PT-symmetric kinks

A. de Souza Dutra,[1,2,*] V. G. C. S. dos Santos,[2] and A. C. Amaro de Faria Jr.[2]

[1]*Abdus Salam ICTP, Strada Costiera 11, Trieste, I-34100 Italy*

[2]*UNESP-Campus de Guaratinguetá-DFQ,*[†] *Departmento de Física e Química, 12516-410 Guaratinguetá SP Brasil*

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Some kinks for non-Hermitian quantum field theories in 1 + 1 dimensions are constructed. A class of models where the soliton energies are stable and real are found. Although these kinks are not Hermitian, they are symmetric under *PT* transformations.

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I. INTRODUCTION

In the last few years, a great deal of interest has appeared on the so-called *PT*-symmetric systems introduced in the seminal paper by Bender and Boettcher [1]. They consist of non-Hermitian Hamiltonians with real eigenvalues, which present a *PT* symmetry or, in other words, when one makes the spatial-temporal inversion (P: $x \longrightarrow -x$; T: $t \longrightarrow -t$, $i \longrightarrow -i$), the Hamiltonian is invariant. An enormous list of works is devoted to develop and understand this new kind of quantum systems [2–11]. Recently, the problem of a Hamiltonian for an oscillator with time-dependent mass and frequency, plus a linear non-Hermitian term with time-dependent coupling, was exactly solved [12]. However, as far as we know, no one did discuss the possibility of existence of topological objects belonging to this new kind of physical environment, where Hermiticity requirement is substituted by a physical demanding of *PT*-symmetric models. In fact, there are an increasing number of very interesting extensions and applications of these ideas for the quantum field theory scenario [13–18].

Although less usual than linear systems, the nonlinear ones and particularly those having solitonic excitations are very interesting and important in modeling many physical, biological, and chemical systems. A very important example shows up in the electrical conductivity of some organic materials, where polarons and other polymer chain solitons are responsible for the appearance of conducting polymers [19]. Another important appearance of solitons is related to electrical conduction through DNA molecules [20].

Usually, topological objects like domain walls, strings, and monopoles appear when the models support at least two degenerate vacua. Nevertheless, there are some models which defy this common sense, like the Liouville model [21,22], the vacuumless (VL) model introduced originally by Cho and Vilenkin [23,24] and, more recently a model where the kink interpolates between two inflection points instead of vacua [25]. In fact, two years ago it was shown that, at least in some cases like that of the VL model, one can introduce an orthodox model, having degenerate vacua, whose limit is precisely the VL one, so giving a kind of regularization of some properties of the model [26].

In this work we intend to combine some of the above ingredients. We explore an unusual class of topological configurations in 1 + 1 dimensions. In this class of systems the Hamiltonian is non-Hermitian but the energy of the field configuration is real and it is stable under small linear perturbations. We show that the usual properties of the Bogomol'nyi-Prasad-Sommerfield (BPS) kinks are still valid in this new and promising class of systems. This work is organized as follows. In Sec. II, we make a brief review of the usual procedure used to obtain the BPS kinks, verifying their stability and the possibility of consistent coupling with fermions. In Sec. III, we analyze some important models introduced in the literature. Section IV is devoted to the study of non-Hermitian versions of the $\lambda\Phi^4$ and Sine-Gordon models. In Sec. V, we construct other models through the deformation technique. In Sec. VI, we discuss the quantum mapping of a class of non-Hermitian kink systems to their Hermitian counterparts. In Sec. VII, the difficulties of the construction of consistent kinks in a model with two coupled scalar fields is studied. Finally, in Sec. VIII, we make our final comments.

II. BPS APPROACH, STABILITY ANALYSIS, AND COUPLING TO FERMIONS

Here we give a short explanation of a number of properties and approaches usually valid for usual nonlinear Hermitian models and discuss the expected changes or features resulting from the extension to quantum non-Hermitian models. The Lagrangian density for one scalar field models, considered here is given by

$$\mathcal{L} = \tfrac{1}{2}\partial_\mu\phi\partial^\mu\phi - V(\phi), \tag{1}$$

where $V(\phi)$ is the potential of the system. In the BPS approach [27], used to present the solution of this and other models studied in this work, one can write the potential in terms of the so-called superpotential, which is given by

$$V(\phi) = \tfrac{1}{2}W_\phi^2, \tag{2}$$

[*]Electronic address: dutra@feg.unesp.br
[†]Permanent institution.

where $W_\phi \equiv \frac{dW}{d\phi}$, and the energy of the static configuration can be obtained as

$$E_{\rm BPS} = \frac{1}{2}\int_{-\infty}^{\infty} dx\left[\left(\frac{d\phi}{dx} - W_\phi\right)^2 + 2W_\phi\frac{d\phi}{dx}\right]. \quad (3)$$

