



On the phase space of free higher-spin theories and conformal transformations

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Received 9 March 2017; received in revised form 5 October 2017; accepted 26 October 2017

Available online 31 October 2017

Editor: Leonardo Rastelli

Abstract

We write a first order action for higher-spin fields and construct a canonical map to Fronsdal theory. The first-order description is defined over complex field configurations and has conformal invariance. We show that it is possible to push forward these transformations to a set of symmetries in Fronsdal theory that satisfies the conformal algebra but is not given by standard conformal change of coordinates.

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1. Introduction

Higher-spin theories have an important application in the AdS/CFT context. When the AdS radius is small, it is conjectured that a subset of large string excitations decouples from the remaining degrees of freedom and is described by an interacting higher-spin theory [1]. Unfortunately, interactions are subtle to construct, but there is a comparatively easier case which we can study: in the $N \rightarrow \infty$ limit, we have free massless higher-spin theories.

There are two known descriptions of free theories, which are referred to as Fronsdal and Penrose formulation. In Fronsdal theory [3,4], we have constrained spacetime tensors that form an irreducible representation of the little group on-shell, while in Penrose theory one uses twistor geometry to construct irreducible representations of the little group [7]. Both theories are well described by an action which is invariant under higher-spin gauge symmetries. It is interesting,

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however, that Penrose formulation *is* invariant under conformal symmetries while Fronsdal formulation *is not* [10].

At first glance, it may seem strange that two descriptions of the same theory have different symmetries. The present paper aims to solve this conflict. After a brief review of the two formulations, we construct a first order action for Penrose theory and show that both theories describe the same classical phase space via a canonical transformation. Using the canonical map, we can push forward the conformal transformations of Penrose theory to a set of non-local conformal symmetries in Fronsdal description.

Plan of this paper We organize our presentation as follows. Section 2 is a brief review, where we explain the two approaches for free massless higher-spin theories.

In section 3 we write an action for Penrose higher-spin theory. To our knowledge, such action for general higher-spins has never appeared before in the literature. First-order formulations, however, were used by Fradkin and Vasiliev in [14] for AdS space, where they were also extended to interactions. More recently, Kirill Krasnov described full self-dual gravity in [2] using an action that resembles ours; but, in our case, this action is defined over complex field configurations, and it describes off-shell a doubled set of the higher-spin modes. In phase space, however, there is a well-defined notion of reality, and it is where we obtain a single copy of the spectrum.

At this point, we look at some examples, so the spins 1, 3/2 and 2 cases are discussed in detail, each of which highlights a particular feature of our construction outlining our strategy for dealing with general spins. The spin s case is done in section 4; our construction is a particular instance of the prescription given in [13], where a set of equations of motion and a presymplectic structure are shown to lift to a well-defined Lagrangian.

With this map, we can investigate conformal invariance. In section 5 we show that Penrose action does have conformal symmetry for every spin s . Therefore one is able to push forward these transformations to the Fronsdal case. For spins lower than 2, these new transformations agree with usual conformal change of coordinates. The first non-trivial case is linearized gravity. We write explicitly the resulting transformation, where one is able to see the difference from standard Lie derivatives.

On notation Our conventions follow those of [8]; we are concerned with 4-dimensional Minkowski space; so, through out the paper, the various indices will always be running over fixed intervals. Small Latin letters, for example, are spacetime indices running from 0 to 3, so that A_m is a spacetime covector. Capital Latin letters, in turn, are spinor indices in Van der Warden notation, that is, dotted and undotted running from 0 to 1. In particular, a Dirac spinor is a two component Weyl and anti-Weyl spinor written like

$$\Psi = \begin{pmatrix} \psi_A \\ \bar{\chi}^{\dot{A}} \end{pmatrix} \quad (1.1)$$

for some chiral spinor ψ_A and anti-chiral $\bar{\chi}^{\dot{A}}$.

Such notation is designed so that there is a correspondence between spacetime and spinor indices where, for instance, m will correspond to the pair $M\dot{M}$. The explicit realization is given by the Pauli matrices with index structure $\sigma_{M\dot{M}}^m$, where

$$\sigma^0 = -1 \quad \text{and} \quad \vec{\sigma} = (\sigma^1, \sigma^2, \sigma^3).$$

The epsilon symbol satisfies $\epsilon_{AB}\epsilon^{BC} = \delta_A^C$ for undotted and dotted indices. This enables one to raise the indices of $\sigma_{M\dot{M}}^m$ to obtain

$$\bar{\sigma}^{m M \dot{M}}, \quad \text{where} \quad \sigma^0 = -1 \quad \text{and} \quad \vec{\sigma} = (-\sigma^1, -\sigma^2, -\sigma^3).$$

Everything is combined to form the Weyl representation of the Dirac matrices:

$$\Gamma^m = \begin{pmatrix} 0 & \sigma^m \\ \bar{\sigma}^m & 0 \end{pmatrix}, \quad (1.2)$$

which satisfy the Clifford algebra

$$\{\Gamma^m, \Gamma^n\} = -2\eta^{mn} \quad (1.3)$$

for the metric signature $(-, +, +, +)$.

2. Review of massless higher-spin formulations

This section is an overview of some background material based on references [3] and [4]. It begins with Fronsdal theory and then proceeds to Penrose description [6].

2.1. Fronsdal theory of free massless higher-spin fields

Let us begin with bosonic spins. Given a totally symmetric tensor of s indices, $h_{m_1 \dots m_s}$, which has higher-spin gauge freedom of the form

$$\delta h_{m_1 \dots m_s} = s \partial_{(m_1} \varepsilon_{m_2 \dots m_s)} \quad (2.1)$$

and is double-traceless:

$$\eta^{m_1 m_2} \eta^{m_3 m_4} h_{m_1 m_2 m_3 m_4 \dots m_s} = 0; \quad (2.2)$$

one can form the so-called Fronsdal tensor:

$$F_{m_1 \dots m_s} = \square h_{m_1 \dots m_s} - s \partial_{(m_1} \partial^p h_{|p| m_2 \dots m_s)} + \frac{s(s-1)}{2} \partial_{(m_1} \partial_{m_2} h^p{}_{|p| m_3 \dots m_s)}. \quad (2.3)$$

A higher-spin theory in flat spacetime is then described by the action

$$S = \frac{(-1)^{s+1}}{2} \int d^4 x \left(h^{m_1 \dots m_s} F_{m_1 \dots m_s} - \frac{s(s-1)}{4} h_n{}^{m_3 \dots m_s} F^p{}_{p m_3 \dots m_s} \right), \quad (2.4)$$

which is symmetric in the higher-spin field $h_{m_1 \dots m_s}$ and gauge invariant under transformations (2.1).

The equations of motion read

$$F_{m_1 \dots m_s} - \frac{s(s-1)}{4} \eta_{(m_1 m_2} F^p{}_{p m_3 \dots m_s)} = 0. \quad (2.5)$$

And these can be further simplified if (2.2) is taken into account. It implies

$$\eta^{m_1 m_2} \eta^{m_3 m_4} F_{m_1 m_2 m_3 m_4 \dots m_s} = 0 \quad (2.6)$$

which, in turn, allows us to cast equation (2.5) as

$$F_{m_1 \dots m_s} = 0. \quad (2.7)$$

We see the Fronsdal tensor fixes $h_{m_1 \dots m_s}$ up to gauge transformations since both have the same number of degrees of freedom. The physical degrees of freedom, however, are obtained once we

gauge fix the above description. It is possible to gauge away the trace part of the higher-spin field $h_{m_1 \dots m_s}$ as well as its divergence. Consider the gauge field ε which satisfies

$$h^p_{pm_3 \dots m_s} = \partial^n \varepsilon_{nm_3 \dots m_s} \quad (2.8)$$

and

$$\partial^p h_{pm_2 \dots m_s} = \square \varepsilon_{m_2 \dots m_s}, \quad (2.9)$$

so that the remaining gauge symmetry obeys

$$\square \varepsilon_{m_2 \dots m_s} = 0, \quad \partial^n \varepsilon_{nm_3 \dots m_s} = 0, \quad \text{and} \quad \varepsilon^p_{pm_3 \dots m_s} = 0. \quad (2.10)$$

Once we choose (2.10), our higher-spin field satisfies

$$\square h_{m_1 \dots m_s} = 0, \quad \partial^p h_{pm_2 \dots m_s} = 0, \quad \text{and} \quad h^p_{pm_3 \dots m_s} = 0; \quad (2.11)$$

thus proving that $h_{m_1 \dots m_s}$ describes a spin s massless particle.

