**IFT**

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# Anti-de Sitter Holography in Different Coordinates

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All my work is dedicated to my family, friends and Estefanía.

"*We fought far under the living earth, where time is not counted.*"

Lord of the Rings, J.R.R Tolkien.

# Resumo

O objetivo deste trabalho é mostrar a importância da escolha do sistema de coordenadas e assinatura para descrever o espaço Anti-de Sitter na Correspondência AdS/CFT. Para fazer isto, mostraremos como calcular correlatores em diferentes coordenadas no lado gravitacional da correspondência. Em particular, apresentamos o procedimiento geral para encontrar o propagador bulk-fronteira para um campo escalar, de gauge e espinorial usando o método de Witten em coordenadas de Poincaré com assinatura euclideana. Também mostramos os cálculos das funçoẽs de $n$ pontos para um campo escalar interagente em AdS a nível de árvore e comparamos os resultados com os vínculos da teoría de campos conformes. Para comparar as implicaçoẽs de trabalhar em outro sistema coordenado refazemos os cálculos dos propagadores para o campo escalar em coordenadas globais. Finalmente mostramos como calcular o propagador bulk-fronteira para um campo 3-forma que vive em $AdS_7$ em coordenadas globais e de Poincaré.

**Palavras Chaves**: Espaço Anti-de Sitter; Teoría de Campos Conformes; Correspondência AdS/CFT

**Áreas do conhecimento**: Relatividade Geral; Teoría Quântica de Campos: Teoría das Cordas

# Abstract

The goal of this work is to show the significance of the choice of coordinates system and signature to describe the Anti-de Sitter space within AdS/CFT Correspondence. In order to do so, we will show how to calculate AdS/CFT correlators in different coordinates from the gravity side of the correspondence. In particular, we present the general procedure to find the bulk-to-boundary propagators for scalar, gauge and spinor fields using Witten's method in Poincaré coordinates with Euclidean signature. We also show the calculations of $n$-point functions for a classical interacting scalar field in AdS at tree level and compare the results with the CFT constraints. To compare the implications of working in different coordinates, we recalculate the propagators for the scalar field in Global coordinates. Finally, we show the calculations of bulk-to-boundary propagator for a 3-form field living in $AdS_7$ in both Poincaré and Global coordinates.

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# Contents

# 1 Introduction

The AdS/CFT Correspondence or Gauge/String duality discovered by Maldacena [1] has been one of the majors breakthroughs in theoretical high energy physics in the past two decades, it is a conjecture that relates a gravitational theory in a $(d+1)$-dimensional space with a conformal field theory defined in a space of lower dimension $d$, that is the reason we use the term *holography* when we refer to some aspects of the correspondence. Its succes is due in part to the fact that it has influenced and motivated the foundation of several research areas within theoretical physics such as condensed matter, quantum chromodynamics, plasma physics, gravity, etc, and it has opened the possibility to deal with some issues such as nonpertubative regime of some field theories by looking at its equivalent theory pertubatively via AdS/CFT, property that has made the correspondence a Strong/Weak coupling relation. A precise prescription to match observables of the two parts of the correspondence was proposed separately by Witten [2] and Gubser, Polyakov and Klebanov [3] shortly after the discovery and it has turned out to be effective, it states the equivalence of the partition function of the gravity theory in Anti-de Sitter space with the partition function of the conformal field theory defined on the boundary of AdS. This allowed the first checks of the correspondence by computing and matching correlators or $n$-point functions, e.g [4], [8], [17], [18], [21], [15]. Despite the fact that the correspondence is not defined in a particular coordinate system for AdS, most of the works were done using Poincaré coordinates with euclidean signature. This happens because people usually want field theories defined on flat spaces, but for instance, we could use global coordinates (see [5]) to describe AdS and the correspondence is still valid. Some people (see [6] and [7]) have worked on describing the new features that appear in holography when using other coordinate system than euclidean Poincaré, for instance, Poincaré AdS with Minkowskian signature. It is important to remark that Poincaré coordinates only covers a part of the whole Anti-de Sitter space but the results working with this system are so neat that it seems people have ignored this issue and that is what this dissertation is about. The main purpose of this work is to show that the relation between holography in Poincaré and global coordinates is not just a change of coordinates and we still do not know why the AdS/CFT correspondence works so well in the Poincaré patch if this one covers only a portion of the entire AdS. In principle all the calculations and checks, such as computing $n$-point functions, should be possible to perform no matter which coordinate system you choose to work with. What we will see is that the choice of coordinates system to describe

AdS plays a role in the AdS/CFT which is simply far from trivial.

This dissertation is organized as follows: The first chapter has to do with Anti-de Sitter space (AdS), we start defining what AdS is within general relativity, how it can be derived as a solution of Einstein equations or by embedding of a hypersurface in certain ambient space, later we present the different coordinates system of our interest to describe this space and analyze the causal structure by using Penrose diagrams, we end this section studying the boundary of AdS and what changes with every coordinates we consider. In the next chapter we present the basics of conformal field theory (CFT), starting from the definitions of conformal transformations of coordinates, its general group properties and how a field theory with conformal invariance is restricted, this is done by studying the constraints imposed on the $n$-point functions working in the coordinate representation. Once we study AdS and CFT separately we are ready to say some words about the AdS/CFT Correspondence. Its discovery and foundations are beyond the scope of the intended work, so we only mention what it means, we present the most known and studied example and we develop the prescription proposed by Witten in [2] which serves as a recipe to match observables of the two sides of the correspondence. Following, we find a chapter treating holography, basically we compute bulk-to-boundary propagators in Poincaré coordinates for scalars, gauge and spinor fields living in AdS, as well as 2-point functions at tree level, which results are compared with functional form dictated by conformal invariance. A general formula to compute $n$-point for scalars at tree level is also found. Besides, in this chapter we check the relations between scaling dimension and the mass according to the AdS/CFT dictionary. In chapter 6 we deal with one of the main topics of this work, holography in other coordinates systems than Euclidean Poincaré coordinates. We start by solving the equation of motion for a free scalar field living on AdS with Minkowski signature and we see that consistency requires the inclusion of new "objects" which were not present in the Euclidean case. In this section we also solve the Klein-Gordon equation with global AdS background and we find the new bulk-to-boundary propagator which relation with the one found in the euclidean Poincaré is, as expected, far from being a simple coordinate transformation. In the last chapter we present an example of correspondence different from the well-known case [1] and we calculate the bulk-to-boundary propagator for a 3-form living in $AdS_7$ in Poincaré coordinates and finally we compute the same object in global coordinates as a modest original contribution from the author.

# 2 Anti-de Sitter Spaces

General relativity is the theory that describes the dynamics of space-time, it is based on the Einstein equations (1) which gives the space-time metric $g_{ab}$ once the content of matter $T_{ab}$ is known.

$$R_{ab} - \frac{1}{2}Rg_{ab} + \Lambda g_{ab} = 8\pi T_{ab} \tag{1}$$

Exact solutions of (1) are more likely found for spaces of high symmetry, in particular we are interested in spaces with constant curvature. To be an exact solution, $T_{ab}$ must be the energy momentum tensor of some configuration of matter which does not violate local causality and it is also restricted to forms in which the energy density measured by any observer would not be negative.

Once a solution is found, we would like to describe physical objects living on the this space-time, so we consider the line element also called the metric:

$$ds^2 = g_{ab}dx^a dx^b \tag{2}$$

We can describe a space as a surface living inside some background space, or embedding space. For instance, the curved $2d$ surface space of the 2-sphere can be regarded as living on the $3d$ euclidean space (3) supplemented with an extra condition (4).

$$ds^2 = dx^2 + dy^2 + dz^2 \tag{3}$$

$$x^2 + y^2 + z^2 = R^2 \tag{4}$$

Then, the 2-dimensional metric is given by:

$$ds^2 = dx^2(1 + \frac{x^2}{R^2 - x^2 - y^2}) + dy^2(1 + \frac{y^2}{R^2 - x^2 - y^2}) + 2dxdy\frac{xy}{R^2 - x^2 - y^2} \tag{5}$$

Another example is the Lobachevski space, which has negative curvature $R < 0$ (two parallel geodesics diverge), this is a 2d space described by embedding the surface (6) into a $3d$ Minkowski space (7).

$$x^2 + y^2 - z^2 = -R^2 \tag{6}$$

$$ds^2 = dx^2 + dy^2 - dz^2 \tag{7}$$

## 2.1 AdS as exact solution of Einstein Equations

Anti-de Sitter space is the simplest, maximally symmetric solution of the Einstein equation with negative curvature. Locally, spacetime metrics of constant curvature are characterized by :

$$R_{abcd} = \frac{1}{d(d-1)} R(g_{ac}g_{bd} - g_{ad}g_{bc}) \tag{8}$$

which is equivalent to:

$$R_{ab} - \frac{1}{2} R g_{ab} = -\frac{1}{4} R g_{ab} \tag{9}$$

Comparing this with Einstein equation we can consistently identify these spaces as solutions with $\Lambda = \frac{1}{4}R$ for an empty space $T_{ab} = 0$. For the case $R < 0$ we obtain the so-called Anti-de Sitter space(AdS) .

## 2.2 AdS as embedding

AdS space is represented by a Lorentzian version of Lobachevski space given by the embedding in $d+1$ dimensions:

$$-X_0^2 + \sum_i^{d-1} X_i^2 - X_{d+1}^2 = -R^2 \tag{10}$$

$$ds^2 = -dX_0^2 + \sum_i^{d-1} dX_i^2 - dX_{d+1}^2 \tag{11}$$

Then, denoting $X_\mu = (X_0, X_{d+1}, X_1, \ldots, X_{d-1})$ we see that by construction it is invariant under $SO(2, d-1)$.

## 2.3 Systems of coordinates and signatures

We can solve the embedding equations with the so called global coordinates:

$$\begin{aligned} X_0 &= R\cosh\rho\cos\tau \\ X_i &= R\sinh\rho\Omega_i \\ X_{d+1} &= R\cosh\rho\sin\tau \end{aligned} \tag{12}$$

where $0 \leq \rho < \infty$, $0 \leq \tau < 2\pi$ and $-1 \leq \Omega_i \leq 1$ with the additional condition $\sum_i \Omega_i = 1$. Then the metric can be written as:

$$ds^2 = R^2 \left(-\cosh^2\rho d\tau^2 + d\rho^2 + \sinh^2\rho d\Omega_{d-1}^2\right) \tag{13}$$

With $\tau \in [0, 2\pi)$ we are covering the space once, that is why these coordinates are called Global. In these coordinates AdS can be represented by the hyperboloid shown in the Figure 1. We also see that the ambient space has two times, but the hyperboloid only one (they are orthogonal to each other). Nonetheless , there is a problem as we approach at $\rho \approx 0$ :

$$ds^2 = R^2 \left(-d\tau^2 + d\rho^2 + \rho^2 d\Omega_{d-1}^2\right) \tag{14}$$

Which means we obtain a space $S^1 \times \mathbf{R}^{d-1}$ and therefore we could have closed time-like curves which are prohibited because of causal concerns. We can solve this by unwrapping the circle, it means letting $-\infty < \tau < \infty$ and we obtain the "universal cover" of global AdS, where we do not have closed time-like curves. So in this work whenever we mention global AdS we refer to its universal covering space. We can also compute the curvature for this metric and check that this space has a constant negative curvature $\Lambda = \frac{-d(d+1)}{R^2}$.

## 2.4 Causal Structure

Penrose diagrams are an important tool developed by Penrose with the aim to understand the causal structure of a spacetime. We are interested in studying AdS spaces and in order to do that it is easier to start with the simpler cases. First, we are going to study the Penrose diagram of flat Minkowski space $\mathbf{R}^{1,p}$ which metric in two dimensions is given by

$$ds^2 = -dt^2 + dx^2 \tag{15}$$

where $-\infty < t, x < \infty$. Introducing new coordinates $u_\pm = t \pm x$ we obtain:

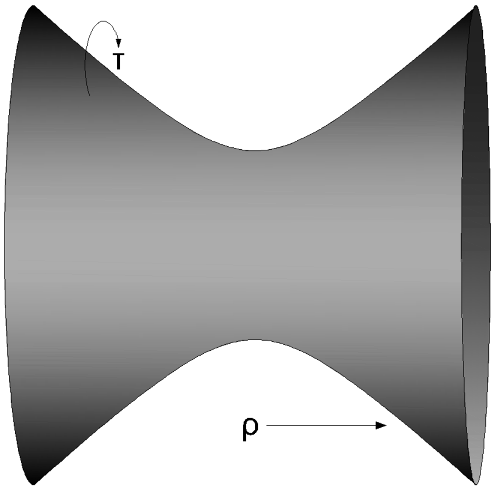


Figure 1: AdS Space in Global coordinates

$$ds^2 = -du_+ du_- \tag{16}$$

where again $-\infty < u_\pm < \infty$. Now, it is easy to show that by making $u_\pm = \tan{(\tau \pm \theta)/2}$ we get:

$$ds^2 = \Omega^2(\tau, \theta)(-d\tau^2 + d\theta^2) \tag{17}$$

with

$$\Omega^2(\tau, \theta) = \frac{1}{4} \sec^2 \frac{1}{2}(\tau + \theta) \ \sec^2 \frac{1}{2}(\tau - \theta) \tag{18}$$

where $|\tau \pm \theta| < \pi$.

Since light ray trajectories ($ds^2 = 0$) are not affected by conformal factors in the metric, we can map Minkowski space into the full diamond region $|\tau \pm \theta| < \pi$ in the Figure 2 and analyze the causal structure of the space. The two corners $i^0$ where $(\tau, \theta) = (0, \pm\pi)$ represent the two spacelike infinities that correspond to $x = \pm\infty$, and $i^\pm$ the future and past timelike infinity. This procedure is used to define asymptotic flatness of a spacetime, a certain spacetime having the same conformal boundary structure as the one found for flat space is called asymptotically flat.

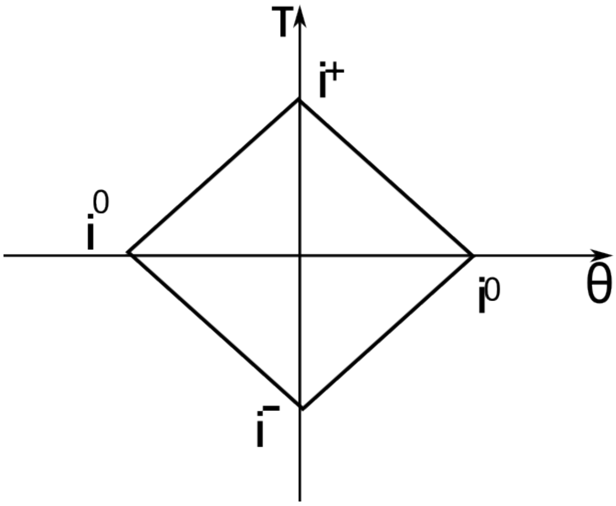


Figure 2: Penrose diagram of $\mathbf{R}^{1,1}$

Now let us analyze the case of Minkowski in arbitrary dimensions $\mathbf{R}^{1,p}$ and $p \geq 2$. In spherical coordinates the metric is given by

$$ds^2 = -dt^2 + dr^2 + r^2 d\Omega_{p-1}^2 \tag{19}$$

where $0 < r < \infty$ and at every point $(r, t)$ there is a sphere $S^{p-1}$, so we can focus on the $(r, t)$ portion. We obtain the same results as in previous case of $\mathbf{R}^{1,1}$ where $|\tau \pm \theta| < \pi$, but now $\theta$ is restricted to $\theta > 0$ and the Penrose diagram is the triangle in the figure 3. Explicitly, the conformal scaled metric will be given by:

$$ds^2 = -d\tau^2 + d\theta^2 + \sin^2\theta \Omega_{p-1}^2 \tag{20}$$

We can analytically continue this space outside the triangle by letting $-\infty < \tau < \infty$ and $0 \leq \theta \leq \pi$ and we obtain the geometry of $\mathbb{R} \times S^p$ which is the Einstein static universe.

Now for the case of AdS in global coordinates first we would like to map $\rho = \infty$ to a finite distance, then we set $\tan\theta = \sinh\rho$, with $0 \leq \theta \leq \pi/2$, except in the 2-dimensional case which is special and allows $-\pi/2 \leq \theta \leq \pi/2$. For this we have:

$$ds^2 = \frac{R^2}{\cos^2\theta}\left(-d\tau^2 + d\theta^2 + \sin^2\theta d\Omega_{d-2}^2\right) \tag{21}$$

Dropping the conformal factor we obtain:

$$ds^2 = -d\tau^2 + d\theta^2 + \sin^2\theta d\Omega_{d-2}^2 \tag{22}$$

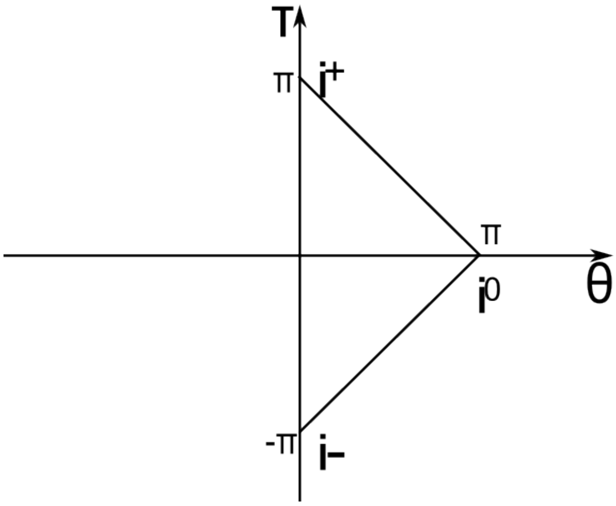


Figure 3: Penrose diagram of $\mathbf{R}^{1,d}$

We notice that this is the same metric of the Einstein static universe found above with a one less dimension. Nevertheless, in this case of AdS the $\theta$ coordinate takes different values, more exactly conformal $AdS_{d+2}$ can be mapped to a half of the Einstein static universe. This can be used to define spaces which are asymptotically AdS: It is a spacetime which is conformal to a region with the same boundary structure of one half of the Einstein static universe.

## 2.5 Boundaries

Let us look at $AdS_3$ which will have a boundary at $\theta = \pi/2$ and we obtain $ds^2 = -d\tau^2 + d\Omega^2$. So the boundary has the topology $\mathbf{R} \times S$, in general the boundary of $AdS_{d+1}$ in global coordinates is $\mathbf{R} \times S^{d-1}$

We would not like to deal with a theory defined on spheres, then we choose coordinates in which the boundary metric is flat space, it can be Minskowski or Euclidean space. The coordinates system chosen is called Poincaré coordinates.

The Poincaré coordinates are defined by:

$$
\begin{aligned}
X_0 &= \frac{z}{2}\left(1 + \frac{1}{z^2}\left(R^2 + \vec{x}^2 - t^2\right)\right) \\
X_{d-1} &= \frac{z}{2}\left(1 - \frac{1}{z^2}\left(R^2 - \vec{x}^2 + t^2\right)\right) \\
X_{d+1} &= \frac{Rt}{z} \\
X_i &= \frac{Rx_i}{z}
\end{aligned}
\tag{23}
$$

We notice that :

$$\frac{1}{z} = \frac{X_0 - X_{d-1}}{R^2} \tag{24}$$

So we can choose the sign of $z$ and that will describe only half the hyperboloid, this equation also tell us that the region $z = 0$ does not belong to AdS space. Usually people work with $z > 0$.