Observing this equation, we note that the field configuration which minimizes the energy obeys the first-order differential equation

$$\frac{d\phi}{dx} = W_\phi(\phi), \quad (4)$$

and its energy is written as

$$E_{\rm BPS} = |W(\phi(\infty)) - W(\phi(-\infty))|. \quad (5)$$

As it is well known, it is not enough to show the existence of kinklike configurations. One must still verify the stability of the topological field configuration. As shown in [28] for the case of coupled scalar fields, the linear stability of the model with one scalar field can be verified by performing small perturbations on the kink solution,

$$\phi(x,t) = \phi_{\rm kink}(x) + \eta(x,t), \quad (6)$$

where $\eta(x,t) \equiv \sum_n \eta_n(x)\cos(\omega_n t)$. By taking into account terms only up to the first order in the perturbation field $\eta(x,t)$, one gets the Schroedinger-like equation for the perturbation field

$$\left(-\frac{d^2}{dx^2} + V_{\phi\phi}(\phi \equiv \phi_{\rm kink}(x))\right)\eta_n(x) = \omega_n^2\eta_n(x). \quad (7)$$

It is not difficult to see that the above equation can be factored by using the following ladder operators,

$$a_\pm \equiv \pm\frac{d}{dx} + W_{\phi\phi}, \quad (8)$$

where $W_{\phi\phi}$ stands for $W_{\phi\phi}(\phi_{\rm kink})$. The corresponding Hamiltonian operators $\hat{H} = a_+a_-$, as shown in [29] for the case of general coupled real scalar fields, have their eigenvalues positive definite and, as a consequence, the models are stable under small quantum fluctuations.

However, as we are facing a case where the superpotential is non-Hermitian, the operators $a_\pm$ are no longer conjugate to each other. As a consequence, the above standard procedure is not necessarily valid. Then, we intend to perform a direct verification of the stability in the next sections. This will be done by calculating explicitly the solution of the ground-state energy and wave function, in each case analyzed.

The bosonic ground state, which is allowed due to the translational invariance, can be obtained from the solution of the equation,

$$a\psi_0(x) = \left(-\frac{d}{dx} + W_{\phi\phi}\right)\psi_0(x) = 0. \quad (9)$$

By substituting the BPS equation (4) into (9), we obtain a relation between the bosonic zero mode and the superpotential [26]

$$\psi_0(x) = N_0 W_\phi, \quad (10)$$

where N_0 is a normalization constant. This allows us to show that the normalization of the zero mode is related to the BPS energy through

$$\int|\psi_0(x)|^2dx = N_0^2\int W_\phi^2 dx = N_0^2\int W_\phi d\phi = N_0^2 E_{\rm BPS} \equiv 1, \quad (11)$$

and finally we get the normalized bosonic zero mode

$$\psi_0(x) = \sqrt{\frac{1}{E_{\rm BPS}}}W_\phi, \quad (12)$$

apart from an arbitrary constant phase factor.

The above procedure is the starting point for some approaches developed for the construction of kinklike systems from the corresponding ground-state solution of the Schroedinger equation [30,31], and it can be used also to generate the nonlinear quantum fields from the known PT-symmetric quantum mechanical ones. It is interesting to note here that, from the above, one can as well compute the BPS energy from the corresponding ground state for the stability Schroedinger-like equation.

Now we look for other possible requirements for the non-Hermitian models, besides the reality of the soliton configuration energy, in such a way that the model be so well behaved as a Hermitian one. We also study some features of the coupling to fermions.

For instance, let us now try to calculate the fermion zero mode. Using in this case, as done in [24,26], the Yukawa coupling given by $f(\phi)\bar{\psi}\psi$, where we choose $f(\phi) = gW_{\phi\phi}$ ($g = 1$, in order to allow a supersymmetric version of the model [32]), we arrive at the following equation for Dirac fermions

$$i\gamma^1\frac{d\Psi}{dx} + f(\phi)\Psi = 0, \ \Psi = \begin{pmatrix}\psi_+\\ \psi_-\end{pmatrix}, \quad (13)$$

and using the representation where $i\gamma^1 \to \sigma_3$ we obtain the following equations for the spinor components,

$$\pm\frac{d\psi_\pm}{dx} + f(\phi)\psi_\pm = 0. \quad (14)$$

The above equations can be expressed as

$$\frac{d\psi_\pm}{\psi_\pm} = \mp f(\phi)dx = \mp gW_{\phi\phi}\frac{d\phi}{W_\phi}, \quad (15)$$

whose integration gives us finally the spinor

$$\Psi = \begin{pmatrix}C_+W_\phi^{-g}\\ C_-W_\phi^{g}\end{pmatrix}, \quad (16)$$

where $C_\pm$ are arbitrary integration constants. However, supposing that the function W_ϕ is well behaved, vanishing at $x \to \pm\infty$, the normalization of the above spinor,

$$\int |\Psi|^2 dx = \int dx[|C_+|^2 W_\phi^{-2g} + |C_-|^2 W_\phi^{2g}] \equiv 1, \quad (17)$$

will impose that one of the above arbitrary constants must be chosen equal to zero. Otherwise, the spinor will not be square integrable and, as a consequence, we are left with two possible solutions, depending on the sign of g,

$$\Psi_+ = C_+ W_\phi^{-g}\begin{pmatrix}1\\0\end{pmatrix}, \quad g<0; \quad \Psi = C_- W_\phi^{g}\begin{pmatrix}0\\1\end{pmatrix}, \quad g>0. \quad (18)$$