There are minor changes if one wants to describe fermions. For a spin $s = h + 1/2$, we have a Majorana spinor $\Psi_{m_1 \dots m_h}$ totally symmetric in its h indices which has gauge freedom

$$\delta \Psi_{m_1 \dots m_h} = h \partial_{(m_1} \chi_{m_2 \dots m_h)}, \quad (2.12)$$

and satisfies the triple Γ -trace condition:

$$\Gamma^{m_1} \Gamma^{m_2} \Gamma^{m_3} \Psi_{m_1 m_2 m_3 \dots m_h} = 0. \quad (2.13)$$

The fermionic Fronsdal tensor,

$$F_{m_1 \dots m_h} = \Gamma^a \partial_a \Psi_{m_1 \dots m_h} - h \partial_{(m_1} \Gamma^a \Psi_{m_2 \dots m_h) a}, \quad (2.14)$$

is the gauge invariant object used to construct the action

$$S = \frac{1}{2} \int d^4 x \left(\bar{\Psi}^{m_1 \dots m_h} F_{m_1 \dots m_h} - \frac{h}{2} \Gamma^p \bar{\Psi}_p^{m_2 \dots m_h} \Gamma^a F_{a m_2 \dots m_h} - \frac{h(h-1)}{4} \bar{\Psi}_q^{m_3 \dots m_h} F^p_{pm_3 \dots m_h} \right) \quad (2.15)$$

where $\bar{\Psi}^{m_1 \dots m_h}$ satisfies the Majorana condition:

$$\bar{\Psi}^{m_1 \dots m_h} = \Psi^T C, \quad \text{and} \quad C = \begin{pmatrix} \epsilon_{BA} & 0 \\ 0 & \epsilon \dot{B} \dot{A} \end{pmatrix} \quad (2.16)$$

is the charge conjugation matrix. The equations of motion are

$$F_{m_1 \dots m_s} - \frac{h}{2} \Gamma_{(m_1} \Gamma^a F_{m_2 \dots m_s) a} - \frac{h(h-1)}{4} \eta_{(m_1 m_2} F^p_{m_3 \dots m_s) p} = 0. \quad (2.17)$$

and they can be simplified once one notices (2.13) implies

$$\Gamma^{m_1} \Gamma^{m_2} \Gamma^{m_3} F_{m_1 m_2 m_3 \dots m_h} = 0, \quad (2.18)$$

which enables one to cast (2.17) in the form

$$F_{m_1 \dots m_h} = 0. \quad (2.19)$$

Notice that, again, the fermionic Fronsdal tensor fixes $\Psi_{m_1 \dots m_h}$ up to gauge transformations. The physical degrees of freedom are obtained from the gauge parameter $\chi_{m_2 \dots m_h}$ that satisfies

$$\Gamma^P \Psi_{pm_2 \dots m_h} = \Gamma^m \partial_m \chi_{m_2 \dots m_h}, \quad (2.20)$$

so that the remaining gauge symmetry obeys

$$\Gamma^m \partial_m \chi_{m_2 \dots m_h} = 0 \quad \text{and} \quad \Gamma^P \chi_{pm_2 \dots m_h} = 0. \quad (2.21)$$

The gauge fixing (2.20) ensures that $\Psi_{m_1 \dots m_h}$ is an irreducible representation of the little group. The on-shell degrees of freedom are then described by a field Ψ which satisfies

$$\Gamma^P \partial_P \Psi_{m_1 \dots m_h} = 0 \quad \text{and} \quad \Gamma^P \Psi_{pm_2 \dots m_h} = 0 \quad (2.22)$$

thus proving $\Psi_{m_1 \dots m_h}$ describes a spin $s = h + 1/2$ representation.

2.2. Penrose theory of free massless higher-spin fields

Penrose's description of massless higher-spin fields is obtained from the Penrose transform. It relates homogeneous functions of definite degree in twistor space to massless higher-spin fields in Minkowski space. For an introduction to twistors, see reference [5] as well as references therein.

Here we describe the integral expressions obtained by Penrose in [6] only to give some context. These integral formulas are not necessary for the rest of this paper. We are only interested in the spacetime fields they define.

Let $Z = (\omega^A, \pi_{\dot{A}})$ be the coordinates of a twistor inside the complex projective line $\mathbf{P}_1$. These are constrained by the twistor equation:

$$\omega^A = x^{A\dot{A}} \pi_{\dot{A}}, \quad (2.23)$$

where $x^{A\dot{A}}$ parametrizes the Minkowski space. Consider also a point $\bar{Z} = (\lambda_A, \mu^{\dot{A}})$ in the dual twistor space and fix two closed cycles of integration: γ inside $\mathbf{P}_1$ and γ^* inside the dual line $\mathbf{P}_1^*$. Define the following spacetime spinors

$$\bar{\phi}_{\dot{A}\dot{B}\dots\dot{D}}(x) = \frac{1}{2\pi i} \int_{\gamma} \underbrace{\pi_{\dot{A}} \pi_{\dot{B}} \dots \pi_{\dot{D}}}_{2s} f(Z) \pi_{\dot{E}} d\pi^{\dot{E}} \quad (2.24a)$$

and

$$\phi_{AB\dots D}(x) = \frac{1}{2\pi i} \int_{\gamma^*} \underbrace{\lambda_A \lambda_B \dots \lambda_D}_{2s} \bar{f}(\bar{Z}) \lambda^A d\lambda_A \quad (2.24b)$$

for some semi-integer number s .

Remark. These integrals are well defined over $\mathbf{P}_1$ if the integrands are homogeneous functions of degree 0. Hence, the complex functions $f(Z)$ and $\bar{f}(\bar{Z})$ must have homogeneity $-2s - 2$ in $\pi_{\dot{A}}$ and λ_A respectively.

These spinors form an irreducible representation of the Lorentz group $SL(2, \mathbb{C})$ and satisfy, by consequence of their definitions, the differential equations

$$\partial^{A\dot{A}} \bar{\phi}_{\dot{A}\dot{B}\dots\dot{D}}(x) = 0 \quad (2.25)$$

and

$$\partial^{AA} \phi_{AB\dots D}(x) = 0. \quad (2.26)$$

In view of the (anti-)self-duality conditions, we can see $\phi^{AB\cdots D}$ and $\bar{\phi}^{\dot{A}\dot{B}\cdots\dot{D}}$ describe right-handed massless free fields of spin s and left-handed massless free fields of spin $-s$ respectively.

Let $a_{\dot{A}B\cdots D}$ be the field given by

$$\bar{\phi}^{\dot{A}\dot{B}\cdots\dot{D}} = \partial^B_{(\dot{B}} \cdots \partial^D_{\dot{D}} a_{\dot{A})B\cdots D}. \quad (2.27)$$

It readily follows that equation (2.25) is automatically satisfied when

$$\partial_{(A} \dot{A} a_{B\cdots D)\dot{A}} = 0. \quad (2.28)$$

Notice, however, that there is an ambiguity. There are gauge symmetries of the form

$$\delta a_{\dot{A}B\cdots D} = \partial_{\dot{A}(B} \xi_{\cdots D)} \quad (2.29)$$

for some symmetric spinor $\xi_{C\cdots D}$ of $2s - 2$ indices. These are the higher-spin gauge symmetries which were also present in Fronsdal theory.

We will always refer to $\phi^{AB\cdots D}$ and $a_{\dot{A}B\cdots D}$ as the fundamental fields of Penrose description. And, for future reference, we call $\phi_{AB\cdots D}$ the curvature spinor and $a_{\dot{A}B\cdots D}$ the gauge field.

3. Higher-spin action in Penrose's description

3.1. Higher-spin action

We suggest the following higher-spin action for a massless spin s particle:

$$S = i \int d^4x \left(\phi^{AB\cdots D} \partial_{A\dot{A}} a^{\dot{A}}_{B\cdots D} \right) \quad (3.1)$$

where $\phi^{AB\cdots D}$ and $a^{\dot{A}}_{B\cdots D}$ have $2s$ and $2s - 1$ undotted indices respectively. Invariance under higher-spin gauge symmetries is respected, because if we consider the variation under (2.29) the action transforms into

$$\delta S = i \int d^4x \left[\phi^{AB\cdots D} \partial_{A\dot{A}} \partial^{\dot{A}}_{(B} \xi_{\cdots D)} \right]. \quad (3.2)$$

From the identity

$$\partial_{A\dot{A}} \partial^{\dot{A}}_B = +\frac{1}{2} \epsilon_{AB} \square, \quad (3.3)$$

we get $\delta S = 0$ since the curvature spinor $\phi^{AB\cdots D}$ is completely symmetric in its indices. The equations of motion obtained from (3.1) are precisely (2.25) and (2.28):

$$\partial_{A\dot{A}} \phi^{AB\cdots D} = 0 \quad \text{and} \quad \partial_{\dot{A}(A} a^{\dot{A}}_{B\cdots D)} = 0.$$

3.2. Reality conditions

It is a good point to make some observations. First, although twistors were used as a motivation for this action, we are not integrating over twistor space. We are only using a spinor basis and it is possible to write this action with usual Lorentz indices too. The convenience of using spinors is the easier treatment of self-duality conditions. Second, and possibly a troublesome point, is that it appears that this action describes just one helicity, but this is not the case.

Let us discuss this point in detail. For the sake of argument, let us specialize our discussion to the spin 1 case. We want to show that the phase space spanned by these equations is equivalent to the phase space of Maxwell's electromagnetism. The natural route is to describe a canonical map. Therefore, given the data (ϕ, a) , we are supposed to construct a map to the Maxwell gauge field A ,

$$H : (\phi, a) \mapsto A, \quad (3.4)$$

where solutions of the (ϕ, a) system are carried to solutions of the Maxwell's equations. In addition, we must verify two things: the kernel of this map must be zero, otherwise there are configurations of ϕ and a which would correspond to zero electromagnetic solution; and the cokernel should also be zero, that is the set of all Maxwell solutions, given by A , should be fully covered.

