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Four-Dimensional Compactification of the Pure Spinor Formalism

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Para Carla,
o amor da minha vida.

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"Forms of thought and forms of society have one thing in common.
They are both forms*"*

A. Sohn-Rethel, *Intellectual and Manual Labor: A Critique of Epistemology*

"A libertação *é um ato histórico*
e não um ato de pensamento"

K. Marx & F. Engels, *A ideologia alemã*

Resumo

A presente tese tem como objetivo a formulação da supercorda compactificada em orbifolds no formalismo de spinores puros. Para isso, a prescrição de espinores puros é adaptada ao cálculo de amplitudes topológicas na teoria de supercordas de tipo II.

Resulta que uma versão mais simples em quatro dimensões do fantasma b é suficiente para obter a resposta correta no cálculo dessas amplitudes, cujo regulador no formalismo não mínimo não precisa ser modificado em genus de ordem superior. Além disso, para que esta prescrição funcione, o espaço de espinores puros deve ser restrito para certas regiões correspondentes a espinores quirais ou antiquirais em quatro dimensões. Os cálculos usando os formalismos de RNS e híbrido, assim como os tópicos necessários para entender o cálculo de amplitudes a varios loops, são discutidos com fins de comparação.

Palavras Chave: Supercordas; Formalismo de Espinores Puros; Amplitudes a vários loops; Amplitudes Topológicas; Compactificações em orbifolds; Formalismo RNS.

Áreas do conhecimento: Física; Física de Altas Energias; Teoria de Supercordas.

Abstract

The purpose of this thesis is to study orbifold compactifications of the pure spinor formalism for the superstring. This is done by adapting the pure spinor prescription to computations of topological amplitudes in type II superstring theory.

It is found that a simpler four-dimensional version of the b ghost suffices to calculate these amplitudes giving the correct answer, where the usual regulator of the non-minimal pure spinor formalism does not need to be modified at higher genus. Also, in order for this prescription to work, one has to restrict to certain patches of pure spinor space of chiral or antichiral spinors in four-dimensions. The computations in the RNS and hybrid formalisms, as well as the necessary topics needed to understand amplitude computations at higher loops, are also discussed for the purpose of comparison.

Keywords: Superstrings; Pure Spinor Formalism; Higher-loop Amplitudes; Topological Amplitudes; Orbifold Compactifications; RNS Formalism.

Contents

Chapter 1

Introduction

String theory is the most prominent theoretical attempt towards a proper understanding of quantum gravity. Its original development, having its roots in the search for an adequate description of the world of hadrons and their interactions under the name of Dual Resonance Models, suddenly became a quest for unification of all fundamental forces of nature, and thus, for a solution to the long-standing problem of reconciling general relativity with quantum mechanics.[1]

In the course of its development, other fascinating areas have emerged from it or shown to be related with previous concepts not in appearance connected to the string world. Two-dimensional conformal field theory and its applications to condensed matter systems and statistical models, the relation between supersymmetry with the geometry and topology of manifolds, string dualities and non-perturbative physics, mirror symmetry, among other subjects, have benefited enormously from and have inspired further advances in string theory.

The basic picture is that of a string moving in spacetime. Many structures appear when this classical system is quantized; moreover, consistency at the quantum level imposes serious restrictions on the possible string theories constructed. One of the well known consequences of these constraints on the theory is dimensionality allowed for the spacetime in which the string is moving. The simplest case of a one-dimensional dynamical object whose embedding in spacetime M is described just by coordinates in M implies critical dimension 26. The restriction is put by the vanishing of an anomaly that potentially appears at the quantum level; an anomaly of a powerful two-dimensional symmetry of the string worldsheet Σ: *conformal invariance*.

The original bosonic action is actually reparametrization invariant. The positions of the string in *target space* are matter fields for a gravity theory on the worldsheet. After gauge-fixing, conformal invariance remains as a residual sym-

[1]For introductions to string theory see [1, 2]. A historical exposition of the first years of development by some of its creators is given in [3].

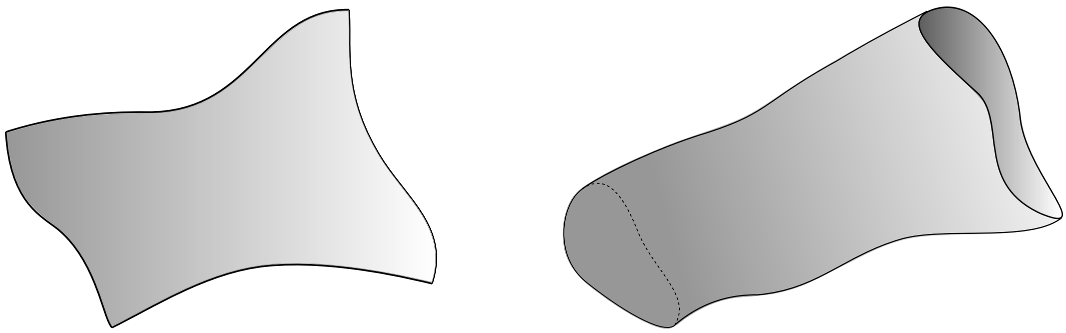

Figure 1.1: Worldsheets for open and closed strings

metry, generated, although, by an infinite set of operators.[2] These operators realize an infinite dimensional Lie algebra known as the Virasoro algebra. Besides, and as is well known, a proper quantization requires the introduction of ghosts associated to the local symmetry of the model. At the end of the day, one is dealing with two conformal systems, the one consisting of matter fields x^m $(m = 1, ..., 26)$ and the other, of reparametrization ghosts (b, c) which are fermionic fields. The potential anomalies of these systems cancel each other precisely for the number of dimensions $D = 26$. The important thing to notice is that, in principle, one could replace the matter system x^m by any other conformal field theory whose anomaly is of the same amount, and exactly cancels the conformal anomaly of the ghosts.

This basic framework can then be further explored by putting more structure into either one of the components Σ or M of bosonic string theory. For example, in order to describe spacetime fermions, it was found necessary to extend the Virasoro algebra of conformal generators to a graded-algebra which contains anticommuting as well as commuting generators. This is precisely one way supersymmetry was discovered in the early days of string theory. In essence, one could either *supersymmetrize* the worldsheet or the target space. Both ways one gets to a more or less satisfying description of the *superstring*. The critical dimension of this theory is 10 instead of 26, and its spectrum is spacetime supersymmetric. Nevertheless, the two ways of constructing such a superstring are very different

[2]Closed strings, one-dimensional loops moving through spacetime, have actually two sets of operators. Their modes are split into right- and left-moving sectors, which more or less decouple, as will be discussed later. For open strings, line segments moving in spacetime, boundary conditions reduce these two sectors into one. For brevity in exposition, in the following, only the left-moving sector of the closed string is discussed explicitly. Analogous considerations hold for the right-moving sector or for the open string, unless otherwise indicated.

from each other in appearance.

Superstring formalisms

Supersymmetrization of the worldsheet leads to a version of the superstring known as the Ramond-Neveu-Schwarz (RNS) formalism. Spacetime supersymmetry is not manifest, and it is not even achieved before a specific projection on the Hilbert space of string states is performed. Nevertheless, conformal field theory techniques allow for covariant quantization of this RNS superstring. Their basic ingredients are worldsheet supermatter x^m, ψ^m ($m = 1, ..., 10$), where ψ^m are fermionic worldsheet fields, reparametrization ghosts (b, c), and superconformal ghosts (β, γ).

Once the spectrum of a quantum string is obtained, interest concentrates on the study of string interactions. This is, of course, carried out computing scattering amplitudes. These can be constructed using the path integral formulation; ultimately, one has to compute two-dimensional correlation functions of *vertex operators* inserted on the worldsheet, to describe interactions in a ten-dimensional target space. The vertex operators carry the information of external string states of the amplitude, so to each state in the string Hilbert space one can associate an appropriate operator.

The most notorious complication with respect to the bosonic string is that now, the superstring spectrum contains spacetime fermions. The spin-statistics relation in M requires these new states to be spacetime spinors, and it is not obvious at first sight how to construct vertex operators for these states from the basic ingredients of RNS. The search for a *fermion vertex* led to the application of bosonization into string theory. The (β, γ) system is related to a mixed system consisting of ϕ and (ξ, η), where ϕ is a chiral boson with negative kinetic term. The construction is made with the use of spin fields for operators constructed out of $\exp(\pm\phi)$ and ψ^m. These spin fields, however, create branch cuts on the worldsheet; fields like ψ^m change boundary conditions around spin field insertions. Fields like ψ^m and β, γ can actually have periodic or antiperiodic boundary conditions along the string, giving rise to two sectors (for left-moving fields, and independently for right-moving counterparts) denoted by R and NS; spin fields interpolate between them.

This phenomenon has another crucial effect. When studying string interactions, one has to take into account all possible geometries of the worldsheet in the path integral. Furthermore, on has to include a sum over all topologies allowed by the string model. For closed (oriented) strings, these correspond to compact Riemann surfaces, two-dimensional surfaces with handles: the sphere (no handles at all), the torus (one handle), and higher *genus* surfaces as well. Starting from the torus, ψ^m and β, γ fields can take antiperiodic boundary conditions around non-contractible loops of these surfaces. A genus g surface contains $2g$ independent loops of this kind, so there are 2^{2g} classes of possible field configurations. These are known as spin structures, and the inclusion of all spin structure contributions to the amplitude, with appropriate relative weights, turns out to be crucial for the spacetime supersymmetric projection mentioned before. The sum over spin structures significantly complicates amplitude computations at higher genus, and this is related to the lack of manifest target space supersymmetry in the RNS formalism.

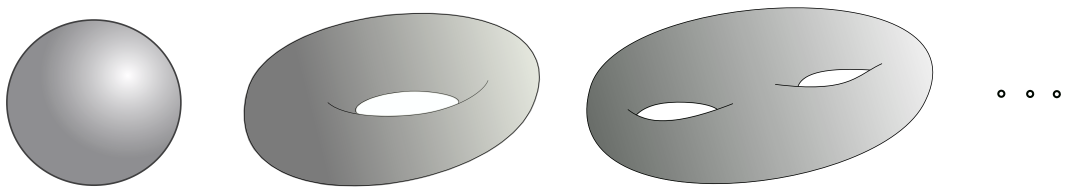

Figure 1.2: Genus expansion of oriented closed Riemann surfaces

This is enough motivation for considering alternative formulations which could maintain target space supersymmetry manifest; the Green-Schwarz (GS) formalism precisely does the job.

In ten dimensions, $N = 1$ superspace contains anticommuting spinors $\theta^{\underline{\alpha}}$ ($\underline{\alpha} = 1, ..., 16$) which are Majorana-Weyl, besides the commuting coordinates x^m. One way to construct closed superstrings is to extend supersymmetry to $N = 2$, having now two Majorana-Weyl spinors and to promote them to worldsheet fields, associating one spinor to the left- and the other to the right-moving sector. Two different superstring theories emerge from this construction: type IIA, where left- and right-moving spinor fields have opposite spacetime chirality[3], $\theta^{\underline{\alpha}}$, $\tilde{\theta}_{\underline{\alpha}}$; type IIB, where both spinors have the same chirality, $\theta^{\underline{\alpha}}$, $\tilde{\theta}^{\underline{\alpha}}$. The classical action in the GS formalism is then supercovariant. It is desirable to bring this action to a conformal gauge consisting of free fields, as in the RNS case. The main difficulty comes

[3]Right-moving worldsheet fields are denoted with tildes, and by the same symbol of the left-moving counterpart, when it exists.

from the fact that the GS string possesses a local fermionic invariance, known as kappa (κ) symmetry, which is important to get the correct number of degrees of freedom for the superstring. Unfortunately, a simple covariant gauge-fixing does not exist for κ symmetry. Although it is possible to do a light-cone gauge fixing, computation of multiloop amplitudes become extremely difficult as the number of external string states and loops in the worldsheet increase.

To quantize the string action, it is convenient to bring it into first order form. One defines the conjugate momenta of the coordinates and write the action as a kinetic term plus constraints imposed by Lagrange multipliers. This is the approach Siegel took when trying to quantize the GS superstring, introducing momentum $p_{\underline{\alpha}}$ conjugate to the fermionic spinor $\theta^{\underline{\alpha}}$. The complicated set of constraints, necessary to describe the superstring at quantum level, cannot be covariantly split into first and second class. This prevents, once more, covariant quantization.

There is, however, one class of systems where this first order approach works acceptably. All the discussion so far focused on string theories defined in a flat ten-dimensional background. Of course, the observable universe is four-dimensional, and the fact that gravity emerges from string theory allows for the possibility that many non-trivial geometries could arise in principle. One such situation is that of a superstring compactified on an *unobservable* six-dimensional manifold K_6, such that the background is expressed as $M_4 \times K_6$, where M_4 can be chosen as the flat Minkowski four-manifold. If one manages to preserve supersymmetry in four dimensions, then it is possible to design a Green-Schwarz-Siegel-like formalism which is $D = 4$ supercovariant; superspace consists of x^μ ($\mu = 1, .., 4$), corresponding to non-compact four dimensions, and $p_\alpha, \theta_\alpha, p_{\dot{\alpha}}, \theta_{\dot{\alpha}}$, ($\alpha, \dot{\alpha} = 1, 2$), Weyl and anti-Weyl spinors of $D = 4$. In effect, this is achieved in Berkovits' hybrid formalism. Here, worldsheet fields associated to the internal manifold are described by RNS-like variables. One subtlety of this formulation is the appearance of a chiral boson whose behavior is analogous to the ϕ boson coming from the β, γ system in RNS. This turns out to reduce its effectiveness when computing higher genus amplitudes. It took several years and many attempts before an ingenious proposal was successful in quantize the superstring in a super-Poincaré covariant manner.

Pure spinors

The left-moving sector of type II superstrings has $N = 1$ supersymmetry generated by a supercharge $\int dz q_{\underline{\alpha}}$, which is a sixteen-component spinor. None of those sixteen supersymmetries is manifest in RNS. Moreover, to construct spin fields out of fermionic matter ψ^m using bosonization, the full $SO(9,1)$ Lorentz invariance, must be broken to $U(5)$ after appropriate Wick rotation. On the other hand four supersymmetries are manifest in the hybrid formalism. Actually, it is possible to construct a $U(5)$ formalism which preserves six of the sixteen supersymmetries. Here it was noticed that $U(5)$ preserves up to scale a special type of spinor, the pure spinor. These pure spinors have interesting geometrical properties and, in particular, they were found to play a crucial role in the description of ten-dimensional super-Yang-Mills (SYM) as coming from integrability along supersymmetric lines.

The Pure Spinor superstring formalism is constructed out of the Siegel description with worldsheet fields $x^m, p_{\underline{\alpha}}, \theta^{\underline{\alpha}}$ by addition of a pure spinor bosonic worldsheet system $w_{\underline{\alpha}}, \lambda^{\underline{\alpha}}$. A pure spinor has eleven independent components due to a set of constraints that it obeys in ten dimensions, $\lambda^{\underline{\alpha}} \gamma^m_{\underline{\alpha}\underline{\beta}} \lambda^{\underline{\beta}} = 0$.[4] These constraints generate gauge transformations of the conjugate momenta $w_{\underline{\alpha}}$ which allows to eliminate five of them; it also has eleven independent components.

The central charge of the combined conformal systems adds up to zero, and there are no reparametrization or superconformal ghosts; it is the pure spinor and its conjugate momenta which are interpreted as the bosonic ghosts of the superstring. This intriguing feature is related to the fact that there is still no satisfactory gauge invariant action from which the pure spinor superstring can be derived by some gauge-fixing procedure. Modern BRST covariant quantization of bosonic and RNS strings allows for a detailed description of the spectrum using a fermionic charge Q_{BRST}. Physical states are defined as those annihilated by Q_{BRST} (closed) modulo states which can be written as Q_{BRST} of something (exact). This cohomological structure is translated in string amplitudes as the requirement of decoupling of BRST-trivial external states. Instead, in the pure spinor formalism a BRST charge Q_{ps} is postulated from the beginning, and reads $Q_{ps} = \int dz \lambda^{\underline{\alpha}} d_{\underline{\alpha}}$ where $d_{\underline{\alpha}}$ is one of the constraints appearing in the Siegel superstring. Using this charge it is possible to derive integrated $\mathcal{U}$ and unintegrated $\mathcal{V}$ vertex operators.

[4]Pure spinors in even dimensions are always Weyl or anti-Weyl. Gamma matrices $\gamma^m_{\underline{\alpha}\underline{\beta}}$ correspond to one of the off-diagonal blocks of $D = 10$ Clifford matrices in the Weyl representation.

Then, tree amplitudes are computed in analogy with the prescription for bosonic strings.

On the other hand, higher-loop amplitudes in the bosonic and RNS cases are known to require insertions of the reparametrization ghost b. At genus $g \geq 1$, infinitesimal diffeomorphisms times Weyl rescalings of the metric do not completely fix it. Variations orthogonal to these gauge transformations exist, and they correspond to the moduli of the Riemann surface Σ_g. The amplitude must contain an integration over the moduli space $\mathcal{M}_g$ of genus g Riemann surfaces Σ_g. In particular, a surface of genus $g \geq 2$ has a moduli space of dimension $3g - 3$. The effect of gauge fixing inside the path integral is the appearance of $3g - 3$ b ghost insertions in the correlator. It is evident now that the pure spinor formalism cannot compute higher-loop amplitudes as it stands. One possible solution is to allow for a composite field which has the main properties as the reparametrization b ghost: being fermionic, satisfying $\{Q_{BRST}, b\} = T$ for the total stress energy tensor T of the conformal model, and so on. The initial formulation does not allow for the construction of such a composite b ghost unless *picture changing*, an important feature of the RNS formalism, is also introduced here. Nevertheless, it is an extension of the original (minimal) pure spinor formalism which is better suited for multiloop amplitude computations.

There is another situation in which computation of multiloop amplitudes requires the insertion of $3g - 3$ copies of a composite fermionic operator to take into account the moduli space integration: the *topological string*. Roughly speaking this string is generally constructed as the coupling of a twisted two-dimensional $N = 2$ superconformal field theory (STCFT) with worldsheet gravity. Already in the RNS superstring, $N = 1$ SCFT appears as a residual symmetry after gauge-fixing. The $N = 2$ case, containing two fermionic generators instead of just one, is not derived here from such process but set out from the beginning. Furthermore, it is not a supersymmetric model but a twisted one. Twisting of supersymmetric $N = 2$ theories is a nice way to construct models whose correlation functions do not depend on the (full) metric put on the manifold where the theory lives. That is the reason such theories are called topological. The composite operator playing the role of the b ghost is one of the fermionic generators G^- of the superconformal algebra; it has spin two and satisfies $\{Q_{BRST}, G^-\} = T$. The amplitude prescription at any loops in these models just mimics that for the bosonic string.

It is nice that after, adding a BRST trivial quartet of fields, the resulting *non-minimal* pure spinor formalism has the structure of a twisted $N = 2$ topological string. The additional fields consist of bosonic $\bar{w}^{\underline{\alpha}}, \bar{\lambda}_{\underline{\alpha}}$, and fermionic $s^{\underline{\alpha}}, r_{\underline{\alpha}}$ systems, where $\bar{\lambda}_{\underline{\alpha}}$ is a pure spinor of opposite chirality with respect to $\lambda^{\underline{\alpha}}$. As for $r_{\underline{\alpha}}$, it is a constrained fermionic field obeying $\bar{\lambda}_{\underline{\alpha}}\gamma_{\underline{m}}^{\underline{\alpha\beta}} r_{\underline{\beta}} = 0$; a constraint which generates a gauge transformation on $s^{\underline{\alpha}}$. Hence, all these variables have, each one, eleven independent components. They decouple due to the well known quartet mechanism, after suitable modification of the BRST charge. A desired composite b ghost appear as the G^{-} generator of the twisted $N = 2$ superconformal algebra so that computation of higher genus scattering amplitudes are possible to define.

One remaining subtlety has to do with integration over bosonic zero modes of $\lambda^{\underline{\alpha}}, w_{\underline{\alpha}}, \bar{\lambda}_{\underline{\alpha}}, \bar{w}^{\underline{\alpha}}$. As expected, the naive integral diverges; however, by defining an appropriate regulator it is possible to render the scattering amplitude well defined.

The non-minimal pure spinor formalism is a powerful framework with which to compute multiloop amplitudes that are extremely difficult to deal with using the RNS formalism. It is manifestly super-Poincaré covariant so that the sum over spin structures is no longer required. Nevertheless, the origin of pure spinors in the superstring remains an open problem, although interesting explanatory attempts already exist. Comparison with computations performed in RNS could be helpful in trying to understand the pure spinor formalism from first principles.

Four dimensions and effective actions

All string theories contain gravity. The first manifestation of this is the appearance of a massless particle of spin two in the closed string spectrum. Of course, this is not enough to guarantee a generally covariant theory. Nonetheless, by analyzing string scattering amplitudes involving gravitons, it is possible to show that this spin two-particle couples to the theory in a gauge invariant way. Another use of string amplitude computations is related to the fact that, at low energies, it is expected for the string theory to reduce to ordinary quantum field theory. The last one will appear as an effective field theory, and its properties could be derived from the corresponding string by taking some low-energy limit, which turns out to be pictured as the limit where strings degenerate to points. A natural way to obtain information about the effective field theory lagrangian, for example, is to look for terms that reproduce field theory limits of corresponding string scattering amplitudes.

On the other hand, four-dimensional physics can be obtained from superstring theory by compactifying on an appropriate internal manifold K_6. Since, the precise nature of this six-dimensional manifold will strongly depend on the features of phenomena in Minkowski space M_4, one could ask which internal model reproduces the observable world. Specifically, the couplings and quantum numbers of particles in the effective field theory will depend in turn on the topology and geometry of the internal manifold. In spite of this task being too far from being fulfilled, string phenomenology was already used as a very rich laboratory where to study aspects of supergravity theories and stringy corrections to four-dimensional physics, as well as to get a better understanding of string theory itself.

One beautiful aspect of string theory is that it modifies and generalizes classical geometry in a consistent (although not well understood) way. Heterotic strings and orbifold target spaces are two examples of this. In heterotic strings, the left- and right-moving sectors are of different nature, the first being supersymmetric $N = 1$, whereas the second is bosonic. So, the left-movers describe a ten-dimensional target space, while right-movers, a space of 26 dimensions. The other example consists of orbifolds. These are spaces which are smooth everywhere except at some singular points (or other lower-dimensional submanifolds); they are obtained as the quotient of a smooth manifold by one of its discrete isometries. The nice behavior of quantum strings moving in this singular spaces has no counterpart in the particle case. All these ingredients appear when studying four-dimensional compactifications of superstring theories.

For phenomenological reasons, initial interest historically appeared in the case of classical background configurations for the heterotic superstring which preserve $N = 1$ supersymmetry in four-dimensional Minkowski spacetime. This requirement puts strong restrictions on the type of internal manifold on which the compactification has to be done; the manifold must be of a very special type: a Calabi-Yau (CY) threefold, which can be further abstracted (and generalized) to a class of $c = (9,9)$, $N = (2,2)$ superconformal field theories on the worldsheet. Since heterotic superstrings are closed strings and the vacua gives $N = 1$ spacetime supersymmetry, one expects the field theory to be consistent with four-dimensional $N = 1$ supergravity. For the Calabi-Yau case, the spectrum contains, besides the gravitational supermultiplet, gauge supermultiplets, a set of matter chiral superfields in the some representations of the heterotic gauge group, and corresponding moduli superfields with flat potential for their scalar components. The low-energy lagrangian is obtained computing appropriate scattering amplitudes.

Tree level (sphere) scattering amplitudes will give the classical part in spacetime of the effective field theory. For instance, the metrics on the space of the different types of scalar fields can be computed by the two-point string amplitude, with appropriate vertex operators for the scalar fields. Then one can use $N = (2,2)$ superconformal invariance to relate the kinetic terms of the moduli with those of the corresponding matter fields. The spacetime loop corrections to this lagrangian are then obtained by computing g-loop (higher genus) string amplitudes.

Although initially, heterotic strings were most intensively studied, type II superstrings seemed more natural for compactifications on $N = (2,2)$ SCFT.

Topological string amplitudes

Topological strings obtained by twisting an $N = 2$ SCFT and coupling to worldsheet gravity comes in two flavors. This is because in the case of closed strings the twisting can be done in the left- and right-moving sectors independently. Twisting consists of changing the spin of fermionic generators using the U(1) current J of the $N = 2$ SCFT, and this can happen in two ways; twisting in one direction gives spin 1 to G^+ and spin 2 to G^-, while twisting in the opposite direction gives the reverse assignment of spins $h(G^+) = 2$ and $h(G^-) = 1$. If the same twisting is performed in the left- and right-moving sectors of the closed string then one calls it the topological B model, for opposite relative twisting one obtains the topological A model.

There is an interesting connection between certain amplitudes in type II superstring theory compactified on a $\hat{c} = 3$ $N = (2,2)$ SCFT and the partition functions of A and B models of the $\hat{c} = 3$ topological string[5]. The result can be stated as follows: the g-loop superstring amplitude of two graviton and $2g - 2$ graviphoton string states reproduces at low energies the effective field theory term $F_g R^2 T^{2g-2}$ where F_g is the genus g partition function of a $\hat{c} = 3$ topological string and R and T are the self-dual components of the curvature and graviphoton field-strength.

As already stated, manifestly target-space supersymmetric formulations of superstrings avoid the necessity of summing over spin structures, simplifying string amplitude computations. Since type II superstrings compactified on a $\hat{c} = 3$ $N = (2,2)$ superconformal field theory have $N = 2$ spacetime supersymmetry in four dimensions, both the hybrid and pure spinor formalisms are suitable to study these type of compactifications.

Although the hybrid formalism has been successfully used to compute these

[5]For an $N = 2$ CFT $\hat{c}$ is related to the conformal central charge $c = 3\hat{c}$. In the case of topological theories, it can be thought simply as the anomaly in the OPE between the two J currents of the twisted SCFT

amplitudes, its chiral bosons ρ, $\tilde{\rho}$ lead to subtleties if one tries to compute other non-topological multiloop amplitudes. These bosons are absent in the pure spinor formalism so it is worth trying to obtain a compactified version of it, so that it can be applied to other amplitudes like multiloop four-boson scattering in four dimensions. The pure spinor formalism includes the 16 θ variables of $D = 10$ superspace, so for orbifold and Calabi-Yau compactifications, it contains many more worldsheet variables than those of the hybrid formalism, which only contains 4 θ variables. Nevertheless, it will be shown in this thesis that a simplified version of the composite b ghost can be constructed for orbifold compactifications if one only requires $D = 4$ super-Poincaré covariance. This four-dimensional version of the b ghost depends on all 16 θ's and is equivalent to the usual ten-dimensional b ghost up to a BRST-exact term, but has a simpler form and only depends on two components of the non-minimal variables. Using this simplified b ghost on an orbifold compactification, it is straightforward to show that the standard pure spinor amplitude prescription correctly computes the $N = 2$ $D = 4$ supersymmetric version of the $F_g R^2 T^{2g-2}$ topological term in the Type II effective action.

Outline

This thesis focus on orbifold compactifications in the pure spinor formalism. The structure is as follows. Chapter 2 gives brief expositions of the basic topics necessary for understanding of the remaining chapters. After reviewing the RNS and Pure Spinor formalisms in sections 2.1 and 2.2, respectively, string scattering amplitude prescriptions are discussed for arbitrary genus in section 2.3. Also, a short review of spinors in even dimensions is given. Then, multiloop correlators are written for different types of conformal systems in section 2.4. This presentation is given in the context of bosonization on higher genus Riemann surfaces, where the concepts of divisors, holomorphic factorization, and twisted line bundles are explained.

Chapter 3 contains material on orbifold compactifications. After discussing the relevant spectrum from the RNS point of view in section 3.1, orbifold compactification of the pure spinor formalism is presented in section 3.2. Breaking of ten dimensional spinors as well as the zero mode structure of twisted orbifold sectors are explained. Also, a change of basis in field space necessary to make contact with the topological string is performed. The spectrum arising from the pure spinor BRST physical condition is the subject of section 3.3.

Topological amplitudes are computed at arbitrary genus in Chapter 4. After a brief explanation of what a topological string is, the computation is done in three formalisms: RNS, hybrid, and pure spinor, in sections 4.1, 4.2 and 4.3, respectively. The amplitude of anti-self-dual gravitons and graviphotons in the RNS formalism is obtained by an explicit computations after giving a heuristic derivation. In the pure spinor computation, the type IIB case is discussed in detail after constructing the four-dimensional version of the b ghost. The modifications necessary for the type IIA, as well as hypermultiplet scattering are also given. The one-loop case is discussed separately.

Finally, conclusions and discussion of future directions of research are presented in Chapter 5. Also, Appendix A contains the proof of BRST equivalence between the non-minimal pure spinor b ghost and the simpler b ghost used in the topological amplitude computation.

Chapter 2

Preliminaries

This chapter contains reviews of the basic ingredients needed for the construction of pure spinor compactifications and topological amplitude computations in next chapters. RNS and pure spinor formalisms are the main superstring formulations used in this thesis. Others like Green-Schwarz and the hybrid formalism are mentioned or treated when necessary. Since multiloop amplitude computations in orbifolds requires correlation functions of twisted conformal (b, c) systems, a general discussion which comprises these cases is included in the context of bosonization on higher genus Riemann-surfaces.

2.1 RNS Formalism

In the year of 1971 (three years after Veneziano published his famous formula which marked the beginning of string theory [4]), two dual models (as the subject was then known) appeared which included fermionic operators in their formulation. The Ramond dual model for fermions [5], inspired by a correspondence principle which promoted Clifford gamma matrices Γ^m (which satisfy $\{\Gamma^m, \Gamma^n\} = 2\eta^{mn}$) to the aforementioned fermionic field operators, and the Neveu-Schwarz dual model for pions [6], which contained only bosonic external states, both had a structure which extended the Virasoro algebra responsible for decoupling of unphysical states in those models [7]. Further investigation on four-fermion scattering showed that, by merging these two models, one can get a theory with no tachyons in the spectrum if one performs a projection in the Hilbert space which, as was seen to be essential for the new model to work well, imposes reality of ten-dimensional spinors that described the fermionic states [8]. The effect of this is to project out a whole sector of Neveu-Schwarz (NS) states and also to project out all spinors of the Ramond (R) *sector* of given chirality. This Ramond-Neveu-Schwarz (RNS) formalism [9] turns out to have ten-dimensional spacetime supersymmetry; thus, describing the ten-dimensional superstring.

By that time, the relevance of the string interpretation of dual models was already established, due to the clarification of the origin of critical dimensions and intercept conditions as coming from consistency in the light-cone quantization of a massless relativistic string moving in spacetime [10]. In the context of the RNS model, an anticommuting extension of the Virasoro algebra is now a manifestation of local supersymmetry of the worldsheet swept out by the string [11, 12]. Nevertheless, only after developments such as the path integral quantization of string actions [13, 14], the machinery of two-dimensional conformal field theory [15], and BRST quantization of non-abelian gauge theories [16, 17], a fully covariant quantization of the RNS formalism was possible, and it allowed for a convenient and systematic computation of superstring scattering amplitudes [18].

These basic ingredients of the RNS formulation of superstring theory will be briefly reviewed.

The fundamental action for string theory is that of a string covariantly coupled to two-dimensional gravity. Since the RNS formalism (for type II superstrings) has $N = (1,1)$ worldsheet supersymmetry, its dynamics can be described using $N = (1,1)$ $d = 2$ superspace $(\sigma^{\hat{m}}, \kappa^{\hat{\mu}})$ [19] where *matter* superfields $X^m(\sigma, \kappa)$ $(m = 1, ..., 10)$ couple to a two-dimensional supervielbein $E_A^{\ M}(\sigma, \kappa)$ through the highly redundant action,[1]

$$S = \frac{1}{2} \int d^2\sigma d^2\kappa E^{-1} E_{\hat{\alpha}}^{\ M} \mathcal{D}_M X^m E^{\hat{\alpha}N} \mathcal{D}_N X_m. \tag{2.1}$$

The vielbein comprises sixteen superfields, while the supergeometry of the worldsheet requires a single superfield for its description. This reduction can be achieved by means of local Lorentz transformations and diffeomorphisms, as well as the imposition of torsion constraints. Actually, the supergravity multiplet $E_A^{\ M}$ reduces to a zweibein $e_{\hat{m}}^{\ \hat{a}}$, a gravitino field $\chi_m^{\ \hat{\alpha}}$, and an extra auxiliary field [20].

For a flat background, the RNS action in component form is

$$S = \int d^2\sigma \sqrt{g} \Big[\frac{1}{2} g^{\hat{m}\hat{n}} \partial_{\hat{m}} x^m \partial_{\hat{n}} x_m + \psi^m \gamma^{\hat{m}} \partial_{\hat{m}} \psi_m \tag{2.2}$$

$$- \psi^m \gamma^{\hat{a}} \gamma^{\hat{m}} \chi_{\hat{a}} \partial_{\hat{m}} x_m - \frac{1}{4} \psi^m \gamma^{\hat{a}} \gamma^{\hat{b}} \chi_{\hat{a}} (\chi_{\hat{b}} \psi_m) \Big] \tag{2.3}$$

where $g^{\hat{m}\hat{n}}$ is the inverse metric on the worldsheet and $\gamma^{\hat{a}}$ are two-dimensional gamma matrices. Besides, x^m and ψ^m are components of the matter superfields, $X^m = x^m + \kappa^{\hat{\alpha}} \psi_{\hat{\alpha}}^\mu + \kappa \bar{\kappa} F^\mu$, where the auxiliary fields F^μ decouple from the original

[1] M is a world-superindex $(\hat{m}, \hat{\mu})$, while tangent indices are represented by $A = (\hat{a}, \hat{\alpha})$.

action.

Local symmetries allows one to achieve a superconformal gauge, $g_{\hat{a}\hat{b}} = \rho\delta_{\hat{a}\hat{b}}, \chi_{\hat{a}} = \gamma_{\hat{a}}\zeta$, that brings the action to the form

$$S_{g.f.} = \frac{1}{2}\int d^2z\left[\partial x^m\bar{\partial}x_m + \psi^m\bar{\partial}\psi_m + \tilde{\psi}^m\partial\tilde{\psi}_m\right] \tag{2.4}$$

where... Or, written in flat two-dimensional $N = (1,1)$ superspace,

$$S_{g.f.} = \frac{1}{2}\int d^2zd^2\kappa\bar{D}X^mDX_m \tag{2.5}$$

where

$$D = \frac{\partial}{\partial\kappa} + \kappa\partial, \qquad \bar{D} = \frac{\partial}{\partial\bar{\kappa}} + \bar{\kappa}\bar{\partial} \tag{2.6}$$

The equations of motion derived from this action imply a separation into holomorphic and antiholomorphic pieces of the matter superfields, comprising the left- and right-moving sectors of the closed string. The generators for the residual $N = (1,1)$ superconformal invariance read[2]

$$\mathcal{T}(z,\kappa) = -\frac{1}{2}DX^mD^2X_m, \qquad \bar{\mathcal{T}}(\bar{z},\bar{\kappa}) = -\frac{1}{2}\bar{D}X^m\bar{D}^2X_m \tag{2.7}$$

The component expansion of $\mathcal{T}(z,\kappa)$, which is left-moving, contains the usual stress tensor T as its κ component, and the fermionic $N = 1$ supercurrent G at lowest order. They are

$$T = -\frac{1}{2}\partial x^m\partial x_m - \frac{1}{2}\psi^m\partial\psi_m, \qquad G = \frac{1}{2}\psi^m\partial x_m \tag{2.8}$$

The modes of these fields form the infinite set of operators that extend the Virasoro algebra to its $N = 1$ supersymmetric completion. An identical copy is present in the right-moving sector for type II superstrings.

The BRST covariant quantization of this model [21, 22] requires the introduction of fermionic reparametrization (b,c), and bosonic superconformal (β,γ) ghosts associated to the generators T and G, respectively. In superconformal gauge, the action for these fields is[3]

$$S_{gh} = \int d^2z\left[b\bar{\partial}c + \beta\bar{\partial}\gamma\right] \tag{2.9}$$

[2] All composite operators are assumed to be normal ordered.

[3] These ghosts can be similarly combined into left- and right-moving superfields in $d = 2$ $N = (1,1)$ superspace.

The corresponding superconformal currents of these combined ghost systems is

$$T = -b\partial c - \partial(bc) - \beta\partial\gamma - \frac{1}{2}\partial(\beta\gamma), \qquad G = \frac{1}{2}b\gamma + \frac{3}{2}\beta\partial c + \frac{1}{2}(\partial\beta)c \tag{2.10}$$

Ghost number current is defined as

$$j_{gh} = -bc - \beta\gamma \tag{2.11}$$

A very important information for amplitude computations is the singular behavior as local fields approach one another on the worldsheet. This is encoded in operator product expansions (OPE's) between fundamental fields of the theory, as follows

$$\partial x^m(y)\partial x^n(z) \longrightarrow -(y-z)^{-2}\eta^{mn}, \qquad \psi^m(y)\psi^n(z) \longrightarrow (y-z)^{-1}\eta^{mn} \tag{2.12}$$

$$\gamma(y)\beta(z) \longrightarrow (y-z)^{-1}, \qquad c(y)b(z) \longrightarrow (y-z)^{-1} \tag{2.13}$$

where η^{mn} is the flat metric in target space.

The OPE between the full RNS stress tensor

$$T_{RNS} = -\frac{1}{2}\partial x^m\partial x_m - \frac{1}{2}\psi^m\partial\psi_m - b\partial c - \partial(bc) - \beta\partial\gamma - \frac{1}{2}\partial(\beta\gamma) \tag{2.14}$$

and any conformal field $\mathcal{O}$ allows one to read its conformal weight as

$$T_{\mathrm{RNS}}(y)\mathcal{O}(z) \longrightarrow \frac{h_{\mathcal{O}}}{(y-z)^2}\mathcal{O}(z) + \frac{1}{y-z}\partial\mathcal{O}(z), \tag{2.15}$$

so conformal weights of basic fields are $h(\partial x^m) = 1$, $h(\psi^m) = \frac{1}{2}$, $h(b) = 2$, $h(c) = -1$, $h(\beta) = \frac{3}{2}$, and $h(\gamma) = -\frac{1}{2}$. Besides, by computing the OPE $T_{RNS}(y)T_{RNS}(z)$ one shows that the conformal anomaly vanishes precisely for $D = 10$.

Covariant quantization crucially rests on the BRST charge (which is nilpotent only in target space dimension $D = 10$) constructed from ghosts and symmetry generators of the model. The RNS BRST charge is

$$Q_{RNS} = \int dz \Big[cT_{RNS} - bc\partial c + \gamma\psi^m\partial x_m + \gamma^2 b\Big] \tag{2.16}$$

Physical external string states in scattering amplitudes are emitted by vertex operators of the two-dimensional superconformal field theory describing the superstring. These vertex operators are required to belong to the cohomology of the BRST operator.

Since fermionic matter fields come from two-dimensional spinors, they are allowed to be double-valued worldsheet fields. The aforementioned NS and R sectors of the superstring come precisely from different choices of periodicity obeyed by these fields. States in the NS sector obey periodic boundary conditions around the origin of the complex plane, while states in the R sector are antiperiodic.

The superconformal algebra implies that Ramond ground states have $h = \frac{5}{8}$ in order to get unbroken worldsheet supersymmetry. Since the lowest state, the vacuum, has $h = 0$, it must belong to the NS sector. All NS states can be generated by acting on the vacuum with conformal superfields of given conformal weight. However, Ramond states cannot be generated this way because those superfields do not change periodicity conditions of worldsheet fermions. Conformal fields responsible to create Ramond states from the NS vacuum are of a new kind; they are called spin fields. Fermionic fields are double-valued around these spin fields, so OPE's between them have an expansion in half integer powers of $(y - z)$.

In the RNS superstring, spin fields which create Ramond ground states from the NS vacuum, collectively transform as a spinor of $SO(10)$, $S^{\underline{A}}$ $(\underline{A} = 1, ..., 32)$. This 32 component spinor splits into Weyl spinors of opposite chirality $S^{\underline{\alpha}}$ and $S_{\underline{\alpha}}$. Short distance behavior is given by[4]

$$\psi^m(y)S_{\underline{\alpha}}(z) \longrightarrow (y-z)^{-\frac{1}{2}}\gamma^m_{\underline{\alpha}\underline{\beta}}S^{\underline{\beta}}(z) \tag{2.17}$$

$$S^{\underline{\alpha}}(y)S_{\underline{\beta}}(z) \longrightarrow (y-z)^{-\frac{5}{4}}\delta^{\underline{\alpha}}_{\underline{\beta}} + \frac{1}{2}(y-z)^{-\frac{1}{4}}(\gamma^{mn})^{\underline{\alpha}}{}_{\underline{\beta}}\psi_m\psi_n \tag{2.18}$$

$$S_{\underline{\alpha}}(y)S_{\underline{\beta}}(z) \longrightarrow (y-z)^{-\frac{3}{4}}\gamma^m_{\underline{\alpha}\underline{\beta}}\psi_m \tag{2.19}$$

To the NS vacuum and all states created from it by the action of an even number of fermionic fields ψ^m, odd GSO parity is assigned; remaining states in this sector are even. Also, ground states in the Ramond sector created by chiral (antichiral) spin field $S^{\underline{\alpha}}$ $(S_{\underline{\alpha}})$ have odd (even) parity; higher Ramond states have GSO assignment consistent to the fact that the action of a single fermionic field changes parity. This way, the superstring is constructed by projecting out all states with odd GSO parity. The resulting massless spectrum of the RNS superstring consists of the gluon and gluino of $D = 10$ $N = 1$ SYM, as is well known.

An analogous situation holds for the superconformal ghosts, (β, γ). Notice that these ghosts are related to variations of the worldsheet gravitino of the original

[4] Here, a specific representation of the Clifford matrices Γ^m is chosen.

action in a similar way as (b, c) ghosts are related to reparametrizations. This means that they have to obey the same boundary conditions as the gravitino $\chi_{\hat{a}}$. But a well defined locally supersymmetric action requires $\chi_{\hat{a}}$ to behave the same as ψ^m due to the way they are coupled in the original lagrangian. All this implies that there are also two sectors, NS and R, for the superconformal ghosts, and that they mimic the behavior of fermionic matter fields. Therefore, the (β, γ) system has its own spin field.

Before introducing vertex operators it is convenient to mention some facts about the (β, γ) system of superconformal ghosts. The Fock space generated by this bosonic conformal system has an infinite set of inequivalent vacua, parametrized by an integer in the NS sector and a half-integer in the R sector. It is not possible to relate these vacua by acting with a finite number of oscillators in the mode expansion of the fields. A convenient way to interpolate between them is obtained through bosonization.

Two-dimensional bosonization establishes an equivalence between a fermionic first order system (b, c) of conformal weights $(\lambda, 1-\lambda)$ and the theory consisting of a compactified boson σ coupled to the two-dimensional curvature through a background charge $Q = 1 - 2\lambda$. The corresponding actions are

$$S_{bc} = \int d^2z \, b\bar{\partial}c \tag{2.20}$$

$$S_\sigma = \frac{1}{2}\int d^2z \Big[\partial\sigma\bar{\partial}\sigma + \frac{i}{4}QR\sigma\Big] \tag{2.21}$$

The correspondence says that fermions in one theory translate to *instantons* of the other as $c = e^{-i\sigma},\ b = e^{i\sigma}$. Then, one can easily show the equivalence between stress energy tensors as well as the anomalous U(1) currents on both theories.

$$j_{bc} = bc = i\partial\sigma = j_\sigma \tag{2.22}$$

$$T_{bc} = -\lambda b\partial c + (1-\lambda)(\partial b)c = -\frac{1}{2}\partial\sigma\partial\sigma + \frac{i}{2}(1-2\lambda)\partial^2\sigma = T_\sigma \tag{2.23}$$

In the case of a bosonic system (β, γ) of conformal weights $(\lambda, 1-\lambda)$, besides a chiral boson ϕ, one needs a fermionic system (η, ξ) of weights $(1, 0)$. Then, $\beta = \partial\xi \exp(-\phi)$, $\gamma = \exp(\phi)\eta$. The Hilbert space associated to ϕ, η, ξ is larger than that of β, γ. On restricts from the large Hilbert space to the small one by requiring that physical states are independent of the zero mode of ξ, or alternative, are annihilated by η.

The stress tensor relation is

$$T_{\beta\gamma} = -\lambda\beta\partial\gamma + (1-\lambda)(\partial\beta)\gamma = -\eta\partial\xi - \frac{1}{2}\partial\phi\partial\phi - \frac{1}{2}(2\lambda-1)\partial^2\phi = T_{\phi\eta\xi} \quad (2.24)$$

Precisely, operators like $\exp(\pm\phi)$ allows to interpolate between inequivalent vacua of the β, γ Hilbert space. Each vacuum determines a picture in which physical states of the superstring are represented. The picture number operator is defined as $P = \int dz(-\partial\phi + \xi\eta)$, so β and γ have zero picture, and $\exp(q\phi)$ has picture $P = q$. Amplitude computations in the RNS formalism will require insertion of vertex operators at different picture.

Using the stress tensor in bosonized form, it is no difficult to show that the conformal weight of an exponential operator is $h(\exp(q\phi)) = -\frac{1}{2}q(q+2)$.

Unintegrated vertex operators for the massless sector can be constructed as follows; focusing first on the open string where only the left-moving sector of the string is relevant.

For a given momentum k^m, $k^2 = 0$, a generic vertex operator for open strings is of ghost number one and written as

$$V = \Phi\, e^{ik\cdot x}(z) \quad (2.25)$$

where Φ has zero conformal weight. In the NS sector, at picture zero, Φ can take the general form

$$\Phi = \gamma\psi^m f_m + c\Big(\partial x^m g_m + \psi^m\psi^n h_{mn}\Big) \quad (2.26)$$

The vertex operator V belongs to the cohomology of Q_{RNS} if

$$a_m \equiv g_m = -f_m, \qquad h_{mn} = \frac{i}{2}k_{[m}f_{n]}, \qquad k^m a_m = 0 \quad (2.27)$$

and the vertex is BRST exact when $a_m = k_m$. So one gets the physical vertex operator for the gluon at zero picture,

$$V_a^{(0)} = \Big[-\gamma\psi^m a_m + c\Big(\partial x^m a_m + i\psi^m\psi^n k_n a_m\Big)\Big]e^{ik\cdot x} \quad (2.28)$$

Alternatively, one could write the vertex at picture -1. A factor of $\exp(-\phi)$ of conformal weight $\frac{1}{2}$ should appear, and the vertex operator becomes

$$V_a^{(-1)} = ce^{-\phi}\psi^m a_m, \qquad k^m a_m = 0 \quad (2.29)$$

The passage from $V_a^{(-1)}$ to $V_a^{(0)}$ is accomplished trough the action of the picture changing operator define as

$$V_a^{(0)} = \left[Q_{RNS}, \xi V_a^{(-1)}\right] \tag{2.30}$$

This corresponds to changing the vacuum for the β, γ system. Although $V_a^{(0)}$ is BRST-exact in appearance, it is just not the case because ξ is not allowed to appear without derivatives in physical operators. It turns out that that at the level of cohomology, state spaces for different picture are all isomorphic, even for half-integer pictures corresponding to states in the Ramond sector. The isomorphism between physical subspaces which differ by integer picture is given by a repeated application of the picture changing operator $Z = \{Q_{RNS}, \xi\}$. While for the superstring, where the GSO projection has already been performed, the bijection between physical subspaces of integer and half-integer picture is provided by spacetime supersymmetry, which is not manifest in the RNS formalism. Before giving the expression for the supersymmetry current, it is convenient to write the vertex operator for the massless state in the Ramond sector at picture $-\frac{1}{2}$.

One expects the presence of the field $\exp(-\frac{1}{2}\phi)$ which has conformal weight $\frac{3}{8}$. So, the vertex operator will be of the form

$$V_\chi^{(-\frac{1}{2})} = ce^{-\frac{1}{2}\phi} S_{\underline{\alpha}} \chi^{\underline{\alpha}} e^{ik\cdot x} \tag{2.31}$$

This vertex operator is BRST closed if the polarization χ satisfies $\chi^{\underline{\alpha}} \gamma^m_{\underline{\alpha}\underline{\beta}} k_m = 0$.

Using the picture changing operator Z, one readily obtains the gluino vertex operator at picture $\frac{1}{2}$,

$$V^{(+1/2)} = ce^{\phi/2}\left[\partial x^m (S\gamma_m \chi) + \frac{1}{4}\psi^n \psi^m (S\gamma_m \partial_n \chi)\right] + e^{3\phi/2} \eta S_\alpha \chi^\alpha \tag{2.32}$$

where $S\gamma^m \chi \equiv S^{\underline{\alpha}} \gamma^m_{\underline{\alpha}\underline{\beta}} \chi^{\underline{\beta}}$.