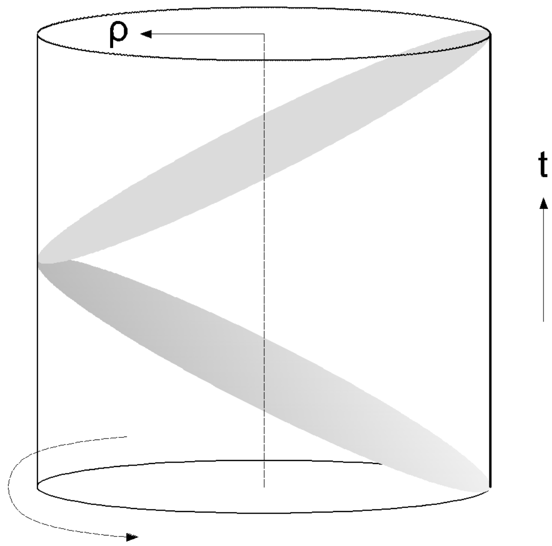


Figure 4: Diagram of the global AdS boundary and the Poincaré patch

We obtain:

$$\begin{aligned} ds^2 &= \frac{R^2}{z^2}(-dt^2 + \sum_i^{d-1} dx_i^2 + dz^2) \\ &= R^2\left(z^2(-dt^2 + \sum_i^{d-1} dx_i^2) + \frac{dz^2}{z^2}\right) \end{aligned} \tag{25}$$

Now the boundary is located at $z = 0$ and it has the topology $\mathbf{R}^{1,d}$. One of the points we would like to stress in this section is that Anti-de Sitter space in Poincaré coordinates is a patch which is a subset of the whole global AdS.

The $SO(2, p+1)$ isometry of AdS can be regarded as a clue towards the AdS/CFT correspondence, we will see that it has the same isometry as for $CFT_{p+1}$,

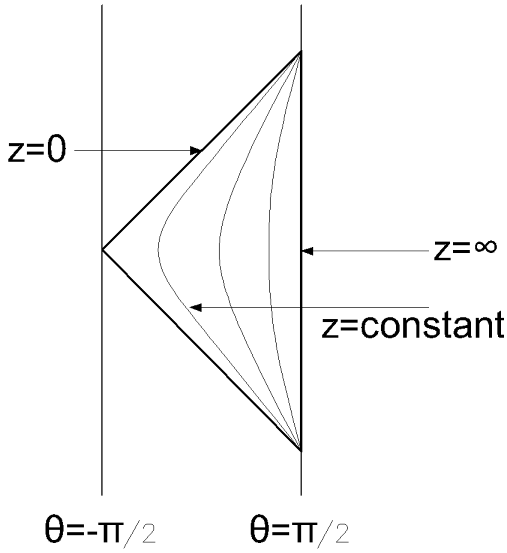


Figure 5: Penrose diagram for Poincaré AdS

in particular we are interested in the case where $p = 3$, and then $SO(2,4)$ is the isometry group of $N = 4$ $SYM$ in 4d.

## 2.6 More properties of AdS spaces

The Penrose diagram of $AdS_2$ (Figure 5) is an infinite strip between $\theta = -\pi/2$ and $\theta = \pi/2$ ($\tau$ is arbitrary) in global coordinates. If we do the same in Poincaré coordinates we realize that in these coordinates the Penrose diagram is only portion (triangle) of the strip.

Now if we analyze the case $d > 2$ we find the Penrose diagram is obtained by rotating the strip around $\theta$ where each point of the revolution circle belongs to a $(d-2)$-dimensional sphere.

# 3 Conformal Field Theory

In this chapter we develop the basics of one of the two sides of the AdS/CFT Correspondence, we will sutdy the definition of conformal transformations, its properties within group theory and the constraints of conformal invariance on field theories. It is relevant to mention that along this work we are going to check the conformally invariant structure of the correlation functions calculated in classical supergravity on AdS background, therefore the natural choice is to work in the coordinate representation.

## 3.1 Conformal group

Let us start by studying the conformal group and its representations. A general coordinate transformation $x^\mu \to x'^\mu$ is such that metric transforms as

$$g'_{\mu\nu}(x') = \frac{\partial x^\alpha}{\partial x'^\mu}\frac{\partial x^\beta}{\partial x'^\nu} g_{\alpha\beta}(x) \tag{26}$$

The conformal group of transformations is a subgroup of the global coordinate transformations such that they leave the metric invariant up to a scale change $\Omega$ that can depend on the coordinates:

$$g'_{\mu\nu}(x') = \Omega(x) g_{\mu\nu}(x) \tag{27}$$

Considering an infinitesimal transformation $x'^\mu = x^\mu + \epsilon^\mu(x)$, which together with (26) leads to:

$$\partial_\mu \epsilon_\nu + \partial_\nu \epsilon_\mu = \frac{2}{d}(\partial \cdot \epsilon)\eta_{\mu\nu} \tag{28}$$

For $d \geq 3$ we find $(d-1)\Box(\partial \cdot \epsilon) = 0$ and we can make the ansatz $\epsilon_\mu(x) = a_\mu + b_{\mu\nu}x^\nu + C_{\mu\nu\rho}x^\nu x^\rho$.

As special cases we have:

$$\begin{aligned} Translations \quad & \epsilon_\mu(x) = a_\mu \\ Rotations \quad & \epsilon_\mu(x) = \omega_\mu^\nu x_\nu \\ Dilations \quad & \epsilon_\mu(x) = \lambda x_\mu \\ Special\ Conformal\ Transformations \quad & \epsilon_\mu(x) = b_\mu x^2 - x_\mu b \cdot x \end{aligned} \tag{29}$$

The correspondent generators are given by:

$$
\begin{aligned}
P_\mu &= -i\partial_\mu \\
M_{\mu\nu} &= i(x_\mu\partial_\nu - x_\nu\partial_\mu) \\
D &= ix^\mu\partial_\mu \\
K_\mu &= -i(x^2\partial_\mu - 2x_\mu x^\nu\partial_\nu)
\end{aligned}
\tag{30}
$$

And the non vanishing commutation rules define the algebra:

$$
\begin{aligned}
[D, P_\mu] &= iP_\mu \\
[D, K_\mu] &= -iK_\mu \\
[K_\mu, P_\nu] &= 2i(\eta_{\mu\nu}D - M_{\mu\nu}) \\
[K_\rho, M_{\mu\nu}] &= i(\eta_{\rho\mu}K_\nu - \eta_{\rho\nu}K_\mu) \\
[P_\rho, M_{\mu\nu}] &= i(\eta_{\rho\mu}P_\nu - \eta_{\rho\nu}P_\mu) \\
[M_{\mu\nu}, M_{\rho\sigma}] &= i(\eta_{\nu\rho}M_{\mu\sigma} - \eta_{\nu\sigma}M_{\mu\rho} + \eta_{\mu\sigma}M_{\nu\rho} - \eta_{\mu\rho}M_{\nu\sigma})
\end{aligned}
\tag{31}
$$

These conform the algebra of the conformal group, we are specially interested in $[D, M_{\mu\nu}] = 0$ which means $D$ is an scalar and also $[D, P_\mu] = iP_\mu$, $[D, K_\mu] = -iK_\mu$ which will help us to construct representations of the conformal group.

We must diagonalize the dilation operator and label states with the eigenvalue $\Delta$, remember these operator represents a scale transformation so we call $\Delta$ scaling dimension.

The commutation relations show that $P_\mu$ raises the dimension of the field and $K_\mu$ lowers it. Unitary representations are constructed with the assumption that there is a state with the lowest scaling dimension $\Delta$, such state is called conformal primary.

$$
\begin{aligned}
[P_\mu, \Psi(x)] &= i\partial_\mu\Psi(x) \\
[M_{\mu\nu}, \Psi(x)] &= [i(x_\mu\partial_\nu - x_\nu\partial_\mu) + \Sigma_{\mu\nu}]\Psi(x) \\
[D, \Psi(x)] &= i(-\Delta + x^\mu\partial_\mu)\Psi(x) \\
[K_\mu, \Psi(x)] &= [i(x^2\partial_\mu - 2x_\mu x^\nu\partial_\nu + 2x_\mu\Delta) - 2x^\nu\Sigma_{\mu\nu}]\Psi(x)
\end{aligned}
\tag{32}
$$

From previous analysis of commutation relations we notice that given $\Psi(x)$ with conformal dimension $\Delta$, this state is not an eigenfunction of the Hamiltonian $H = P_0$ neither has a mass $M$ associated. Then if we want to label states belonging to some representation of the conformal group, we might use the scaling dimension $\Delta$, the Lorentz numbers associated with $M_{\mu\nu}$ and the additional internal quantum numbers if there are any extra internal symmetries.

## 3.2 Conformal Invariance

A field theory with conformal invariance must fulfill several requirements, among them: $\phi_j$ a set of fields called "quasi-primary" that under global conformal transformations, considering spinless field for simplicity:

$$\phi_j(x) \to \left|\frac{\partial x'}{\partial x}\right|^{\Delta_j/d} \phi_j(x') \tag{33}$$

where $\Delta_j$ represents the conformal(scaling) dimension of the field $\phi_j$. Then correlation functions must transform as:

$$\langle\phi_1(x_1)\phi_2(x_2)\ldots\phi_n(x_n)\rangle = \left|\frac{\partial x'}{\partial x}\right|_{x=x_1}^{\Delta_1/d} \left|\frac{\partial x'}{\partial x}\right|_{x=x_2}^{\Delta_2/d} \cdots \left|\frac{\partial x'}{\partial x}\right|_{x=x_n}^{\Delta_n/d} \langle\phi_1(x_1')\phi_2(x_2')\ldots\phi_n(x_n')\rangle \tag{34}$$

Because of the group invariance we find that two quasi-primary fields are correlated if they have the same scaling dimension $\Delta$.

As a result of conformal invariance, the functional forms of $n$-point functions of quasi-primary fields are severely restricted. For simplicity, let us first analyze the case of spinless objects, namely conformal invariant correlators for scalar fields: Translation invariance dictates that $n$-point functions only depend on the distances $x_i - x_j$, rotational invariance tell us that they can only depend on $r_{ij} = |x_i - x_j|$, scale invariance says that correlation can only depend on the ratios $r_{ij}/r_{lm}$ and special conformal transformations impose the dependence on the so called cross-ratios $r_{ij}r_{lm}/r_{il}r_{jm}$. [14]

For the case of the 2-point function we get $\langle\phi_1(x_1)\phi_2(x_2)\rangle = C_{12}r_{12}^a$, with $C_{12}$ a constant and $a$ a power to be determined, and conformal invariance requires:

$$\langle\phi_1(x_1)\phi_2(x_2)\rangle = \left|\frac{\partial x'}{\partial x}\right|^{\Delta_1/d} \left|\frac{\partial x'}{\partial x}\right|^{\Delta_2/d} \langle\phi_1(x_1')\phi_2(x_2')\rangle \tag{35}$$

If the conformal transformation is a scale transformation $\Omega = \lambda^{-2}$ then $\left|\frac{\partial x'}{\partial x}\right| = \Omega^{-d/2} = \lambda^d$ we obtain:

$$\langle\phi_1(x_1)\phi_2(x_2)\rangle = \lambda^{\Delta_1+\Delta_2}C_{12}\ r_{12}'^{a} = \lambda^{\Delta_1+\Delta_2}\lambda^a C_{12}\ r_{12}^a \tag{36}$$

So we must have $-a = \Delta_1 + \Delta_2$ and so:

$$\langle\phi_1(x_1)\phi_2(x_2)\rangle = \frac{C_{12}}{r_{12}^{\Delta_1+\Delta_2}} \tag{37}$$

We can restrict even more our correlator by using invariance under special conformal transformations. We have that:

$$(r'_{12})^2 = \frac{(r_{12})^2}{(1+2b\cdot x_1+b^2x_1^2)(1+2b\cdot x_2+b^2x_2^2)} \tag{38}$$

and also:

$$\frac{C_{12}}{r_{12}^{\Delta_1+\Delta_2}} = \frac{1}{(1+2b\cdot x_1+b^2x_1^2)^{\Delta_1}}\frac{1}{(1+2b\cdot x_2+b^2x_2^2)^{\Delta_2}}\frac{C_{12}}{r_{12}^{\prime\,\Delta_1+\Delta_2}} \tag{39}$$

Combining the two we get:

$$1 = (1+2b\cdot x_1+b^2x_1^2)^{\frac{\Delta_1+\Delta_2-2\Delta_1}{2}}(1+2b\cdot x_2+b^2x_2^2)^{\frac{\Delta_1+\Delta_2-2\Delta_2}{2}} \tag{40}$$

Therefore, if $C_{12} \neq 0$ then $\Delta_1 = \Delta_2 = \Delta$ and so we obtain:

$$\langle \phi_1(x_1)\phi_2(x_2)\rangle = \frac{C_{12}}{(r_{12})^{2\Delta}} \tag{41}$$

and the constant can be determined by the normalization condition of the fields.

The 3-point function is similarly restricted by conformal invariance, in general we have

$$\langle \phi_1(x_1)\phi_2(x_2)\phi_3(x_3)\rangle = \sum_{a,b,c}\frac{C_{abc}}{(r_{12})^a(r_{23})^b(r_{13})^c} \tag{42}$$

In this case scale transformations invariance says $a+b+c = \Delta_1+\Delta_2+\Delta_3$. Special conformal transformations gives $a = \Delta_1+\Delta_2-\Delta_3$, $b = \Delta_2+\Delta_3-\Delta_1$ and $c = \Delta_1+\Delta_3+\Delta_2$. So the general form is:

$$\langle \phi_1(x_1)\phi_2(x_2)\phi_3(x_3)\rangle = \frac{C_{123}}{(r_{12})^{\Delta_1+\Delta_2-\Delta_3}(r_{23})^{\Delta_2+\Delta_3-\Delta_1}(r_{13})^{\Delta_1+\Delta_3+\Delta_2}} \tag{43}$$

The functional forms are more complicated por $n \geq 4$ because the correlators begin to depend on the cross-ratios invariants defined before. In general for the 4-point function we obtain:

$$\langle \phi_1(x_1)\phi_2(x_2)\phi_3(x_3)\phi_4(x_4)\rangle = F\left(\frac{r_{12}r_{34}}{r_{13}r_{24}},\frac{r_{12}r_{34}}{r_{23}r_{14}}\right)\prod_{i<j} r_{ij}^{-(\Delta_i+\Delta_j)+\Delta/3} \tag{44}$$

where $\Delta = \Delta_1+\Delta_2+\Delta_3+\Delta_4$.

It is important to remark that these results obtained in euclideanized space can be analitycally continued to minkowski signature, therefore these functional dependence of the correlators are valid in both Minkowski and Euclidean space.

Later we will see that it is important to notice that conformal transformations can be generated only by combinations of translations, rotations and inversions. Despite the fact that inversions are not elements connected to the identity we can generate special conformal transformations by combining inversions and translations. Until now we have shown only the infinitesimal form of the conformal transformations because we were interested in its group and algebra properties, but all of them have its finite form. In particular the finite form of a special conformal transformation can be written as:

$$x'^{\mu} = \frac{x^{\mu} - b^{\mu}x^2}{1 - 2b \cdot x - b^2x^2} \tag{45}$$

now this transformation can be rewritten as:

$$\frac{x'^{\mu}}{x'^2} = \frac{x^{\mu}}{x^2} - b^{\mu} \tag{46}$$

And in this form we see that SCT can be regarded as a inversion followed by a translation followed by another inversion.

# 4 The AdS/CFT Correspondence

The AdS/CFT Correspondence or Maldacena's Conjecture is a relation between a gravity theory defined in Anti-de Sitter space and a conformal field theory living on a space of lower dimension than the AdS space. Despite the fact that the main issue of the work concerns the Maldacena conjecture, we do not intend to study its foundations. We will limit ourselves to describe the correspondence from the gravity side in a geometrical sense, starting with the idea that AdS is a space with a holographic boundary at spatial infinity. As we studied in the previous sections, the two theories has the same isometry group SO(2,d).

## 4.1 Physics in AdS is holographic

Among the peculiar properties of Anti-de Sitter spaces we find that it is possible to reach the boundary in finite time. Let us consider a massless geodesics $ds^2 = 0$, in global coordinates it is:

$$0 = -\cosh^2 \rho d\tau^2 + d\rho^2 \longrightarrow d\tau = \frac{d\rho}{\cosh \rho} \tag{47}$$

This implies that light rays can reach the boundary and bounce back (see Figure 6). In other words, we can send a light signal to infinity and it will come back in finite time.

This allows some kind of realization of Holography, we can know what is happening in AdS by looking at its boundary: In the AdS/CFT context it means calculating observables like $\langle \mathcal{O}(x) \rangle$, $\langle \mathcal{O}(x)\mathcal{O}(y) \rangle$, etc.

## 4.2 Example

The most known and well-studied example corresponds to the relation between $\mathcal{N} = 4$ Super Yang-Mills theory in 4 dimensions, which has a gauge group $SU(N)$ and coupling constant $g_{YM}$, and a type IIB superstring theory on the background $AdS_5 \times S_5$ with coupling constant $g_s$. At large N, string theory is weakly coupled and we can use supergravity on $AdS_5 \times S_5$ as an approximation of this theory, supergravity can be regarded as a "free" or low energy limit of the string theory.

The action of Super-Yang-Mills in 4 dimensions is given by:

$$\begin{aligned} S = -2 \int d^4x \;\; Tr(&-\frac{1}{4}F_{\mu\nu}^2 - \frac{1}{2}\bar{\psi}_i \not{D}\psi^i - \frac{1}{2}D_\mu \phi_{ij} D^\mu \phi^{ij} \\ &+ ig\bar{\psi}^i[\phi_{ij}, \psi^j] - g^2[\phi_{ij}, \phi_{kl}][\phi^{ij}, \phi^{kl}]) \end{aligned} \tag{48}$$

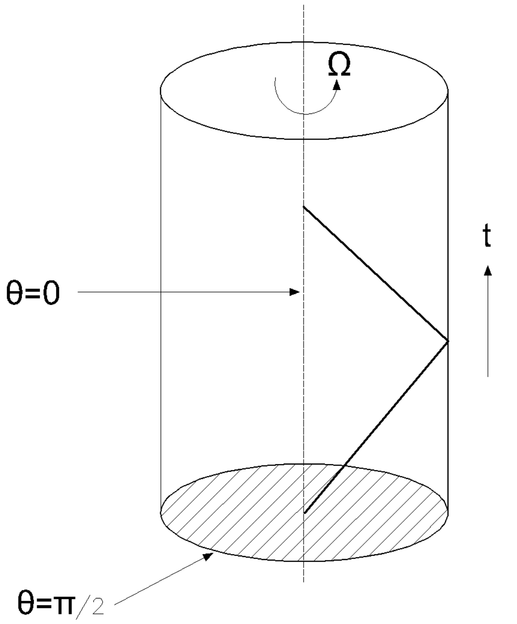


Figure 6: Massless geodesic in AdS

We present this action in order to show that this theory contains the kind of fields which propagators will be studied, we have scalar fields represented by $\phi_{ij}$, gauge fields $A^a_\mu$ and spinors $\psi_i$.