The canonical map H is constructed as follows. Given the equation of motion (2.25), locally by the Poincaré lemma, we can write ϕ as

$$\phi = d\bar{a} \quad (3.5)$$

with some possible ambiguity given by the addition of a closed form. The second equation of motion, (2.28), is the statement that a does not contribute to the self-dual part, hence it must describe the anti-self-dual piece. It becomes natural to define

$$A = a + \bar{a} \quad (3.6)$$

since it satisfies Maxwell's equations as a consequence of self-duality:

$$\begin{aligned} d \star dA &= d \star d(a + \bar{a}) \\ &= d \star (da + d\bar{a}) \\ &= i d(da - d\bar{a}) \\ &= 0. \end{aligned} \quad (3.7)$$

Notice that the kernel of (3.6) indeed vanishes. One takes $-a + d\alpha = \bar{a}$, for some α , and, by consequence of (2.28), $\phi = 0$, which forces a to be pure gauge. That the cokernel vanishes is a more subtle point. Because the Hodge star operator $\star$ satisfies $\star^2 = -1$ in four dimensions, it splits the bundle Λ^2 , of two-forms in Minkowski space, into a direct sum,

$$\Lambda^2 = \Lambda_+^2 \oplus \Lambda_-^2, \quad (3.8)$$

where $\Lambda_{\pm}^2$ are the $\pm i$ eigenspaces of $\star$. Thus, any two form can be written as

$$F = \phi + \bar{\phi} \quad (3.9)$$

and, by the Poincaré lemma, we locally have the decomposition (3.6).

The analysis of this construction is special to the 4-dimensional Minkowski space and it carries through only for the equations of motion. It is not true that the action (3.1) is off-shell equivalent to the Maxwell action. One way to understand this is to notice that the action (3.1) is not real. In general, equation (3.1) is defined over some complex infinite-dimensional manifold.

Such consideration raises the question if whether the map (3.6) defines a real A or not. It turns out that, in phase space, complex conjugation acts as an involution, where the complex conjugation map, denoted $\mathbb{C} \cdot \mathbb{C} \cdot$, is

$$\mathbb{C} \cdot \mathbb{C} \cdot \begin{pmatrix} a \\ \phi \end{pmatrix} = \begin{pmatrix} d^{-1}\bar{\phi} \\ d\bar{a} \end{pmatrix}. \quad (3.10)$$

It has fixed point given by

$$\overline{\phi_{AB}} = \overline{\phi}_{\dot{A}\dot{B}} = \partial_{C(\dot{A}} a^C_{\dot{B})}, \quad (3.11)$$

from where we see that the complex conjugate of a is $\bar{a}$ and vice-versa. To summarize our results: the action (3.1) is complex, but in phase space – that is, the space of classical solutions – there is a well-defined notion of reality, which is given by the fixed point of the involution (3.10), namely equation (3.11). Only in this submanifold, the two theories classically agree.

Outside the fixed point, the complex theory describes two photons. Self-duality of ϕ allows one to write

$$\phi = F + i \star F \quad (3.12)$$

for a real 2-form F . Hence, the equation of motion $d\phi = 0$ implies Maxwell's equations:

$$dF = 0 \quad \text{and} \quad d \star F = 0. \quad (3.13)$$

On the other hand, the gauge field a on-shell gives an anti-self-dual 2-form:

$$da = G - i \star G \quad (3.14)$$

from where the second Maxwell equations come:

$$dG = 0 \quad \text{and} \quad d \star G = 0. \quad (3.15)$$

The reality conditions (3.11) impose $F = G$.

3.3. Making action real

Consider the real part of the action¹ (3.1):

$$S = \int (\phi \wedge da + \bar{\phi} \wedge d\bar{a}). \quad (3.16)$$

It turns out that the equations of motion are unchanged. To see this, consider the variation of this action under the real and imaginary parts of a , it gives

$$d(\phi + \bar{\phi}) = 0 \quad \text{and} \quad d(\phi - \bar{\phi}) = 0 \quad (3.17)$$

respectively. Self-duality of ϕ does not allow us to vary its real and imaginary parts independently, therefore we have a single equation of motion:

$$d(a + \bar{a}) + i \star d(a - \bar{a}) = 0. \quad (3.18)$$

Inspection shows that the real and imaginary parts of a satisfy the Maxwell's equations while ϕ again satisfies $d\phi = 0$. The two copies of the Maxwell theory can be identified with the reality condition (3.11). It is surprising that the addition of complex conjugation does not change the field content of the theory.

¹ We would like to thank Arkady Tseytlin for suggesting this idea.

3.4. Symplectic structure

We wish to establish the above correspondence for every spin s field. The above consideration can be rephrased using the notion of symplectic structure.² In this language, although the action is defined for complex field configurations, there is a real submanifold inside the phase space where the restriction of the symplectic form derived from (3.1) is non-degenerate. Then, we will construct a map H that becomes a canonical transformation to the phase space of Fronsdal.

The symplectic structure for action (3.1) is

$$\Omega = i \int_C \delta\phi^{AB\cdots D} \wedge \delta a^{\dot{A}}_{B\cdots D} \wedge d^3x_{A\dot{A}}, \quad \text{where} \quad n_{A\dot{A}} d^3x = d^3x_{A\dot{A}}, \quad (3.19)$$

for a normal vector $n_{A\dot{A}}$ to the spacelike contour C . It is δ -closed and invariant under deformations of C , because

$$\partial_{A\dot{A}} \left(\delta\phi^{AB\cdots D} \wedge \delta a^{\dot{A}}_{B\cdots D} \right) = 0 \quad (3.20)$$

once we use the equations of motion. However, note that this symplectic structure is also degenerate. Degeneracies indicate the presence of gauge symmetries in the action. In our case, if we let

$$V = \partial_{\dot{A}(B} \xi_{\cdots D)}(x) \frac{\delta}{\delta a^{\dot{A}}_{B\cdots D}(x)} \quad (3.21)$$

be a tangent vector field along gauge trajectories, we get

$$\begin{aligned} \iota_V \Omega &= i \int_C \partial^{\dot{A}}_{(B} \xi_{\cdots D)} \delta\phi^{AB\cdots D} d^3x_{A\dot{A}} \\ &= i \int_C \partial^{\dot{A}}_{(B} \left[\xi_{\cdots D)} \delta\phi^{AB\cdots D} \right] d^3x_{A\dot{A}} - i \int_C \xi_{(C\cdots D} \partial_{B)}^{\dot{A}} \delta\phi^{ABC\cdots D} d^3x_{A\dot{A}} \\ &= i \int_C \partial^{\dot{A}}_{(B} \left[\xi_{\cdots D)} \delta\phi^{AB\cdots D} \right] d^3x_{A\dot{A}} = 0, \end{aligned} \quad (3.22)$$

where the last line vanishes due to C being a closed contour. Degenerate symplectic structures descend to a reduced phase space. If we define $\ker \Omega$ to be the set of gauge generators, then the reduced phase space is given by the factor $M / \ker \Omega$. On-shell gauge-invariant functions are points in this space and they coincide with physical observables.

It still remains to be checked whether this symplectic structure is real over the fixed point defined by the involution.³ The fixed point can be written as

$$\overline{\phi_{AB\cdots D}} = \overline{\phi_{\dot{A}\dot{B}\cdots\dot{D}}} = \partial_{(\dot{B}}^B \cdots \partial_{\dot{D}}^D a_{\dot{A})B\cdots D} \quad (3.23)$$

and it follows that

² For a brief review of these terms, see [Appendix A](#).

³ See paragraph above equation (3.11).

$$\begin{aligned}
\overline{\Omega} &= -i \int_C \delta \overline{\phi}^{\dot{A}\dot{B}\dots\dot{D}} \wedge \delta \overline{a}^A_{\dot{B}\dots\dot{D}} \wedge d^3 x_{A\dot{A}} \\
&= -i \int_C \partial^{\dot{B}}_{\dot{B}} \partial^{\dot{C}}_{\dot{C}} \dots \partial^{\dot{D}}_{\dot{D}} \delta a^{\dot{A}}_{B\dots D} \wedge \delta \overline{a}^A_{\dot{B}\dots\dot{D}} \wedge d^3 x_{A\dot{A}} \\
&= (-)^{2s+1} i \int_C \delta a^{\dot{A}B\dots D} \wedge \partial^{\dot{B}}_{(B} \partial^{\dot{C}}_{\dot{C}} \dots \partial^{\dot{D}}_{\dot{D}} \delta \overline{a}_{A)\dot{B}\dots\dot{D}} \wedge d^3 x_{\dot{A}}{}^A \\
&= -i \int_C \delta a^{\dot{A}}_{B\dots D} \wedge \delta \phi^{AB\dots D} \wedge d^3 x_{A\dot{A}} \\
&= +\Omega,
\end{aligned} \tag{3.24}$$

thus proving that indeed the symplectic structure is real.

Having the symplectic structure for Penrose theory, it remains to construct the canonical map which will relate the two descriptions. In doing so, we are ready to prove that the two phase spaces agree.

4. Canonical map between descriptions

It is instructive to consider some examples before treating the general case. We specialize our discussion to Rarita–Schwinger and linearized gravity in the next two subsections. Each case will serve to emphasize the introduction of a new tool for the analysis.

In the Rarita–Schwinger case, for example, we will see how the splitting of the gauge field into self-dual and anti-self-dual connection – as it has already happened in electromagnetic case – comes about in the symplectic structure. The main objective is to demonstrate, on the real slice given by (3.23), that the canonical map indeed preserves the symplectic structure.

In linearized gravity, we show how the analysis can be made rather straightforward once we pass to momentum space. It will avoid dealing with integration by parts when we show that the symplectic structures agree.