The supersymmetry current can be extracted from the fermion vertex at zero momentum,

$$j_{\underline{\alpha}} = e^{-\frac{1}{2}\phi} S_{\underline{\alpha}} \tag{2.33}$$

Vertex operators for closed strings can be written as0 left-right products of those for open strings. For example, using the gluon vertex operator one gets, at zero picture,

$$c\tilde{c}h_{mn}\left(\partial x^m + i\psi^m(\psi\cdot k)\right)\left(\bar{\partial}x^n + i\tilde{\psi}^n(\tilde{\psi}\cdot k)\right)e^{ik\cdot x} \tag{2.34}$$

From this expression one can extract NS-NS the vertex operators for the graviton, dilation, and axion in the usual way; for example, the graviton is given by a symmetric traceless h_{mn} satisfying $k^m h_{mn} = 0$. One can similarly get vertex operators in the other sectors R-NS, NS-R and RR.

2.2 Pure Spinor Formalism

The pure spinor formalism [23] has the advantage over all other formulations of the ten-dimensional superstring in that it allows for manifestly super-Poincaré covariant computation of its amplitudes. Basically one adds to the usual Green-Schwarz-Siegel $D = 10$ superspace worldsheet variables, bosonic ghosts which, from the target-space point of view are pure (or simple) spinors. Pure spinors exist in any dimension, and in the particular case of $D = 10$ they can be defined as Weyl spinors satisfying some set of algebraic relations, namely

$$\lambda^{\underline{\alpha}}\gamma^m_{\underline{\alpha}\underline{\beta}}\lambda^{\underline{\beta}} = 0 \tag{2.35}$$

where $\underline{\alpha} = 1, ...16$, and $m = 1, ..., 10$. These constraints reduce the number of independent components of the pure spinor to eleven. A few words about spinors in even (and in particular, ten) dimensions are in order

2.2.1 Spinors in even dimensions

Algebraically, spinors are elements of a vector space that carries a representation of the Clifford algebra $Cl(g)$ [24]. This important algebra is generated by a complex vector space W of dimension $D = 2d$ provided with a scalar product g. Elements of the Clifford algebra have a well known realization in terms of complex matrices Γ^m of dimension $2^d \times 2^d$ satisfying $\{\Gamma^m, \Gamma^n\} = 2g^{mn}\mathbb{1}$, with $\mathbb{1}$ being the identity matrix $\mathbb{1}_{2^d\times 2^d}$. The space W can be though of as the complexification of the target space for superstrings in a flat background; the scalar product is then reduced to Minkowski (or Euclidean) metric in ten dimensions, after a *real slice* is taken in W. The representation space is known as the complex spinor space, S.

Spinors have a nice geometrical interpretation in terms of complex planes in

W. One can associate to each spinor $\zeta \in S$ a subspace of W which is totally null. This means that every vector in W has zero norm, and every pair of such vectors is orthogonal. This totally null space is defined as

$$M = \{u \in W | u\zeta = 0\} \tag{2.36}$$

where u acts on ζ through a representation of the Clifford algebra; one can write u as $u = u_m\Gamma^m$ where $\{\Gamma^m\}$, $m = 1, ..., D$ is one basis for the generators of the representation of $Cl(g)$ acting on S. Then $uv + vu = g(u, v))$. The maximal dimension M can have is d, and when this happens M is called a maximal totally null (MTN) space; pure spinors are such ζ's which define MTN's.

Pure spinors exist in every (even) dimension. To see this consider the following null basis of W constructed from the standard basis $\{\Gamma^m\}$.

$$n^a = \frac{1}{2}(\Gamma^{2a-1} - \Gamma^{2a}), \qquad a = 1, ..., d \tag{2.37}$$

$$p^a = \frac{1}{2}(\Gamma^{2a-1} + \Gamma^{2a}), \qquad a = 1, ..., d \tag{2.38}$$

These elements satisfy

$$n^a n^b + n^b n^a = 0, \qquad p^a p^b + p^b p^a = 0, \qquad n^a p^b + p^b n^a = \delta^{ab} \tag{2.39}$$

The spaces $N = span\{n^1, ..., n^d\}$ and $P = span\{p^1, ..., p^d\}$ are both maximal totally null subspaces orthogonal to each other ($N \cap P = \{0\}$, $W = N \oplus P$). The existence of pure spinors comes from the faithfulness of $Cl(g)$ in S, since this implies that there exists a spinor $\chi \in S$ such that $\omega \equiv n^1 n^2 ... n^d \chi$ is nonzero. Then, $M(\omega) = N$ and ω is pure [25].

Notice that the commutation relations between n^a and p^a are those satisfied by the annihilation and creation operators of states with Fermi statistics. Hence, the pure spinor ω can get the interpretation of a vacuum state $\omega = |\Omega\rangle$ annihilated by all the n^a [26]. A basis of S is constructed starting from this vacuum by the action of ordered products of creation operators p^a,

$$|\Omega\rangle;\ p^1|\Omega\rangle,\ ...,\ p^d|\Omega\rangle;\ p^1 p^2|\Omega\rangle,\ ...,\ p^{d-1}p^d|\Omega\rangle;\ ...;\ p^1 ... p^d|\Omega\rangle. \tag{2.40}$$

Indeed, this is a set of 2^d linearly independent spinors. Furthermore, it is inmediate to see that all spinors in this basis are pure. Generic elements of S are called Dirac spinors and they can be expanded in the previous basis

$$\zeta = \left(\zeta^+ + \zeta_a p^a + \frac{1}{2!}\zeta_{ab}p^a p^b + ... + \frac{1}{d!}\zeta_{a_1...a_d}p^1...p^d\right)|\Omega\rangle \tag{2.41}$$

In even dimensions there is a splitting of spinor space into Weyl (or chiral) S^+ and anti-Weyl (or anti-chiral) S^- subspaces characterized by the eigenvalue of the operator $\Gamma = \prod_{m=1}^{2d}\Gamma^m$.

Elements of the Clifford algebra can be regarded as endomorphisms of S.[5] In dual spinor space S^*, there exists a representation defined by the transposed endomorphisms $(\Gamma^m)^{tr}$. The fact that $Cl(g)$ has the structure of a simple algebra implies that there is an isomorphism between S and S^*, $B : S \to S^*$. Since, S^* is the space of linear functionals on S, one can defined the application of an element χ^* of S^* on an element ζ of S as $\langle\chi^*, \zeta\rangle \in \mathbb{C}$. Then, the isomorphism B allows a definition of a natural bilinear product on S, $(\chi, \zeta) = \langle B\chi, \zeta\rangle$. Using this bilinear product it is possible now to define a sequence of *multivectors* $B_k(\chi, \zeta)$, $k = 0, 1, ..., 2d$,

$$B_k^{m_1,...,m_k}(\chi, \zeta) = (\chi, \Gamma^{m_k}...\Gamma^{m_1}\zeta) \tag{2.42}$$

with $1 \leq m_1 \leq m_k \leq 2d$. Then it is possible to give an algebraic characterization of pure spinors; a non-zero Weyl (or anti-Weyl) spinor λ is pure if it satisfies $B_k(\lambda, \lambda) = 0$ for $k \neq d$.[6]

It is worth mentioning that for a pair of Weyl spinors χ and ζ, the multivector $B_k(\chi, \zeta)$ vanishes whenever $k \equiv d + 1(\mathrm{mod}2)$.

Also, defining

$$\Gamma^{m_1...m_r} = \frac{1}{r!}\Gamma^{[m_1}...\Gamma^{m_r]}, \tag{2.43}$$

then $(\chi, \Gamma^{m_1...m_r}\zeta)$ is symmetric in $\{\chi, \zeta\}$ if $2d - 2r \equiv 0(\mathrm{mod}8)$, and antisymmetric if $2d - 2r \equiv 4(\mathrm{mod}8)$ [27][28, Appendix of].

With these ingredients it is possible to reduce the conditions for a spinor being pure. For example, given a chiral spinor λ in eight dimensions ($d = 4$), then $B_1(\lambda, \lambda)$ and $B_3(\lambda, \lambda)$ are automatically zero. Furthermore, $B_2(\lambda, \lambda)$ can be written in terms of $B_0(\lambda, \lambda) = (\lambda, \lambda)$ and $(\lambda, \Gamma^{m_1 m_2}\lambda)$. The last expression vanishes due to antisymmetry, moreover all quantities B_k for $k > 4$ are related by duality using

[5] With S an ordinary vector space, an endomorphism is just a linear transformation on S.

[6] Actually, one can show that for even dimension every pure spinor is Weyl (or anti-Weyl). The equivalence of the algebraic characterization with the original geometrical definition is given in [25].

the Levi-Civita symbol $\varepsilon^{m_1 \cdots m_{2d}}$. Thus, the only pure spinor constraint in eight dimensions is

$$(\lambda, \lambda) = 0 \tag{2.44}$$

In ten dimensions ($d = 5$), $B_0(\lambda, \lambda)$, $B_2(\lambda, \lambda)$ and $B_4(\lambda, \lambda)$ are identically zero; B_3 reduces to B_1 and $(\lambda, \Gamma^{m_1 m_2 m_3} \lambda)$, where the last vanishes again because of antisymmetry. Then, the pure spinor constraints in ten dimensions are

$$B_1^m(\lambda, \lambda) = (\lambda, \Gamma^m \lambda) = 0 \tag{2.45}$$

The matrices Γ^m can be written in a Weyl representation with only off-diagonal non-zero blocks

$$\Gamma = \begin{pmatrix} 0 & \gamma \\ \gamma' & 0 \end{pmatrix} \tag{2.46}$$

and such that a generic Dirac spinor ζ with 32 components (2.41) splits as a sum of a Weyl and an anti-Weyl spinor, $\zeta = \zeta^{(+)}$ and $\zeta^{(-)}$ each of which corresponds to the first and second half with 16 components

$$\zeta = \begin{pmatrix} \zeta^{(+)} \\ \zeta^{(-)} \end{pmatrix} \tag{2.47}$$

So, consistently with notation used in previous sections, chiral spinors are denoted by $\zeta^{(+)} \to \zeta^{\underline{\alpha}}$, whereas antichiral spinors are written as $\zeta^{(-)} \to \zeta_{\underline{\alpha}}$. The blocks in Γ have components denoted by $\gamma \to \gamma^{m\underline{\alpha}\underline{\beta}}$, $\gamma' \to \gamma^m_{\underline{\alpha}\underline{\beta}}$. Also, in ten dimensions the isomorphism B can be chosen such that, for chiral λ the pure spinor conditions become $(\lambda, \Gamma^m \lambda) = \lambda^{\underline{\alpha}} \gamma^m_{\underline{\alpha}\underline{\beta}} \lambda^{\underline{\beta}} = 0$ as written at the beginning of the section.

Using the Fock space construction of spinor space, one recognizes in the algebra of p^a and n^a bases for operators in an exterior algebra corresponding to the wedge product by a one-form, and contraction by a vector. Spinors themselves are sums of p-forms of different degree; a *polyform* applied to the vaccum $|\Omega\rangle$.
Changing notation as

$$p^a \to \frac{1}{\sqrt{2}} \gamma^a, \qquad n^a \to \frac{1}{\sqrt{2}} \gamma_a \tag{2.48}$$

and appropriately scaling the spinor components, a generic Dirac spinor is written as

$$\zeta = \left(\zeta^{(0)} + \zeta^{(1)} + \zeta^{(2)} + \zeta^{(3)} + \zeta^{(4)} + \zeta^{(5)}\right)|\Omega\rangle \tag{2.49}$$

where

$$\zeta^{(0)} = \zeta^{+}, \quad \zeta^{(1)} = \zeta_a \gamma^a, \quad \zeta^{(2)} = \frac{1}{2!}\zeta_{ab}\gamma^a \wedge \gamma^b, \quad \zeta^{(3)} = \frac{1}{3!}\zeta_{abc}\gamma^a \wedge \gamma^b \wedge \gamma^c \tag{2.50}$$

$$\zeta^{(4)} = \frac{1}{4!}\zeta_{abcd}\gamma^a \wedge \gamma^b \wedge \gamma^c \wedge \gamma^d, \quad \zeta^{(5)} = \frac{1}{5!}\zeta_{abcde}\gamma^a \wedge \gamma^b \wedge \gamma^c \wedge \gamma^d \wedge \gamma^e \tag{2.51}$$

Chiral spinors, $\zeta^{\underline{\alpha}}$ are made of even p-forms,

$$\zeta^{(+)} = \left(\zeta^{(0)} + \zeta^{(2)} + \zeta^{(4)}\right)|\Omega\rangle \tag{2.52}$$

while antichiral spinors $\zeta_{\underline{\alpha}}$ are made of odd p-forms,

$$\zeta^{(-)} = \left(\zeta^{(1)} + \zeta^{(3)} + \zeta^{(5)}\right)|\Omega\rangle \tag{2.53}$$

Then, contractions between two arbitrary spinors are defined with the help of the *measure* given by

$$\langle\Omega|\gamma^a \wedge \gamma^b \wedge \gamma^c \wedge \gamma^d \wedge \gamma^e|\Omega\rangle \equiv \langle\gamma^a \wedge \gamma^b \wedge \gamma^c \wedge \gamma^d \wedge \gamma^e\rangle = \varepsilon^{abcde} \tag{2.54}$$

and $\left\langle \chi^{(p)} \right\rangle = 0$ for any p-form with $p \neq 5$.

From this it is easy to see that spinor contractions will be non-vanishing only between chiral and antichiral spinors. Contact with previous notation for the bilinear product (χ, ζ) is obtained by means of the *reversion* linear operator [29]

$$R\left[\zeta^{(p)}|\Omega\rangle\right] = (-1)^{p(p-1)/2}\langle\Omega|\zeta^{(p)} \tag{2.55}$$

Then, for chiral χ and antichiral ζ,

$$(\chi, \zeta) = \chi^{\underline{\alpha}}\zeta_{\underline{\alpha}} = R[\chi]\zeta \tag{2.56}$$

These formulas make it simple to translate between covariant and $U(5)$ notation using exterior algebra. For example, the previous contraction is expanded as

$$\chi^{\underline{\alpha}}\zeta_{\underline{\alpha}} = \left\langle \chi^{(0)} \wedge \zeta^{(5)} - \chi^{(2)} \wedge \zeta^{(3)} + \chi^{(4)} \wedge \zeta^{(1)} \right\rangle = \chi^{+}\zeta^{-} - \frac{1}{2}\chi_{ab}\zeta^{ab} + \chi^a\zeta_a \tag{2.57}$$

where $\zeta^{-} = \frac{1}{5!}\varepsilon^{abcde}\zeta_{abcde}$, $\zeta^{ab} = \frac{1}{3!}\varepsilon^{abcde}\zeta_{cde}$ and $\chi^a = \frac{1}{4!}\varepsilon^{abcde}\chi_{bcde}$.

To calculate expressions like $B_1^m(\chi,\xi)$ with χ and ξ both chiral, it is better to contract first with some vector v_m; it is easy to see that

$$v_m B_1^m(\chi,\xi) = v_m \chi^{\underline{\alpha}} \gamma^m_{\underline{\alpha}\underline{\beta}} \xi^{\underline{\beta}} = R[\chi] \wedge v_a \gamma^a \wedge \xi + R[\chi] \wedge v^a \gamma_a \xi, \tag{2.58}$$

so the action of a vector on a spinor through the Clifford algebra is split into the wedge product with a one-form and the contraction i_v with a $U(5)$ *vector*,

$$v^a \gamma_a \xi = 2 i_v \xi. \tag{2.59}$$

Terms in (2.58) give

$$R[\chi] \wedge v_a \gamma^a \wedge \xi = \left\langle \chi^{(0)} \wedge v_a \gamma^a \wedge \xi^{(4)} - \chi^{(2)} \wedge v_a \gamma^a \wedge \xi^{(2)} + \chi^{(4)} \wedge v_a \gamma^a \wedge \xi^{(0)} \right\rangle \tag{2.60}$$

$$= v_a \left(\chi^+ \xi^a - \frac{1}{4} \varepsilon^{abcde} \chi_{bc} \xi_{de} + \chi^a \xi^+ \right) \tag{2.61}$$

$$R[\chi] \wedge v^a \gamma_a \xi = \left\langle \chi^{(4)} \wedge v^a \gamma_a \xi^{(2)} - \chi^{(2)} \wedge v^a \gamma_a \xi^{(4)} \right\rangle \tag{2.62}$$

$$= 2 v^a (\chi_{ab} \xi^b + \chi^b \xi_{ab}) \tag{2.63}$$

These formulas allow one to write the pure spinor constraints in $U(5)$ notation,

$$\lambda^+ \lambda^a = \frac{1}{8} \varepsilon^{abcde} \lambda_{bc} \lambda_{de}, \qquad \lambda_{ab} \lambda^b = 0 \tag{2.64}$$

One can always solve this constraints by going to a patch in pure spinor space where $\lambda^+ \neq 0$; then, solving the first equation for λ^a the second one automatically holds. Here it becomes obvious that the number of independent components of a pure spinor in ten dimensions is eleven.

One can find analogous formulae for $B_1(\chi,\xi)$ with antichiral χ and ξ,

$$\chi_{\underline{\alpha}} \gamma^{a\underline{\alpha}\underline{\beta}} \xi_{\underline{\beta}} = -\chi_a \xi^{ab} - \chi^{ab} \xi_b \tag{2.65}$$

$$\chi_{\underline{\alpha}} \gamma_a^{\underline{\alpha}\underline{\beta}} \xi_{\underline{\beta}} = 2 \chi_a \xi^- - \frac{1}{2} \varepsilon_{abcde} \chi^{bc} \xi^{de} + 2 \chi^- \xi_a \tag{2.66}$$

2.2.2 The formalism

As mentioned in the introduction, to define a super-Poincaré covariant superstring, a conformal system consisting of a pure spinor worldsheet field $\lambda^{\underline{\alpha}}$ and its conjugate momentum $w_{\underline{\alpha}}$ needs to be added to the Siegel superstring describing superspace $x^m, p_{\underline{\alpha}}, \theta^{\underline{\alpha}}$. The pure spinor has zero conformal weight whereas the conjugate $w_{\underline{\alpha}}$ of opposite chirality has conformal weight equal to one.[7] This conventionally corresponds to the left-moving sector of the closed superstring. For type IIB, the corresponding right-moving pure spinor has the same chirality as the left-moving one, while for type IIA it has opposite chirality.

The pure spinor constraint generates a gauge transformation on $w_{\underline{\alpha}}$, which reads $\delta w_{\underline{\alpha}} = \Lambda_m(\gamma^m\lambda)_{\underline{\alpha}}$. In $U(5)$ notation,

$$\delta w^- = \Lambda_a\lambda^a, \qquad \delta w_a = \Lambda_a\lambda^+ - 2\Lambda^b\lambda_{ab}, \tag{2.67}$$

$$\delta w^{ab} = \frac{1}{2}\varepsilon^{abcde}\Lambda_c\lambda_{de} - 2\Lambda^{[a}\lambda^{b]} \tag{2.68}$$

Hence, in the patch where $\lambda^+ \neq 0$, one can choose the gauge parameters in such a way as to set $w_a = 0$. This way, it is immediate to see that $w_{\underline{\alpha}}$ has eleven independent components.

The pure spinor term of the worldsheet action in the left-moving sector is

$$\int d^2z\; w_{\underline{\alpha}}\bar{\partial}\lambda^{\underline{\alpha}}. \tag{2.69}$$

The computation of OPE's in this system is subtle due to the gauge invariance $\delta w_{\underline{\alpha}}$. It is possible to work in the patch where $\lambda^+ \neq 0$ and write an action that, altough breaks Lorentz invariance, is free on the worldsheet; then, OPE's are as usual. However, notice that expressions are well defined only when $w_{\underline{\alpha}}$ appears in gauge invariant combinations of fields. The relevant expressions are precisely the Lorentz current $N_{mn} = \frac{1}{2}w\gamma_{mn}\lambda$, the ghost current $J_\lambda = w_{\underline{\alpha}}\lambda^{\underline{\alpha}}$, and the stress tensor $T_\lambda = w_{\underline{\alpha}}\partial\lambda^{\underline{\alpha}}$. Their invariance are implied by pure spinor constraints and the antisymmetry of $\gamma^{mnp}_{\underline{\alpha}\underline{\beta}}$. Going to the $U(5)$ gauge one can compute all OPE's in a simple manner to find out, at the end, that all formulas can be rewritten in Lorentz covariant form. One gets

$$T_\lambda(y)T_\lambda(z) \longrightarrow \frac{11}{(y-z)^4} + \frac{2T_\lambda(z)}{(y-z)^2} + \frac{\partial T_\lambda(z)}{y-z} \tag{2.70}$$

$$N_{mn}(y)\lambda^{\underline{\alpha}}(z) \longrightarrow \frac{1}{2}(y-z)^{-1}(\gamma_{mn}\lambda)^{\underline{\alpha}} \tag{2.71}$$

among other relations.

[7] A classic review of the pure spinor formalism for the superstring is [30]. For a study of the pure spinor system as a *beta-gamma* system as well as certain of its mathematical aspects, see [31].

The first OPE shows that the central charge of the combined x^m ($c = 10$), $p_{\underline{\alpha}}, \theta^{\underline{\alpha}}$ ($c = -32$) and $w_{\underline{\alpha}}, \lambda^{\underline{\alpha}}$ systems, vanishes so the formalism does not have a conformal anomaly. OPE's for fermionic matter is, as usual,

$$p_{\underline{\alpha}}(y)\theta^{\underline{\beta}}(z) \longrightarrow (y-z)^{-1}\delta^{\underline{\beta}}_{\underline{\alpha}} \tag{2.72}$$

For open superstrings, the physical spectrum is correctly reproduced by the cohomology of the pure spinor BRST charge

$$Q = \int dz \lambda^{\underline{\alpha}} d_{\underline{\alpha}} \tag{2.73}$$

where

$$d_{\underline{\alpha}} = p_{\underline{\alpha}} - \frac{1}{2}(\gamma^m\theta)_{\underline{\alpha}}\partial x_m - \frac{1}{8}(\gamma^m\theta)_{\underline{\alpha}}(\theta\gamma_m\partial\theta) \tag{2.74}$$

is the current which generates supersymmetric derivatives in spacetime. Although it was initially not known how Q appeared as a result of BRST quantization of some gauge invariant theory, one solution to this problem was considered in [32].

The massless physical states of the open string are given by ghost number one (unintegrated) vertex operators with zero conformal weight at zero momentum,

$$V(k, a, \xi) = \lambda^{\underline{\alpha}} A_{\underline{\alpha}}(\theta) e^{ik\cdot x} \tag{2.75}$$

where a_m and $\xi^{\underline{\alpha}}$ are gluon and gluino polarizations, respectively; they appear in the $\theta^{\underline{\alpha}}$ expansion of A. The BRST condition implies then that $A_{\underline{\alpha}}$ satisfies the linearized ten-dimensional super-Yang-Mills superfield equations of motion with the correct gauge invariances coming from BRST exactness of trivial states. In a specific gauge, the first terms in the component expansion of $A_{\underline{\alpha}}$ are

$$A_{\underline{\alpha}}(\theta) = \frac{1}{2}a_m(\gamma^m\theta)_{\underline{\alpha}} - \frac{1}{3}(\xi\gamma_m\theta)(\gamma^m\theta)_{\underline{\alpha}} - \frac{1}{32}f_{mn}(\gamma_p\theta)_{\underline{\alpha}}(\theta\gamma^{mnp}\theta) + ... \tag{2.76}$$

where $f_{mn} = k_{[m}a_{n]}$. Thus, as a consequence of manifest supersymmetry in the pure spinor formalism, both R and NS sectors can be grouped into a single superfield vertex operator.

Similarly, one can define unintegrated vertex operators for type IIA and type IIB superstrings and define tree level scattering amplitudes. The vertex operators can be as usual obtained as left-right products of vertex operators for the open superstring. In the massless sector,

$$V = \lambda^{\underline{\alpha}} \tilde{\lambda}^{\underline{\beta}} A_{\underline{\alpha}\underline{\beta}}(\theta, \tilde{\theta}) e^{ik\cdot x}, \qquad \text{type IIB} \tag{2.77}$$

$$V = \lambda^{\underline{\alpha}} \tilde{\lambda}_{\underline{\beta}} A_{\underline{\alpha}}^{\ \underline{\beta}}(\theta, \tilde{\theta}) e^{ik\cdot x}, \qquad \text{type IIA} \tag{2.78}$$

The BRST charge for the right-moving sector also depends on which type II superstring one is dealt with, being $\tilde{Q} = \int d\bar{z} \tilde{\lambda}^{\underline{\alpha}} \tilde{d}_{\underline{\alpha}}$ for type IIB, and $\tilde{Q} = \int d\bar{z} \tilde{\lambda}_{\underline{\alpha}} \tilde{d}^{\underline{\alpha}}$ for type IIA. BRST physical conditions now hold for both left and right sectors independently, and it can be shown that massless states describe linearized type IIB or type IIA supergravity, all contained in a single *superfield* vertex operator. For future reference, the supersymmetry current is given here,

$$j_{\underline{\alpha}} = p_{\underline{\alpha}} + \frac{1}{2}(\gamma^m \theta)_{\underline{\alpha}} \partial x_m + \frac{1}{24}(\gamma^m \theta)_{\underline{\alpha}} (\theta \gamma_m \partial \theta). \tag{2.79}$$

As mentioned in the introduction, higher loop amplitudes in the RNS formalism require $3g - 3$ insertions of the reparametrization b-ghost for a genus g worldsheet. The problem when trying to define a composite b ghost that could play this role is the requirement for this operator to have ghost number -1. In the minimal pure spinor formalism, the only ghost number -1 field, $w_{\underline{\alpha}}$ appears exclusively, as discussed before, in gauge invariant combinations all of which has ghost number zero, so no composite b ghost is possible here. One way to overcome this problem is to introduce picture changing operators analogous to RNS ones, in the manner of [33]. Nevertheless, this method breaks Lorentz invariance at intermediate steps. To preserve this symmetry manifest, it is necessary to include extra worldsheet fields in the formalism. They must not affect the cohomology given by the original pure spinor formalism, so they can only appear as BRST trivial pairs which always decouple due to the usual BRST quartet mechanism [16]. The resulting *non-minimal* pure spinor formalism [34] opened the possibility for computing arbitrary g-loop superstring amplitudes in a manifest super-Poincaré invariant manner.

The bosonic non-minimal variables added to the minimal formalism are again a pure spinor, $\bar{\lambda}_{\underline{\alpha}}$, of opposite ten-dimensional chirality with respect to the minimal pure spinor $\lambda^{\underline{\alpha}}$, and the corresponding conjugate field $\bar{w}^{\underline{\alpha}}$. Each of these has also eleven independent components. To complete the quartet one has to introduce fermionic variables which match the number of new bosonic degrees of freedom. This is accomplished by introducing a constrained fermionic spinor $r_{\underline{\alpha}}$ obeying

$\bar{\lambda}_{\underline{\alpha}}\gamma_m^{\underline{\alpha}\underline{\beta}} r_{\underline{\beta}} = 0$, which has eleven independent components, and its conjugate field $s^{\underline{\alpha}}$, which also have eleven components due to the gauge invariance generated by the constraint on $r_{\underline{\alpha}}$.

Decoupling of these new degrees of freedom requires the following modification to the minimal BRST charge,

$$Q_{\text{n.m.}} = \int dz \Big(\lambda^{\underline{\alpha}} d_{\underline{\alpha}} + \bar{w}^{\underline{\alpha}} r_{\underline{\alpha}} \Big) \tag{2.80}$$

and similar considerations work for the right-moving sector of type II theories.

Now, allowing the appearance of inverse powers of $\bar{\lambda}_{\underline{\alpha}}\lambda^{\underline{\alpha}}$, it is possible to write a fermionic composite operator $b_{\text{n.m.}}$ of ghost number -1 satisfying

$$\{Q_{\text{n.m.}}, b_{\text{n.m.}}\} = T_{\text{n.m.}} \tag{2.81}$$

where $T_{\text{n.m.}} = T + \bar{w}^{\underline{\alpha}}\partial\bar{\lambda}_{\underline{\alpha}} - s^{\underline{\alpha}}\partial r_{\underline{\alpha}}$ is the non-minimal stress-energy tensor of the formalism; the minimal stress tensor is $T = -\frac{1}{2}\partial x^m \partial x_m - p_{\underline{\alpha}}\partial\theta^{\underline{\alpha}} + w_{\underline{\alpha}}\partial\lambda^{\underline{\alpha}}$.

The structure of b is given by a chain of operators starting from T,

$$(T, G^{\underline{\alpha}}, H^{[\underline{\alpha}\underline{\beta}]}, K^{[\underline{\alpha}\underline{\beta}\underline{\gamma}]}, L^{[\underline{\alpha}\underline{\beta}\underline{\gamma}\underline{\delta}]}) \tag{2.82}$$

satisfying the following relations:

$$\{Q, G^{\underline{\alpha}}\} = \lambda^{\underline{\alpha}} T, \quad [Q, H^{[\underline{\alpha}\underline{\beta}]}] = \lambda^{[\underline{\alpha}} G^{\underline{\beta}]}, \quad \{Q, K^{[\underline{\alpha}\underline{\beta}\underline{\gamma}]}\} = \lambda^{[\underline{\alpha}} H^{\underline{\beta}\underline{\gamma}]}, \tag{2.83}$$

$$[Q, L^{[\underline{\alpha}\underline{\beta}\underline{\gamma}\underline{\delta}]}] = \lambda^{[\underline{\alpha}} K^{\underline{\beta}\underline{\gamma}\underline{\delta}]}, \qquad \lambda^{[\underline{\alpha}} L^{\underline{\beta}\underline{\gamma}\underline{\delta}\underline{\kappa}]} = 0 \tag{2.84}$$

where Q is the minimal pure spinor BRST charge, and the explicit form of operators in the chain is [33]

$$\begin{aligned} G^{\underline{\alpha}} &= \frac{1}{2}\Pi^m (\gamma_m d)^{\underline{\alpha}} - \frac{1}{4} N_{mn} (\gamma^{mn}\partial\theta)^{\underline{\alpha}} - \frac{1}{4} J\partial\theta^{\underline{\alpha}}, \\ H^{[\underline{\alpha}\underline{\beta}]} &= \frac{1}{192}(\gamma^{mnk})^{\underline{\alpha}\underline{\beta}} (d\gamma_{mnk} d + 24 N_{mn}\Pi_k), \\ K^{[\underline{\alpha}\underline{\beta}\underline{\gamma}]} &= -\frac{1}{96}(\gamma_m d)^{[\underline{\alpha}} (\gamma^{mnk})^{\underline{\beta}\underline{\gamma}]} N_{nk}, \\ L^{[\underline{\alpha}\underline{\beta}\underline{\gamma}\underline{\delta}]} &= -\frac{1}{128}\frac{1}{4!}(\gamma_{mnp})^{[\underline{\alpha}\underline{\beta}} (\gamma^{pqr})^{\underline{\gamma}\underline{\delta}]} N^{mn} N_{qr}. \end{aligned} \tag{2.85}$$

with $\Pi^m = \partial x^m - \theta^{\underline{\alpha}}\gamma^m_{\underline{\alpha}\underline{\beta}}\partial\theta^{\underline{\beta}}$ the spacetime supersymmetric momentum.

The composite b field is given by

$$b_{\text{n.m.}} = s^{\underline{\alpha}}\partial\bar{\lambda}_{\underline{\alpha}} + \frac{\bar{\lambda}_{\underline{\alpha}} G^{\underline{\alpha}}}{\bar{\lambda}\lambda} - \frac{\bar{\lambda}_{\underline{\alpha}} r_{\underline{\beta}} H^{[\underline{\alpha}\underline{\beta}]}}{(\bar{\lambda}\lambda)^2} - \frac{\bar{\lambda}_{\underline{\alpha}} r_{\underline{\beta}} r_{\underline{\gamma}} K^{[\underline{\alpha}\underline{\beta}\underline{\gamma}]}}{(\bar{\lambda}\lambda)^3} + \frac{\bar{\lambda}_{\underline{\alpha}} r_{\underline{\beta}} r_{\underline{\gamma}} r_{\underline{\delta}} L^{[\underline{\alpha}\underline{\beta}\underline{\gamma}\underline{\delta}]}}{(\bar{\lambda}\lambda)} \tag{2.86}$$

where $\bar{\lambda}\lambda \equiv \bar{\lambda}_{\underline{\alpha}}\lambda^{\underline{\alpha}}$.

However, the presence of inverse powers of $\bar{\lambda}\lambda$ brings in further subtleties when computing amplitudes. As the number of loops increases in number, higher inverse powers of $\bar{\lambda}\lambda$ appears in the correlators through b ghosts insertions. In the path integral picture, integration over pure spinor zero modes has a measure factor $\int d^{11}\lambda d^{11}\bar{\lambda}$;thus, divergences can appear when $\lambda^{\underline{\alpha}} \to 0$. Since the path integral measure factor converges like $(\bar{\lambda}\lambda)^{11}$, the poles in the b ghost can cause divergences if the poles accumulate to order $(\bar{\lambda}\lambda)^{-11}$ or worse.

This restriction to poles of lower order than $(\bar{\lambda}\lambda)^{-11}$ can also be understood from BRST cohomology arguments since it happens that $\{Q_{\text{n.m.}}, \xi\} = 1$ where

$$\xi = \frac{\bar{\lambda}_{\underline{\alpha}}\theta^{\underline{\alpha}}}{\lambda^{\underline{\alpha}}\bar{\lambda}_{\underline{\alpha}} + r_{\underline{\alpha}}\theta^{\underline{\alpha}}} = \frac{\bar{\lambda}_{\underline{\alpha}}\theta^{\underline{\alpha}}}{\lambda^{\underline{\alpha}}\bar{\lambda}_{\underline{\alpha}}} + ... + \frac{(\bar{\lambda}_{\underline{\alpha}}\theta^{\underline{\alpha}})(r_{\underline{\alpha}}\theta^{\underline{\alpha}})^{10}}{(\lambda^{\underline{\alpha}}\bar{\lambda}_{\underline{\alpha}})^{11}}.$$

So allowing states with $(\bar{\lambda}\lambda)^{-11}$ dependence like ξ into the Hilbert space would trivialize the BRST cohomology since $Q(\xi V) = V$ whenever $QV = 0$.

2.3 Superstring Amplitudes

Superstring scattering amplitudes in the RNS and pure spinor formalism will be reviewed in this section. It is worth mentioning that there is also an amplitude prescription for the hybrid formulation of the superstring that will also be used in topological amplitude computations. This prescription, though, will be described when needed in Chapter 4.

2.3.1 RNS prescription

String scattering amplitude prescription for both formalisms (RNS and Pure Spinor) will be reviewed in this section. It is convenient to recall first what does it mean to compute a scattering amplitude in the bosonic string.

Given a string moving in spacetime, it was already explained that the classical action for that system is that of a surface (worldsheet) embedded in the target space (spacetime). A string amplitude *à la Polyakov* is defined as the functional integral over all geometries of the worldsheet, and over all embeddings of this surface in spacetime. Worldsheet geometry is encoded in a two-dimensional metric $g_{\hat{m}\hat{n}}(\sigma)$, while embeddings are given by D worldsheet scalars $x^m(\sigma)$ $(m = 1, ..., D)$, where D is the target space dimension, which is 26 for the bosonic string to have vanishing conformal anomaly.

In general, scattering amplitudes involving several external states will have cylinders associated to them where a particular state in the string Hilbert space is attached to one end of each cylinder, while the other end merge with the rest of the worldsheet. For on-shell scattering amplitudes, the cylinders are pulled off to infinity. Then, by a conformal transformation, the state at infinity can be brought to an arbitrary small circle surrounding the center of the complex plane. The state-operator correspondence allows one, thus, to replace this state by an insertion at the center on the plane. For closed (oriented) strings, all this means that string scattering amplitudes can be given by path integrals over oriented compact closed surfaces with punctures, and embeddings on target space. Operator insertions on the punctures are translated to vertex operator insertions $\mathcal{V}(k, \zeta)$ in the path integral language; here k^m is the momentum carried by the state, and ζ is its polarization (in case it has a tensor structure). Also, consideration of inequivalent worldsheet topologies express the scattering amplitude as a genus expansion. For n external states, all this translates to

$$\mathcal{A}_n = \sum_{genus} \int \mathfrak{D}g\mathfrak{D}x^m e^{-S_{string}[g,x^m]} \prod_{i=1}^{n} \mathcal{V}_i \tag{2.87}$$

Diffeomorphism of gauge invariance of the string action S_{string} allows simplification of this expression. Actually, the relevant information is carried by the conformal class of the worldsheet metric. So, Riemann surfaces, oriented surfaces endowed with conformal structures, play a crucial role in string theory. At genus $g \geq 2$, the set of all inequivalent Riemann surfaces Σ_g is a complex space parametrized by $3g - 3$ moduli m_s $(s = 1, ..., 3g - 3)$.

BRST quantization of the string can be used to show that the string path-integral without vertex operator insertions generates a complex $3g - 3$-form on the space of all metrics modulo Weyl transformations $\mathcal{J}$, which serves as a measure of integration [35]. This space is infinite dimensional; however, diffeomorphism invariance allows for the appearance of a measure in the finite dimensional quotient $\mathcal{M}_g = \mathcal{J}/\mathbb{D}$, where $\mathbb{D}$ is the group of orientation-preserving diffeomorphisms. $\mathcal{M}_g$ is precisely the moduli space of Riemann surfaces of genus g. This also extend to the case when vertex operator insertions are present in the path-integral.

After introducing diffeomorphism ghosts b and c, BRST quantization allows for a conformal gauge where the genus g (or g-loop) amplitude takes the form [36]

$$\mathcal{A}_{g,n} = \int_{\mathcal{M}_g} \left\langle \left| \prod_{i=1}^{3g-3} (\mu_i, b) \right|^2 \prod_{j=1}^{n} \mathcal{V}_j \right\rangle_g \tag{2.88}$$

where μ_i $(i = 1, ..., 3g-3)$ is a basis of Beltrami differentials, $(\mu_i, b) = \int dz \mu_i(z) b(z)$, left-right products are denoted by $|\ |^2$, and the $\mathcal{V}_j$ are given by integrated vertex operators $\mathcal{V}_j = \int d^2 z U_j(z)$. Also, the correlator stands for path-integration over worldsheet fields living on Riemann surfaces of genus g:

$$\langle ... \rangle = \int \mathfrak{D} x^m |\mathfrak{D} b \mathfrak{D} c|^2 e^{-S[x^m, b, c, \tilde{b}, \tilde{c}]} ... \tag{2.89}$$

In the RNS formalism, the situation is analogous, although much more complicated at higher genus. Recall that, for type II superstrings, the worldsheet has actually $N = (1,1)$ supergeometry. Then, scattering amplitudes will involve supersymmetric generalizations of the objects appearing in the bosonic string.

Instead of on the moduli space of Riemann surfaces $\mathcal{M}_g$, the RNS path-integral localizes on the supermoduli space of super-Riemann surfaces, $s\mathcal{M}_g$, a space parametrized by $3g-3$ commuting, and $2g-2$ anticommuting moduli. Then one has to consider each relevant object as a supermultiplet of $N = (1,1)$ $d = 2$ superspace; for example, there will be a superghost system which contains as components both the reparametrization and superconformal ghosts, (b, c) and (β, γ), respectively. A rigorous treatment of superstring perturbation theory would require to work in this framework [35]. Nevertheless, for the present purposes it suffices to follow the path where everything is reduced to the language of ordinary moduli space and Riemann surfaces.

Integration over supermoduli and absorption of bosonic ghost zero-modes turn out to be equivalent to the insertion of $2g-2$ picture-changing operators $Z = e^{\phi} G$ at arbitrary points on the Riemann surface [37].[8] On the other hand, recall that worldsheet fields ψ^m, β, γ are allowed to satisfy antiperiodic boundary conditions around spin fields. The same situation is encountered along non-contractible loops of a genus g $(g \geq 1)$ Riemann surface. At genus g one can define a canonical basis for non-contractible loops. These are the non-trivial homology cycles a_i, b_i

[8]Recall that ϕ is the negative energy chiral boson coming from bosonization of the (β, γ) system and G is the total worldsheet supercurrent of the RNS formalism.

$(i = 1, ..., g)$. So one must specify the boundary conditions, periodic or antiperiodic, satisfied by field configuration around these cycles. A spin structure on Σ_g is the information about this periodicity assignment. There are, hence, 2^{2g} spin structure at genus g. For each of them, there is a possible contribution to the scattering amplitude, and one must sum over all spin structures. The relative weight of different contributions is ambiguous in principle. However, it can be shown that requirements of modular invariance[9] and unitarity can fix them [38]. Moreover, the sum over spin structures is closely related to the GSO projection of RNS superstrings.

Therefore, the prescription for g-loop n-point scattering amplitudes in the RNS formalism is

$$\mathcal{A}_{g,n} = \int_{\mathcal{M}_g} \sum_{\boldsymbol{\alpha}} \epsilon_{\boldsymbol{\alpha}} \left\langle \left| \prod_{i=1}^{3g-3} (\mu_i, b) \right|^2 \left| \prod_{j}^{2g-2} Z(z_j) \right|^2 \prod_{k=1}^{n} \mathcal{V}_k \right\rangle_{g,\boldsymbol{\alpha}} \tag{2.90}$$

where, this time, the correlator means path-integration over RNS worldsheet fields,

$$\langle ... \rangle = \int \mathfrak{D}x^m |\mathfrak{D}\psi^m \mathfrak{D}b \mathfrak{D}c \mathfrak{D}\beta \mathfrak{D}\gamma|^2 e^{-S_{RNS}} ... \tag{2.91}$$

at given genus and for spin structure $\boldsymbol{\alpha}$. Here, $\epsilon_{\boldsymbol{\alpha}}$ are the spin structure weights for the supersymmetric string. Also, the sum of the pictures of all vertex operator insertions must be zero.

Separate expressions should be given for the contributions at genus 0 (the sphere) and 1 (the torus).

On the sphere, c has three zero modes (which together with the three zero modes of $\tilde{c}$ correspond to the six conformal Killing vectors of the sphere), while b has none (since the sphere has no moduli). For bosonic strings, this translates to the tree amplitude prescription

$$\mathcal{A}_{0,n} = \left\langle V_1(z_1) V_2(z_2) V_3(z_3) \prod_{i=4}^{n} \mathcal{V}_i \right\rangle_{tree} \tag{2.92}$$

where V_i are unintegrated vertex operators, and their positions z_1, z_2, z_3 are arbitrary, and can be set to any three specific values using Möbius transformations; the

[9]Modular transformations are reparametrizations which preserves the conformal gauge of the worldsheet metric, known also as large diffeomorphism, since they are not connected to the identity of the reparametrization group. Modular invariance provides crucial criteria for the construction of consistent superstring theories; they are also at the heart of the theory of Riemann surfaces discussed in the next section.

amplitude is independent of this choice. Also, at tree level there is no moduli so determinants of differential operators on the worldsheet are just constants which can be grouped into a proportionality constant.

The torus has one complex moduli, so b has one zero mode, and it has two (real) Killing vectors corresponding to translations, so c has also one zero-mode. The one-loop amplitude for bosonic strings is

$$\mathcal{A}_{1,n} = \int_{\mathcal{M}_1} \sum_{\alpha} \epsilon_{\alpha} \left\langle |(\mu, b)|^2 V_1(z_1) \prod_{i=2}^{n} \mathcal{V}_i \right\rangle_{1-loop,\alpha} \tag{2.93}$$

with arbitrary z_1 (which can be set to zero by translation).

2.3.2 Pure spinor prescription

To compute N-point open superstring scattering amplitudes at tree level the minimal pure spinor formalism is good enough. Both integrated $\int dzU$ and unintegrated V vertex operators have to be inserted into the correlator, and amplitudes mimic the bosonic string prescription

$$\mathcal{A}_{0,n} = \left\langle V_1(z_1) V_2(z_2) V_3(z_3) \prod_{i=4}^{n} \int d^2 z_i U_i(z_i) \right\rangle_{tree} \tag{2.94}$$

To perform computation of amplitudes one has to make the path integration over all zero- and non-zero modes of every worldsheet field in the formalism. This is equivalent to three calculations: in the first place, one has to compute determinants of differential operators $\bar{\partial}$ appearing in the worldsheet action; secondly, non-zero modes are integrated out using Wick's theorem; finally one has to perform integration over the zero modes.

Notice that, except for bosonic x^m, all remaining systems present in the pure spinor formalism consist of pairs having conformal weights 1 and 0. The number of zero modes of conjugate fields is a consequence of the Riemann-Roch theorem that will be discussed in the next section; the result is that on the sphere, conformal weight one fields have no zero modes, while each field of zero conformal weight has a single zero mode. Moreover, as mentioned in the previous subsection, determinants of differential operators give constants. After integrating out non-zero modes, amplitudes have the general form

$$\mathcal{A}_{0,n} = \prod_{i=4}^{n} \int d^2 z_i \left\langle \left| \lambda^{\underline{\alpha}} \lambda^{\underline{\beta}} \lambda^{\underline{\gamma}} \right|^2 f_{\underline{\alpha}\underline{\beta}\underline{\gamma}}(\theta, \tilde{\theta}; z_i) \right\rangle \tag{2.95}$$

The prescription for zero-mode integration was originally postulated as

$$\big\langle (\lambda\gamma^m\theta)(\lambda\gamma^n\theta)(\lambda\gamma^p\theta)(\theta\gamma_{mnp}\theta) \big\rangle = 1 \tag{2.96}$$

and a similar expression for the right-moving sector. It is convenient to rewrite this measure as

$$\left\langle \mathcal{T}_{((\underline{\alpha}\underline{\beta}\underline{\gamma}))[\underline{\kappa}_1\underline{\kappa}_2\underline{\kappa}_3\underline{\kappa}_4\underline{\kappa}_5]} \lambda^{\underline{\alpha}} \lambda^{\underline{\beta}} \lambda^{\underline{\gamma}} \theta^{\underline{\kappa}_1} \theta^{\underline{\kappa}_2} \theta^{\underline{\kappa}_3} \theta^{\underline{\kappa}_4} \theta^{\underline{\kappa}_5} \right\rangle \tag{2.97}$$

where $\mathcal{T}$ is symmetric and γ-traceless in its first three indices, and antisymmetric in the other five.[10]

This measure gets justified in the non-minimal pure spinor formalism. Here, to deal with integration over non-compact bosonic zero modes, some regularization is needed. A natural regulator would be $\exp(-\rho\bar{\lambda}\lambda)$, for some positive cut-off ρ. Actually, to maintain BRST invariance it is convenient to choose a regulator of the form $\mathcal{N} = exp\big(\{Q,\chi\}\big)$ for an appropriately chosen fermionic operator χ. Since $\mathcal{N}$ is of the form $1 + Q\mathcal{O}$, decoupling of BRST trivial states guarantee that the amplitude is independent of the choice of χ.

The simplest choice is $\chi = -\rho\bar{\lambda}_{\underline{\alpha}}\theta^{\underline{\alpha}}$; thus

$$\mathcal{N} = \exp\Big(-\rho\big(\bar{\lambda}_{\underline{\alpha}}\lambda^{\underline{\alpha}} + r_{\underline{\alpha}}\theta^{\underline{\alpha}}\big)\Big) \tag{2.98}$$

Actually, the limit $\rho \to \infty$ allows this to make contact with the minimal pure spinor prescription with picture changing operators mentioned before. From now on, the limit $\rho \to 1$ is taken.

The non-minimal prescription for tree amplitudes is therefore

$$\mathcal{A}_{0,n} = \left\langle |\mathcal{N}(y)|^2\, V_1(z_1)V_2(z_2)V_3(z_3) \prod_{i=4}^{n} \int d^2 z_i U_i(z_i) \right\rangle_{tree} \tag{2.99}$$

with the regulator $\mathcal{N}(y)$ inserted anywhere on the worldsheet.

Here, after integrating out non-zero modes the amplitude takes the general

[10] γ-traceless means zero after contraction with $\gamma_m^{\underline{\alpha}\underline{\beta}}$, that is, $\gamma_m^{\underline{\alpha}\underline{\beta}} \mathcal{T}_{((\underline{\alpha}\underline{\beta}\underline{\gamma}))\ldots} = 0$.

form

$$\mathcal{A}_{0,n} = \prod_{i=4}^{n} \int d^2 z_i \left| \int [d\lambda][d\bar{\lambda}][dr] d^{16}\theta \right|^2 \left| \mathcal{N} \lambda^{\underline{\alpha}} \lambda^{\underline{\beta}} \lambda^{\underline{\gamma}} \right|^2 f_{\underline{\alpha}\underline{\beta}\underline{\gamma}}(\theta, \tilde{\theta}; z_i) \tag{2.100}$$

The measure for $\theta^{\underline{\alpha}}$ integration has the usual form; however measures for the remaining variables have a more complicated form due to their constrained nature.

