pQ\sin\sigma'_z)\\
\frac{r}{a'}\frac{\partial^3 c}{\partial v'\partial\eta^2} &= -4\alpha(W_1\sin\sigma_z + W_2\cos\sigma_z)\\
&\quad\times(\cos pQ\cos\sigma_z - \sin pQ\sin\sigma_z)\\
\frac{r}{a'}\frac{\partial^2 c}{\partial v'^2} &= -\alpha(W_1\cos pQ + W_2\sin pQ) \qquad (16)
\end{aligned}
$$

where $\alpha = a/a'$ (recall that a' is a constant). From Eqs. (6) we also have

$$
\begin{aligned}
\frac{1}{a'}\frac{\partial r'}{\partial e'} &= -\cos pQ\cos\sigma' + \sin pQ\sin\sigma'\\
\frac{\partial v'}{\partial e'} &= 2(\cos pQ\sin\sigma' + \sin pQ\cos\sigma')\\
\frac{1}{a'}\frac{\partial^2 r'}{\partial e'^2} &= 1 - \cos 2pQ\cos 2\sigma' + \sin 2pQ\sin 2\sigma'\\
\frac{\partial^2 v'}{\partial e'^2} &= \frac{5}{2}(\cos 2pQ\sin 2\sigma' + \sin 2pQ\cos 2\sigma') \qquad (17)
\end{aligned}
$$

and the computation of the derivatives is almost completed.

It is worth noting that Eqs. (15) can be written in the form

$$
\begin{aligned}
W_1 &= g_2\cos(p+q)Q - g_3\sin(p+q)Q\\
W_2 &= g_2\sin(p+q)Q + g_3\cos(p+q)Q \qquad (18)
\end{aligned}
$$

where g_2, g_3 are the functions

$$
\begin{aligned}
g_2 &= \frac{r}{a}\cos(\varphi-\lambda)\\
g_3 &= \frac{r}{a}\sin(\varphi-\lambda) \qquad (19)
\end{aligned}
$$

with $\varphi = v + \varpi$. Thus, using the notation of FMS89 (see their Eqs. 11, 12, 14 and 37) we introduce the definitions

$$
\begin{aligned}
g_1 &= W_1^2 + W_2^2\\
&= g_2^2 + g_3^2\\
g_4 &= W_1\cos pQ + W_2\sin pQ\\
&= g_2\cos qQ - g_3\sin qQ\\
g_5 &= W_2\cos pQ - W_1\sin pQ\\
&= g_2\sin qQ + g_3\cos qQ. \qquad (20)
\end{aligned}
$$

Recall that g_4 differs by a factor α and g_5 differs by a factor 2α to their counterparts in FMS89. In addition, we define the functions

$$
\begin{aligned}
g_6 &= W_1\cos pQ - W_2\sin pQ\\
&= g_2\cos(q+2p)Q - g_3\sin(q+2p)Q\\
g_7 &= W_2\cos pQ + W_1\sin pQ\\
&= g_2\sin(q+2p)Q + g_3\cos(q+2p)Q. \qquad (21)
\end{aligned}
$$

Finally, after some algebra, we obtain the following results for both the direct and indirect parts.

3.1. Term of zero order

It is

$$\mathscr{R}_0 = (f_0 + f_0^1) \qquad (22)$$

where

$$
\begin{aligned}
f_0 &= (1 + \alpha^2 g_1 - 2\alpha g_4)^{-1/2}\\
f_0^1 &= -\alpha g_4 \qquad (23)
\end{aligned}
$$

(compare with Eqs. 7 and 13 of FMS89).

3.2. Terms of $\mathscr{O}(e')$

The term in e' is:

$$\frac{\partial\mathscr{R}}{\partial e'} = (f_1 + f_1^1)\cos\sigma' + (f_2 + f_2^1)\sin\sigma' \qquad (24)$$

where

$$
\begin{aligned}
f_1 &= f_0^3\,[(1-\alpha g_4)\cos pQ + 2\alpha g_5\sin pQ]\\
f_2 &= -f_0^3\,[(1-\alpha g_4)\sin pQ - 2\alpha g_5\cos pQ] \qquad (25)
\end{aligned}
$$

$$
\begin{aligned}
f_1^1 &= -2\alpha(g_4\cos pQ + g_5\sin pQ)\\
f_2^1 &= 2\alpha(g_4\sin pQ - g_5\cos pQ) \qquad (26)
\end{aligned}
$$

(compare with Eq. 36 of FMS89). The term in η^2 is:

$$
\begin{aligned}
\frac{\partial^2\mathscr{R}}{\partial\eta^2} &= (f_3 + f_3^1) + (f_4 + f_4^1)\cos 2\sigma_z\\
&\quad + (f_5 + f_5^1)\sin 2\sigma_z \qquad (27)
\end{aligned}
$$

where

$$
\begin{aligned}
f_3 &= -f_0^3 2\alpha g_4\\
f_4 &= f_0^3 2\alpha g_6\\
f_5 &= -f_0^3 2\alpha g_7 \qquad (28)
\end{aligned}
$$

$$
\begin{aligned}
f_3^1 &= 2\alpha g_4\\
f_4^1 &= -2\alpha g_6\\
f_5^1 &= 2\alpha g_7. \qquad (29)
\end{aligned}
$$

3.3. Higher order terms

The term in e'^2 is:

$$
\begin{aligned}
\frac{\partial^2\mathscr{R}}{\partial e'^2} &= (f_6 + f_6^1) + (f_7 + f_7^1)\cos 2\sigma'\\
&\quad + (f_8 + f_8^1)\sin 2\sigma' \qquad (30)
\end{aligned}
$$

where

$$
\begin{aligned}
f_6 &= -f_0^3\left(\frac{3}{2} + \alpha g_4\right) + \frac{3}{2}f_0^5\left[(1-\alpha g_4)^2 + (2\alpha g_5)^2\right]\\
f_7 &= f_0^3\left[\left(\frac{1}{2} + \alpha g_4\right)\cos 2pQ + \frac{1}{2}\alpha g_5\sin 2pQ\right]
\end{aligned}
$$

$$+\frac{3}{2}f_0^5\left[\left((1-\alpha g_4)^2-(2\alpha g_5)^2\right)\cos 2pQ + 4(1-\alpha g_4)\alpha g_5\sin 2pQ\right]$$

$$\begin{aligned} f_8 &= -f_0^3\left[\left(\frac{1}{2}+\alpha g_4\right)\sin 2pQ - \frac{1}{2}\alpha g_5\cos 2pQ\right] \\ &\quad -\frac{3}{2}f_0^5\left[\left((1-\alpha g_4)^2-(2\alpha g_5)^2\right)\sin 2pQ - 4(1-\alpha g_4)\alpha g_5\cos 2pQ\right] \end{aligned} \tag{31}$$

$$\begin{aligned} f_6^1 &= \alpha g_4 \\ f_7^1 &= -7\alpha g_4\cos 2pQ - \frac{13}{2}\alpha g_5\sin 2pQ \\ f_8^1 &= 7\alpha g_4\sin 2pQ - \frac{13}{2}\alpha g_5\cos 2pQ \end{aligned} \tag{32}$$