In fact, the normalization of the spinor implies additional conditions over the constant g. Let us return to the normalization integral, now given by

$$\int |\Psi_\pm|^2 dx = |C_\pm|^2 \int dx W_\phi^{(\mp 2g)} = |C_\pm|^2 \int W_\phi^{(\mp 2g-1)} d\phi. \quad (19)$$

Once again, we note that $|g| \leq \frac{1}{2}$ or the integration may diverge. At this point, however, some differences can appear depending on the model being considered. In order to be quite clear on this point, let us take for instance the limiting case $g = \pm\frac{1}{2}$, where we have

$$|C_\pm|^2 \int W_\phi^{(\mp 2g-1)} d\phi = |C_\pm|^2 \int d\phi = |C_\pm|^2[\phi(+\infty) - \phi(\infty)]. \quad (20)$$

It is evident that in models like the vacuumless, the zero-mode fermion is not normalizable, due to the divergence of the kink profile [24,26]. However, for any usual topological model with two different finite vacua, the above case is absolutely admissible.

The models we are going to discuss in this work admit normalizable fermion zero-modes states even for the case where the coupling constant is given by $g = \pm\frac{1}{2}$, provided the difference between the asymptotic fields is real. So, not only the energy of the topological configuration, but also the difference $\phi(\infty) - \phi(-\infty)$ must be real, at least if one cares about coupling the kinks with fermions. Now, considering the cases where a supersymmetric extension of the model is allowed [32], $g = \pm 1$. The normalization of the fermionic zero mode becomes quite similar to its bosonic counterpart,

$$C_\pm = \sqrt{\frac{1}{E_{\mathrm{BPS}}}}. \quad (21)$$

III. AN ANALYSIS OF SOME MODELS APPEARING IN THE LITERATURE

Let us begin our quest by studying some models in quantum field theory like those analyzed by Bender, Brody, and Jones for self-interacting scalar fields [13,14]. In this section we are going to verify the possibility of existence of topological solutions of those models. The first case we consider is the one with a cubic complex interaction [14], characterized by the potential

$$V(\phi) = ig\phi^3 + \frac{\mu^2}{2}\phi^2, \quad (22)$$

with a superpotential given by

$$W_\phi = \phi\sqrt{\mu^2 + 2ig\phi}. \quad (23)$$

Then, by applying the approach explained in the last section, one can obtain the solution

$$\phi_{\mathrm{cl}}(x) = i\frac{\mu^2}{2g}\mathrm{sech}^2\left(\frac{\mu x}{2}\right), \quad (24)$$

where ϕ_{cl} stands for the classical solution. That is a field configuration known as *lump* [25], given that its asymptotic behavior is nontopological $[\phi_{\mathrm{cl}}(+\infty) = 0 = \phi_{\mathrm{cl}}(-\infty)]$.

It is important to emphasize that the above solution is a valid solution for the Eq. (4) if $x < 0$, but not if $x > 0$. However, our primary goal is to achieve a solution of the equation

$$\left(\frac{d\phi}{dx}\right)^2 = W_\phi^2, \quad (25)$$

which is the one obtained from the second-order differential equation governing the system. In fact, for this last equation, the above solution is correct for the entire x axis.

This field configuration connects the only real local minimum existing in the potential to itself. Usually the lumps are unstable, however, as it was emphasized in the previous section, the new situation of non-Hermitian systems allows one to seek for other possibilities. Then we still try to see if there is any possibility of getting a stable lump in a non-Hermitian system. For this we compute the corresponding stability equation (7), given by

$$\left[-\frac{d^2}{dx^2} + \mu^2\left(3\tanh\left(\frac{\mu x}{2}\right)^2 - 2\right)\right]\eta_n(x) = \omega_n^2\eta_n(x),$$

whose unnormalized ground-state wave function and energy are given, respectively, by

$$\eta_0(x) = N_0 \mathrm{sech}\left(\frac{\mu x}{2}\right)^3, \qquad \omega_0^2 = -\frac{5}{4}\mu^2, \quad (27)$$

where N_0 is a normalization constant. From the above we can conclude that, as it is usual, the lump solution is unstable. It is interesting to note that, despite the fact that the lump is still unstable, its stability potential is a real one.