The correct measures are [34, 39]

$$[d\lambda]\lambda^{\underline{\alpha}}\lambda^{\underline{\beta}}\lambda^{\underline{\gamma}}\mathcal{T}_{((\underline{\alpha}\underline{\beta}\underline{\gamma}))[\underline{\kappa}_1...\underline{\kappa}_5]} = \varepsilon_{\underline{\kappa}_1...\underline{\kappa}_{16}} d\lambda^{\underline{\kappa}_6}...d\lambda^{\underline{\kappa}_{16}} \tag{2.101}$$

$$[d\bar{\lambda}]\bar{\lambda}_{\underline{\alpha}}\bar{\lambda}_{\underline{\beta}}\bar{\lambda}_{\underline{\gamma}} = (\varepsilon\mathcal{T})^{\underline{\kappa}_1...\underline{\kappa}_{11}}_{\underline{\alpha}\underline{\beta}\underline{\gamma}} d\bar{\lambda}_{\underline{\kappa}_1}...d\bar{\lambda}_{\underline{\kappa}_{11}} \tag{2.102}$$

$$[dr]\mathcal{T}_{((\underline{\alpha}\underline{\beta}\underline{\gamma}))[\underline{\kappa}_1...\underline{\kappa}_5]} = \bar{\lambda}_{\underline{\alpha}}\bar{\lambda}_{\underline{\beta}}\bar{\lambda}_{\underline{\gamma}}\varepsilon_{\underline{\kappa}_1...\underline{\kappa}_{16}}\partial_{r_{\kappa_6}}...\partial_{r_{\kappa_{16}}} \tag{2.103}$$

with $\mathcal{T}$ introduced above; when inserted in the tree amplitude (2.100) they give a result that agrees with the minimal computation (2.95) provided (2.96) holds.

At one-loop, a single insertion of the composite b ghost is needed, so here one needs to use the non-minimal pure spinor formalism. The prescription is similar to the RNS case without necessity of summing over spin structures,

$$\mathcal{A}_{1,n} = \int_{\mathcal{M}_1} \left\langle |\mathcal{N}(y)(\mu, b)|^2 V_1(z_1) \prod_{i=2}^{n} \int d^2 z_i U_i(z_i) \right\rangle_{1-loop} \tag{2.104}$$

Starting from genus one, worldsheet fields of conformal field $h = 1$ have zero modes, so to regularize bosonic zero-mode integration it is necessary to write a regulator which includes them; it is convenient to choose

$$\mathcal{N} = \exp\left(-\bar{\lambda}_{\underline{\alpha}}\lambda^{\underline{\alpha}} - r_{\underline{\alpha}}\theta^{\underline{\alpha}} - \left[\frac{1}{2}N_{mn}\bar{N}^{mn} + J\bar{J} + \frac{1}{4}S_{mn}d\gamma^{mn}\lambda + S\lambda^{\underline{\alpha}}d_{\underline{\alpha}}\right]\right) \tag{2.105}$$

where $\bar{N}_{mn}$, $\bar{J}$, S_{mn} and S are gauge invariant objects constructed out of non-minimal fields,

$$\bar{N}_{mn} = \frac{1}{2}(\bar{w}\gamma_{mn}\bar{\lambda} - s\gamma_{mn}r), \qquad \bar{J} = \bar{w}^{\underline{\alpha}}\bar{\lambda}_{\underline{\alpha}}, \tag{2.106}$$

$$S_{mn} = \frac{1}{2}s\gamma_{mn}\bar{\lambda}, \qquad S = s^{\underline{\alpha}}\bar{\lambda}_{\underline{\alpha}} \tag{2.107}$$

Here it is necessary to include the correct measures for the new zero modes of $w_{\underline{\alpha}}$, $\bar{w}^{\underline{\alpha}}$, $d_{\underline{\alpha}}$ and $s^{\underline{\alpha}}$,

$$[dw] = \lambda^{\underline{\alpha}}\lambda^{\underline{\beta}}\lambda^{\underline{\gamma}}(\varepsilon\mathcal{T})^{\underline{\kappa}_1\ldots\underline{\kappa}_{11}}_{\underline{\alpha}\underline{\beta}\underline{\gamma}}dw_{\underline{\kappa}_1}\ldots dw_{\underline{\kappa}_{11}} \tag{2.108}$$

$$[d\bar{w}]\mathcal{T}_{((\underline{\alpha}\underline{\beta}\underline{\gamma})))[\underline{\kappa}_1\ldots\underline{\kappa}_5]} = \bar{\lambda}_{\underline{\alpha}}\bar{\lambda}_{\underline{\beta}}\bar{\lambda}_{\underline{\gamma}}\varepsilon_{\underline{\kappa}_1\ldots\underline{\kappa}_{16}}d\bar{w}^{\underline{\kappa}_6}\ldots d\bar{w}^{\underline{\kappa}_{16}} \tag{2.109}$$

$$[ds] = \frac{\lambda^{\underline{\alpha}}\lambda^{\underline{\beta}}\lambda^{\underline{\gamma}}}{(\bar{\lambda}\lambda)^3}(\varepsilon\mathcal{T})^{\underline{\kappa}_1\ldots\underline{\kappa}_{11}}_{\underline{\alpha}\underline{\beta}\underline{\gamma}}\partial_{s^{\underline{\kappa}_1}}\ldots\partial_{s^{\underline{\kappa}_{11}}} \tag{2.110}$$

The g-loop N-point closed string amplitude prescription in the nonminimal pure spinor formalism is

$$\mathcal{A} = \int_{\mathcal{M}}\left\langle\left|\mathcal{N}(y)\prod_{i=1}^{3g-3}b(\mu_i)\right|^2\prod_{i=1}^{N}\int d^2z_iU_i(z_i)\right\rangle \tag{2.111}$$

where $\mathcal{N}$ is chosen as

$$\mathcal{N} = \exp\left(-\bar{\lambda}_{\boldsymbol{\alpha}}\lambda^{\boldsymbol{\alpha}} - r_{\boldsymbol{\alpha}}\theta^{\boldsymbol{\alpha}} - \sum_{\Xi=1}^{g}\left[\frac{1}{2}N^{\Xi}_{mn}\bar{N}^{mn\Xi} + J^{\Xi}\bar{J}^{\Xi} + \frac{1}{4}S^{\Xi}_{mn}d^{\Xi}\gamma^{mn}\lambda + S^{\Xi}\lambda^{\boldsymbol{\alpha}}d^{\Xi}_{\boldsymbol{\alpha}}\right]\right) \tag{2.112}$$

In a genus g worldsheet, fields of conformal weight 1 have g zero modes distinguished by the index $\Xi = 1,\ldots,g$. Fields of zero conformal weight have only one zero mode. The measures have the same form as before, with a copy for each zero mode of conformal weight one fields. Although the regulator seems to be appropriate, the potential appearance of poles that accumulate to order $(\bar{\lambda}\lambda)^{-11}$ is problematic, and would demand a more complicated regularization as in [40].

2.4 Bosonization on higher genus Riemann surfaces

In all formulations of the superstring first order systems are introduced. These can be either fermionic (like fermionic matter fields or reparametrization ghosts), or bosonic (like superconformal ghosts or pure spinors). Non-chiral systems that live on a genus g compact Riemann surface Σ_g have actions of the form

$$S = \int_{\Sigma_g}\left[b\bar{\partial}c + \tilde{b}\partial\tilde{c}\right] \tag{2.113}$$

As seen in the previous section, evaluation of scattering amplitudes requires computing correlation functions of these fields on Σ_g. A very convenient way to derive these correlators uses bosonization techniques, which can be better understood after introducing some relevant mathematical concepts. It turns out that the main characterization of a world-sheet field in string theory is given by the line bundle of which it is a holomorphic section. A fermionic field b is defined as a section of a holomorphic line bundle ξ over Σ_g.

A fiber bundle can be though of as a generalization of a product space $M \times F$ to non-trivial topology. M plays the role of a base manifold while an entire copy of the space F is attached to each point of M. The bundle is not equivalent to $M \times V$ but looks like a product space in every open neighborhood that belongs to some covering of M. The global information of the bundle is encoded in *gluing functions* that relates different local descriptions of a fiber in the intersection of two neighborhoods. When the fiber F is the complex line $\mathbb{C}$, one is talking about line bundles; when gluing functions are holomorphic, about holomorphic line bundles. In string applications one takes, of course, $M = \Sigma_g$. An important line bundle on Σ_g is the cotangent (canonical) bundle K whose fiber at a point $P \in \sigma_g$ is the space of holomorphic one-forms at P.

Recall that objects integrated over the Riemann surface must be $(1,1)$-forms, so the field c must be defined as a section of $K \times \xi^{-1}$.[11] Since $\bar{\partial}$ is the Cauchy-Riemann operator transforming any $(q,0)$-form into a $(q,1)$-form, the integrand has the correct nature. In general, the corresponding differential operator acting on sections of ξ will be denoted by $\bar{\partial}_\xi$; its adjoint is written as $\bar{\partial}^\dagger_\xi$.

The Riemann-Roch theorem says about the relation between the number of zero-modes of these differential operators, $\bar{\partial}_\xi$ and $\bar{\partial}^\dagger_\xi$. They are the number of zero modes of b and c, respectively.

$$n(\xi) - n(K \otimes \xi^{-1}) = \deg \xi + 1 - g \tag{2.114}$$

where deg ¸ is the degree of the line bundle ξ, defined as the first Chern class of ξ.

Take, for example, the system of reparametrization ghosts, where b has conformal weight 2 and c has conformal weight -1. The line bundle which describes b is given by the tensor product $\mathcal{L}_b = K \otimes K$. The degree of K is $2g-2$, hence, that of $\mathcal{L}_b$ is $4g-4$. The Riemann-Roch theorem states that $n(b) - n(c) = 3g-3$. More generally, a fermionic (b,c) system of conformal weights $(\lambda, 1-\lambda)$ with integer λ, has $\xi = \mathcal{L}_b = K^{\otimes\lambda}$, the λ-th tensor product of the canonical line bundle with itself.

[11]The tensor product bundle $\xi_1 \otimes \xi_2$ over M is defined as the bundle whose fibers at each point P are tensor products of the fibers of ξ_1 and ξ_2 at P.

Thus,

$$n(b) - n(c) = (2\lambda - 1)(g - 1) \tag{2.115}$$

In the case $\lambda = 1$, as with fermionic matter systems $(p_{\underline{\alpha}}, \theta^{\underline{\alpha}})$ of the pure spinor formalism in a flat background, $\mathcal{L}_b = K$, and c is just a section of the trivial line bundle over Σ_g. Thus, c has 1 zero mode (the constant function) and b has g zero modes, which can be specified by a basis $\{\omega_i,\ i = 1, ..., g\}$ of holomorphic 1-differentials on the Riemann surface. These objects are chosen to satisfy

$$\int_{a_i} \omega_j = \delta_{ij} \tag{2.116}$$

around the g homology a-cycles of Σ_g. The integrals around b-cycles define the *period matrix*

$$\tau_{ij} \equiv \int_{b_i} \omega_j. \tag{2.117}$$

which is symmetric and has positive definite imaginary part.

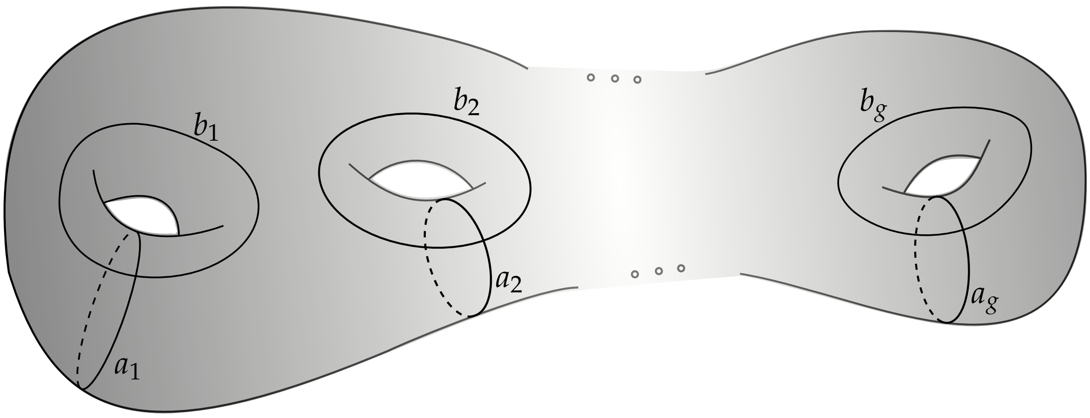

Figure 2.1: Basis of homology cycles on a genus g Riemann surface

Bosonization states that a generic (b, c) system of conformal weights $(\lambda, 1 - \lambda)$ (together with their right-moving partners) is equivalent to the theory of a real compact scalar whose action is determined, apart from the kinetic free term, by symmetry requirements and the fact that it must include information about λ and the spin structures when λ is half-integer [41].

Assuming first that there are no zero-modes for $\bar{\partial}^{\dagger}_{\xi}$ (which is the same as the absence of c zero-modes), the identification of correlation functions in both theories is

$$\left\langle \prod_{i=1}^{p} \|b(P_i)\|^2 \right\rangle_{fermion} = \left\langle \prod_{i=1}^{p} e^{i\varphi(P_i)} \right\rangle_{boson} \tag{2.118}$$

where $p = (2\lambda - 1)(g - 1)$ is the number of independent zero-modes of b; p insertions of b is the minimum necessary in order to have non-vanishing fermionic integration over zero-modes of b. P_i, $(i = 1, ..., p)$ denote points on Σ_g. In the left hand side, the norm is defined as $\|b(P)\|^2 = \rho^{-\lambda}(P)b(P)\tilde{b}(P)$, where $\rho(P)$ is the conformally flat metric $ds^2 = \rho dz d\bar{z}$.

The fermion correlator gives

$$\left\langle \prod_{i=1}^{p} \|b(P_i)\|^2 \right\rangle_{fermion} = \frac{\det' \bar{\partial}_\xi^\dagger \bar{\partial}_\xi}{\det(u_i, u_j)} \|\det(u_i(P_j))\|^2 \tag{2.119}$$

Here $\bar{\partial}_\xi^\dagger \bar{\partial}_\xi$ is the laplacian acting on sections of the holomorphic line bundle ξ whose zero modes are excluded from the determinant $\det'$, and (u_i, u_j) is the inner product defined on the space of zero modes.

To compute the bosonic correlator it is better to split the scalar field into a topologically nontrivial piece which obeys the equations of motion, and a fluctuation which is topologically trivial, $\varphi = \varphi_{inst} + \varphi_{fluct}$. Details of the computation can be found in [41].

The first piece give the instanton contribution to the correlator

$$Z_{inst} = (\det(\mathrm{Im}\tau))^{1/2} \mathcal{N}(z), \tag{2.120}$$

where the function $\mathcal{N}$ will be shortly described; its argument in (2.120) depends on the line bundle ξ associated to b, and on the positions of $\exp(i\varphi)$ insertions in the correlation function.

Digression on divisors

Correlation functions of fields in the (b, c) system turns out to be meromorphic functions of the positions of its insertions,

$$A(z_1, ..., w_M) = \left\langle \prod_{i=1}^{M+p} b(z_i) \prod_{j=1}^{M} c(w_j) \right\rangle \tag{2.121}$$

From the conformal behavior of b and c one gets,

$$A(z'_1, ..., w'_M) = \prod_{i=1}^{M+p} \left(\frac{dz_i}{dz'_i}\right)^{\lambda} \prod_{j=1}^{M} \left(\frac{dw_j}{dw'_j}\right)^{1-\lambda} A(z_1, ..., w_M) \tag{2.122}$$

As a function of, say, z_1, A has zeros at $z_i \neq z_1$ and poles at w_i; this is a consequence of the OPE's between b and c. $A(z_1)$ is called a meromorphic λ-differential and it is determined by the position of all its zeroes and poles. These positions define the *divisor* as a formal sum

$$D_A = \sum_{i=2}^{M+p} z_i - \sum_{j=1}^{M} w_j \tag{2.123}$$

The problem of determining the correlation function is reformulated as the problem of characterizing all possible divisors of a meromorphic λ-differential [42].

To a given line bundle ξ one can associate a divisor to each of its meromorphic sections. Since the quotient of two λ-differentials is a meromorphic function, the difference of their divisors is just the divisor of a meromorphic function. By defining the group of divisor classes as the set of all divisors modulo the divisors of meromorphic functions, one gets a one-to-one correspondence between inequivalent line bundles and divisor classes.

The degree of D_A is defined as the number of zeros minus the number of poles, including multiplicities. It is a known fact that, for a compact Σ_g, the degree of D_A for A a meromorphic function is zero. This means that all divisors associated to a line bundle have the same degree, which turns out to be the same line bundle degree defined earlier. The line bundle of λ-differentials for integer λ, has a divisor class $[D_\lambda] = \lambda[D_1]$ where $[D_1]$ is the divisor class for the canonical line bundle K (often denoted by the same letter K).

An important case is that of half-integer λ. Here, as already mentioned, one has to distinguish between different spin structures on the Riemann surface. For each spin structure $\boldsymbol{\alpha}$ there is a spin bundle $S_{\boldsymbol{\alpha}}$ such that $S_{\boldsymbol{\alpha}} \otimes S_{\boldsymbol{\alpha}} = K$. Then, the corresponding divisor class satisfies $2[D_{\boldsymbol{\alpha}}] = K$, and it has degree $g-1$. For general half-integer λ, $[D_{\lambda,\boldsymbol{\alpha}}] = (\lambda - \frac{1}{2})K + [D_{\boldsymbol{\alpha}}]$.

Given a divisor of zero degree $D = A - B$, where both A and B are positive divisors, one can associate to it a complex g-dimensional vector z using the so called Jacobi map

$$\mathbb{I}: D \longrightarrow z_i = \int_B^A \omega_i \tag{2.124}$$

Due to (2.116), (2.117) the image is actually an element of the complex torus $J(\Sigma_g) = \mathbb{C}/(\mathbb{Z}^g + \tau\mathbb{Z}^g)$, which is known as the jacobian variety of Σ_g. Abel's theorem states that two divisors are mapped onto the same point in the jacobian variety if and only if they are equivalent. This way, one gets a precise correspondence between degree zero line bundles and elements of the jacobian variety $J(\Sigma)$.

The argument of the function $\mathcal{N}$ in (2.120) is precisely a specific element of $J(\Sigma_g)$. The function itself is defined as[12]

$$\mathcal{N}(z) = e^{-2\pi \mathrm{Im} z\cdot(\mathrm{Im}\tau)^{-1}\cdot \mathrm{Im} z}|\Theta(z|\tau)|^2 \tag{2.125}$$

where the Riemann theta function $\Theta(z|\tau)$ is

$$\Theta(z|\tau) = \sum_{n\in\mathbb{Z}^g} \exp(i\pi n_i \tau_{ij} n_j + 2\pi i n_i z_i). \tag{2.126}$$

Given a point P_0 on Σ, consider the function

$$f(P) = \Theta\left(z + \int_{P_0}^P \omega \middle| \tau\right). \tag{2.127}$$

The Riemann Vanishing Theorem [43] states that $f(P)$ either vanishes identically for all P on Σ, or it has exactly g zeroes $\{P_i, i = 1, ..., g\}$ on Σ_g. In the latter case, there exists a vector Δ which depends only on the reference point P_0 so that

$$z + \sum_{i=1}^g \int_{P_0}^{P_i} \omega = \Delta \tag{2.128}$$

Δ is known as the *vector of Riemann constants*; the divisor class associated to it, known as the Riemann class (here denoted also by Δ). This class has degree $g-1$, and it satisfies $2\Delta = K$, so it corresponds to one of the spin bundles, denoted by S_0.

With all these ingredients, it is possible to write the specific value of the argu-

[12]Given two vectors x, y, and a $g\times g$ matrix A, then $x\cdot A\cdot y$ is understood as $x^{tr}Ay$.

ment in $\mathcal{N}$.[13] It is

$$z = \mathbb{I}\left[\xi \otimes \mathcal{O}(-D_{\text{ins}}) \otimes \Delta^{-1}\right] \tag{2.129}$$

where the Jacobi map is actually applied to the divisor associated to the line bundle written here. $\mathcal{O}(-D_{\text{ins}})$ is the line bundle whose divisor class contains $-D_{\text{ins}}$, the negative of the divisor of insertion points on Σ_g, $D_{\text{ins}} = \sum_{i=1}^{p} P_i$. The third factor in the tensor product could actually be any even spin bundle L_0, with $L_0 \otimes L_0 = K$; $L_0 = \Delta$ is a convenient choice, the final results being independent of it.

The contribution to bosonic correlation functions coming from fluctuations φ_{fluct} is obtained just by gaussian integration and gives, after appropriate regularization,

$$Z_{fluct} = \left(\frac{\det'\bar{\partial}^{\dagger}\bar{\partial}}{A_{\Sigma}}\right)^{-1/2} \prod_{i,j=1}^{p} G(P_i, P_j) \tag{2.130}$$

where $\bar{\partial}^{\dagger}\bar{\partial}$ is the usual wave operator for a scalar in two dimensions, A is the of Σ_g, and $G(P_i, P_j)$ is the Green's function for $\bar{\partial}^{\dagger}\bar{\partial}$, whose explicit form depends on the metric chosen on the Riemann surface.

Thus, the bosonization formula gives

$$\frac{\det'\bar{\partial}_{\xi}^{\dagger}\bar{\partial}_{\xi}}{\det(u_i, u_j)} \|\det(u_i(P_j))\|^2 = \left(\frac{\det'\bar{\partial}^{\dagger}\bar{\partial}}{\det(\text{Im}\tau) A_{\Sigma}}\right)^{-1/2} \mathcal{N}(\xi \otimes \mathcal{O}(-D_{\text{ins}}) \otimes \Delta^{-1}) \times \prod_{i,j=1}^{p} G(P_i, P_j) \tag{2.131}$$

Holomorphic factorization

It is convenient to get bosonization formulas for just the chiral sector of a given $(b, c, \tilde{b}, \tilde{c})$ system. Nonetheless, a theory of a chiral boson which reproduces the correlators in the fermionic side is not known. However, if one manages to eliminate all appearances of the conformal metric on both sides of the bosonization formula, then one can take the holomorphic square root of the resulting expression. In order to be able to do that, it is necessary that all anholomorphic factors of the form $\exp[k\text{Im}z{\cdot}(\text{Im}\tau)^{-1}{\cdot}\text{Im}z]$ cancel out in the formula.

A convenient choice of metric is the so called Arakelov metric [44], where the

[13] The specific form of the bundle is determined from modular invariance and consistency with the fermionic field theory side.

factor $\rho(P)$ takes the form

$$\rho(P) = |\sigma(P)|^{4/g} \exp\left[\frac{4\pi}{g(g-1)}\mathrm{Im}\Delta^P \cdot (\mathrm{Im}\tau)^{-1} \cdot \mathrm{Im}\Delta^P\right]. \tag{2.132}$$

Here, $\Delta^P = \int_{(g-1)P}^{\Delta} \omega$, and $\sigma(P)$ is a $\frac{g}{2}$-differential with no zeroes or poles [42].

With this metric, for points $P_i \neq P_j$ the Arakelov Green's function is

$$G(P_i, P_j)^2 = \rho(P_i)^{1/2}\rho(P_j)^{1/2} F(P_i, P_j) \tag{2.133}$$

where $F(P_i, P_j)$ is

$$F(P_i, P_j) = \exp\left[-2\pi \mathrm{Im}\int_{P_j}^{P_i} \omega \cdot (\mathrm{Im}\tau)^{-1} \cdot \mathrm{Im}\int_{P_j}^{P_i} \omega\right] |E(P_i, P_j)|^2 \tag{2.134}$$

In the last formula, $E(P_i, P_j)$ is the *prime form*: a holomorphic $-\frac{1}{2}$-differential in each of its arguments, which, geometrically, functions as a generalization of $z_{P_i} - z_{P_j}$ on the Riemann sphere.

For coincident points, the regulated Green's function for the Arakelov metric is just equal to one.

With all these formulas one can get holomorphic factorization for the case $\lambda > 1$ with $\lambda \in \mathbb{Z}$. In these cases, $\xi = \mathcal{L}_{b,\lambda} \equiv K^{\otimes \lambda}$. Then, the point in the jacobian variety is[14]

$$z = \mathbb{I}\left[K^{\otimes\lambda} \otimes \mathcal{O}(-D_{\mathrm{ins}}) \otimes \Delta^{-1}\right] = -\sum_{i=1}^{p} P_i + (2\lambda - 1)\Delta \tag{2.135}$$

Substituting all necessary definitions into the bosonization formula (2.4) it is easy to check that all anholomorphic factors cancel giving rise to the nice formula

$$\frac{\det' \bar{\partial}^{\dagger}_{\mathcal{L}_{b,\lambda}} \bar{\partial}_{\mathcal{L}_{b,\lambda}}}{\det(u_i, u_j)} |\det(u_i(P_j))|^2 = |Z_1|^{-1} |\Theta(z|\tau)|^2 \prod_{i<j}^{p} |E(P_i, P_j)|^2 \prod_{i=1}^{p} |\sigma(P_i)|^{2(2\lambda-1)} \tag{2.136}$$

where Z_1 can be considered as the partition function of the chiral scalar field, satisfying

$$|Z_1|^{-1} = \left(\frac{\det' \bar{\partial}^{\dagger} \bar{\partial}}{\det(\mathrm{Im}\tau) A_{\Sigma}}\right)^{-1/2} \tag{2.137}$$

[14] For brevity, the image of a divisor D under the Jacobi map is denoted by the same symbol D.

So, given a chiral system (b, c) of conformal weights $(\lambda, 1-\lambda)$ the correlation function

$$\langle b(P_1)...b(P_p)\rangle = \int \mathfrak{D}b\mathfrak{D}c\, b(P_1)...b(P_p)e^{-S[b,c]} \equiv Z_\lambda \det(u_i(P_j)) \tag{2.138}$$

is given by

$$Z_\lambda \det(u_i(P_j)) = Z_1^{-1/2}\Theta\Big(-\sum_{i=1}^{p} P_i + (2\lambda-1)\Delta\Big|\tau\Big)\prod_{i<j}^{p} E(P_i, P_j)\prod_{i=1}^{p}\sigma(P_i)^{2\lambda-1} \tag{2.139}$$

Bosonization formula for $\xi = K$ (that is, $\lambda = 1$) is slightly different because one has to take into account the constant zero-mode of c (denoted simply by 1) which requires at least one insertion in the correlator. The analogue to (2.4) is

$$\frac{\det'\bar{\partial}_K^\dagger\bar{\partial}_K}{\det(\omega_i,\omega_j)(1,1)}\|\det(\omega_i(P_j))\|^2 = |Z_1|^{-1}\mathcal{N}(\mathcal{O}(-D_{\text{ins}})\otimes\Delta) \tag{2.140}$$

$$\times\frac{\prod_{i,j=1}^{g} G(P_i,P_j)G(Q,Q)}{\prod_{i=1}^{g} G(P_i,Q)^2} \tag{2.141}$$

where $D_{ins} = \sum_{i=1}^{g} P_i - Q$, and Q is an arbitrary point on the surface. Since $\det'\bar{\partial}_K^\dagger\bar{\partial}_K = \det'\bar{\partial}^\dagger\bar{\partial}$, $\det(\omega_i,\omega_j) = \det(\text{Im}\tau)$, and $(1,1) = A_\Sigma$, the left hand side also contains a power of $|Z_1|$. This allows one to solve for an explicit expression of $|Z_1|$. With the Arakelov metric, one has

$$|Z_1|^3 = \frac{\mathcal{N}(\mathcal{O}(-D_{\text{ins}})\otimes\Delta)}{\|\det(\omega_i(P_j))\|^2}\frac{\prod_{i<j}^{g} G(P_i,P_j)^2}{\prod_{i=1}^{g} G(P_i,Q)^2} \tag{2.142}$$

Making the necessary substitutions to express everything in terms of theta function, prime form, and holomorphic section σ (apart from one-differentials ω_i), one gets again a consistent holomorphic factorization; thus

$$Z_1^{3/2} = \frac{\Theta(-\sum_{i=1}^{g} P_i + Q + \Delta \mid \tau)}{\det(\omega_i(P_j))}\frac{\prod_{i<j}^{g} E(P_i,P_j)}{\prod_{i=1}^{g} E(P_i,Q)}\frac{\prod_{i=1}^{g}\sigma(P_i)}{\sigma(Q)} \tag{2.143}$$

Twisted bundles

Non-trivial bundles of zero degree will play an important role in the computation of amplitudes for twisted sectors of an orbifold compactification. A very convenient way to characterize them is through holonomy. Given a general line bundle, if one parallel-transports an element v of the fiber at some point $P \in \Sigma_g$ around a closed loop $\mathcal{C}$, then after one turn, one reaches another element v' of the same fiber, which is just $\mathbb{C}$, so that these elements are related by

$$v' = ve^{2\pi i H(\mathcal{C},\xi)}. \tag{2.144}$$

The holonomy $H(\mathcal{C},\xi)$ turns out to be real, and its exponential in the previous equation, well defined under change of trivialization [41]. Moreover, the holonomy changes under a deformation of $\mathcal{C}$ to a nearby curve $\mathcal{C}'$ by a quantity proportional to the integral of the curvature R over the surface whose boundary is $\mathcal{C}' - \mathcal{C}$. Thus, for a flat bundle ψ,[15] H depends only on the homology class of $\mathcal{C}$. This means that holonomy defines a real cohomology class modulo an integral class, $H(\psi) \in H^1(\Sigma_g;\mathbb{R})/H^1(\Sigma_g;\mathbb{Z})$; flat line bundles are characterized by phases $\boldsymbol{\phi} = \{\phi_1^i,\ \phi_2^i,\ i=1,...,g\}$ around the non-trivial homology a- and b-cycles of Σ_g, respectively. Besides, the vanishing of $H(\psi_{\boldsymbol{\phi}})$ is equivalent to the triviality of ψ as a line bundle.

An important result is that the Jacobi map of a divisor of ψ is given in terms of these phases,

$$\mathbb{I}[D_{\psi_\phi}] = \phi_2 + \tau \cdot \phi_1 \tag{2.145}$$

It is possible then to consider a more general bosonization formula. As will be explained in the next chapter, compactification of the superstring on orbifolds leads to the introduction of novel sectors in the string Hilbert space. This *twisted* sectors induce the introduction of field configurations in the path integral which have non-trivial boundary conditions around homology a- and b-cycles on the Riemann surface. One such system of fields is a twisted $(1,0)$ (b,c) chiral theory. In these cases, although $\lambda = 1$, c has no zero modes (in contrast to the usual *untwisted* case), since the only holomorphic function possible in a compact Riemann surface is the constant function, which does not obviously satisfy the required boundary conditions. Hence, by the Riemann-Roch theorem, b has $g-1$ zero modes, which are given by $\boldsymbol{\phi}$-*twisted* one-differentials $\{\omega_{\boldsymbol{\phi},i},\ i = 1,...,g-1\}$. The line bundle $\mathcal{L}_b \equiv \xi_{\boldsymbol{\phi}}$ has the form $K \otimes \psi_{\boldsymbol{\phi}}$, where the flat line bundle completely specifies the twisting around homology cycles.

[15] The Chern number of the bundle can be represented by $\deg\xi = -\frac{1}{2\pi i}\int_{\Sigma_g} R$; hence, flat metrics imply that the bundle have zero degree. Moreover, every degree-zero bundle admits a flat metric, which is unique up to a constant. These bundles are thus called, flat bundles.

The general bosonization formula is

$$\frac{\det'\bar{\partial}^{\dagger}_{\xi_\phi}\bar{\partial}_{\xi_\phi}}{\det(u_i,u_j)}\|\det(u_i(P_j))\|^2 = |Z_1|^{-1}\mathcal{N}(\xi_\phi\otimes\mathcal{O}(-D_{\text{ins}})\otimes\Delta^{-1})\prod_{i<j}^{p}G(P_i,P_j)^2 \tag{2.146}$$

where $G(P_i,P_j)$ is the Green's function for any metric, not necessary the Arakelov metric.

Replacing $\xi_h = K\otimes\psi_h$ in the argument of $\mathcal{N}$, and taking into account that p is set to $g-1$, then

$$z = \mathbb{I}\Big[\psi_\phi\otimes\mathcal{O}(-D_{\text{ins}})\otimes\Delta\Big] = \phi_2+\tau\cdot\phi_1-\sum_{i=1}^{g-1}P_i+\Delta \tag{2.147}$$

The better way to show holomorphic factorization here is to divide the equation (2.146) for $K\otimes\psi_\phi$ by equation (2.142) for K [45].

Since insertion points are arbitrary, those for the twisted case are taken to be $P_i,\ i=1,...,g-1$, while those for the case $\xi=K$ are conveniently taken as $P_{i'},\ i'=1,...,g$, with $P_{i'}=P_i$ for $i=i'=1,...,g-1$, and $P_{i'=g}\equiv P_g$ is still left arbitrary. One gets,

$$\frac{\det'\bar{\partial}^{\dagger}_{\xi_\phi}\bar{\partial}_{\xi_\phi}}{\det(\omega_{\phi,i},\omega_{\phi,j})}|\det(\omega_{\phi,i}(P_j))|^2 = \rho(P_g)^{-1}|\det\omega_{i'}(P_{j'})|^2|Z_1|^2 \tag{2.148}$$

$$\times\frac{\mathcal{N}(\phi_2+\tau\cdot\phi_1-\sum_{i=1}^{g-1}P_i+\Delta)}{\mathcal{N}(-\sum_{i'=1}^{g}P_{i'}+Q+\Delta)}\frac{\prod_{i'=1}^{g}G(P_{i'},Q)^2}{\prod_{i=1}^{g-1}G(P_i,P_g)^2} \tag{2.149}$$

The right hand side is independent of Q so one can take the limit $Q\to P_g$. The function $\mathcal{N}$ in the denominator contains the holomorphic square of

$$\Theta\Big(-\sum_{i'=1}^{g}P_{i'}+Q+\Delta\,\Big|\,\tau\Big) = \Theta\Big(-\sum_{i=1}^{g-1}P_i+\Delta\,\Big|\,\tau\Big) \tag{2.150}$$

$$+(z_Q-z_{P_g})\sum_{i'=1}^{g}\omega_{i'}(P_g)\partial_{i'}\Theta\Big(-\sum_{i=1}^{g-1}P_i+\Delta\,\Big|\,\tau\Big)+... \tag{2.151}$$

The first term in the right-hand-side is zero due to the *Riemann vanishing theorem*, while higher order terms will vanish as $Q\to P_g$. Then, in the bosonization formula there remains the square of the factor

$$\lim_{Q\to P_g} \frac{G(Q,P_g)^2}{|z_Q - z_{P_g}|^2} \tag{2.152}$$

This is a regular expression as can be seen by choosing the Arakelov metric. In fact, with this choice

$$G(Q,P_g)^2 = \rho(Q)^{1/2}\rho(P_g)^{1/2}\exp\left[-2\pi\mathrm{Im}\int_{P_g}^{Q}\omega\cdot(\mathrm{Im}\tau)^{-1}\cdot\mathrm{Im}\int_{P_g}^{Q}\omega\right]|E(Q,P_g)|^2 \tag{2.153}$$

and, since

$$\lim_{Q\to P_g} \frac{|E(Q,P_g)|^2}{|z_Q - z_{P_g}|^2} = 1 \tag{2.154}$$

the resulting limit is

$$\lim_{Q\to P_g} \frac{G(Q,P_g)^2}{|z_Q - z_{P_g}|^2} = \rho(P_g) \tag{2.155}$$

Therefore,

$$\begin{aligned}\frac{\det'\bar{\partial}^\dagger_{\xi_{\boldsymbol{\phi}}}\bar{\partial}_{\xi_{\boldsymbol{\phi}}}}{\det(\omega_{\boldsymbol{\phi},i},\omega_{\boldsymbol{\phi},j})}|\det(\omega_{\boldsymbol{\phi},i}(P_j))|^2 &= |\det\omega_{i'}(P_{j'})|^2|Z_1|^2\exp\left[2\pi\mathrm{Im}z\cdot(\mathrm{Im}\tau)^{-1}\cdot\mathrm{Im}z\right] \\ &\times \frac{\mathcal{N}(\phi_2+\tau\cdot\phi_1+z)}{|\sum_{i'=1}^{g}\omega_{i'}(P_g)\partial_{i'}\Theta(z|\tau)|^2}\end{aligned} \tag{2.156}$$

with $z = -\sum_{i=1}^{g-1} P_i + \Delta$.

Fortunately, anholomorphic factors cancel out so holomorphic factorization is possible. One defines the theta function with characteristics $\boldsymbol{\phi} \equiv (\phi_1,\phi_2)$ as

$$\Theta[\boldsymbol{\phi}](z|\tau) = \exp[i\pi\phi_1\cdot(\tau\cdot\phi_1 + 2(z+\phi_2))]\Theta(\phi_2+\tau\cdot\phi_1+z|\tau). \tag{2.157}$$

so that it entirely contains all information of twist structue $\boldsymbol{\phi}$. The chiral partition function of a twisted $(1,0)$ fermionic system is

$$Z_{1,\boldsymbol{\phi}}\det(\omega_{\boldsymbol{\phi},i}(P_j)) = Z_1\det(\omega_{i'}(P_{j'}))\frac{\Theta[\boldsymbol{\phi}](z|\tau)}{\sum_{i'=1}^{g}\omega_{i'}(P_g)\partial_{i'}\Theta(z|\tau)} \tag{2.158}$$

Of course, this expression is equivalent to

$$Z_{1,\boldsymbol{\phi}}\det(\omega_{\boldsymbol{\phi},i}(P_j)) = Z^{-1/2}\Theta[\boldsymbol{\phi}]\Big(-\sum_{i=1}^{g-1}P_i+\Delta\Big|\tau\Big)\prod_{i<j}^{g-1}E(P_i,P_j)\prod_{i=1}^{g-1}\sigma(P_i) \quad (2.159)$$

as can be seen by using the explicit form of Z_1 and taking appropriate limit.

Finally, there is an important class of bundles L which have degree $g-1$ but which not necessarily satisfy $L\otimes L = K$. The latter corresponds to spin bundles $S_{\boldsymbol{\alpha}}$ with spin structure $\boldsymbol{\alpha}$ introduced earlier. One of these bundles is the Riemann class Δ, which depends on the choice of a- and b-cycles, or *marking*, on Σ_g. General twisted spin bundles are thus parametrized with respect to this special spin bundle as

$$L_{\boldsymbol{\phi}} = \Delta\otimes\psi_{\boldsymbol{\phi}} \quad (2.160)$$

Spin bundles are a special case of this where $\boldsymbol{\phi}=\boldsymbol{\alpha}=(\alpha_1,\alpha_2)$, and $\boldsymbol{\alpha}\in(\frac{1}{2}\mathbb{Z}/\mathbb{Z})^{2g}$. The Riemann class corresponds to $\boldsymbol{\alpha}=\mathbf{0}$, so it can be denoted by $\Delta=S_{\mathbf{0}}$.

Theta functions which were just introduced, when they have characteristics $\boldsymbol{\alpha}$, satisfy

$$\Theta[\boldsymbol{\alpha}](-z|\tau) = (-1)^{4\alpha_1\cdot\alpha_2}\Theta[\boldsymbol{\alpha}](z|\tau) \quad (2.161)$$

Parity of spin structures is defined as the parity of their associated theta functions. There are $2^{g-1}(2^g+1)$ even and $2^{g-1}(2^g-1)$ odd spin structures. The special bundle Δ has even spin structure.

For twisted spin bundles generically there are no zero modes for b and c, so in these cases one can take no insertions in the correlation function. The bosonization formula is simpler and reads

$$\det\bar{\partial}^\dagger_{L_{\boldsymbol{\phi}}}\bar{\partial}_{L_{\boldsymbol{\phi}}} = |Z_1|^{-1}\mathcal{N}(L_{\boldsymbol{\phi}}\otimes\Delta^{-1}) = |Z_1|^{-1}\mathcal{N}(\phi_2+\tau\cdot\phi_1) \quad (2.162)$$

Holomorphic factorization gives

$$Z_{1/2,\boldsymbol{\phi}} = Z_1^{-1/2}\Theta[\boldsymbol{\phi}](0|\tau) \quad (2.163)$$

When there are zero modes, which always happens for odd spin structures, then $Z_{1/2,\boldsymbol{\phi}}$ vanishes, and additional insertions are necessary.

All previous formulas can be generalized to the case in which more than the minimal number of insertions is present, as long as background charge cancellation is obeyed, as in (2.121). The modifications are contained entirely in the choice of bundle D_{ins} in the argument of the function $\mathcal{N}$ and the inclusion of extra factors of Green's function coming from contractions between additional insertions in the correlator. Holomorphic factorization implies that, for chiral systems, one needs to include factors of the prime form $E(P,Q)$ to take account of these contractions.

Take, for example, a complex system with $\lambda = \frac{1}{2}$, usually denoted by $(\bar{\psi}, \psi)$. The correlator must have an equal number of $\bar{\psi}$ and ψ insertions. Hence, in the absence of zero modes for $\bar{\psi}, \psi$,

$$\left\langle \prod_{i=1}^{M} \bar{\psi}(P_i) \prod_{j=1}^{M} \psi(Q_j) \right\rangle_{\boldsymbol{\phi}} = Z_1^{-1/2} \Theta[\boldsymbol{\phi}]\Big(-\sum_{i=1}^{M} P_i + \sum_{j=1}^{M} Q_j \,\Big|\, \tau \Big) \tag{2.164}$$

$$\times \frac{\prod_{i<j}^{M} E(P_i, P_j) \prod_{i<j}^{M} E(Q_i, Q_j)}{\prod_{i,j=1}^{M} E(P_i, Q_j)} \tag{2.165}$$

The propagator can be obtained from this formula by specializing to the two-point correlation function

$$\mathcal{P}_{\boldsymbol{\phi}}(P,Q) = \frac{\langle \bar{\psi}(P)\psi(Q)\rangle_{\boldsymbol{\phi}}}{\langle 1 \rangle_{\boldsymbol{\phi}}} = \frac{\langle \bar{\psi}(P)\psi(Q)\rangle_{\boldsymbol{\phi}}}{Z_{1/2,\boldsymbol{\phi}}} = \frac{\Theta[\boldsymbol{\phi}](-P+Q|\tau)}{\Theta[\boldsymbol{\phi}](0|\tau)} \frac{1}{E(P,Q)} \tag{2.166}$$

One can further generalize correlators in order to include spin fields. To relate these to the basic $\bar{\psi}$ and ψ, one expresses the latter as *solitons* of the bosonic corresponding theory,

$$\bar{\psi} = e^{i\varphi}, \qquad \psi = e^{-i\varphi} \tag{2.167}$$

Spin fields of this system are defined as

$$S^{\pm} = e^{\pm \frac{i}{2}\varphi}. \tag{2.168}$$

So, OPE's can be easily computed:

$$S^{+}(y)S^{-}(z) \longrightarrow (y-z)^{-\frac{1}{4}} \tag{2.169}$$

while $S^{\pm}(y)S^{\pm}(z)$ starts at order $(y-z)^{1/4}$.

The general correlator then is given by[16].

$$\left\langle \prod_{i=1}^{M_1} \bar{\psi}(P_i) \prod_{j=1}^{M_2} \psi(Q_j) \prod_{k=1}^{M_3} S^+(P'_k) \prod_{l=1}^{M_4} S^-(Q'_l) \right\rangle_{\boldsymbol{\phi}} = Z_1^{-1/2} \Theta[\boldsymbol{\phi}](-D_{ins}|\tau) \tag{2.170}$$

$$\times \frac{\prod_{i<j} E(P_i, P_j) \prod_{i<j} E(Q_i, Q_j)}{\prod_{i,j=1} E(P_i, Q_j)} \frac{\prod_{i<j} E(P'_i, P'_j)^{1/4} \prod_{i<j} E(Q'_i, Q'_j)^{1/4}}{\prod_{i,j=1} E(P'_i, Q'_j)^{1/4}} \tag{2.171}$$

$$\times \frac{\prod_{i,j=1} E(P_i, P'_j)^{1/2} \prod_{i,j=1} E(Q_i, Q'_j)^{1/2}}{\prod_{i,j=1} E(P_i, Q'_j)^{1/2} \prod_{i,j=1} E(Q_i, P'_j)^{1/2}} \tag{2.172}$$

where

$$D_{ins} = \sum_{i=1}^{M_1} P_i - \sum_{j=1}^{M_2} Q_j + \frac{1}{2} \sum_{k=1}^{M_3} P'_k - \frac{1}{2} \sum_{l=1}^{M_1} Q'_l, \tag{2.173}$$

and M_i $(i = 1, .., 4)$ must satisfy the relation $M_1 - M_2 + \frac{1}{2}M_3 - \frac{1}{2}M_4 = 0$ in order to respect background charge cancellation (which in this system is just zero).

$\beta\gamma$ systems

Finally, one needs to get expressions for correlators of bosonic (β, γ) systems like the superconformal ghosts of RNS formalism or bosonic ghosts of the pure spinor formalism. Since this is a bosonic first order system, the presence of zero modes will give infinities unless one inserts appropriate operators to absorb them. The correlation functions involving exclusively β and γ are of the form

$$\left\langle \prod_{i=1}^{p} \delta(\beta(P'_i)) \prod_{j=1}^{M} \beta(P_j) \prod_{k=1}^{M} \gamma(Q_k) \right\rangle \tag{2.174}$$

where $p = (2\lambda - 1)(g - 1)$.

As discussed in [47], bosonization of this type of systems at higher genus requires dealing with some subtleties concerning divergences coming from the ϕ instanton sum, and the presence of η zero modes. Although it is possible to show bosonization by taking care of these issues carefully, some bosonic correlation functions which are relevant in this work can be directly obtained from the results in last subsections on (b, c) systems.

Since (b, c) and (β, γ) systems with the same λ have identical eigenmode

[16] This spin field correlator was computed without bosonization in [46]

expansions, differing only by statistics, the following relation holds (with obvious modification when c, respectively γ, zero modes exist):[17]

$$\left\langle \prod_{i=1}^{p} \delta(\beta(P_i)) \right\rangle_{(\beta,\gamma)} = \left\langle \prod_{i=1}^{p} b(P_i) \right\rangle_{(b,c)}^{-1} \tag{2.175}$$

$$= \frac{Z_1^{1/2}}{\Theta\left(\sum_{i=1}^{p} P_i - (2\lambda - 1)\Delta \middle| \tau \right)} \frac{1}{\prod_{i<j}^{p} E(P_i, P_j)} \frac{1}{\prod_{i=1}^{p} \sigma(P_i)^{2\lambda-1}} \tag{2.176}$$

After bosonization one can check using OPE's that $\delta(\beta) = e^{\phi}$. Then making use of the OPE's

$$e^{a\phi}(y)e^{b\phi}(z) \longrightarrow (y-z)^{-ab}e^{(a+b)\phi} \tag{2.177}$$

one can write an expression for the correlator involving spin fields $\exp(-\frac{1}{2}\phi)$ in number such that background charge $-(2\lambda-1)(g-1)$ is always canceled:

$$\left\langle \prod_{i=1}^{p+M} e^{\phi}(P_i) \prod_{j=1}^{2M} e^{-\frac{1}{2}\phi}(Q_j) \right\rangle = \frac{Z_1^{1/2}}{\Theta(D_{ins} - (2\lambda-1)\Delta|\tau)} \tag{2.178}$$

$$\times \frac{\prod_{i,j=1} E(P_i, Q_j)^{1/2}}{\prod_{i<j} E(P_i, P_j) \prod_{i<j} E(Q_i, Q_j)^{1/4}} \frac{\prod_{i=1} \sigma(Q_i)^{\lambda-\frac{1}{2}}}{\prod_{i=1} \sigma(P_i)^{2\lambda-1}} \tag{2.179}$$

where $D_{ins} = \sum_{i=1}^{p+M} P_i - \frac{1}{2}\sum_{j=1}^{2M} Q_j$. Factors os σ can be trace back from the non-chiral relation between bosonic and fermionic correlators; recall that they appear both in the norm $\|b()\|^2$ as well as in the Green's function. For half-integer λ, the theta function in the denominator appears with characteristics $\boldsymbol{\alpha}$.

[17]The relative minus sign in the argument of the theta function compared to fermionic formula is due to the negative kinetic energy of ϕ, and the change of sign in the background charge.

Chapter 3

Superstrings on N=1 Orbifolds

Compactification on orbifolds is a good setting to start studying aspects of four-dimensional low-energy effective field theory for the superstring. This is so due to the fact that orbifolds are much easy to deal with than general Calabi-Yau manifolds. The common property of these internal models is that they reduce supersymmetry in four-dimensions compared to, for example, toroidal compactifications.

Recall that the closed superstring in a flat background has two $D = 10$ spacetime supersymmetries coming from left- and right-moving sectors of the worldsheet model. Each supersymmetry current is a Weyl (or anti-Weyl) spinor in ten dimensions, so there are 32 supersymmetries for closed superstrings. Toroidal compactifications preserve all of them; thus when compactifying to four dimensions, the resulting field theory has to be $N = 8$ $D = 4$ supergravity. The low energy limit was studied first in [48], where four-point amplitudes were computed both for Type I and type II superstrings.