(compare with Eqs. 40, 41, 42 and 43 of FMS89). The term in η'^2 is almost the same as Eq. (27):

$$\begin{aligned} \frac{\partial^2\mathscr{R}}{\partial\eta'^2} &= (f_3+f_3^1)+(f_4+f_4^1)\cos 2\sigma_z' \\ &\quad +(f_5+f_5^1)\sin 2\sigma_z'. \end{aligned} \tag{33}$$

The term in η^4 has no indirect part:

$$\begin{aligned} \frac{\partial^4\mathscr{R}}{\partial\eta^4} &= f_9+f_{10}\cos 2\sigma_z+f_{11}\sin 2\sigma_z \\ &\quad +f_{12}\cos 4\sigma_z+f_{13}\sin 4\sigma_z \end{aligned} \tag{34}$$

where

$$\begin{aligned} f_9 &= 18f_0^5\alpha^2\left(g_1+2g_4^2\right) \\ f_{10} &= -72f_0^5\alpha^2 g_4 g_6 \\ f_{11} &= 72f_0^5\alpha^2 g_4 g_7 \\ f_{12} &= 18f_0^5\alpha^2\left(g_6^2-g_7^2\right) \\ f_{13} &= -36f_0^5\alpha^2 g_6 g_7. \end{aligned} \tag{35}$$

And finaly, the mixed term in $e'\eta^2$ is:

$$\begin{aligned} \frac{\partial^3\mathscr{R}}{\partial e'\partial\eta^2} &= \left[(f_{14}+f_{14}^1)+(f_{15}+f_{15}^1)\cos 2\sigma_z + (f_{16}+f_{16}^1)\sin 2\sigma_z\right]\cos\sigma' \\ &\quad +\left[(f_{17}+f_{17}^1)+(f_{18}+f_{18}^1)\cos 2\sigma_z + (f_{19}+f_{19}^1)\sin 2\sigma_z\right]\sin\sigma' \end{aligned} \tag{36}$$

where

$$\begin{aligned} f_{14} &= f_0^3\alpha\left(3W_1-g_4\cos pQ-g_5\sin pQ\right)-6f_0^2f_1\alpha g_4 \\ f_{15} &= -f_0^3\alpha\left(3W_1-g_6\cos pQ+g_7\sin pQ\right)+6f_0^2f_1\alpha g_6 \\ f_{16} &= f_0^3\alpha\left(3W_2-g_6\sin pQ-g_7\cos pQ\right)-6f_0^2f_1\alpha g_7 \\ f_{17} &= -f_0^3\alpha\left(3W_2-g_4\sin pQ+g_5\cos pQ\right)-6f_0^2f_2\alpha g_4 \\ f_{18} &= -f_0^3\alpha\left(3W_2+g_6\sin pQ+g_7\cos pQ\right)+6f_0^2f_2\alpha g_6 \\ f_{19} &= -f_0^3\alpha\left(3W_1+g_6\cos pQ-g_7\sin pQ\right)-6f_0^2f_2\alpha g_7 \end{aligned} \tag{37}$$

$$\begin{aligned} f_{14}^1 &= -2f_1^1 \\ f_{15}^1 &= -f_{19}^1 = -4\alpha(g_6\cos pQ-g_7\sin pQ) \\ f_{16}^1 &= f_{18}^1 = 4\alpha(g_6\sin pQ+g_7\cos pQ) \\ f_{17}^1 &= 2f_2^1. \end{aligned} \tag{38}$$

All these terms are of $\mathscr{O}(e'^2)$. There is also a mixed term in $\eta'\eta$ which is of $\mathscr{O}(e'^{3/2})$; it is:

$$\begin{aligned} \frac{\partial^2\mathscr{R}}{\partial\eta'\partial\eta} &= \left[(f_{20}+f_{20}^1)\cos\sigma_z'+(f_{21}+f_{21}^1)\sin\sigma_z'\right]\sin\sigma_z \\ &\quad +\left[(f_{22}+f_{22}^1)\cos\sigma_z'+(f_{23}+f_{23}^1)\sin\sigma_z'\right]\cos\sigma_z \end{aligned} \tag{39}$$

where

$$\begin{aligned} f_{20} &= f_0^3 4\alpha W_1\sin pQ = f_0^3 2\alpha(g_7-g_5) \\ f_{21} &= f_0^3 4\alpha W_1\cos pQ = f_0^3 2\alpha(g_4+g_6) \\ f_{22} &= f_0^3 4\alpha W_2\sin pQ = f_0^3 2\alpha(g_4-g_6) \\ f_{23} &= f_0^3 4\alpha W_2\cos pQ = f_0^3 2\alpha(g_7+g_5) \end{aligned} \tag{40}$$

$$\begin{aligned} f_{20}^1 &= -4\alpha W_1\sin pQ = -2\alpha(g_7-g_5) \\ f_{21}^1 &= -4\alpha W_1\cos pQ = -2\alpha(g_4+g_6) \\ f_{22}^1 &= -4\alpha W_2\sin pQ = -2\alpha(g_4-g_6) \\ f_{23}^1 &= -4\alpha W_2\cos pQ. = -2\alpha(g_7+g_5). \end{aligned} \tag{41}$$

With these calculations we have completed the first part of the expansion.

4. Expansion in k, h and α

We know that, in the resonance, the critical argument σ librates around a certain value σ_0, and both the eccentricity and the semimajor axis librate around e_0 and a_0 respectively. So we could make an expansion of the disturbing function around these points. Since e and σ are coupled through the D'Alembert rule, any term with the argument $n\sigma$ is proportional to e^n. Thus any expansion around a point $e_0 \neq 0$ is singular at $e = 0$. In order to avoid this non-analyticity at the origin it is useful to replace the pair of variables e, σ by the regular variables

$$\begin{aligned} k &= e\cos\sigma \\ h &= e\sin\sigma \end{aligned} \tag{42}$$

and to expand R around the point (k_0, h_0).