Now, we consider the next example [13], where

$$V(\phi) = \tfrac{1}{2}\phi^2(\mu^2 + 2ig\phi^\varepsilon), \quad (28)$$

where ε is, in principle, an arbitrary parameter and

$$W_\phi \equiv \phi\sqrt{\mu^2 + 2ig\phi^\varepsilon}, \qquad (29)$$

with the following exact solution

$$\phi_{\rm cl}(x) = \left[i\frac{\mu^2}{2g}\,{\rm sech}\left(\frac{\mu\varepsilon x}{2}\right)^2\right]^{1/\varepsilon}. \qquad (30)$$

Note that, as in the previous case, this is a solution valid along the whole axis for the Eq. (25) and not for the Eq. (4).

Unfortunately, we obtained another lump no matter what the degree of the field potential $V(\phi)$, which is determined by the value of the parameter ε. It is simple to show that in this case the stability potential is represented by

$$V(x) = \mu^2\left[1 - \frac{(1+\varepsilon)(2+\varepsilon)}{2}\,{\rm sech}\left(\frac{\mu\varepsilon x}{2}\right)^2\right], \qquad (31)$$

and one can obtain two solutions for the ground-state wave function and energy which are

$$\eta_0^{(1)}(x) = N_0{\rm sech}\left(\frac{\mu\varepsilon x}{2}\right)^{-2((1+\varepsilon)/\varepsilon)}, \qquad \omega_0^2 = -\mu^2\varepsilon(2+\varepsilon) \qquad (32)$$

and

$$\eta_0^{(2)}(x) = N_0{\rm sech}\left(\frac{\mu\varepsilon x}{2}\right)^{((2+\varepsilon)/\varepsilon)}, \qquad \omega_0^2 = -\frac{\mu^2\varepsilon}{4}(4+\varepsilon). \qquad (33)$$

In order to keep the soliton stable, $\eta_0(x)$ cannot be divergent on the boundaries. Furthermore the ground state must also be positive definite, so rendering the field configuration stable. These conditions lead us to conclude that the second solution must be discarded. However, in contrast with the usual behavior of the lumps, if we restrict ourselves to the range $-1 < \varepsilon < 0$ this field configuration is stable. Furthermore, these stable lumps occur for nonsingular field potentials in this range. This is a remarkable nontrivial result which, as far as we know, was never observed for the usual Hermitian systems. It is interesting to observe that the stability of this lump happens in a range of the parameters where the PT symmetry is broken and the spectrum is partially complex [1].

We also have investigated the possible existence of kinks in the quantum field theory of the models introduced in [9] but, once again, as far as we could verify there are no kinks either in that case.

However, in this work we are primarily interested in finding topological stable field configurations. Particularly our quest is for the so-called BPS kinks. So, we must go forward looking for such kind of structures.

IV. A NON-HERMITIAN VERSION OF THE $\lambda\phi^4$ AND SINE-GORDON MODELS

We begin this section trying to construct a kink configuration by means of an extension of the well-known $\lambda\phi^4$ model. A trial superpotential can be proposed as

$$W_\phi(\phi) = \phi^2 + 2ia\phi - b^2, \qquad (34)$$

whose BPS kink expression is given by

$$\phi_{\rm kink}(x) = -ia + \alpha\tanh(\alpha x),$$

where $\alpha = \sqrt{b^2 - a^2}$ and, despite the fact that it is a complex function and having also complex asymptotic values $[\phi_{\rm kink}(\infty) = ia - \alpha$ and $\phi_{\rm kink}(-\infty) = ia + \alpha]$, the corresponding BPS energy for this field configuration is real for $b^2 > a^2$ and given by

$$E_{\rm BPS} = \frac{4}{3}\alpha^3, \qquad (35)$$

which has precisely the same behavior of the BPS energy of the well-known Hermitian $\lambda\phi^4$ model in the limit where $a \rightarrow 0$. It is important to stress at this point that, with the above features, the coupling of the field with Dirac fermions can be done in the spirit of Sec. II without further problems. In other words, once the difference $\phi(\infty) - \phi(-\infty)$ is real, the zero-mode fermion states are admitted.

Now, we proceed with the stability analysis of this PT-symmetric kink. Its stability potential can be written as

$$V(x) \equiv V_{\phi\phi} = 2\alpha^2[2 - 3{\rm sech}(\alpha x)^2]. \qquad (36)$$

Once more we have obtained an exactly solvable potential whose nondivergent solution is given by $\eta_0 = N_0{\rm sech}(\alpha x)^2$ and, in this case, the ground-state energy is zero. This leads us to conclude that the model is stable.