4.1. Rarita–Schwinger case

The Rarita–Schwinger theory is obtained when $h = 1$ in Section 2.1. We have the Majorana spinor

$$\Psi_m = \begin{pmatrix} \psi_{Am} \\ \overline{\psi}_m^{\dot{A}} \end{pmatrix} \tag{4.1}$$

with higher-spin gauge symmetries $\delta \Psi_m = \partial_m \varepsilon$ and gauge-invariant action

$$S = \int d^4 x \left(\overline{\Psi}^m F_m + \frac{1}{2} \overline{\Psi}_p \Gamma^p \Gamma^m F_m \right). \tag{4.2}$$

The equations of motion read

$$F_m = \Gamma^n \partial_n \Psi_m - \partial_m \Gamma^n \Psi_n = 0. \tag{4.3}$$

For our applications, it will be useful to consider the gauge-invariant combination

$$R_{mn} = \partial_m \Psi_n - \partial_n \Psi_m, \tag{4.4}$$

in order to make contact with the curvature spinors ϕ^{ABC} and $\bar{\phi}^{\dot{A}\dot{B}\dot{C}}$. To see how, let us introduce the following spinor counterpart of R_{mn} :

$$R_{M\dot{M}N\dot{N}} = d\Psi_{(\dot{M}\dot{N})} \epsilon_{MN} - d\Psi_{(MN)} \epsilon_{\dot{M}\dot{N}}, \quad (4.5)$$

where abbreviations have been used:

$$\partial^A_{(\dot{N}} \Psi_{\dot{M})A} = d\Psi_{(\dot{M}\dot{N})} = \begin{pmatrix} d\psi_{(\dot{M}\dot{N})B} \\ d\bar{\psi}^{\dot{B}}_{(\dot{M}\dot{N})} \end{pmatrix} \quad (4.6a)$$

and

$$\partial_{\dot{A}(M} \Psi_{N)}^{\dot{A}} = d\Psi_{(MN)} = \begin{pmatrix} d\psi_{(MN)B} \\ d\bar{\psi}^{\dot{B}}_{(MN)} \end{pmatrix}. \quad (4.6b)$$

It enables us to rewrite the equations of motion in the form

$$\Gamma^m R_{mn} = 0 \mapsto \begin{pmatrix} 0 & \delta_B^M \delta_{\dot{B}}^{\dot{M}} \\ \epsilon^{MB} \epsilon_{\dot{M}\dot{B}} & 0 \end{pmatrix} \begin{pmatrix} d\psi_{\dot{M}\dot{N}B} \epsilon_{MN} - d\psi_{MN\dot{B}} \epsilon_{\dot{M}\dot{N}} \\ d\bar{\psi}^{\dot{B}}_{\dot{M}\dot{N}} \epsilon_{MN} - d\bar{\psi}^{\dot{B}}_{MN} \epsilon_{\dot{M}\dot{N}} \end{pmatrix} = 0, \quad (4.7)$$

from where we obtain

$$d\psi_{\dot{A}\dot{N}N} - d\psi_{NC}^C \epsilon_{\dot{A}\dot{N}} = 0 \quad (4.8a)$$

and

$$d\bar{\psi}^{\dot{C}}_{\dot{C}\dot{N}} \epsilon_{AN} + d\bar{\psi}^{\dot{N}}_{AN} = 0. \quad (4.8b)$$

A quick inspection shows the only possible solutions for (4.8a) are

$$d\psi_{\dot{A}\dot{N}N} = 0 \quad \text{and} \quad d\psi_{NC}^C = 0 \quad (4.9)$$

since the first term is symmetric in $\dot{A}\dot{N}$ while the second one is anti-symmetric in $\dot{A}\dot{N}$. The same type of reasoning leads us to the solutions of (4.8b):

$$d\bar{\psi}^{\dot{C}}_{\dot{C}\dot{N}} = 0 \quad \text{and} \quad d\bar{\psi}^{\dot{N}}_{AN} = 0. \quad (4.10)$$

These solutions annihilate any components with dotted and undotted indices. Moreover they completely symmetrize the self-dual and anti-self-dual part. The remaining components split R_{mn} into

$$R_{mn} \mapsto -d\psi_{(MNA)} \epsilon_{\dot{M}\dot{N}} - d\bar{\psi}_{(\dot{A}\dot{M}\dot{N})} \epsilon_{MN} \quad (4.11)$$

and we can identify

$$-d\psi_{(MNA)} \quad \text{as} \quad \phi_{AMN}, \quad (4.12)$$

and

$$-d\bar{\psi}_{\dot{A}\dot{M}\dot{N}} \quad \text{as} \quad \bar{\phi}_{\dot{A}\dot{M}\dot{N}}. \quad (4.13)$$

This procedure occurs for other spins as well. One defines a gauge-invariant combination, and once the equations of motion are imposed the spinors $\phi^{AB\cdots D}$ and $\bar{\phi}^{\dot{A}\dot{B}\cdots \dot{D}}$ are the only remaining components. Notice that

$$\partial_{[m} R_{np]} = 0 \quad (4.14)$$

is trivially satisfied in the presence of Ψ_m . As soon as we change pictures and use the curvature spinors, this equation turns into an equation of motion. The anti-symmetry is equivalent to a contraction of spinor indices, and so we recover (2.25) and (2.26):

$$\partial^{\dot{A}A}\phi_{AMN} = 0 \quad \text{and} \quad \partial^{A\dot{A}}\bar{\phi}_{\dot{A}\dot{M}\dot{N}} = 0.$$

The Penrose description splits the gauge field $h_{m_1\dots m_s}$ into anti-self-dual and self-dual parts treating the self-dual part via the curvature while the anti-self-dual part is described with the anti-self-dual gauge field.

In the Rarita–Schwinger case, the gauge field $a_{\dot{A}BC}$ is mapped to the anti-chiral part $\bar{\psi}^{m\dot{A}}$ with the *ansatz*

$$\bar{\psi}^{m\dot{A}} = i\bar{\sigma}^{m\dot{E}E} \left(\partial^{\dot{A}C} a_{\dot{E}CE} + \frac{1}{2} \partial_{\dot{E}}^C a^{\dot{A}}_{CE} \right) \quad (4.15)$$

where the coefficients are fixed by requiring the higher-spin gauge symmetries to coincide. For consistency, it is also possible, with this choice, to check that $\bar{\psi}^{m\dot{A}}$ satisfies the equations of motion when $a_{\dot{A}BC}$ does. We should point out that this map is the non-trivial piece of our correspondence. For other higher-spins, it has to be constructed with the right coefficients case by case.

One can derive the symplectic structure from action (4.2) and it reads:

$$\begin{aligned} \Omega = \int & \left(2\delta\psi_m \wedge \sigma^m \bar{\sigma}^{np} \delta\bar{\psi}_p + 2\delta\bar{\psi}_m \wedge \bar{\sigma}^m \sigma^{np} \delta\psi_p \right. \\ & \left. + \delta\psi_m \wedge \sigma^n \delta\bar{\psi}^m + \delta\bar{\psi}_m \wedge \bar{\sigma}^n \delta\psi^m - \delta\psi^n \wedge \sigma^m \delta\bar{\psi}_m - \delta\bar{\psi}^n \wedge \bar{\sigma}^m \delta\psi_m \right) \wedge d^3x_n. \end{aligned} \quad (4.16)$$

If we intend to describe the spin 3/2 piece, we are allowed to use the gauge

$$\Gamma^m \Psi_m = 0 \quad (4.17)$$

so the symplectic structure collapses to

$$\Omega = 2 \int \delta\psi_m \wedge \sigma^n \delta\bar{\psi}^m \wedge d^3x_n. \quad (4.18)$$

In Penrose case, the symplectic structure follows from (3.1), and it is

$$\Omega = i \int \delta\phi^{ABC} \wedge \delta a^{\dot{A}}_{BC} \wedge d^3x_{\dot{A}\dot{A}}. \quad (4.19)$$

Notice the gauge condition implies

$$\partial^{\dot{B}A}\psi_{\dot{E}EA} = 0, \quad (4.20)$$

and by consequence of (4.15):

$$\partial_{\dot{A}}^B a^{\dot{A}}_{BC} = 0. \quad (4.21)$$

When substitute our *ansatz* into the symplectic structure (4.18), we obtain

$$\Omega = +i \int \delta\psi^{\dot{E}EA} \wedge \partial_{\dot{E}}^C \delta a^{\dot{A}}_{\dot{A}EC} \wedge d^3x_A^{\dot{A}} \quad (4.22)$$

and there is a subtlety we must highlight. Despite the advantage of being able to use the equations of motion when dealing with a symplectic structure, we are not allowed to integrate by parts indiscriminately. If we assume, for the moment, that we can make such integration, then we would get the desired result:

$$\Omega = +i \int \delta\psi^{\dot{E}EA} \wedge \partial_{\dot{E}}^C \delta a_{\dot{A}EC} \wedge d^3x_A^{\dot{A}} = -i \int \partial_{\dot{E}}^C \delta\psi^{\dot{E}EA} \wedge \delta a_{\dot{A}EC} \wedge d^3x_A^{\dot{A}}, \quad (4.23)$$

because, by the equations of motion, the $d\psi$ term is symmetric in the pair CE but also in EA – thus being symmetric in all of its indices – and we have

$$\Omega = -i \int \partial_{\dot{E}}^C \delta\psi^{\dot{E}EA} \wedge \delta a_{\dot{A}EC} \wedge d^3x_A^{\dot{A}} = -i \int \delta\phi^{CEA} \wedge \delta a_{\dot{A}EC} \wedge d^3x_A^{\dot{A}}. \quad (4.24)$$

The integration by parts is justified if we show that the two terms differ by an exact form. Consider

$$\begin{aligned} \int \partial_m \delta X^{[mn]} \wedge d^3x_n &= \int \partial_{\dot{E}C} \delta X^{[\dot{E}C|\dot{A}A]} \wedge d^3x_{\dot{A}A} \\ &= - \int \partial_{\dot{E}}^C \left(\delta\psi^{\dot{E}AE} \wedge \delta a_{\dot{A}CE} - \delta\psi_{\dot{A}}^E \wedge \delta a^{\dot{E}A}_E \right) \wedge d^3x_A^{\dot{A}} \end{aligned} \quad (4.25)$$

and notice that (4.25) is exactly what we want:

$$- \int \left(\partial_{\dot{E}}^C \delta\psi^{\dot{E}EA} \wedge \delta a_{\dot{A}EC} \wedge d^3x_A^{\dot{A}} + \delta\psi^{\dot{E}EA} \wedge \partial_{\dot{E}}^C \delta a_{\dot{A}EC} \wedge d^3x_A^{\dot{A}} \right), \quad (4.26)$$

since all other terms cancel after we use (4.20) together with the equation of motion for the gauge field $a_{\dot{A}B\dots D}$:

$$\partial^{\dot{A}}_{(A} a_{BC)\dot{A}} = 0. \quad (4.27)$$

In all other cases, the integration by parts will be the main issue. We circumvent the difficulty of finding appropriate exact forms by working in momentum space.