It is worth noting that the expressions obtained in Sect. 3 depend on k, h and α only through the coefficients f_i and f_i^1, and more precisely, they depend on k, h only through the functions W_1 and W_2 (or equivalently, the functions g_2 and g_3). Thus, the second part of the expansion consists in expanding each of these coefficients in powers of $(h-h_0)$, $(k-k_0)$ and $(\alpha-\alpha_0)$.

We assume that all Δk, Δh and $\Delta\alpha$ are of order $\mathscr{O}(e')$ (i.e., $\Delta e \simeq \mathscr{O}(e')$ and $\Delta\alpha \simeq 0.05$). So, we expand the term $\mathscr{R}_0$ up to second the order in e':

$$\mathscr{R}_0 = \mathscr{R}_{00} + \frac{\partial\mathscr{R}_0}{\partial k}(k-k_0) + \frac{\partial\mathscr{R}_0}{\partial h}(h-h_0)$$

$$+\frac{\partial \mathscr{R}_0}{\partial \alpha}(\alpha-\alpha_0)+\frac{1}{2}\frac{\partial^2 \mathscr{R}_0}{\partial k^2}(k-k_0)^2$$
$$+\frac{1}{2}\frac{\partial^2 \mathscr{R}_0}{\partial h^2}(h-h_0)^2+\frac{1}{2}\frac{\partial^2 \mathscr{R}_0}{\partial \alpha^2}(\alpha-\alpha_0)^2$$
$$+\frac{\partial^2 \mathscr{R}_0}{\partial k \partial h}(k-k_0)(h-h_0)+\frac{\partial^2 \mathscr{R}_0}{\partial h \partial \alpha}(h-h_0)(\alpha-\alpha_0)$$
$$+\frac{\partial^2 \mathscr{R}_0}{\partial k \partial \alpha}(k-k_0)(\alpha-\alpha_0). \quad (43)$$

The terms of $\mathscr{O}(e')$ are expanded only up to the first order in e' obtaining

$$\begin{aligned}
\frac{\partial \mathscr{R}}{\partial e'} &= \left(\frac{\partial \mathscr{R}}{\partial e'}\right)_0+\frac{\partial}{\partial k}\frac{\partial \mathscr{R}}{\partial e'}(k-k_0) \\
&\quad +\frac{\partial}{\partial h}\frac{\partial \mathscr{R}}{\partial e'}(h-h_0)+\frac{\partial}{\partial \alpha}\frac{\partial \mathscr{R}}{\partial e'}(\alpha-\alpha_0) \\
\frac{\partial^2 \mathscr{R}}{\partial \eta^2} &= \left(\frac{\partial^2 \mathscr{R}}{\partial \eta^2}\right)_0+\frac{\partial}{\partial k}\frac{\partial^2 \mathscr{R}}{\partial \eta^2}(k-k_0) \\
&\quad +\frac{\partial}{\partial h}\frac{\partial^2 \mathscr{R}}{\partial \eta^2}(h-h_0)+\frac{\partial}{\partial \alpha}\frac{\partial^2 \mathscr{R}}{\partial \eta^2}(\alpha-\alpha_0). \quad (44)
\end{aligned}$$

The remaining terms are just evaluated at k_0, h_0 and α_0.

The calculation of these derivatives is almost simple. We show below some of the most cumbersome derivations of the direct part. Denoting by X, Y any of the variables k, h or α, we have

$$\begin{aligned}
\frac{\partial f_0}{\partial X} &= -\frac{1}{2}f_0^3\left[\frac{\partial(\alpha^2 g_1)}{\partial X}-2\frac{\partial(\alpha g_4)}{\partial X}\right] \\
\frac{\partial^2 f_0}{\partial X \partial Y} &= 3f_0^{-1}\frac{\partial f_0}{\partial X}\frac{\partial f_0}{\partial Y} \\
&\quad -\frac{1}{2}f_0^3\left[\frac{\partial^2(\alpha^2 g_1)}{\partial X \partial Y}-2\frac{\partial^2(\alpha g_4)}{\partial X \partial Y}\right] \\
\frac{\partial f_1}{\partial X} &= 3f_0^{-1}\frac{\partial f_0}{\partial X}f_1 \\
&\quad +f_0^3\left[-\frac{\partial(\alpha g_4)}{\partial X}\cos pQ+2\frac{\partial(\alpha g_5)}{\partial X}\sin pQ\right] \\
\frac{\partial f_3}{\partial X} &= -6f_0^2\frac{\partial f_0}{\partial X}\alpha g_4-2f_0^3\frac{\partial(\alpha g_4)}{\partial X}. \quad (45)
\end{aligned}$$

The explicit form of these coefficients can be found in Roig (1997).

5. The functions W_1 and W_2

The evaluation of the functions W_1 and W_2 as well as of their derivatives at the point k_0, h_0 can be done in two different ways as already described in FMS89. We can use the relations (18) together with the classical elliptic expansions of the functions g_2 and g_3 (see Sect. 7 of FMS89), which leads to the so-called *high-eccentricity* (HE) asymmetric expansion, valid only for $e_0 < 0.6627$, i.e., inside the radius of convergence of the cited expansions of g_2 and g_3. Alternatively, we can write the functions W_1 and W_2 in the form

$$\begin{aligned}
W_1 &= \cos(u-\sigma)-k+hT \\
W_2 &= \sin(u-\sigma)+h+kT \quad (46)
\end{aligned}$$

where

$$T=\frac{1}{e}\left(\sqrt{1-e^2}-1\right)\sin u \quad (47)$$

and we can calculate the derivatives with the use of closed formulas which are not singular at $e=0$ (see Sects. 3.1, 3.2 and 3.3 of FMS89 and pay attention to some obvious misprints in Eqs. 28 and 32, and in the formula after Eq. 22). This later method leads to the so-called *very-high-eccentricity* (VHE) asymmetric expansion, valid for any eccentricity e_0.

It is worth noting that in Eq. (46), W_1 and W_2 depend on the eccentric anomaly u. Thus, we must use the relation

$$\ell=\sigma+(p+q)Q \quad (48)$$

and to solve Kepler's equation in order to obtain the dependence of W_1, W_2 on Q.