Proceeding further on the extension for non-Hermitian systems, we consider now a generalization of the popular Sine-Gordon model. This is done with the trial superpotential

$$W_\phi(\phi) = \cos(\phi) + ia\sin(\phi) + b,$$

whose topological solution is written as

$$\phi_{\rm kink}(x) = 2\arctan\left[\frac{-ia - 2\beta\tanh(\beta x)}{b-1}\right], \qquad (37)$$

where $2\beta \equiv \sqrt{1-(a^2+b^2)}$.

So, if we want to keep the usual kink profile, we shall restrict the potential parameters in order to keep β real. In this example, the field configuration energy is real but its expression has a quite complicated form which, due to this fact, will not be written explicitly here. Despite this, one can verify numerically that they are in fact real in the full range of validity of the potential parameters. Finally, we plot the real and imaginary parts of the PT-symmetric Sine-Gordon kink in Fig. 1, where it can be observed that the real part of the topological solution has a kinklike

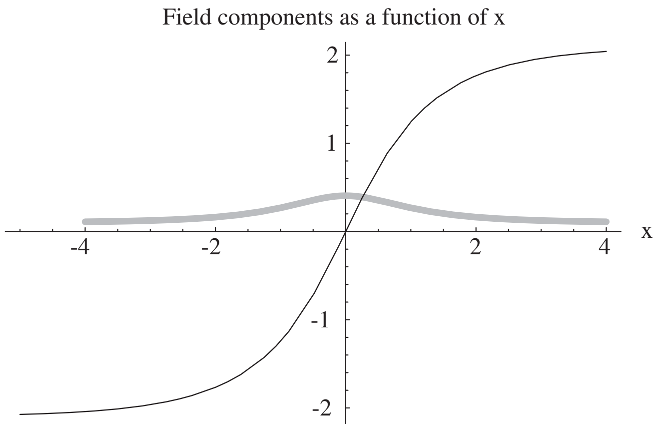

FIG. 1. The real (thin line) and imaginary (thick line) parts of the *PT*-symmetric extension of the Sine-Gordon model ($a = 0.1$ and $b = 0.5$).

profile. Again, the difference $\phi(-\infty) - \phi(\infty)$ is a real quantity, which implies in the localized fermion zero-mode states, in concordance with the discussion addressed in Sec. II.

V. NEW NON-HERMITIAN KINK MODELS COMING FROM THE METHOD OF DEFORMATIONS

Before going further on the construction of new solitonic models for *PT*-symmetric Hamiltonians, let us begin by introducing another systematic way of generating kinks, which appear in $1 + 1$ dimensions. This approach starts with nothing but the usual equation coming from calculation of the BPS solutions [27]

$$\frac{d\phi}{dx} = \sqrt{2V(\phi)}. \tag{38}$$

In fact it is easy to see that, starting with the second-order differential equation for the static configuration

$$\frac{d^2\phi}{dx^2} = \frac{dV(\phi)}{d\phi}, \tag{39}$$

and multiplying both sides by $\frac{d\phi}{dx}$, one can write it as

$$\frac{d}{dx}\left[\frac{1}{2}\left(\frac{d\phi}{dx}\right)^2\right] = \frac{d}{dx}[V(\phi)], \tag{40}$$

which allows us to recover the Eq. (38), but an arbitrary integration constant associated with the arbitrariness in the choice of the minimum value of the potential energy. Thus we can write a kind of invariant, responsible for the ultimate classical connection between the kinks of different models [33]

$$\int \frac{d\phi}{\sqrt{2V(\phi)}} = x - x_0 = \int \frac{d\varphi}{\sqrt{2V(\varphi)}}. \tag{41}$$

In general one can rewrite the above equation simply as $\phi = f(\varphi)$, as has been done originally by Bazeia and collaborators in what they [25] have called method of deformations.

Then we can try to generate a new non-Hermitian model with real energy kinklike configurations starting, for example, from the non-Hermitian $\lambda\phi^4$ model established in the previous section. For this we can use, for instance, the following deformation function

$$f(\varphi) = \sinh(\varphi). \tag{42}$$

By doing so, we finish with the following superpotential [25]

$$W(\varphi) = \sinh(\varphi) + 2ia\ln[\cosh(\varphi)] - 2(1 + b^2)\tanh^{-1}\left[\tanh\left(\frac{\varphi}{2}\right)\right], \tag{43}$$

with the solution

$$\varphi_{\text{kink}}(x) = \sinh^{-1}(-ia + \alpha a\tanh(kax)), \tag{44}$$

where, once more, $\alpha \equiv \sqrt{b^2 - a^2}$. The corresponding BPS configuration energy can be put in the form

$$E_{\text{BPS}} = \left| -2a\arctan\left[\frac{2\alpha a^2}{(1 + (\alpha^2 - 1)a^2)}\right] - 2a\alpha + \alpha^2\{\arg[1 + \tan(h_+)] - \arg[1 + \tanh(h_-)] - \arg[1 - \tan(h_+)] + \arg[1 - \tan[h_-]]\}\right|. \tag{45}$$

where $h_\pm \equiv \frac{1}{2}\arcsin[a(1 \pm i\alpha)]$ and $\arg[z]$ gives the phase angle of z in radians.