4.2. Linearized gravity case

When $s = 2$ in section 2.1 we have linearized Einstein theory of gravity. The field h_{mn} has gauge invariance of the form

$$\delta_{\xi} h_{mn}(x) = \partial_m \xi_n(x) + \partial_n \xi_m(x) \quad (4.28)$$

and is described by the flat space action

$$S = -\frac{1}{2} \int d^4x \left(h^{mn} R_{mn} - \frac{1}{2} h^p_p R^q_q \right). \quad (4.29)$$

The R_{mn} and R^p_p represent the Ricci tensor and Ricci scalar respectively. Both can be obtained from the linearized curvature given by

$$R_{mnpq} = 4 \partial_{[m} h_{n][p} \overleftarrow{\partial}_{q]}. \quad (4.30)$$

The equations of motion are the linearized Einstein field equations

$$R_{mn} = 0 \quad (4.31)$$

and the symplectic structure is

$$\Omega = -\frac{1}{2} \int \left(2\delta h^m_n \wedge \partial^p \delta h_p^n - \delta h_{pn} \wedge \partial^m \delta h^{pn} + \delta h_p^p \wedge \partial^m \delta h_n^n - \partial_n \delta h^{mn} \wedge \delta h_p^p + \partial^p \delta h^n_n \wedge \delta h_p^m \right) \wedge d^3 x_m. \quad (4.32)$$

In order to change to Penrose description, we need to identify the (ϕ, a) fields. The self-dual part of R_{mnpq} gives ϕ_{MNPQ} via

$$\phi_{MNPQ} = \partial_{\dot{M}(M} \partial_{|\dot{N}|N} h^{\dot{M}\dot{N}}{}_{PQ)}, \quad (4.33)$$

while the anti-self-dual piece is described by the map

$$h_{M\dot{M}N\dot{N}} = -i \partial_{\dot{M}}{}^C a_{\dot{N}CMN} - i \partial_{\dot{N}}{}^C a_{\dot{M}CMN}. \quad (4.34)$$

Again, (4.34) is an *ansatz*. It is constructed by requiring gauge symmetries to coincide. An interesting feature we should stress is that h comes traceless since a is completely symmetric in its undotted indices. This is not a problem. In Fronsda theory these degrees of freedom are pure gauge.

We will demonstrate that the phase spaces of these descriptions agree. In this on-shell counting, let us go into Fourier space and fix the only non-zero component of the momentum to be $p_2^{\dot{2}}$. From the spinor description, we have then

$$\partial^{\dot{A}A} \phi_{ABCD} = 0 \implies p^{\dot{2}1} \phi_{1BCD} = 0, \quad (4.35)$$

which implies that every term with an 1 index vanishes. The only non-zero component of ϕ thus is ϕ_{2222} . For the gauge field a , we have

$$\partial_{(A}{}^{\dot{A}} a_{BCD)\dot{A}} = 0 \implies p_{(2}{}^{\dot{2}} a_{BCD)\dot{2}} = 0, \quad (4.36)$$

which means that every a with a $\dot{2}$ and a 2 index vanishes. The only remaining degrees of freedom are $a_{\dot{1}BCD}$. However, we should account for the gauge invariance:

$$\delta a_{\dot{A}BCD} = \partial_{\dot{A}(B} \xi_{CD)} \implies \delta a_{\dot{1}2CD} = p_{\dot{1}(2} \xi_{CD)}, \quad (4.37)$$

which makes the only non-zero component $a_{\dot{1}111}$. Finally the symplectic structure for spin 2 Penrose theory is

$$\Omega = i \int \delta \phi^{1111} \wedge \delta a^{\dot{2}}{}_{111} \wedge d^3 x_{12}. \quad (4.38)$$

Let us turn to Fronsda theory. Fix a gauge where h_{mn} is traceless, so the symplectic structure (4.32) reduces to

$$\Omega = -\frac{1}{2} \int \left(2\delta h^m_n \wedge \partial^p \delta h_p^n - \delta h_{pn} \wedge \partial^m \delta h^{pn} \right) \wedge d^3 x_m. \quad (4.39)$$

The degrees of freedom of the self-dual part are fixed by Einstein's equation since ϕ is written in terms of h . For spin 2:

$$R_{(M\dot{M}|N\dot{N})} = p^2 h_{(M\dot{M}|N\dot{N})} + P_{(M\dot{M}} P^a h_{|a|N\dot{N})} = 0, \quad (4.40)$$

which gives, after we impose $p^2 = 0$,

$$p_{(M\dot{M}}h_{N\dot{N})1\dot{2}} = 0. \quad (4.41)$$

The general solution of this equation is

$$h_{(1\dot{2}|M\dot{M})} = 0. \quad (4.42)$$

So, for the self-dual part of the curvature, we have then

$$\phi_{22CD} = -p_{(2}{}^{\dot{2}}p_{\dot{2}}{}^{\dot{2}}h_{CD)\dot{2}\dot{2}} \implies \phi_{2222} = -p_2{}^{\dot{2}}p_{\dot{2}}{}^{\dot{2}}h_{22\dot{2}\dot{2}}. \quad (4.43)$$

To connect the two descriptions, we split the gravitational field h into a self-dual and anti-self-dual part. The self-dual piece is already described by Einstein's equations while the anti-self-dual part is given by the *ansatz* (4.21). It implies:

$$h_{(1\dot{1}|1\dot{1})} = p_i{}^1a_{i111} + p_{\dot{i}}{}^1a_{\dot{i}111} = +2p_i{}^1a_{i111}. \quad (4.44)$$

These considerations collapse the symplectic structure to

$$\begin{aligned} \Omega &= -\frac{i}{2} \int \left(2\delta h_n^m \wedge p^{1\dot{2}}\delta h_{1\dot{2}}{}^n \right) \wedge d^3x_m - \left(\delta h_{pn} \wedge p^{1\dot{2}}\delta h^{pn} \right) \wedge d^3x_{1\dot{2}} \\ &= +\frac{i}{2} \int \left(\delta h_{pn} \wedge p^{1\dot{2}}\delta h^{pn} \right) \wedge d^3x_{1\dot{2}} \\ &= +i \int \left(p_i{}^1\delta a_{i111} \wedge p^{1\dot{2}}\delta h^{i1\dot{1}1} \right) \wedge d^3x_{1\dot{2}} \\ &= +i \int \left(\delta a_{i111} \wedge p_{\dot{i}}{}^1p^{1\dot{2}}\delta h^{i1\dot{1}1} \right) \wedge d^3x_{1\dot{2}} \\ &= -i \int \left(\delta a_{i111} \wedge p_2{}^{\dot{2}}p_{\dot{2}}{}^{\dot{2}}\delta h_{22\dot{2}\dot{2}} \right) \wedge d^3x_{1\dot{2}} \\ &= +i \int \left(\delta a_{i111} \wedge \delta \phi_{2222} \right) \wedge d^3x_{1\dot{2}}. \end{aligned} \quad (4.45)$$

This computation highlights the usefulness of momentum space. We can work directly with physical degrees of freedom as it is suggested when dealing with symplectic structures.

4.3. Canonical map between formulations for general spin s

In order to relate the two descriptions in general case, we split the Fronsdal field $h_{m_1\dots m_s}$ into self-dual and anti-self-dual components. The anti-self-dual part is described by the gauge field $a_{\dot{M}A\dots N}$ via

$$h_{M_1\dot{M}_1\dots M_s\dot{M}_s} = (-i)^{2s-1} \partial_{(\dot{M}_s}{}^{N_s} \dots \partial_{\dot{M}_2}{}^{N_2} a_{\dot{M}_1)N_2\dots N_s M_1\dots M_s}, \quad (4.46)$$

while the self-dual degrees of freedom are given by the curvature $\phi_{A\dots D}$, which should come from the gauge-invariant tensor

$$R_{[m_1n_1]\dots[m_sn_s]} = \partial_{[n_s]}\partial_{[n_{s-1}]\dots}\partial_{[n_1}h_{m_1]|\dots|m_{s-1}]}|m_s]. \quad (4.47)$$

Once Fronsdal equations are imposed, we expect⁴

$$\phi_{M_1N_1\dots M_sN_s} = R_{(M_1N_1\dots M_sN_s)} = \partial_{(N_s}{}^{\dot{N}_s} \dots \partial_{N_1}{}^{\dot{N}_1} h_{M_1\dots M_s)\dot{N}_1\dots\dot{N}_s}. \quad (4.48)$$

⁴ Remember, to a spacetime index m there corresponds a pair $M\dot{M}$.