6. Final expression and average

Now, we can write the final expression of the expansion. Denoting $\mathscr{R}_i=f_i+f_i^1$ and $\mathscr{R}_{iX}=\dfrac{\partial \mathscr{R}_i}{\partial X}$, we have

$$\begin{aligned}
R = \frac{\mathrm{G}m'}{a'}\Big[&\mathscr{R}_0+\mathscr{R}_{0h}(h-h_0)+\mathscr{R}_{0k}(k-k_0) \\
&+\mathscr{R}_{0\alpha}(\alpha-\alpha_0)+\left[\mathscr{R}_1\cos\sigma'+\mathscr{R}_2\sin\sigma'\right]e' \\
&+\left[\mathscr{R}_3+\mathscr{R}_4\cos 2\sigma_z+\mathscr{R}_5\sin 2\sigma_z\right]\frac{1}{2}\eta^2 \\
&+\mathscr{R}_{0hh}\frac{1}{2}(h-h_0)^2+\mathscr{R}_{0kk}\frac{1}{2}(k-k_0)^2 \\
&+\mathscr{R}_{0\alpha\alpha}\frac{1}{2}(\alpha-\alpha_0)^2+\mathscr{R}_{0hk}(h-h_0)(k-k_0) \\
&+\mathscr{R}_{0h\alpha}(h-h_0)(\alpha-\alpha_0)+\mathscr{R}_{0k\alpha}(k-k_0)(\alpha-\alpha_0) \\
&+[\mathscr{R}_{1h}\cos\sigma'+\mathscr{R}_{2h}\sin\sigma']e'(h-h_0) \\
&+[\mathscr{R}_{1k}\cos\sigma'+\mathscr{R}_{2k}\sin\sigma']e'(k-k_0) \\
&+[\mathscr{R}_{1\alpha}\cos\sigma'+\mathscr{R}_{2\alpha}\sin\sigma']e'(\alpha-\alpha_0) \\
&+[\mathscr{R}_{3h}+\mathscr{R}_{4h}\cos 2\sigma_z+\mathscr{R}_{5h}\sin 2\sigma_z]\frac{1}{2}\eta^2(h-h_0) \\
&+[\mathscr{R}_{3k}+\mathscr{R}_{4k}\cos 2\sigma_z+\mathscr{R}_{5k}\sin 2\sigma_z]\frac{1}{2}\eta^2(k-k_0) \\
&+[\mathscr{R}_{3\alpha}+\mathscr{R}_{4\alpha}\cos 2\sigma_z+\mathscr{R}_{5\alpha}\sin 2\sigma_z]\frac{1}{2}\eta^2(\alpha-\alpha_0) \\
&+[\mathscr{R}_6+\mathscr{R}_7\cos 2\sigma'+\mathscr{R}_8\sin 2\sigma']\frac{1}{2}e'^2 \\
&+\left[\mathscr{R}_3+\mathscr{R}_4\cos 2\sigma_z'+\mathscr{R}_5\sin 2\sigma_z'\right]\frac{1}{2}\eta'^2 \\
&+[\mathscr{R}_9+\mathscr{R}_{10}\cos 2\sigma_z+\mathscr{R}_{11}\sin 2\sigma_z]\frac{1}{24}\eta^4 \\
&+[\mathscr{R}_{12}\cos 4\sigma_z+\mathscr{R}_{13}\sin 4\sigma_z]\frac{1}{24}\eta^4 \\
&+[\mathscr{R}_{14}+\mathscr{R}_{15}\cos 2\sigma_z+\mathscr{R}_{16}\sin 2\sigma_z]\cos\sigma'\frac{1}{2}\eta^2 e' \\
&+[\mathscr{R}_{17}+\mathscr{R}_{18}\cos 2\sigma_z+\mathscr{R}_{19}\sin 2\sigma_z]\sin\sigma'\frac{1}{2}\eta^2 e'
\end{aligned}$$

$$+\left[\mathscr{R}_{20}\cos\sigma'_z + \mathscr{R}_{21}\sin\sigma'_z\right]\eta'\eta\sin\sigma_z$$
$$+\left[\mathscr{R}_{22}\cos\sigma'_z + \mathscr{R}_{23}\sin\sigma'_z\right]\eta'\eta\cos\sigma_z \Big] . \qquad (49)$$

Since each of the coefficients $\mathscr{R}_i$ depends on the fast variable Q, this expansion can be averaged by solving numerically the integral

$$R^* = \frac{1}{2\pi}\int_0^{2\pi} R\left[u(Q), Q\right]\, dQ. \qquad (50)$$

The calculation of this integral requires the solution of Kepler's equation as an intermediate step (see Sect. 5). It is easy to show that this step can be avoided by performing the average directly over the eccentric anomaly (cf. Moons, 1993). From Eq. (48) we write

$$Q = \frac{(u - e_0\sin u) - \sigma_0}{(p+q)} \qquad (51)$$

where e_0, σ_0 are the values at the center of the expansion; then Eq. (50) can be replaced by

$$R^* = \frac{1}{2\pi(p+q)}\int_0^{2\pi(p+q)} (1 - e_0\cos u)\, R\left[u, Q(u)\right]\, du. \qquad (52)$$

6.1. Parity rules

There are some special cases in which the coefficients $\mathscr{R}_i$ have a well defined parity with respect to the variable Q (or u). This property can be used to predict some zero averages (for odd coefficients) and reduce to half the interval of integration in the calculation of even coefficients.

The case of particular interest is the one in which we use a center of expansion on the k-axis, i.e. $h_0 = 0$. Such an expansion is useful to study first-order resonances ($q = 1$), since, in this case, libration occurs around $\sigma_0 = 0$. Then, we can see from Eq. (51) that the variable Q is an odd function of u, so the parity of any function with respect to Q is the same as with respect to u. This property can be used together with the parities of the functions W_i (and their derivatives) with respect to u, shown in Table 1, to predict the parity of each coefficient.