At the end, as an example we will present a simple calculation concerning a 3-form that belongs to a string theory on $AdS_7 \times S_4$ which is thought to be related via AdS/CFT to a 6 dimensional theory known as (2,0) CFT. In this example we have some kind of relation coming from $\mathbb{R}^{11} \rightarrow AdS_7 \times S_4$ via Kaluza-Klein reduction, and the gauged supergravity lagrangian has the form:

$$\mathcal{L} = \sqrt{g}\left(\frac{1}{4}R - \frac{1}{48}F_{\mu\nu\rho\sigma}F^{\mu\nu\rho\sigma}\right) + \frac{1}{72}A \wedge F \wedge F + \ fermions \qquad (49)$$

where $F = dA$.

The main technical difficulty to match observables comes from reading off the terms on the supergravity action that contributes to some particular calculation.

## 4.3 Witten's prescription

A specific and precise correspondence was proposed by Witten and later works are based on this prescription. The recipe consists of assuming that partition functions of the two theories are equivalent in the sense that observables obtained from one side must correspond to observables on the other side, GPKW formula:

$$Z_{CFT}(h) = Z_S(h) \tag{50}$$

In the gravity side, the partition function is defined by the supergravity action in terms of the field $\phi$, until now we have not stated if those fields are scalars, spinors, forms or any other kind.

$$Z_S(h) = \exp\left[-I_S(\phi)\right] \tag{51}$$

According to the prescription, the boundary value $\phi_0$ of those fields living in the bulk(AdS) must act as sources of the operators $\mathcal{O}$ of the CFT.

$$Z_{CFT}(h) = \left\langle \exp \int \phi_0 \mathcal{O} \right\rangle \tag{52}$$

This involves that correlators of the CFT corresponds to correlators of the Supergravity theory.

In general, denoting $x = (x_0, \vec{x}) = (x_0, \mathbf{b})$, where $\mathbf{b}$ represents the coordinates on the boundary, we can obtain the field $\phi$ on the bulk by propagating the source $\phi_0$ via the bulk-to-boundary propagator denoted by $G_{B\partial}(x, \mathbf{b})$:

$$\phi(x) = \int_{\partial AdS} d\mathbf{b} G_{B\partial}(x, \mathbf{b}) \phi_0(\mathbf{b}) \tag{53}$$

and so

$$\langle \mathcal{O}(\mathbf{b_1}) \ldots \mathcal{O}(\mathbf{b_n}) \rangle = \int \prod_{i=1}^{n} \left[dx_i \; G_{B\partial}(x_i, \mathbf{b_i})\right] \langle \phi(x_1) \ldots \phi(x_n) \rangle \tag{54}$$

In the Lorentzian case we have

$$\phi(x) = \int_{\partial AdS} d\mathbf{b} G_{B\partial}(x, \mathbf{b}) \phi_0(\mathbf{b}) + \phi_N(x) \tag{55}$$

The AdS/CFT dictionary states that fields $\phi^A$ in AdS correspond to local operators $\mathcal{O}^A$ in the CFT, masses $m$ of a certain scalar field correspond to scaling dimension of the operators in CFT and gauge fields $A_\mu$ are dual to currents $J_\mu$. Until now, this proposal has passed a great deal of tests. Besides all the calculations on correlation functions to be done in this work will be at tree level we know that the are all corrected by higher order terms and those corrections found in one side(Supergravity) have its own correspondent corrected quantity on the other side(CFT). Namely, corrections to the mass(renormalization) of a field of the supergravity theory will be translated as corrections to the scaling dimension of operators on the CFT.

A generalization of the AdS/CFT correspondence is called the Gauge/String duality. The well studied case states that string theory in AdS is dual to a conformal field theory living on the boundary of AdS, but people are working on finding new correspondences for theories defined on more general backgrounds.

## 4.4 Extra

So far, most of the work that has been done in AdS/CFT was developed in the Poincaré patch, but we have seen that Anti-de Sitter space is larger. Let us consider the case of $AdS_5$. If we work on global coordinates we know that this system covers all the space. The duality will tell us that gravity theory on this space will be equivalent to a CFT defined on its boundary, which in this case is the space $\mathbf{R} \times S^3$ which is conformally related to $\mathbf{R}^4$, the boundary of Poincaré AdS.

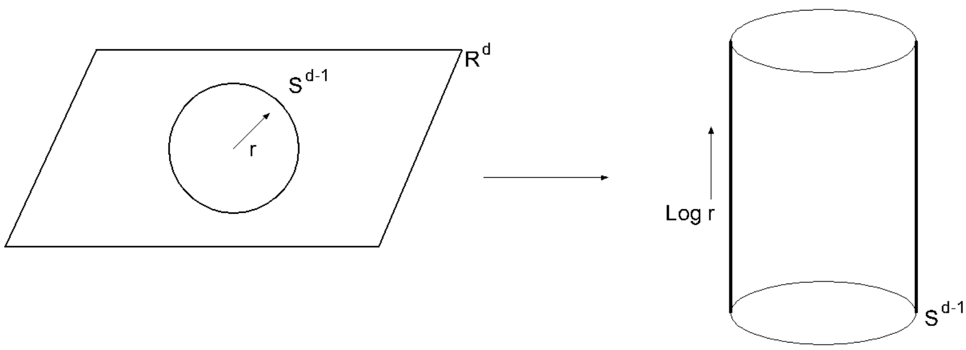


Figure 7: Diagram of the conformal mapping

In global coordinates the metric of $AdS_5$ can be written as:

$$ds^2 = \frac{R^2}{\cos^2\theta}(-d\tau^2 + d\theta^2 + \sin^2\theta d\Omega_3^2) \tag{56}$$

And as we approach the boundary $\theta \to \pi/2$ we get:

$$ds^2 = \frac{R^2}{\cos^2\theta}(-d\tau^2 + d\Omega_3^2) \tag{57}$$

On the other hand, in Poincaré coordinates $AdS$ metric:

$$ds^2 = \frac{R^2}{x_0^2}(dx_0^2 + d\vec{x}^2) \tag{58}$$

So, we can write:

$$ds^2 = f(x)dx^2 = g(x)(dx'^2 + x'^2 d\Omega_3) = h(x)(\frac{dx'^2}{x'^2} + d\Omega_3^2) \tag{59}$$

finally,

$$ds^2 = h(x)((d\ln x')^2 + d\Omega_3^2) \sim (d\tau^2 + d\Omega_3^2) \tag{60}$$

So, as expected we get that in the euclidean case the boundary of global AdS is conformally related to Poincaré AdS. (see Figure 7)

# 5 Holography

By holography we mean the way to relate or encode whatever is happening in the bulk into information on the boundary. In this case the main tool is the bulk-to-boundary propagator. The first checks of the correspondence via $n$-point functions were originally done in Poincaré coordinates with Euclidean signature, in this chapter we will follow this approach and later we will try to understand how the duality works in other coordinates. We will see that in this case, where we are calculating tree-level graphs in supergravity, the external legs of the graphs represent the boundary value of fields $\phi_0$ rather than asymptotic states as we were used to in conventional Feynman diagrams.

The relation between mass $m$ and scaling dimension $\Delta$ for a vast kind of operators/fields has been determined in several works, the majority has been done by calculating $n$-point functions in classical supergravity at tree level. Some of the results that we will check here are the following:

- Scalars: $\Delta_\pm = \frac{1}{2}\left(d \pm \sqrt{d^2 + 4m^2}\right)$
- Vectors: $\Delta_\pm = \frac{1}{2}\left(d \pm \sqrt{(d-2)^2 + 4m^2}\right)$
- Spinors: $\Delta = \frac{1}{2}\left(d + 2|m|\right)$
- $p$-Forms: $\Delta_\pm = \frac{1}{2}\left(d \pm \sqrt{(d-2p)^2 + 4m^2}\right)$

## 5.1 Holography for Scalar Fields

The prescription to calculate bulk-to-boundary propagators was proposed by Witten and it has been applied in several cases to compute $n-$point functions for classical Supergravity in a AdS background. The results have been checked with the results coming from conformal invariance of some field theories.

The bulk-to-boundary propagator is defined as follows: If $F(x_a)$ is the asymptotic behavior dependence of some field $\phi(x)$ as $x_a \to boundary$, it means that

$$\phi(x) \overset{x_a \to boundary}{\longrightarrow} F(x_a) f(b) \tag{61}$$

where $b$ represents the coordinates on the boundary, then the bulk-to-boundary propagator $K_{B\partial}(b, x)$ is defined such that

$$\phi(x) = \int db f(b) K_{B\partial}(b, x) \tag{62}$$

Now, we are going to calculate the propagator for a massless scalar field in Euclidean Poincaré coordinates by using Witten's method, which exploits the symmetries of AdS and its boundary. The equation of motion is given by

$$\frac{1}{\sqrt{-g}}\partial_\mu\left(\sqrt{-g}g^{\mu\nu}\partial_\nu\phi(x)\right)=0 \tag{63}$$

The boundary of Euclidean Poincaré $AdS_{d+1}$ is $R^d$ plus a point at infinity, then let the boundary point $\vec{x}'$ represent $p$ at infinity, so $K(x;p)=K(x_0,\vec{x};p)$. The propagator must be invariant under AdS isometry group, and translation invariance dictates it cannot depend on $\vec{x}$:

$$\partial_0\left(\sqrt{g}\partial^0 K_\infty(x_0)\right)=0 \tag{64}$$

$$\frac{d}{dx_0}\left(x_0^{-d+1}\frac{d}{dx_0}K_\infty(x_0)\right)=0 \tag{65}$$

Letting $K_\infty(x_0)=c(x_0)^p$

$$\frac{d}{dx_0}\left(p(x_0)^{p-d}\right)=0 \tag{66}$$

then we should have $p(p-d)=0$, which non-trivial solution is given by $p=d$.

Now, the propagator must be a delta function at a point $x$ on the boundary and it should vanish on the boundary $x_0$ for any $\vec{x}$, so let us map the point $p$ to the origin $\vec{x}'=0$ with a coordinate transformation belonging to the isometry group:

$$x_i\longrightarrow\frac{x_i}{x_0^2+\sum x_j^2} \tag{67}$$

With this transformation the new propagator holds:

$$K(x_0,\vec{x};p)\longrightarrow K(x_0,\vec{x};\vec{0})=\frac{c(x_0)^d}{(x_0^2+\vec{x}^2)^d} \tag{68}$$

Then, from translation invariance on the boundary $\vec{x}\to\vec{x}-\vec{x}'$ we have:

$$K(x_0,\vec{x};\vec{x}')=\frac{c(x_0)^d}{\left(x_0^2+(\vec{x}-\vec{x}')^2\right)^d} \tag{69}$$

We can check that our propagator has the expected delta function behavior: For $x_0\to 0^+$ the propagator behaves as $K\sim\delta^d(\vec{x}-\vec{x}')$ and for $\vec{x}\neq\vec{x}'$ $K\to 0$ as $x_0\to 0$, so the equation (69) represents the bulk-to-boundary propagator for a massless scalar field living on Euclidean Poincaré AdS.

Now, let us calculate the propagator for a massive scalar field. The equation of motion is given by:

$$\left(\frac{1}{\sqrt{g}}\partial_\alpha(\sqrt{g}g^{\alpha\beta}\partial_\beta) - m^2\right)\phi(x) = 0 \tag{70}$$

We find that a massive scalar of mass $m$ in $AdS_{d+1}$ couples to an operator $\mathcal{O}$ in the boundary with $\Delta = \Delta_+$. $\Delta(\Delta - d) = m^2$

$$\Delta_\pm = \frac{d}{2} \pm \sqrt{\left(\frac{d}{2}\right)^2 + m^2} \tag{71}$$

Again we look for a propagator vanishing at $x_0 = 0$ but a delta function at $p : x_0 = \infty$, it must be independent of $\vec{x}$, so $K(x;p) = K(x_0, \vec{x}; p) = K_\infty(x_0)$ and the equation of motion gives:

$$\left(-x_0^{n+1}\frac{d}{dx_0}x_0^{-d+1}\frac{d}{dx_0} + m^2\right)K_\infty(x_0) = 0 \tag{72}$$

with the ansatz $K_\infty(x_0) = c(x_0)^{\lambda+d}$ we obtain

$$-(\lambda + d)\lambda + m^2 = 0 \tag{73}$$

Then $\lambda_\pm = \frac{1}{2}\left(-d \pm \sqrt{d^2 + 4m^2}\right)$ and the only solution which gives a vanishing propagator for $x_0$ is $\lambda_+$.

Taking into account $d + \lambda_+ = d - \lambda_- = \Delta_+$ and using the same isometry properties of the space as in the massless case we find the bulk-to-boundary propagator for the massive scalar field.

$$K(x_0, \vec{x}; \vec{x}') = \frac{c^*(x_0)^{\Delta_+}}{(x_0^2 + (\vec{x} - \vec{x}')^2)^{\Delta_+}} \tag{74}$$

To fix the constant $c^*$ we look at the asymptotic behavior of the propagator $K$ as $x_0$ goes to zero. We find that $K \to c^* x_0^{\Delta_+}$ if $\vec{x} \neq \vec{x}'$ and $K \to c^* x_0^{-\Delta_+}$ if $\vec{x} = \vec{x}'$. Then $K \to c^*\epsilon^{\Delta_-}\delta^d(\vec{x} - \vec{x}')$.

We fix $c^*$ such that: $1 = \int d^dx\epsilon^{-\Delta_-}K(\epsilon, \vec{x}; 0)$, then:

$$1 = \int d^dx \frac{c^*\epsilon^{-\Delta_-}\epsilon^{\Delta_+}}{(\epsilon^2 + \vec{x}^2)^{\Delta_+}} = \frac{\epsilon^{\Delta_+ - \Delta_- + d}}{\epsilon^{2\Delta_+}}c^* \int d^dx \frac{1}{(1 + \vec{x}^2)^{\Delta_+}} = c^* \frac{\Gamma(\Delta_+)}{\pi^{d/2}\Gamma(\Delta_+ - d/2)} \tag{75}$$

and finally:

$$K(x_0, \vec{x}; \vec{x}') = \frac{\pi^{d/2}\Gamma(\Delta_+ - d/2)}{\Gamma(\Delta_+)} \frac{(x_0)^{\Delta_+}}{(x_0^2 + (\vec{x} - \vec{x}')^2)^{\Delta_+}} \tag{76}$$

## 5.2 $n$-point functions for Scalar Fields

In the functional formulation of field theory we can calculate easily the $n$-point functions by using the generating functional or partition function. The boundary values $\phi_0(x)$ of scalar fields serve as sources for operators $O(x)$ of the conformal field theory living on the boundary.

$$Z_O[\phi_0] = \int D(fields) \exp\left(-S_{SYM} + \int d^4x O(x)\phi_0(x)\right) \tag{77}$$

Then, by functional derivatives we can compute the $n$-point functions on the CFT:

$$\langle O(x_1) \dots O(x_n) \rangle = \frac{\delta^n}{\delta\phi_0(x_1) \dots \delta\phi_0(x_n)} Z_O[\phi_0]|_{\phi_0} \tag{78}$$

According to the AdS/CFT correspondence the partition function of the CFT is equivalent to the generating functional of a supergravity theory in AdS, so we can compute correlators from the gravity side and match them with results coming from the conformal invariance on the other side of the correspondence. In the AdS side we have:

$$Z[\phi_0] = \exp\left(-S_{sugra}\left[\phi[\phi_0(x)]\right]\right) \tag{79}$$

Given a source scalar field living on the boundary $\phi_0(\vec{x})$ we can find its "propagated field" in the bulk through the bulk-to-boundary propagator found in the last section:

$$\phi(\vec{x}, x_0) = \int d^d y K_{B\partial}(\vec{x}, x_0; \vec{y})\phi_0(\vec{y}) \tag{80}$$

The idea is to find the classical solution $\phi[\phi_0]$ and replace it in $S_{sugra}$.

$$S_{sugra} = \int d^{d+1}x\sqrt{g}\left(\partial_\mu\phi(\vec{x}, x_0)\partial^\mu\phi(\vec{x}, x_0) + \dots\right) \tag{81}$$

Then we can try to calculate the two-point function by using brute force:

$$\langle O(x_1)O(x_2)\rangle = \left.\frac{-\delta^2 S_{sugra}}{\delta\phi_0(x_1)\delta\phi_0(x_2)}\right|_{\phi_0=0} \tag{82}$$

We are taking two derivatives of $\phi_0$ and then making $\phi_0 = 0$, so we can neglect the interacting terms in the *Sugra* action, the two-point function has nothing to do with dynamics. So we will obtain:

$$\langle O(x_1)O(x_2)\rangle = \frac{-\delta^2}{\delta\phi_0(x_1)\delta\phi_0(x_2)} \int d^{d+1}x\sqrt{g}\partial_\mu \left(\int d^d y K_{B\partial}(\vec{x},x_0;\vec{y})\phi_0(\vec{y})\right) \times \partial^\mu \left(\int d^d y' K_{B\partial}(\vec{x},x_0;\vec{y}')\phi_0(\vec{y}')\right) = -\int d^{d+1}x\sqrt{g}\partial_\mu K_{B\partial}(\vec{x},x_0;\vec{x}_1)\partial^\mu K_{B\partial}(\vec{x},x_0;\vec{x}_2) \tag{83}$$

That is the general approach one can use for a $n$-point function. To compute higher point functions we must keep non-linear terms and consider bulk-to-bulk propagators as well as bulk-to-boundary propagators.