Phenomenological motivations led to the search for manifolds which are less trivial than the six-torus, and which could reduce the large amount of supersymmetry in four dimensions. For the heterotic string [49, 50], with target space of the form $M_4 \times K_6$, the condition on the internal K_6 is that it must be a complex manifold with SU(3) holonomy [51]; this is precisely one way of characterize Calabi-Yau manifolds. In these non-trivial backgrounds, the ten dimensional supersymmetry charge is no longer conserved, except for some components, those corresponding to the four-dimensional sector [52].

Some orbifolds can be interpreted as coming from CY manifolds on singular limits. In any case, they can also reduce the initial 32 supersymmetries to a subset of them. Toroidal orbifolds, which will be the relevant ones in this thesis, are constructed from the six-torus T^6 after a quotient by some discrete isometry. In the case of $SU(3)$ holonomy, it is convenient to use $U(3) \equiv SU(3) \times U(1)_{\text{int}}$ indices for internal directions of the worldsheet fields which come from vectors of target space. The six coordinates of the parent torus combine to give complex coordinates x^I

and x_I ($I = 1, 2, 3$), transforming in the (anti-)fundamental of $SU(3)$, and carrying internal $U(1)_{\text{int}}$ charge $+1$ (-1). In other words, $x^{I=1} = \frac{1}{\sqrt{2}}(x^5 + ix^6)$, $x_{I=1} = \frac{1}{\sqrt{2}}(x^5 - ix^6)$, etc. Equivalent notation is $(x^I, x^{\bar{I}})$ where the upper index $\bar{I}$ can be lowered by use of a hermitian metric $g_{I\bar{J}}$, which in the case of orbifolds is just the flat metric, $g_{I\bar{J}} = \delta_{I\bar{J}}$.

Orbifolds in string theory were originally introduced in [53, 54]. The internal manifold of the model is a quotient of the six-torus T^6 by a discrete subgroup of its isometry group, G. That is, different points of T^6, $\{x^I, x_I\}$ and $\{x'^I, x'_I\}$, are identified if they are related by the action of $g \in G$, $x \sim x' = gx$. Fixed points under G become singularities in the quotient, but as mentioned in the introduction, string theory is still well-defined. For simplicity only abelian orbifolds are considered, in which the action of the group on the complex coordinates x^I, x_I is given by

$$x^I \longrightarrow e^{2\pi i \phi_I} x^I, \qquad x_I \longrightarrow e^{-2\pi i \phi_I} x_I \tag{3.1}$$

Boundary conditions for other worldsheet fields which have tangent indices I are similarly defined by requiring that all conserved currents of the theory are well-defined. For instance, internal fermions for RNS can be paired up into complex ψ^I, ψ and required to transform as

$$\psi^I \longrightarrow e^{2\pi i \phi_I} \psi^I, \qquad \psi_I \longrightarrow e^{-2\pi i \phi_I} \psi_I \tag{3.2}$$

in order for the $N = 1$ superconformal generator to be invariant under the orbifold action. Remaining worldsheet fields (spacetime fields and superconformal ghosts) in the RNS formalism have trivial behavior.

Physical states of this theory are grouped into two types of sectors. Those inherited from the parent toroidal compactification are said to belong to the untwisted sector of the Hilbert space. Then there are twisted sectors where strings are not closed in the parent T^6, but are closed after orbifold identification. In both of these sectors, only G-invariant states are allowed and all non-invariant states with respect to G are projected out.

Vertex operators are constructed out of fields which obey periodic or twisted boundary conditions. Insertions of *twisted* vertex operators satisfy *twisted* correlation functions which obey the appropriate monodromies around their insertions; this is similar to the case of spin field insertions on the worldsheet. For amplitudes computed in this thesis, all vertex operators belong to the untwisted sector, so it is not necessary to discuss twisted correlation functions.

Nevertheless, for higher genus amplitudes one should consider, even for scattering of untwisted states, nontrivial boundary conditions around each homology cycle of the corresponding genus g Riemann surface Σ_g. Each possible choice of boundary conditions is called a twist structure, and the amplitude computations have to include a sum over all twist structures, including the trivial one: the structure where boundary conditions of the worldsheet fields are periodic around all a_j and b_j cycles. Untwisted worldsheet fields have the usual integer mode expansions and twisted ones are generally expanded in non-integer modes.

Bosonic orbifold coordinates x^I and x_I are defined such that

$$x^I(z+a_i) = e^{2\pi i\phi^i_{I,1}} x^I(z), \quad x^I(z+b_i) = e^{2\pi i\phi^i_{I,2}} x^I(z), \tag{3.3}$$

$$x_I(z+a_i) = e^{-2\pi i\phi^i_{I,1}} x_I(z), \quad x_I(z+b_i) = e^{-2\pi i\phi^i_{I,2}} x_I(z), \tag{3.4}$$

where $\{\phi^i_{I,1}, \phi^i_{I,2},\ i = 1, ..., g\} \equiv \boldsymbol{\phi}_I$ specify the orbifold group elements that twist the boundary conditions along the homology cycles of Σ_g. Similarly internal fermions with twist structure $\boldsymbol{\phi}_I$ satisfy

$$\psi^I(z+a_i) = e^{2\pi i\phi^i_{I,1}} \psi^I(z), \quad \psi^I(z+b_i) = e^{2\pi i\phi^i_{I,2}} \psi^I(z), \tag{3.5}$$

$$\psi_I(z+a_i) = e^{-2\pi i\phi^i_{I,1}} \psi_I(z), \quad \psi_I(z+b_i) = e^{-2\pi i\phi^i_{I,2}} \psi_I(z). \tag{3.6}$$

For closed strings (type II) similar twisted boundary conditions hold for the right-moving sector. Type II spectrum on orbifold compactifications depends, of course, on the details of the group G; however it is possible to study the general structure of the spectrum by noticing that they are realizations of the compactification on superconformal structures to four dimensions. This also allows to get basic information about the more complicated cases of Calabi-Yau manifolds. The focus will be on type II superstring compactifications; as mentioned before the original formulation was made in the framework of heterotic strings, while type I orbifold compactifications are also possible [55].

3.1 Type II spectrum in 4D Compactifications

The general setting to describe the four-dimensional compactifications relevant to this work is the type II superstring compactified on $N = (2, 2)$ superconformal field theory (SCFT) with central charge $c = 9$ [56]. They abstract non-linear sigma models on Calabi-Yau three-manifolds as well as theories on six-dimensional orbifolds.

Effective field theories arising from these compactifications have $D = 4$ $N = 2$ spacetime supersymmetry. The charges of these extended supersymmetric theories come from both right and left-moving worldsheet sectors of the closed type II superstring. To analyze the massless spectrum of the superstring one can start with the bosonic NS-NS states and obtain the others making use of the spectral flow characteristic to two-dimensional $N = 2$ SCFT's [57].

The simplest operator, which always exists in superconformal theories, is the identity operator $\mathbb{1}$ with conformal weights $(h, \bar{h}) = (0, 0)$ and charges $(q, \bar{q}) = (0, 0)$. This operator leads to the graviton, dilaton, and antisymmetric tensor, all of which belongs to the compactification independent, or universal, sector of the theory. Operators of positive conformal weight can lead to massless states only when $h = \frac{1}{2}$ or $\bar{h} = \frac{1}{2}$, but those with $(h, \bar{h}) = (0, \frac{1}{2})$ or $(h, \bar{h}) = (\frac{1}{2}, 0)$ give rise to free fermions in the CFT, thus increasing the amount of supersymmetry beyond $N = 2$. So, the restriction is over SCFT's where these operators cannot appear. The remaining possibility is $(h, \bar{h}) = (\frac{1}{2}, \frac{1}{2})$. It is possible to show that GSO projection requires $N = 2$ chiral and antichiral operators to be primaries, so the relation with superconformal charge is $q = \pm 2h$ as well as $\bar{q} = \pm 2\bar{h}$. These operators belong to the compactification dependent, or non-universal, sector of the theory.

It is convenient to recall here the spectrum of $D = 4$ $N = 2$ supergravity. There are basically three types of multiplets. The gravitational multiplet G contains the graviton, a vector called *graviphoton*, and two gravitini. A scalar multiplet, or hypermultiplet Φ, contains four scalars and a Dirac fermion. Finally, a vector multiplet V contains a vector, a complex scalar, and a Dirac fermion.

Vertex operators corresponding to theses states are constructed now in the RNS formalism.

In the universal sector, the gravitational multiplet starts with the graviton, whose vertex operator is easily obtained by dimensional reduction of the ten-dimensional version. At zero picture, it is

$$V_g^{(0)}(k, h) = h_{\mu\nu}(\partial x^\mu + ik \cdot \psi\psi^\mu)(\bar{\partial} x^\nu + ik \cdot \tilde{\psi}\tilde{\psi}^\nu)\mathbb{1}e^{ik \cdot x} \tag{3.7}$$

where $h_{\mu\nu}$, $\mu, \nu = 0, ..., 3$ is the polarization, a symmetric traceless tensor obeying $k^\mu h_{\mu\nu} = 0$, k is the four-dimensional momentum and x^μ denotes the spacetime components of x^m, $m = 0, ..., 9$.

At picture -1 the graviton vertex operator is

$$V_g^{(-1)}(k,h) = h_{\mu\nu} e^{-(\phi+\tilde{\phi})} \psi^\mu \tilde{\psi}^\nu e^{ik\cdot x} \tag{3.8}$$

As mentioned before, the vertex operator for the graviphoton is obtained by means of spectral flow. The explicit form of the operator depends of which type II superstring is dealing with, since the flows are in the opposite directions for type IIA and in the same direction for type IIB. This means that, for type IIA,

$$\mathbb{1} \longrightarrow \Sigma^0(z,\bar{z}) \tag{3.9}$$

where Σ^0 is the unique operator in the internal SCFT with conformal weights $(h,\bar{h}) = (\frac{3}{8},\frac{3}{8})$ and charges $(q,\bar{q}) = (\frac{3}{2},-\frac{3}{2})$. On the other hand, for type IIB,

$$\mathbb{1} \longrightarrow \Xi^0(z,\bar{z}) \tag{3.10}$$

where Ξ^0 is the unique operator with conformal weights $(h,\bar{h}) = (\frac{3}{8},\frac{3}{8})$ and charges $(q,\bar{q}) = (\frac{3}{2},\frac{3}{2})$. Consistency requires that spectral flow acts also on space-time fermionic vectors and superconformal ghosts, schematically as

$$e^{-(\phi+\tilde{\phi})} \psi^\mu \tilde{\psi}^\nu \longrightarrow e^{-\frac{1}{2}(\phi+\tilde{\phi})} S^\alpha \tilde{S}^\beta \tag{3.11}$$

where S^α (and $S^{\dot{\alpha}}$) are spin fields constructed from ψ^μ through bosonization.

Therefore, type IIA graviphoton vertex operator at picture $-\frac{1}{2}$ is given by

$$V_T^{(-\frac{1}{2})}(k,a) = e^{-\frac{1}{2}(\phi+\tilde{\phi})} k_\nu a_\mu \left[S^\alpha (\sigma^{\mu\nu})_\alpha^{\ \beta} \tilde{S}_\beta \Sigma^0 + S_{\dot{\alpha}} (\bar{\sigma}^{\mu\nu})^{\dot{\alpha}}_{\ \dot{\beta}} \tilde{S}^{\dot{\beta}} \bar{\Sigma}^0 \right] e^{ik\cdot x} \tag{3.12}$$

The type IIB vertex operators is obtained after replacement $\Sigma^0 \longrightarrow \Xi^0$.

The other multiplet in the universal sector, the hypermultiplet, is constructed starting from the dilaton and axion vertices,

$$V_\phi^{(-1)}(k) = e^{-(\phi+\tilde{\phi})} \psi^\mu \tilde{\psi}_\mu e^{ik\cdot x} \tag{3.13}$$

$$V_B^{(-1)}(k,b) = b_{\mu\nu} e^{-(\phi+\tilde{\phi})} \psi^\mu \tilde{\psi}^\nu e^{ik\cdot x} \tag{3.14}$$

where $b_{\mu\nu}$ is an antisymmetric tensor.

Spectral flow gives rise to a couple of real scalars (or one complex scalar), which for type IIA read

$$V_Z^{(-\frac{1}{2})}(k) = e^{-\frac{1}{2}(\phi+\tilde{\phi})} k_\mu S^\alpha \sigma^\mu_{\alpha\dot{\beta}} \tilde{S}^{\dot{\beta}} \Xi^0 e^{ik\cdot x} \tag{3.15}$$

$$V_{\bar{Z}}^{(-\frac{1}{2})}(k) = e^{-\frac{1}{2}(\phi+\tilde{\phi})} k^\mu S_{\dot{\alpha}} \bar{\sigma}_\mu^{\dot{\alpha}\beta} \tilde{S}_\beta \bar{\Xi}^0 e^{ik\cdot x} \tag{3.16}$$

Corresponding vertex operators for type IIB are obtained by making the replacement $\Xi^0 \longrightarrow \Sigma^0$.

As already seen, the other type of states that could appear in the massless spectrum of this kind of compactifications are related to operators in the SCFT with quantum numbers $h = \bar{h} = \frac{1}{2},\ q = \pm\bar{q} = 1$. In a generic $N = (2,2)$ SCFT there will be n_1 operators Δ^A, $A = 1, ..., n_1$ with $(h, \bar{h}) = (\frac{1}{2}, \frac{1}{2}),\ (q, \bar{q}) = (1, 1)$, and n_2 operators Λ^B, $B = 1, ..., n_2$ with $(h, \bar{h}) = (\frac{1}{2}, \frac{1}{2}),\ (q, \bar{q}) = (1, -1)$. The nature of multiplets in the $D = 4$ $N = 2$ field effective field theory depends on whether the superstring is of type IIA or of type IIB. In fact, for type IIA, GSO projection requires that operators Δ^A gives rise to hypermultiplets, whereas operators Λ^B give rise to vector multiplets. The situation for type IIB is reversed so there are n_1 vector multiplets and n_2 hypermultiplets instead.

Vertex operators for type IIA are now constructed. NS-NS scalars in vector multiplets V^B have vertex operators:

$$V^{B(-1)}(k) = e^{-(\phi+\tilde{\phi})} \Lambda^B e^{ik\cdot x} \tag{3.17}$$

$$V^{\bar{B}(-1)}(k) = e^{-(\phi+\tilde{\phi})} \bar{\Lambda}^B e^{ik\cdot x} \tag{3.18}$$

Spectral flow maps

$$\Lambda^B \longrightarrow \Sigma^B \tag{3.19}$$

where Σ^B are operators with $(h, \bar{h}) = (\frac{3}{8}, \frac{3}{8}),\ (q, \bar{q}) = (-\frac{1}{2}, \frac{1}{2})$. Thus, R-R vectors in these multiplets are given by

$$V^{B(-\frac{1}{2})}(k, \zeta) = e^{-\frac{1}{2}(\phi+\tilde{\phi})} k_\nu \zeta_\mu \left[S^\alpha (\sigma^{\mu\nu})_\alpha^{\ \beta} \tilde{S}_\beta \Sigma^B + S_{\dot{\alpha}} (\bar{\sigma}^{\mu\nu})^{\dot{\alpha}}_{\ \dot{\beta}} \tilde{S}^{\dot{\beta}} \bar{\Sigma}^B \right] e^{ik\cdot x} \tag{3.20}$$

NS-NS scalars in hypermultiplets Φ^A have vertex operators

$$V^{A(-1)}(k) = e^{-(\phi+\tilde{\phi})} \Delta^A e^{ik\cdot x} \tag{3.21}$$

$$V^{\bar{A}(-1)}(k) = e^{-(\phi+\tilde{\phi})} \bar{\Delta}^A e^{ik\cdot x} \tag{3.22}$$

Spectral flow

$$\Delta^A \longrightarrow \Xi^A \tag{3.23}$$

where Ξ^A are operators with $(h, \bar{h}) = (\frac{3}{8}, \frac{3}{8})$, $(q, \bar{q}) = (-\frac{1}{2}, -\frac{1}{2})$, allows one to write vertex operators for R-R scalars,

$$V^{A(-\frac{1}{2})}(k) = e^{-\frac{1}{2}(\phi+\tilde{\phi})} k_\mu S^\alpha \sigma^\mu_{\alpha\dot{\beta}} \tilde{S}^{\dot{\beta}} \Xi^A e^{ik\cdot x} \tag{3.24}$$

$$V^{\bar{A}(-\frac{1}{2})}(k) = e^{-\frac{1}{2}(\phi+\tilde{\phi})} k^\mu S_{\dot{\alpha}} \bar{\sigma}^{\dot{\alpha}\beta}_\mu \tilde{S}_\beta \bar{\Xi}^A e^{ik\cdot x} \tag{3.25}$$

Type IIB vertex operators are obtained from all these ones by making substitutions

$$\Lambda^B \longleftrightarrow \Delta^A, \qquad \Sigma^B \longleftrightarrow \Xi^A. \tag{3.26}$$

3.2 Pure Spinor Formalism on an Orbifold

In the pure spinor formalism, four-dimensional compactifications will be obtained starting from an appropriate splitting of the ten-dimensional objects in such a way as to reflect the nature of target space. Obviously, bosonic coordinates x^m are written again in the form x^μ, x^I, x_I. And the orbifold action is just as in the RNS formalism.

Instead of fermionic vectors, the pure spinor formalism contains both fermionic and bosonic ten-dimensional spinors, so it is also necessary to decompose each sixteen component object using four-dimensional and internal indices.

Recall that a chiral spinor in $D = 10$ can be written using U(5) components as $\zeta^{\underline{\alpha}} \to (\zeta^+,\ \zeta_{ab},\ \zeta_{abcd})$, and that this construction can be interpreted in terms of elements in a Fock space through the action of creation operators γ^a. Besides, operators γ^1, γ_1 and γ^2, γ_2 can be thought of as generating the four-dimensional part of the spinors, as they are related to the first four diagonal components of the metric defining the Clifford algebra g^{mn}. Therefore, if the index a is split as (i, I), the components of $\zeta^{\underline{\alpha}}$ are

$$\zeta^+, \qquad \zeta_{ab} = (\zeta_{ij},\ \zeta_{iI},\ \zeta_{IJ}), \qquad \zeta_{abcd} = (\zeta_{ijIJ},\ \zeta_{iIJK}) \tag{3.27}$$

The four-dimensional chiral and antichiral spinors can be identified, thus, as singlets of the internal U(3); they are ζ^+, ζ_{ij} and ζ_{iIJK}. To see the relation with more common notation ζ_α, $\zeta_{\dot{\alpha}}$, it is useful to recall that the same construction of ten-dimensional spinors can be performed in $D = 4$.

The Clifford algebra in four dimensions is $\{\gamma^\mu, \gamma^\nu\} = 2\eta^{\mu\nu}\mathbb{1}$. The γ^μ are grouped into creation (γ^i, $i = 1,2$) and annihilation (γ_i, $i = 1,2$) operators as before. The spinor space is spanned by the vacuum $|\Omega\rangle$ and all states generated by application of the creation operators on it.

$$|\Omega\rangle, \quad \gamma^i|\Omega\rangle \quad \gamma^i\gamma^j|\Omega\rangle \tag{3.28}$$

Thus, a Dirac spinor ζ can be written as

$$\zeta = \zeta^+|\Omega\rangle + \zeta_i\gamma^i|\Omega\rangle + \frac{1}{2}\zeta_{ij}\gamma^i\gamma^j|\Omega\rangle = (\zeta^{(0)} + \zeta^{(1)} + \zeta^{(2)})|\Omega\rangle \tag{3.29}$$

Since indices take only two values, and the γ^i's anticommute, there is only one independent component for ζ_{ij} which can be denoted as $\zeta_{12} = \zeta^-$. Then a chiral spinor is $\zeta^\alpha \to (\zeta^+, \zeta^-)$ and an antichiral one is $\zeta^{\dot\alpha} \to \zeta_i$. Furthermore, one can compute the usual covariant objects constructed from these spinors in the same way as in $D = 10$. In four dimensions, contractions are non-vanishing between spinors of the same chirality. Defining the measure as in subsection 2.2.1, $\langle\gamma^i \wedge \gamma^j\rangle = \varepsilon^{ij}$, one gets the contraction between two chiral spinors $\chi^\alpha, \zeta^\alpha$,

$$(\chi, \zeta) = \chi^\alpha\zeta_\alpha = R[\chi]\zeta = \langle\chi^{(0)} \wedge \zeta^{(2)} - \chi^{(2)} \wedge \zeta^{(0)}\rangle, \tag{3.30}$$

from where one can obtain

$$\chi^\alpha\zeta_\alpha = \chi^+\zeta^- - \chi^-\zeta^+, \tag{3.31}$$

whereas for antichiral spinors antichiral $\chi^{\dot\alpha}$, $\zeta^{\dot\alpha}$, their contraction is

$$\chi_{\dot\alpha}\zeta^{\dot\alpha} = \varepsilon^{ij}\chi_i\zeta_j \tag{3.32}$$

Moreover, it is easy to show that the following expressions hold

$$\chi^\alpha\sigma^i_{\alpha\dot\alpha}\zeta^{\dot\alpha} = \chi^+\varepsilon^{ij}\zeta_j, \qquad \chi^\alpha\sigma_{i\alpha\dot\alpha}\zeta^{\dot\alpha} = -2\chi^-\zeta_i \tag{3.33}$$

Going back to ten dimensions, and changing notation as follows: $\zeta_{IJ} = \varepsilon_{IJK}\zeta^{+K}$, $\zeta_{ijIJ} = \varepsilon_{ij}\varepsilon_{IJK}\zeta^{-K}$, $\zeta_{iIJK} = \varepsilon_{IJK}\zeta_i$, a chiral spinor $\zeta^{\underline{\alpha}}$ can be written in components as

$$(\zeta_\alpha, \zeta_{\dot\alpha}, \zeta^I_\alpha, \zeta_{\dot\alpha I}). \tag{3.34}$$

Similarly, an antichiral spinor $\bar{\zeta}_{\underline{\alpha}}$ has components

$$\bar{\zeta}_a = (\bar{\zeta}_i,\ \bar{\zeta}_I), \qquad \bar{\zeta}_{abc} = (\bar{\zeta}_{ijI},\ \bar{\zeta}_{iIJ},\ \bar{\zeta}_{IJK}), \qquad \bar{\zeta}_{abcde} = \bar{\zeta}_{ijIJK} \tag{3.35}$$

and, redefining $\bar{\zeta}_I = \bar{\zeta}_I^+$, $\bar{\zeta}_{ijI} = \varepsilon_{ij}\bar{\zeta}_I^-$, $\bar{\zeta}_{iIJ} = \varepsilon_{IJK}\bar{\zeta}_i^K$, it becomes

$$(\bar{\zeta}_\alpha, \bar{\zeta}_{\dot{\alpha}}, \bar{\zeta}_{\alpha I}, \bar{\zeta}^I_{\dot{\alpha}}). \tag{3.36}$$

It is convenient to have expressions for different contractions between spinors and gamma matrices in terms of these split components. From the formulas in subsection 2.2.1 it is straightforward to deduce

$$\zeta^{\underline{\alpha}}\bar{\zeta}_{\underline{\alpha}} = \zeta^\alpha\bar{\zeta}_\alpha - \zeta_{\dot{\alpha}}\bar{\zeta}^{\dot{\alpha}} - \zeta^{\alpha I}\bar{\zeta}_{\alpha I} + \zeta_{\dot{\alpha}I}\bar{\zeta}^{\dot{\alpha}I}, \tag{3.37}$$

$$\zeta^{\underline{\alpha}}\gamma^\mu_{\underline{\alpha}\underline{\beta}}\xi^{\underline{\beta}} = \zeta^\alpha\sigma^\mu_{\alpha\dot{\beta}}\xi^{\dot{\beta}} + \xi^\alpha\sigma^\mu_{\alpha\dot{\beta}}\zeta^{\dot{\beta}} - \zeta^{\alpha I}\sigma^\mu_{\alpha\dot{\beta}}\xi^{\dot{\beta}}{}_I - \xi^{\alpha I}\sigma^\mu_{\alpha\dot{\beta}}\zeta^{\dot{\beta}}{}_I, \tag{3.38}$$

$$\zeta^{\underline{\alpha}}\gamma^I_{\underline{\alpha}\underline{\beta}}\xi^{\underline{\beta}} = \zeta^\alpha\xi_\alpha{}^I - \zeta^{\alpha I}\xi_\alpha + \varepsilon^{IJK}\zeta_{\dot{\alpha}J}\xi^{\dot{\alpha}}{}_K, \tag{3.39}$$

$$\zeta^{\underline{\alpha}}\gamma_{I\underline{\alpha}\underline{\beta}}\xi^{\underline{\beta}} = -2\zeta_{\dot{\alpha}I}\xi^{\dot{\alpha}} + 2\zeta_{\dot{\alpha}}\xi^{\dot{\alpha}}_I - 2\varepsilon_{IJK}\zeta^{\alpha J}\xi_\alpha{}^K, \tag{3.40}$$

$$\bar{\zeta}_{\underline{\alpha}}\gamma^{\mu\underline{\alpha}\underline{\beta}}\bar{\xi}_{\underline{\beta}} = -\bar{\zeta}^\alpha\sigma^\mu_{\alpha\dot{\beta}}\bar{\xi}^{\dot{\beta}} - \bar{\xi}^\alpha\sigma^\mu_{\alpha\dot{\beta}}\bar{\zeta}^{\dot{\beta}} + \bar{\zeta}^\alpha_I\sigma^\mu_{\alpha\dot{\beta}}\bar{\xi}^{\dot{\beta}I} + \bar{\xi}^\alpha_I\sigma^\mu_{\alpha\dot{\beta}}\bar{\zeta}^{\dot{\beta}I}, \tag{3.41}$$

$$\bar{\zeta}_{\underline{\alpha}}\gamma^{I\underline{\alpha}\underline{\beta}}\bar{\xi}_{\underline{\beta}} = -\bar{\zeta}_{\dot{\alpha}}\bar{\xi}^{\dot{\alpha}I} + \bar{\zeta}_{\dot{\alpha}}^{\ I}\bar{\xi}^{\dot{\alpha}} - \varepsilon^{IJK}\bar{\zeta}^\alpha_J\bar{\xi}_{\alpha K}, \tag{3.42}$$

$$\bar{\zeta}_{\underline{\alpha}}\gamma_I^{\underline{\alpha}\underline{\beta}}\bar{\xi}_{\underline{\beta}} = 2\bar{\zeta}^\alpha_I\bar{\xi}_\alpha - 2\bar{\zeta}^\alpha\bar{\xi}_{\alpha I} + 2\varepsilon_{IJK}\bar{\zeta}_{\dot{\alpha}}^{\ J}\bar{\xi}^{\dot{\alpha}K}, \tag{3.43}$$

where $\sigma^\mu_{\alpha\dot{\alpha}}$ are four-dimensional Pauli matrices.

To get the correct action of the orbifold group on θ^I_α and $\theta_{\dot{\alpha}I}$ (together with the corresponding right-moving worldsheet fields) it is necessary to require the correct amount of supersymmetry to be preserved in $D = 4$. This means that components j_α and $j_{\dot{\alpha}}$ are invariant under the orbifold action. These currents, together with their right moving analogs, define the $D = 4$ $N = 2$ supersymmetry of the model and their explicit expressions are

$$j_\alpha = p_\alpha + \frac{1}{2}\sigma^\mu_{\alpha\dot{\beta}}\theta^{\dot{\beta}}\partial x_\mu + \frac{1}{2}\theta^I_\alpha\partial x_I + \frac{1}{24}\sigma^\mu_{\alpha\dot{\beta}}\theta^{\dot{\beta}}(\theta^{\underline{\alpha}}\gamma_{\mu\underline{\alpha}\underline{\beta}}\partial\theta^{\underline{\beta}}) + \frac{1}{24}\theta^I_\alpha(\theta^{\underline{\alpha}}\gamma_{I\underline{\alpha}\underline{\beta}}\partial\theta^{\underline{\beta}}), \tag{3.44}$$

$$j_{\dot{\alpha}} = p_{\dot{\alpha}} + \frac{1}{2}\theta^\beta\sigma^\mu_{\beta\dot{\alpha}}\partial x_\mu - \theta_{\dot{\alpha}I}\partial x^I + \frac{1}{24}\theta^\beta\sigma^\mu_{\beta\dot{\alpha}}(\theta^{\underline{\alpha}}\gamma_{\mu\underline{\alpha}\underline{\beta}}\partial\theta^{\underline{\beta}}) - \frac{1}{12}\theta_{\dot{\alpha}I}(\theta^{\underline{\alpha}}\gamma^I_{\underline{\alpha}\underline{\beta}}\partial\theta^{\underline{\beta}}). \tag{3.45}$$

These currents must be well-defined also when worldsheet fields have non-trivial twist structures on Σ_g. It is immediate to see that for these currents j_α, $j_{\dot\alpha}$ to be single-valued on Σ_g, one needs to impose

$$\theta^I_\alpha(z+a_i) = e^{2\pi i\phi^i_{I,1}}\theta^I_\alpha(z), \qquad \theta_{\dot\alpha I}(z+a_i) = e^{-2\pi i\phi^i_{I,1}}\theta_{\dot\alpha I}(z), \tag{3.46}$$

$$\theta^I_\alpha(z+b_i) = e^{2\pi i\phi^i_{I,2}}\theta^I_\alpha(z), \qquad \theta_{\dot\alpha I}(z+b_i) = e^{-2\pi i\phi^i_{I,2}}\theta_{\dot\alpha I}(z), \tag{3.47}$$

Here, $U(1)_{\text{int}}$ charge conservation is almost enough to imply single-valuedness. The only additional requirement comes from the following terms in the supersymmetry currents:

$$j_\alpha = ... - \frac{1}{12}\varepsilon_{IJK}\theta^I_\alpha\theta^{\beta J}\partial\theta^K_\beta, \qquad j_{\dot\alpha} = ... - \frac{1}{12}\varepsilon^{IJK}\theta_{\dot\alpha I}\theta_{\dot\beta J}\partial\theta^{\dot\beta}_K. \tag{3.48}$$

These terms are not single-valued unless $\exp(2\pi i(\phi^i_{(I=1)} + \phi^i_{(I=2)} + \phi^i_{(I=3)})) = 1$, which is a well-known constraint, together with the right-moving analog, to obtain $N = 2$ $D = 4$ spacetime supersymmetry from orbifold compactifications of type II superstrings. In particular, one can choose

$$\boldsymbol{\phi}_1 + \boldsymbol{\phi}_2 + \boldsymbol{\phi}_3 = 0. \tag{3.49}$$

Notice that the other components of the initial ten-dimensional supersymmetry current, $j_{\alpha I}$ and $j^I_{\dot\alpha}$, are not single-valued when all $\boldsymbol{\phi}_I \neq 0$, so the amount of supersymmetry is reduced to $N = 2$ $D = 4$, as required. On the other hand, the trivial twist structure where $\boldsymbol{\phi}_1 = \boldsymbol{\phi}_2 = \boldsymbol{\phi}_3 = 0$ preserves all 32 supersymmetry currents, giving rise to a subsector with $N = 8$ spacetime supersymmetry. Another possible situation is where one of the $\boldsymbol{\phi}'s$ is trivial, say, $\boldsymbol{\phi}_1 = 0$, and the other two $\boldsymbol{\phi}$'s satisfying $\boldsymbol{\phi}_3 = -\boldsymbol{\phi}_2$, are non-zero. In this case, besides j_α and $j_{\dot\alpha}$, the currents $j_{\alpha 1}$ and $j^1_{\dot\alpha}$ are also single-valued, so the amount of supersymmetry is only reduced to $N = 4$ $D = 4$. It will be shown in the next chapter that sectors preserving $N = 4$ or $N = 8$ supersymmetry do not contribute to topological amplitudes since there are too many fermionic zero modes to be absorbed.

Concerning the remaining worldsheet fields, boundary conditions for the conjugate momentum should be chosen so as to imply single-valuedness of the worldsheet action. Thus,

$$p^I_{\dot\alpha}(z+a_i) = e^{2\pi i\phi^i_{I,1}}p^I_{\dot\alpha}(z), \qquad p_{\alpha I}(z+a_i) = e^{-2\pi i\phi^i_{I,1}}p_{\alpha I}(z). \tag{3.50}$$

with similar expressions to hold along b-cycles.

Furthermore, the requirement that the minimal pure spinor BRST current $\lambda^{\underline{\alpha}} d_{\underline{\alpha}}$ is single-valued implies that

$$\lambda_\alpha^I(z+a_i) = e^{2\pi i \phi^i_{I,1}} \lambda_\alpha^I(z), \qquad \lambda_{\dot\alpha I}(z+a_i) = e^{-2\pi i \phi^i_{I,1}} \lambda_{\dot\alpha I}(z), \tag{3.51}$$

and that its conjugate variables satisfy the boundary conditions

$$w_{\dot\alpha}^I(z+a_i) = e^{2\pi i \phi^i_{I,1}} w_{\dot\alpha}^I(z), \qquad w_{\alpha I}(z+a_i) = e^{-2\pi i \phi^i_{I,1}} w_{\alpha I}(z). \tag{3.52}$$

Notice that condition (3.49) is crucial for the pure spinor constraint to be well defined, independently of the requirement of supersymmetry in four-dimensional spacetime. Using (3.38), (3.39), and (3.40), the pure spinor conditions $\lambda^{\underline{\alpha}} \gamma^m_{\underline{\alpha}\underline{\beta}} \lambda^{\underline{\beta}} = 0$ decompose into three equations

$$\lambda^\alpha \lambda^{\dot\beta} - \lambda^{\alpha I} \lambda_I^{\dot\beta} = 0, \tag{3.53}$$

$$\lambda^\alpha \lambda_\alpha^I + \frac{1}{2} \varepsilon^{IJK} \lambda_{\dot\alpha J} \lambda_K^{\dot\alpha} = 0, \tag{3.54}$$

$$\lambda_{\dot\alpha I} \lambda^{\dot\alpha} + \frac{1}{2} \varepsilon_{IJK} \lambda^{\alpha J} \lambda_\alpha^K = 0, \tag{3.55}$$

and the last two equations only make sense if (3.49) is satisfied.

3.2.1 Zero modes of twisted fields

Twisted boundary conditions on a genus g Riemann surface change the zero mode structure of the worldsheet fields. Notice that the zero mode of a field having conformal weight zero cannot satisfy twisted boundary conditions because it should be a constant. For the case of compactified spacetime coordinates, (x^I, x_I), their zero modes are constrained to be the fixed points at the orbifold singularities.

Recall that worldsheet systems are fermionic or bosonic field theories with an action of the form,

$$S = \int_{\Sigma_g} (b\bar\partial c + \tilde b \partial \tilde c). \tag{3.56}$$

For untwisted fields, it is well known that the number of zero modes of b and c

are related by the Riemann-Roch theorem [41]

$$n(b) - n(c) = (2\lambda - 1)(g - 1) \tag{3.57}$$

where (b, c) have conformal weights $(\lambda, 1 - \lambda)$. As reviewed in section 2.4, the same formula can be applied for twisted systems (they correspond to twisted line bundles on Σ_g). So for the twisted case when (b, c) have conformal weights $(1, 0)$, c has no zero modes and b has $g - 1$ zero modes which are given by a basis of the so-called h-twisted one-differentials $\omega_{h,i}$ for $i = 1, ..., g - 1$ [58].

In the twisted sector of the orbifold compactified model, fermionic variables $\theta^{\underline{\alpha}}$ will split into superspace coordinates $(\theta^\alpha, \theta^{\dot{\alpha}})$ that are always untwisted, and internal coordinates $(\theta^I_\alpha, \theta_{\dot{\alpha}I})$. So there are four untwisted $(1, 0)$ systems, $(p_\alpha, \theta^\alpha)$ and $(p_{\dot{\alpha}}, \theta^{\dot{\alpha}})$, six $\boldsymbol{\phi}_I$-twisted $(1, 0)$ systems $(p^\alpha_I, \theta^I_\alpha)$, and six $-\boldsymbol{\phi}_I$-twisted $(1, 0)$ systems $(p^{\dot{\alpha}I}, \theta_{\dot{\alpha}I})$, where the distinction between $\boldsymbol{\phi}_I$-twisted or $-\boldsymbol{\phi}_I$-twisted can be read from the position of the I index. Concerning the zero modes, untwisted systems contribute four constant zero modes contained in $(\theta^\alpha, \theta^{\dot{\alpha}})$ and $4g$ zero modes contained in $(d_\alpha, d_{\dot{\alpha}})$, $-\boldsymbol{\phi}_I$-twisted systems contribute $6g - 6$ zero modes contained in $d^{\dot{\alpha}I}$ and no zero modes for $\theta_{\dot{\alpha}I}$, and $\boldsymbol{\phi}_I$-twisted systems contribute $6g - 6$ zero modes contained in d^α_I and no zero modes for θ^I_α.

In the case of constrained variables, the situation is more involved. As in the previous case, the pure spinor field $\lambda^{\underline{\alpha}}$ has four untwisted, six $\boldsymbol{\phi}_I$-twisted, and six $-\boldsymbol{\phi}_I$-twisted components; however, as it must be the case for a pure spinor, only 11 of these 16 components are independent. To figure out the actual number of untwisted and twisted components, it is necessary to solve the pure spinor constraint in such a way that Lorentz symmetry in four dimensions is preserved.

Recall that if one breaks $SO(10)$ to $U(5)$, the pure spinor decomposes as $(\lambda^+, \lambda_{ab}, \lambda^a)$ where $a = 1, ..., 5$. The pure spinor space can be described using sixteen patches where each patch corresponds to the subset where a specific component is required to be nonzero. For example, in the patch where $\lambda^+ \neq 0$, one can solve the constraints by expressing the five components λ^a in terms of λ^+ and λ_{ab} as

$$\lambda^a = \frac{1}{8\lambda^+}\varepsilon^{abcde}\lambda_{bc}\lambda_{de}. \tag{3.58}$$

To preserve $D = 10$ Lorentz covariance, all 16 patches of pure spinor space must be considered. However, if one requires no more than $D = 4$ Lorentz invariance, it is sufficient to only consider the two patches where $\lambda^{\dot{\alpha}} \neq 0$, that is, the patch $\lambda^{\dot{1}} \neq 0$ and the patch $\lambda^{\dot{2}} \neq 0$. If $\xi_{\dot{\alpha}}\lambda^{\dot{\alpha}} \neq 0$ for some $\xi_{\dot{\alpha}}$, one can use the pure spinor constraints (3.53) and (3.55) to solve for the two components of λ^α and for the three components of $(\lambda_{\dot\alpha}\lambda_I^{\dot\alpha})$ in terms of the other 11 components as

$$\lambda_\alpha = \frac{\lambda_\alpha^I\left(\xi_{\dot\beta}\lambda_I^{\dot\beta}\right)}{\xi_{\dot\gamma}\lambda^{\dot\gamma}}, \quad \lambda_{\dot\alpha}\lambda_I^{\dot\alpha} = \frac{1}{2}\varepsilon_{IJK}\lambda^{\alpha J}\lambda_\alpha^K. \tag{3.59}$$

So the 11 independent components of the pure spinor can be chosen as $\lambda_{\dot\alpha}$, $\lambda^{\alpha I}$ and $\xi_{\dot\beta}\lambda_I^{\dot\beta}$. To recover $D=4$ Lorentz covariance, the spinor $\xi_{\dot\alpha}$ will be later replaced by the components $\bar\lambda_{\dot\alpha}$ of the non-minimal pure spinor $\bar\lambda_{\underline\alpha}$. Since $\lambda_{\dot\alpha}$ is always untwisted, the zero modes in their two components are the only independent zero modes in the twisted sectors.

Due to constraints (3.53), (3.54), (3.55), the model has a gauge invariance that allows to set five components of $w_{\underline\alpha}$ equal to zero. In the patch where $\xi_{\dot\alpha}\lambda^{\dot\alpha}\neq 0$, it is possible to gauge away w_α and $\xi_{\dot\alpha}w^{\dot\alpha I}$. Therefore, the pure spinor sector consists of 11 pairs of $(1,0)$ chiral bosons: two untwisted, $(w^{\dot\alpha},\lambda_{\dot\alpha})$, six $\boldsymbol{\phi}_I$-twisted, $(w_{\alpha I},\lambda^{\alpha I})$, and three $-\boldsymbol{\phi}_I$-twisted, $(\lambda_{\dot\alpha}w^{\dot\alpha I},\ (\xi_{\dot\beta}\lambda^{\dot\beta})^{-1}\xi_{\dot\alpha}\lambda_I^{\dot\alpha})$. So there are $2g$ $w_{\dot\alpha}$, $6g-6$ $w_{\alpha I}$ and $3g-3$ $\lambda_{\dot\alpha}w^{\dot\alpha I}$ zero modes.

Alternatively, one could have preserved $D=4$ Lorentz covariance by only considering patches where $\xi_\alpha\lambda^\alpha\neq 0$ for some ξ_α (and then replacing this ξ_α by the non-minimal spinor $\bar\lambda_\alpha$). In this case, the 11 pairs of $(1,0)$ chiral bosons are: two untwisted, $(w^\alpha,\lambda_\alpha)$, six $-\boldsymbol{\phi}_I$-twisted, $(w^{\dot\alpha I},\lambda_{\dot\alpha I})$, and three $\boldsymbol{\phi}_I$-twisted, $(\lambda^\alpha w_{\alpha I},\ (\xi^\beta\lambda_\beta)^{-1}\xi^\alpha\lambda_\alpha^I)$. And one gets two zero modes from λ_α, $2g$ zero modes from w_α, $6g-6$ zero modes from $w^{\dot\alpha I}$, and $3g-3$ zero modes from $\lambda_{\dot\alpha}w^{\dot\alpha I}$.

In general backgrounds the composite b ghost of the pure spinor formalism has a very complicated expression; fortunately, since orbifold backgrounds are constructed from free worldsheet theories, the b ghost takes just the same form as in flat space, equation (2.86). One only needs to split vectors and spinors as has been done so far.

Then, boundary conditions for non-minimal variables in twisted sectors are implied by single-valuedness of this composite b ghost and the pure spinor BRST charge (2.80),

$$\bar\lambda_{\dot\alpha}^I(z+a_i) = e^{2\pi i\phi_{I,1}^i}\bar\lambda_{\dot\alpha}^I(z), \qquad \bar\lambda_{\alpha I}(z+a_i) = e^{-2\pi i\phi_{I,1}^i}\bar\lambda_{\alpha I}(z), \tag{3.60}$$

$$\bar w_\alpha^I(z+a_i) = e^{2\pi i\phi_{I,1}^i}\bar w_\alpha^I(z), \qquad \bar w_{\dot\alpha I}(z+a_i) = e^{-2\pi i\phi_{I,1}^i}\bar w_{\dot\alpha I}(z), \tag{3.61}$$

$$r_{\dot\alpha}^I(z+a_i) = e^{2\pi i\phi_{I,1}^i}r_{\dot\alpha}^I(z), \qquad r_{\alpha I}(z+a_i) = e^{-2\pi i\phi_{I,1}^i}r_{\alpha I}(z), \tag{3.62}$$

$$s_{\alpha}^{I}(z+a_i)=e^{2\pi i\phi_{I,1}^{i}}s_{\alpha}^{I}(z),\qquad s_{\dot\alpha I}(z+a_i)=e^{-2\pi i\phi_{I,1}^{i}}s_{\dot\alpha I}(z). \tag{3.63}$$

and similarly along homology b-cycles. When $\bar\lambda_{\dot\alpha}\lambda^{\dot\alpha}\neq 0$, one can use the same arguments as for the pure spinor $\lambda^{\underline\alpha}$ and solve all components of $\bar\lambda_{\underline\alpha}$ in terms of $\bar\lambda_{\dot\alpha}$, $\bar\lambda_{\alpha I}$ and $\lambda_{\dot\alpha}\bar\lambda^{\dot\alpha I}$, and the independent bosonic and fermionic systems are $(\bar w^{\dot\alpha},\bar\lambda_{\dot\alpha})$, $(\bar w^{\alpha I},\bar\lambda_{\alpha I})$, $((\bar\lambda_{\dot\beta}\lambda^{\dot\beta})^{-1}\bar\lambda_{\dot\alpha}\bar w_{I}^{\dot\alpha},\ \lambda_{\dot\alpha}\bar\lambda^{\dot\alpha I})$ and $(s_{\dot\alpha},r^{\dot\alpha})$, $(s^{\alpha I},r_{\alpha I})$, $((\bar\lambda_{\dot\beta}\lambda^{\dot\beta})^{-1}\bar\lambda_{\dot\alpha}s_{I}^{\dot\alpha},\ \lambda_{\dot\alpha}r^{\dot\alpha I})$. Also, when $\bar\lambda_{\dot\alpha}\lambda^{\dot\alpha}\neq 0$ in a twisted sector, the bosonic zero modes include two from $\bar\lambda_{\dot\alpha}$, $2g$ from $\bar w_{\dot\alpha}$, $6g-6$ from $\bar w^{\alpha I}$, and $3g-3$ from $(\bar\lambda_{\dot\beta}\lambda^{\dot\beta})^{-1}\bar\lambda_{\dot\alpha}\bar w_{I}^{\dot\alpha}$. And the fermionic zero modes include two from $r_{\dot\alpha}$, $2g$ from $s_{\dot\alpha}$, $6g-6$ from $s^{\alpha I}$, and $3g-3$ from $(\bar\lambda_{\dot\beta}\lambda^{\dot\beta})^{-1}\bar\lambda_{\dot\alpha}s_{I}^{\dot\alpha}$.

3.2.2 A four-dimensional b ghost

Restricting to patches where $\lambda^{\dot\alpha}\neq 0$, or equivalently, $\bar\lambda_{\dot\alpha}\lambda^{\dot\alpha}\neq 0$, allows for interesting possibilities concerning the composite pure spinor b ghost. Recall that in flat ten-dimensional background, the b ghost (2.86) is constructed out of operators $(T, G^{\underline\alpha},...)$, obeying a chain of relations like $\{Q,G^{\underline\alpha}\}=\lambda^{\underline\alpha}T$, $\{Q,H^{\underline\alpha\underline\beta}\}=\lambda^{[\underline\alpha}G^{\underline\beta]}$, and so on. The reasoning is that, since $\{Q,b\}=T$ must hold, one could write the b ghost as

$$b=\frac{\bar\lambda_{\underline\alpha}G^{\underline\alpha}}{\bar\lambda_{\underline\alpha}\lambda^{\underline\alpha}}+... \tag{3.64}$$

which satisfies $\{Q,b\}=T+...$; extra terms are then canceled by further terms in b with higher order poles $\bar\lambda_{\underline\alpha}\lambda^{\underline\alpha}$.