With these rules, the coefficient f_0 as well as any power of it are always even functions of u. Consequently, for a first-order resonance the averaged expansion is reduced to the following expression:

$$\begin{aligned}
R^* = \frac{\mathrm{G}m'}{a'} \Big[& \mathscr{R}^*_0 + \mathscr{R}^*_{0k}(k-k_0) + \mathscr{R}^*_{0\alpha}(\alpha-\alpha_0) \\
& + \mathscr{R}^*_1\, e'\cos\sigma' + \left[\mathscr{R}^*_3 + \mathscr{R}^*_4\cos 2\sigma_z\right]\frac{1}{2}\eta^2 \\
& + \mathscr{R}^*_{0hh}\frac{1}{2}h^2 + \mathscr{R}^*_{0kk}\frac{1}{2}(k-k_0)^2 \\
& + \mathscr{R}^*_{0\alpha\alpha}\frac{1}{2}(\alpha-\alpha_0)^2 + \mathscr{R}^*_{0k\alpha}(k-k_0)(\alpha-\alpha_0) \\
& + \mathscr{R}^*_{2h}\, e'\sin\sigma' h + \mathscr{R}^*_{1k}\, e'\cos\sigma'(k-k_0) \\
& + \mathscr{R}^*_{1\alpha}\, e'\cos\sigma'(\alpha-\alpha_0) + \mathscr{R}^*_{5h}\frac{1}{2}\eta^2\sin 2\sigma_z h \\
& + [\mathscr{R}^*_{3k} + \mathscr{R}^*_{4k}\cos 2\sigma_z]\frac{1}{2}\eta^2(k-k_0) \\
& + [\mathscr{R}^*_{3\alpha} + \mathscr{R}^*_{4\alpha}\cos 2\sigma_z]\frac{1}{2}\eta^2(\alpha-\alpha_0) \\
& + [\mathscr{R}^*_6 + \mathscr{R}^*_7\cos 2\sigma']\frac{1}{2}e'^2 \\
& + \left[\mathscr{R}^*_3 + \mathscr{R}^*_4\cos 2\sigma'_z\right]\frac{1}{2}\eta'^2 \\
& + [\mathscr{R}^*_9 + \mathscr{R}^*_{10}\cos 2\sigma_z + \mathscr{R}^*_{12}\cos 4\sigma_z]\frac{1}{24}\eta^4 \\
& + [\mathscr{R}^*_{14} + \mathscr{R}^*_{15}\cos 2\sigma_z]\frac{1}{2}\eta^2 e'\cos\sigma' \\
& + \mathscr{R}^*_{19}\frac{1}{2}\eta^2\sin 2\sigma_z\, e'\sin\sigma' + \mathscr{R}^*_{21}\,\eta\sin\sigma_z\,\eta'\sin\sigma'_z \\
& + \mathscr{R}^*_{22}\,\eta\cos\sigma_z\,\eta'\cos\sigma'_z \Big]
\end{aligned} \qquad (53)$$

where $\mathscr{R}^*_i = \dfrac{1}{\pi(p+q)}\displaystyle\int_0^{\pi(p+q)} (1 - e_0\cos u)\,\mathscr{R}_i\, du.$

In the general case, when the center of the expansion is at a point with $\sigma_0 \neq n\pi/2$ (n integer) the functions W_1, W_2 as well as the variable Q do not have well-defined parities with respect to u. Thus, the full expression Eq. (49) shall be considered.

7. The oblateness of the central body

In resonant satellite problems, we must take into account the part of the disturbing function arising from the oblateness of the primary body. Using the classical Legendre expansion, we can write this oblateness potential in a first approximation as

$$R^J = -\frac{\mathrm{G}MJ_2b^2}{a^3}\left(\frac{r}{a}\right)^{-3} P_2(\sin\phi) + \cdots \qquad (54)$$

where M, b and J_2 are the mass, the equatorial radius and the ellipticity factor of the potential of the central body; r, ϕ and a are the planetocentric distance, the latitude over the equator and the semimajor axis of the disturbed body. P_2 is the Legendre polynomial

$$P_2(x) = \frac{3}{2}x^2 - \frac{1}{2}. \qquad (55)$$

If the equator of the primary is the reference plane for definition of inclinations, the following relation holds:

$$\sin\phi = \sin I\sin(v+\omega) \qquad (56)$$

which in terms of Eqs. (1) and (4) can be written as

$$\sin\phi = 2\eta\sqrt{1-\eta^2}\sin\left((v-\sigma)+\sigma_z\right). \qquad (57)$$

We now substitute Eqs. (55) and (57) in Eq. (54) and obtain

$$\begin{aligned}
R^J = -\frac{\mathrm{G}MJ_2b^2}{a'^3} \Big[& \mathscr{R}^J_0 + \frac{1}{2}\eta^2\left(1-\eta^2\right) \\
& \times \left(\mathscr{R}^J_1 + \mathscr{R}^J_2\cos 2\sigma_z + \mathscr{R}^J_3\sin 2\sigma_z\right) \Big]
\end{aligned} \qquad (58)$$

Table 1. Parities with respect to u for the case $q = 1$.

Even	$W_1, W_{1k}, W_{2h}, W_{1hh}, W_{1kk}, W_{2hk}$
Odd	$W_2, W_{1h}, W_{2k}, W_{2hh}, W_{2kk}, W_{1hk}$

where a' is the semimajor axis of the disturbing body. Taking into account Eqs. (15) and (20), we have

$$\mathscr{R}_0^J = -\frac{1}{2}\left(\alpha\, g_1^{1/2}\right)^{-3} \tag{59}$$

(compare with Eq. 72 of FMS89). In addition, we introduce the definitions

$$\begin{aligned} g_8 &= W_1^2 - W_2^2 \\ g_9 &= 2W_1W_2 \end{aligned} \tag{60}$$

and we can write

$$\begin{aligned} \mathscr{R}_1^J &= -12\mathscr{R}_0^J \\ \mathscr{R}_2^J &= 12\mathscr{R}_0^J\, g_1^{-1} g_8 \\ \mathscr{R}_3^J &= -12\mathscr{R}_0^J\, g_1^{-1} g_9. \end{aligned} \tag{61}$$

Now, we expand each term $\mathscr{R}_i^J$ around k_0, h_0 and α_0 as in Sect. 4. The calculation of the derivatives is simple and the final expression reads