Now let us calculate in a different and shorter way the 2-point function for the case of the scalar field. We take:

$$S = \frac{\eta}{2}\int d^{d+1}x\sqrt{g}\left(\partial_\mu\phi\partial^\mu\phi + m^2\phi^2\right) \tag{84}$$

Integrating by parts we get:

$$S = \frac{\eta}{2}\int d^{d+1}x\sqrt{g}\phi\left(-\nabla\phi + m^2\phi^2\right) + \frac{\eta}{2}\int d^{d+1}x\sqrt{g}\phi\partial_\mu\left(\phi\partial^\mu\phi\right) \tag{85}$$

The remaining term vanish everywhere except in the $x_0$ direction where we have the boundary. Then, we evaluate at the boundary $x_0 = \epsilon$:

$$S = \frac{\eta}{2}\int_\partial d^d x\sqrt{h}\left(\phi\vec{n}\cdot\vec{\nabla}\phi\right) \tag{86}$$

We have $\sqrt{h} = x_0^{-d}$ and $\vec{n}\cdot\vec{\nabla}\phi = x_0\frac{\partial}{\partial x_0}\phi$. Also $\lim_{\epsilon\to 0}\phi(\vec{x},x_0) = \phi_0(\vec{x})$
Then, we obtain:

$$x_0\frac{\partial}{\partial x_0}\phi = x_0\frac{\partial}{\partial x_0}\int d^4x' K_{B\partial}(\vec{x},x_0;\vec{x}')\phi_0(\vec{x}') \tag{87}$$

As $x_0$ goes to zero, we get:

$$x_0\frac{\partial}{\partial x_0}\phi = c^*\Delta \times x_0 x_0^{\Delta-1}\int d^4x' \frac{\phi_0(\vec{x}')}{(\vec{x}-\vec{x}')^{2\Delta}} \tag{88}$$

And so:

$$S = \frac{\eta}{2}\lim_{\epsilon\to 0}\int d^d x d^d x' \epsilon^{\Delta-d} c^*\Delta \times \frac{\phi_\epsilon(\vec{x})\phi_0(\vec{x}')}{(\vec{x}-\vec{x}')^{2\Delta}} \tag{89}$$

Being aware of $\phi_0(\vec{x}) = \epsilon^{\Delta-d}\phi_\epsilon(\vec{x})$ we finally obtain:

$$\langle O(x_1)O(x_2)\rangle = -\frac{c^*\eta\Delta}{2} \times \frac{1}{|\vec{x}-\vec{x}'|^{2\Delta}} \tag{90}$$

In this treatment $\phi_0$ must be a small perturbation around a CFT, otherwise there is going to be a backreaction on the geometry. This is required in order to use perturbation theory.

A general action for scalar field in $d+1$ dimension is given by:

$$I[\phi] = \int_\Omega d^{d+1}x\sqrt{-g}\left[\frac{1}{2}\left[(\nabla\phi)^2 + m^2\phi^2\right] + \sum_{n\geq 3}\frac{\lambda_n}{n!}\phi^n\right] \tag{91}$$

The equation of motion is then:

$$\left(\nabla^2 - m^2\right)\phi = \sum_{n\geq 3}\frac{\lambda_n}{(n-1)!}\phi^{n-1} \tag{92}$$

And so the covariant Green's function will satisfy:

$$\left(\nabla^2 - m^2\right)G(x,y) = \delta(x,y) = \frac{1}{\sqrt{-g}}\delta(x-y) \tag{93}$$

and the boundary condition $G(x,y)|_{x\in\partial\Omega}$.

The formal solution is:

$$\phi(x) = \int_{\partial\Omega} d^dy\sqrt{h}\eta^\mu\partial_{\mu,y}G(x,y)\phi(y) + \int_\Omega d^{d+1}y\sqrt{g}G(x,y)\sum_{n\geq 3}\frac{\lambda_n}{(n-1)!}\phi^{n-1}(y) \tag{94}$$

For the n-point functions ($n \geq 3$) at tree level we will only have contributions from the interaction term in the action:

$$I[\phi] = \sum_{n\geq 3}\frac{\lambda_n}{n!}\int_\Omega d^{d+1}x\sqrt{g}\left(\phi^{(0)}\right)^n \tag{95}$$

It turns out that at tree-level one can take the limit $\epsilon \to 0$ beforehand and so:

$$I^1[\phi] = \sum_{n\geq 3}\frac{\lambda_n}{n!}\int_\Omega d^{d+1}x\sqrt{g}\left(c^*\int d^dy\phi(\vec{y})\left(\frac{x_0}{x_0^2+|\vec{x}-\vec{y}|^2}\right)^\Delta\right)^n \tag{96}$$

where $\sqrt{g} = (x_0)^{d+1}$.

$$\begin{aligned} I^1[\phi] = \sum_{n\geq 3}\frac{c^n\lambda_n}{n!}&\int d^dy_1d^dy_2\cdots d^dy_n\phi(\vec{y}_1)\cdots\phi(\vec{y}_n) \\ &\times\int d^{d+1}x\frac{(x_0)^{n\Delta-(d+1)}}{\left[(x_0^2+|\vec{x}-\vec{y}_1|^2)\cdots(x_0^2+|\vec{x}-\vec{y}_n|^2)\right]^\Delta} \end{aligned} \tag{97}$$

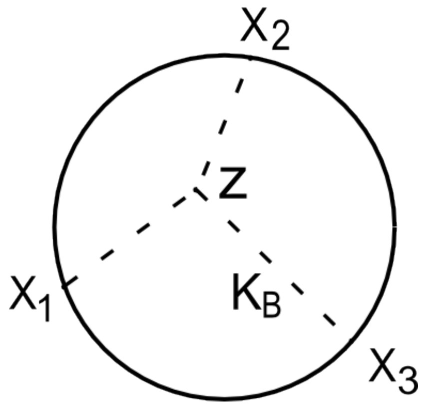


Figure 8: Witten diagram for 3-point function at tree level

Calling:

$$I_n(x_1,\cdots,x_n)=\int d^{d+1}x\frac{(x_0)^{n\Delta-(d+1)}}{[(x_0^2+|\vec{x}-\vec{y}_1|^2)\cdots(x_0^2+|\vec{x}-\vec{y}_n|^2)]^{\Delta}} \tag{98}$$

The connected part of the tree-level n-point functions is given by:

$$\langle\phi(x_1)\cdots\phi(x_n)\rangle=-\lambda_n c^n I_n(x_1,\cdots,x_n) \tag{99}$$

The task is to evaluate the integral $I_n$. We need the useful formula so-called Feynman parameterization:

$$\frac{1}{A_1^{\alpha_1}\cdots A_n^{\alpha_n}}=\frac{\Gamma(\sum\alpha_i)}{\prod_i\Gamma(\alpha_i)}\int du_1\dots du_n\frac{\delta(\sum u_k-1)u_1^{\alpha_1-1}\dots u_n^{\alpha_n-1}}{(u_1A_1+\cdots+u_nA_n)^{\sum\alpha_k}} \tag{100}$$

Then, we can rewrite $I_n$ as:

$$I_n=\frac{\Gamma(n\Delta)}{\Gamma^n(\Delta)}\int du_1\dots du_n\delta(\sum u_k-1)\prod u_i^{\Delta-1}\int d^{d+1}x\frac{(x_0)^{n\Delta-(d+1)}}{[\sum_i u_i\,(x_0^2+(\vec{x}-\vec{y}_i)^2)]^{n\Delta}} \tag{101}$$

We will use $\sum_i u_i=1$ as well as the volume of the unitary d-dimensional sphere $\Omega_d=\frac{2\pi^{d/2}}{\Gamma(d/2)}$ and:

$$\int_0^\infty dk\frac{k^{l-1}}{(k^2+v^2)^m}=(v^2)^{l/2-m}\frac{\Gamma(l/2)\Gamma(m-l/2)}{2\Gamma(m)} \tag{102}$$

Integrating the $x_0$ variable: $l=n\Delta-d$, $m=n\Delta$

$$I_n=\frac{\Gamma(n\Delta)\Gamma(\frac{n\Delta}{2}-\frac{d}{2})\Gamma(\frac{n\Delta}{2}+\frac{d}{2})}{2\Gamma(n\Delta)\Gamma^n(\Delta)}\int du_i\delta(\sum u_k-1)\prod u_i^{\Delta-1}\int d^dx\frac{1}{[\sum_i u_i(\vec{x}-\vec{y}_i)^2]^{\frac{n\Delta}{2}+\frac{d}{2}}} \tag{103}$$

Calling:

$$\begin{aligned} I_i &= \int d^d x \frac{1}{[\sum_i u_i(\vec{x}-\vec{y}_i)^2]^{\frac{n\Delta}{2}+\frac{d}{2}}} \\ &= \int d^d x \frac{1}{[x^2 - 2\sum_i u_i \vec{x}\cdot\vec{y}_i + \sum_i u_i \vec{y}_i^2]^{\frac{n\Delta}{2}+\frac{d}{2}}} \\ &= \int d^d x \frac{1}{[x^2 - 2\sum_i u_i \vec{x}\cdot\vec{y}_i + (\sum_i u_i\vec{y}_i)^2 - (\sum_i u_i\vec{y}_i)^2 + \sum_i u_i \vec{y}_i^2]^{\frac{n\Delta}{2}+\frac{d}{2}}} \end{aligned} \tag{104}$$

changing variables $\vec{x} \to \vec{x} + \sum_i u_i \vec{y}_i$:

$$\begin{aligned} I_i &= \int d^d x \frac{1}{[x^2 + \sum_i u_i \vec{y}_i^2 - (\sum_i u_i \vec{y}_i)^2]^{\frac{n\Delta}{2}+\frac{d}{2}}} \\ &= \frac{\Omega_d \Gamma(\frac{n\Delta}{2})\Gamma(\frac{d}{2})}{2\Gamma(\frac{n\Delta}{2}+\frac{d}{2})} \times \frac{1}{[\sum_i u_i \vec{y}_i^2 - (\sum_i u_i\vec{y}_i)^2]^{\frac{n\Delta}{2}}} \end{aligned} \tag{105}$$

and we can rewrite the denominator as:

$$\begin{aligned} \sum_i u_i \vec{y}_i^2 - (\sum_i u_i \vec{y}_i)^2 &= \sum_{i,j} u_i u_j \vec{y}_i^2 - \sum_{i,j} u_i u_j \vec{y}_i \cdot \vec{y}_j \\ &= \frac{1}{2}\sum_{i,j} u_i u_j (\vec{y}_i - \vec{y}_j)^2 \\ &= \sum_{i<j} u_i u_j (\vec{y}_i - \vec{y}_j)^2 \end{aligned} \tag{106}$$

So back in $I_n$ we have:

$$I_n = \frac{\pi^{d/2}\Gamma(\frac{n\Delta}{2}-\frac{d}{2})\Gamma(\frac{n\Delta}{2})}{2\Gamma^n(\Delta)} \int du_i \frac{\delta(\sum u_k - 1)\prod u_i^{\Delta-1}}{\left[\sum_{i<j} u_i u_j r_{ij}^2\right]^{\frac{n\Delta}{2}}} \tag{107}$$

We can make $u_i \to \beta_1 \beta_i$ and $\beta_1 = u_1$:

$$I_n = \frac{\pi^{d/2}\Gamma(\frac{n\Delta}{2}-\frac{d}{2})\Gamma(\frac{n\Delta}{2})}{2\Gamma^n(\Delta)} \int_0^\infty d\beta_2 \ldots d\beta_n \frac{\prod_{i=2}^n \beta_i^{\Delta-1}}{\left[\sum_{i=2}\beta_i(r_{1i}^2 + \sum_{j>1}\beta_j r_{ij}^2)\right]^{\frac{n\Delta}{2}}} \tag{108}$$

So, we have obtained the general formula to compute $n$-point functions for scalars. Now, we are going to use the previous result to calculate the 3-point function:

$$\langle\phi(x_1)\phi(x_2)\phi(x_3)\rangle = \frac{-c^3\lambda_3}{2} I_3(x_1,x_2,x_3) \tag{109}$$

Notice that without our reduced formula, the first diagram calculation will be:

$$\langle\phi(x_1)\phi(x_2)\phi(x_3)\rangle = \int d^d z \lambda_3 K_{B\partial}(x_1,z)K_{B\partial}(x_2,z)K_{B\partial}(x_3,z) \tag{110}$$

But now we have a simpler calculation starting from

$$\langle\phi(x_1)\phi(x_2)\phi(x_3)\rangle = \frac{-c^3\lambda_3}{2}\frac{\pi^{d/2}\Gamma(\frac{3\Delta}{2}-\frac{d}{2})\Gamma(\frac{3\Delta}{2})}{2\Gamma^3(\Delta)}\int_0^\infty d\beta_2 d\beta_3 \frac{\beta_2^{\Delta-1}\beta_3^{\Delta-1}}{\left[\beta_2 r_{12}^2+\beta_2\beta_3 r_{23}^2+\beta_3 r_{13}^2\right]^{\frac{3\Delta}{2}}} \tag{111}$$

To perform this integral we need:

$$\int_0^\infty \frac{x^m dx}{(a+bx)^{n+1/2}} = 2^{m+1}m!\frac{(2n-2m-3)!!}{(2n-1)!!}\frac{a^{m-n+1/2}}{b^{m+1}} \tag{112}$$

and calling:

$$\begin{aligned} I_\beta &= \int_0^\infty d\beta_2 d\beta_3 \frac{\beta_2^{\Delta-1}\beta_3^{\Delta-1}}{\left[\beta_2 r_{12}^2+\beta_2\beta_3 r_{23}^2+\beta_3 r_{13}^2)\right]^{\frac{3\Delta}{2}}} \\ &= \frac{2^\Delta\Gamma(\Delta)(\Delta-2)!!}{(3\Delta-2)!!}\int_0^\infty d\beta_3 \frac{\beta_3^{\Delta/2-1}}{r_{13}^\Delta\left(r_{12}^2+\beta_3 r_{23}^2\right)^\Delta} \end{aligned} \tag{113}$$

Then:

$$I_\beta = \frac{2^{3\Delta/2}\Gamma(\Delta)\Gamma(\Delta/2)(\Delta-2)!!(\Delta-2)!!}{(3\Delta-2)!!(2\Delta-2)!!}\frac{1}{\left(r_{13}r_{12}r_{23}\right)^\Delta} \tag{114}$$

And taking into account $(2k)!! = k!2^k$ and $c = \frac{\Gamma(\Delta)}{\pi^{d/2}\Gamma(\Delta-d/2)}$ we finally find:

$$\langle\phi(x_1)\phi(x_2)\phi(x_3)\rangle = \frac{-\lambda_3\Gamma^3(\Delta/2)}{2\pi^d\left(r_{13}r_{12}r_{23}\right)^\Delta}\frac{\Gamma(\frac{3\Delta}{2}-\frac{d}{2})\Gamma(3\Delta/2)\Gamma^3(\Delta)}{\Gamma^3(\Delta-d/2)\Gamma^3(\Delta)\Gamma(3\Delta/2)} \tag{115}$$

We can rewrite the result with $\alpha = \Delta - d/2$:

$$\langle\phi(x_1)\phi(x_2)\phi(x_3)\rangle = \frac{-\lambda_3\Gamma(\frac{\Delta}{2}+\alpha)}{2\pi^d}\left[\frac{\Gamma(\frac{\Delta}{2})}{\Gamma(\alpha)}\right]^3\frac{1}{\left(r_{13}r_{12}r_{23}\right)^\Delta} \tag{116}$$

We can compare the result with the constraints imposed by conformal invariance and that we have presented in previous sections and realized that the dynamics in this case depends on the conformal dimension of the scalar field. We can also have terms of the kind $\phi\partial\phi\partial\phi$ and we will obtain contributions with the same functional form and different coefficient.

Now, let us calculate the 4-point function at tree level, in this case we have 2 diagrams with the same order:

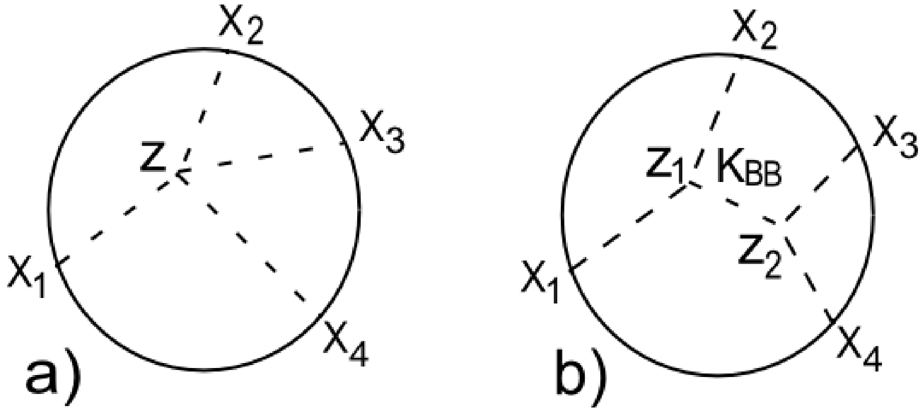


Figure 9: Witten diagrams for 4-point function at tree level

Again, if we were to calculate by brute force the first diagram (part $a$ of Figure 9):

$$\langle\phi(x_1)\phi(x_2)\phi(x_3)\phi(x_4)\rangle_{First\ diagram} = \int d^d z \lambda_4 K_{B\partial}(x_1,z)K_{B\partial}(x_2,z)K_{B\partial}(x_3,z)K_{B\partial}(x_4,z) \tag{117}$$

The other contribution corresponding to the second diagram (part $b$) will have the form:

$$\begin{aligned}\langle\phi(x_1)\phi(x_2)\phi(x_3)\phi(x_4)\rangle_{Second\ diagram} = \int d^d z_1 d^d z_2 (\lambda_3)^2 \quad & K_{B\partial}(x_1,z_1)K_{B\partial}(x_2,z_1) \\ & \times K_{BB}(z_1,z_2)K_{B\partial}(x_3,z_2)K_{B\partial}(x_4,z_2)\end{aligned} \tag{118}$$

Here we will only calculate the first contribution. So, by using the previous results we obtain:

$$\langle\phi(x_1)\phi(x_2)\phi(x_3)\phi(x_4)\rangle = \frac{-c^4\lambda_4}{2} I_4(x_1,x_2,x_3,x_4) \tag{119}$$

Explicitly,

$$\langle\phi(x_1)\phi(x_2)\phi(x_3)\phi(x_4)\rangle_{First\ diagram} = \frac{-c^4\lambda_4}{2}\frac{\pi^{d/2}\Gamma(2\Delta-\frac{d}{2})\Gamma(2\Delta)}{2\Gamma^4(\Delta)}$$
$$\times\int_0^\infty d\beta_2 d\beta_3 d\beta_4 \frac{\beta_2^{\Delta-1}\beta_3^{\Delta-1}\beta_4^{\Delta-1}}{[\beta_2 r_{12}^2+\beta_3 r_{13}^2+\beta_4 r_{14}^2+\beta_2\beta_3 r_{23}^2+\beta_2\beta_4 r_{24}^2+\beta_3\beta_4 r_{34}^2]^{2\Delta}} \tag{120}$$

First, we proceed to calculate the integral over $\beta_4$:

$$\begin{aligned} I_4 =& k\int_0^\infty d\beta_2 d\beta_3 d\beta_4 \frac{\beta_2^{\Delta-1}\beta_3^{\Delta-1}\beta_4^{\Delta-1}}{[\beta_2 r_{12}^2+\beta_3 r_{13}^2+\beta_4 r_{14}^2+\beta_2\beta_3 r_{23}^2+\beta_2\beta_4 r_{24}^2+\beta_3\beta_4 r_{34}^2]^{2\Delta}} \\ =& k\frac{\Delta!(\Delta-3/2)!}{(2\Delta-3/2)!}\int_0^\infty d\beta_2 d\beta_3 \frac{\beta_2^{\Delta-1}\beta_3^{\Delta-1}}{[(\beta_2 r_{12}^2+\beta_3 r_{13}^2+\beta_2\beta_3 r_{23}^2)(r_{14}^2+\beta_2 r_{24}^2+\beta_3 r_{34}^2)]^{\Delta}} \end{aligned} \tag{121}$$

This last integral has the form (page 314 in [22]):

$$\int_0^\infty x^{\nu-1}(\alpha+x)^{-\mu}(\gamma+x)^{-\rho}dx = \alpha^{-\mu}\gamma^{\nu-\rho}B(\nu,\mu-\nu+\rho)\,{}_2F_1(\mu,\nu;\mu+\rho;1-\frac{\gamma}{\alpha}) \tag{122}$$

Taking $\alpha = \frac{\beta_2 r_{12}^2}{(r_{13}^2+\beta_2 r_{23}^2)}$ and $\gamma = \frac{(r_{14}^2+\beta_2 r_{24}^2)}{r_{34}^2}$, $\mu=\nu=\rho=\Delta$ we can perform the integral and it holds:

$$I_4 = k\frac{\Delta!(\Delta-3/2)!}{(2\Delta-3/2)!}\frac{B(\Delta,\Delta)}{r_{12}^{2\Delta}r_{34}^{2\Delta}}\int_0^\infty \frac{d\beta_2}{\beta_2}\,{}_2F_1\left(\Delta,\Delta;2\Delta;1-\frac{(r_{14}^2+\beta_2 r_{24}^2)(r_{13}^2+\beta_2 r_{23}^2)}{\beta_2 r_{12}^2 r_{34}^2}\right) \tag{123}$$

As we want to compare the calculations with the results form conformal invariant theories, we can rewrite the last result in terms of the harmonic ratios:

Taking:

$$\eta = \frac{r_{12}r_{34}}{r_{14}r_{23}} \tag{124}$$

$$\zeta = \frac{r_{12}r_{34}}{r_{13}r_{24}} \tag{125}$$

$$\beta_2 = \frac{r_{13}r_{14}}{r_{23}r_{24}}e^{2z} \tag{126}$$

We get $\frac{d\beta_2}{\beta_2} = 2dz$ and $r_{12}r_{34} = \eta r_{14}r_{23} = \zeta r_{13}r_{24}$.