Despite the fact that this new case of a non-Hermitian kink configuration has considerably more lengthy calculations, its energy is real as required to be an observable. In this case the stability potential is given by

$$V(x) = \frac{(1 + a^2 + \alpha^2(a^2 - 1) - a^2\alpha^2\text{sech}(a\alpha x)^2)}{\sqrt{1 - a^2[1 + i\alpha\tanh(a\alpha x)]^2}}, \tag{46}$$

which is *PT* symmetric but unfortunately, at least as far as we know, it is not from a class of non-Hermitian exactly solvable potentials. So, in order to check the stability of the kink we shall try to solve it numerically. This is going to be done in a future step of this research. However, as long as an approximation is valid, where α is small and $a < 1$, one can approximate the above potential as

$$U(x) \simeq U_2x^2 + iU_1x + U_0, \tag{47}$$

with $U_0 \equiv \frac{1+a^2-\alpha^2}{\sqrt{1-a^2}}$, $U_1 \equiv \frac{a^3\alpha^2(1+a^2-\alpha^2)}{(1-a^2)^{3/2}}$, and $U_2 \equiv \frac{a^4\alpha^4[1+a^2(2\alpha^2-7)+\alpha^2]}{2(1-a^2)^{5/2}}$. For small values of α, there is a range of a which leads to a positive U_2, so that we deal with a *PT*-symmetric oscillator with real frequency. In this situation, it is not difficult to conclude that certainly, at least for a small range of validity for the parameter a, we have real energies (if $\alpha = 0.01$ for instance, this happens for $0 < a < 0.36$). As a consequence, we can guarantee the stability of this topological configuration. Furthermore, in this model the difference $\phi(\infty) - \phi(-\infty)$ is real, and the coupling of the kinks with fermions can be done in a consistent way, in the spirit of Sec. II.

VI. QUANTUM MAPPING INTO A HERMITIAN SYSTEM

In this section we show how to make a connection between a non-Hermitian system and a Hermitian one. Here we look for a quantum version [33] of the classical equivalence accomplished in [25]. This is done by starting with the Lagrangian density of the original model given by

$$L = \tfrac{1}{2}\partial_\mu \phi \partial^\mu \phi - V(\phi), \tag{48}$$

which is a key ingredient of the Feynman path integral quantization. By performing in the Lagrangian density the same transformation $\phi = f(\varphi)$, used in the case of the differential equation, one obtains

$$L = \tfrac{1}{2} f_\varphi^2 \partial_\mu \varphi \partial^\mu \varphi - V[f(\varphi)], \tag{49}$$

and its Hamiltonian density will look like

$$H = \frac{1}{2 f_\varphi^2} \Pi_\mu \Pi^\mu + V[f(\varphi)], \tag{50}$$

where $\Pi_\mu = f_\varphi^2 \partial_\mu \varphi$. It is important, for the sake of comparison and interpretation, to remember that when the transformation is introduced directly on the differential equation as in the usual treatment for the deformed systems [25], the deformed Lagrangian density takes the form

$$L_d = \frac{1}{2}\partial_\mu \varphi \partial^\mu \varphi - \frac{V[f(\varphi)]}{f_\varphi^2}. \tag{51}$$

From above, we see that for a class of transformations where the imaginary part of the transformation is constant, the resulting Lagrangian density can be taken as a real one. By computing the energy of the static configurations one easily obtains

$$E = \frac{1}{2}\int_{-\infty}^{\infty} dx \left\{ f_\varphi \frac{d\varphi}{dx} - \sqrt{2V[f(\varphi)]} \right\}^2 + \int_{-\infty}^{\infty} dx W_\varphi \frac{d\varphi}{dx}, \tag{52}$$

where

$$W_\varphi = \frac{dW}{d\varphi} \equiv f_\varphi \sqrt{2V[f(\varphi)]}, \tag{53}$$

and the minimum of the energy will correspond to the case where

$$\frac{d\varphi}{dx} = \frac{\sqrt{2V[f(\varphi)]}}{f_\varphi}, \tag{54}$$

whose corresponding second-order differential equation is given by

$$\frac{d^2\varphi}{dx^2} + \left(\frac{f_{\varphi\varphi}}{f_\varphi}\right)\left(\frac{d\varphi}{dx}\right)^2 - \frac{V_\varphi}{f_\varphi^2} = 0. \tag{55}$$

From above, it is not difficult to show that the solutions of the first order differential equation (54) are solutions of the corresponding second-order one. Regarding the BPS energy of the solution, one can verify from Eq. (52) that

$$E_{\mathrm{BPS}} = \int d\varphi (f_\varphi \sqrt{2V[f(\varphi)]})|_{-\infty}^{\infty} = \sqrt{2V[(f[\varphi(x)])}|_{-\infty}^{\infty}. \tag{56}$$

Finally, as a consequence of the transformation $\phi(x) = f[\varphi(x)]$, we conclude that the energy of both systems will be equal when the transformation is done directly in the Lagrangian density.