$$\begin{aligned} R^J = -\frac{\mathrm{G}MJ_2b^2}{a'^3}\Big[&\mathscr{R}_0^J + \mathscr{R}_{0h}^J(h-h_0) + \mathscr{R}_{0k}^J(k-k_0) \\ &+\mathscr{R}_{0\alpha}^J(\alpha-\alpha_0) + \frac{1}{2}\mathscr{R}_{0hh}^J(h-h_0)^2 \\ &+\frac{1}{2}\mathscr{R}_{0kk}^J(k-k_0)^2 + \frac{1}{2}\mathscr{R}_{0\alpha\alpha}^J(\alpha-\alpha_0)^2 \\ &+\mathscr{R}_{0hk}^J(h-h_0)(k-k_0) + \mathscr{R}_{0h\alpha}^J(h-h_0)(\alpha-\alpha_0) \\ &+\mathscr{R}_{0k\alpha}^J(k-k_0)(\alpha-\alpha_0) + \frac{1}{2}\eta^2\left(1-\eta^2\right) \\ &\quad\times\left(\mathscr{R}_1^J + \mathscr{R}_2^J\cos 2\sigma_z + \mathscr{R}_3^J\sin 2\sigma_z\right) \\ &+\frac{1}{2}\eta^2(h-h_0) \\ &\quad\times\left(\mathscr{R}_{1h}^J + \mathscr{R}_{2h}^J\cos 2\sigma_z + \mathscr{R}_{3h}^J\sin 2\sigma_z\right) \\ &+\frac{1}{2}\eta^2(k-k_0) \\ &\quad\times\left(\mathscr{R}_{1k}^J + \mathscr{R}_{2k}^J\cos 2\sigma_z + \mathscr{R}_{3k}^J\sin 2\sigma_z\right) \\ &+\frac{1}{2}\eta^2(\alpha-\alpha_0) \\ &\quad\times\left(\mathscr{R}_{1\alpha}^J + \mathscr{R}_{2\alpha}^J\cos 2\sigma_z + \mathscr{R}_{3\alpha}^J\sin 2\sigma_z\right)\Big]. \end{aligned} \tag{62}$$

The average is computed in the same way as already described in Sect. 6, and parity rules apply when $h_0 = 0$.

8. Comparisons

In order to test the precision of our asymmetric expansion, we made several comparisons with the exact averaged resonant disturbing function, computed numerically using the formulas of

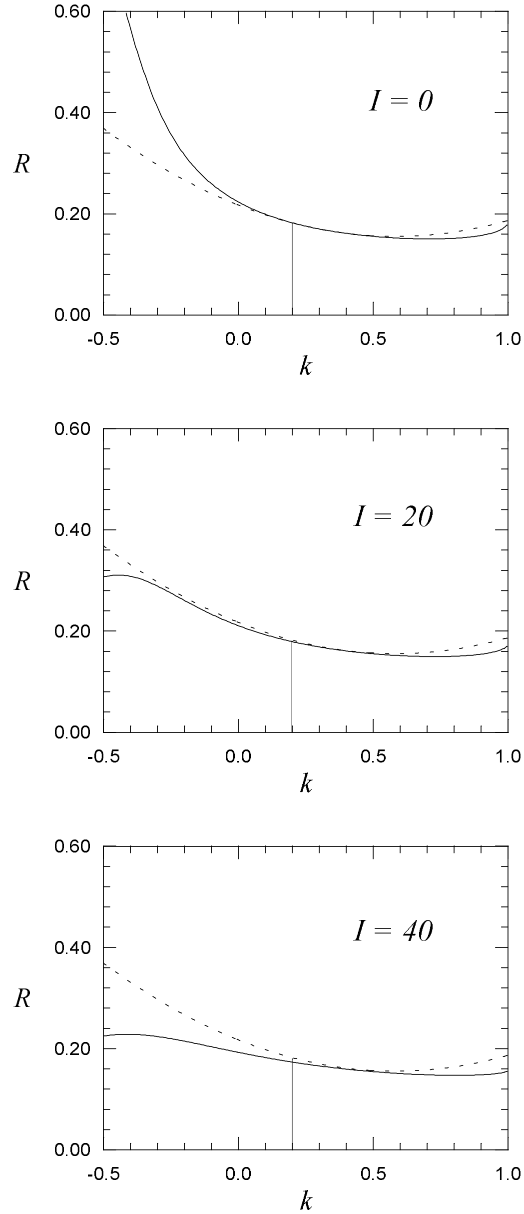

Fig. 1. Comparison at $\sigma = 0$ between the exact averaged disturbing function (continous line) and the asymmetric expansion (dotted line), expanded around the point $k_0 = 0.2$, $h_0 = 0$ and $a_0 = 3.275$ AU (2/1 resonance) for different values of I, considering $\sigma_z = 0$. In all cases we take $a' = 5.2$ AU, $e' = 0.05$, $I' = 1.2°$, and $\sigma' = \sigma'_z = 0$. The expansion is almost the same in all cases. The agreement at $I = 20°$ is better than at $I = 0°$ due to the fact that the planar disturbing function has a singularity at $k \simeq -0.5$.

Moons (1993). These comparisons were made for the 2/1 resonance and in all cases we considered

$$\begin{aligned} a' &= 5.2 \text{ AU} \\ e' &= 0.05 \\ I' &= 1.2° \\ \sigma' &= \sigma'_z = 0 \end{aligned} \tag{63}$$

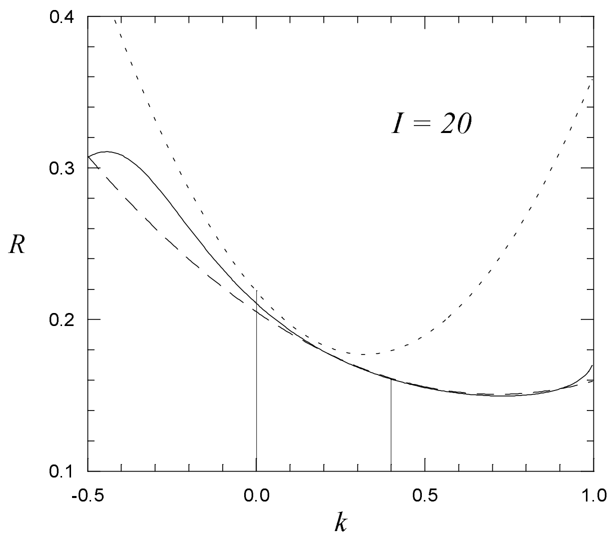

Fig. 2. Comparison at $\sigma = 0$ between the exact disturbing function (continuous line) and two expansions around the points $k_0 = h_0 = 0$ (dotted line) and $k_0 = 0.4$, $h_0 = 0$ (dashed line), with $a_0 = 3.275$ AU in both cases. The comparison is made for $I = 20°$ and $\sigma_z = 0$. The variables of the disturbing body have the same values as in Fig. 1. Note that at $k = 0$ the difference between each expansion and the exact function is almost the same. However, the second expansion gives better approximation for other values of k (see comment in the text).

as fixed parameters. The coefficients of the expansion were evaluated using the closed formulas as in the VHE expansion. The average was computed using a fourty points Gauss quadrature.