So

$$\frac{\eta\zeta}{(r_{12}r_{34})}\prod_{i<j} r_{ij} = (r_{12}r_{34})^2 \tag{127}$$

Then we can write:

$$\begin{aligned} I_4 =& k\frac{\Delta!(\Delta-3/2)!}{(2\Delta-3/2)!}\frac{2B(\Delta,\Delta)}{(\eta\zeta\prod_{i<j} r_{ij})^{2\Delta/3}} \\ & \times\int_0^\infty dz\ {}_2F_1\left(\Delta,\Delta;2\Delta;1-\frac{(\eta+\zeta)^2}{(\eta\zeta)^2}-\frac{4}{\eta\zeta}\sinh^2 z\right) \end{aligned} \tag{128}$$

We can check that this result has the functional form dictated by conformal invariance found in the chapter 3. The 4-point function only depends on the cross-invariants ratios. Beforehand we know that the second contribution coming from the exchange diagram must have the same functional structure as obtained for the first one, it means the same dependence on the ratios and can only differ in numerical factors.

## 5.3 Holography for Gauge Fields

In general, if the theory in AdS has a gauge group $G$ with gauge fields $A^a$, $a = 1, \dots, k$ with $k$ the dimension of the group, then according to the correspondence the symmetry associated is equivalent to a global symmetry of the Conformal theory on the boundary that acts on currents $J_a$. Along this subsection we will use the notation introduced by Witten in [2].

For some fixed $a$ we can call $A_0^a$ the value on the boundary of some field $A^a$ living in AdS, the conjecture states that it is source of some $J^a$ on the boundary and then according to the previous definitions, the prescription is given by:

$$Z_S(A_0^a) = \left\langle \exp(\int_{\partial AdS} J_a A_0^a) \right\rangle_{CFT} \tag{129}$$

We will consider a field $A_\mu(x)$ belonging to a free $U(1)$ gauge theory and look for a solutions of equations of motion for the bulk-to-bulk propagator in AdS space, the simpler one with a singularity at only one point in the boundary, it means a delta function source. We write the classical action for the gauge field with a AdS background in Poincaré coordinates:

$$I[A] = \int d^{d+1}x\sqrt{-g}\left(-\frac{1}{2}F_{\mu\nu}F^{\mu\nu}\right) \tag{130}$$

Then taking into account $F_{\mu\nu} = \partial_\mu A_\nu - \partial_\nu A_\mu$ we find the equation of motion for the gauge field:

$$\frac{1}{\sqrt{-g}}\partial_\mu\left(\sqrt{-g}F^{\mu\nu}\right) = 0 \tag{131}$$

In a similar way, we consider the single point $p = \infty$ at the boundary($x_0' = 0$) representing $\vec{x}'$ and so we expect a propagator independent of $\vec{x}$ because of translation invariance. Then $A(x; x') = A(x; p) = A(x_0, \vec{x}; p)$ and we look for a solution by a 1-form $A = f(x_0)dx^i$, where $i$ is a fixed component($i \geq 1$). The ansatz must satisfy the equation of motion and so: $F_{0i} = \frac{\partial f}{\partial x_0} \to F^{0i} = g^{0k}g^{il}F_{kl} = (x_0)^4\frac{\partial f}{\partial x_0}$ and considering $\sqrt{-g} = (x_0)^{-d-1}$ (we are in $AdS_{d+1}$) we obtain the equation of motion:

$$\frac{d}{dx_0}[\sqrt{g}F^{0i}] = \frac{d}{dx_0}[(x_0)^{-d+3}\frac{\partial f}{\partial x_0}] = 0 \tag{132}$$

Then the quantity inside the brackets [ ] must be independent of $x_0$ and we obtain up to a constant factor $\frac{\partial f}{\partial x_0} = (x_0)^{d-3}$ or $f(x_0) = (x_0)^{d-2}$ and we can write $A$ in the form:

$$A = \frac{d-1}{d-2}(x_0)^{d-2}dx^i \tag{133}$$

With the same arguments used in the scalar case the we must show that the singularity of the new solution at the infinity is a delta function. As we did before we map the point $p$ to the origin $\vec{x}' = 0$ with the inversion $x_i \to x_i/(x_0^2 + \vec{x}^2)$ and then the propagator takes the form:

$$A = \frac{d-1}{d-2}\left(\frac{x_0}{x_0^2+\vec{x}^2}\right)^{d-2} d\left(\frac{x^i}{x_0^2+\vec{x}^2}\right) \tag{134}$$

Despite the fact that it does not look like an usual propagator, we must remember that it is "propagating" a delta source on the boundary, simply $A(x_0, \vec{x}; \vec{0}) = \int d\vec{x}' A(x_0, \vec{x}; \vec{x}')\delta(\vec{x}')$ from $\vec{x}' = 0$ on the boundary to $(x_0, \vec{x})$ in the bulk. To simplify the calculations we can make use of the gauge invariance of the theory and subtract from the propagator the exterior derivative of

$$\frac{1}{d-2}\left(\frac{x^i x_0^{d-2}}{(x_0^2+\vec{x}^2)^{d-1}}\right) \tag{135}$$

This is possible since $\delta A = d\Lambda$ represents a gauge transformation. In this gauge we evade working with a propagator with components in all directions. The results we are to present have been obtained in another gauge choices. The new propagator is:

$$A = \frac{1}{d-2}\left[(d-1)\left(\frac{x_0}{x_0^2+\vec{x}^2}\right)^{d-2}\frac{dx^i}{x_0^2+\vec{x}^2} + (d-1)\left(\frac{x_0}{x_0^2+\vec{x}^2}\right)^{d-2} x^i d\left(\frac{1}{x_0^2+\vec{x}^2}\right) \right. \\ \left. - d\left(\frac{x^i x_0^{d-2}}{(x_0^2+\vec{x}^2)^{d-1}}\right)\right] \tag{136}$$

$$A = \frac{1}{d-2}\left[(d-1)\frac{x_0^{d-2}dx^i}{(x_0^2+\vec{x}^2)^{d-1}} + x^i x_0^{d-2} d\left(\frac{1}{(x_0^2+\vec{x}^2)^{d-1}}\right) \right. \\ \left. - d\left(\frac{x^i x_0^{d-2}}{(x_0^2+\vec{x}^2)^{d-1}}\right)\right] \tag{137}$$

We finally obtain:

$$
\begin{aligned}
A(x_0,\vec{x};\vec{0}) =& \frac{1}{d-2}\left[(d-1)\frac{x_0^{d-2}dx^i}{(x_0^2+\vec{x}^2)^{d-1}} - \frac{x_0^{d-2}dx^i}{(x_0^2+\vec{x}^2)^{d-1}} - (d-2)\frac{x_0^{d-3}x^i dx_0}{(x_0^2+\vec{x}^2)^{d-1}}\right] \\
=& \frac{x_0^{d-2}dx^i}{(x_0^2+\vec{x}^2)^{d-1}} - \frac{x_0^{d-3}x^i dx_0}{(x_0^2+\vec{x}^2)^{d-1}}
\end{aligned}
\tag{138}
$$

We wanted a solution such that on the boundary ($x_0 = 0$) we have $A_0 = \sum a_i dx^i$ and we can check this behavior in our previous result. First, we use translation invariance in the bulk $\vec{x} \to (\vec{x} - \vec{x}')$ and we can write for an arbitrary source on the boundary $a_i(\vec{x}')$:

$$
\begin{aligned}
A(x_0,\vec{x}) =& \int d^d x' A(x_0,\vec{x};\vec{x}') a_i(\vec{x}') \\
=& x_0^{d-2}\int d^d x' \left[\frac{a_i(\vec{x}')dx^i}{(x_0^2+(\vec{x}-\vec{x}')^2)^{d-1}}\right] - x_0^{d-3}dx_0 \int d^d x' \left[\frac{a_i(\vec{x}')(x-x')^i}{(x_0^2+(\vec{x}-\vec{x}')^2)^{d-1}}\right]
\end{aligned}
\tag{139}
$$

The first term is a delta function as $x_0$ goes to zero and the second vanishes.

Now we can calculate the field strength $F = dA = (dx^0\partial_0 + dx^i\partial_i)A$:

$$
\begin{aligned}
F = (d-2)x_0^{d-3}dx^0 \wedge & \int d^d x' \left[\frac{a_i(\vec{x}')dx^i}{(x_0^2+(\vec{x}-\vec{x}')^2)^{d-1}}\right] \\
-2(d-1)x_0^{d-1}dx^0 \wedge & \int d^d x' \left[\frac{a_i(\vec{x}')dx^i}{(x_0^2+(\vec{x}-\vec{x}')^2)^{d}}\right] \\
+2(d-1)x_0^{d-3}dx^j \wedge dx_0 & \int d^d x' \left[\frac{a_i(\vec{x}')(x-x')^j(x-x')^i}{(x_0^2+(\vec{x}-\vec{x}')^2)^{d}}\right] \\
-x_0^{d-3}dx^i \wedge dx^0 & \int d^d x' \left[\frac{a_i(\vec{x}')}{(x_0^2+(\vec{x}-\vec{x}')^2)^{d-1}}\right] + \dots
\end{aligned}
\tag{140}
$$

where the dots indicate terms with no $dx^0$.

In form language we can rewrite the classical action by using $F = dA$ and the equation of motion $d \star F = 0$:

$$
I[A] = \frac{1}{2}\int_{AdS_{n+1}} F \wedge \star F = \frac{1}{2}\int_{AdS_{n+1}} d(A \wedge \star F) \tag{141}
$$

and this will not vanish only in $x_0 = 0$ where we have the boundary of AdS space:

$$
I[A] = \frac{1}{2}\int_{\partial AdS} (A \wedge \star F) \tag{142}
$$

Now we see the reason we forgot about the terms with no $dx^0$ in the calculation of the field strength: We only need the components $i = 1, \ldots, d$ of $(A \wedge \star F)$ and according with the definition of the $\star$ operation: $\star T^{\mu_1 \cdots \mu_p} = \frac{1}{(D-p)!} \epsilon^{\mu_1 \mu_2 \cdots \mu_{d+1}} T_{\mu_{p+1} \cdots \mu_{d+1}}$ for a $i$ component of $A$ then $\star F$ has no 0 or that $i$ component. So we need only the $0i$ components of $F$ and our action will take the form (note 1/2 factor disappear because of $F^{0i} = -F^{i0}$):

$$I[A] = \frac{1}{(d-1)} \int d^d x \sqrt{h} (n^0 A^j F_{0j}) \tag{143}$$

where the $(d-1)$ factor appears from $(D-p) = (d+1-2)$, $n^0$ is the zero component of a unit vector normal $(g_{\mu\nu} n^\mu n^\nu = 1)$ to the boundary which we can define it as $n^\mu = (x_0, \ldots, 0)$ and the $h_{ij}$ is the "residual" metric in the boundary of the AdS space which holds:

$$h_{ij} = \frac{1}{x_0^2} \delta_{ij} \tag{144}$$

The boundary is $d-$dimensional, therefore $\sqrt{h} = x_0^{-d}$. Using $dx_0 \wedge dx_i = -dx_i \wedge dx_0$ we can rewrite the field strength:

$$\begin{aligned} F = (d-1) x_0^{d-3} dx^0 \wedge \int d^d x' \left[ \frac{a_i(\vec{x}') dx^i}{(x_0^2 + (\vec{x} - \vec{x}')^2)^{d-1}} \right] \\ -2(d-1) x_0^{d-1} dx^0 \wedge \int d^d x' \left[ \frac{a_i(\vec{x}') dx^i}{(x_0^2 + (\vec{x} - \vec{x}')^2)^{d}} \right] \\ -2(d-1) x_0^{d-3} dx_0 \wedge \int d^d x' \left[ \frac{a_i(\vec{x}')(x-x')^j (x-x')^i dx^j}{(x_0^2 + (\vec{x} - \vec{x}')^2)^{d}} \right] + \ldots \end{aligned} \tag{145}$$

Now

$$\begin{aligned} I[A] &= \frac{1}{(d-1)} \int d^d x \; x_0^{-d} [x^0 h^{ij} A_i(x_0, \vec{x}) F_{0j}(x_0, \vec{x})] \\ &= \frac{1}{(d-1)} \int d^d x \; x_0^{-d+3} [\delta^{ij} A_i(x_0, \vec{x}) F_{0j}(x_0, \vec{x})] \end{aligned} \tag{146}$$

Taking the limit $x_0 \to 0$ the leading term in $F$ will be $x_0^{d-3}$ and $A_i \to a_i(\vec{x})$:

$$\begin{aligned} I[A] = \frac{1}{(d-1)} \int d^d x \; d^d x' \delta^{ij} \Bigg( (d-1) \left[ \frac{A_i(x_0, \vec{x}) a_j(\vec{x}')}{(x_0^2 + (\vec{x} - \vec{x}')^2)^{d-1}} \right] \\ - 2(d-1) \left[ \frac{A_i(x_0, \vec{x})(x-x')_j a_k(\vec{x}')(x-x')^k}{(x_0^2 + (\vec{x} - \vec{x}')^2)^{d}} \right] \Bigg) \end{aligned} \tag{147}$$

$$I[A] = \int d^dx \; d^dx' \; a_i(\vec{x})a_j(\vec{x}') \left[ \frac{\delta^{ij}}{(\vec{x}-\vec{x}')^{2d-2}} - \frac{2(x-x')^i(x-x')^j}{(\vec{x}-\vec{x}')^{2d}} \right] \tag{148}$$

And so the 2-point function is given by:

$$\left\langle J^i(\vec{x})J^j(\vec{x}')\right\rangle = \frac{1}{(\vec{x}-\vec{x}')^{2d-2}} \left[ \delta^{ij} - \frac{2(x-x')^i(x-x')^j}{(\vec{x}-\vec{x}')^{2}} \right] \tag{149}$$

We check that this result has the form:

$$\left\langle J^i(\vec{x})J^j(\vec{x}')\right\rangle = \frac{1}{(\vec{x}-\vec{x}')^{2\Delta}} R^{ij}(\vec{x},\vec{x}') \tag{150}$$

where we can identify the scaling dimension of this gauge field $\Delta = d-1$ and we also obtain the vectorial representation matrix $R^{ij}(\vec{x},\vec{x}')$:

$$R^{ij}(\vec{x},\vec{x}') = \delta^{ij} - \frac{2(x-x')^i(x-x')^j}{(\vec{x}-\vec{x}')^{2}} \tag{151}$$

## 5.4 Holography for Spinor Fields

The case of spinor fields is a bit more involved because the free action vanishes on-shell and so we have to supplement it with surface terms on the boundary. This fact is known for fields satisfying first order equations of motion and in order to make sense of 2-point function for these fields within AdS/CFT correspondence, the inclusion of boundary terms was the proposed solution and people have worked on some other cases by using the same procedure. Besides, as we need to couple the background with the spinor field we must use the formalism of vielbein and spin connection.

We will write the action as:

$$I[\psi] = \int_{AdS} d^{d+1}x \sqrt{g}\, \bar{\psi}(\not{D} - m)\psi + k \int_{\partial AdS} d^d x \sqrt{h}\, \bar{\psi}\psi \tag{152}$$

The relative parameter $k$ will be fixed by the theory we are defining in the $AdS$ space. It can depend, for instance, on gauge invariance or supersymmetry.

As we mentioned, we have to choose a local Lorentz frame, it means $g_{\mu\nu} = e^a_\mu e^b_\nu \eta_{ab}$, and considering the metric of AdS space in Poincaré coordinates the natural choice of the vielbein is:

$$e^a_\mu = \frac{\delta^a_\mu}{x_0} \tag{153}$$

and we can define the spin connection as:

$$\omega_i^{0j} = -\omega_i^{j0} = \frac{\delta^j_i}{x_0} \tag{154}$$

Then, the slashed covariant derivative is given by:

$$\begin{aligned}
\not{D} = \Gamma^\mu \left(\partial_\mu + \frac{1}{4}\omega_\mu^{ab}\Gamma_{ab}\right) &= e^\mu_a \Gamma^a \left(\partial_\mu + \frac{1}{4}\omega_\mu^{bc}\Gamma_{bc}\right) \\
&= x_0\Gamma^0\partial_0 + x_0\vec{\Gamma}\cdot\nabla + \frac{1}{2}x_0\delta^i_a\Gamma^a\omega_i^{0j}\Gamma_{0j} \\
= x_0\Gamma^0\partial_0 + x_0\vec{\Gamma}\cdot\nabla + \frac{1}{2}\delta^i_a\delta^j_i\Gamma^a\Gamma_{0j} &= x_0\Gamma^0\partial_0 + x_0\vec{\Gamma}\cdot\nabla + \frac{1}{2}\Gamma^j\Gamma_{0j} \\
&= x_0\Gamma^0\partial_0 + x_0\vec{\Gamma}\cdot\nabla - \frac{d}{2}\Gamma^0
\end{aligned} \tag{155}$$

Where we have used $\Gamma_{ab} = \frac{1}{2}[\Gamma_a, \Gamma_b]$ and $\{\Gamma_a, \Gamma_b\} = 2\eta_{ab}$.