Suppose now that one restricts to a patch in pure spinor space where either $\bar\lambda_{\alpha}\lambda^{\alpha}\neq 0$ or $\bar\lambda_{\dot\alpha}\lambda^{\dot\alpha}\neq 0$. Since $\{Q_0,G^{\alpha}\}=\lambda^{\alpha}T_0$ and $\{Q_0,G^{\dot\alpha}\}=\lambda^{\dot\alpha}T_0$ where $Q_0=\int\lambda^{\underline\alpha}d_{\underline\alpha}$ and $T_0=-\frac{1}{2}\partial x^m\partial x_m-p_{\underline\alpha}\partial\theta^{\underline\alpha}+w_{\underline\alpha}\partial\lambda^{\underline\alpha}$ are the minimal BRST charge and stress-tensor, one can choose on these patches the first term of the four-dimensional b ghost as either

$$b^{(c)}=\frac{\bar\lambda_{\alpha}G^{\alpha}}{\bar\lambda_{\beta}\lambda^{\beta}}+...,\qquad \text{or}\qquad b^{(a)}=\frac{\bar\lambda_{\dot\alpha}G^{\dot\alpha}}{\bar\lambda_{\dot\beta}\lambda^{\dot\beta}}+.... \tag{3.65}$$

Applying the non-minimal BRST charge to the first term in $b^{(a)}$, one gets

$$\left\{Q_0+\int\bar w^{\underline\alpha}r_{\underline\alpha},\frac{\bar\lambda_{\dot\alpha}G^{\dot\alpha}}{\bar\lambda_{\dot\gamma}\lambda^{\dot\gamma}}\right\}=T_0-\frac{r_{\dot\alpha}G^{\dot\alpha}}{\bar\lambda_{\dot\gamma}\lambda^{\dot\gamma}}+\frac{\bar\lambda_{\dot\alpha}G^{\dot\alpha}r_{\dot\beta}\lambda^{\dot\beta}}{(\bar\lambda_{\dot\gamma}\lambda^{\dot\gamma})^2}. \tag{3.66}$$

To cancel the second and third term in the right hand side of the last equation one proceeds as in the ten dimensional case, picking other operators in the chain. Notice that there is no need for components other than $H^{[\dot{\alpha}\dot{\beta}]}$, which has actually only one independent component since dotted indices take just two values. Using $\left[Q, H^{[\dot{\alpha}\dot{\beta}]}\right] = \lambda^{[\dot{\alpha}}G^{\dot{\beta}]}$,

$$\left\{ Q_0 + \int \bar{w}^{\underline{\alpha}} r_{\underline{\alpha}}, -\frac{\bar{\lambda}_{\dot{\alpha}} r_{\dot{\beta}} H^{[\dot{\alpha}\dot{\beta}]}}{(\bar{\lambda}_{\dot{\delta}}\lambda^{\dot{\delta}})^2} \right\} = \frac{r_{\dot{\alpha}} G^{\dot{\alpha}}}{\bar{\lambda}_{\dot{\delta}}\lambda^{\dot{\delta}}} - \frac{\bar{\lambda}_{\dot{\alpha}} G^{\dot{\alpha}} r_{\dot{\beta}} \lambda^{\dot{\beta}}}{(\bar{\lambda}_{\dot{\delta}}\lambda^{\dot{\delta}})^2} + \frac{r_{\dot{\alpha}} r_{\dot{\beta}} H^{[\dot{\alpha}\dot{\beta}]}}{(\bar{\lambda}_{\dot{\delta}}\lambda^{\dot{\delta}})^2} - \frac{2 r_{\dot{\gamma}} \lambda^{\dot{\gamma}} \bar{\lambda}_{\dot{\alpha}} r_{\dot{\beta}} H^{[\dot{\alpha}\dot{\beta}]}}{(\bar{\lambda}_{\dot{\delta}}\lambda^{\dot{\delta}})^3}. \tag{3.67}$$

The last two terms in (3.67) cancel each other because of the identity $r_{\dot{\alpha}} r_{\dot{\beta}} = \frac{1}{2}\varepsilon_{\dot{\alpha}\dot{\beta}} r_{\dot{\gamma}} r^{\dot{\gamma}}$, and the first two cancel those appearing in (3.66), so we can define the b-ghost to be

$$b^{(a)} = \frac{\bar{\lambda}_{\dot{\alpha}} G^{\dot{\alpha}}}{\bar{\lambda}_{\dot{\alpha}}\lambda^{\dot{\alpha}}} - \frac{\bar{\lambda}_{\dot{\alpha}} r_{\dot{\beta}} H^{[\dot{\alpha}\dot{\beta}]}}{(\bar{\lambda}_{\dot{\alpha}}\lambda^{\dot{\alpha}})^2} + s^{\underline{\alpha}} \partial \bar{\lambda}_{\underline{\alpha}} \tag{3.68}$$

where the last term is necessary to get the non-minimal terms in $T = T_0 + \bar{w}^{\underline{\alpha}} \partial \bar{\lambda}_{\underline{\alpha}} - s^{\underline{\alpha}} \partial r_{\underline{\alpha}}$.

Explicitly,

$$b^{(a)} = s^{\underline{\alpha}} \partial \bar{\lambda}_{\underline{\alpha}} - \frac{\bar{\lambda}_{\dot{\alpha}}}{\bar{\lambda}_{\dot{\alpha}}\lambda^{\dot{\alpha}}} \left(\frac{1}{2} \Pi_I d^{\dot{\alpha} I} + \frac{1}{2} \Pi^{\mu} \tilde{\sigma}_{\mu}^{\dot{\alpha}\beta} d_{\beta} \right) - w_{\dot{\alpha}} \partial \theta^{\dot{\alpha}} - w_I^{\alpha} \partial \theta_{\alpha}^I$$

$$- \frac{1}{\bar{\lambda}_{\dot{\alpha}}\lambda^{\dot{\alpha}}} \left(\lambda_{\dot{\alpha}} w^{\dot{\alpha} I} \bar{\lambda}_{\dot{\beta}} \partial \theta_I^{\dot{\beta}} - \bar{\lambda}_{\dot{\beta}} \lambda_I^{\dot{\beta}} w_{\dot{\alpha}}^I \partial \theta^{\dot{\alpha}} - w_{\dot{\alpha}} \lambda^{\dot{\alpha}} \bar{\lambda}_{\dot{\beta}} \partial \theta^{\dot{\beta}} \right) + \frac{\bar{\lambda}_{\dot{\alpha}} r^{\dot{\alpha}} d^{\alpha} d_{\alpha}}{4(\bar{\lambda}_{\dot{\alpha}}\lambda^{\dot{\alpha}})^2} - \frac{\bar{\lambda}_{\dot{\alpha}} r^{\dot{\alpha}} \lambda_{\dot{\beta}} w^{\dot{\beta} I} \Pi_I}{(\bar{\lambda}_{\dot{\alpha}}\lambda^{\dot{\alpha}})^2}. \tag{3.69}$$

This four-dimensional version satisifes $b^{(a)} = b + Q\Lambda$ where Λ is well-defined when $\bar{\lambda}_{\dot{\alpha}}\lambda^{\dot{\alpha}} \neq 0$, so they are BRST equivalent. In fact, it is shown in the Appendix that this simpler $b^{(a)}$ is related to the original one by

$$b = \frac{\bar{\lambda}_{\dot{\alpha}} G^{\dot{\alpha}}}{\bar{\lambda}_{\dot{\alpha}}\lambda^{\dot{\alpha}}} - \frac{\bar{\lambda}_{\dot{\alpha}} r_{\dot{\beta}} H^{[\dot{\alpha}\dot{\beta}]}}{(\bar{\lambda}_{\dot{\alpha}}\lambda^{\dot{\alpha}})^2} + s^{\underline{\alpha}} \partial \bar{\lambda}_{\underline{\alpha}} \tag{3.70}$$

$$+ Q \left(\frac{\bar{\lambda}_{\dot{\alpha}} \bar{\lambda}_{\beta'} H^{[\dot{\alpha},\beta']}}{(\bar{\lambda}_{\dot{\alpha}}\lambda^{\dot{\alpha}})(\bar{\lambda}_{\underline{\alpha}}\lambda^{\underline{\alpha}})} + \frac{\bar{\lambda}_{\dot{\alpha}} \bar{\lambda}_{\beta'} r_{\underline{\gamma}} K^{[\dot{\alpha}\beta'\underline{\gamma}]}}{\bar{\lambda}_{\dot{\alpha}}\lambda^{\dot{\alpha}} (\bar{\lambda}_{\underline{\alpha}}\lambda^{\underline{\alpha}})^2} + \frac{\bar{\lambda}_{\dot{\alpha}} \bar{\lambda}_{\beta'} r_{\dot{\gamma}} K^{[\dot{\alpha}\beta'\dot{\gamma}]}}{(\bar{\lambda}_{\dot{\alpha}}\lambda^{\dot{\alpha}})^2 \bar{\lambda}_{\underline{\alpha}}\lambda^{\underline{\alpha}}} \right. \tag{3.71}$$

$$\left. - \frac{\bar{\lambda}_{\dot{\alpha}} \bar{\lambda}_{\beta'} r_{\underline{\gamma}} r_{\delta'} L^{[\dot{\alpha}\beta'\underline{\gamma}\delta']} + \bar{\lambda}_{\dot{\alpha}} \bar{\lambda}_{\beta'} r_{\gamma'} r_{\dot{\delta}} L^{[\dot{\alpha}\beta'\gamma'\dot{\delta}]}}{\bar{\lambda}_{\dot{\alpha}}\lambda^{\dot{\alpha}} (\bar{\lambda}_{\underline{\alpha}}\lambda^{\underline{\alpha}})^3} - \frac{\bar{\lambda}_{\dot{\alpha}} \bar{\lambda}_{\beta'} r_{\gamma'} r_{\dot{\delta}} L^{[\dot{\alpha}\beta'\gamma'\dot{\delta}]}}{(\bar{\lambda}_{\dot{\alpha}}\lambda^{\dot{\alpha}})^2 (\bar{\lambda}_{\underline{\alpha}}\lambda^{\underline{\alpha}})^2} \right) \tag{3.72}$$

where the convention used for spinor indices is the following: $\dot{\alpha}$ denotes antichiral spinors in four dimensions, and α' denotes any of the other components in the ten-dimensional quantity, that is, $\alpha' = (\alpha, \alpha I, \dot\alpha I)$.

One can similarly define a four-dimensional version $b^{(c)}$ satisfying $\{Q, b^{(c)}\} = T$ if one restricts to the patches where $\lambda^\alpha \neq 0$ as

$$b^{(c)} = s^{\underline{\alpha}}\partial\bar\lambda_{\underline{\alpha}} + \frac{\bar\lambda_\alpha G^\alpha}{\bar\lambda_\alpha\lambda^\alpha} - \frac{\bar\lambda_\alpha r_\beta H^{[\alpha\beta]}}{(\bar\lambda_\alpha\lambda^\alpha)^2} \tag{3.73}$$

$$= s^{\underline{\alpha}}\partial\bar\lambda_{\underline{\alpha}} + \frac{\bar\lambda_\alpha}{\bar\lambda_\alpha\lambda^\alpha}\left(\frac{1}{2}\Pi^I d_I^\alpha + \frac{1}{2}\Pi^\mu\bar\sigma_\mu^{\dot\alpha\alpha}d_{\dot\alpha}\right) + w^\alpha\partial\theta_\alpha + w^I_{\dot\alpha}\partial\theta^{\dot\alpha}_I$$
$$-\frac{1}{\bar\lambda_\alpha\lambda^\alpha}\left(\lambda^\alpha w_{\alpha I}\bar\lambda^\beta\partial\theta^I_\beta - \bar\lambda^\beta\lambda^I_\beta w^\alpha_I\partial\theta_\alpha - w^\alpha\lambda_\alpha\bar\lambda^\beta\partial\theta_\beta\right) + \frac{\bar\lambda^\alpha r_\alpha d_{\dot\alpha}d^{\dot\alpha}}{4(\bar\lambda_\alpha\lambda^\alpha)^2} - \frac{\bar\lambda^\alpha r_\alpha\lambda^\beta w_{\beta I}\Pi^I}{(\bar\lambda_\alpha\lambda^\alpha)^2}. \tag{3.74}$$

Although $b^{(a)}$ is simpler than the original b ghost, the restriction to patches $\lambda^{\dot\alpha} \neq 0$ allows one to define

$$\xi^{(a)} = \frac{\bar\lambda_{\dot\alpha}\theta^{\dot\alpha}}{\lambda^{\dot\alpha}\bar\lambda_{\dot\alpha} + r_{\dot\alpha}\theta^{\dot\alpha}} = \frac{\bar\lambda_{\dot\alpha}\theta^{\dot\alpha}}{\lambda^{\dot\alpha}\bar\lambda_{\dot\alpha}} - \frac{(\bar\lambda_{\dot\alpha}\theta^{\dot\alpha})(r_{\dot\alpha}\theta^{\dot\alpha})}{(\lambda^{\dot\alpha}\bar\lambda_{\dot\alpha})^2},$$

which satisfies $\{Q, \xi^{(a)}\} = 1$ and only diverges like $(\lambda^{\dot\alpha}\bar\lambda_{\dot\alpha})^{-2}$. Furthermore, in the twisted sector, $\lambda^{\underline{\alpha}}$ has only two independent zero modes so the path integral $\int d^2\lambda d^2\bar\lambda$ converges only like $(\lambda^{\dot\alpha}\bar\lambda_{\dot\alpha})^2$. So unlike the original non-minimal formalism where the Hilbert space allows states with poles less divergent than $(\lambda^{\underline{\alpha}}\bar\lambda_{\underline{\alpha}})^{-11}$, the non-minimal formalism restricted to patches $\lambda^{\dot\alpha} \neq 0$ only allows states with poles less divergent than $(\lambda^{\dot\alpha}\bar\lambda_{\dot\alpha})^{-2}$.

A simple way to obtain $b^{(a)}$ is to start with the ten-dimensional b ghost of (2.86) and rescale the non-minimal variables as

$$\bar\lambda_{\alpha I} \to c\bar\lambda_{\alpha I}, \qquad \lambda^{\dot\alpha}\bar\lambda^I_{\dot\alpha} \to c\lambda^{\dot\alpha}\bar\lambda^I_{\dot\alpha}, \tag{3.75}$$

$$r_{\alpha I} \to c r_{\alpha I}, \qquad \lambda^{\dot\alpha}r^I_{\dot\alpha} \to c\lambda^{\dot\alpha}r^I_{\dot\alpha}, \tag{3.76}$$

whereas $\bar\lambda_{\dot\alpha}$ and $r_{\dot\alpha}$ are kept invariant. From the pure spinor constraints, this implies that

$$\bar\lambda_\alpha \to c^2\bar\lambda_\alpha, \qquad \bar\lambda^{\dot\alpha}\bar\lambda^I_{\dot\alpha} \to c^2\bar\lambda^{\dot\alpha}\bar\lambda^I_{\dot\alpha}, \tag{3.77}$$

$$r_\alpha \to c^2 r_\alpha, \qquad \bar\lambda^{\dot\alpha}r^I_{\dot\alpha} \to c^2\bar\lambda^{\dot\alpha}r^I_{\dot\alpha}. \tag{3.78}$$

To leave the worldsheet action and BRST operator invariant, the conjugate non-minimal variables must be rescaled as

$$\bar{w}^{\alpha I} \to c^{-1}\bar{w}^{\alpha I}, \quad \bar{\lambda}_{\dot{\alpha}}\bar{w}_I^{\dot{\alpha}} \to c^{-1}\bar{\lambda}_{\dot{\alpha}}\bar{w}_I^{\dot{\alpha}} \tag{3.79}$$

$$s^{\alpha I} \to c^{-1}s^{\alpha I}, \quad \bar{\lambda}_{\dot{\alpha}}s_I^{\dot{\alpha}} \to c^{-1}\bar{\lambda}_{\dot{\alpha}}s_I^{\dot{\alpha}} \tag{3.80}$$

whereas $\bar{w}^{\dot{\alpha}}$ and $s^{\dot{\alpha}}$ are kept invariant.

Under this rescaling of the non-minimal variables, one can easily verify that in the limit where $c \to 0$, the b ghost of (2.86) goes to $b^{(a)}$ of (3.69). Since the non-minimal variables do not appear in the BRST-invariant vertex operators, the only other effect of this rescaling is to change the definition of $\Lambda = \bar{\lambda}_{\underline{\alpha}}\theta^{\underline{\alpha}} + \ldots$ which appears in the regulator $\mathcal{N} = \exp(Q\Lambda)$. But since $\mathcal{N} = 1 + Q(\Lambda + \frac{1}{2}\Lambda Q\Lambda + \ldots)$, changing the definition of Λ is a BRST-trivial operation and does not affect scattering amplitudes. For this reason, one is free to take the limit $c \to 0$ when computing scattering amplitudes, and thus, to replace the b ghost of (2.86) with $b^{(a)}$. Of course, there is a different rescaling of the non-minimal variables which instead replaces b with $b^{(c)}$.

3.2.3 A convenient change of variables

In the patch where $\bar{\lambda}_{\dot{\alpha}}\lambda^{\dot{\alpha}} \neq 0$, it is possible to make the following field transformation,

$$\psi^I = (\bar{\lambda}_{\dot{\beta}}\lambda^{\dot{\beta}})^{-1}\bar{\lambda}_{\dot{\alpha}}p^{\dot{\alpha}I}, \qquad \chi^I = \lambda_{\dot{\alpha}}p^{\dot{\alpha}I}, \tag{3.81}$$

$$\psi_I = -\lambda_{\dot{\alpha}}\theta_I^{\dot{\alpha}}, \qquad \chi_I = (\bar{\lambda}_{\dot{\beta}}\lambda^{\dot{\beta}})^{-1}\bar{\lambda}_{\dot{\alpha}}\theta_I^{\dot{\alpha}}, \tag{3.82}$$

which is invertible when $\bar{\lambda}_{\dot{\alpha}}\lambda^{\dot{\alpha}} \neq 0$ and whose jacobian in the path integral is 1.

Recall that the fermionic part of the action in the pure spinor formalism is

$$\int d^2z \left[p_{\underline{\alpha}}\bar{\partial}\theta^{\underline{\alpha}}\right] \tag{3.83}$$

and

$$p_{\underline{\alpha}}\bar{\partial}\theta^{\underline{\alpha}} = -p^{\alpha}\bar{\partial}\theta_{\alpha} + p_{\dot{\alpha}}\bar{\partial}\theta^{\dot{\alpha}} + p_I^{\alpha}\bar{\partial}\theta_{\alpha}^I - p_{\dot{\alpha}}^I\bar{\partial}\theta_I^{\dot{\alpha}} \tag{3.84}$$

Under the previous field transformation the term $-p_{\dot{\alpha}}^I\bar{\partial}\theta_I^{\dot{\alpha}}$ takes the form

$$\psi^I\bar{\partial}\psi_I + \chi^I\bar{\partial}\chi_I - \Big(\lambda_{\dot{\alpha}}\psi^I\chi_I + \frac{\bar{\lambda}_{\dot{\alpha}}}{\bar{\lambda}_{\dot{\gamma}}\lambda^{\dot{\gamma}}}(\psi^I\psi_I - \chi^I\chi_I)\Big)\bar{\partial}\lambda^{\dot{\alpha}} + \frac{\bar{\lambda}_{\dot{\alpha}}}{(\bar{\lambda}_{\dot{\gamma}}\lambda^{\dot{\gamma}})^2}\chi^I\psi_I\bar{\partial}\bar{\lambda}^{\dot{\alpha}} \tag{3.85}$$

The pure spinor action is

$$-\int d^2z \left[w_{\underline{\alpha}}\bar{\partial}\lambda^{\underline{\alpha}}\right] \tag{3.86}$$

where

$$-w_{\underline{\alpha}}\bar{\partial}\lambda^{\underline{\alpha}} = w^{\alpha}\bar{\partial}\lambda_{\alpha} - w_{\dot{\alpha}}\bar{\partial}\lambda^{\dot{\alpha}} - w_I^{\alpha}\bar{\partial}\lambda_{\alpha}^I + w_{\dot{\alpha}}^I\bar{\partial}\lambda_I^{\dot{\alpha}} \tag{3.87}$$

Since it is possible to use the gauge invariance under $\delta w_{\underline{\alpha}}$ to gauge away w_{α} and $\bar{\lambda}_{\dot{\alpha}}w^{\dot{\alpha}I}$, the term $w^{\alpha}\bar{\partial}\lambda_{\alpha}$ will be absent in the action and

$$w_{\dot{\alpha}}^I\bar{\partial}\lambda_I^{\dot{\alpha}} = -\phi^I\bar{\partial}\phi_I - \frac{\bar{\lambda}_{\dot{\alpha}}}{\bar{\lambda}_{\dot{\gamma}}\lambda^{\dot{\gamma}}}\phi^I\phi_I\bar{\partial}\lambda^{\dot{\alpha}} + \frac{\bar{\lambda}_{\dot{\alpha}}}{(\bar{\lambda}_{\dot{\gamma}}\lambda^{\dot{\gamma}})^2}\phi^I\lambda_{\dot{\beta}}\lambda_I^{\dot{\beta}}\bar{\partial}\bar{\lambda}^{\dot{\alpha}} \tag{3.88}$$

where

$$\phi^I = \lambda_{\dot{\alpha}}w^{\dot{\alpha}I}, \qquad \phi_I = (\bar{\lambda}_{\dot{\gamma}}\lambda^{\dot{\gamma}})^{-1}\bar{\lambda}_{\dot{\alpha}}\lambda_I^{\dot{\alpha}} \tag{3.89}$$

Terms of higher order in the fields can be absorbed by a redefinition of $w_{\dot{\alpha}}$ and $\bar{w}_{\dot{\alpha}}$ as

$$w_{\dot{\alpha}} \longrightarrow w_{\dot{\alpha}} + \lambda_{\dot{\alpha}}\psi^I\chi_I + \frac{\bar{\lambda}_{\dot{\alpha}}}{\bar{\lambda}_{\dot{\gamma}}\lambda^{\dot{\gamma}}}(\psi^I\psi_I - \chi^I\chi_I) + \frac{\bar{\lambda}_{\dot{\alpha}}}{\bar{\lambda}_{\dot{\gamma}}\lambda^{\dot{\gamma}}}\phi^I\phi_I \tag{3.90}$$

$$\bar{w}_{\dot{\alpha}} \longrightarrow \bar{w}_{\dot{\alpha}} - \frac{\bar{\lambda}_{\dot{\alpha}}}{(\bar{\lambda}_{\dot{\gamma}}\lambda^{\dot{\gamma}})^2}\chi^I\psi_I - \frac{\bar{\lambda}_{\dot{\alpha}}}{(\bar{\lambda}_{\dot{\gamma}}\lambda^{\dot{\gamma}})^2}\phi^I\lambda_{\dot{\beta}}\lambda_I^{\dot{\beta}} \tag{3.91}$$

The resulting (minimal) action is still free and given by

$$\int d^2z\Big[-p^{\alpha}\bar{\partial}\theta_{\alpha} + p_{\dot{\alpha}}\bar{\partial}\theta^{\dot{\alpha}} + p_I^{\alpha}\bar{\partial}\theta_{\alpha}^I + \psi^I\bar{\partial}\psi_I + \chi^I\bar{\partial}\chi_I - w_{\dot{\alpha}}\bar{\partial}\lambda^{\dot{\alpha}} - w_I^{\alpha}\bar{\partial}\lambda_{\alpha}^I - \phi^I\bar{\partial}\phi_I\Big] \tag{3.92}$$

It turns out that this action allows one to make contact with topological strings when computing the relevant amplitudes in the next chapter. However, the precise meaning of this field redefinition deserves further investigation.

3.3 Spectrum in the Pure Spinor Formalism

In section 3.1, a description of the spectrum in the RNS formalism was discussed. Actually, the discussion there was general enough to include abstract compactifications on an $N = (2,2)$ SCFT. These generic backgrounds and in particular, Calabi-Yau compactifications, need a formulation in the language of pure spinors that is still lacking. For orbifolds, however, once again it is possible to take advantage of the fact that the corresponding worldsheet theory is free. Then, the

BRST charge takes the same form as in flat space and physical state conditions arise in an analogous way.

Equations of motion for compactification independent states come precisely from the pure spinor BRST condition generalized to the orbifold background, if one initially assumes that vertex operators are independent of the internal x, θ coordinates.

For simplicity, the following discussion is given first in the context of open strings. In flat background, the pure spinor BRST condition gives linearized equations of motion for $N = 1$ $D = 10$ super-Yang-Mills (SYM), and it is well known that, upon toroidal compactification, these equations reduce to $N = 4$ SYM in $D = 4$. Furthermore, both $N = 1$ $D = 10$ and $N = 4$ $D = 4$ SYM can be written in terms of $N = 1$ superfields [59]. Precisely, it will now be shown that, assuming independence of the internal coordinates, the BRST condition $QV = 0$ implies the linearized superfield equations of motion for a four-dimensional $N = 1$ super-Maxwell multiplet plus three $N = 1$ chiral multiplets.

The pure spinor BRST supercharge written in four dimensional plus internal coordinates notation reads

$$Q_{ps} = \int dz(\lambda^{\alpha} d_{\alpha} - \lambda_{\dot{\alpha}} d^{\dot{\alpha}} - \lambda^{\alpha I} d_{\alpha I} + \lambda_{\dot{\alpha} I} d^{\dot{\alpha} I}), \tag{3.93}$$

and a general vertex operator of ghost number one is

$$V = \lambda^{\alpha} A_{\alpha} - \lambda_{\dot{\alpha}} A^{\dot{\alpha}} - \lambda^{\alpha I} A_{\alpha I} + \lambda_{\dot{\alpha} I} A^{\dot{\alpha} I} \tag{3.94}$$

The independent conditions for physical states are easily shown to be

$$\lambda^{\alpha}\lambda^{\beta}(D_{\alpha}A_{\beta} + D_{\beta}A_{\alpha}) = 0, \qquad \lambda^{\dot{\alpha}}\lambda^{\dot{\beta}}(D_{\dot{\alpha}}A_{\dot{\beta}} + D_{\dot{\beta}}A_{\dot{\alpha}}) = 0 \tag{3.95}$$

$$\lambda^{\alpha}\lambda_I^{\dot{\beta}}(D_{\alpha}A^I_{\dot{\beta}} + D^I_{\dot{\beta}}A_{\alpha}) = 0, \qquad \lambda^{\dot{\alpha}}\lambda^{\beta I}(D_{\dot{\alpha}}A_{\beta I} + D_{\beta I}A_{\dot{\alpha}}) = 0 \tag{3.96}$$

$$\lambda^{\alpha}\lambda^{\dot{\beta}}(D_{\alpha}A_{\dot{\beta}} + D_{\dot{\beta}}A_{\alpha}) + \lambda^{\alpha I}\lambda_J^{\dot{\beta}}(D_{\alpha I}A^J_{\dot{\beta}} + D^J_{\dot{\beta}}A_{\alpha I}) = 0 \tag{3.97}$$

$$\lambda^{\alpha}\lambda^{\beta I}(D_{\alpha}A_{\beta I} + D_{\beta I}A_{\alpha}) - \frac{1}{2}\lambda_I^{\dot{\alpha}}\lambda_J^{\dot{\beta}}(D^I_{\dot{\alpha}}A^J_{\dot{\beta}} + D^J_{\dot{\beta}}A^I_{\dot{\alpha}}) = 0 \tag{3.98}$$

$$\lambda^{\dot{\alpha}}\lambda_I^{\dot{\beta}}(D_{\dot{\alpha}}A^I_{\dot{\beta}} + D^I_{\dot{\beta}}A_{\dot{\alpha}}) - \frac{1}{2}\lambda^{\alpha I}\lambda^{\beta J}(D_{\alpha I}A_{\beta J} + D_{\beta J}A_{\alpha I}) = 0 \tag{3.99}$$

where $D_{\underline{\alpha}}$ is the spacetime supersymmetric derivative.

The first four equations immediately imply

$$D_\alpha A_\beta + D_\beta A_\alpha = 0, \qquad D_{\dot\alpha} A_{\dot\beta} + D_{\dot\beta} A_{\dot\alpha} = 0 \tag{3.100}$$

$$D_\alpha A^I_{\dot\beta} + D^I_{\dot\beta} A_\alpha = 0, \qquad D_{\dot\alpha} A_{\beta I} + D_{\beta I} A_{\dot\alpha} = 0 \tag{3.101}$$

To obtain solutions of the remaining equations one must use the pure spinor constraints $\lambda^{\underline{\alpha}} \gamma^m_{\underline{\alpha}\underline{\beta}} \lambda^{\underline{\beta}} = 0$, which are written here again in split components for convenience,

$$\lambda^\alpha \lambda^{\dot\beta} - \lambda^{\alpha I} \lambda_I^{\dot\beta} = 0 \tag{3.102}$$

$$\lambda^\alpha \lambda^I_\alpha + \frac{1}{2} \varepsilon^{IJK} \lambda_{\dot\alpha J} \lambda^{\dot\alpha}_K = 0 \tag{3.103}$$

$$\lambda_{\dot\alpha I} \lambda^{\dot\alpha} + \frac{1}{2} \varepsilon_{IJK} \lambda^{\alpha J} \lambda^K_\alpha = 0 \tag{3.104}$$

This constrains the symmetric parts (with respect to indices α, β or $\dot\alpha, \dot\beta$) of the expressions inside parenthesis in (3.98) and (3.99) to be zero, while the antisymmetric parts allow for the appearance of arbitrary fields $A_{\alpha\dot\beta}, A_I, A^I$.

$$D_\alpha A_{\dot\beta} + D_{\dot\beta} A_\alpha = A_{\alpha\dot\beta}, \qquad D_{\alpha I} A^J_{\dot\beta} + D^J_{\dot\beta} A_{\alpha I} = -\delta^J_I A_{\alpha\dot\beta} \tag{3.105}$$

$$\varepsilon^{\alpha\beta}(D_\alpha A_{\beta I} + D_{\beta I} A_\alpha) = 2A_I, \qquad \varepsilon^{\dot\alpha\dot\beta}(D^I_{\dot\alpha} A^J_{\dot\beta} + D^J_{\dot\beta} A^I_{\dot\alpha}) = -2\varepsilon^{IJK} A_K \tag{3.106}$$

$$\varepsilon^{\dot\alpha\dot\beta}(D_{\dot\alpha} A^I_{\dot\beta} + D^I_{\dot\beta} A_{\dot\alpha}) = 2A^I, \qquad \varepsilon^{\alpha\beta}(D_{\alpha I} A_{\beta J} + D_{\beta J} A_{\alpha I}) = 2\varepsilon_{IJK} A^K \tag{3.107}$$

These equations could also be obtained from the $D = 10$ BRST conditions together with the obvious splitting of ten-dimensional superfields, and using the known definition of the superfield A_m. Accordingly, linearized $D = 10$ super-Yang-Mills equations of motion [30] can be written using $D = 4$ superfields as

$$D_\alpha A^\mu - \sigma^\mu_{\alpha\dot\beta} W^{\dot\beta} = \partial^\mu A_\alpha, \qquad D_{\dot\alpha} A^\mu - W^\beta \sigma^\mu_{\beta\dot\alpha} = \partial^\mu A_{\dot\alpha} \tag{3.108}$$

$$D_{\alpha I} A^\mu - \sigma^\mu_{\alpha\dot\beta} W^{\dot\beta}_I = \partial^\mu A_{\alpha I}, \qquad D^I_{\dot\alpha} A^\mu - W^{\beta I} \sigma^\mu_{\beta\dot\alpha} = \partial^\mu A^I_{\dot\alpha} \tag{3.109}$$

$$D_\alpha A^I - W^I_\alpha = \partial^I A_\alpha, \qquad D_{\dot\alpha} A^I = \partial^I A_{\dot\alpha} \tag{3.110}$$

$$D_{\alpha I} A^J - \delta^J_I W_\alpha = \partial^J A_{\alpha I}, \quad \varepsilon_{JKL} D^I_{\dot\alpha} A^J - \delta^I_K W_{\dot\alpha L} + \delta^I_L W_{\dot\alpha K} = \varepsilon_{JKL} \partial^J A^I_{\dot\alpha} \tag{3.111}$$

$$D_\alpha A_I = \partial_I A_\alpha, \qquad D_{\dot\alpha} A_I + 2W_{\dot\alpha I} = \partial_I A_{\dot\alpha} \tag{3.112}$$

$$\varepsilon^{JKL} D_{\alpha I} A_J - 2\delta^K_I W^L_\alpha + 2\delta^L_I W^K_\alpha = \varepsilon^{JKL} \partial_J A_{\alpha I}, \quad D^I_{\dot\alpha} A_J + 2\delta^I_J W_{\dot\alpha} = \partial_J A^I_{\dot\alpha} \tag{3.113}$$

where $(A_{\underline{\alpha}}, A_m)$ are the spinor and vector superpotentials of SYM, and $W^{\underline{\alpha}}$ is the spinor superfield whose lowest component is the gaugino field strength.

As mentioned before, in order to describe the four-dimensional gauge sector of an N=1 orbifold compactification, independence of internal coordinates should be assumed, so all partial derivatives ∂_I, ∂^I are set to zero. Hence, the previous equations reduce to

$$D_\alpha A^\mu - \sigma^\mu_{\alpha\dot{\beta}} W^{\dot{\beta}} = \partial^\mu A_\alpha, \qquad D_{\dot{\alpha}} A^\mu - W^\beta \sigma^\mu_{\beta\dot{\alpha}} = \partial^\mu A_{\dot{\alpha}} \tag{3.114}$$

$$D_{\alpha I} A^\mu - \sigma^\mu_{\alpha\dot{\beta}} W^{\dot{\beta}}_I = \partial^\mu A_{\alpha I}, \qquad D^I_{\dot{\alpha}} A^\mu - W^{\beta I} \sigma^\mu_{\beta\dot{\alpha}} = \partial^\mu A^I_{\dot{\alpha}} \tag{3.115}$$

$$D_\alpha A^I = W^I_\alpha, \qquad D_{\dot{\alpha}} A^I = 0 \tag{3.116}$$

$$D_{\alpha I} A^J = \delta^J_I W_\alpha, \qquad D^I_{\dot{\alpha}} A^J = -\varepsilon^{IJK} W_{\dot{\alpha}K}, \tag{3.117}$$

$$D_\alpha A_I = 0, \qquad D_{\dot{\alpha}} A_I = -2W_{\dot{\alpha}I} \tag{3.118}$$

$$D_{\alpha I} A_J = -2\varepsilon_{IJK} W^K_\alpha, \qquad D^I_{\dot{\alpha}} A_J = -2\delta^I_J W_{\dot{\alpha}} \tag{3.119}$$

The content of these equations is actually $N = 4$ $D = 4$ SYM, containing the gauge superfields A^μ, A^I, A_I. To describe the $N = 1$ supersymmetric field theory, a truncation, which will be the effect of the orbifold nature of the internal sector, must be imposed. This sets to zero the physical $D = 4$ superfields which do not belong to the $N = 1$ gauge supermultiplet. The lowest components of A_I, A^I, which depend only on four-dimensional superspace coordinates are set to zero:

$$A^I| = 0, \quad A_I| = 0, \tag{3.120}$$

where the notation $\Phi|$ means

$$\Phi| = \Phi|_{\theta^I_{\dot{\alpha}}=0,\ \theta_{\dot{\alpha}I}=0} \tag{3.121}$$

Similarly, those $D = 4$ spinor superfield strengths containing extra gauginos as first components are set to zero:

$$W^I_\alpha| = 0, \quad W_{\dot{\alpha}I}| = 0 \tag{3.122}$$

Therefore one can write

$$W^\alpha| = -\frac{1}{4} D_{\dot{\beta}} A^{\dot{\beta}\alpha}| + \frac{1}{4}\partial^{\dot{\beta}\alpha} A_{\dot{\beta}}|, \qquad W^{\dot{\alpha}}| = -\frac{1}{4} D_\beta A^{\dot{\alpha}\beta}| + \frac{1}{4}\partial^{\dot{\alpha}\beta} A_\beta| \tag{3.123}$$

$$D_{\alpha I}A^{\mu}| = \partial^{\mu}A_{\alpha I}|, \qquad D^I_{\dot\alpha}A^{\mu}| = \partial^{\mu}A^I_{\dot\alpha}| \tag{3.124}$$

$$D_{\alpha I}A^J| = \delta^J_I W_\alpha|, \quad D^I_{\dot\alpha}A^J| = 0, \quad D_{\alpha I}A_J| = 0, \quad D^I_{\dot\alpha}A_J| = -2\delta^I_J W_{\dot\alpha}| \tag{3.125}$$

Some important identities, which do not depend on the restriction ...| but hold when independence on internal coordinates is assumed, are in order,

$$D^I_{\dot\alpha}W_{\dot\beta J} = -\delta^I_J D_{\dot\beta}W_{\dot\alpha}, \qquad D_{\alpha I}W_{\dot\beta J} = -\varepsilon_{IJK}D_{\dot\beta}W^K_\alpha, \tag{3.126}$$

$$D^I_{\dot\alpha}W^J_\beta = \varepsilon^{IJK}D_\beta W_{\dot\alpha K}, \qquad D_{\alpha I}W^J_\beta = -\delta^J_I D_\beta W_\alpha. \tag{3.127}$$

They are easily shown by means of the algebra of covariant derivatives,

$$\{D_\alpha, D_\beta\} = 0, \qquad \{D_{\dot\alpha}, D_{\dot\beta}\} = 0, \qquad \{D_\alpha, D_{\dot\beta}\} = -\partial_{\alpha\dot\beta}, \tag{3.128}$$

$$\{D_\alpha, D_{\beta I}\} = -\varepsilon_{\alpha\beta}\partial_I, \quad \{D_{\dot\alpha}, D^I_{\dot\beta}\} = -\varepsilon_{\dot\alpha\dot\beta}\partial^I, \quad \{D_\alpha, D^I_{\dot\beta}\} = 0, \quad \{D_{\dot\alpha}, D_{\beta I}\} = 0, \tag{3.129}$$

$$\{D_{\alpha I}, D_{\beta J}\} = -2\varepsilon_{\alpha\beta}\varepsilon_{IJK}\partial^K, \quad \{D^I_{\dot\alpha}, D^J_{\dot\beta}\} = -2\varepsilon_{\dot\alpha\dot\beta}\varepsilon^{IJK}\partial_K, \quad \{D_{\alpha I}, D^J_{\dot\beta}\} = -\delta^J_I\partial_{\alpha\dot\beta}, \tag{3.130}$$

with the restriction $\partial_I = 0, \partial^I = 0$.

For the uncompactified open superstring, the on-shell integrated vertex operator is

$$V = \int dz(\partial\theta^{\underline{\alpha}}A_{\underline{\alpha}}(x,\theta) + \Pi^m A_m(x,\theta) + d_{\underline{\alpha}}W^{\underline{\alpha}}(x,\theta) + N_{mn}F^{mn}(x,\theta)) \tag{3.131}$$

where, F^{mn} is the superfield whose lowest component is the gauge vector field strength.

To obtain the integrated vertex operator for the on-shell $D = 4$ SYM multiplet from (3.131), one simply requires all superfields to satisfy the N=1 orbifold constraints deduced for the unintegrated one. Although the resulting vertex operator will depend on all 16 θ's, it will be independent of the zero modes of the internal coordinates (x^I, x_I) for $I = 1$ to 3.

As seen in section 3.1, the spectrum of type II superstrings compactified on an orbifold to four dimensions consists of a gravitational supermultiplet which contains the graviton and the graviphoton, a universal hypermultiplet which contains the dilaton, the axion and two Ramond-Ramond scalars, and sets of hypermultiplets and vector supermultiplets which, in the case of Calabi-Yau compactifications, are related to the cohomology of the internal manifold [60].

The closed string vertex operator for these multiplets is obtained from the left-right product of two open superstring vertex operators and has the form

$$V = \int d^2z (\partial\theta^{\alpha}\bar{\partial}\tilde{\theta}^{\underline{\beta}} A_{\underline{\alpha\beta}}(x,\theta,\tilde{\theta}) + ...) \tag{3.132}$$

where $A_{\underline{\alpha\beta}}$ is the left-right product of $A_{\underline{\alpha}}$ and $A_{\underline{\beta}}$ and ... involves similar left-right products of the other superfields.

As will be shown in the next chapter, the only terms in the closed superstring vertex operator (3.132) which will contribute to topological amplitudes are the terms

$$\int d^2z \left[d^{\alpha}\tilde{d}^{\beta}P_{\alpha\beta} + d^{\dot{\alpha}}\tilde{d}^{\dot{\beta}}P_{\dot{\alpha}\dot{\beta}} + d^{\alpha}\tilde{d}^{\dot{\beta}}Q_{\alpha\dot{\beta}} + d^{\dot{\alpha}}\tilde{d}^{\beta}Q_{\dot{\alpha}\beta} \right] \tag{3.133}$$

where $(P_{\alpha\beta}, P_{\dot{\alpha}\dot{\beta}})$ are superfields whose lowest components are the graviphoton anti-self-dual and self-dual field strengths, and $(Q_{\alpha\dot{\beta}}, Q_{\dot{\alpha}\beta})$ are superfields whose lowest components are derivatives of the complex Ramond-Ramond scalar. These $N = 2$ $D = 4$ superfields can be understood as the left-right product of chiral and anti-chiral photino $N = 1$ $D = 4$ superfields, i.e. $P_{\alpha\beta}$ is obtained from the left-right product of W_{α} with W_{β}, $P_{\dot{\alpha}\dot{\beta}}$ is obtained from the left-right product of $W_{\dot{\alpha}}$ with $W_{\dot{\beta}}$, $Q_{\alpha\dot{\beta}}$ is obtained from the left-right product of W_{α} with $W_{\dot{\beta}}$, and $Q_{\dot{\alpha}\beta}$ is obtained from the left-right product of $W_{\dot{\alpha}}$ with W_{β}. Of course, the complete integrated vertex operator will contain additional terms to those of (3.133), but it will be argued that only the terms in (3.133) will contribute to the topological amplitudes.[1] Moreover, vertex operators for compactification-dependent states will not be needed in the next chapter, so they will not be discussed here.

[1]This is similar to the situation in the hybrid formalism [60, 71, 72], but the difference here is that the vertex operators in the pure spinor formalism depend on all 16 θ and 16 $\tilde{\theta}$ variables.

Chapter 4

Superstring topological amplitudes

The type of orbifold compactifications of type II superstrings described in the previous chapter preserves $N = 2$ supersymmetry in $D = 4$. A crucial role is played by the superconformal field theory induced by the internal model [61]. With focus in the left-moving sector of the string, this SCFT is generated by the following currents

$$T, \quad G^+, \quad G^-, \quad J \tag{4.1}$$

satisfying the OPE's

$$T(y)T(z) \longrightarrow \frac{3\hat{c}}{2(y-z)^4} + \frac{2T(z)}{(y-z)^2} + \frac{\partial T(z)}{y-z} \tag{4.2}$$

$$T(y)G^{\pm}(z) \longrightarrow \frac{3G^{\pm}(z)}{2(y-z)^2} + \frac{\partial G^{\pm}(z)}{y-z}, \qquad T(y)J(z) \longrightarrow \frac{J(z)}{(y-z)^2} + \frac{\partial J(z)}{y-z} \tag{4.3}$$

$$G^+(y)G^-(z) \longrightarrow \frac{2\hat{c}}{(y-z)^3} + \frac{2J(z)}{(y-z)^2} + \frac{2T(z) + \partial J(z)}{z-w} \tag{4.4}$$

$$J(y)G^{\pm}(z) \longrightarrow \pm\frac{G^{\pm}(z)}{y-z}, \qquad J(y)J(z) \longrightarrow \frac{\hat{c}}{(y-z)^2} \tag{4.5}$$

where $\hat{c} = 3$, and from which one can see that $G^{\pm}$ and J are conformal primaries of weights $h = 3/2$ for $G^{\pm}$ and $h = 1$ for J. Also, the central charge is $c = 3\hat{c} = 9$, and there is a U(1) current anomaly with value $\hat{c}$.

Topological quantum field theories were first developed as a way to understand, from a physical point of view, some topological properties of four-dimensional manifolds. Topological Yang-Mills (TYM) theory [62] is a generally covariant quantum field theory in which general covariance is unbroken. This means, in particular, that it contains no graviton and correlation functions of this theory is independent of geometrical details of the four-manifold in which the theory lives. TYM was found to be related to $N = 2$ $D = 4$ SYM through a redefinition of the spins of supersymmetric currents. One of these becomes a BRST-like current which then allows to define *observables* of the theory. The shift on spins is, of course, equivalent to a redefinition of the stress tensor. Similarly, one could twist the $N = 2$ supersymmetric structure in two-dimensions. Starting from $N = 2$ supersymmetric nonlinear sigma models one gets a topological theory in two dimensions [63]. The change in the stress tensor is made using the U(1) current of the $N = 2$ superconformal algebra, and it can be made in two directions

$$T \longrightarrow T \pm \frac{1}{2}\partial J \tag{4.6}$$

The effect of the $(+)$ twisting is to set the conformal weight of G^+ equal to one, while that of G^- becomes $h = 2$. The $(-)$ twist gives the reverse assignment of conformal weights. Furthermore, with either one of the new stress tensor, the new central charge of the algebra vanishes, while the U(1) anomaly stays the same.

Twisted $N = 2$ SCFT can then be coupled to two-dimensional gravity to construct the topological string theory. There are two possibilities for closed strings. If the left- and right-moving sectors are twisted in the same direction, the theory is called the B model, while if the twistings are performed in opposite directions, one is talking about the A model; both models are topological strings [64]. In particular, the partition function at genus $g \geq 2$ for the B model is given by

$$F_g^B = \int_{\mathcal{M}_g} \left\langle \left| \prod_{i=1}^{3g-3} (G^-, \mu_i) \right|^2 \right\rangle \tag{4.7}$$

while the partition function for the A model is

$$F_g^A = \int_{\mathcal{M}_g} \left\langle \prod_{i=1}^{3g-3} (G^-, \mu_i) \prod_{j=1}^{3g-3} (\tilde{G}^+, \bar{\mu}_j) \right\rangle \tag{4.8}$$

In type II superstrings compactified to $D = 4$ while preserving $N = 2$ space-time supersymmetry, some amplitudes are related to terms in the low-energy effective field theory whose couplings are precisely these partition functions of topological strings.[1]

This chapter presents a computation of topological amplitudes using a compactification of the pure spinor formalism as presented in [66]. Before discussing it, RNS and hybrid calculations are reviewed in some detail.

[1]For a brief review of the relation between topological strings and physical couplings in the superstring, see [65].

4.1 RNS computation

That certain type II genus g superstring amplitudes are given by the topological string partition function at the same genus, F_g, was first demonstrated in [67]. There the authors used the RNS formalism to compute g-loop amplitudes for the scattering of $2g-2$ graviphotons and two gravitons coming from the superstring compactified on an internal $c = 9, N = (2,2)$ superconformal theory. As shown in the previous chapter, these compactifications have $N = 2$ supergravity coupled to matter and vector multiplets as its four-dimensional low energy effective field theory. Although the amplitudes were calculated both in orbifold and Calabi-Yau compactifications, only the orbifold case will be considered here.

Recall that the spectrum of these theories consist of a gravitational supermultiplet which contains the graviton and the graviphoton, a universal hypermultiplet which contains the dilaton and the axion, and sets of hypermultiplets and vector multiplets which, in the case of Calabi-Yau compactifications, are related to the cohomology of the internal manifold. Actually, to see which type of amplitudes lead to the topological partition functions a heuristic argument contained in [68] will be presented first. Then, an explicit calculation will be given in detail.

4.1.1 Heuristic derivation

For convenience, the argument will focus on the left-moving sector of the superstring. The goal is to find a relation between untwisted and twisted $N = 2$ SCFT's.

In the untwisted (superstring) side there are more worldsheet fields than in the twisted one, and there must be some cancellations to get only the twisted field content contributing to the path integral. For example, in addition to the untwisted $\hat{c} = 3$, $N = 2$ superconformal field theory for the internal sector, the RNS superstring contains fermionic reparametrization ghosts (b, c) of conformal weights $(2, -1)$ and bosonic superconformal ghosts (β, γ) with conformal weights $(3/2, -1/2)$. There is also the spacetime sector, with bosonic fields (written in complex notation) $x^i, x^{\bar{i}}$ with $h = 0$ and fermionic ones $\psi^i, \psi^{\bar{i}}$ $(i = 1, 2)$ with $h = 1/2$.

The important thing to observe is that all these extra fields come in boson-fermion pairs, and their nonzero modes would tend to cancel in the path integral if each field in a pair had equal conformal weight. One thus needs to change conformal weights of $\beta, \gamma, \psi^i, \psi^{\bar{i}}$ appropriately as well as the fermions in the internal $N = 2$ SCFT. Moreover, this last shift would be just the twisting of the internal $N = 2$ SCFT by the U(1) current.

One should recall that the conformal weight of a field can be changed by adding to the worldsheet action of the theory a background charge, that is, the coupling of the bosonized field (call it φ) to the scalar curvature, R. The term to be added is

$$\pm \frac{1}{2} \int R\varphi \tag{4.9}$$

which changes conformal weight by $1/2$, and the sign depends on how the shift is done between a field and its conjugate momenta; that is, which one has its spin increased and which one, decreased. In a genus g surface one can choose the scalar curvature to have delta function support at $2g - 2$ points. Each such point would give rise to a factor in the path integral

$$e^{\frac{1}{2} \sum_a (-1)^{n_a} \varphi_a} \tag{4.10}$$

where a runs over all pairs of conjugate fields, one needs to twist, and $n_a = 0, 1$, giving the direction in which the shift is done. Now, some details of the argument will depend on the specific type of superstring theory.

The graviphoton vertex operator at picture $-1/2$ reads

$$V_T^{(-1/2)} = e^{-\frac{1}{2}(\phi+\tilde{\phi})} \left[k_{\alpha\dot{\gamma}} a_\beta^{\ \dot{\gamma}} S^{(\alpha} \tilde{S}^{\beta)} \Sigma + k_{\gamma\dot{\alpha}} a^\gamma_{\ \dot{\beta}} S^{(\dot{\alpha}} \tilde{S}^{\dot{\beta})} \bar{\Sigma} \right] e^{ik\cdot x}, \qquad \text{type IIA} \tag{4.11}$$

and the same expression with Σ ($\bar{\Sigma}$) replaced by Ξ ($\bar{\Xi}$) for type IIB. The two terms correspond to the anti-self-dual and self-dual parts of the graviphoton operator, respectively.