We also expect that any component of $R_{m_1 n_1 \dots m_s n_s}$ which contains mixed dotted and undotted indices should vanish. In what follows, we will prove that this is indeed the case.

For the moment, we should stress interesting features of this map. The anti-self-dual component gives a traceless $h_{m_1 \dots m_s}$. But this is not a problem since these degrees of freedom are pure gauge. Moreover, in order to show that the symplectic structures match, one does not need all coefficients in the anti-self-dual map. The Fronsdal equations will restrict these to a single component each.

4.4. Equivalent symplectic structures: Fourier counting

We proceed to the symplectic structures. We circumvent the need to look for exact forms by going to momentum space, which also makes straightforward to work only with physical degrees of freedom.

Let us choose a non-zero $p_2^{\dot{2}}$ component. Hence, the equation of motion for $a_{\dot{A}B\dots D}$ collapses into

$$p_2^{\dot{2}} a_{\dot{2}B_2\dots B_{2s}} = 0, \quad (4.49)$$

and we can see the only non zero component is $a_{\dot{1}B\dots D}$. We can restrict further using the gauge transformations:

$$\delta a_{\dot{1}2\dots D} = p_{\dot{1}(2} \xi^{\dot{2}} \dots D), \quad (4.50)$$

from where the only physical component which remains is $a_{\dot{1}1\dots 1}$. Thus, the map we described in (4.46) gives $h_{\dot{1}1\dots 11}$ component of the Fronsdal gauge field.

The degrees of freedom which the curvature spinor describes are obtained from the Fronsdal equation. Together with the condition $p^2 = 0$, they imply

$$p_{(M_1 \dot{M}_1} h_{\dot{1}2 M_3 \dot{M}_3 \dots M_s \dot{M}_s)} = 0, \quad (4.51)$$

since our map describes a traceless $h_{m_1 \dots m_s}$ field. This equation forces $h_{\dot{1}2\dots} = 0$, which also annihilates any component with mixed dotted and undotted indices, and so we have

$$\phi_{22\dots 22} = i^s p_2^{\dot{2}} \dots p_2^{\dot{2}} h_{2\dots 22\dots 2\dot{2}}. \quad (4.52)$$

Such considerations are in line with the usual formulation of Fronsdal theory, where the degrees of freedom contained in the trace and divergence of $h_{m_1 \dots m_s}$ can be gauged away.

We combine all of such considerations to show the symplectic structures agree. Note that we are allowed to discard terms of the type

$$\int \delta h_{\dots} \wedge \partial^p h_{p\dots} \quad \text{and} \quad \int \delta h^p_{p\dots} \wedge \delta h_{\dots}$$

because $h_{\dot{1}2\dots}$ vanishes and our canonical map gives a traceless $h_{m_1 \dots m_s}$. Thus the only allowed combination for the bosonic case is of the form

$$\Omega = \int (\delta h_{n_1 \dots n_s} \wedge \partial^m \delta h^{n_1 \dots n_s}) \wedge d^3 x_m \quad (4.53)$$

and if we apply our results to (4.53) we obtain

$$\begin{aligned}
\Omega &= \int \left(\delta h_{(1i|\dots|1i)} \wedge p_2^{\dot{2}} \delta h^{(1i|\dots|1i)} \right) \wedge d^3 x_2^2 \\
&= (-i)^{s-1} \int \left(p^1_{\dot{1}} \dots p^1_{\dot{i}} \delta a_{i1\dots 1} \wedge p_2^{\dot{2}} \delta h_{2\dots 2\dot{2}\dots\dot{2}} \right) \wedge d^3 x_2^2 \\
&= (-)^{s-1} (-i)^{s-1} \int \left(\delta a_{i1\dots 1} \wedge p_2^{\dot{2}} \dots p_2^{\dot{2}} \delta h_{2\dots 2\dot{2}\dots\dot{2}} \right) d^3 x_2^2 \\
&= (-)^{s-1} (-i)^{s-1} (-i)^s \int \left(\delta a_{i1\dots 1} \wedge \delta \phi_{22\dots 22} \right) \wedge d^3 x_2^2 \\
&= -i \int \left(\delta a_{i1\dots 1} \wedge \delta \phi_{22\dots 22} \right) \wedge d^3 x_2^2
\end{aligned} \tag{4.54}$$

thus proving the desired result.

5. Conformal invariance

The conformal generator v^c is

$$v^c = a^c + \omega^{cb} x_b + \alpha x^c + 2(\rho \cdot x) x^c - \rho^c (x \cdot x), \tag{5.1}$$

where the first two terms are the usual Poincaré transformations; the third one describes dilations and the last two generate special conformal transformations.

5.1. Lie derivation of spinors

In treating Penrose action, we are going to need to vary spinor fields under conformal transformations. The Lie derivative of a spinor field is not widely used when compared with the usual tensor variations. This subsection explains briefly this terminology before applying it to our case.

In geometry, given a vector field v^c and a vector density u^b , the Lie derivative of u^b with respect to v^c is defined as

$$\mathcal{L}_v u^b = v^a \partial_a u^b - u^a \partial_a v^b + w_u (\partial_a v^a) u^b, \tag{5.2}$$

where w_u is the density weight of u^b . When u^b is null, it can be written as product of two spinors, $u^b = \mu^B \bar{\mu}^{\dot{B}}$, and so we can use equation (5.2) to define the Lie derivative of μ^B .

Following this procedure, a general spinor density [11,5] μ^A flows along the flux of v^c such that its infinitesimal change is given by

$$\delta_v \mu^A = \mathcal{L}_v \mu^A = v^m \partial_m \mu^A + \mu^B f^A_B + w_\mu (\partial_m v^m) \mu^A; \tag{5.3}$$

in here w_μ denotes the density weight of the μ field and f^A_B is the self-dual part of v^c :

$$f_{AB} = -\frac{1}{2} \partial_{\dot{C}(A} v_{B)}^{\dot{C}}. \tag{5.4}$$

In deriving (5.3) from (5.2), we must impose that v^c is a conformal generator. Indeed, the second term in (5.2) gives a contribution of the form:

$$\begin{aligned}
-u^a \partial_a v_b &= -\mu^A \bar{\mu}^{\dot{A}} \partial_{A\dot{A}} v_{B\dot{B}} \\
&= -\mu^A \bar{\mu}^{\dot{A}} \partial_{[A\dot{A}} v_{B\dot{B}]} - \mu^A \bar{\mu}^{\dot{A}} \partial_{(A\dot{A}} v_{B\dot{B})} \\
&= -\mu^A \bar{\mu}^{\dot{A}} (f_{AB} \epsilon_{\dot{A}\dot{B}} + \bar{f}_{\dot{A}\dot{B}} \epsilon_{AB}) - \mu^A \bar{\mu}^{\dot{A}} \partial_{(A\dot{A}} v_{B\dot{B})} \\
&= \bar{\mu}_{\dot{B}} \mu^A f_{AB} + \mu_B \bar{\mu}^{\dot{A}} \bar{f}_{\dot{A}\dot{B}} - \mu^A \bar{\mu}^{\dot{A}} \partial_{(A\dot{A}} v_{B\dot{B})},
\end{aligned} \tag{5.5}$$

in which the last term does not split into something dependent of B and $\dot{B}$ separately. It is precisely when v^c is a conformal generator, that is

$$\partial_{(A\dot{A}} v_{B\dot{B})} = \left(\frac{1}{2} \partial_m v^m \right) \epsilon_{AB} \epsilon_{\dot{A}\dot{B}}, \quad (5.6)$$

that we can identify the desired contributions to each spinor.

In our applications, of special interest is the self-dual part of the special conformal transformations. We write it explicitly for future use:

$$f_{AB} = -2 \rho \dot{C}_{(A} x_{B)}^{\dot{C}}. \quad (5.7)$$

5.2. Weight conventions

The weight of a density is a geometrical quantity, that is, it has fixed value independent of which transformation is made; and usually we would have

$$\mathcal{L}_v \epsilon_{AB} = \frac{\lambda}{2} \epsilon_{AB}. \quad (5.8)$$

However, there is still freedom if we define ϵ_{AB} to be a density instead of a tensor. We choose the weight of ϵ_{AB} such that

$$\mathcal{L}_v \epsilon_{AB} = 0. \quad (5.9)$$

From definition (5.3):

$$\begin{aligned} \mathcal{L}_v \epsilon_{AB} = 0 &= \frac{\lambda}{2} \epsilon_{AB} + w_\epsilon \partial_m v^m \epsilon_{AB} \\ &= \left(\frac{1}{2} + 2w_\epsilon \right) \epsilon_{AB} \end{aligned} \quad (5.10)$$

we see this amounts choosing $w_\epsilon = -1/4$. Consistency, however, requires ϵ^{AB} to have weight $w^\epsilon = +1/4$. Hence, given an arbitrary spinor μ^A , in our conventions it is true that

$$\mathcal{L}_v \mu_A = \epsilon_{AB} \mathcal{L}_v \mu^B, \quad (5.11)$$

which is equivalent to state that a spinor and its dual have the same conformal weight. All considerations apply equally for dotted indices.