Now we can try to solve the equation of motion for the propagator by using the same considerations that we have used in the previous cases. Let us call $G(x_0, \vec{x}; x'_0, \vec{x}')$ the propagator and we choose the point $p = \infty$ on the boundary so it does not depend on the value of $\vec{x}'$ and $\vec{x}$ because of translation invariance in the bulk. So our ansatz should depend only on $x_0$. Then we propose $G_\infty(x_0) = f(x_0)x_0\Gamma^0$ so

$$(-x_0\partial_0 + \frac{d}{2} - m\Gamma^0)G_\infty(x_0) = 0 \tag{156}$$

gives $G_\infty(x_0) = (x_0)^{\frac{d}{2}-m\Gamma^0}\Gamma^0$.

Now we can bring the point $p = \infty$ to an arbitrary finite point on the boundary by the isometry group transformation $x_i \to x_i/(x_0^2 + \vec{x}^2)$, but this has to be accompanied by a local Lorentz transformation $e^a_\mu = \Lambda^a_b e^b_\mu$ to preserve the choice of the vielbein.

Instead of the same strategy to find the boundary-to-bulk propagator we will show another method developed on [23] based on the transformation property of the Dirac operator under inversion.

First we must find an operator $U(x)$ which will represent the inversion that will transform the vectorial index in the spinorial representation, it is:

$$U^{-1}(x)\Gamma_\mu U(x) = -J_{\mu\nu}(x)\Gamma^\nu \tag{157}$$

where $J_{\mu\nu}(x)$ is the jacobian of the operation. To see this let us consider the inversion $\hat{x}_\mu = x_\mu/x^2$, then we find

$$\frac{\partial \hat{x}_\mu}{\partial x_\nu} = \frac{1}{x^2}(\delta_{\mu\nu} - 2\frac{x_\mu x_\nu}{x^2}) = \frac{1}{x^2}J_{\mu\nu}(x) \tag{158}$$

also

$$\frac{\partial}{\partial x_\nu} = \frac{1}{x^2}J_{\mu\nu}(x)\frac{\partial}{\partial \hat{x}_\nu} \tag{159}$$

$$J_{\mu\nu}(x)J_{\nu\alpha}(x) = J_{\mu\alpha}(x) \tag{160}$$

The solution is given by

$$U(x) = \frac{\Gamma^\mu x_\mu}{\sqrt{x_0}} \tag{161}$$

Let us check that this is solution

$$
\begin{aligned}
\Gamma_\mu U(x) &= (-\Gamma_\nu\Gamma_\mu + 2\delta_{\mu\nu})\frac{x_\nu}{\sqrt{x_0}} = \frac{2x_\mu}{\sqrt{x_0}} - U(x)\Gamma_\mu \\
&= \frac{2x_\mu}{\sqrt{x_0}x^2}\delta_{\alpha\beta}x^\alpha x^\beta - U(x)\Gamma_\mu \\
&= \frac{2x_\mu}{\sqrt{x_0}x^2}\Gamma_\alpha\Gamma_\beta x^\alpha x^\beta - U(x)\Gamma_\mu = -U(x)(\Gamma_\mu - \frac{2x_\mu}{x^2}\Gamma_\alpha x^\alpha) \\
&= -U(x)J_{\mu\nu}(x)\Gamma^\nu
\end{aligned} \tag{162}
$$

Finally we get:

$$
U^{-1}(x)\Gamma_\mu U(x) = -J_{\mu\nu}(x)\Gamma^\nu \tag{163}
$$

And we will derive another property:

$$
\begin{aligned}
D_\mu U(x) &= x_0\partial_\mu(\frac{\Gamma^\nu x_\nu}{\sqrt{x_0}}) = \frac{\Gamma_\mu X_0}{\sqrt{x_0}} - \frac{U(x)\delta_{\mu 0}}{2} \\
&= \frac{\Gamma_\mu x_0}{\sqrt{x_0}} - \frac{\Gamma_\mu\Gamma_0 U(x)}{2} = \frac{\Gamma_\mu x_0}{\sqrt{x_0}} - \frac{\Gamma_\mu\Gamma_0\Gamma^0 x_0}{2\sqrt{x_0}} - \frac{\Gamma_\mu\Gamma_0\Gamma^i x_i}{2\sqrt{x_0}} \\
&= \frac{1}{2}\Gamma_\mu U(x)\Gamma_0
\end{aligned} \tag{164}
$$

From these it is straightforward to obtain the inversion of the Dirac operator:

$$
\not{D}(\hat{x}) = U^{-1}(x)\not{D}(x)U(x) + \frac{1}{2}\Gamma_0 \tag{165}
$$

Now, we must remember that we are looking for a propagator which is a solution of $(\not{D}(x) - m)\Sigma = 0$ and rewriting the last result we obtain:

$$
-U(x)[\not{D}(\hat{x}) + m - \frac{1}{2}\Gamma_0]U^{-1}(x) = \not{D}(x) - m \tag{166}
$$

Finally it is easy to see the solution for the bulk-to-boundary propagator:

$$
\Sigma(x) = U(x)\hat{x}_0^{\frac{d+1}{2}-m\Gamma_0} = (x_0\Gamma^0 + \vec{x}\cdot\vec{\Gamma})(x_0^2 + \vec{x}^2)^{m\Gamma_0 - \frac{d+1}{2}}x_0^{\frac{d}{2}-m\Gamma_0} \tag{167}
$$

Then after using translation invariance we find:

$$
\psi(x_0,\vec{x}) = \int d^dx' \left(x_0\Gamma^0 + (\vec{x}-\vec{x}')\cdot\vec{\Gamma}\right)\left(x_0^2 + (\vec{x}-\vec{x}')^2\right)^{m\Gamma^0 - \frac{d+1}{2}} x_0^{-m\Gamma^0 + d/2}\psi_0(\vec{x}') \tag{168}
$$

and similarly

$$\bar{\psi}(x_0,\vec{x}) = \int d^dx'\bar{\psi}_0(\vec{x}')x_0^{m\Gamma^0+d/2}\left(x_0^2+(\vec{x}-\vec{x}')^2\right)^{-m\Gamma^0-\frac{d+1}{2}}\left(x_0\Gamma^0+(\vec{x}-\vec{x}')\cdot\vec{\Gamma}\right) \tag{169}$$

Now we must determine and "choose" the behavior of the solutions near the boundary. By calling $\Gamma^0\psi_\pm(\vec{x}') = \pm\psi_\pm(\vec{x}')$ and $\bar{\psi}_\pm(\vec{x}')\Gamma^0 = \pm\bar{\psi}_\pm(\vec{x}')$, we decompose the solutions:

$$\lim_{x_0\to 0}(x_0)^{m-\frac{d}{2}}\psi(x_0,\vec{x}) = -k\psi_-(\vec{x}) + \int d^dx'|\vec{x}-\vec{x}'|^{2m-d-1}\vec{\Gamma}\cdot(\vec{x}-\vec{x}')\psi_+(\vec{x}') \tag{170}$$

$$\lim_{x_0\to 0}(x_0)^{m-\frac{d}{2}}\bar{\psi}(x_0,\vec{x}) = k\bar{\psi}_+(\vec{x}) + \int d^dx'\bar{\psi}_-(\vec{x}')|\vec{x}-\vec{x}'|^{2m-d-1}\vec{\Gamma}\cdot(\vec{x}-\vec{x}') \tag{171}$$

Now, we must impose boundary conditions on the solutions. Notice from the first order differential equation that we are solving, half the components can be fixed. In general, there is not a natural choice of these components, but it has been shown that $\psi_+(\vec{x}') = 0$ and $\bar{\psi}_-(\vec{x}') = 0$ seem to work .

Now, replacing the results in the surface term added the action we find:

$$\begin{aligned}
Z[\psi] = e^{-I_1} &= \exp\left(-\lim_{\epsilon\to 0}\int d^dx\sqrt{\Sigma}\bar{\psi}\psi\right)\\
&= \exp\left(-\lim_{\epsilon\to 0}\int d^dx\int d^dx'\int d^dx''\epsilon^{-d}\epsilon^{m+d/2}\left(\epsilon^2+(\vec{x}-\vec{x}'')^2\right)^{-m-\frac{d+1}{2}}\right.\\
&\left.\times(\vec{x}-\vec{x}'')\cdot\vec{\Gamma}\left(\epsilon^2+(\vec{x}-\vec{x}')^2\right)^{-m-\frac{d+1}{2}}\epsilon^{m+d/2+1}\right)\\
&= \exp\left(-\int d^dx'\int d^dx''\ \bar{\psi}_+(\vec{x}')D(\vec{x}',\vec{x}'')\psi_-(\vec{x}'')\right)
\end{aligned} \tag{172}$$

where we identify $D(\vec{x}',\vec{x})$ with the 2-point function:

$$D(\vec{x}'',\vec{x}') = \lim_{\epsilon\to 0}\int d^dx\epsilon^{2m+1}\left[\epsilon^2+|\vec{x}''-\vec{x}|^2\right]^{-m-\frac{d+1}{2}}(\vec{x}''-\vec{x}')\cdot\Gamma\left[\epsilon^2+|\vec{x}'-\vec{x}|^2\right]^{-m-\frac{d+1}{2}} \tag{173}$$

and so:

$$\begin{aligned}
D(\vec{x}'',\vec{x}') &= k\frac{(\vec{x}''-\vec{x}')\cdot\Gamma}{|\vec{x}''-\vec{x}'|^{2m+d+1}} = k\frac{(\vec{x}''-\vec{x}')\cdot\Gamma}{|\vec{x}''-\vec{x}'|}\frac{1}{|\vec{x}''-\vec{x}'|^{2(m+d/2)}}\\
&= k\frac{(\vec{x}''-\vec{x}')\cdot\Gamma}{|\vec{x}''-\vec{x}'|}\frac{1}{|\vec{x}''-\vec{x}'|^{2\Delta}}
\end{aligned} \tag{174}$$

Then the 2-point function for a primary spinor operator $\mathcal{O}^\alpha$ with scaling dimension $\Delta = m + d/2$ is given by

$$\left\langle \mathcal{O}^\beta(\vec{x}'')\mathcal{O}^\alpha(\vec{x}')\right\rangle = k\frac{(\vec{x}'' - \vec{x}') \cdot (\Gamma)^{\beta\alpha}}{|\vec{x}'' - \vec{x}'|}\frac{1}{|\vec{x}'' - \vec{x}'|^{2\Delta}} \tag{175}$$

Again, we check that this result coincides with the conformal invariance constraints.

# 6 Holography in different Coordinates

Because of the different topologies that we obtain for the boundary of AdS in different coordinates, the holography, namely the correlations functions, vacuum expectation values and $n$-point functions turn out to be really different in each case. Here we are going to study 3 cases: Euclidean AdS in Poincaré coordinates, Minkowskian Poincaré AdS and Global Lorentzian AdS. We would do so by solving the wave equations in each one of the cases.

The simplest case, i.e, where calculations and analysis are easier to perform is Euclidean Poincaré AdS. In this case, the metric is given by:

$$ds^2 = \frac{R^2}{x_0^2}(dx_0^2 + dx_i^2) \tag{176}$$

whose boundary lies at $x_0 \to 0$.

It is found [18] that the bulk to bulk propagator is:

$$G(x,y) = (x_0 y_0)^{\frac{d}{2}} \int \frac{d^d k}{(2\pi)^d} e^{i\vec{k}\cdot(\vec{x}-\vec{y})} I_\nu(k x_0^<) K_\nu(k x_0^>) \tag{177}$$

where $x_0^<(x_0^>)$ is the smaller(bigger) among $x_0$ and $y_0$.

We also have that:

$$\phi(\vec{x}, x_0) = \int d^d y K(\vec{x}, x_0; \vec{y}) \phi_0(\vec{y}) \tag{178}$$

In Lorentzian signature, quantization conditions and consistency require the inclusion of normalizable modes as solutions of the wave equation. This modes are added in the bulk and do not affect the source:

$$\phi(\vec{x}, x_0) = \phi_n(\vec{x}, x_0) + \int d^d y K(\vec{x}, x_0; \vec{y}) \phi_0(\vec{y}) \tag{179}$$

Now, in global Lorentzian AdS, the metric is:

$$ds^2 = \frac{R^2}{\cos^2\rho}(-d\tau^2 + d\rho^2 + \sin^2\rho d\Omega_{d-1}^2) \tag{180}$$

We will see later that the bulk-to-bulk propagator in these coordinates is given by:

$$K_{B\partial}(\vec{y}, x) = k \left[ \frac{\cos^2\rho}{[\cos(\tau-\tau') - \sin\rho\Omega_i\Omega_i']^2 + i\epsilon} \right]^{h_+} \tag{181}$$

As before, non-normalizable modes correspond to sources for the CFT operators and normalizable modes correspond to states on the cylinder at special frequencies.

## 6.1 Scalar fields in Lorentzian Poincaré Coordinates

Solving the equation of motion for $\phi(x_0, \vec{x}) = e^{ik\cdot x} f_k(x_0)$ near the boundary $x_0 \to 0$ and supposing $f_k(x_0) = x_0^{\Delta}$ we find

$$\Delta_{\pm} = \frac{d}{2} \pm \sqrt{\left(\frac{d}{2}\right)^2 + m^2} \tag{182}$$

we note $\Delta_+ + \Delta_- = d$.

The Berezin operator is defined as

$$\nabla^2 = \frac{1}{\sqrt{g}} \partial_\alpha (\sqrt{g} g^{\alpha\beta} \partial_\beta) \tag{183}$$

Back to the wave equation $(\nabla^2 - m^2)\phi(x_0, \vec{x}) = 0$ we have for the ansatz $\phi(x_0, \vec{x}) = e^{ik\cdot x} x_0^{d/2} \chi(x_0)$

$$(x_0^{d+1} \partial_\nu (x_0^{1-d} \partial_\nu) - x_0^2 k^2 - m^2) x_0^{d/2} \chi(x_0) = 0 \tag{184}$$

$$x_0^2\, \partial_0^2 \chi + x_0\, \partial_0 \chi - \left( (m^2 + \frac{d^2}{4}) + x_0^2 k^2 \right) \chi = 0 \tag{185}$$

Solutions for this equation are the Bessel Functions and the kind depends on the sign of $k^2 = \vec{k}^2 - \omega^2$.

Then we obtain for $k^2 = \vec{k}^2 - \omega^2 > 0$ and $\nu = \sqrt{m^2 + \frac{d^2}{4}}$ the modified Bessel functions as solutions:

$$\chi(x_0) = A_K K_\nu(kx_0) + A_I I_\nu(kx_0) \tag{186}$$

At $x_0 \to \infty$ we have the asymptotic behavior $K_\nu(kx_0) \to e^{-kx_0}$ and $I_\nu(kx_0) \to e^{kx_0}$, so in the interior of AdS $I_\nu$ is not well behaved and is eliminated to regularize the solution. It holds

$$\phi(x_0, \vec{x}) = A_K e^{ik\cdot x} x_0^{d/2} K_\nu(kx_0) \tag{187}$$

Now for $k^2 = \vec{k}^2 - \omega^2 < 0$ we get the Bessel functions of first kind as solutions:

$$\chi(x_0) = A_+ J_{\pm\nu}(|k|x_0) \tag{188}$$

If $\nu$ is integral both solutions are equivalent and we have $J_{\pm\nu}(|k|x_0) = Y_\nu(|k|x_0)$, but in general.

$$\phi^{\pm}(x_0, \vec{x}) = A_K e^{ik\cdot x} x_0^{d/2} J_{\pm\nu}(|k|x_0) \tag{189}$$

If we look at the asymptotic expansion of Bessel functions we notice that according to the value of $\nu$, $\phi^{\pm}$ can be normalizable or non-normalizable. In particular for $\nu > 1$ $\phi^{+}$ is normalizable and $\phi^{-}$ is non-normalizable, for $\nu < 1$ both $\phi^{\pm}$ are normalizable.

The new normalizable modes that appear in the Lorentzian formulation cannot be discarded because in order to define a quantum formulation we need to expand operators in terms of those modes as well, ultimately these modes form the Hilbert space of the bulk theory.

The idea is that non-normalizable modes still correspond to operators as they do for the Euclidean case and the new normalizable modes appearing as solutions are fluctuations in the bulk. The AdS/CFT correspondence states then that non-normalizable modes on AdS are dual to sources in the CFT and the normalizable modes are dual to states.

In the work [6] and [7] the authors investigated the correspondence in Lorentzian signature and they proposed that the new normalizable modes could fluctuate and they describe the Hilbert space of physical states on the bulk theory. They also stated the correspondence in the classical and quantum regimes, classical bulk field configurations correspond to expectation values of operators on the boundary and at the quantum level they proposed a correspondence relation between operator expansions of bulk and boundary fields.

## 6.2 Scalar fields in Global Coordinates

Now we are going to find solutions to the scalar wave equation in global coordinates $(\tau, \rho, \Omega_i)$ and find the bulk-to-bulk and bulk-to-boundary propagators. As we studied in the chapter 4, AdS/CFT is also defined in global coordinate system and we will see that the treatment is more involved. Of course, the CFT defined in the boundary will now live in a "sphere" but we saw that this theory will be related by a conformal transformation with the one defined in Euclidean space.