It is clear now that the $2g - 2$ insertions required to change the conformal weight of appropriate worldsheet fields can be taken as self-dual (or anti-self-dual) graviphoton insertions in particular kinematic configurations (choices of components in the spacetime spin fields). But there is another possibility; what if one takes Ξ as the operator for the internal sector in type IIA and similarly, take Σ as the corresponding operator in type IIB superstring? Then one needs to take four dimensional spinors of opposite chirality for the left- and right-moving sectors. The vertex operators arising this way precisely describe the RR bosons in the universal hypermultiplet.

$$V_Z^{(-1/2)} = e^{-\frac{1}{2}(\phi+\tilde{\phi})} k_{\alpha\dot{\beta}} S^{\alpha} \tilde{S}^{\dot{\beta}} \Xi e^{ik\cdot x}, \qquad \text{type IIA} \tag{4.12}$$

$$V_{\bar{Z}}^{(-1/2)} = e^{-\frac{1}{2}(\phi+\tilde{\phi})} k^{\dot{\alpha}\beta} S_{\dot{\alpha}} \tilde{S}_{\beta} \bar{\Xi} e^{ik\cdot x}, \qquad \text{type IIA} \tag{4.13}$$

and similar expressions with Ξ ($\bar{\Xi}$) replaced by Σ ($\bar{\Sigma}$) for type IIB.

So the amplitudes of interest must involve $2g-2$ graviphotons or $2g-2$ universal RR scalars. Besides, to have conformal invariance each of these insertions have to be integrated over the worldsheet. This can be seen as integration over the new moduli arising from punctures on the Riemann surface. These insertions thus play an important role in the cancellation of non-zero modes, and it is known that this feature is crucial in order to reduce (or localize) the theory to its topological sector.

The effect of Σ insertions is to twist spins in opposite directions for the left- and right-moving sectors, so it is expected to give as a result the partition function for the topological A-model, F_g^A. Similarly, Ξ insertions would be related to the partition function of the topological B model, F_g^B. Therefore, we expect amplitudes involving graviphotons in type IIA (IIB) superstring theory to reduce to the topological A (B) model partition function, and amplitudes involving universal RR scalars in type IIA (IIB) superstring theory to reduce to the topological B (A) model partition function.

However, to show this we still need to take care of the zero modes present in the superstring side computation. After the twisting produced by the insertions, there are $3g-3$ b, $3g-3$ β, g ψ^i (for each i), and 1 $\psi^{\bar{i}}$ (for each i) zero modes. These zero modes need to be absorbed in order for the partition function not to vanish.

It is well-known that one must insert in the superstring measure a factor of

$$\left| \prod_{i=1}^{3g-3} (\mu_i, b) \right|^2 \tag{4.14}$$

Besides, to absorb the β zero modes in an amplitude with $2g-2$ insertions of picture $-1/2$ graviphoton (or RR scalar) vertex operators, one should insert a total of $3g-3$ factors of $\delta(\beta)G$, where G is the supersymmetry current of the full theory. One can choose the same basis of Beltrami differentials as before to fold with those factors.

One uses now charge conservation of the internal $N=2$ theory to see that not every term in G contributes to the amplitude since there is some internal $U(1)$ charge present in the vertex operator insertions. For example, in type IIB theory amplitudes, involving anti-self-dual graviphotons, the relevant contribution from the $\delta(\beta)G$ factors is just[2]

$$\left|\prod_{i=1}^{3g-3} \delta(\beta) G^-_{\text{int}}(\mu_i)\right|^2 \tag{4.15}$$

since each graviphoton vertex carry $+3/2$ U(1) charge for both left- and right-moving sectors, due to the presence of the Ξ operator. The zero modes of b and $\delta(\beta)$ give opposite contributions to the path integral and, therefore, the b, c, β and γ ghosts completely decouple. Besides, we would nicely get the amplitude prescription for the topological B model after integrating over the moduli space of genus-g Riemann surfaces.

For type IIA, the relevant expression is

$$\prod_{j=1}^{3g-3} \delta(\beta) G^-_{\text{int}}(\mu_j) \prod_{k=1}^{3g-3} \delta(\tilde{\beta}) G^+_{\text{int}}(\bar{\mu}_k) \tag{4.16}$$

giving rise to the prescription for the topological A model.

It is also necessary to absorb the spacetime fermion zero modes. It can be shown that the two zero modes $\psi^{\bar{i}}$ and two of the zero modes ψ^i (and the corresponding zero modes for the right moving sector) can be absorbed by inserting two anti-self-dual graviton vertex operators when one has graviphoton insertions or two dilaton-axion vertex operators when one has universal RR scalar insertions.

It remains to absorb $2g-2$ zero modes of ψ^i. The way this happens can be better seen by going to a particular kinematic configuration for the graviphoton vertex operators. Take, for instance the $2g-2$ anti-self-dual graviphoton insertions in the kinematic configuration where all spacetime spin fields appear as $S^{++}\tilde{S}^{++}$. Then one absorbs the remaining $2g-2$ zero modes of ψ^i by inserting $g-1$ operators $\psi^1\psi^2\tilde{\psi}^1\tilde{\psi}^2$ at $g-1$ of the points given by the delta-function curvature singularity. This has the effect of transforming $g-1$ graviphoton vertex operators into those with kinematics $S^{--}\tilde{S}^{--}$, without changing self-duality property of the vertex.

Thus, the amplitude involving two anti-self-dual graviton and $2g-2$ anti-self-dual graviphoton vertex operators reproduces the topological string partition function of the $\hat{c}=3$ twisted $N=2$ internal theory, and the type of topological model depends on the type II superstring theory.

[2]Either $(\mu_i, \mathcal{O})$ or $\mathcal{O}(\mu_i)$ is used as abbreviation for $\int dz \mu_i(z)\mathcal{O}(z)$, interchangeably.

In fact, the authors in [67] started computing the amplitude for a specific kinematic configuration and then argued, using four-dimensional Lorentz invariance and $N = 2$ supersymmetry, that the generic string computation gives two different expressions corresponding to spacetime terms in the low energy effective action of the superstring compactification. The result they found is that the field theory limit of those amplitudes can be obtained from the effective action

$$S_{\text{eff}} = \int d^4x\, gF_g \left[R^2 (T^2)^{g-1} + 2(g-1)(RT)^2 (T^2)^{g-2} \right] \tag{4.17}$$

where R is short for the anti-self dual Riemann curvature and T for the anti-self-dual field strength of the graviphoton, and

$$R^2 = R_{\mu\nu\rho\sigma} R^{\mu\nu\rho\sigma}, \qquad T^2 = T_{\mu\nu} T^{\mu\nu}, \tag{4.18}$$

$$(RT)^2 = R_{\mu\nu\rho\sigma} R^{\mu\nu\lambda\omega} T^{\rho\sigma} T_{\lambda\omega} \tag{4.19}$$

The amplitude involving two dilaton-axions ($\partial\partial S$) and $2g-2$ universal RR scalars (Z) was also shown to reproduce the topological string amplitude (in sort of a complementary way compared to the graviton-graviphoton amplitude), the corresponding low-energy effective action being

$$S_{\text{eff}} = \int d^4x\, g\tilde{F}_g \left[(\partial\partial S)^2 ((\partial Z)^2)^{g-1} + 2(g-1)(\partial\partial S \partial Z)^2 ((\partial Z)^2)^{g-2} \right] \tag{4.20}$$

where

$$(\partial\partial S)^2 = (\partial_\mu \partial_\nu S)(\partial^\mu \partial^\nu S), \qquad (\partial Z)^2 = \partial_\mu Z \partial^\mu Z \tag{4.21}$$

$$(\partial\partial S \partial Z)^2 = (\partial_\mu \partial_\nu S \partial^\nu Z)(\partial^\mu \partial^\rho S \partial_\rho Z) \tag{4.22}$$

Here $\tilde{F}_g$ denotes the topological string partition function for the B (A) model if F_g obtained from the graviton-graviphoton amplitude corresponds to the A (B) model.

4.1.2 Explicit computation

At g loops, the $2g$-point scattering amplitude between two gravitons and $2g-2$ graviphoton external states in type IIB superstring is now explicitly calculated.

If all graviphotons insertions have picture $-1/2$, and both gravitons are at picture zero, then $3g-3$ insertions of picture changing operators are needed to cancel the background charge of ϕ. Therefore, the amplitude reads[3]

$$\int_{\mathcal{M}_g} \sum_{\boldsymbol{\alpha}} \epsilon_{\boldsymbol{\alpha}} \left\langle \left| \prod_{i=1}^{3g-3} b(\mu_i) \right|^2 \int d^2z V_g^1(z) \int d^2z' V_g^2(z') \prod_{i=1}^{2g-2} \int d^2z_i V_T^i(z_i) \left| \prod_{i=1}^{3g-3} Z(y_i) \right|^2 \right\rangle_{g,\boldsymbol{\alpha}} \tag{4.23}$$

To evaluate the anti-self-dual amplitude it is necessary to extract the corresponding anti-self-dual parts of the graviphoton and graviton in their vertex operators.

The field strength $f_{\mu\nu}$ of a vector field a_μ can be decomposed as

$$f_{\mu\nu} = (\sigma_{\mu\nu})_\alpha^{\ \beta} \varepsilon^{\alpha\gamma} f_{\beta\gamma} + (\sigma_{\mu\nu})^{\dot\alpha}_{\ \dot\beta} \varepsilon^{\dot\gamma\dot\beta} f_{\dot\beta\dot\gamma} \tag{4.24}$$

where $f_{\alpha\beta} = k_{(\alpha|\dot\gamma} a^{\dot\gamma}_{|\beta)} = k_\mu a_\nu (\sigma^{\mu\nu})_\alpha^{\ \gamma} \varepsilon_{\gamma\beta}$, and similarly $f_{\dot\alpha\dot\beta} = k_{\gamma(\dot\alpha} a^{\gamma}_{\ \dot\beta)}$. Anti-self-dual graviphotons are described using the usual vertex operator by setting $f_{\dot\alpha\dot\beta}$ to zero. That is

$$V_T^{(-1/2)} = e^{-\frac{1}{2}(\phi+\tilde\phi)} f_{\alpha\beta} S^\alpha \tilde S^\beta \Xi^0 \, e^{ik\cdot x} \tag{4.25}$$

with the usual constraint for BRST invariance, $k^\mu a_\mu = 0$.

A similar restriction is made to describe anti-self-dual graviton states.

The linearized Riemann tensor $R_{\mu\mu'\nu\nu'}$ is given by

$$R_{\mu\nu\mu'\nu'} = \frac{1}{2}(k_\nu k_{\mu'} h_{\mu\nu'} - k_\mu k_{\mu'} h_{\nu\nu'} - k_\nu k_{\nu'} h_{\mu\mu'} + k_\mu k_{\nu'} h_{\nu\mu'}) \tag{4.26}$$

The anti-self-dual part of this tensor is described by a multispinor $R_{\alpha\beta\gamma\delta}$ as[4]

$$R^{antiSD}_{\mu\nu\mu'\nu'} = \varepsilon^{\alpha\rho}\varepsilon^{\gamma\omega} (\sigma_{\mu\nu})_\rho^{\ \beta} (\sigma_{\mu'\nu'})_\omega^{\ \delta} \, R_{\alpha\beta\gamma\delta} \tag{4.27}$$

and setting all other components to zero. The part of the vertex which is quartic on fermions reduces to

$$\frac{1}{2} R^{\alpha\ \ \gamma}_{\ \ \beta\ \ \delta} (\sigma_{\mu\nu})_\alpha^{\ \beta} (\sigma_{\mu'\nu'})_\gamma^{\ \delta} \psi^\mu \psi^\nu \tilde\psi^{\mu'} \tilde\psi^{\nu'} \tag{4.28}$$

To compute the scattering amplitudes, correlators for different fields are taken from the formulas of section 2.4.

[3] Sum over orbifold sectors are implicit in this expression.
[4] ρ and ω are four-dimensional chiral spinor indices.

The correlators

There are essentially six correlators to compute in this amplitude; those for

- $D = 4$ spacetime coordinate fields x^μ, which gives the usual Koba-Nielsen factors;
- $D = 4$ spcatime fermions ψ^μ and their chiral spin fields S^α;
- reparametrization ghosts b, c,
- superconformal ghosts, which involve the field ϕ;
- fermions in the internal sector ψ^I, ψ_I which can be twisted or untwisted; and
- bosonic internal coordinates x^I, x_I which can also be twisted or untwisted.

The basic contribution from spacetime coordinate x^μ, which are non-compact bosonic fields, is known to be

$$\left\langle e^{ik\cdot x}(z)e^{ik'\cdot x}(z')\prod_{i=1}^{2g-2}e^{ik_i\cdot x}(z_i)\right\rangle = \tag{4.29}$$

$$= (\det(\mathrm{Im}\tau))^{-2}|Z_1|^{-4}\delta\Big(k+k'+\sum_{i=1}^{2g-2}k_i\Big)\Pi_{2g}(z_j;k_j) \tag{4.30}$$

where $\Pi_{2g}(z_i, k_i)$ is the Koba-Nielsen factor whose low energy limit is just one. This is the relevant limit to study the lowest order couplings of the effective supergravity field theory. Higher order terms won't be considered in this thesis. Also, the delta function imposing momentum conservation will be understood when writing the final form of the amplitude.

There are in principle contributions of the form $\langle \partial x^\mu \prod \exp(ik_i\cdot x)\rangle$, and so on; these correlators don't appear in non-vanishing terms of the amplitudes, as will be shown later.

The b, c correlator is immediate from equation (2.139) with $\lambda = 2$:

$$\left\langle\prod_{i=1}^{3g-3}b(w_i)\right\rangle = Z_1^{-1/2}\Theta\Big(-\sum_{i=1}^{3g-3}w_i+3\Delta\Big|\tau\Big)\prod_{i<j}^{3g-3}E(w_i,w_j)\prod_{i=1}^{3g-3}\sigma(w_i)^3 \tag{4.31}$$

The ϕ correlator is obtained from (2.179) with $\lambda = \frac{3}{2}$, and $M = g - 1$:

$$\left\langle \prod_{i=1}^{3g-3} e^{\phi}(y_i) \prod_{j=1}^{2g-2} e^{-\frac{1}{2}\phi}(z_j) \right\rangle_{\alpha} = \frac{Z_1^{1/2}}{\Theta[\boldsymbol{\alpha}](\sum_{i=1}^{3g-3} y_i - \frac{1}{2}\sum_{j=1}^{2g-2} z_j - 2\Delta|\tau)} \tag{4.32}$$

$$\times \frac{\prod_{i,j=1} E(y_i, z_j)^{1/2}}{\prod_{i<j} E(y_i, y_j) \prod_{i<j} E(z_i, z_j)^{1/4}} \frac{\prod_{i=1} \sigma(z_i)}{\prod_{i=1} \sigma(y_i)^2} \tag{4.33}$$

Sectors with spin 1/2

To compute the remaining factors, one should notice that not all terms in the picture changing operators contribute. Recall that the RNS supercurrent, for an orbifold compactification is simply written as

$$G = \psi^{\mu}\partial x_{\mu} + \psi^{I}\partial x_{I} + \psi_{I}\partial x^{I} + G_{ghost} \tag{4.34}$$

On the other hand each graviphoton contains a copy of the internal operator Ξ^0, whose left-moving part reads (for an orbifold CFT)

$$\exp(\frac{i}{2}\sum_{I=1}^{3}\varphi_I) \tag{4.35}$$

where φ_I appear in the bosonization of the internal fermions as

$$\psi_I = e^{i\varphi_I}, \qquad \psi^I = e^{-i\varphi_I} \tag{4.36}$$

Since each of these fields has $\lambda = \frac{1}{2}$, hence, vanishing background charge, there must be a total cancellation from the insertions in the correlators for each $I = 1, 2, 3$. Graviphotons give a total charge of $g - 1$ for each φ_I; it is easy to see that, in order to cancel this charge, each picture changing operator must contribute exclusively with the term $\psi^I \partial x_I$. So amplitudes with ψ^{μ} insertions at positions y_i vanish.

Then the relevant correlators for ψ^{μ} and S^{α} are

$$\left\langle (\sigma_{\mu\nu})_{\alpha}{}^{\beta} \psi^{\mu}\psi^{\nu}(z) (\sigma_{\mu'\nu'})_{\gamma}{}^{\delta} \psi^{\mu'}\psi^{\nu'}(z') \prod_{i=1}^{2g-2} S^{\alpha_i}(z_i) \right\rangle_{\alpha}, \tag{4.37}$$

$$\left\langle (\sigma_{\mu\nu})_\alpha{}^\beta \psi^\mu \psi^\nu(w) \prod_{i=1}^{2g-2} S^{\alpha_i}(z_i) \right\rangle_{\boldsymbol{\alpha}}, \qquad \left\langle \prod_{i=1}^{2g-2} S^{\alpha_i}(z_i) \right\rangle_{\boldsymbol{\alpha}} \tag{4.38}$$

while those for the internal fermions in the sector with twist structure $\boldsymbol{\phi}$ have the form

$$\left\langle \prod_{j=1}^{g-1} e^{-i\varphi}(u_j) \prod_{i=1}^{2g-2} e^{\frac{i}{2}\varphi}(z_i) \right\rangle_{\boldsymbol{\alpha}} = Z_1^{-1/2} \Theta[\boldsymbol{\phi}+\boldsymbol{\alpha}]\Big(\sum_{j=1}^{g-1} u_j - \frac{1}{2} \sum_{i=1}^{2g-2} z_i \Big| \tau \Big) \tag{4.39}$$

$$\times \frac{\prod_{i<j} E(u_i,u_j) \prod_{i<j} E(z_i,z_j)^{1/4}}{\prod_{i,j=1} E(u_i,z_j)^{1/2}} \tag{4.40}$$

where this applies for each of the orbifold planes $I = 1,2,3$. Take, for example, a partition of the positions y_i of picture changing operators into three sets as $U_I = \{u_{iI}, i = 1, ..., g-1\}$, then the total contribution from internal fermions is

$$Z_1^{-3/2} \prod_{I=1}^{3} \Theta[\boldsymbol{\phi}_I+\boldsymbol{\alpha}]\Big(\sum_{j=1}^{g-1} u_{jI} - \frac{1}{2} \sum_{i=1}^{2g-2} z_i \Big| \tau \Big) \frac{\prod_{i<j} E(z_i,z_j)^{3/4} \prod_{I=1}^{3} \prod_{i<j} E(u_{iI},u_{jI})}{\prod_{i,j=1} E(y_i,z_j)^{1/2}} \tag{4.41}$$

Finally, with this partition of the positions y_i, the correlators for the bosonic internal coordinates is

$$\prod_{I=1}^{3} \left\langle \prod_{i=1}^{g-1} \partial x_I(u_{iI}) \right\rangle \tag{4.42}$$

Notice that non-zero modes of ∂x_I only can contract with x^I; hence, only their zero-modes contribute in the correlator.

The correlator involving ψ^μ,S^α fields will give sums of terms, each one containing two factors of theta functions as $\Theta[\boldsymbol{\alpha}](-D_1|\tau)\Theta[\boldsymbol{\alpha}](-D_2|\tau)$. This is due to the fact that two complex fermion systems are involved here. Recall that to bosonize the fermions, one has to define the complex combinations $\psi^1 \pm i\psi^2$, $\psi^3 \pm i\psi^4$. Then, one defines for $a = 1,2$

$$\psi^a = e^{i\sigma_a}, \qquad \psi_a = e^{-i\sigma_a} \tag{4.43}$$

Sum over spin structures

Before attempting to compute the correlator, it is convenient to analyze the sum over spin structures in the total amplitude. Notice that all information about spin structure is contained in the theta functions with characteristics arising from

(ψ^μ, S^α), ϕ, and φ_I correlators. That is, it is entirely contained in expressions like

$$\frac{\Theta[\boldsymbol{\alpha}](-D_1|\tau)\Theta[\boldsymbol{\alpha}](-D_2|\tau)\prod_{I=1}^{3}\Theta[\boldsymbol{\phi}_I+\boldsymbol{\alpha}]\left(\sum_{j=1}^{g-1}u_{jI}-\frac{1}{2}\sum_{i=1}^{2g-2}z_i\middle|\tau\right)}{\Theta[\boldsymbol{\alpha}](\sum_{i=1}^{3g-3}y_i-\frac{1}{2}\sum_{j=1}^{2g-2}z_j-2\Delta|\tau)} \tag{4.44}$$

To perform the sum over spin structures it will be convenient to take the last expression into a form suitable for applying the Riemann theta identity [69]

$$\sum_{\boldsymbol{\alpha}}\prod_{i=0}^{3}\Theta[\boldsymbol{\phi}_i+\boldsymbol{\alpha}](e_i|\tau)=2^g\prod_{i=0}^{3}\Theta[\boldsymbol{\phi}'_i](e'_i|\tau) \tag{4.45}$$

where

$$e'_0 = \frac{1}{2}(e_0+e_1+e_2+e_3) \tag{4.46}$$
$$e'_1 = \frac{1}{2}(e_0+e_1-e_2-e_3) \tag{4.47}$$
$$e'_2 = \frac{1}{2}(e_0-e_1+e_2-e_3) \tag{4.48}$$
$$e'_3 = \frac{1}{2}(e_0-e_1-e_2+e_3) \tag{4.49}$$

and $\boldsymbol{\phi}_i \to \boldsymbol{\phi}'_i$ is the same change of basis, component-wise. The expression (4.44), points at the specific case where $\boldsymbol{\phi}_0 = \mathbf{0}$ and $\boldsymbol{\phi}_i = \boldsymbol{\phi}_I$, for $i = 1, 2, 3$. Also, using the relation valid for supersymmetric orbifolds, $\sum_{I=1}^{3}\boldsymbol{\phi}_I = 0$, one can see that the previous change of basis does nothing to the twist structures $\{\mathbf{0}, \boldsymbol{\phi}_I\}$.

Then, the sum over spin structures adapted to the case of interest is

$$\sum_{\boldsymbol{\alpha}}\Theta[\boldsymbol{\alpha}](e_0|\tau)\prod_{I=1}^{3}\Theta[\boldsymbol{\phi}_I+\boldsymbol{\alpha}](e_I|\tau)=2^g\Theta(e'_0|\tau)\prod_{I=1}^{3}\Theta[\boldsymbol{\phi}_I](e'_I|\tau) \tag{4.50}$$

To apply this formula, one can set the argument of the theta function in the denominator equal to one of the divisors arising in the ψ^μ, S^α sector, by choosing conveniently the positions of the picture changing operator insertions, say

$$\sum_{i=1}^{3g-3}y_i=\frac{1}{2}\sum_{j=1}^{2g-2}z_j+2\Delta-D_1 \tag{4.51}$$

This is always possible by means of the *Jacobi inversion theorem* [42].[5]

[5]The theorem says that, given a divisor D on Σ_g, there is generically a unique set of points

Then, the sum over spin structures is written as

$$\sum_{\boldsymbol{\alpha}} \epsilon_{\boldsymbol{\alpha}} \Theta[\boldsymbol{\alpha}](-D_2|\tau) \prod_{I=1}^{3} \Theta[\boldsymbol{\phi}_I + \boldsymbol{\alpha}]\Big(\sum_{j=1}^{g-1} u_{jI} - \frac{1}{2} \sum_{i=1}^{2g-2} z_i \Big| \tau \Big) \tag{4.52}$$

where the constants $\epsilon_{\boldsymbol{\alpha}}$, which are just phases, are still undetermined. To get their correct value on needs to check the monodromy of this expression around different homology cycles of Σ_g. This way it is possible to see that, once one of the $\epsilon_{\boldsymbol{\alpha}}$ is fixed to, say, one, all the other are also one.

By setting

$$e_0 = -D_2, \qquad e_I = \sum_{i=1}^{g-1} u_{iI} - \frac{1}{2} \sum_{i=1}^{2g-2} z_i \tag{4.53}$$

then, using $\sum_{I=1}^{3} \sum_{i=1}^{g-1} u_{iI} = \sum_{i=1}^{3g-3} y_i$, one gets

$$e'_0 = \frac{1}{2}\Big(-D_1 - D_2 - \sum_{i=1}^{2g-2} z_i \Big) + \Delta \tag{4.54}$$

$$e'_I = \frac{1}{2}(D_1 - D_2) + \sum_{i=1}^{g-1} u_{iI} - \Delta \tag{4.55}$$

Divisors D_1 and D_2 will depend on specific correlators in (4.37) and (4.38). In any case, after summing over spin structures the result is

$$2^g \Theta\Big(\frac{1}{2}\Big(-D_1 - D_2 - \sum_{i=1}^{2g-2} z_i \Big) + \Delta \Big| \tau \Big) \prod_{I=1}^{3} \Theta[\boldsymbol{\phi}_I] \Big(\frac{1}{2}(D_1 - D_2) + \sum_{i=1}^{g-1} u_{iI} - \Delta \Big| \tau \Big) \tag{4.56}$$

Chiral spin fields of ψ^μ are given by $S^\alpha = (S^{++}, S^{--})$, where

$$S^{++} = e^{\frac{i}{2}(\sigma_1 + \sigma_2)}, \qquad S^{--} = e^{-\frac{i}{2}(\sigma_1 + \sigma_2)} \tag{4.57}$$

Taking first,

$$\Big\langle \prod_{i=1}^{2g-2} S^{\alpha_i}(z_i) \Big\rangle_{\boldsymbol{\alpha}} \tag{4.58}$$

it is immediate to see that background charge cancellation implies that only correlators with an equal number of S^{++} and S^{--} insertions could give non-vanishing results. Denoting the $g-1$ positions of S^{++} as v_i and those $g-1$ of S^{--} as v'_i, this

$P_1, \ldots, P_g$ such that $[D] = \sum_{i=1}^{g} P_i$. Thus, by choosing conveniently $2g-3$ y_i points in (4.51), the remaining g y_i points are completely determined.

means that divisors D_1 and D_2 are

$$D_1 = D_2 = \frac{1}{2}\sum_{i=1}^{g-1}(v_i - v_i') \tag{4.59}$$

Substitution into (4.56) gives

$$2^g \Theta\Big(-\sum_{i=1}^{g-1} v_i + \Delta\Big|\tau\Big)\prod_{I=1}^{3}\Theta[\boldsymbol{\phi}_I]\Big(\sum_{i=1}^{g-1} u_{iI} - \Delta\Big|\tau\Big) = 0 \tag{4.60}$$

where the first theta function gives a factor zero as a consequence of the Riemann Vanishing Theorem.

As for the remaining correlators it is convenient to express the fermion bilinear insertion in terms of complex ψ^a, ψ_a. Taking two arbitrary chiral spinors $\chi^\alpha, \lambda^\alpha$, one can express

$$\chi^\alpha (\sigma_{\mu\nu})_\alpha^{\ \beta}\varepsilon_{\beta\gamma}\lambda^\gamma\psi^\mu\psi^\nu = -\chi^{++}\lambda^{++}\psi_1\psi_2 - 4\chi^{--}\lambda^{--}\psi^1\psi^2 \tag{4.61}$$

$$+\,2(\chi^{++}\lambda^{--} + \chi^{--}\lambda^{++})(\psi^1\psi_1 + \psi^2\psi_2) \tag{4.62}$$

Thus, the only three combinations of fermion which appear in the anti-self-dual graviton vertex operators are $\psi^1\psi^2$, $\psi_1\psi_2$, and $\psi^1\psi_1 + \psi^2\psi_2$. Each one of these picks different components of the anti-self-dual Riemann tensor to be contracted with remaining factors in the amplitude.

Concerning only one bilinear insertion, it is easy to see that the correlator

$$\left\langle \psi^1\psi_1(w)\prod_{i=1}^{2g-2} S^{\alpha_i}(z_i)\right\rangle_{\boldsymbol{\alpha}} \tag{4.63}$$

vanishes. One needs to replace the insertion $\psi^1\psi_1(w)$ by its point-splitting definition $\lim_{w'\to w}\left[\psi^1(w')\psi_1(w) - (w'-w)^{-1}\right]$, and compute the respective correlators. The same result holds for $\psi^2\psi_2(w)$.

The next correlator

$$\left\langle \psi^1\psi^2(w)\prod_{i=1}^{2g-2} S^{\alpha_i}(z_i)\right\rangle_{\boldsymbol{\alpha}} \tag{4.64}$$

is non-vanishing only if the number of S^{++} and S^{--} insertions is $g-2$ and g, respectively. The corresponding divisors are

$$D_1 = D_2 = w + \frac{1}{2}\sum_{i=1}^{g-2} v_i - \frac{1}{2}\sum_{i=1}^{g} v'_i \tag{4.65}$$

Then, after summing over spin structures the result contains the theta function

$$\Theta\Big(-w - \sum_{i=1}^{g-2} v_i + \Delta \Big| \tau\Big) = 0 \tag{4.66}$$

which again is zero due to the Riemann Vanishing Theorem. The last correlator of this type containing $\psi_1\psi_2(w)$ also vanishes. This can be shown by noticing that $\Theta[\boldsymbol{\alpha}](-D_1|\tau)\Theta[\boldsymbol{\alpha}](-D_2|\tau) = \Theta[\boldsymbol{\alpha}](D_1|\tau)\Theta[\boldsymbol{\alpha}](D_2|\tau)$, and taking an appropriate definition of $\sum_{i=1}^{3g-3} y_i$ (with D_1 instead of $-D_1$) in the original choice.

From all this discussion, the conclusion is that terms in the graviton vertex operators proportional to ∂x^μ (or $\bar{\partial} x^\mu$) do not contribute to the scattering amplitude.

Thus, the contribution could only come from

$$\Big\langle (\sigma_{\mu\nu})_\alpha{}^\beta \psi^\mu\psi^\nu(z)(\sigma_{\mu'\nu'})_\gamma{}^\delta \psi^{\mu'}\psi^{\nu'}(z')\prod_{i=1}^{2g-2} S^{\alpha_i}(z_i)\Big\rangle_{\boldsymbol{\alpha}} \tag{4.67}$$

When expanding in terms of ψ^a, ψ_a, many terms give zero contribution due to the Riemann Vanishing Theorem. Take for example

$$\Big\langle \psi^1\psi^2(z)\psi^1\psi^2(z')\prod_{i=1}^{2g-2} S^{\alpha_i}(z_i)\Big\rangle \tag{4.68}$$

Background charge cancellation implies that there must be $g-3$ v_i and $g+1$ v'_i (which automatically excludes a possible contribution for $g=2$). It is easy to see that the divisors are such that, after summing over spin structures there is a factor of

$$\Theta\Big(-z - z' - \sum_{i=1}^{g-3} v_i + \Delta \Big| \tau\Big) = 0 \tag{4.69}$$

The first non-vanishing correlator is

$$\Big\langle \psi^1\psi^2(z)\psi_1\psi_2(z')\prod_{i=1}^{2g-2} S^{\alpha_i}(z_i)\Big\rangle \tag{4.70}$$

This time, there must be an equal number of S^{++} and S^{--} insertions. The result is

$$\left\langle \psi^1\psi^2(z)\psi_1\psi_2(z')\prod_{i=1}^{g-1}S^{++}(v_i)\prod_{j=1}^{g-1}S^{--}(v'_j)\right\rangle_{\boldsymbol{\alpha}} = \tag{4.71}$$

$$= Z_1^{-1}\Big[\Theta[\boldsymbol{\alpha}]\Big(-z+z'-\frac{1}{2}\sum_{i=1}^{g-1}v_i+\frac{1}{2}\sum_{i=1}^{g-1}v'_i\Big|\tau\Big)\Big]^2\frac{1}{E(z,z')^2} \tag{4.72}$$

$$\times\frac{\prod_{i<j}E(v_i,v_j)^{1/2}\prod_{i<j}E(v'_i,v'_j)^{1/2}}{\prod_{i,j=1}E(v_i,v'_j)^{1/2}}\frac{\prod_{i=1}E(z,v_i)\prod_{i=1}E(z',v'_i)}{\prod_{i=1}E(z,v'_i)\prod_{i=1}E(z',v_i)} \tag{4.73}$$

After summing over spin structures, the resulting product of theta functions is

$$2\Theta\Big(-z+z'-\sum_{i=1}^{g-1}v_i+\Delta\Big|\tau\Big)\prod_{I=1}^{3}\Theta[\boldsymbol{\phi}_I]\Big(\sum_{i=1}^{g-1}u_{iI}-\Delta\Big|\tau\Big) \tag{4.74}$$

Putting it all together

Assembling all previous results, one gets a kinematic factor involving components of $f_{\alpha\beta}$ and $R_{\alpha\beta\gamma\delta}$, times

$$2^g Z_1^{-5/2}\Theta\Big(-\sum_{i=1}^{3g-3}w_i+3\Delta\Big|\tau\Big)\Theta\Big(-z+z'-\sum_{i=1}^{g-1}v_i+\Delta\Big|\tau\Big)\prod_{I=1}^{3}\Theta[\boldsymbol{\phi}_I]\Big(\sum_{i=1}^{g-1}u_{iI}-\Delta\Big|\tau\Big) \tag{4.75}$$

$$\times\prod_{i<j}E(w_i,w_j)\prod_i\sigma(w_i)^3\prod_{i<j}E(v_i,v_j)\prod_{i<j}E(v'_i,v'_j) \tag{4.76}$$

$$\times\frac{\prod_i E(z,v_i)\prod_i E(z',v'_i)}{E(z,z')^2\prod_i E(z,v'_i)\prod_i E(z',v_i)}\frac{\prod_I\prod_{i<j}E(u_{iI},u_{jI})}{\prod_{i<j}E(y_i,y_j)}\frac{\prod_i\sigma(z_i)}{\prod_j\sigma(y_i)^2} \tag{4.77}$$

In order to further simplify this expression, it is convenient to trade the positions of the b ghost insertions w_i for the positions of the picture changing operators y_i. This is easily done by dividing the bosonization formula (4.31) for $\left\langle\prod_{i=1}^{3g-3}b(w_i)\right\rangle$ by the same formula for b insertions at y_i. Denoting by h_i a basis for zero modes of $\mathcal{L}_b$, that is, for the space of holomorphic quadratic differentials, one obtains the following equation

$$\Theta\Big(-\sum_{i=1}^{3g-3}w_i+3\Delta\Big|\tau\Big)\frac{\prod_{i<j}E(w_i,w_j)\prod_i\sigma(w_i)^3}{\prod_{i<j}E(y_i,y_j)\prod_i\sigma(y_i)^2} = \tag{4.78}$$

$$= \frac{\det(h_i(w_j))}{\det(h_i(y_j))}\Theta\Big(-\sum_{i=1}^{3g-3} y_i + 3\Delta\Big|\tau\Big)\prod_{i=1}^{3g-3}\sigma(y_i) \tag{4.79}$$

substituting this into (4.75), making use of the choice (4.51) which was already made for $D_1 = z - z' + \frac{1}{2}\sum_{i=1}^{g-1} v_i - \frac{1}{2}\sum_{i=1}^{g-1} v_i'$, and using the relation $\Theta[\boldsymbol{\phi}](-z|\tau) = \Theta[-\boldsymbol{\phi}](z|\tau)$, gives

$$\begin{aligned}
&-2^g Z_1^{-1/2}\Theta\Big(-z-\sum_{i=1}^{g-1} v_i + z' + \Delta\Big|\tau\Big)\frac{\prod_i E(z,v_i)\prod_{i<j}E(v_i,v_j)}{E(z,z')\prod_i E(v_i,z')}\frac{\prod_i\sigma(v_i)\prod_i\sigma(z)}{\prod_i\sigma(z')}\\
&\times Z_1^{-1/2}\Theta\Big(-z'-\sum_{i=1}^{g-1} v_i' + z + \Delta\Big|\tau\Big)\frac{\prod_i E(z',v_i')\prod_{i<j}E(v_i',v_j')}{E(z',z)\prod_i E(v_i',z)}\frac{\prod_i\sigma(v_i')\prod_i\sigma(z')}{\prod_i\sigma(z)}\\
&\times\prod_{I=1}^{3}\left[Z_1^{-1/2}\Theta[-\boldsymbol{\phi}_I]\Big(-\sum_{i=1}^{g-1} u_{iI} + \Delta\Big|\tau\Big)\prod_{i<j}E(u_{iI},u_{jI})\prod_{i=1}\sigma(u_{iI})\right]\frac{\det(h_i(w_j))}{\det(h_i(y_j))}
\end{aligned} \tag{4.80}$$

Here one can recognize the presence of partition functions Z_1, and $Z_{1,-\boldsymbol{\phi}}$. The amplitude now corresponds effectively to two fermionic untwisted $(1,0)$, and three fermionic $\boldsymbol{\phi}_I$-twisted $(1,0)$ systems.

Using (2.143) and (2.159), including the contribution from x^μ and x^I, x_I correlators, an the right-sector, (which have a completely analogous form for type IIB superstrings) the result is proportional to

$$\int_{\mathcal{M}_g}\left|\int dw\mu(w)\right|^2\int d^2z\, d^2z'\prod_i^{2g-2} d^2z_i\,\frac{|\det(\omega_i(v_j,z))\det(\omega_i(v_j',z'))|^2}{(\det(\mathrm{Im}\tau))^2} \tag{4.81}$$

$$\times\left|\frac{\det(h_i(w_j))}{\det(h_i(y_j))}\right|^2\left|\prod_{I=1}^{3}\left[Z_{1,-\boldsymbol{\phi}_I}\det(\omega_{-\boldsymbol{\phi}_I,i}(u_{jI}))\left\langle\prod_{i=1}^{g-1}\partial x_I(u_{iI})\right\rangle\right]\right|^2 \tag{4.82}$$

The zero mode part of the insertion ∂x_I can be expanded in terms of a basis of $g-1$ $\boldsymbol{\phi}$-twisted holomorphic one-differentials $\omega_{\boldsymbol{\phi}_I,i}$. This means that objects like $\omega_{-\boldsymbol{\phi}_I,i}\partial x_I$ appearing in the last form of the amplitude have trivial monodromy around the homology cycles of Σ_g; hence they are ordinary holomorphic quadratic differentials. On the other hand, the expression obtained corresponds to a fixed partition $u_{i,I}$ of the insertion points y_j $(i = 1, ..., 3g-3)$. The total contribution is given by summing over all possible partitions with appropriate antisymmetrization. The effect of this is the appearance of a factor $|\det(h_i(y_j))|^2$ which will cancel all y_i dependence of the amplitude. Then, it is allowed to perform integration over all the positions of the vertex operators using $\mathrm{Im}\tau_{ij} = \int d^2z\omega_i(z)\bar{\omega}_j(\bar{z})$, to get

$$\int_{\mathcal{M}_g} \left|\int dw\mu(w)\right|^2 \left|\det(h_i(w_j))\right|^2 \left|\prod_{I=1}^{3}\left[Z_{1,-\boldsymbol{\phi}_I}\left\langle\prod_{i=1}^{g-1} p_{iI}\right\rangle\right]\right|^2 \tag{4.83}$$

where p_{iI} denotes the coefficients in the zero mode expansion of ∂x_I.

Now it is possible to *undo* a path integral which would give rise to the determinant $Z_{1,-\boldsymbol{\phi}_I}$, by introducing *twisted* fermions ψ^I, ψ_I of spins $(1,0)$; this way, the integrand in the amplitude can be rewritten as products of $\psi^I\partial x_I$ contributing with zero modes, as

$$\int_{\mathcal{M}_g} \left\langle \prod_{i=1}^{3g-3} G^-_{int}(\mu_i) \right\rangle \tag{4.84}$$

where $G^-_{int} = \psi^I\partial x_I$, and the correlator means path integration over x^I, x_I, ψ^I, ψ_I. Therefore, the result is the topological string partition function for the B model, as expected from the heuristic derivation, times the kinematic factor which can now be covariantize to reproduce (4.17). Amplitudes for type IIA, as well as those involving RR scalar are computed in an entirely analogous way.

4.2 Derivation using the hybrid formalism

Of course the previous results could have been obtained in a more direct way had one used a formalism which manifestly preserve the symmetries ($D = 4$ super-Poincaré) of the model. Although the covariant Green-Schwarz formalism which has these symmetries manifest in ten dimensions does not allow for straightforward quantization, in orbifold compactifications only four-dimensional Lorentz and $N = 2$ spacetime supersymmetry need to be manifest. Then, going to a first order lagrangian description it is possible to write four-dimensional Green-Schwarz-Siegel-like variables to describe type II superstrings. As mentioned in the introduction, this was successfully done in the hybrid formalism [70] by adding chiral bosons of negative energy $\rho, \tilde{\rho}$, to the set of usual superspace variables $x^\mu, \theta^\alpha, p^\alpha, \bar{\theta}^{\dot{\alpha}}, \bar{p}^{\dot{\alpha}}, \tilde{\theta}^\alpha, \tilde{p}^\alpha, \bar{\tilde{\theta}}^{\dot{\alpha}}, \bar{\tilde{p}}^{\dot{\alpha}}$. The internal variables can be treated in the same way as in the RNS formalism. The formalism has $N = 2$ superconformal symmetry generated by T, G^+, G^-, J (and similarly for the right-moving sector), that can be used to show equivalence with the light-cone Green-Schwarz formalism.

Because of manifest spacetime supersymmetry, vertex operators can be written as $N = 2$ $D = 4$ superfields, and they will not have square roots with the supersymmetry generators; therefore, GSO projection is automatic, and there is no need to sum over spin structures. Indeed, in the original RNS computation of the previous section, the effective twisting of the fields is only achieved after summing products of theta functions over all spin structures of the Riemann surface; in the hybrid formalism, twisting is immediate once one consider the graviphoton or hypermultiplet insertions.

The $N = 2$ superconformal algebra for the hybrid left-moving sector reads

$$T = -\frac{1}{2}\partial x^\mu \partial x_\mu - p^\alpha \partial\theta_\alpha - \bar{p}_{\dot{\alpha}}\partial\bar{\theta}^{\dot{\alpha}} - \frac{1}{2}\partial\rho\partial\rho + T_{\text{int}} \tag{4.85}$$

$$G^+ = -\frac{1}{4\sqrt{2}}e^{-\rho}\bar{d}_{\dot{\alpha}}\bar{d}^{\dot{\alpha}} + G^+_{\text{int}}, \qquad G^- = \frac{1}{4\sqrt{2}}e^{\rho}d^\alpha d_\alpha + G^-_{\text{int}}, \qquad J = \partial\rho + J_{\text{int}} \tag{4.86}$$

and it has $\hat{c} = 2$.[6]

A nice feature of the hybrid formalism is that all vertex operators of the universal sector can be written exclusively in terms of four-dimensional spacetime variables. Actually, a single scalar superfield $V(x, \theta, \bar{\theta}, \tilde{\theta}, \tilde{\bar{\theta}})$ can be used to describe the massless universal sector of compactified type II superstrings. In supergravity language this is just the prepotential for the $N = 2$ gravitational and tensor multiplets [60]. Furthermore, in order for this V to be physical it must be a conformal primary of the $N = 2$ superconformal algebra for both the right and left moving sector, and must have zero conformal weight and zero $U(1)$ charge. All this implies

$$D^2 V = \bar{D}^2 V = \tilde{D}^2 V = \tilde{\bar{D}}^2 V = \partial_\mu \partial^\mu V = 0 \tag{4.87}$$

The integrated vertex operator is obtained by computing

$$U = \int d^2 z \{\tilde{G}^-, [\tilde{G}^+, \{G^-, [G^+, V]\}]\}, \tag{4.88}$$

and this operator is invariant under gauge transformations

$$\delta V = D^2\Lambda + \bar{D}^2\Lambda + \tilde{D}\Lambda + \tilde{\bar{D}}^2\Lambda \tag{4.89}$$

which allows reducing the component fields to the physical ones. More explicitly,

[6]This in contrast to the non-minimal pure spinor formalism, which has $\hat{c} = 3$. The spacetime sector of the hybrid formalism contribute with $\hat{c} = -1$, which then adds to the $\hat{c} = 3$ of the internal sector.

in the Wess-Zumino gauge one can derive the expression [60, 71]

$$V_{WZ} = h_{\mu\nu}(\theta\sigma^{\mu}\bar{\theta})(\tilde{\theta}\sigma^{\nu}\tilde{\bar{\theta}}) + \left[T_{\alpha\beta}\theta^{\alpha}\tilde{\theta}^{\beta}\bar{\theta}_{\dot{\gamma}}\bar{\theta}^{\dot{\gamma}}\tilde{\bar{\theta}}_{\dot{\delta}}\tilde{\bar{\theta}}^{\dot{\delta}} + \text{h.c.}\right] \tag{4.90}$$

$$+ \left[(\partial_{\alpha\dot{\beta}}Z + ...)\theta^{\alpha}\theta_{\dot{\gamma}}\theta^{\dot{\gamma}}\tilde{\bar{\theta}}^{\dot{\beta}}\tilde{\theta}^{\delta}\tilde{\theta}_{\delta} + \text{h.c.}\right] + ... \tag{4.91}$$

where $h_{\mu\nu}$ contains the graviton, the dilaton and the antisymmetric tensor, $T_{\alpha\beta}$ ($T_{\dot{\alpha}\dot{\beta}}$) is the antiself-dual (self-dual) graviphoton field strength, and $Z, \bar{Z}$ are the two RR scalars of the universal hypermultiplet.

On the other hand one can construct chiral, $P_{\alpha\beta}$, and twisted-chiral, $Q_{\alpha\dot{\beta}}$, $N = 2$ superfields describing the gravitational multiplet and the universal hypermultiplet, respectively. These read

$$P_{\alpha\beta} = \bar{D}^2 D_{\alpha}\tilde{\bar{D}}^2\tilde{D}_{\beta}V, \qquad Q_{\alpha\dot{\beta}} = \bar{D}^2 D_{\alpha}\tilde{D}^2\tilde{\bar{D}}_{\dot{\beta}}V \tag{4.92}$$

Both of these superfield already appear in the integrated vertex operator as

$$U = \int d^2z\left[d^{\alpha}\tilde{d}^{\beta}P_{\alpha\beta} + d^{\alpha}\tilde{\bar{d}}^{\dot{\beta}}Q_{\alpha\dot{\beta}}\right] + \text{h.c.} + ... \tag{4.93}$$

where the dots include all other terms obtained after the computation of (4.88).

Now, one is interested in reproducing the results of the previous section using this formalism. In order to do that, it is necessary to review the appropriate amplitude prescription for the hybrid formalism. By construction the computation can be performed in a manifestly supersymmetric way, and the amplitudes involve in general the entire integrated vertex operator U.

Hybrid amplitude prescription

Since the theory has a $\hat{c} = 2$ $N = 2$ superconformal symmetry, one can extend it to an $N = 4$ superconformal symmetry, and it is possible to use the $N = 4$ topological string prescription of [72] to compute the amplitude.[7]

The extra generators which provide a completion to the $N = 4$ algebra are given by

$$J^{\pm\pm} = e^{\pm(\rho + \int J_{\text{int}})} \tag{4.94}$$

$$\breve{G}^{+} = -\frac{1}{4\sqrt{2}}e^{2\rho+\int J_{\text{int}}}d^{\alpha}d_{\alpha} - e^{\rho}G^{++}_{\text{int}} \tag{4.95}$$

[7] It happens that the hybrid formalism also allows for computation of non-topological superstring amplitudes, as in [73].

$$\breve{G}^- = -\frac{1}{4\sqrt{2}} e^{-2\rho - \int J_{\text{int}}} \bar{d}_{\dot{\alpha}} \bar{d}^{\dot{\alpha}} - e^{-\rho} G^{--}_{\text{int}} \tag{4.96}$$

where $G^{\pm\pm}_{\text{int}} = \oint e^{\pm J_{\text{int}}}(G^{\mp}_{\text{int}})$. The structure of the set $\{J, J^{++}, J^{--}\}$ is that of an $SU(2)$ current algebra. The rising (J^{++}) and lowering (J^{--}) operators have been used to construct the two new fermionic generators through

$$\breve{G}^+ = \int J^{++}(G^-), \qquad \breve{G}^- = \int J^{--}(G^+) \tag{4.97}$$

The amplitude prescription is the following. First twist the superconformal algebra in such a way that the generators $G^+, \breve{G}^+$ both have conformal weight $h = 1$ and the generators $G^-, \breve{G}^-$ both have conformal weight $h = 2$ (The corresponding right moving generators must be twisted in the same way for type IIB superstrings, and in the opposite way for type IIA). Then, physical fields Φ in this twisted $N = 4$ are defined by $G^+\Phi = \breve{G}^+\Phi = 0$, $\Phi \sim \Phi + G^+\breve{G}^+\chi$, where the identification is made for any field χ. In order to take advantage of the usual $N = 2$ topological prescription, which requires insertion of $3g - 3$ G^- operators folded with a basis of Beltrami differentials, one can insert $\oint \breve{G}^+$ around each a-cycle of the genus g Riemann surface (and the corresponding right-moving insertion). This restrict states flowing through the cycles to those belonging to the reduced Hilbert space of states annihilated by $\breve{G}^+$. Then one is left to take care of the other condition $G^+\Phi = 0$, which can be thought of as a twisted $N = 2$ physical condition in the reduced Hilbert space. Finally, to decouple only trivial states of the $N = 4$ theory, that is, those of the form $G^+\breve{G}^+\chi$, one should insert $\int J\tilde{J}$ in the correlation function.