First, it is known that the planar asymmetric expansion becomes singular when the path of integration over Q includes the point of collision of the bodies. The set of points in the k_0, h_0 plane where this singularity occurs is the so-called *collision curve* (see Ferraz-Mello et al., 1993). The non-planar asymmetric expansion presented here is the sum of the planar expansion plus terms related with the inclinations; so, this spatial expansion has embedded the presence of the collision curve of the planar problem. If we choose the center of expansion (k_0, h_0) close to the collision curve, the spatial expansion will try to reproduce a false singularity, leading to a very bad representation of the disturbing function, even if a collision may occur only at the mutual node of the orbits.

However, when the center (k_0, h_0) is far away from the collision curve, the expansion can be used to obtain a very good representation of the non-planar disturbing function. This representation can be even better than that of the planar disturbing function (FMS89), due to the fact that, in general, the non-planar function does not have a collision curve. We can see in Fig. 1 that, for a given center of expansion k_0, h_0, there seems to be a certain inclination for which the expansion works the best.

It is worth noting that the best agreement between the asymmetric expansion and the exact disturbing function at a given point in the plain k, h is, sometimes, obtained using a center of expansion k_0, h_0 other than that point. This does not only depend on the values of k_0, h_0, but also on the values of I, e' and I' at which the expansion is evaluated. For example, we see in Fig. 2 that, around the origin ($k = 0$), the asymmetric expansion (for which $k_0 = 0.4$) gives an approximation as good as the Laplacian expansion (for which $k_0 = 0$). However, the asymmetric one gives better approximation for other values of k, so it can be applyed to problems in which the center of libration is at $k = 0.4$ as well as at $k = 0$. Note that, in this figure, neither the asymmetric expansion nor the Laplacian one coincide with the exact function when $k = k_0$. The difference with the exact function at this point is because both expansions are made around $e' = I' = I = 0$, but they are being evaluated at $e' = 0.05$, $I' = 1.2°$ and $I = 20°$. The difference is greater at $k_0 = 0$ because we are nearer the collision curve and the expansion becomes worse.

We also perfomed some comparisons for different values of σ. To do this, we used an asymmetric expansion around $a_0 = 3.275$ AU, $h_0 = 0$ and a given k_0 between 0 and 0.8. Then we fixed a, I and σ_z and calculated the relative difference between this expansion and the exact function at each point of a grid in the plane e, σ. The results were plotted as level curves in that plane. Some of these comparisons, for three different values of the inclination ($I = 0°, I = 15°, I = 30°$) are shown in Fig. 3, using centers of expansion around $k_0 = 0.2$ and around $k_0 = 0.6$. In all these examples we fixed $a = 3.275$ AU and $\sigma_z = 0$.

In those figures the brightest level represents the region where the relative difference is below 1%. The center of expansion is marked with a cross. We see that for inclinations $\sim 30°$ (in the chosen limit for the truncation of the expansion), the approximation becomes bad, and can be better in other region than around the center of expansion. For smaller inclinations we obtained a good agreement even for amplitudes of libration of about 60° or 90°. It is worth noting that the evaluation of the averaged asymmetric expansion in a grid of 100×200 points in the plane e, σ is about ten times faster than the numerical evaluation of the exact averaged disturbing function in the same grid.

Although the asymmetric expansion gives a good approximation of the exact function in a wide region of the phase space, it is worth stressing the fact that the most important feature for the dynamics is to be able to reproduce with the expansion the location of the equilibrium solutions of the system (pericentric branch, corotation solutions, etc.). For example, we can see in Fig. 1 that for $I = 20°$ the location of the minimum of the expansion does not coincide with the minimum of the exact function. As this minimum is associated with the location of the Kozai resonance, we will not obtain a good representation of the dynamics for $e > 0.5$ with this expansion. It is obvious that, in this case, we have to change the center of expansion and take a value of k_0 nearer to the minimum of the exact function. This is clear in Fig. 2, where the expansion for $k_0 = 0.4$ gives a better location of that minimum.

In this sense, the asymmetric expansion should be combined with a sort of iterative procedure to locate the position of these equilibrium points. Starting with a given center (k_0, h_0), we estimate the location of such an equilibrium point, then we can use this point as a new center to recalculate the coefficients of