We must solve the wave equation:

$$\left(\frac{1}{\sqrt{g}}\partial_\alpha(\sqrt{g}g^{\alpha\beta}\partial_\beta) - m^2\right)\Psi = 0 \tag{190}$$

in the background:

$$ds^2 = \frac{1}{\cos^2\rho}(-d\tau^2 + d\rho^2 + \sin^2\rho d\Omega_i^2) \tag{191}$$

We make the ansatz:

$$\Psi = e^{-i\omega\tau}Y_{lm}(\Omega)\chi(\rho) \tag{192}$$

We are working in $AdS_{d+1}$ so $Y_l$ are spherical harmonics living on $S^{d-1}$ and so:

$$\nabla^2_{S^{d-1}} Y_l = -l(l+d-2)Y_l \tag{193}$$

The equation of motion is given by:

$$\left(\frac{1}{\sqrt{g}}\partial_\rho(\sqrt{g}g^{\rho\rho}\partial_\rho) + \frac{1}{\sqrt{g}}\partial_\tau(\sqrt{g}g^{\tau\tau}\partial_\tau) + \frac{1}{\sqrt{g}}\partial_i(\sqrt{g}g^{ii}\partial_i) - m^2\right)\Psi = 0 \tag{194}$$

where $\sqrt{g} = \sec^2\rho\tan^{d-1}\rho$, $g^{\rho\rho} = \sec^{-2}\rho$, $g^{\tau\tau} = -\sec^{-2}\rho$ and $g^{ii} = \tan^{-2}\rho$. and so we find:

$$\frac{1}{\tan^{d-1}\rho}\partial_\rho[\tan^{d-1}\rho\partial_\rho]\chi + [\omega^2 - l(l+d-2)\csc^2\rho - m^2\sec^2\rho]\chi = 0 \tag{195}$$

To examine regularity and boundary behavior of the solutions it is convenient to write $\chi$ in terms of $\cos\rho$ and $\sin\rho$. So we will solve the last equation for

$$\chi(\rho) = (\cos\rho)^{2h}(\sin\rho)^{2b}f(\rho) \tag{196}$$

So, calling $y = \sin^2\rho$: $\frac{d\chi}{d\rho} = \frac{d\chi}{dy}\frac{dy}{d\rho}$ and

$$\frac{dy}{d\rho} = 2\sin\rho\cos\rho = 2y^{1/2}(1-y)^{1/2} \tag{197}$$

Then with $\chi = y^b(1-y)^h f$, $(\tan\rho)^{d-1} = y^{\frac{d}{2}-\frac{1}{2}}(1-y)^{\frac{1}{2}-\frac{d}{2}}$ :

$$\begin{aligned}
&\frac{1}{\tan^{d-1}\rho}\frac{dy}{d\rho}\partial_y\left[\tan^{d-1}\rho\frac{dy}{d\rho}\partial_y\right]y^b(1-y)^h f = \frac{1}{(\tan\rho)^{d-1}}\frac{dy}{d\rho}\partial_y\left[(\tan\rho)^{d-1}\frac{dy}{d\rho}\left[by^{b-1}(1-y)^h f\right.\right.\\
&\left.\left.-hy^b(1-y)^{h-1}f + y^b(1-y)^h\partial_y f\right]\right]\\
&= \frac{1}{(\tan\rho)^{d-1}}\frac{dy}{d\rho}\partial_y\left[2y^{\frac{d}{2}}(1-y)^{1-\frac{d}{2}}\left[by^{b-1}(1-y)^h f - hy^b(1-y)^{h-1}f + y^b(1-y)^h\partial_y f\right]\right]\\
&= 4y^{1-\frac{d}{2}}(1-y)^{\frac{d}{2}}\partial_y\left[by^{b+\frac{d}{2}-1}(1-y)^{h-\frac{d}{2}+1}f - hy^{b+\frac{d}{2}}(1-y)^{h-\frac{d}{2}}f + y^{b+\frac{d}{2}}(1-y)^{h-\frac{d}{2}+1}\partial_y f\right]\\
&= 4y^{1-\frac{d}{2}}(1-y)^{\frac{d}{2}}\left[b(b+\frac{d}{2}-1)y^{b+\frac{d}{2}-2}(1-y)^{h-\frac{d}{2}+1} - b(h-\frac{d}{2}+1)y^{b+\frac{d}{2}-1}(1-y)^{h-\frac{d}{2}}f\right.\\
&+ by^{b+\frac{d}{2}-1}(1-y)^{h-\frac{d}{2}+1}\partial_y f - h(b+\frac{d}{2})y^{b+\frac{d}{2}-1}(1-y)^{h-\frac{d}{2}}f + h(h-\frac{d}{2})y^{b+\frac{d}{2}}(1-y)^{h-\frac{d}{2}-1}f\\
&- hy^{b+\frac{d}{2}}(1-y)^{h-\frac{d}{2}}\partial_y f + (b+\frac{d}{2})y^{b+\frac{d}{2}-1}(1-y)^{h-\frac{d}{2}+1}\partial_y f - (h-\frac{d}{2}+1)y^{b+\frac{d}{2}}(1-y)^{h-\frac{d}{2}}\partial_y f\\
&\left.+y^{b+\frac{d}{2}}(1-y)^{h-\frac{d}{2}+1}\partial_y^2 f\right]
\end{aligned} \tag{198}$$

Then

$$
\begin{aligned}
&\frac{1}{\tan^{d-1}\rho}\frac{dy}{d\rho}\partial_y\left[\tan^{d-1}\rho\frac{dy}{d\rho}\partial_y\right]\chi = \\
&= 4\left[b(b+\frac{d}{2}-1)y^{b-1}(1-y)^{h+1} - b(h-\frac{d}{2}+1)y^b(1-y)^h f\right. \\
&+ by^b(1-y)^{h+1}\partial_y f - h(b+\frac{d}{2})y^b(1-y)^h f + h(h-\frac{d}{2})y^{b+1}(1-y)^{h-1}f \\
&- hy^{b+1}(1-y)^h\partial_y f + (b+\frac{d}{2})y^b(1-y)^{h+1}\partial_y f - (h-\frac{d}{2}+1)y^{b+1}(1-y)^h\partial_y f \\
&+y^{b+1}(1-y)^{h+1}\partial_y^2 f] \\
&= 4\left[b(b+\frac{d}{2}-1)y^{b-1}(1-y)^{h+1} + h(h-\frac{d}{2})y^{b+1}(1-y)^{h-1}f\right. \\
&- [b(h-\frac{d}{2}+1)+h(b+\frac{d}{2})]y^b(1-y)^h f + y^{b+1}(1-y)^{h+1}\partial_y^2 f \\
&\left.+(2b+\frac{d}{2})y^b(1-y)^{h+1}\partial_y f - (2h-\frac{d}{2}+1)y^{b+1}(1-y)^h\partial_y f\right]
\end{aligned}
\tag{199}
$$

Matching powers back in the wave equation we find:

$$
y(1-y)\partial_y^2 f + \left[2b+\frac{d}{2}-(2h+2b+1)y\right]\partial_y f - \left[(h+b)^2-\frac{\omega^2}{4}\right]f = 0 \tag{200}
$$

and

$$
h\left(h-\frac{d}{2}\right) = \frac{m^2}{4} \tag{201}
$$

$$
b(b-\frac{d}{2}-1) = \frac{l(l+d-1)}{4} \tag{202}
$$

Solving these two equations:

$$
h_\pm = \frac{d \pm \sqrt{d^2+4m^2}}{4} \tag{203}
$$

$$
b = \frac{l}{2}\ ,\ \frac{1}{2}(2-d-l) \tag{204}
$$

As $h_+ + h_- = d/2$, we then notice that we have two independent solutions for each value of $b$. We could have chosen $y = \cos^2\rho$ and we would have obtained another hypergeometric equation with two independent solutions for two values of $h$.

Choosing $h = h_+$ we find that the only solution that is regular at the origin $\rho = 0$ corresponds to the choice $b = l/2$ and:

$$\Psi = e^{-i\omega\tau} Y_l(\Omega) (\cos\rho)^{2h_+} (\sin\rho)^l \;\; {}_2F_1(h_+ + \frac{l}{2} + \frac{\omega}{2}, h_+ + \frac{l}{2} - \frac{\omega}{2}; l + \frac{d}{2}; \sin^2\rho) \quad (205)$$

Now we must study the behavior of the solutions at the boundary and so it is convenient to write them in terms of $\cos^2\rho$, we find:

$$\Phi^+ = e^{-i\omega\tau} Y_l(\Omega) (\cos\rho)^{2h_+} (\sin\rho)^l \;\; {}_2F_1(h_+ + \frac{l}{2} + \frac{\omega}{2}, h_+ + \frac{l}{2} - \frac{\omega}{2}; 2h_+ + 1 - \frac{d}{2}; \cos^2\rho) \quad (206)$$

and

$$\Phi^- = e^{-i\omega\tau} Y_l(\Omega) (\cos\rho)^{2h_-} (\sin\rho)^l \;\; {}_2F_1(h_- + \frac{l}{2} + \frac{\omega}{2}, h_- + \frac{l}{2} - \frac{\omega}{2}; 2h_- + 1 - \frac{d}{2}; \cos^2\rho) \quad (207)$$

Notice that we chose $h = h_+$ in $\Psi$ but in general it is a combination of the two solutions $h = h_\pm$ and so the general solution will have the form:

$$\Psi = C^+\Phi^+ + C^-\Phi^- \quad (208)$$

Defining $\nu = \frac{1}{2}\sqrt{d^2 + 4m^2}$, we can write $C^\pm$ as:

$$C^+ = \frac{\Gamma(l + \frac{d}{2})\Gamma(-\nu)}{\Gamma(h_+ + \frac{l}{2} + \frac{\omega}{2})\Gamma(h_- + \frac{l}{2} - \frac{\omega}{2})} \quad (209)$$

$$C^- = \frac{\Gamma(l + \frac{d}{2})\Gamma(\nu)}{\Gamma(h_+ + \frac{l}{2} + \frac{\omega}{2})\Gamma(h_+ + \frac{l}{2} - \frac{\omega}{2})} \quad (210)$$

The coefficient $C^-$ must vanish because of regularity in the boundary, so we must have a zero or negative argument in one of the gamma functions. This gives the quantization condition:

$$\omega_n = \pm(2h_\pm + l + 2n) \quad (211)$$

With these conditions we can rewrite the solutions in terms of Jacobi polynomials:

$$\Phi^\pm = e^{-i\omega\tau} Y_l(\Omega) (\cos\rho)^{2h_\pm} (\sin\rho)^l \;\; P_n^{l+d/2-1,\nu}(\cos 2\rho) \quad (212)$$

The point we want to stress here is that depending on the value of $\nu$ we have both normalizable and non-normalizable solutions of the equation of motion, in that sense it is similar to the Lorentzian case.

For completeness, we are going to calculate the bulk-to-boundary propagator for the scalar field in these coordinates. The bulk-to-bulk propagator was found by [24] :

$$iK_B(x,x') = C_b(\cosh^2\frac{s}{R})^{-h_+}\ {}_2F\left(h_+, h_+ + \frac{1}{2}; \nu+1; \frac{1}{\cosh^2 s/R}\right) \tag{213}$$

where $\cosh(\frac{s}{R}) = 1 + \frac{\sigma}{R^2}$, being $\sigma$ the chordal(geodetic) distance in AdS.

$$\sigma = -R^2 + \frac{R^2}{\cos\rho\cos\rho'}\left[\cos(\tau-\tau') - \sin\rho\sin\rho'\ \Omega\cdot\Omega'\right] \tag{214}$$

and so:

$$\cosh(\frac{s}{R}) = \left[\frac{\cos(\tau-\tau') - \sin\rho\sin\rho'\ \Omega\cdot\Omega'}{\cos\rho\cos\rho'}\right] \tag{215}$$

To construct the bulk-to-boundary propagator $K_{B\partial}$ we follow the method proposed by Witten. We notice the asymptotic behavior of the non-normalizable solutions as we approach the boundary, it is as $\rho\to\pi/2$:

$$\phi(x) \overset{\rho\to\pi/2}{\longrightarrow} (\cos\rho)^{2h_-} f(b) \tag{216}$$

where we have denoted $b$ the coordinates on the boundary. The bulk-to-bulk propagator is a solution of the equation $(\Box - m^2)G_B(x,x') = -\delta(x,x')$, so we can write:

$$\begin{aligned}\phi'(x') &= -\int_V dV\phi(x)(\Box - m^2)G_B(x,x') = -\int_V dV\left[\partial^\mu(\phi\partial_\mu G_B) - \partial^\mu\phi\partial_\mu G_B - m^2\phi G_B\right]\\ &= -\int_V dV\left[\partial^\mu(\phi\partial_\mu G_B) - \partial^\mu(G_B\partial_\mu\phi) + G_B(\Box - m^2)\phi\right]\\ &= -\int_V dV\partial^\mu[\phi(x)\overleftrightarrow{\partial_\mu}G_B(x,x')]\end{aligned} \tag{217}$$

This integral vanishes everywhere but in the boundary $\rho\to\pi/2$:

$$\phi'(x') = -\int_{\partial V} d\Sigma^\mu\ \phi(x)\overleftrightarrow{\partial_\mu}G_B(x,x') \tag{218}$$

And we also have

$$G_B(x,x') \overset{\rho\to\pi/2}{\longrightarrow} (\cos\rho)^{2h_+} G(b,x') \tag{219}$$

We must compute $d\Sigma^\mu$:

$n^\rho$ is the $\rho$ component of a unit vector normal ($g_{\mu\nu}n^\mu n^\nu = 1$) to the boundary which we can define it as $n^\mu = (0, \cos\rho, \ldots, 0)$ and the $h_{ij}$ is the "residual" metric in the boundary of the AdS space which holds:

$$ds^2 = \frac{1}{\cos^2\rho}(-d\tau^2 + \sin^2\rho d\Omega_i^2) \tag{220}$$

The boundary is $d-$dimensional, therefore $\sqrt{h} = \sec\rho\tan^{d-1}\rho$.

$$\begin{aligned}\phi'(x') &= -\int_{\partial V} d^d x\sqrt{h}n^\rho\ \phi(x)\ \overleftrightarrow{\partial_\rho}\ G_B(x,x') \\ &= -\int db\ f(b)\ G(b,x')\ \lim_{\rho\to\pi/2} \tan^{d-1}\rho\left[(\cos\rho)^{2h_-}\ \overleftrightarrow{\partial_\rho}\ (\cos\rho)^{2h_+}\right]\end{aligned} \tag{221}$$

Taking into account $2h_\pm = \frac{d}{2} \pm \nu$, we find:

$$\phi'(x') = 2\nu\int db\ f(b)\ G(b,x') \tag{222}$$

Finally, according to the prescription given by Witten to construct the bulk-to-boundary propagator:

$$K_{B\partial}(b,x') = 2\nu \lim_{\rho\to\pi/2}(\cos\rho)^{-2h_+} K_B(x,x') \tag{223}$$

In this limit the hypergeometric funtion in the bulk-to-bulk propagator goes to 1 and we obtain:

$$K_{B\partial}(b,x') = C_{B\partial}\left[\frac{\cos\rho'}{\cos(\tau-\tau') - \sin\rho'\ \Omega\cdot\Omega'}\right]^{2h_+} \tag{224}$$

which is the bulk-to-boundary propagator for the scalar field in global coordinates.

# 7 Propagators for $p$-forms in $AdS_{2p+1}$: An Example

Until now we have done calculations with the simpler fields to work with, where the methods first proposed by Witten were useful and effective. Now, the idea is to find the 2-point function for a $p$-form. We know that it is not straightforward to solve the wave equation for fields of higher rank other than scalars, gauge fields and spinors, so in this case it is necessary to implement new techniques and so we are going to present a different approach to solve the problem.

Allen and Jacobson developed a method to calculate $n$-point functions, it consists of constructing every possibly symmetric bitensors in terms of the geodesic or chordal distance, they are called bitensors because they are given in terms of two points $x$ and $x'$ that are connected by the chordal distance. First, we are going to work in Poincaré coordinates, reproducing the results found by Bena, Nastase and Vaman and later we present some results obtained by the author working in global coordinates.

As before, the first step is to calculate the bulk-to-boundary propagator for this field living in Anti-de Sitter background in Poincaré coordinates. Just to remember, we have that the metric of Euclidean Poincaré Anti-de Sitter space is given by:

$$ds^2 = \frac{1}{z_0^2}(dz_0^2 + d\vec{z}^2) \tag{225}$$

As we already mentioned, the procedure is based on the fact that invariant functions and tensors are easily expressed in terms of the chordal distance $u$, which in this case for Euclidean Poincaré is given by:

$$\mathcal{U} = \frac{1}{2}(x - x')^2 = -\frac{1}{2}(X_0 - X_0')^2 - \frac{1}{2}(X_{d+1} - X_{d+1}')^2 + \frac{1}{2}(X_i - X_i')^2 \tag{226}$$

where $X$ are the coordinates in the embedding space.

$$u = \frac{(z_0 - w_0)^2 + (z_i - w_i)^2}{2z_0 w_0} \tag{227}$$

To prove this we must use the relations

$$\begin{aligned}
X_0 &= \frac{u}{2}\left(1+\frac{1}{u^2}\left(R^2+\vec{x}^2-t^2\right)\right)\\
X_{d-1} &= \frac{u}{2}\left(1-\frac{1}{u^2}\left(R^2-\vec{x}^2+t^2\right)\right)\\
X_{d+1} &= \frac{Rt}{u}\\
X_i &= \frac{Rx_i}{u}
\end{aligned} \tag{228}$$

And so

$$\begin{aligned}
u = &-\frac{1}{8}\left[\left(u+\frac{1}{u}\left(R^2+\vec{x}^2-t^2\right)\right)-\left(u'+\frac{1}{u'}\left(R^2+\vec{x}'^2-t'^2\right)\right)\right]^2\\
&+\frac{1}{8}\left[\left(u-\frac{1}{u}\left(R^2-\vec{x}^2+t^2\right)\right)-\left(u'-\frac{1}{u'}\left(R^2-\vec{x}'^2+t'^2\right)\right)\right]^2\\
&-\frac{1}{2}\left[\frac{Rt}{u}-\frac{Rt'}{u'}\right]^2+\frac{1}{2}\left[\frac{Rx_i}{u}-\frac{Rx_i'}{u'}\right]^2\\
&=-\frac{1}{8}\left[2\left(u-\frac{1}{u}\left(R^2-\vec{x}^2+t^2\right)\right)\left(u'-\frac{1}{u'}\left(R^2-\vec{x}'^2+t'^2\right)\right)\right.\\
&\qquad -2\left(u+\frac{1}{u}\left(R^2+\vec{x}^2-t^2\right)\right)\left(u'+\frac{1}{u'}\left(R^2+\vec{x}'^2-t'^2\right)\right)\\
&+\left.\left(u+\frac{1}{u}\left(R^2+\vec{x}^2-t^2\right)\right)^2-\left(u-\frac{1}{u}\left(R^2-\vec{x}^2+t^2\right)\right)^2+(unprimed\leftrightarrow primed)\right]\\
&-\frac{1}{2}\left[\frac{Rt}{u}-\frac{Rt'}{u'}\right]^2+\frac{1}{2}\left[\frac{Rx_i}{u}-\frac{Rx_i'}{u'}\right]^2
\end{aligned} \tag{229}$$

Then,

$$\begin{aligned}
u = &-\frac{1}{8}\left[\frac{2}{uu'}\left(R^2+\vec{x}^2-t^2\right)\left(R^2+\vec{x}'^2-t'^2\right)-\frac{2}{uu'}\left(R^2-\vec{x}^2+t^2\right)\left(R^2-\vec{x}'^2+t'^2\right)\right.\\
&\left.+8R^2+4\frac{R^2}{u^2}(\vec{x}^2-t^2)+4\frac{R^2}{u'^2}(\vec{x}'^2-t'^2)\right]\\
&-\frac{1}{2}\left[\frac{Rt}{u}-\frac{Rt'}{u'}\right]^2+\frac{1}{2}\left[\frac{Rx_i}{u}-\frac{Rx_i'}{u'}\right]^2+\frac{R^2}{2}\left(\frac{u}{u'}+\frac{u'}{u}\right)
\end{aligned} \tag{230}$$

Organizing terms,

$$
\begin{aligned}
u = &-\frac{1}{8}\left[\frac{-4}{uu'}R^2\left(x^2 - t^2 + x'^2 - t'^2\right) + 8R^2 + 4\frac{R^2}{u^2}(\vec{x}^2 - t^2) + 4\frac{R^2}{u'^2}(\vec{x}'^2 - t'^2)\right] \\
&- \frac{1}{2}\left[\frac{Rt}{u} - \frac{Rt'}{u'}\right]^2 + \frac{1}{2}\left[\frac{Rx_i}{u} - \frac{Rx'_i}{u'}\right]^2 + \frac{R^2}{2}\left(\frac{u}{u'} + \frac{u'}{u}\right) \\
&= \frac{R^2}{2}\left[\frac{u}{u'} + \frac{u'}{u} - 2 + \left(\frac{x_i}{u} - \frac{x'_i}{u'}\right)^2 - \left(\frac{t}{u} - \frac{t'}{u'}\right)^2 - \frac{1}{u^2}(\vec{x}^2 - t^2) - \frac{1}{u'^2}(\vec{x}'^2 - t'^2)\right. \\
&\left. + \frac{1}{uu'}(x^2 - t^2) + \frac{1}{uu'}(x'^2 - t'^2)\right]
\end{aligned} \tag{231}
$$

Finally, we find:

$$
u = R^2\left[\frac{(u-u')^2 + (\vec{x} - \vec{x}')^2 - (t-t')^2}{2uu'}\right] \tag{232}
$$

Then by going to a euclideanized space $t \to it$ and taking $R = 1$ by simplicity we prove that the chordal distance in Euclidean Poincaré AdS is given by:

$$
u = \frac{(z_0 - w_0)^2 + (z_i - w_i)^2}{2z_0 w_0} \tag{233}
$$

As we said, the idea is to write an ansatz for the propapator in terms of the chordal distance and the basis bitensors, then plug it in the equation of motion and solve it to find teh bulk-to-bulk propagator.