The prescription for type IIB scattering amplitude of $2g$ integrated vertices at genus g is thus

$$\mathcal{A}_g = \int_{\mathcal{M}_g} \frac{1}{\det(\text{Im}\tau)} \prod_{i=1}^{g} \int d^2 v_i \left\langle \left| \prod_{j=1}^{g-1} \breve{G}^+(v_j) J(v_g) \prod_{k=1}^{3g-3} G^-(\mu_k) \right|^2 \prod_{l=1}^{2g} U_l \right\rangle \tag{4.98}$$

where τ is the period matrix of Σ_g, and one has used the relation

$$\left| \prod_{i=1}^{g} \oint_{a_i} \breve{G}^+ \right|^2 \propto \frac{1}{\det(\text{Im}\tau)} \prod_{i=1}^{g} \int d^2 v_i \left| \prod_{j=1}^{g} \breve{G}^+(v_j) \right|^2 \tag{4.99}$$

(an extra factor of $\int \breve{G}$ can be obtained by writing one of the G^- present there as $\oint \breve{G}^+ J^{--}$ and pulling the contour off the surface until it hits the J insertion).

It is also convenient to use the relation [71]

$$\prod_{i=1}^{g}\int d^2v_i\left|\breve{G}^+(v_i)\right|^2=\frac{\det(\mathrm{Im}\tau)}{|\det(\omega_j(\hat{v}_k))|^2}\left|\prod_{k=1}^{g}\breve{G}^+(\hat{v}_k)\right|^2 \tag{4.100}$$

where $\hat{v}_k$ are g arbitrary points on the Riemann surface, to recast the amplitude to the form

$$\mathcal{A}_g=\int_{\mathcal{M}_g}\frac{1}{|\det(\omega_i(\hat{v}_j))|^2}\left\langle\left|\prod_{j=1}^{g-1}\breve{G}^+(\hat{v}_j)J(\hat{v}_g)\prod_{k=1}^{3g-3}G^-(\mu_k)\right|^2\prod_{l=1}^{2g}U_l\right\rangle \tag{4.101}$$

One feature of the $N=4$ topological nature of the hybrid formalism is that one actually has a family of $N=4$ models parametrized by harmonic variables u_1,u_2,u_1^*,u_2^* related by complex conjugation as $\bar{u}_1=u_2^*,\bar{u}_2=-u_1^*$, and satisfying $|u_1|^2+|u_2|^2=1$. The general expression for the generators is

$$\widehat{\breve{G}^+}=u_1\breve{G}^++u_2G^+ \tag{4.102}$$

$$\widehat{G^-}=u_1G^--u_2\breve{G}^- \tag{4.103}$$

$$\widehat{\breve{G}^-}=u_2^*\breve{G}^--u_1^*G^- \tag{4.104}$$

$$\widehat{G^+}=u_2^*G^++u_1^*\breve{G}^+ \tag{4.105}$$

These are $SU(2)$ rotations of the superconformal generators which are not realized as a symmetry of the algebra. Then, they should give different partition functions and amplitudes which can be thought of as realizing vector representations of $SU(2)$. In particular, making a similar construction for the right-moving sector, instead of a partition function this topological $N=4$ model provides one with a partition tensor $F_g^{n,m}$ where $n,m=2-2g,3-2g...,2g-2$.

By making the replacements $G^-\to\widehat{G^-}$, $\breve{G}^+\to\widehat{\breve{G}^+}$ in the expression for the genus g amplitude (4.98), one gets a homogenoeus polynomial in the $u_i,\tilde{u}_i$ variables,

$$\mathcal{A}_g=\sum_{n,m=2-2g}^{2g-2}\binom{4g-4}{2g-2-n}\binom{4g-4}{2g-2-m}\mathcal{A}_g^{n,m}u_1^{2g-2+n}u_2^{2g-2-n}\tilde{u}_1^{2g-2+m}\tilde{u}_2^{2g-2-m} \tag{4.106}$$

It turns out that the amplitudes involving the anti-self-dual graviphoton, as well as the amplitude with self-dual graviphotons, and both possible amplitudes with universal RR scalars, appear as different components of this amplitude tensor $\mathcal{A}_g^{n,m}$. In order to show this one has to consider the R-transformations of the model. This transformation is generated by

$$R = \int dz \Big[\partial\rho + \frac{1}{2}\theta^\alpha d_\alpha - \frac{1}{2}\bar{\theta}_{\dot{\alpha}} d^{\dot{\alpha}} \Big] \tag{4.107}$$

There is a field redefinition between hybrid and RNS variables which shows that this is the RNS picture number operator expressed in terms of hybrid fields. The background charge for this operator is $1-g$ and must be canceled for an amplitude to be non-vanishing. Since insertions $G^\pm$ carry no R-charge, and insertions $\check{G}^\pm$ carry R-charge ∓ 1, it is easy to show that in the generic component $\mathcal{A}_g^{n,m}$, these insertions always contribute a charge of $(g-1-n, g-1-m)$. Thus, vertex operators must contribute a total charge (n,m) in non-vanishing amplitudes.

Hybrid computation

For graviphoton or universal RR scalars, the relevant amplitudes can be immediately determined. For example, since the $2g$ anti-self-dual graviphoton superfields carry $(1/2, 1/2)$ R-charge and the expected result involves integration over half the superspace, $\int d^2\theta d^2\tilde{\theta}$, vertex operators will violate R-charge by $(g-1, g-1)$; thus, the relevant amplitude is $\mathcal{A}_g^{g-1,g-1}$. Similarly, anti-self-dual scattering is computed by $\mathcal{A}_g^{1-g,1-g}$. Finally, universal hypermultiplets carry R-charge $(1/2, -1/2)$ or $(-1/2, 1/2)$ and its scattering is computed by the amplitude $\mathcal{A}_g^{g-1,1-g}$ or $\mathcal{A}_g^{1-g,g-1}$, respectively.

It turns out that for each of these amplitudes, only specific terms in the insertions contribute so that each time a simpler amplitude expression must be computed. In order to see this one should look for ρ and $\int J_{\text{int}}$ background charge cancellation separately.[8]

First, ρ background charge cancellation requires $\mathcal{A}_g^{g-1,g-1}$ to be

[8] J background charge cancellation give no further restriction. Also, J_{int} must be conserved because of symmetry of the internal Calabi-Yau model.

$$\mathcal{A}_g^{g-1,g-1} = \int_{\mathcal{M}_g} \frac{1}{|\det(\omega_i(\hat{v}_j))|^2} \left\langle \left| \prod_{j=1}^{m} e^{\rho} G_{\text{int}}^{++}(\hat{v}_j) \prod_{j=m+1}^{g-1} e^{-\rho} \bar{d}_{\dot{\alpha}} \bar{d}^{\dot{\alpha}}(\hat{v}_j) J(\hat{v}_g) \right. \right.$$
$$\left. \left. \times \prod_{k=1}^{m} (e^{-2\rho - \int J_{\text{int}}} \bar{d}_{\dot{\alpha}} \bar{d}^{\dot{\alpha}})(\mu_k) \prod_{k=m+1}^{3g-3} G_{\text{int}}^{-}(\mu_k) \right|^2 \prod_{l=1}^{2g} U_l \right\rangle \tag{4.108}$$

where U_l are vertex operators for graviphoton superfields.

In order to obtain the non-vanishing contribution to this amplitude one must have enough zero modes coming from the correlation to be absorbed by the corresponding fermionic integrations. On a Riemann surface of genus g, fields of conformal weight $h = 1$ have g zero modes, and its conjugate fields of conformal weight $h = 0$ have only one zero mode corresponding to the constant function. It is easy to see that the $2g$ zero modes in d_α must come from the vertex operators, from the specific terms of the form $\int d^2 z d_\alpha \tilde{d}_\beta P^{\alpha\beta}$. Besides, the $d_{\dot{\alpha}}$ zero modes are not saturated in the last expression, and the functional integration of the field ρ is not well-defined as it stands. One way to overcome this problems is to express one of the vertex operators as $U = |\int dz e^{-\rho} \bar{d}_{\dot{\alpha}} \bar{d}^{\dot{\alpha}}(z)|^2 \int d^2 w |e^{\rho} d^{\alpha} D_\alpha|^2 V$, where V is the prepotential. Then, pulling the first factor off the vertex it only hits $J(\hat{v}_g)$.

One last thing to do before actually computing the amplitude is to appropriately fix the g arbitrary points $(\hat{v}_j)$ to be at the same position as g of the arguments in Beltrami differentials. This is possible only because when fusing the correspondent operators one does not encounter poles nor zeros in the OPE's.

The amplitude becomes

$$\mathcal{A}_g^{g-1,g-1} = \int_{\mathcal{M}_g} \int \prod_{j=1}^{g} d^2 z_j \frac{1}{|\det(\omega_i(z_j))|^2} \left\langle (\mu_j e^{-\rho} G_{\text{int}}^{-} \bar{d}_{\dot{\alpha}} \bar{d}^{\dot{\alpha}})(z_j) \prod_{l=1}^{2g-3} G_{\text{int}}^{-}(\mu_l) \right.$$
$$\left. \times \prod_{r=1}^{2r-1} \int d^2 y_r d_\alpha \tilde{d}_\beta P^{\alpha\beta} \int d^2 y_{2g} |e^{\rho} d^{\alpha} D_\alpha|^2 V \right\rangle \tag{4.109}$$

All zero modes of $d_{\dot{\alpha}}$ are now saturated and, besides the usual functional integration over the x^μ fields, one only needs to explicitly compute the correlator involving ρ. The regulated correlation function turns out to be

$$\left\langle \prod_{j=1}^{g} e^{-\rho(z_j)} e^{\rho(y_{2g})} \right\rangle = \frac{1}{Z_1^2 \det(\omega_i(z_j))} \tag{4.110}$$

which is independe of the position y_{2g}. Thus, the contribution of the $\rho, \tilde{\rho}$ functional integration is $|Z_1|^{-4}|\det(\omega_i(z_j))|^{-2}$.

To obtain the low-energy effective field theory at the lowest order, it is enough to consider only the x^μ zero-mode dependence of the vertex operators. Functional integration over x^μ then gives a factor of $|Z_1|^{-4}(\det(\mathrm{Im}\tau))^{-2}$.

Now it is time to evaluate the functional integral over d_α, θ_α. Since all of them contribute only with its zero modes, they give, besides integration over zero modes of θ, $\int d^2\theta d^2\tilde{\theta}$, a factor $|Z_1|^4|\det(\omega_I(y_r))|^4$, which can be safely integrated over all the positions of vertex operators, producing $|Z_1|^4(\det(\mathrm{Im}\tau))^2$. One also gets contractions between the various spinor indices in the vertex operators. Similarly, integration over fields $\bar{\theta}_{\dot{\alpha}}, \bar{d}_{\dot{\alpha}}$ gives $|Z_1|^4|\det(\omega_i(z_j))|^4$ and there remains an integral $\int d^2\bar{\theta}d^2\tilde{\bar{\theta}}$.

Putting all this together one arrives, after many cancellations, at

$$\mathcal{A}_g^{g-1,g-1} = \int d^2\theta d^2\tilde{\theta} d^2\bar{\theta} d^2\tilde{\bar{\theta}} (P_{\alpha\beta}P^{\alpha\beta})^{g-1} P_{\gamma\delta} D^\delta \tilde{D}^\gamma V \int_{\mathcal{M}_g} \left\langle \prod_{i=1}^{3g-3} |G^-_{\mathrm{int}}(\mu_i)|^2 \right\rangle \tag{4.111}$$

Performing the integration over $\bar{\theta}$ and $\tilde{\bar{\theta}}$, and recognizing the partition function of the topological B model, F_g^B, the result is

$$\mathcal{A}_g^{g-1,g-1} = F_g^B \int d^2\theta d^2\tilde{\theta} (P_{\alpha\beta}P^{\alpha\beta})^g \tag{4.112}$$

Notice that if one expands the superfield $P_{\alpha\beta}$ in components and performs the $\theta, \tilde{\theta}$ integrations the RNS result is correctly reproduced. For type IIA superstrings the twisting in the right-moving sector must be done in the opposite direction, and the coupling is F_g^A as expected. The other amplitudes can be evaluated in a similar way; for example the scattering of universal hypermultiplets (Z) in type IIB gives

$$\mathcal{A}^{g-1,1-g} = F_g^A \int d^2\theta d^2\tilde{\bar{\theta}} (Q_{\alpha\dot{\beta}}Q^{\alpha\dot{\beta}})^g \tag{4.113}$$

In this section, a super-Poincaré covariant computation of topological amplitudes was performed using the hybrid formalism. It is worth noting that the explicit form of the internal G^-_{int} was never used, so this result automatically extends to Calabi-Yau compactifications. This is in contrast with RNS, where the explicit calculation works in an straightforward manner only for orbifolds. For generic Calabi-Yau manifolds in the internal sector, the underlying $N = 2$ superconformal theory is used, noticing that the spin structure information resides entirely in the U(1) current J_{int}. It is very suggestive that this is the only object that mixes the spacetime and internal sectors in the field redefinition that relates the RNS and hybrid formalisms.

4.3 Pure spinor computation

As in a flat $D = 10$ background, for $g \geq 2$, the g-loop n-point amplitude prescription on an orbifold compactification is taken to be

$$\mathcal{A}_{g,n} = \int_{\mathcal{M}_g} \left\langle \left| \mathcal{N}(y) \prod_{i=1}^{3g-3} b(\mu_i) \right|^2 \prod_{j=1}^{n} U_j \right\rangle_g , \tag{4.114}$$

while for $g = 1$,

$$\mathcal{A}_{1,n} = \int_{\mathcal{M}_1} \left\langle |\mathcal{N}(y) b(\mu)|^2 \prod_{j=1}^{n} U_j \right\rangle_{1-loop} \tag{4.115}$$

$\mathcal{N}$ is an appropriately chosen BRST-invariant regulator of the form $\mathcal{N} = \exp(Q\Lambda)$ which is inserted anywhere on the Riemann surface and resolves the divergences coming from integration over the non-compact bosonic zero modes. For example, one can define for amplitudes at any loop

$$\mathcal{N} = \exp\left(- \bar{\lambda}_{\underline{\alpha}} \lambda^{\underline{\alpha}} - r_{\underline{\alpha}} \theta^{\underline{\alpha}} - \left[\frac{1}{2} N_{mn} \bar{N}^{mn} + J\bar{J} + \frac{1}{4} S_{mn} d\gamma^{mn} \lambda + S \lambda^{\underline{\alpha}} d_{\underline{\alpha}} \right] \right) \tag{4.116}$$

where

$$N_{mn} = \frac{1}{2} w \gamma_{mn} \lambda, \qquad \bar{N}_{mn} = \frac{1}{2} (\bar{w} \gamma_{mn} \bar{\lambda} - s \gamma_{mn} r), \qquad J = w_{\underline{\alpha}} \lambda^{\underline{\alpha}}, \tag{4.117}$$

$$\bar{J} = \bar{w}^{\underline{\alpha}} \bar{\lambda}_{\underline{\alpha}} - s^{\underline{\alpha}} r_{\underline{\alpha}}, \qquad S_{mn} = \frac{1}{2} s \gamma_{mn} \bar{\lambda}, \qquad S = s^{\underline{\alpha}} \bar{\lambda}_{\underline{\alpha}}. \tag{4.118}$$

Since the simpler four-dimensional b ghosts defined in chapter 3 are BRST equivalent to the original b ghost in the restricted patches $\bar{\lambda}_{\dot{\alpha}} \lambda^{\dot{\alpha}} \neq 0$, $\bar{\lambda}_{\alpha} \lambda^{\alpha} \neq 0$, they can be used in the amplitude prescriptions if there are no subtleties concerning regularization. In fact, for special choices of the external states in the vertex operators U_j, this will actually work well. For external states corresponding to anti-self-dual topological amplitudes, one can use the $b^{(a)}$ version since the poles from these $b^{(a)}$ ghosts will not accumulate to poles of order $(\bar{\lambda}_{\dot{\alpha}} \lambda^{\dot{\alpha}})^{-2}$. And for external states corresponding to self-dual topological amplitudes, one can use the

$b^{(c)}$ version since the poles from these $b^{(c)}$ ghosts will not accumulate to poles of order $(\bar{\lambda}_\alpha \lambda^\alpha)^{-2}$. The dangerous terms using these four-dimensional b ghosts appear when the $b^{(a)}$ ghosts contribute $r_{\dot{\alpha}} r^{\dot{\alpha}}$ or when the $b^{(c)}$ ghosts contribute $r_\alpha r^\alpha$ to the amplitude, and it will be shown from zero-mode counting that these dangerous terms cannot contribute to the topological amplitudes.

4.3.1 Type IIB multiloop scattering of anti-self-dual graviphotons

The first amplitude which will be computed is the g-loop type IIB superstring scattering of $2g-2$ anti-self-dual graviphotons and 2 anti-self-dual gravitons which contributes to the $R^2 T^{2g-2}$ term in the low-energy effective action. This comes from the g-loop $2g$-point pure spinor amplitude. The computation of the case $g=1$ is slightly different and will be considered later.

The multiloop amplitude will be computed in the patch $\bar{\lambda}_{\dot{\alpha}} \lambda^{\dot{\alpha}}$, and using the $b^{(a)}$ ghost of (3.69). Because of the integral over the $d_{\underline{\alpha}}$ fermionic zero modes, the only term in the closed string vertex operators U_i which contributes to this amplitude is the term $\int d^2 z d_\alpha \tilde{d}_\beta P^{\alpha\beta}(x^\mu, \theta^{\underline{\alpha}}, \tilde{\theta}^{\underline{\alpha}})$, and the only term in the $b^{(a)}$ ghosts which contributes is

$$b^{(a)} = -\frac{\bar{\lambda}_{\dot{\alpha}} \Pi_I d^{\dot{\alpha} I}}{\bar{\lambda}_{\dot{\alpha}} \lambda^{\dot{\alpha}}}.$$

To see this, focus on the zero modes of the left-moving fermionic variables and note that the vertex operators U_i can only contribute the zero modes of the 2 components d_α whereas the ghosts $b^{(a)}$ can only contribute the zero modes of the 3 components $\bar{\lambda}_{\dot{\alpha}} d^{\dot{\alpha} I}$ and the 2 components d_α. The zero modes of the other 11 components of $d_{\underline{\alpha}}$ (i.e. $d_{\dot{\alpha}}$, $d_{\alpha I}$ and $\lambda_{\dot{\alpha}} d^{\dot{\alpha} I}$) must all come from the regulator $\mathcal{N}$. Since there are $3g-3$ $b^{(a)}$ ghosts, the $3g-3$ zero modes of $\bar{\lambda}_{\dot{\alpha}} d^{\dot{\alpha} I}$ must all come from the $b^{(a)}$ ghosts and the $2g$ zero modes of d_α must all come from the U_i vertex operators. Furthermore, only the twisted sectors of the worldsheet variables contribute to this amplitude since, in the untwisted sector, $\bar{\lambda}_{\dot{\alpha}} d^{\dot{\alpha} I}$ would have $3g$ zero modes which cannot be obtained from the $3g-3$ $b^{(a)}$ ghosts. For the same reason, orbifold sectors which preserve $N=4$ $D=4$ supersymmetry cannot contribute to this amplitude since, in this case, one of the three x^I's in (3.3) would be untwisted and $\bar{\lambda}_{\dot{\alpha}} d^{\dot{\alpha} I}$ would have $3g-2$ zero modes.

In the twisted sector assuming that $\bar{\lambda}_{\dot{\alpha}} \lambda^{\dot{\alpha}} \neq 0$, the only fermionic zero modes of $r_{\underline{\alpha}}$ are the two components $r_{\dot{\alpha}}$ and the only fermionic zero modes of $\theta^{\underline{\alpha}}$ are the 4 components θ^α and $\theta^{\dot{\alpha}}$. In addition to providing the fermionic zero modes of 11 components of $d_{\underline{\alpha}}$, the regulator $\mathcal{N}$ also provides the fermionic zero modes of the 11 components of $s^{\underline{\alpha}}$, and the fermionic zero modes of the 2 components of $r_{\dot{\alpha}}$ and $\theta^{\dot{\alpha}}$. The remaining zero modes of the two components of θ^{α} must come from the vertex operators U_i.

Integration over d_α and $\tilde{d}_\alpha$ zero modes produces index contractions between the $P^{\alpha\beta}$ superfields, giving the expression

$$(P_{\alpha\beta}P^{\alpha\beta})^g[\det(\omega_i(z_j))]^2[\det(\bar{\omega}_i(\bar{z}_j))]^2,$$

where $[\det(\omega_i(z_j))]^2$ denotes the sum of terms of the form $\det(\omega_i(z_j))\det(\omega_i(z_{j'}))$ with z_j denoting g of the $2g$ positions of vertex operators and $z_{j'}$ denoting the other g positions. Using the relation

$$\prod_{i=1}^{g}\int d^2z_i\,|\det(\omega_j(z_k))|^2 = \det(\mathrm{Im}\tau), \tag{4.119}$$

and integrating over the zero modes of $(x^\mu, \theta^\alpha, \tilde{\theta}^\alpha, p_\alpha, \tilde{p}_\alpha)$, the vertex operators therefore contribute in the low energy limit

$$(\det(\mathrm{Im}\tau))^2\int d^4x\int d^2\theta\int d^2\tilde{\theta}(P_{\alpha\beta}(x,\theta,\tilde{\theta})P^{\alpha\beta}(x,\theta,\tilde{\theta}))^g.$$

To separate the $\bar{\lambda}_{\dot{\alpha}}d^{\dot{\alpha}I}$ zero modes appearing in the $b^{(a)}$ ghosts from the $d^{\dot{\alpha}I}$ zero modes appearing in the regulator $\mathcal{N}$, it is convenient to make the change of basis in field space introduced in subsection 3.2.3:

$$\psi^I = (\bar{\lambda}_{\dot{\beta}}\lambda^{\dot{\beta}})^{-1}\bar{\lambda}_{\dot{\alpha}}p^{\dot{\alpha}I}, \qquad \chi^I = \lambda_{\dot{\alpha}}p^{\dot{\alpha}I}, \tag{4.120}$$

$$\psi_I = \lambda_{\dot{\alpha}}\theta_I^{\dot{\alpha}}, \qquad \chi_I = (\bar{\lambda}_{\dot{\beta}}\lambda^{\dot{\beta}})^{-1}\bar{\lambda}_{\dot{\alpha}}\theta_I^{\dot{\alpha}}, \tag{4.121}$$

which is invertible when $\bar{\lambda}_{\dot{\alpha}}\lambda^{\dot{\alpha}} \neq 0$ and whose jacobian in the path integral is 1.

So after integration over the zero modes of the worldsheet variables, the multi-loop amplitude reduces to

$$\mathcal{A}_g = \int_{\mathcal{M}_g}(\det(\mathrm{Im}\tau))^2\int d\psi_0^I\int d\tilde{\psi}_0^I\left\langle\left|\prod_{i=1}^{3g-3}G^-_{int}(\mu_i)\right|^2\right\rangle'\int d^4x\int d^2\theta\int d^2\tilde{\theta}(P_{\alpha\beta}P^{\alpha\beta})^g \tag{4.122}$$

where $\int d^4x\int d^2\theta\int d^2\tilde{\theta}(P_{\alpha\beta}P^{\alpha\beta})^g$ is the $N=2$ $D=4$ supersymmetric expression containing the term $\int d^4xR^2T^{2g-2}$, $\int d\psi_0^I\int d\tilde{\psi}_0^I$ denotes integration over the zero modes of ψ^I and $\tilde{\psi}^I$, $\langle ... \rangle'$ means path integration over non-zero modes of all remaining worldsheet fields as well as the lattice sum coming from the (x^I, x_I) path integral, and the only term contributing from $b^{(a)}$ is

$$G^-_{int} \equiv -\psi^I \partial x_I.$$

Note that the explicit expression for Π_I in the $b^{(a)}$ ghost is

$$\Pi_I = \partial x_I - \theta_{\dot{\alpha} I} \partial \theta^{\dot{\alpha}} - \theta_{\dot{\alpha}} \partial \theta^{\dot{\alpha}}_I - \varepsilon_{IJK} \theta^{\alpha J} \partial \theta^K_\alpha, \tag{4.123}$$

but since all $d_{\underline{\alpha}}$'s must contribute with zero modes, the only term in Π_I which can contribute is ∂x_I. For the same reason, the θ and $\tilde{\theta}$ dependence of the anti-self-dual graviphoton superfield in the amplitude reduces to $P^{\alpha\beta}(x^\mu, \theta^\alpha, \tilde{\theta}^\alpha)$. Note also that (4.122) holds up to a proportionality factor that comes from integration over pure spinor zero modes. This proportionality factor will not be considered in this thesis but can be computed using the methods of [74].

Non-zero mode integration

Recall that for a fermionic $(1, 0)$ chiral system (b, c) on a Riemann surface of genus g, the result for the path integral is

$$\int \mathcal{D}b\mathcal{D}c\, b(z_1)...b(z_g)c(y)e^{-S[b,c]} = Z_1 \det(\omega_i(z_j)) \tag{4.124}$$

where $\{\omega_i, i = 1, ..., g\}$ is a basis for holomorphic 1-differentials on the genus g Riemann surface Σ_g. For bosonic $(1, 0)$ chiral systems, the result of the path integral is the inverse $[Z_1 \det(\omega_i(z_j))]^{-1}$.

For the case of worldsheet fields defined within a given non-trivial twist structure $\boldsymbol{\phi}_I$, the path integral is instead

$$\int \mathcal{D}b\mathcal{D}c\, b(z_1)...b(z_{g-1})e^{-S[b,c]} = Z_{1,\boldsymbol{\phi}_I} \det(\omega_{h_I,i}(z_j)) \tag{4.125}$$

where $\{\omega_{\boldsymbol{\phi}_I,i}, i = 1, ..., g-1\}$ is a basis of h_I-twisted holomorphic 1-differentials and $Z_{1,\boldsymbol{\phi}_I}$ is the partition function coming from non-zero mode integration for twisted fields. Analogously, the path integral of the bosonic $\boldsymbol{\phi}_I$-twisted $(1, 0)$ system is $[Z_{1,\boldsymbol{\phi}_I} \det(\omega_{\boldsymbol{\phi}_I,i}(z_j))]^{-1}$.

The fermionic variables $(d_{\underline{\alpha}}, \theta^{\underline{\alpha}})$ involve four untwisted $(1, 0)$ systems $(d_\alpha, \theta^\alpha)$ and $(d_{\dot{\alpha}}, \theta^{\dot{\alpha}})$, two copies of the three $\boldsymbol{\phi}_I$-twisted $(I = 1, 2, 3)$ $(1, 0)$ systems, $(d^I_{\dot{\alpha}}, \theta^{\dot{\alpha}}_I)$, and two copies of the three $-\boldsymbol{\phi}_I$-twisted $(1,0)$ systems, $(d_{\alpha I}, \theta^{\alpha I})$. Integration over the non-zero modes of $(d_{\underline{\alpha}}, \theta^{\underline{\alpha}})$ therefore gives

$$(Z_1)^4 \prod_{I=1}^{3} \left[(Z_{1,\boldsymbol{\phi}_I})^2 (Z_{1,-\boldsymbol{\phi}_I})^2 \right]. \tag{4.126}$$

So after including the contribution from the right moving sector $(\tilde{d}_{\underline{\alpha}}, \tilde{\theta}^{\underline{\alpha}})$, one gets

$$|Z_1|^8 \prod_{I=1}^{3} \left[|Z_{1,\boldsymbol{\phi}_I}|^4 |Z_{1,-\boldsymbol{\phi}_I}|^4 \right]. \tag{4.127}$$

Since the bosonic pure spinor variables $(w_{\underline{\alpha}}, \lambda^{\underline{\alpha}})$ are equivalent to eleven bosonic $(1,0)$ chiral systems, nine of them twisted by $\boldsymbol{\phi}_I$ or $-\boldsymbol{\phi}_I$ depending on the position of the index I in the corresponding conformal weight one field, their contribution to the amplitude after including the left and right-moving sectors is

$$|Z_1|^{-4} \prod_{I=1}^{3} \left[|Z_{1,\boldsymbol{\phi}_I}|^{-2} |Z_{1,-\boldsymbol{\phi}_I}|^{-4} \right]. \tag{4.128}$$

Finally, the contribution from the non-zero modes of x^μ is $|Z_1|^{-4}(\det(\mathrm{Im}\tau))^{-2}$, and the contribution from the bosonic non-minimal variables $(\bar{w}^{\underline{\alpha}}, \bar{\lambda}_{\underline{\alpha}})$ cancels the contribution from the fermionic non-minimal variables $(s^{\underline{\alpha}}, r_{\underline{\alpha}})$.

Including these contributions from the non-zero modes in (4.122), one therefore obtains

$$\mathcal{A}_g = \int_{\mathcal{M}_g} \prod_{I=1}^{3} |Z_{1,\boldsymbol{\phi}_I}|^2 \int d\psi_0^I \int d\tilde{\psi}_0^I \left\langle \left| \prod_{i=1}^{3g-3} G_{int}^-(\mu_i) \right|^2 \right\rangle \int d^4x \int d^2\theta \int d^2\tilde{\theta} (P_{\alpha\beta} P^{\alpha\beta})^g. \tag{4.129}$$

Finally, one can replace $\prod_{I=1}^3 |Z_{1,\boldsymbol{\phi}_I}|^2 \int d\psi_0^I \int d\tilde{\psi}_0^I$ with the path integral over ψ^I and $\tilde{\psi}^I$, and sum over all non-trivial twist structures to get the formula

$$\mathcal{A}_g = \int_{\mathcal{M}_g} \left\langle \left| \prod_{i=1}^{3g-3} G_{int}^-(\mu_i) \right|^2 \right\rangle_{top} \int d^4x \int d^2\theta \int d^2\tilde{\theta} (P_{\alpha\beta} P^{\alpha\beta})^g. \tag{4.130}$$

where $\langle ... \rangle_{top}$ denotes the path integral over all modes of x^I, x_I, ψ^I, ψ_I. Thus, the coupling to the supersymmetric term containing $\int d^4x R^2 T^{2g-2}$ is precisely the partition function $F_g^B = \int_{\mathcal{M}_g} \left\langle \left| \prod_{i=1}^{3g-3} G_{int}^-(\mu_i) \right|^2 \right\rangle_{top}$ of the B-model topological string theory at genus g when $g > 1$. This same result will now be shown to also occur when $g = 1$.

4.3.2 One-loop amplitude

The non-minimal pure spinor prescription for one-loop amplitudes involving two external states is

$$\mathcal{A}_1 = \int_{\mathcal{M}_1} \left\langle |\mathcal{N} b(\mu)|^2 \; U \, V(z) \right\rangle , \tag{4.131}$$

where U is the integrated vertex operator for one of the states,

$$V(z) = \lambda^{\dot{\alpha}} \tilde{\lambda}^{\dot{\beta}} A_{\dot{\alpha}\dot{\beta}}(x, \theta, \tilde{\theta})$$

is the BRST-invariant unintegrated vertex operator for the other state which is inserted anywhere on the surface, and $A_{\dot{\alpha}\dot{\beta}}(x, \theta, \tilde{\theta})$ is the bispinor prepotential superfield appearing in (3.132). As before, the four-dimensional version of $b^{(a)}$ can be substituted for the b ghost when computing the one-loop scattering of two anti-self-dual gravitons.

Because of the integration over fermionic zero modes, only the twisted sector will contribute to this one-loop amplitude and the fermionic zero modes of the left-moving variables are $(\theta^{\alpha}, \bar{\theta}^{\dot{\alpha}}, r_{\dot{\alpha}})$ and $(p_{\alpha}, \bar{p}_{\dot{\alpha}}, s^{\dot{\alpha}})$. The two zero modes of p_{α} must come from U and $b^{(a)}$ where $b^{(a)}$ of (3.69) contributes $-\frac{\bar{\lambda}_{\dot{\alpha}}}{2\bar{\lambda}_{\dot{\gamma}}\lambda^{\dot{\gamma}}}\left(\partial x^{\mu} \sigma_{\mu}^{\dot{\alpha}\beta} p_{\beta}\right)$, the θ^{α} zero modes come from V and U, and the remaining zero modes come from the regulator $\mathcal{N}$.

To relate this one-loop amplitude with the one-loop topological partition function, use the OPE $b(y)J_g(z) \to (y-z)^{-1}b(z)$, where $J_g = w_{\underline{\alpha}}\lambda^{\underline{\alpha}} + r_{\underline{\alpha}}s^{\underline{\alpha}}$ is the ghost-number current, to write the $b^{(a)}$ ghost as the contour integral of $b^{(a)}$ around J_g, that is,

$$b^{(a)}(\mu) = [\oint b^{(a)}, J_g(\mu)]. \tag{4.132}$$

The contour integral can be pulled off of J_g and since the commutator of $\oint b^{(a)}$ with U and $\mathcal{N}$ does not contain enough zero modes of p_{α}, the only contribution comes from the commutator with V. Performing the same operation with the right-moving $\tilde{b}^{(a)}$ ghost, one can write the one-loop amplitude as

$$\mathcal{A}_1 = \int_{\mathcal{M}_1} \left\langle |\mathcal{N} J_g(\mu)|^2 \; U \oint b^{(a)} \oint \tilde{b}^{(a)} V(z) \right\rangle . \tag{4.133}$$

In the gauge where $b^{(a)}$ and $\tilde{b}^{(a)}$ have no double poles with $V = \lambda^{\dot\alpha}\tilde\lambda^{\dot\beta}A_{\dot\alpha\dot\beta}$, the superfields $P^{\alpha\beta}$ and $A_{\dot\alpha\dot\beta}$ are related by

$$P^{\alpha\beta} = (\sigma^{\mu})^{\alpha\dot\alpha}(\sigma^{\nu})^{\beta\dot\beta}\partial_{\mu}\partial_{\nu}A_{\dot\alpha\dot\beta}. \tag{4.134}$$

So after integrating over the fermionic zero modes of $(\theta^{\alpha}, \tilde\theta^{\alpha})$ and $(d_{\alpha}, \tilde{d}_{\alpha})$, and using (4.134), (4.133) reduces to

$$\mathcal{A}_1 = \int_{\mathcal{M}_1}\left\langle |\mathcal{N}J_g(\mu)|^2\right\rangle \int d^4x \int d^2\theta \int d^2\tilde\theta (P_{\alpha\beta}P^{\alpha\beta}) \tag{4.135}$$

where $\langle\ \rangle$ denotes the functional integral over all worldsheet variables except for the four-dimensional variables $(x^{\mu}, \theta^{\alpha}, \tilde\theta^{\alpha}, p_{\alpha}, \tilde{p}_{\alpha})$.

Finally, one can relate $\int_{\mathcal{M}_1}\left\langle |\mathcal{N}J_g(\mu)|^2\right\rangle$ to the one-loop topological partition function by using the field redefinition of (4.120) and shifting $w_{\dot\alpha}$ and $\bar{w}^{\dot\alpha}$ as in (3.90), (3.91). Since ψ_I and ψ^I carry ghost-number charge $+1$ and -1,

$$J_g = K + \psi^I\psi_I \tag{4.136}$$

where $K = J_g - \psi^I\psi_I$ is independent of (ψ^I, ψ_I) and is constructed from the pure spinor variables and (χ^I, χ_I). Note that J_g satisfies the OPE $J_g(y)J_g(z) \to 3(y-z)^{-2}$, so the OPE of K with K has no double pole. One can therefore define $K = \partial\sigma$ where σ is a null boson and construct new fermionic variables

$$\zeta^I = \psi^I e^{+\frac{1}{3}\sigma}, \quad \zeta_I = \psi_I e^{-\frac{1}{3}\sigma}, \tag{4.137}$$

which satisfy the same OPE's and twistings as ψ^I and ψ_I, and satisfy

$$J_g = \zeta^I\zeta_I. \tag{4.138}$$

The integration over all fermionic and bosonic variables in (4.135) except for the internal $(x^I, x_I, \zeta^I, \zeta_I, \tilde\zeta^I, \tilde\zeta_I)$ variables cancels out, so the one-loop amplitude can be expressed as

$$\mathcal{A}_1 = F_1^B \int d^4x \int d^2\theta \int d^2\tilde\theta (P_{\alpha\beta}P^{\alpha\beta}) \tag{4.139}$$

where F_1^B is the one-loop partition function of the topological B-model [75]

$$F_1^B = \int_{\mathcal{M}_1}\left\langle \left|\zeta^I\zeta_I(\mu)\right|^2\right\rangle_{top}$$

and $\langle\ \rangle_{top}$ denotes integration over the internal variables $(x^I, x_I, \zeta^I, \zeta_I, \tilde{\zeta}^I, \tilde{\zeta}_I)$.

4.3.3 Other topological amplitudes

In the previous subsection, the Type IIB scattering of $2g-2$ anti-self-dual graviphotons and 2 anti-self-dual gravitons was computed using the non-minimal pure spinor prescription of (4.114) with $b^{(a)}$ and $\tilde{b}^{(a)}$ ghosts to obtain the amplitude

$$F_g^B \int d^4x\, d^2\theta d^2\tilde{\theta}\, (P_{\alpha\beta}P^{\alpha\beta})^g. \tag{4.140}$$

where $F_g^B = \int_{\mathcal{M}_g} \left\langle \left| \prod_{i=1}^{3g-3} G_{int}^-(\mu_i) \right|^2 \right\rangle_{top}$ is the topological B-model partition function. Using identical reasoning, one can compute the Type IIB scattering of $2g-2$ self-dual graviphotons and 2 self-dual gravitons using the prescription of (4.114) with $b^{(c)}$ and $\tilde{b}^{(c)}$ ghosts. In this case, one restricts to patches where $\bar{\lambda}_\alpha \lambda^\alpha \neq 0$, and the resulting amplitude is

$$\bar{F}_g^B \int d^4x\, d^2\bar{\theta} d^2\tilde{\bar{\theta}}\, (P_{\dot{\alpha}\dot{\beta}}P^{\dot{\alpha}\dot{\beta}})^g \tag{4.141}$$

where $\bar{F}_g^B$ is the complex conjugate of F_g^B defined by

$$\bar{F}_g^B = \int_{\mathcal{M}_g} \left\langle \left| \prod_{i=1}^{3g-3} G_{int}^+(\mu_i) \right|^2 \right\rangle_{top}, \tag{4.142}$$

$G_{int}^+ \equiv \psi_I \partial x^I$ is the contribution from the $b^{(c)}$ ghost, and $\psi_I = (\bar{\lambda}^\beta \lambda_\beta)^{-1} \bar{\lambda}^\alpha p_{\alpha I}$.

One can also compute the Type IIB scattering of $2g-2$ Ramond-Ramond hypermultiplet scalars and 2 NS-NS hypermultiplet scalars with the prescription of (4.114) if one uses either $b^{(a)}$ ghosts in the left-moving sector and $\tilde{b}^{(c)}$ ghosts in the right-moving sector or $b^{(c)}$ ghosts in the left-moving sector and $\tilde{b}^{(a)}$ ghosts in the right-moving sector. In the first case, the amplitude is

$$F_g^A \int d^4x \int d^2\theta \int d^2\tilde{\bar{\theta}}\, (Q_{\alpha\dot{\beta}}Q^{\alpha\dot{\beta}})^g. \tag{4.143}$$

where F_g^A is the partition function at genus g for the A model topological string defined as

$$F_g^A = \int_{\mathcal{M}_g} \left\langle \prod_{i=1}^{3g-3} G_{int}^-(\mu_i) \prod_{j=1}^{3g-3} \tilde{G}_{int}^+(\bar{\mu}_j) \right\rangle_{top}. \tag{4.144}$$

In the second case, one obtains the complex conjugate of (4.143) which is

$$\bar{F}_g^A \int d^4x \int d^2\bar{\theta} \int d^2\tilde{\theta}\ (Q_{\dot{\alpha}\beta} Q^{\dot{\alpha}\beta})^g, \tag{4.145}$$

where

$$\bar{F}_g^A = \int_{\mathcal{M}_g} \left\langle \prod_{i=1}^{3g-3} G^+_{int}(\mu_i) \prod_{j=1}^{3g-3} \tilde{G}^-_{int}(\bar{\mu}_j) \right\rangle_{top}. \tag{4.146}$$

For type IIA topological amplitudes, the results are complementary and the only difference comes from the opposite chirality of the right-moving sector as compared with type IIB. For the anti-self-dual graviphoton amplitude, the right-moving b-ghost now contributes with $(\bar{\tilde{\lambda}}_{\dot{\alpha}} \tilde{\lambda}^{\dot{\alpha}})^{-1} \bar{\partial} x^I \tilde{p}_I^{\dot{\alpha}}$ and the appropriate change of variables that has to be performed to make contact with the topological string description of the internal model is

$$\tilde{\psi}_I = (\bar{\tilde{\lambda}}_{\dot{\beta}} \tilde{\lambda}^{\dot{\beta}})^{-1} \bar{\tilde{\lambda}}_{\dot{\alpha}} \tilde{p}_I^{\dot{\alpha}}, \qquad \tilde{\chi}_I = \tilde{\lambda}_{\dot{\alpha}} \tilde{p}_I^{\dot{\alpha}}, \tag{4.147}$$

$$\tilde{\psi}^I = \tilde{\lambda}_{\dot{\alpha}} \tilde{\theta}^{\dot{\alpha} I}, \qquad \tilde{\chi}^I = (\bar{\tilde{\lambda}}_{\dot{\beta}} \tilde{\lambda}^{\dot{\beta}})^{-1} \bar{\tilde{\lambda}}_{\dot{\alpha}} \tilde{\theta}^{\dot{\alpha} I}, \tag{4.148}$$

The effect of this is to change the relative topological twisting in the right-moving sector of the internal model, and the corresponding terms in the Type IIA effective action are

$$\begin{aligned} S \;=\; & F_g^A \int d^4x \int d^2\theta \int d^2\tilde{\theta}\ (P_{\alpha\beta} P^{\alpha\beta})^g + \bar{F}_g^A \int d^4x \int d^2\bar{\theta} \int d^2\bar{\tilde{\theta}}\ (P_{\dot{\alpha}\dot{\beta}} P^{\dot{\alpha}\dot{\beta}})^g \\ & + F_g^B \int d^4x \int d^2\theta \int d^2\bar{\tilde{\theta}}\ (Q_{\alpha\dot{\beta}} Q^{\alpha\dot{\beta}})^g + \bar{F}_g^B \int d^4x \int d^2\bar{\theta} \int d^2\tilde{\theta}\ (Q_{\dot{\alpha}\beta} Q^{\dot{\alpha}\beta})^g. \end{aligned} \tag{4.149}$$

Chapter 5

Conclusions and future directions

This thesis contains a first step towards a pure spinor formulation of superstrings compactified on six-dimensional manifolds by studying topological amplitude computations of type II superstring theory with an orbifold as the internal sector. It was shown that, as in topological amplitude computations using the hybrid formalism, the pure spinor computation preserves manifest $N = 2$ $D = 4$ supersymmetry and do not require summing over spin structures. Although the pure spinor vertex operators depend on all 16 θ^{α} and 16 $\tilde{\theta}^{\alpha}$ worldsheet variables, the restriction to patches where $\bar{\lambda}^{\dot{\alpha}} \neq 0$ and the construction of a four-dimensional version of the b ghost simplify the computations. Moreover, the multiloop calculations using the non-minimal pure spinor formalism do not suffer from the subtleties of the hybrid formalism coming from negative-energy chiral bosons.

Topological strings contain fermionic worldsheet variables with spins 1 and 0. They can be though of as twisted RNS fermionic matter which, before twisting, has spin $\frac{1}{2}$. Regarding the topological amplitude calculation in the RNS formalism, two phenomena can be identified. First of all, as seen in the heuristic derivation, the *twisting* of fermionic matter is nothing else that a redefinition of the background charge for different fields; this change is induced by the presence of insertions of suitable external states on the worldsheet. Nevertheless, in the actual computation, the effective twisting is a consequence of a sum over spin structures which traded sums of products of Θ-functions with half-characteristics, for Θ-functions without them. The conformal systems which provided the first ones where the conformal ghosts (β, γ), fermionic spacetime matter ψ^{μ}, and fermionic internal matter ψ^{I}, ψ_{I}.

In fact, the relation between hybrid and RNS formalisms is known to be given by a field redefinition which, in the internal sector reads

$$T_{hybr} = T_{RNS} + \frac{3}{2}(\partial\phi + \eta\xi)^2 - (\partial\phi + \eta\xi)J_{RNS} \tag{5.1}$$

$$G^{+}_{hybr} = e^{\phi}\eta G^{+}_{RNS}, \qquad G^{-}_{hybr} = \xi e^{-\phi} G^{-}_{RNS}, \qquad J_{hybr} = J_{RNS} + 3(\partial\phi + \eta\xi) \tag{5.2}$$

For an orbifold compactification characterized by fermions ψ^I, ψ_I, this relation comes from the field redefinition

$$\Gamma_I = \gamma\psi_I, \qquad \Gamma^I = \frac{1}{\gamma}\psi^I \tag{5.3}$$

which effectively twists the $N = 2$ superconformal algebra of the orbifold SCFT. The field redefinition is more complicated for supermatter variables in the $D = 4$ sector, while, obviously, the internal sector of the hybrid formalism allows from the beginning a treatment of the more general Calabi-Yau compactification. Notice that fermionic spacetime spinors of the hybrid formalism are given as

$$\theta^\alpha = c\xi e^{-\frac{1}{2}(3\phi+\int^z J_{RNS})} S^\alpha, \qquad \theta^{\dot\alpha} = e^{\frac{1}{2}(\phi+\int^z J_{RNS})} S^{\dot\alpha} \tag{5.4}$$

The only information from the internal sector that appears in this field redefinition is the U(1) current, and this fact must have something to do with the extension of the RNS computation to generic Calabi-Yau compactifications given in [67]. It will be interesting to see how this feature translates to the pure spinor formalism, in order to deal with these more generic Calabi-Yau manifolds.

On the other hand, twisted fermions for the internal sector are already contained in *higher* components of $p_{\underline{\alpha}}, \theta^{\underline{\alpha}}$. In this thesis, the relation was given only for free fermions. It would be obviously interesting to extend the pure spinor computation to include Calabi-Yau backgrounds based on the field redefinition (4.120)-(4.121). Notice that this field redefinition, which resembles the twisting considered in [76], would give untwisted fermions if the conformal weight of $\lambda_{\dot\alpha}$ were shifted by $1/2$; the extra untwisted fields χ^I, χ_I would call for an appropriate interpretation. Besides, a simplified b ghost for flat background was given in [77] in terms of fermionic vectors. This was done at the level of the worldsheet action by introducing the conjugate pair $(\bar\Gamma^m, \Gamma^m)$, and adding a constraint which trivializes their kinetic term as $\bar\Gamma^m = 0$. Then, a suitable similarity transformation transforms this constraint into the more complicated

$$\bar\Gamma^m - \frac{\bar\lambda\gamma^m d}{2\bar\lambda_{\underline{\alpha}}\lambda^{\underline{\alpha}}} + \frac{(\bar\lambda\gamma^{mnp}r)N_{np}}{8(\bar\lambda_{\underline{\alpha}}\lambda^{\underline{\alpha}})^2} = 0 \tag{5.5}$$

which, nevertheless, allow to write a much simpler ten-dimensional b ghost

$$b = \Pi^m\bar\Gamma_m - \frac{\lambda\gamma^{mn}r}{4\bar\lambda_{\underline{\alpha}}\lambda^{\underline{\alpha}}}\bar\Gamma_m\bar\Gamma_n + s^{\underline{\alpha}}\partial\bar\lambda_{\underline{\alpha}} + w_{\underline{\alpha}}\partial\theta^{\underline{\alpha}} - \frac{(w\gamma_m\bar\lambda)(\lambda\gamma^m\partial\theta)}{2\bar\lambda_{\underline{\alpha}}\lambda^{\underline{\alpha}}} \tag{5.6}$$

The natural thing to do is to introduce fermionic vector variables Γ^I, Γ_I exclusively for internal SU(3) indices in the compactified pure spinor formalism, and see the consequences on the form of the b ghost and the relevant amplitude computations. This is work in progress.

In any case, there is a clear simplification of the pure spinor computation of topological amplitudes with respect to other formalisms; this resides in the substitution of the b ghost by the *4d* version which is BRST equivalent to the original composite operator. This is allowed only if a subset of the 16 patches of pure spinor space are taken to be permitted in the orbifold compactification; of course, this situation deserves further study. Moreover, this restriction has effects in the maximum number of inverse powers of $\bar{\lambda}\lambda$ which won't give divergences at $\lambda \to 0$. In fact, for topological amplitudes the contributions from b ghost insertions are such that these inverse powers do not accumulate. Nevertheless, accumulation of singularities become relevant when computing non-topological amplitudes.