5.3. Conformal invariance of Penrose action

In this section we will state the conformal invariance of the action (3.1). This in turn ensures the existence of a set of conformal symmetries in Fronsdal description.

Let us begin with dilatations. The higher-spin fields vary under it according to

$$\delta_v \phi^{AB\cdots D} = \alpha x^m \partial_m \phi^{AB\cdots D} + 4\alpha w_\phi \phi^{AB\cdots D} \quad (5.12a)$$

and

$$\delta_v a^{\dot{A}}_{B\cdots D} = \alpha x^m \partial_m a^{\dot{A}}_{B\cdots D} + 4\alpha w_a a^{\dot{A}}_{B\cdots D}. \quad (5.12b)$$

These change the action by

$$\delta_v S = \int d^4 x \left(\alpha x^m \partial_m \phi^{AB\cdots D} + 4\alpha w_\phi \phi^{AB\cdots D} \right) \partial_{A\dot{A}} a^{\dot{A}}{}_{B\cdots D} \\ + \phi^{AB\cdots D} \partial_{A\dot{A}} \left(\alpha x^m \partial_m a^{\dot{A}}{}_{B\cdots D} + 4\alpha w_a a^{\dot{A}}{}_{B\cdots D} \right). \quad (5.13)$$

After a few simplifications, we get

$$\delta_v S = \int d^4 x \left\{ \alpha [-3 + 4(w_\phi + w_a)] \phi^{AB\cdots D} \partial_{A\dot{A}} a^{\dot{A}}{}_{B\cdots D} \right\}, \quad (5.14)$$

which vanishes only when

$$w_\phi + w_a = \frac{3}{4}. \quad (5.15)$$

As we can see, dilatations are unable to fix completely the conformal weights. The remaining condition comes from the special conformal transformations.

Under special conformal transformations, generated by

$$v^m = 2(\rho \cdot x) x^m - (x \cdot x) \rho^m, \quad (5.16)$$

the spin fields $\phi^{AB\cdots D}$ and $a^{\dot{A}}{}_{B\cdots D}$ vary according to

$$\delta_v \phi^{AB\cdots D} = v^m \partial_m \phi^{AB\cdots D} + 2s \phi^{C(AB\cdots} f^D)_{C} + 8w_\phi (\rho \cdot x) \phi^{AB\cdots D}, \quad (5.17a)$$

and

$$\delta_v a^{\dot{A}}{}_{B\cdots D} = v^m \partial_m a^{\dot{A}}{}_{B\cdots D} + \bar{f}^{\dot{A}}{}_{\dot{C}} a^{\dot{C}}{}_{B\cdots D} - (2s - 1) f^C{}_{(B} a^{\dot{A}}{}_{\cdots D)C} + 8w_a (\rho \cdot x) a^{\dot{A}}{}_{B\cdots D}. \quad (5.17b)$$

The action becomes

$$\delta_v S = \int d^4 x \left(v^m \partial_m \phi^{AB\cdots D} \partial_{A\dot{A}} a^{\dot{A}}{}_{B\cdots D} + 2s \phi^{C(AB\cdots} f^D)_{C} \partial_{A\dot{A}} a^{\dot{A}}{}_{B\cdots D} + 8w_\phi (\rho \cdot x) \phi \partial a \right. \\ + \phi^{AB\cdots D} \partial_{A\dot{A}} v^m \partial_m a^{\dot{A}}{}_{B\cdots D} + \phi^{AB\cdots D} v^m \partial_m \partial_{A\dot{A}} a^{\dot{A}}{}_{B\cdots D} + \phi^{AB\cdots D} \partial_{A\dot{A}} \bar{f}^{\dot{A}}{}_{\dot{C}} a^{\dot{C}}{}_{B\cdots D} \\ + \phi^{AB\cdots D} \bar{f}^{\dot{A}}{}_{\dot{C}} \partial_{A\dot{A}} a^{\dot{C}}{}_{B\cdots D} - (2s - 1) \phi^{AB\cdots D} \partial_{A\dot{A}} f^C{}_{(B} a^{\dot{A}}{}_{\cdots D)C} \\ \left. - (2s - 1) \phi^{AB\cdots D} f^C{}_{(B} \partial_{A\dot{A}} a^{\dot{A}}{}_{\cdots D)C} + 8w_a \phi \rho a + 8w_a (\rho \cdot x) \phi \partial a \right). \quad (5.18)$$

In the second line, we open $\partial_a v_m$ in its symmetric and anti-symmetric pieces and integrate by parts ∂_m in $\partial_{A\dot{A}} \partial_m a^{\dot{A}}{}_{B\cdots D}$. Then we obtain

$$\phi^{AB\cdots D} \partial_{A\dot{A}} v^m \partial_m a^{\dot{A}}{}_{B\cdots D} = \phi^{AB\cdots D} \partial_{(A\dot{A}} v_m) \partial^m a^{\dot{A}}{}_{B\cdots D} + \phi^{AB\cdots D} \partial_{[A\dot{A}} v_m] \partial^m a^{\dot{A}}{}_{B\cdots D} \\ = 2(\rho \cdot x) \phi \partial a + \phi^{AB\cdots D} f_{AM} \partial^M a^{\dot{A}}{}_{B\cdots D} \\ + \phi^{AB\cdots D} \bar{f}_{\dot{A}\dot{M}} \partial^{\dot{M}} a^{\dot{A}}{}_{B\cdots D} \quad (5.19)$$

and

$$\phi^{AB\cdots D} v^m \partial_m \partial_{A\dot{A}} a^{\dot{A}}{}_{B\cdots D} = -v^m \partial_m \phi^{AB\cdots D} \partial_{A\dot{A}} a^{\dot{A}}{}_{B\cdots D} - \partial_m v^m \phi^{AB\cdots D} \partial_{A\dot{A}} a^{\dot{A}}{}_{B\cdots D} \\ = -v^m \partial_m \phi^{AB\cdots D} \partial_{A\dot{A}} a^{\dot{A}}{}_{B\cdots D} - 8(\rho \cdot x) \phi \partial a. \quad (5.20)$$

When we substitute everything back into the action, the only remaining terms are

$$\delta_v S = \int d^4 x \left\{ [8(w_\phi + w_a) - 6] (\rho_{\dot{A}\dot{B}} \phi^{AB\cdots D} \partial_{AA} \dot{a}^{\dot{A}}{}_{B\cdots D}) \right. \\ \left. + (8w_a - 3) \phi^{AB\cdots D} \rho_{AA} \dot{a}^{\dot{A}}{}_{B\cdots D} - (2s - 1) \phi^{AB\cdots D} \partial_{AA} \dot{f}^C{}_{(B} \dot{a}^{\dot{A}}{}_{\cdots D)C} \right\}. \quad (5.21)$$

We can use (5.7) so that

$$\partial_{AA} \dot{f}^C{}_{\dot{B}} = -\rho_{\dot{A}}{}^C \epsilon_{AB} - \rho_{\dot{A}\dot{B}} \delta_A{}^C. \quad (5.22)$$

At the end, we get two relations involving the weights. They are

$$8(w_\phi + w_a) - 6 = 0 \quad (5.23a)$$

and

$$8w_a + 2s - 4 = 0. \quad (5.23b)$$

If we use (5.15), the first equation, (5.23a), is an identity. It gives no new information. However, the second equation fixes the weight of the gauge field. Finally, we have

$$w_a = \frac{2 - s}{4} \quad (5.24)$$

and

$$w_\phi = \frac{s + 1}{4}. \quad (5.25)$$

The following table lists a few values for weights given different spin s theories.

	w_ϕ	w_a
$s = 0$	$1/4$	$1/2$
$s = 1/2$	$3/8$	$3/8$
$s = 1$	$1/2$	$1/4$
$s = 3/2$	$5/8$	$1/8$
$s = 2$	$3/4$	0
$s = 5/2$	$7/8$	$-1/8$

5.4. The structure of conformal transformations

Penrose theory is described by the set (ϕ, a) while Fronsdal theory is described by h . We have defined a map, which we name H , that takes one description into another:

$$H : h_{m_1 \cdots m_s} \longmapsto (\phi^{AB\cdots D}, a_{\dot{A}\dot{B}\cdots D}).$$

It was shown that this map preserves phase space, i.e., it is a canonical transformation.

A map between symplectic structures also carries through symmetries of one description to another. If a symplectic structure admits an action, then its symmetries must be also symmetries of the action. Therefore it is natural to define a conformal transformation of the form

$$\delta_v h_{m_1 \cdots m_s} = H^{-1} \mathcal{L}_v H h_{m_1 \cdots m_s}, \quad (5.26)$$

where v is the conformal generator (5.1). It can act non-trivially; its action, as equation (5.26) shows, is not obtained from standard Lie derivations. Moreover, additional complications may

appear due to H^{-1} , which involves inverting derivatives, as (4.46) illustrates. For spins running from $s = 1/2$ to $s = 3/2$, it can be shown to agree with usual conformal transformations obtained by change of coordinates. At spin $s = 2$, however, since Fronsdal theory *is not* conformal invariant, our transformation exhibits the non-local behaviour.