As a consequence, one example capable of generating many non-Hermitian systems with real BPS energies is that where $\phi = \varphi + ia + b$. For this special choice of transformation, one gets for both formulations that

$$L_d = \tfrac{1}{2}\partial_\mu \varphi \partial^\mu \varphi - V(\varphi + ia + b). \tag{57}$$

In conclusion, we have started with a Hermitian model, and got a non-Hermitian one, which is equivalent to the Hermitian one even at quantum level (there are no quantum corrections for this linear transformation), with the very same real energy as its "parent" model.

VII. TWO INTERACTING SCALAR FIELDS MODEL

In this section we are going to present an extension of the above calculations for the case where the system is represented by two scalar interacting fields. We follow a recently published approach used to get complete exact solutions for some classes of two interacting fields [34].

In order to deal with the problem, following the common procedure to get BPS [27] solutions for nonlinear systems, one can specify the form of the Lagrangian density

$$L = \tfrac{1}{2}(\partial_\mu \phi)^2 + \tfrac{1}{2}(\partial_\mu \chi)^2 - V(\phi, \chi), \tag{58}$$

by imposing that the potential must be written in terms of a superpotential like

$$V(\phi, \chi) = \frac{1}{2}\left[\frac{\partial W(\phi, \chi)}{\partial \phi}\right]^2 + \frac{1}{2}\left[\frac{\partial W(\phi, \chi)}{\partial \chi}\right]^2. \tag{59}$$

The energy of the so-called BPS states can be calculated straightforwardly, giving

$$E_{\mathrm{BPS}} = \frac{1}{2}\int_{-\infty}^{\infty} dx \left[\left(\frac{d\phi}{dx}\right)^2 + \left(\frac{d\chi}{dx}\right)^2 + W_\phi^2 + W_\chi^2\right], \tag{60}$$

which leads us to

$$E_{\rm BPS} = \frac{1}{2}\int_{-\infty}^{\infty} dx\bigg[\bigg(\frac{d\phi}{dx} - W_\phi\bigg)^2 + \bigg(\frac{d\chi}{dx} - W_\chi\bigg)^2 + 2\bigg(W_\chi\frac{d\chi}{dx} + W_\phi\frac{d\phi}{dx}\bigg)\bigg]. \quad (61)$$

In this case, one can see easily that solutions with minimal energy of the second-order differential equation for the static solutions in $1+1$ dimensions can be solved from the corresponding first-order coupled nonlinear equations

$$\frac{d\phi}{dx} = W_\phi(\phi, \chi), \qquad \frac{d\chi}{dx} = W_\chi(\phi, \chi), \quad (62)$$

where $W_\phi \equiv \frac{\partial W}{\partial \phi}$ and $W_\chi \equiv \frac{\partial W}{\partial \chi}$. So that one can finally obtain the BPS energy as given by

$$E_B = |W(\phi_j, \chi_j) - W(\phi_i, \chi_i)|, \quad (63)$$

where ϕ_i and χ_i correspond to the ith vacuum state of the model [35].

At this point, it is important to remark that the BPS solutions settle into vacuum states asymptotically. In other words, the vacuum states act as implicit boundary conditions of the BPS equations.

Now, instead of applying the usual trial-orbit approach [35–41], we note that it is possible to write the following equation:

$$\frac{d\phi}{W_\phi} = dx = \frac{d\chi}{W_\chi}, \quad (64)$$

where dx is a kind of invariant. So, one obtains

$$\frac{d\phi}{d\chi} = \frac{W_\phi}{W_\chi}. \quad (65)$$

This last equation is, in general, a nonlinear differential equation relating the scalar fields of the model. Now, if one is able to solve it completely, the function $\phi(\chi)$ can be used to eliminate one of the fields, thus leaving the Eqs. (62) disentangled. Finally, this disentangled first-order nonlinear differential equation can be solved in general, even if numerically.