To test the pure spinor prescription for amplitudes in orbifold compactifications, it is simpler to start at one-loop. A relevant case to study is type I superstrings which preserve $N = 1$ supersymmetry in four dimensions. One-loop, four-gluon scattering was computed in [78, 79]. Gluon external states belong to the untwisted sector of the orbifold compactification. As in type II superstrings, depending on boundary conditions (or twistings) along homology cycles, amplitudes get contributions from sectors preserving different amounts of supersymmetry. Since twistings must obey $\boldsymbol{\phi}_1 + \boldsymbol{\phi}_2 + \boldsymbol{\phi}_3 = 0$, type I amplitudes have three sectors: $N = 4$, $N = 2$, and $N = 1$, determined by (i) $\boldsymbol{\phi}_I = 0$ $(I = 1, 2, 3)$, (ii) $\boldsymbol{\phi}_3 = 0$, $\boldsymbol{\phi}_1 = -\boldsymbol{\phi}_2$ (or the other possibilities), and (iii) $\boldsymbol{\phi}_I \neq 0$ $(I = 1, 2, 3)$.

The simplest sector is $N = 4$, where each component coming from $d_{\underline{\alpha}}$ has one zero-mode. For amplitudes up to three-point it is easy to see that, since the four-dimensional b ghost can contribute at most with two d zero-modes there are not enough insertions to absorb all sixteen of them; then, these amplitudes vanish. The first non-trivial process is the four-point scattering whose amplitude reads

$$\int_0^\infty \frac{dT}{T} \left\langle \mathcal{N}(y) b^{(c)}(\mu) \lambda^{\underline{\alpha}} A^1_{\underline{\alpha}}(z_1) \prod_{i=2}^{4} \int dz_i U_i(z_i) \right\rangle \tag{5.7}$$

where T is the modulus for the corresponding surface contributing at one-loop (annulus and Möbius strip).

However, when trying to compute this amplitude with the usual one-loop regulator $\mathcal{N}$, the result for the $N = 4$ sector of $N = 1$ orbifold compactification vanishes. The problem appears to be related with the fact that the prescription as it stands cannot give a gauge invariant answer. To see this, notice that the term in $b^{(c)}$ which contributes is proportional to $(\bar{\lambda}\lambda)^{-2}$. Under gauge invariance, $\delta V = Q\Omega$, and when trying to pull off the contour integral of Q of the surface, there appears a pole $(\bar{\lambda}\lambda)^{-2}$ which, as already discussed, is not allowed in well-defined computations. A way out of this unfortunate circumstance resides in using the more complicated regularization developed in [40, 80].

Extending to more general situations, it would also be desirable to consider scattering of compactification dependent states. For orbifolds, this requires the knowledge of twist field correlators, those involving string state in the twisted sectors of the Hilbert space; those were studied in [81, 82].

In any case, the results established in this thesis seems to open a path towards generic pure spinor prescriptions for four-dimensional compactifications.

Appendix A

BRST equivalence of b ghosts

In this section of the appendix, the b ghost of (2.86) will be shown to satisfy $b = b^{(a)} + Q\Lambda^{(a)}$ for some $\Lambda^{(a)}$. Note that terms in b have denominators which are powers of $\bar{\lambda}_{\underline{\alpha}}\lambda^{\underline{\alpha}}$, while those in $b^{(a)}$ have powers of $\bar{\lambda}_{\dot{\alpha}}\lambda^{\dot{\alpha}}$. The strategy will be to manipulate terms in b order by order in $(\bar{\lambda}_{\underline{\alpha}}\lambda^{\underline{\alpha}})^{-1}$ in such a way as to trade them for terms with $(\bar{\lambda}_{\dot{\alpha}}\lambda^{\dot{\alpha}})^{-1}$. This will produce some BRST trivial terms and extra non-trivial terms which will be canceled by expressions coming from manipulations at next order in the analysis. In the end, all non-trivial terms will cancel each other.

The convention used for spinor indices is the following: $\dot{\alpha}$ denotes antichiral spinors in four dimensions, and α' denotes any of the other components in the ten-dimensional quantity, $\alpha' = (\alpha, \alpha I, \dot{\alpha} I)$ where the position of I depends on the chirality of the ten-dimensional spinor.

First term

To analyze the first term in the b ghost of (2.86), use the relation

$$Q(\bar{\lambda}_{\dot{\alpha}}\bar{\lambda}_{\alpha'}H^{[\dot{\alpha},\alpha']}) = \bar{\lambda}_{\dot{\alpha}}\lambda^{\dot{\alpha}}\bar{\lambda}_{\alpha'}G^{\alpha'} - \bar{\lambda}_{\alpha'}\lambda^{\alpha'}\bar{\lambda}_{\dot{\alpha}}G^{\dot{\alpha}} - \bar{\lambda}_{\alpha'}r_{\dot{\alpha}}H^{[\dot{\alpha},\alpha']} - \bar{\lambda}_{\dot{\alpha}}r_{\alpha'}H^{[\dot{\alpha},\alpha']} \quad \text{(A.1)}$$

to express

$$\bar{\lambda}_{\alpha'}G^{\alpha'} = \frac{1}{\bar{\lambda}_{\dot{\alpha}}\lambda^{\dot{\alpha}}}\left[Q(\bar{\lambda}_{\dot{\alpha}}\bar{\lambda}_{\alpha'}H^{[\dot{\alpha},\alpha']}) + \bar{\lambda}_{\alpha'}\lambda^{\alpha'}\bar{\lambda}_{\dot{\alpha}}G^{\dot{\alpha}} + \bar{\lambda}_{\alpha'}r_{\dot{\alpha}}H^{[\dot{\alpha},\alpha']} + \bar{\lambda}_{\dot{\alpha}}r_{\alpha'}H^{[\dot{\alpha},\alpha']}\right], \quad \text{(A.2)}$$

where the patch in pure spinor space is $\bar{\lambda}_{\dot{\alpha}}\lambda^{\dot{\alpha}} \neq 0$. Then the first term in the b-ghost can be written as

$$b_1 = \frac{\bar{\lambda}_{\underline{\alpha}}G^{\underline{\alpha}}}{\bar{\lambda}_{\underline{\alpha}}\lambda^{\underline{\alpha}}} = \frac{1}{\bar{\lambda}_{\underline{\alpha}}\lambda^{\underline{\alpha}}}(\bar{\lambda}_{\dot{\alpha}}G^{\dot{\alpha}} + \bar{\lambda}_{\alpha'}G^{\alpha'}) =$$

$$= \frac{\bar{\lambda}_{\dot{\alpha}}G^{\dot{\alpha}}}{\bar{\lambda}_{\dot{\alpha}}\lambda^{\dot{\alpha}}} + Q\left(\frac{\bar{\lambda}_{\dot{\alpha}}\bar{\lambda}_{\alpha'}H^{[\dot{\alpha},\alpha']}}{(\bar{\lambda}_{\dot{\alpha}}\lambda^{\dot{\alpha}})(\bar{\lambda}_{\underline{\alpha}}\lambda^{\underline{\alpha}})}\right) - \bar{\lambda}_{\dot{\alpha}}\bar{\lambda}_{\alpha'}H^{[\dot{\alpha},\alpha']}Q\left(\frac{1}{(\bar{\lambda}_{\dot{\alpha}}\lambda^{\dot{\alpha}})(\bar{\lambda}_{\underline{\alpha}}\lambda^{\underline{\alpha}})}\right)$$

$$+ \frac{\bar\lambda_{\dot\alpha} r_{\alpha'} H^{[\dot\alpha,\alpha']} - \bar\lambda_{\alpha'} r_{\dot\alpha} H^{[\alpha',\dot\alpha]}}{(\bar\lambda_{\dot\alpha}\lambda^{\dot\alpha})(\bar\lambda_{\underline{\alpha}}\lambda^{\underline{\alpha}})}. \tag{A.3}$$

Second term

Similarly, from the relation

$$Q(\bar\lambda_{\dot\alpha}\bar\lambda_{\beta'} r_{\dot\gamma} K^{[\dot\alpha,\beta',\dot\gamma]}) = -\bar\lambda_{\dot\alpha}\lambda^{\dot\alpha}\bar\lambda_{\beta'} r_{\dot\gamma} H^{[\beta',\dot\gamma]} - \bar\lambda_{\beta'}\lambda^{\beta'}\bar\lambda_{\dot\alpha} r_{\dot\gamma} H^{[\dot\gamma,\dot\alpha]} - r_{\dot\gamma}\lambda^{\dot\gamma}\bar\lambda_{\dot\alpha}\bar\lambda_{\beta'} H^{[\dot\alpha,\beta']}$$
$$- \bar\lambda_{\beta'} r_{\dot\alpha} r_{\dot\gamma} K^{[\dot\alpha,\beta',\dot\gamma]} - \bar\lambda_{\dot\alpha} r_{\beta'} r_{\dot\gamma} K^{[\dot\alpha,\beta',\dot\gamma]} \tag{A.4}$$

it follows that

$$\bar\lambda_{\beta'} r_{\dot\gamma} H^{[\beta',\dot\gamma]} = \frac{1}{\bar\lambda_{\dot\alpha}\lambda^{\dot\alpha}} \Big[\bar\lambda_{\beta'}\lambda^{\beta'}\bar\lambda_{\dot\alpha} r_{\dot\gamma} H^{[\dot\alpha,\dot\gamma]} - r_{\dot\gamma}\lambda^{\dot\gamma}\bar\lambda_{\dot\alpha}\bar\lambda_{\beta'} H^{[\dot\alpha,\beta']}$$
$$-\bar\lambda_{\beta'} r_{\dot\alpha} r_{\dot\gamma} K^{[\dot\alpha,\beta',\dot\gamma]} - \bar\lambda_{\dot\alpha} r_{\beta'} r_{\dot\gamma} K^{[\dot\alpha,\beta',\dot\gamma]} - Q(\bar\lambda_{\dot\alpha}\bar\lambda_{\beta'} r_{\dot\gamma} K^{[\dot\alpha,\beta',\dot\gamma]})\Big]. \tag{A.5}$$

Again, from

$$Q(\bar\lambda_{\dot\alpha}\bar\lambda_{\beta'} r_{\gamma'} K^{[\dot\alpha,\beta',\gamma']}) = -\bar\lambda_{\dot\alpha}\lambda^{\dot\alpha}\bar\lambda_{\beta'} r_{\gamma'} H^{[\beta',\gamma']} - \bar\lambda_{\beta'}\lambda^{\beta'}\bar\lambda_{\dot\alpha} r_{\gamma'} H^{[\gamma',\dot\alpha]} - r_{\gamma'}\lambda^{\gamma'}\bar\lambda_{\dot\alpha}\bar\lambda_{\beta'} H^{[\dot\alpha,\beta']}$$
$$- \bar\lambda_{\beta'} r_{\dot\alpha} r_{\gamma'} K^{[\dot\alpha,\beta',\gamma']} - \bar\lambda_{\dot\alpha} r_{\beta'} r_{\gamma'} K^{[\dot\alpha,\beta',\gamma']}, \tag{A.6}$$

one can express

$$\bar\lambda_{\beta'} r_{\gamma'} H^{[\beta',\gamma']} = \frac{1}{\bar\lambda_{\dot\alpha}\lambda^{\dot\alpha}} \Big[\bar\lambda_{\beta'}\lambda^{\beta'}\bar\lambda_{\dot\alpha} r_{\gamma'} H^{[\dot\alpha,\gamma']} - r_{\gamma'}\lambda^{\gamma'}\bar\lambda_{\dot\alpha}\bar\lambda_{\beta'} H^{[\dot\alpha,\beta']}$$
$$-\bar\lambda_{\beta'} r_{\dot\alpha} r_{\gamma'} K^{[\dot\alpha,\beta',\gamma']} - \bar\lambda_{\dot\alpha} r_{\beta'} r_{\gamma'} K^{[\dot\alpha,\beta',\gamma']} - Q(\bar\lambda_{\dot\alpha}\bar\lambda_{\beta'} r_{\gamma'} K^{[\dot\alpha,\beta',\gamma']})\Big]. \tag{A.7}$$

Expanding

$$- \bar\lambda_{\underline{\alpha}} r_{\underline{\beta}} H^{[\underline{\alpha},\underline{\beta}]} = -\bar\lambda_{\dot\alpha} r_{\dot\beta} H^{[\dot\alpha,\dot\beta]} - \bar\lambda_{\dot\alpha} r_{\beta'} H^{[\dot\alpha,\beta']} - \bar\lambda_{\alpha'} r_{\dot\beta} H^{[\alpha',\dot\beta]} - \bar\lambda_{\alpha'} r_{\beta'} H^{[\alpha',\beta']} \tag{A.8}$$

and denoting

$$A = -\bar\lambda_{\dot\alpha} r_{\dot\beta} H^{[\dot\alpha,\dot\beta]} - \bar\lambda_{\alpha'} r_{\dot\beta} H^{[\alpha',\dot\beta]}, \tag{A.9}$$

$$B = -\bar\lambda_{\dot\alpha} r_{\beta'} H^{[\dot\alpha,\beta']} - \bar\lambda_{\alpha'} r_{\beta'} H^{[\alpha',\beta']}, \tag{A.10}$$

the second term in the b-ghost can be written as

$$b_2 = \frac{A+B}{(\bar{\lambda}_{\underline{\alpha}}\lambda^{\underline{\alpha}})^2}. \tag{A.11}$$

Plugging in equations (A.5) and (A.7) in A and B, respectively, one gets

$$A = \frac{1}{\bar{\lambda}_{\dot{\alpha}}\lambda^{\dot{\alpha}}}\Big[-\bar{\lambda}_{\underline{\alpha}}\lambda^{\underline{\alpha}}\bar{\lambda}_{\dot{\alpha}}r_{\dot{\beta}}H^{[\dot{\alpha},\dot{\beta}]} + r_{\dot{\gamma}}\lambda^{\dot{\gamma}}\bar{\lambda}_{\dot{\alpha}}\bar{\lambda}_{\beta'}H^{[\dot{\alpha},\beta']}$$

$$+\bar{\lambda}_{\beta'}r_{\dot{\alpha}}r_{\dot{\gamma}}K^{[\dot{\alpha},\beta',\dot{\gamma}]} + \bar{\lambda}_{\dot{\alpha}}r_{\beta'}r_{\dot{\gamma}}K^{[\dot{\alpha},\beta',\dot{\gamma}]} + Q\Big(\bar{\lambda}_{\dot{\alpha}}\bar{\lambda}_{\beta'}r_{\dot{\gamma}}K^{[\dot{\alpha},\beta',\dot{\gamma}]}\Big)\Big], \tag{A.12}$$

$$B = \frac{1}{\bar{\lambda}_{\dot{\alpha}}\lambda^{\dot{\alpha}}}\Big[-\bar{\lambda}_{\underline{\alpha}}\lambda^{\underline{\alpha}}\bar{\lambda}_{\dot{\alpha}}r_{\beta'}H^{[\dot{\alpha},\beta']} + r_{\gamma'}\lambda^{\gamma'}\bar{\lambda}_{\dot{\alpha}}\bar{\lambda}_{\beta'}H^{[\dot{\alpha},\beta']}$$

$$+\bar{\lambda}_{\beta'}r_{\dot{\alpha}}r_{\gamma'}K^{[\dot{\alpha},\beta',\gamma']} + \bar{\lambda}_{\dot{\alpha}}r_{\beta'}r_{\gamma'}K^{[\dot{\alpha},\beta',\gamma']} + Q\Big(\bar{\lambda}_{\dot{\alpha}}\bar{\lambda}_{\beta'}r_{\gamma'}K^{[\dot{\alpha},\beta',\gamma']}\Big)\Big]. \tag{A.13}$$

The first term in B after being multiplied by $(\bar{\lambda}_{\underline{\alpha}}\lambda^{\underline{\alpha}})^{-2}$ cancels one of the last terms in (A.3). The first term in A times $(\bar{\lambda}_{\underline{\alpha}}\lambda^{\underline{\alpha}})^{-2}$ combines with the last term in (A.3) to produce $A(\bar{\lambda}_{\dot{\alpha}}\lambda^{\dot{\alpha}})^{-1}(\bar{\lambda}_{\underline{\alpha}}\lambda^{\underline{\alpha}})^{-1}$, and this is

$$-\frac{\bar{\lambda}_{\dot{\alpha}}r_{\dot{\beta}}H^{[\dot{\alpha},\dot{\beta}]}}{(\bar{\lambda}_{\dot{\alpha}}\lambda^{\dot{\alpha}})^2} + \frac{1}{(\bar{\lambda}_{\dot{\alpha}}\lambda^{\dot{\alpha}})^2\bar{\lambda}_{\underline{\alpha}}\lambda^{\underline{\alpha}}}\Big[r_{\dot{\gamma}}\lambda^{\dot{\gamma}}\bar{\lambda}_{\dot{\alpha}}\bar{\lambda}_{\beta'}H^{[\dot{\alpha},\beta']}$$

$$+\bar{\lambda}_{\beta'}r_{\dot{\alpha}}r_{\dot{\gamma}}K^{[\dot{\alpha},\beta',\dot{\gamma}]} + \bar{\lambda}_{\dot{\alpha}}r_{\beta'}r_{\dot{\gamma}}K^{[\dot{\alpha},\beta',\dot{\gamma}]} + Q\Big(\bar{\lambda}_{\dot{\alpha}}\bar{\lambda}_{\beta'}r_{\dot{\gamma}}K^{[\dot{\alpha},\beta',\dot{\gamma}]}\Big)\Big]. \tag{A.14}$$

The second term in A combines with the second term in B to produce

$$\frac{r_{\underline{\alpha}}\lambda^{\underline{\alpha}}\bar{\lambda}_{\dot{\alpha}}\bar{\lambda}_{\beta'}H^{[\dot{\alpha},\beta']}}{\bar{\lambda}_{\dot{\alpha}}\lambda^{\dot{\alpha}}(\bar{\lambda}_{\underline{\alpha}}\lambda^{\underline{\alpha}})^2}. \tag{A.15}$$

This term and the one in $A(\bar{\lambda}_{\dot{\alpha}}\lambda^{\dot{\alpha}})^{-1}(\bar{\lambda}_{\underline{\alpha}}\lambda^{\underline{\alpha}})^{-1}$, that is,

$$\frac{r_{\dot{\gamma}}\lambda^{\dot{\gamma}}\bar{\lambda}_{\dot{\alpha}}\bar{\lambda}_{\beta'}H^{[\dot{\alpha},\beta']}}{(\bar{\lambda}_{\dot{\alpha}}\lambda^{\dot{\alpha}})^2\bar{\lambda}_{\underline{\alpha}}\lambda^{\underline{\alpha}}}, \tag{A.16}$$

will cancel the entire term containing $Q\left(\frac{1}{(\bar{\lambda}_{\dot{\alpha}}\lambda^{\dot{\alpha}})(\bar{\lambda}_{\underline{\alpha}}\lambda^{\underline{\alpha}})}\right)$, as can be seen from

$$Q\left(\frac{1}{(\bar{\lambda}_{\dot{\alpha}}\lambda^{\dot{\alpha}})(\bar{\lambda}_{\underline{\alpha}}\lambda^{\underline{\alpha}})}\right) = \frac{r_{\dot{\alpha}}\lambda^{\dot{\alpha}}}{(\bar{\lambda}_{\dot{\alpha}}\lambda^{\dot{\alpha}})^2\bar{\lambda}_{\underline{\alpha}}\lambda^{\underline{\alpha}}} + \frac{r_{\underline{\alpha}}\lambda^{\underline{\alpha}}}{\bar{\lambda}_{\dot{\alpha}}\lambda^{\dot{\alpha}}(\bar{\lambda}_{\underline{\alpha}}\lambda^{\underline{\alpha}})^2}. \tag{A.17}$$

Summarizing all this,

$$
\begin{aligned}
b_1 + b_2 = {}& \frac{\bar\lambda_{\dot\alpha} G^{\dot\alpha}}{\bar\lambda_{\dot\alpha}\lambda^{\dot\alpha}} - \frac{\bar\lambda_{\dot\alpha} r_{\dot\beta} H^{[\dot\alpha,\dot\beta]}}{(\bar\lambda_{\dot\alpha}\lambda^{\dot\alpha})^2} \\
& + Q\left(\frac{\bar\lambda_{\dot\alpha}\bar\lambda_{\beta'} H^{[\dot\alpha,\beta']}}{(\bar\lambda_{\dot\alpha}\lambda^{\dot\alpha})(\bar\lambda_{\underline{\alpha}}\lambda^{\underline{\alpha}})} + \frac{\bar\lambda_{\dot\alpha}\bar\lambda_{\beta'} r_{\underline{\gamma}} K^{[\dot\alpha,\beta',\underline{\gamma}]}}{\bar\lambda_{\dot\alpha}\lambda^{\dot\alpha}(\bar\lambda_{\underline{\alpha}}\lambda^{\underline{\alpha}})^2} + \frac{\bar\lambda_{\dot\alpha}\bar\lambda_{\beta'} r_{\dot\gamma} K^{[\dot\alpha,\beta',\dot\gamma]}}{(\bar\lambda_{\dot\alpha}\lambda^{\dot\alpha})^2 \bar\lambda_{\underline{\alpha}}\lambda^{\underline{\alpha}}} \right) \\
& + \frac{\bar\lambda_{\beta'} r_{\dot\alpha} r_{\underline{\gamma}} K^{[\dot\alpha,\beta',\underline{\gamma}]} + \bar\lambda_{\dot\alpha} r_{\beta'} r_{\underline{\gamma}} K^{[\dot\alpha,\beta'.\underline{\gamma}]}}{\bar\lambda_{\dot\alpha}\lambda^{\dot\alpha}(\bar\lambda_{\underline{\alpha}}\lambda^{\underline{\alpha}})^2} + \frac{\bar\lambda_{\beta'} r_{\dot\alpha} r_{\dot\gamma} K^{[\dot\alpha,\beta',\dot\gamma]} + \bar\lambda_{\dot\alpha} r_{\beta'} r_{\dot\gamma} K^{[\dot\alpha,\beta',\dot\gamma]}}{(\bar\lambda_{\dot\alpha}\lambda^{\dot\alpha})^2 \bar\lambda_{\underline{\alpha}}\lambda^{\underline{\alpha}}} \\
& - \bar\lambda_{\dot\alpha}\bar\lambda_{\beta'} r_{\underline{\gamma}} K^{[\dot\alpha,\beta',\underline{\gamma}]} Q\left(\frac{1}{\bar\lambda_{\dot\alpha}\lambda^{\dot\alpha}(\bar\lambda_{\underline{\alpha}}\lambda^{\underline{\alpha}})^2}\right) - \bar\lambda_{\dot\alpha}\bar\lambda_{\beta'} r_{\dot\gamma} K^{[\dot\alpha,\beta',\dot\gamma]} Q\left(\frac{1}{(\bar\lambda_{\dot\alpha}\lambda^{\dot\alpha})^2 \bar\lambda_{\underline{\alpha}}\lambda^{\underline{\alpha}}}\right).
\end{aligned}
\tag{A.18}
$$

Third term

From the expression

$$
\begin{aligned}
Q(\bar\lambda_{\dot\alpha}\bar\lambda_{\beta'} r_{\hat\gamma} r_{\hat\delta} L^{[\dot\alpha,\beta',\hat\gamma,\hat\delta]}) = {}& \bar\lambda_{\dot\alpha}\lambda^{\dot\alpha}\bar\lambda_{\beta'} r_{\hat\gamma} r_{\hat\delta} K^{[\beta',\hat\gamma,\hat\delta]} - \bar\lambda_{\beta'}\lambda^{\beta'}\bar\lambda_{\dot\alpha} r_{\hat\gamma} r_{\hat\delta} K^{[\dot\alpha,\hat\gamma,\hat\delta]} \\
& + r_{\hat\gamma}\lambda^{\hat\gamma}\bar\lambda_{\dot\alpha}\bar\lambda_{\beta'} r_{\hat\delta} K^{[\dot\alpha,\beta',\hat\delta]} + r_{\hat\delta}\lambda^{\hat\delta}\bar\lambda_{\dot\alpha}\bar\lambda_{\beta'} r_{\hat\gamma} K^{[\dot\alpha,\beta',\hat\gamma]} \\
& - \bar\lambda_{\beta'} r_{\dot\alpha} r_{\hat\gamma} r_{\hat\delta} L^{[\dot\alpha,\beta',\hat\gamma,\hat\delta]} - \bar\lambda_{\dot\alpha} r_{\beta'} r_{\hat\gamma} r_{\hat\delta} L^{[\dot\alpha,\beta',\hat\gamma,\hat\delta]},
\end{aligned}
\tag{A.19}
$$

where the equation holds for every formula obtained by replacing indices like $\hat\gamma$ by $\dot\gamma$ or γ', one can write

$$
\begin{aligned}
\bar\lambda_{\beta'} r_{\hat\gamma} r_{\hat\delta} K^{[\beta',\hat\gamma,\hat\delta]} = {}& \frac{1}{\bar\lambda_{\dot\alpha}\lambda^{\dot\alpha}} \Big[\bar\lambda_{\beta'}\lambda^{\beta'}\bar\lambda_{\dot\alpha} r_{\hat\gamma} r_{\hat\delta} K^{[\dot\alpha,\hat\gamma,\hat\delta]} - r_{\hat\gamma}\lambda^{\hat\gamma}\bar\lambda_{\dot\alpha}\bar\lambda_{\beta'} r_{\hat\delta} K^{[\dot\alpha,\beta',\hat\delta]} - r_{\hat\delta}\lambda^{\hat\delta}\bar\lambda_{\dot\alpha}\bar\lambda_{\beta'} r_{\hat\gamma} K^{[\dot\alpha,\beta',\hat\gamma]} \\
& + \bar\lambda_{\beta'} r_{\dot\alpha} r_{\hat\gamma} r_{\hat\delta} L^{[\dot\alpha,\beta',\hat\gamma,\hat\delta]} + \bar\lambda_{\dot\alpha} r_{\beta'} r_{\hat\gamma} r_{\hat\delta} L^{[\dot\alpha,\beta',\hat\gamma,\hat\delta]} + Q(\bar\lambda_{\dot\alpha}\bar\lambda_{\beta'} r_{\hat\gamma} r_{\hat\delta} L^{[\dot\alpha,\beta',\hat\gamma,\hat\delta]}) \Big].
\end{aligned}
\tag{A.20}
$$

For the particular case where $(\hat\gamma,\hat\delta) = (\dot\gamma,\dot\delta)$ the previous formula reduces to

$$
\bar\lambda_{\beta'} r_{\dot\gamma} r_{\dot\delta} K^{[\beta',\dot\gamma,\dot\delta]} = -\frac{2}{\bar\lambda_{\dot\alpha}\lambda^{\dot\alpha}} r_{\dot\gamma}\lambda^{\dot\gamma}\bar\lambda_{\dot\alpha}\bar\lambda_{\beta'} r_{\dot\delta} K^{[\dot\alpha,\beta',\dot\delta]}.
\tag{A.21}
$$

The third term in the b-ghost is

$$b_3 = -\frac{\bar{\lambda}_{\underline{\alpha}} r_{\underline{\beta}} r_{\underline{\gamma}} K^{[\underline{\alpha},\underline{\beta},\underline{\gamma}]}}{(\bar{\lambda}_{\underline{\alpha}} \lambda^{\underline{\alpha}})^3} \tag{A.22}$$

which can be written as $b_3 = (C + D + E)(\bar{\lambda}_{\underline{\alpha}} \lambda^{\underline{\alpha}})^{-3}$, where

$$C = -\bar{\lambda}_{\dot{\alpha}} r_{\underline{\beta}} r_{\gamma'} K^{[\dot{\alpha},\underline{\beta},\gamma']} - \bar{\lambda}_{\alpha'} r_{\underline{\beta}} r_{\gamma'} K^{[\alpha',\underline{\beta},\gamma']}, \tag{A.23}$$

$$D = -\bar{\lambda}_{\dot{\alpha}} r_{\beta'} r_{\dot{\gamma}} K^{[\dot{\alpha},\beta',\dot{\gamma}]} - \bar{\lambda}_{\alpha'} r_{\beta'} r_{\dot{\gamma}} K^{[\alpha',\beta',\dot{\gamma}]}, \tag{A.24}$$

$$E = -\bar{\lambda}_{\alpha'} r_{\dot{\beta}} r_{\dot{\gamma}} K^{[\alpha',\dot{\beta},\dot{\gamma}]}. \tag{A.25}$$

Focusing first on terms containing expressions of the form $\bar{\lambda}_{\hat{\alpha}} r_{\hat{\beta}} r_{\hat{\gamma}} K^{[\hat{\alpha},\hat{\beta},\hat{\gamma}]}$ after making the substitutions (A.20) and (A.21), one gets several such terms. The terms in $(C + D)(\bar{\lambda}_{\underline{\alpha}} \lambda^{\underline{\alpha}})^{-3}$ of this form are

$$-\frac{\bar{\lambda}_{\dot{\alpha}} r_{\underline{\beta}} r_{\gamma'} K^{[\dot{\alpha},\underline{\beta},\gamma']}}{\bar{\lambda}_{\dot{\alpha}} \lambda^{\dot{\alpha}} (\bar{\lambda}_{\underline{\alpha}} \lambda^{\underline{\alpha}})^2} - \frac{\bar{\lambda}_{\dot{\alpha}} r_{\beta'} r_{\dot{\gamma}} K^{[\dot{\alpha},\beta',\dot{\gamma}]}}{\bar{\lambda}_{\dot{\alpha}} \lambda^{\dot{\alpha}} (\bar{\lambda}_{\underline{\alpha}} \lambda^{\underline{\alpha}})^2} \tag{A.26}$$

which, together with terms having same denominator in (A.18), give

$$\frac{D}{\bar{\lambda}_{\dot{\alpha}} \lambda^{\dot{\alpha}} (\bar{\lambda}_{\underline{\alpha}} \lambda^{\underline{\alpha}})^2} - \frac{\bar{\lambda}_{\beta'} r_{\dot{\alpha}} r_{\dot{\gamma}} K^{[\beta',\dot{\alpha},\dot{\gamma}]}}{\bar{\lambda}_{\dot{\alpha}} \lambda^{\dot{\alpha}} (\bar{\lambda}_{\underline{\alpha}} \lambda^{\underline{\alpha}})^2}. \tag{A.27}$$

The first term in the last formula contains

$$-\frac{\bar{\lambda}_{\dot{\alpha}} r_{\beta'} r_{\dot{\gamma}} K^{[\dot{\alpha},\beta',\dot{\gamma}]}}{(\bar{\lambda}_{\dot{\alpha}} \lambda^{\dot{\alpha}})^2 \bar{\lambda}_{\underline{\alpha}} \lambda^{\underline{\alpha}}} \tag{A.28}$$

which kills one term in (A.18) with the same denominator. At the end one has in $b_1 + b_2 + b_3$,

$$-\bar{\lambda}_{\alpha'} r_{\dot{\beta}} r_{\dot{\gamma}} K^{[\alpha',\dot{\beta},\dot{\gamma}]} \left(\frac{1}{(\bar{\lambda}_{\underline{\alpha}} \lambda^{\underline{\alpha}})^3} + \frac{1}{\bar{\lambda}_{\dot{\alpha}} \lambda^{\dot{\alpha}} (\bar{\lambda}_{\underline{\alpha}} \lambda^{\underline{\alpha}})^2} + \frac{1}{(\bar{\lambda}_{\dot{\alpha}} \lambda^{\dot{\alpha}})^2 \bar{\lambda}_{\underline{\alpha}} \lambda^{\underline{\alpha}}} \right). \tag{A.29}$$

Using relation (A.21) one eliminates all appearance of terms containing $\bar{\lambda}_{\hat{\alpha}} r_{\hat{\beta}} r_{\hat{\gamma}} K^{[\hat{\alpha},\hat{\beta},\hat{\gamma}]}$.

It is not difficult to check that all terms containing expressions like $(r\lambda)\bar{\lambda}\bar{\lambda} rK$

cancel and the remaining terms contain only the operator $L^{[\underline{\alpha},\underline{\beta},\underline{\gamma},\underline{\delta}]}$. One gets

$$b_1 + b_2 + b_3 = \frac{\bar{\lambda}_{\dot{\alpha}} G^{\dot{\alpha}}}{\bar{\lambda}_{\dot{\alpha}} \lambda^{\dot{\alpha}}} - \frac{\bar{\lambda}_{\dot{\alpha}} r_{\dot{\beta}} H^{[\dot{\alpha},\dot{\beta}]}}{(\bar{\lambda}_{\dot{\alpha}} \lambda^{\dot{\alpha}})^2}$$

$$+Q\left(\frac{\bar{\lambda}_{\dot{\alpha}} \bar{\lambda}_{\beta'} H^{[\dot{\alpha},\beta']}}{(\bar{\lambda}_{\dot{\alpha}} \lambda^{\dot{\alpha}})(\bar{\lambda}_{\underline{\alpha}} \lambda^{\underline{\alpha}})} + \frac{\bar{\lambda}_{\dot{\alpha}} \bar{\lambda}_{\beta'} r_{\underline{\gamma}} K^{[\dot{\alpha},\beta',\underline{\gamma}]}}{\bar{\lambda}_{\dot{\alpha}} \lambda^{\dot{\alpha}} (\bar{\lambda}_{\underline{\alpha}} \lambda^{\underline{\alpha}})^2} + \frac{\bar{\lambda}_{\dot{\alpha}} \bar{\lambda}_{\beta'} r_{\dot{\gamma}} K^{[\dot{\alpha},\beta',\dot{\gamma}]}}{(\bar{\lambda}_{\dot{\alpha}} \lambda^{\dot{\alpha}})^2 \bar{\lambda}_{\underline{\alpha}} \lambda^{\underline{\alpha}}} \right.$$

$$\left. - \frac{\bar{\lambda}_{\dot{\alpha}} \bar{\lambda}_{\beta'} r_{\underline{\gamma}} r_{\delta'} L^{[\dot{\alpha},\beta',\underline{\gamma},\delta']} + \bar{\lambda}_{\dot{\alpha}} \bar{\lambda}_{\beta'} r_{\gamma'} r_{\dot{\delta}} L^{[\dot{\alpha},\beta',\gamma',\dot{\delta}]}}{\bar{\lambda}_{\dot{\alpha}} \lambda^{\dot{\alpha}} (\bar{\lambda}_{\underline{\alpha}} \lambda^{\underline{\alpha}})^3} - \frac{\bar{\lambda}_{\dot{\alpha}} \bar{\lambda}_{\beta'} r_{\gamma'} r_{\dot{\delta}} L^{[\dot{\alpha},\beta',\gamma',\dot{\delta}]}}{(\bar{\lambda}_{\dot{\alpha}} \lambda^{\dot{\alpha}})^2 (\bar{\lambda}_{\underline{\alpha}} \lambda^{\underline{\alpha}})^2} \right)$$

$$- \frac{(\bar{\lambda}_{\dot{\alpha}} r_{\beta'} + \bar{\lambda}_{\beta'} r_{\dot{\alpha}}) r_{\gamma'} r_{\delta'} L^{[\dot{\alpha},\beta',\gamma',\delta']} + 2(\bar{\lambda}_{\dot{\alpha}} r_{\beta'} + \bar{\lambda}_{\beta'} r_{\dot{\alpha}}) r_{\gamma'} r_{\dot{\delta}} L^{[\dot{\alpha},\beta',\gamma',\dot{\delta}]}}{\bar{\lambda}_{\dot{\alpha}} \lambda^{\dot{\alpha}} (\bar{\lambda}_{\underline{\alpha}} \lambda^{\underline{\alpha}})^3}$$

$$- \frac{(\bar{\lambda}_{\dot{\alpha}} r_{\beta'} + \bar{\lambda}_{\beta'} r_{\dot{\alpha}}) r_{\gamma'} r_{\dot{\delta}} L^{[\dot{\alpha},\beta',\gamma',\dot{\delta}]}}{(\bar{\lambda}_{\dot{\alpha}} \lambda^{\dot{\alpha}})^2 (\bar{\lambda}_{\underline{\alpha}} \lambda^{\underline{\alpha}})^2}$$

$$+ \left(\bar{\lambda}_{\dot{\alpha}} \bar{\lambda}_{\beta'} r_{\underline{\gamma}} r_{\delta'} L^{[\dot{\alpha},\beta'\underline{\gamma},\delta']} + \bar{\lambda}_{\dot{\alpha}} \bar{\lambda}_{\beta'} r_{\gamma'} r_{\dot{\delta}} L^{[\dot{\alpha},\beta',\gamma',\dot{\delta}]} \right) Q \left(\frac{1}{\bar{\lambda}_{\dot{\alpha}} \lambda^{\dot{\alpha}} (\bar{\lambda}_{\underline{\alpha}} \lambda^{\underline{\alpha}})^3} \right)$$

$$+ \bar{\lambda}_{\dot{\alpha}} \bar{\lambda}_{\beta'} r_{\gamma'} r_{\dot{\delta}} L^{[\dot{\alpha},\beta',\gamma',\dot{\delta}]} Q \left(\frac{1}{(\bar{\lambda}_{\dot{\alpha}} \lambda^{\dot{\alpha}})^2 (\bar{\lambda}_{\underline{\alpha}} \lambda^{\underline{\alpha}})^2} \right). \tag{A.30}$$

Fourth term

The next term of the b-ghost is

$$b_4 = \frac{\bar{\lambda}_{\underline{\alpha}} r_{\underline{\beta}} r_{\underline{\gamma}} r_{\underline{\delta}} L^{[\underline{\alpha},\underline{\beta},\underline{\gamma},\underline{\delta}]}}{(\bar{\lambda}_{\underline{\alpha}} \lambda^{\underline{\alpha}})^4} \tag{A.31}$$

where the operator $L^{[\underline{\alpha},\underline{\beta},\underline{\gamma},\underline{\delta}]}$ obeys $\lambda^{[\underline{\alpha}} L^{\underline{\beta},\underline{\gamma},\underline{\delta},\underline{\rho}]} = 0$. These relations can be written as

$$\lambda^{[\alpha'} L^{\beta',\gamma',\delta',\rho']} = 0, \tag{A.32}$$

$$\lambda^{\dot{\alpha}} L^{[\beta',\gamma',\delta',\rho']} - \lambda^{\beta'} L^{[\dot{\alpha},\gamma',\delta',\rho']} + \lambda^{\gamma'} L^{[\dot{\alpha},\beta',\delta',\rho']} - \lambda^{\delta'} L^{[\dot{\alpha},\beta',\gamma',\rho']} + \lambda^{\rho'} L^{[\dot{\alpha},\beta',\gamma',\delta']} = 0, \tag{A.33}$$

$$\lambda^{\dot{\alpha}} L^{[\dot{\beta},\gamma',\delta',\rho']} - \lambda^{\dot{\beta}} L^{[\dot{\alpha},\gamma',\delta',\rho']} + \lambda^{\gamma'} L^{[\dot{\alpha},\dot{\beta},\delta',\rho']} - \lambda^{\delta'} L^{[\dot{\alpha},\dot{\beta},\gamma',\rho']} + \lambda^{\rho'} L^{[\dot{\alpha},\dot{\beta},\gamma',\delta']} = 0, \tag{A.34}$$

$$\lambda^{\dot{\alpha}} L^{[\dot{\beta},\dot{\gamma},\delta',\rho']} - \lambda^{\dot{\beta}} L^{[\dot{\alpha},\dot{\gamma},\delta',\rho']} + \lambda^{\dot{\gamma}} L^{[\dot{\alpha},\dot{\beta},\delta',\rho']} = 0, \tag{A.35}$$

which implies

$$\bar\lambda_{\beta'}r_{\gamma'}r_{\delta'}r_{\rho'}L^{[\beta',\gamma',\delta',\rho']}=\frac{1}{\bar\lambda_{\dot\alpha}\lambda^{\dot\alpha}}\Big(\bar\lambda_{\beta'}\lambda^{\beta'}\bar\lambda_{\dot\alpha}r_{\gamma'}r_{\delta'}r_{\rho'}L^{[\dot\alpha,\gamma',\delta',\rho']}-3r_{\gamma'}\lambda^{\gamma'}\bar\lambda_{\dot\alpha}\bar\lambda_{\beta'}r_{\delta'}r_{\rho'}L^{[\dot\alpha,\beta',\delta',\rho']}\Big), \tag{A.36}$$

$$\bar\lambda_{\gamma'}r_{\dot\beta}r_{\delta'}r_{\rho'}L^{[\dot\beta,\gamma',\delta',\rho']}=\frac{1}{\bar\lambda_{\dot\alpha}\lambda^{\dot\alpha}}\Big(-\bar\lambda_{\gamma'}\lambda^{\gamma'}\bar\lambda_{\dot\alpha}r_{\dot\beta}r_{\delta'}r_{\rho'}L^{[\dot\alpha,\dot\beta,\delta',\rho']}$$

$$+r_{\dot\beta}\lambda^{\dot\beta}\bar\lambda_{\dot\alpha}\bar\lambda_{\gamma'}r_{\delta'}r_{\rho'}L^{[\dot\alpha,\gamma',\delta',\rho']}-2r_{\rho'}\lambda^{\rho'}\bar\lambda_{\dot\alpha}\bar\lambda_{\gamma'}r_{\dot\beta}r_{\delta'}L^{[\dot\alpha,\dot\beta,\gamma',\delta']}\Big), \tag{A.37}$$

$$\bar\lambda_{\delta'}r_{\dot\beta}r_{\dot\gamma}r_{\rho'}L^{[\delta',\dot\beta,\dot\gamma,\rho']}=\frac{2}{\bar\lambda_{\dot\alpha}\lambda^{\dot\alpha}}r_{\dot\beta}\lambda^{\dot\beta}\bar\lambda_{\dot\alpha}\bar\lambda_{\delta'}r_{\dot\gamma}r_{\rho'}L^{[\dot\alpha,\dot\gamma,\delta',\rho']}. \tag{A.38}$$

The fourth term of the b-ghost can be written also keeping track of the four dimensional chiral spinor index $\dot\alpha$, as follows:

$$b_4=\frac{X+Y+Z}{(\bar\lambda_{\underline\alpha}\lambda^{\underline\alpha})^4} \tag{A.39}$$

where

$$X=\bar\lambda_{\alpha'}r_{\beta'}r_{\gamma'}r_{\delta'}L^{[\alpha',\beta',\gamma',\delta']}+\bar\lambda_{\dot\alpha}r_{\beta'}r_{\gamma'}r_{\delta'}L^{[\dot\alpha,\beta',\gamma',\delta']}, \tag{A.40}$$

$$Y=3\bar\lambda_{\alpha'}r_{\dot\beta}r_{\gamma'}r_{\delta'}L^{[\alpha',\dot\beta,\gamma',\delta']}+3\bar\lambda_{\dot\alpha}r_{\dot\beta}r_{\gamma'}r_{\delta'}L^{[\dot\alpha,\dot\beta,\gamma',\delta']}, \tag{A.41}$$

$$Z=3\bar\lambda_{\alpha'}r_{\dot\beta}r_{\dot\gamma}r_{\delta'}L^{[\alpha',\dot\beta,\dot\gamma,\delta']}. \tag{A.42}$$

Using relations (A.36), (A.37) and (A.38), one can simplify $b_1+b_2+b_3+b_4$. Let's focus again on terms containing $\bar\lambda_{\hat\alpha}r_{\hat\beta}r_{\hat\gamma}r_{\hat\delta}L^{[\hat\alpha,\hat\beta,\hat\gamma,\hat\delta]}$. From X, one gets

$$\frac{\bar\lambda_{\dot\alpha}r_{\beta'}r_{\gamma'}r_{\delta'}L^{[\dot\alpha,\beta',\gamma',\delta']}}{\bar\lambda_{\dot\alpha}\lambda^{\dot\alpha}(\bar\lambda_{\underline\alpha}\lambda^{\underline\alpha})^3} \tag{A.43}$$

which just cancels one of the terms in $b_1+b_2+b_3$. From Y, one gets

$$\frac{3\bar\lambda_{\dot\alpha}r_{\dot\beta}r_{\gamma'}r_{\delta'}L^{[\dot\alpha,\dot\beta,\gamma',\delta']}}{\bar\lambda_{\dot\alpha}\lambda^{\dot\alpha}(\bar\lambda_{\underline\alpha}\lambda^{\underline\alpha})^3}. \tag{A.44}$$

This adds to two terms in $b_1+b_2+b_3$ to produce

$$\frac{Y}{3\bar\lambda_{\dot\alpha}\lambda^{\dot\alpha}(\bar\lambda_{\underline\alpha}\lambda^{\underline\alpha})^3}. \tag{A.45}$$

Using (A.37), it is seen that this expression contains

$$\frac{\bar\lambda_{\dot\alpha} r_{\dot\beta} r_{\gamma'} r_{\delta'} L^{[\dot\alpha,\dot\beta,\gamma',\delta']}}{(\bar\lambda_{\dot\alpha}\lambda^{\dot\alpha})^2(\bar\lambda_{\underline\alpha}\lambda^{\underline\alpha})^2} \tag{A.46}$$

which cancels another term in $b_1 + b_2 + b_3$. What is left is just

$$\bar\lambda_{\alpha'} r_{\dot\beta} r_{\dot\gamma} r_{\delta'} L^{[\alpha',\dot\beta,\dot\gamma,\delta']}\left(\frac{3}{(\bar\lambda_{\underline\alpha}\lambda^{\underline\alpha})^4} + \frac{2}{(\bar\lambda_{\dot\alpha}\lambda^{\dot\alpha})(\bar\lambda_{\underline\alpha}\lambda^{\underline\alpha})^3} + \frac{1}{(\bar\lambda_{\dot\alpha}\lambda^{\dot\alpha})^2(\bar\lambda_{\underline\alpha}\lambda^{\underline\alpha})^2}\right). \tag{A.47}$$

Using equation (A.38) one eliminates all appearances of terms containing $\bar\lambda_{\hat\alpha} r_{\hat\beta} r_{\hat\gamma} r_{\hat\delta} L^{[\hat\alpha,\hat\beta,\hat\gamma,\hat\delta]}$. One ends up with terms having in the numerator expressions like $(r\lambda)\bar\lambda\bar\lambda rrL$. Collecting all these terms, it is easy to see that they all cancel out.

Result

So the final result for $b = b_1 + b_2 + b_3 + b_4 + s^{\underline\alpha}\partial\bar\lambda_{\underline\alpha}$ is

$$b = \frac{\bar\lambda_{\dot\alpha} G^{\dot\alpha}}{\bar\lambda_{\dot\alpha}\lambda^{\dot\alpha}} - \frac{\bar\lambda_{\dot\alpha} r_{\dot\beta} H^{[\dot\alpha,\dot\beta]}}{(\bar\lambda_{\dot\alpha}\lambda^{\dot\alpha})^2} + s^{\underline\alpha}\partial\bar\lambda_{\underline\alpha} \tag{A.48}$$

$$+Q\left(\frac{\bar\lambda_{\dot\alpha}\bar\lambda_{\beta'} H^{[\dot\alpha,\beta']}}{(\bar\lambda_{\dot\alpha}\lambda^{\dot\alpha})(\bar\lambda_{\underline\alpha}\lambda^{\underline\alpha})} + \frac{\bar\lambda_{\dot\alpha}\bar\lambda_{\beta'} r_{\underline\gamma} K^{[\dot\alpha,\beta',\underline\gamma]}}{\bar\lambda_{\dot\alpha}\lambda^{\dot\alpha}(\bar\lambda_{\underline\alpha}\lambda^{\underline\alpha})^2} + \frac{\bar\lambda_{\dot\alpha}\bar\lambda_{\beta'} r_{\dot\gamma} K^{[\dot\alpha,\beta',\dot\gamma]}}{(\bar\lambda_{\dot\alpha}\lambda^{\dot\alpha})^2\bar\lambda_{\underline\alpha}\lambda^{\underline\alpha}}\right.$$

$$\left.-\frac{\bar\lambda_{\dot\alpha}\bar\lambda_{\beta'} r_{\underline\gamma} r_{\delta'} L^{[\dot\alpha,\beta',\underline\gamma,\delta']} + \bar\lambda_{\dot\alpha}\bar\lambda_{\beta'} r_{\gamma'} r_{\dot\delta} L^{[\dot\alpha,\beta',\gamma',\dot\delta]}}{\bar\lambda_{\dot\alpha}\lambda^{\dot\alpha}(\bar\lambda_{\underline\alpha}\lambda^{\underline\alpha})^3} - \frac{\bar\lambda_{\dot\alpha}\bar\lambda_{\beta'} r_{\gamma'} r_{\dot\delta} L^{[\dot\alpha,\beta',\gamma',\dot\delta]}}{(\bar\lambda_{\dot\alpha}\lambda^{\dot\alpha})^2(\bar\lambda_{\underline\alpha}\lambda^{\underline\alpha})^2}\right).$$

So $b = b^{(a)} + Q\Lambda^{(a)}$ in the patch where $\bar\lambda_{\dot\alpha}\lambda^{\dot\alpha} \neq 0$. The derivation of $b = b^{(c)} + Q\Lambda^{(c)}$ in the patch where $\bar\lambda_\alpha\lambda^\alpha \neq 0$ is completely analogous.

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