We can work out this case explicitly. For special conformal transformations, if we plug the variation (5.17b) inside (4.34), we obtain

$$\delta_v h_{(M\dot{M}|N\dot{N})} = \mathcal{L}_v h_{(M\dot{M}|N\dot{N})} + 2(\rho \cdot x) h_{(M\dot{M}|N\dot{N})} + 6\rho_{(\dot{M}}{}^E a_{\dot{N})MNE}(h), \quad (5.27)$$

where $\mathcal{L}_v$, in this case, denotes the diffeomorphism Lie derivative and ρ is the special conformal parameter. The last term shows the non-local behaviour since it involves rewriting equation (5.17b) for $a_{\dot{M}MNE}$ in terms of $h_{M\dot{M}N\dot{N}}$, giving inverse powers of ∂_a . Notice that the conformal weight obtained from this expression, which reads $w = +1/4$, does not agree with the usual Fronsdal theory, which is dilatation invariant for $w = -1/4$ at every spin [10].

These differences may appear problematic. They raise suspicion whether this transformation satisfies the conformal algebra or not. The simplest way to answer this question is to notice that (5.26) is a conjugation; therefore, if H is well-defined, they must satisfy the same algebra of the vector field v in question.

The issue of conformal invariance is unrelated to the on-shell phase space, but is rather related to the off-shell description. The connection field of Fronsdal theory is not a representation of the conformal algebra, but the curvatures used in Penrose theory transform covariantly under the action of the conformal group. Then, it could be said that the non-local nature of the transformation follows from expressing the Fronsdal gauge field in terms of its curvature, as is usual in spin 2 case when one uses Riemann normal coordinates. It should be stressed though that, in this formalism, we still have a gauge field, $a_{\dot{M}MNE}$ in gravity case, and moreover, our map (4.34) is not obtained by a change of coordinates, it is the definition of a .⁵

The conformal change of coordinates preserves the action (3.1) and its symplectic structure. We have shown that the two symplectic structures agree, so our transformations should be a symmetry of Fronsdal action. This analysis is straightforward for free theories. It is possible that this symmetry is not preserved by arbitrary interaction terms. It would be interesting to understand what kind of interactions, if there is any, would preserve these symmetries.

6. Conclusions

We have defined an action for Penrose theory and constructed its symplectic structure. This action appears to be simpler than the usual one obtained by Fronsdal. Moreover, it depends only on the epsilon symbol, being possible to examine how it should extend to curved spaces. In particular, this action might be the description of the singular limit suggested in [12] for the AdS_4 higher-spin action.

We showed that this theory describes the same classical phase space as of Fronsdal. Moreover, the action of conformal change of coordinates can be push-forwarded from one Lagrangian to another. It, in turn, leads us to conjecture a set of non-trivial conformal symmetries for the Fronsdal higher-spin fields $h_{m_1 \dots m_s}$. These are not generated by usual coordinate changes, although to lower spins – those which run from $1/2$ to $3/2$ – it is possible to show that both symmetries agree. The non-local behaviour appears only at spin 2.

⁵ We would like to thank the referee for pointing this out.

The construction of conformal higher-spin theories in four-dimensional flat space was developed by Fradkin and Tseytlin in [15], and it was generalized for arbitrary curved backgrounds by Segal in [9]. These theories involve higher-derivatives and additional fields; so, it should be stressed that having identified a non-local realization of conformal symmetries is not enough to argue that this gives a non-trivial relation between conformal higher-spin theories and Fronsdal theory.

Acknowledgements

I am indebted to Andrei Mikhailov and Alexei Rosly for suggesting this problem and for numerous helpful discussions. I also would like to thank Arkady Tseytlin for reading the manuscript and for useful suggestions. This work was supported by CAPES grant 33015015001P7 and by FAPESP grant 2014/18634-9.

Appendix A. A mini-introduction to the geometry of classical mechanics

This appendix explains the terminology used in this work. We briefly review basic aspects of the geometry of classical mechanics.

The classical phase space, M , is the set of all classical trajectories. This space is naturally an infinite-dimensional symplectic manifold, that is, a pair (M, Ω) consisting a smooth manifold M and a non-degenerate closed 2-form Ω called symplectic structure.

Let us explain how to obtain the symplectic structure from the action. Fix

$$S[\phi(x)] = \int \mathcal{L}(\phi(x), \partial_m \phi(x)) d^4x \quad (\text{A.1})$$

for a given field $\phi(x)$ and let the classical configuration be denoted $\phi_{cl}(x)$. Then, under arbitrary infinitesimal changes in field configuration, for example $\delta\phi(x)$, the action changes around the classical path according to

$$S[\phi_{cl}(x) + \delta\phi(x)] - S[\phi_{cl}(x)] = \int_C \frac{\partial \mathcal{L}}{\partial(\partial_m \phi)} n_m d^3x \wedge \delta\phi(x), \quad (\text{A.2})$$

where $n_m d^3x$ defines a 3-form in Minkowski space to be integrated over C , a 3-dimensional closed surface.

Remark. Here we have the de Rham complex with exterior derivative d and the variational complex with differentiation δ ; the previous variation $\delta\phi(x)$ may be interpreted as a differential form on the space of field configurations. When dealing with d and δ , we will use the following rules:

$$d\delta = -\delta d \quad \text{and} \quad \delta\phi(x) \wedge dx^m = -dx^m \wedge \delta\phi(x).$$

The variation $\delta\phi$ descends to the phase space once we take it to satisfy the equations of motion. One then can consider formally the symplectic structure to be

$$\Omega = \int_C \delta \left(\frac{\partial \mathcal{L}}{\partial(\partial_m \phi)} \right) \wedge \delta\phi(x) \wedge n_m d^3x, \quad (\text{A.3})$$

since it defines a closed 2-form on phase space. Such differential form is also independent of C . To see this, consider for example two contours, C_1 and C_2 . And let Ω_1 and Ω_2 represent the respective symplectic structures. We want to show that

$$\Omega_1 - \Omega_2 = 0 \quad (\text{A.4})$$

in M . Define Σ to be the 4-dimensional surface whose boundary is $C_1 - C_2$, then by Stokes' theorem

$$\begin{aligned} \Omega_1 - \Omega_2 &= \int_{\Sigma} \partial_m \left(\delta \frac{\partial \mathcal{L}}{\partial (\partial_m \phi)} \wedge \delta \phi(x) \right) \wedge d^4x \\ &= \int_{\Sigma} \delta \left(\partial_m \frac{\partial \mathcal{L}}{\partial (\partial_m \phi)} \right) \wedge \delta \phi(x) \wedge d^4x + \delta \frac{\partial \mathcal{L}}{\partial (\partial_m \phi)} \wedge \delta \partial_m \phi(x) \wedge d^4x \\ &= \int_{\Sigma} \delta \frac{\partial \mathcal{L}}{\partial \phi} \wedge \delta \phi(x) \wedge d^4x + \delta \frac{\partial \mathcal{L}}{\partial (\partial_m \phi)} \wedge \delta \partial_m \phi(x) \wedge d^4x \\ &= \int_{\Sigma} \delta^2 \mathcal{L} \wedge d^4x = 0 \end{aligned} \quad (\text{A.5})$$

with the help of Euler–Lagrange equations. In this computation, and in all of those which involve a symplectic structure, we stress that we are free to use the equations of motion because we are in phase space.

In classical mechanics, a symplectic structure defines a Poisson bracket. For example, one can consider, in a local basis, a bivector which is the inverse matrix of the symplectic form. This bivector, by definition, maps functions into functions and satisfies the Jacobi identity – a consequence of the closeness of Ω .

Examples.

Spin $s = 0$. The action is

$$S = \int d\phi \wedge \star d\phi \quad (\text{A.6})$$

and the symplectic structure obtained is

$$\Omega = \int_C \delta \phi \wedge \star d\delta \phi. \quad (\text{A.7})$$

One can choose the surface C to be $t = \text{constant}$ and the symplectic structure turns into

$$\Omega = \int \delta \phi \wedge \delta \dot{\phi} \wedge d^3 \mathbf{x}. \quad (\text{A.8})$$

Its inverse gives rise to the Poisson Bracket

$$\{F, G\} = \int \left(\frac{\delta F}{\delta \phi(\mathbf{x})} \frac{\delta G}{\delta \dot{\phi}(\mathbf{x})} - \frac{\delta G}{\delta \phi(\mathbf{x})} \frac{\delta F}{\delta \dot{\phi}(\mathbf{x})} \right) d^3 \mathbf{x}. \quad (\text{A.9})$$

Spin $s = 1/2$. The action is

$$S = \int \bar{\psi}^A \partial_{AA} \psi^A \quad (\text{A.10})$$

and the symplectic structure obtained is

$$\Omega = \int_C \delta \bar{\psi}^{\dot{A}} \wedge \delta \psi^A \wedge d^3 x_{AA}. \quad (\text{A.11})$$

One can choose the surface C to be $t = \text{constant}$ and the symplectic structure turns into

$$\Omega = \int \delta \bar{\psi}^{\dot{A}} \wedge \psi^A \wedge \sigma_{AA}^0 d^3 \mathbf{x}. \quad (\text{A.12})$$

Its inverse gives rise to the Poisson Bracket

$$\{F, G\} = \int \left(\frac{\delta F}{\delta \bar{\psi}^{\dot{A}}(\mathbf{x})} \frac{\delta G}{\delta \psi^A(\mathbf{x})} - \frac{\delta G}{\delta \bar{\psi}^{\dot{A}}(\mathbf{x})} \frac{\delta F}{\delta \psi^A(\mathbf{x})} \right) \bar{\sigma}^{0A\dot{A}} d^3 \mathbf{x}. \quad (\text{A.13})$$

The set of transformations that preserve the symplectic structure will also preserve the Poisson bivector. These are usually called canonical transformations.

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