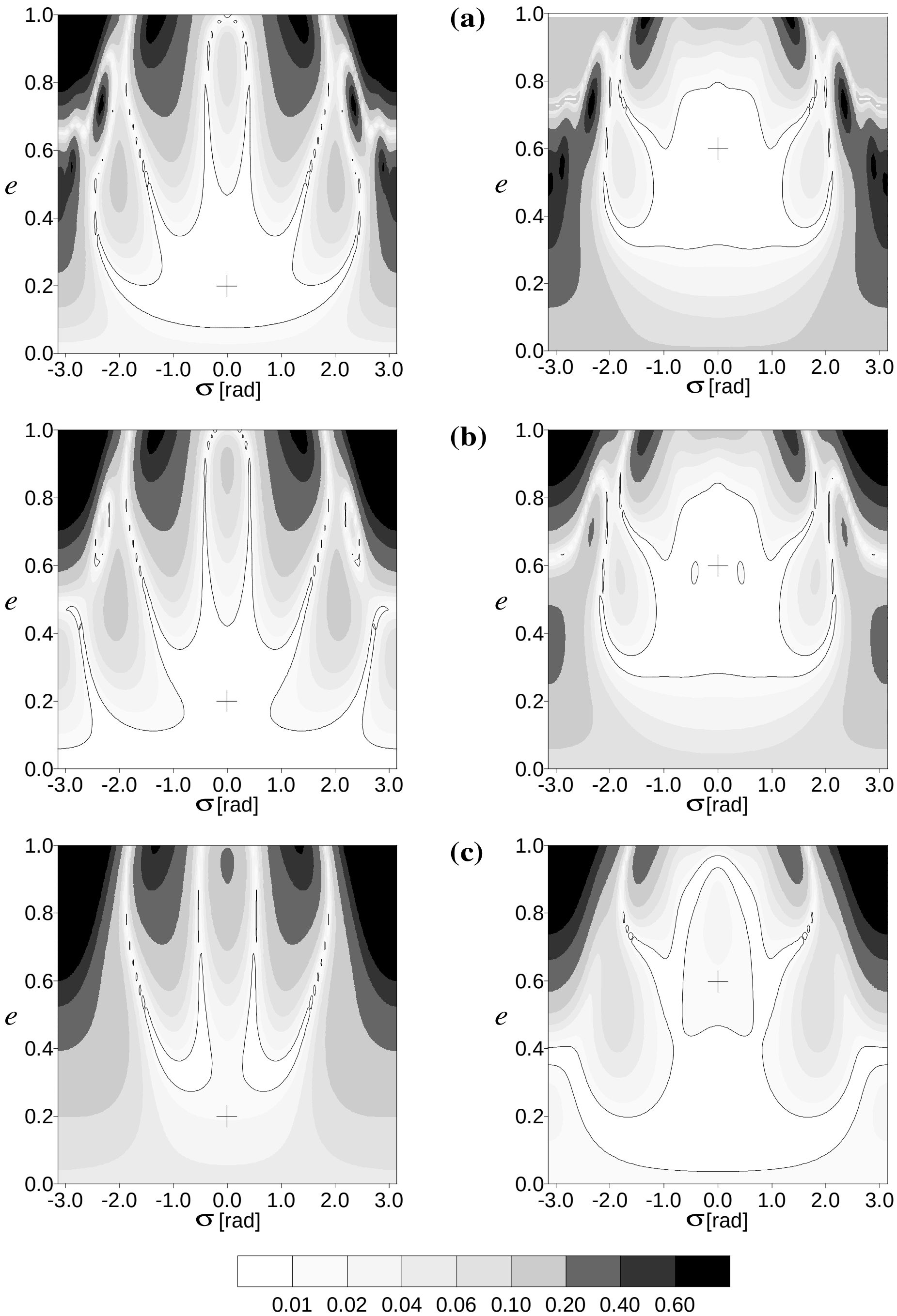

Fig. 3a–c. Relative difference in the plane e, σ between the exact disturbing function and the asymmetric expansion for two different centers of expansion, corresponding to the 2/1 resonance. Left: $k_0 = 0.2$, $h_0 = 0$, $a_0 = 3.275$ AU. Rigth: $k_0 = 0.6$, $h_0 = 0$, $a_0 = 3.275$ AU (the cross indicates the center of expansion). The results are presented for three different inclinations, considering $\sigma_z = 0$: **a** Case $I = 0°$, **b** Case $I = 15°$, and **c** Case $I = 30°$. In all figures, the variables of the disturbing body have the same values as in Fig. 1. The white region corresponds to a difference below 1%. Note that the approximation becomes poor as the inclination increases, and can be better in other region than the center of expansion.

the expansion, and so on. This kind of procedure was already applied, for example, by Ferraz-Mello et al. (1993) to compute the locus of corotation solutions in many asteroidal resonances, using the planar expansion. We stress the fact that, even if we can locate the exact position of a corotation, the expansion will be valid only in neighbourhood of that corotation (which is the actual center of the expansion). Thus, we should not expect to obtain a good representation of the whole secular resonance.

In order to reproduce the locus of secular and secondary resonances inside first-order mean motion resonances, some tests were recently made by Nesvorný (1997), using the spatial asymmetric expansion of this paper. He combined the expansion with the perturbative method of Henrard (1990) and succeded in obtain the position of some secular and secondary resonances, as well as the Kozai resonance, inside the 2/1, 3/2 and 4/3 resonances. As an example, we presented in Fig. 4 the location of the stable equilibrium of the Kozai resonance (together with its separatrixes), at the pericentric branch of the 2/1 resonance. It is also shown the position of the ν_{16} secular resonance. These calculations were made taking different asymmetric expansions around points (k_0, a_0) over the pericentric branch. We can compare this figure with Fig. 6 of Morbidelli & Moons (1993) or

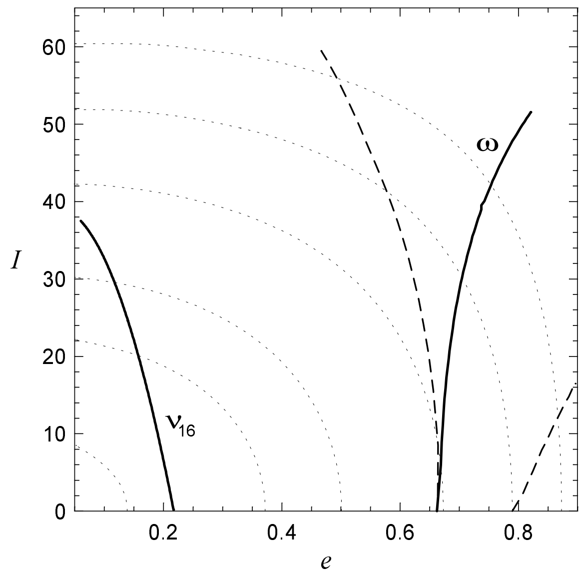

Fig. 4. The continous lines represent the location of the stable equilibrium points of the Kozai resonance (ω curve) and the ν_{16} secular resonance, at the pericentric branch of the 2/1 mean motion resonance ($\sigma = 0$ and $\sigma_z = 0$). The dashed lines are the separatrixes of the Kozai resonance. The calculations were made using the asymmetric expansion of this paper and considering $a' = 5.2$ AU, $e' = 0.048$, $I' = 1.2°$, and $\sigma' = \sigma'_z = 0$. Dotted lines are different levels of the constant $N = (\mu a)^{1/2}(2 - (1 - e^2)^{1/2} \cos I)$.

with Fig. 3 of Henrard et al. (1995) and see that the agreement is very good.

9. Conclusions

We have presented the asymmetric expansion of the averaged resonant disturbing function in the non-planar problem. Since this expansion is done around a fixed point in the k, h plane and truncated at $\mathcal{O}(e'^3)$, it approximates well the disturbing function only in a neighborhood of that point, and we have shown that this neighborhood around the center of expansion can be wide enough. Thus, we can apply it to problems in which the amplitude of libration is moderate. Recall that any application must be also limited in inclination, since we have expanded the function around zero inclinations.

Currently, this asymmetric expansion is being used in the construction of a non-planar symplectic mapping for the first-order asteroidal resonances, which extends the planar mapping used by Ferraz-Mello (1997). It is also being used in the formulation of an analytic theory of the high-eccentricity libration dynamics (Ferraz-Mello et al., 1997).

The asymmetric expansion has been already shown to be an useful tool for the expansion of the disturbing potential acting on a resonant asteroid, and the expansion presented in this paper may be useful in the analytic studies of the spatial resonant problems in high eccentricities.

FORTRAN and C codes to compute the coefficients of this expansion are available by anonymous ftp at: ftp://chaos1.iagusp.usp.br/users/ftp/pub/highecc.

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