From now on we choose a particular model which can be used for modeling a number of systems [35], in order to exemplify the method in a concrete situation. In fact we show that for this situation, the Eq. (65) can be mapped into a linear differential equation, from which it is possible to obtain the general solution. In this case the superpotential is chosen as

$$W(\phi, \chi) = -\lambda\phi + \frac{\lambda}{3}\phi^3 + \mu\phi\chi^2, \quad (66)$$

where $\mu = \mu_R + i\mu_I$ and Eq. (65) looks like

$$\frac{d\phi}{d\chi} = \frac{\lambda(\phi^2 - 1) + \mu\chi^2}{2\mu\phi\chi}. \quad (67)$$

At this point one can verify that, by performing the transformation $\phi^2 = \rho + 1$, the above equation can be written as

$$\frac{d\rho}{d\chi} - \frac{\lambda}{\mu\chi}\rho = \chi, \quad (68)$$

a typical inhomogeneous linear differential equation. It is interesting to observe that its particular solution corresponds to the result usually presented in the literature ($\mu_I = 0$) [35]. The general solution is easily obtained, giving

$$\rho(\chi) = \phi^2 - 1 = c_0\chi^{\lambda/\mu} - \frac{\mu}{\lambda - 2\mu}\chi^2, \quad (69)$$

where c_0 is an arbitrary integration constant. By substituting these solutions in one of the Eqs. (62), and solving it, we obtain a generalized solution for the system. In general it is not possible to solve χ in terms of ϕ from the above solutions, but the contrary is always granted. By substituting $\phi(\chi)$ in the equation for the field χ, one obtains

$$\frac{d\chi}{dx} = \pm 2\mu\chi\sqrt{1 + c_0\chi^{\lambda/\mu} - \bigg(\frac{\mu}{\lambda - 2\mu}\bigg)\chi^2}. \quad (70)$$

We have been able to solve the above equation both in the $c_0 = 0$ case and $c_0 = -2$ with $\mu_R = 1$, $\mu_I = 1$, and $\lambda = 1$. In the first one we have found essentially lump solutions both for the first field χ and for the second field ϕ (Figs. 2 and 3). In the second case ($c_0 \neq 0$) we have found kinks for both fields.

Unfortunately, the BPS energy is complex in this last case, probably due to the fact that the imaginary part of the fields also has a kink profile, in distinction with all the other considered non-Hermitian cases with real BPS energy, whose solitonic imaginary part exhibits a lumplike profile.

In fact we have tried a great number of other possibilities for the superpotential and, up to now, we have been unable to get any example of well-behaved *PT*-symmetric kinklike configurations in the case of two coupled fields.

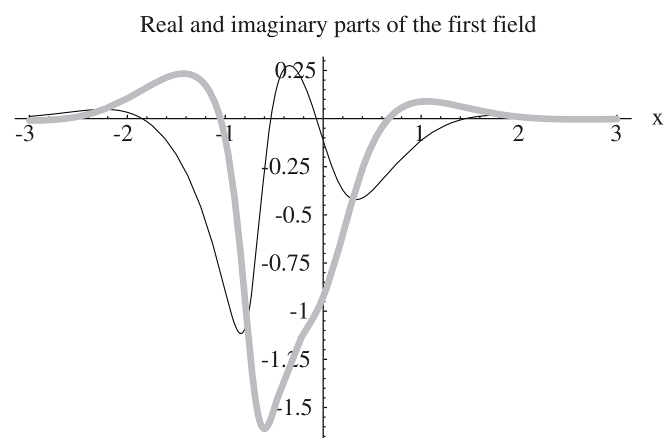

FIG. 2. The real (thin line) and imaginary parts of the field χ ($\lambda = \mu_R = \mu_I = 1$ and $c_0 = 0$).

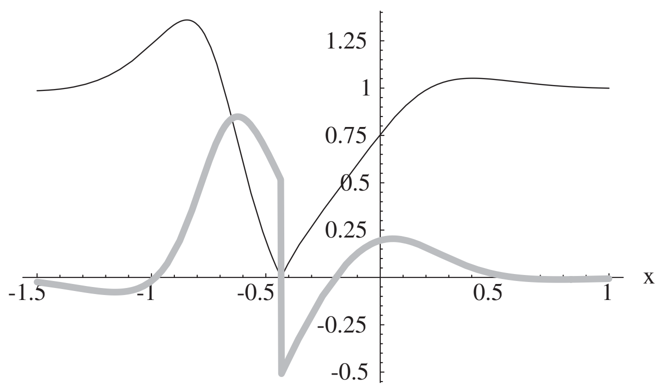

FIG. 3. The real (thin line) and imaginary parts of the field ϕ ($\lambda = \mu_R = \mu_I = 1$ and $c_0 = 0$).

VIII. CONCLUSIONS

In conclusion, we have observed that non-Hermitian systems preserving *PT* symmetry can exhibit topological classical configurations with real total energy, which are stable under small linear perturbations and can couple with fermions preserving supersymmetry. This is one important step in order to consider the so-called *PT*-symmetric systems as consistent and, as a consequence, physically acceptable candidates in quantum field theory. Based on these successful results, we intend to expand the present analysis by searching for consistent vortices and instantons in *PT*-symmetric systems.

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