It is found that for $d = 2p$ p-forms can satisfy two types of equations, the one that will be used here is known as self duality in odd dimensions:

$$
\frac{m}{p!}\epsilon_{\mu_1\mu_2\ldots\mu_p}{}^{\mu_{p+1}\ldots\mu_{d+1}}\partial_{\mu_{p+1}}A_{\mu_{p+2}\ldots\mu_{d+1}} = m^2 A_{\mu_1\mu_2\ldots\mu_p} - J_{\mu_1\mu_2\ldots\mu_p} \tag{234}
$$

In euclidean signature, the equation satisfied by the propagator is:

$$
\begin{aligned}
im\epsilon_{\mu_1\mu_2\ldots\mu_p}{}^{\mu_{p+1}\ldots\mu_{d+1}}D_{\mu_{p+1}}G_{\mu_{p+2}\ldots\mu_{d+1};\mu'_1\mu'_2\ldots\mu'_p} &= m^2 G_{\mu_1\mu_2\ldots\mu_p;\mu'_1\mu'_2\ldots\mu'_p} \\
&- \delta(w,z)(g_{[\mu_1\mu'_1}g_{\mu_2\mu'_2}\ldots g_{\mu_p]\mu'_p})
\end{aligned} \tag{235}
$$

The $i$ factor appears as a result of the analytical continuation to Euclidean space where the $\epsilon$ symbol stays the same. The propagator is a maximally symmetric bitensor, then according to the method described above, it can be written in terms of a basis of antisymmetric bitensors:

$$T^1_{\mu_1\mu_2\ldots\mu_p;\mu'_1\mu'_2\ldots\mu'_p} = \partial_{\mu_1}\partial_{\mu'_1}u\ldots\partial_{\mu_p}\partial_{\mu'_p}u + antipermutations\ of\ primed\ indices \tag{236}$$

$$T^2_{\mu_1\mu_2\ldots\mu_p;\mu'_1\mu'_2\ldots\mu'_p} = \frac{1}{(p-1)!}(\partial_{\mu_1}u\,\partial_{\mu'_1}u\,\partial_{\mu_2}\partial_{\mu'_2}u\ldots\partial_{\mu_p}\partial_{\mu'_p}u + antipermutations\ of\ all\ indices) \tag{237}$$

$$T^3_{\mu_1\mu_2\ldots\mu_p;\mu'_1\mu'_2\ldots\mu'_p} = \epsilon_{\mu_1\mu_2\ldots\mu_p}{}^{\mu_{p+1}\ldots\mu_{d+1}}\partial_{\mu_{p+1}}\partial_{\mu'_1}u\ldots\partial_{\mu_d}\partial_{\mu'_p}u\;\partial_{\mu_{d+1}}u \tag{238}$$

The propagator can be expanded as:

$$G(u) = T^1[g(u) + ph(u)] + T^2h'(u) + T^3k(u) \tag{239}$$

where we have suppressed the indices. In form language we can write the equation of motion as $*dG = -imG$ (we are interested in correlators between different points), where the operations are defined as:

$$(*B)^{\mu_1\ldots\mu_p} = \frac{1}{(d-p)!}\epsilon^{\mu_1\ldots\mu_p\mu_{p+1}\ldots\mu_{d+1}}B_{\mu_{p+1}\ldots\mu_{d+1}} \tag{240}$$

and

$$(dB)_{\mu_1\ldots\mu_p} = \frac{1}{(p-1)!}D_{[\mu_1}B_{\mu_2\ldots\mu_p]} \tag{241}$$

Then we obtain

$$\begin{aligned}
&*\,dT^1g = (-1)^pT^3g' \\
&d(pT^1h + T^2h') = 0 \\
&*\,d(T^3k) = T^1[u(2+u)k' + (d-p+1)(1+u)k] - T^2[(1+u)k' + (d-p+1)k]
\end{aligned} \tag{242}$$

The second relation is the reason why the ansatz was written in such way.

By plugging the ansatz into the equation of motion and using the fact that bitensors are independent, it is found:

$$\begin{aligned}
&(-1)^pg' = (-im)k \\
&u(2+u)k' + (d-p+1)(1+u)k = (-im)(g+ph) \\
&-k'(1+u) - (d-p+1)k = (-im)h'
\end{aligned} \tag{243}$$

from these we get

$$(-1)^p m^2 h = g'(1+u) + (d-p)g \tag{244}$$

By combining again we obtain

$$u(u+2)g'' + (d+1)(1+u)g' + [p(d-p) + (-1)^p m^2]g = 0 \tag{245}$$

the solution is a hypergeometric function:

$$g = \frac{C(-1)^p \Gamma(\frac{d-1}{2})}{4\pi^{\frac{d+1}{2}} u^{\frac{d-1}{2}}} \frac{{}_2F_1(m+1/2, m-p+1, 2m+1, 2/(2+u))}{(2+u)^{m+1/2}} \tag{246}$$

By looking at the relations obtained it is easy to see that we can find $h$ and $k$ by differentiating $g$.

Now, we have the main tool, namely, the bulk-to-bulk propagator and we want to compute correlators for a 3-form $S_{\mu\nu\rho}$ living on $AdS_7$. According to the AdS/CFT correspondence it is thought that these forms couple to a trace operator $tr(H_{ijk}\phi^A) = \mathcal{O}^A_{ijk}$ of the 6d (2,0) CFT.

In particular for 7 dimensions, $p = 3$ we have :

$$g = c \cdot u^{-5/2}(2+u)^{-m-1/2} {}_2F_1(m+1/2, m-2, 2m+1, 2/(2+u)) \tag{247}$$

We are going to derive the bulk-to-boundary propagator from the bulk-to-bulk propagator:

$$S_{\mu_1\mu_2\mu_3}(w^0, \vec{w}) = \lim_{\epsilon\to 0} \int_{M_\epsilon} d^6 z \sqrt{h(z)} G_{\mu_1\mu_2\mu_3;\mu_1'\mu_2'\mu_3'}(w,z)(z^0)^{-m} s^{\mu_1'\mu_2'\mu_3'}(\vec{z}) \tag{248}$$

Substituting the functional forms of $g(u)$, $h(u)$ and $k(u)$ as $\epsilon \to 0$, so $u \to \infty$: $g(u) \to cu^{-m-3}$, $h(u) \to (-c/m)u^{-m-3}$ and $k(u) \to (i(m+3)c/m)u^{-m-4}$.

In our case the basis bitensors are given by:

$$T^1_{\mu_1\mu_2\mu_3;\mu_1'\mu_2'\mu_3'} = \partial_{\mu_1}\partial_{\mu_1'}u\ \partial_{\mu_2}\partial_{\mu_2'}u\ \partial_{\mu_3}\partial_{\mu_3'}u + antipermutations\ of\ primed\ indices \tag{249}$$

$$T^2_{\mu_1\mu_2\mu_3;\mu_1'\mu_2'\mu_3'} = \frac{1}{2}(\partial_{\mu_1}u\ \partial_{\mu_1'}u\ \partial_{\mu_2}\partial_{\mu_2'}u\ \partial_{\mu_3}\partial_{\mu_3'}u + antipermutations\ of\ all\ indices) \tag{250}$$

$$T^3_{\mu_1\mu_2\mu_3;\mu'_1\mu'_2\mu'_3} = \epsilon_{\mu_1\mu_2\mu_3}{}^{\mu_4\mu_5\mu_6\mu_7}\partial_{\mu_4}\partial_{\mu'_1}u\ \partial_{\mu_5}\partial_{\mu'_2}u\ \partial_{\mu_6}\partial_{\mu'_3}u\ \partial_{\mu_7}u \tag{251}$$

Just to give an idea on how these calculations are done, let us use the expressions in the appendix C and write explicitly the imaginary contribution, it means the $T^3k(u)$ term:

$$\begin{aligned}
T^3k(u) &= \frac{i(m+3)c}{m}\frac{1}{u^{m+4}}\epsilon_{ijk}{}^{lmnp}\ \ \partial_l\partial_{i'}u\ \partial_m\partial_{j'}u\ \partial_n\partial_{k'}u\ \partial_p u\\
&= \frac{i(m+3)c}{mu^{m+4}}\epsilon_{ijk}{}^{lmnp}\left(\frac{-1}{z_0^4w_0^3}\right)\left[\delta_{li'} + \frac{(z-w)_l\delta_{0i'}}{w_0} - \frac{(z-w)_{i'}\delta_{0l}}{z_0} - u\delta_{0i'}\delta_{0l}\right]\\
&\times\left[\delta_{mj'} + \frac{(z-w)_m\delta_{0j'}}{w_0} - \frac{(z-w)_{j'}\delta_{0m}}{z_0} - u\delta_{0j'}\delta_{0m}\right]\\
&\times\left[\delta_{nk'} + \frac{(z-w)_n\delta_{0k'}}{w_0} - \frac{(z-w)_{k'}\delta_{0n}}{z_0} - u\delta_{0k'}\delta_{0n}\right]\times\left[\frac{(z-w)_p}{w_0} - u\delta_{0p}\right]
\end{aligned} \tag{252}$$

By putting everything together we find:

$$\begin{aligned}
S_{ijk}(w) =& c\int_{M_\epsilon} d^6z\left[6\frac{3+m}{m}2^{3+m}\frac{(w^0)^m}{[(w^0)^2+(\vec{z}-\vec{w})^2]^{m+3}}s_{ijk}\right.\\
&+\frac{i(3+m)}{m}2^{3+m}\frac{(w^0)^{m+2}}{[(w^0)^2+(\vec{z}-\vec{w})^2]^{m+4}}\epsilon_{ijki'j'k'}s_{i'j'k'}\\
&-\frac{i(3+m)}{m}2^{3+m}\frac{(w^0)^m}{[(w^0)^2+(\vec{z}-\vec{w})^2]^{m+3}}\epsilon_{ijki'j'k'}s_{i'j'k'}\\
&-18\frac{(3+m)}{m}2^{4+m}(\vec{w}-\vec{z})_i(\vec{w}-\vec{z})_{i'}\frac{(w^0)^m}{[(w^0)^2+(\vec{z}-\vec{w})^2]^{m+4}}s_{i'jk}\\
&\left.+3\frac{i(3+m)}{m}2^{4+m}(\vec{w}-\vec{z})_{i'}(\vec{w}-\vec{z})_l\frac{(w^0)^m}{[(w^0)^2+(\vec{z}-\vec{w})^2]^{m+4}}\epsilon_{ijkj'k'l}s_{i'j'k'}\right]
\end{aligned} \tag{253}$$

Now we can rewrite by using the Schouten identity:

$$\begin{aligned}
\epsilon_{ijki'j'k'}s_{li'j'}(\vec{w}-\vec{z})_l(\vec{w}-\vec{z})_{k'} =& \epsilon_{ljki'j'k'}s_{li'j'}(\vec{w}-\vec{z})_i(\vec{w}-\vec{z})_{k'}\\
&+\frac{1}{3}\epsilon_{ijki'j'l}s_{i'j'l}(\vec{w}-\vec{z})^2
\end{aligned} \tag{254}$$

Then it is found:

$$S_{ijk}(w^0,\vec{w}) = 6c\int_{M_\epsilon} d^6z \left(\frac{3+m}{m}\frac{2^{3+m}(w^0)^m}{[(w^0)^2+(\vec{z}-\vec{w})^2]^{m+3}}\right)\times$$
$$\left[\left(s_{ijk}+\frac{i}{3!}\epsilon_{ijki'j'k'}s_{i'j'k'}\right)-6\frac{(\vec{z}-\vec{w})_i(\vec{z}-\vec{w})_{i'}}{[(w^0)^2+(\vec{z}-\vec{w})^2]}\left(s_{i'jk}+\frac{i}{3!}\epsilon_{i'jklmn}s_{lmn}\right)\right] \tag{255}$$

And now, the value of $S_{ijk}(w)$ at the boundary $w^0\to 0$ we obtain:

$$\lim_{\epsilon\to 0}\epsilon^m S_{ijk}(\epsilon,\vec{w}) = 6c\pi^3\frac{2^{3+m}}{m(m+1)(m+2)}\left(s_{ijk}(\vec{w})+\frac{i}{3!}\epsilon_{ijki'j'k'}s_{i'j'k'}(\vec{w})\right) \tag{256}$$

From that we obtain the bulk-to-boundary propagator:

$$G_{\mu\nu\rho;lmn}(w,\vec{z}) = c_1\left[\frac{w^0}{(w-\vec{z})^2}\right]^m \partial_{[\mu}\frac{(w-\vec{z})_i}{[(w-\vec{z})^2]^3}\delta^{i'|\mu\rho]}_{lmn} \tag{257}$$

As in the case of the spinor field that we considered in a previous chapter, this field obeys a first order differential equation and therefore the action vanishes on-shell. In order to give sense to the 2-point function of those kind of fields the solution proposed is to add a boundary term to the action.

$$\mathcal{S}_\infty = \frac{m}{4}\lim_{\epsilon\to 0}\int_{M_\epsilon} d^6z\sqrt{h(z)}S_{\mu_1\mu_2\mu_3}(z)S^{\mu_1\mu_2\mu_3}(z) \tag{258}$$

Then by using the bulk-to-boundary propagator we get:

$$\mathcal{S}_\infty = \frac{m}{4}c\lim_{\epsilon\to 0}\int_{M_\epsilon} d^6z\sqrt{h(z)}S_{ijk}(z)\epsilon^m\int d^6w\frac{1}{[\epsilon^2+(\vec{z}-\vec{w})^2]^{m+3}}\times$$
$$\left[s^\dagger_{ijk}(w)-6\frac{(\vec{z}-\vec{w})_i(\vec{z}-\vec{w})_{i'}}{[\epsilon^2+(\vec{z}-\vec{w})^2]}s^\dagger_{i'jk}(w)\right] \tag{259}$$

$$\mathcal{S}_\infty = \frac{\pi^3mc^2}{4(m+1)(m+2)(m+3)}\int d^6z\int d^6w s_{ijk}(\vec{z})\frac{1}{[(\vec{z}-\vec{w})^2]^{m+3}}\times$$
$$\left[s^\dagger_{ijk}(\vec{w})-6\frac{(\vec{z}-\vec{w})_i(\vec{z}-\vec{w})_{i'}}{[(\vec{z}-\vec{w})^2]}s^\dagger_{i'jk}(\vec{w})\right] \tag{260}$$

And so the 2-point function is :

$$\left\langle\mathcal{O}_{ijk}(\vec{z})\mathcal{O}^\dagger_{lmn}(\vec{w})\right\rangle = \frac{1}{[(\vec{z}-\vec{w})^2]^{m+3}}\left[\delta^{lmn}_{ijk}-6\frac{(\vec{z}-\vec{w})_{[i}(\vec{z}-\vec{w})_{i'}}{[(\vec{z}-\vec{w})^2]}\delta^{lmn}_{i'jk]}\right] \tag{261}$$

remember the scaling dimension $\Delta = m + p = m + 3$.

Now we are going to try to calculate the same propagator in global coordinates $(\rho, \tau, \Omega)$ . First we need the chordal distance in these new coordinates:

$$u = \frac{1}{2}(x - x')^2 = -\frac{1}{2}(X_0 - X_0')^2 - \frac{1}{2}(X_{d+1} - X_{d+1}')^2 + \frac{1}{2}(X_i - X_i')^2 \tag{262}$$

$$\begin{aligned} X_0 &= R\cosh\rho\cos\tau \\ X_i &= R\sinh\rho\Omega_i \\ X_{d+1} &= R\cosh\rho\sin\tau \end{aligned} \tag{263}$$

Then

$$\begin{aligned} u = &\frac{R^2}{2}\left[-(\cosh\rho\cos\tau - \cosh\rho'\cos\tau')^2 - (\cosh\rho\sin\tau - \cosh\rho'\sin\tau')^2\right. \\ &\left.+(\sinh\rho\Omega_i - \sinh\rho'\Omega_i')^2\right] \\ = &\frac{-R^2}{2}\left[\cosh^2\rho\cos^2\tau + \cosh^2\rho'\cos^2\tau' - 2\cosh\rho\cosh\rho'\cos\tau\cos\tau'\right. \\ &+ \cosh^2\rho\sin^2\tau + \cosh^2\rho'\sin^2\tau' - 2\cosh\rho\cosh\rho'\sin\tau\sin\tau' \\ &\left.- \sinh^2\rho\Omega_i^2 - \sinh^2\rho'\Omega_i'^2 + 2\sinh\rho\sinh\rho'\Omega_i\Omega_i'\right] \\ = &\frac{-R^2}{2}\left[2 - 2\cosh\rho\cosh\rho'\cos(\tau - \tau') + 2\sinh\rho\sinh\rho'\Omega_i\Omega_i'\right] \end{aligned} \tag{264}$$

Now, by making the change of coordinates $\tan\theta = \sinh\rho$ and so $\sec\theta = \cosh\rho$, we find:

$$u = \frac{1}{\cos\rho\cos\rho'}\left[-R^2\cos\rho\cos\rho' + R^2\cos(\tau - \tau') - \sin\rho\sin\rho'\Omega\cdot\Omega'\right] \tag{265}$$

Just to have an idea of what we intend to calculate, we are looking for something like:

$$S_{\mu_1\mu_2\mu_3}(\rho, \tau, \hat{e}) = k\lim_{\rho'\to\pi/2}\int d\Sigma\sqrt{h}G_{\mu_1\mu_2\mu_3\mu_1'\mu_2'\mu_3'}(\rho, \tau, \hat{e}; \rho', \tau', \hat{e}')S^{\mu_1'\mu_2'\mu_3'}(\tau', \hat{e}') \tag{266}$$

We know that at the boundary the chordal distance behaves as $u \to \infty$, in Poincaré coordinates was $z_0 \to 0$ and in global coordinates the boundary is located at $\rho \to \pi/2$.

And we find:

$$\partial_\tau u = -R^2 \frac{\sin(\tau - \tau')}{\cos\rho\cos\rho'} \tag{267}$$

$$\partial_{\tau'} u = R^2 \frac{\sin(\tau - \tau')}{\cos\rho\cos\rho'} \tag{268}$$

$$\partial_{\Omega_i} u = -R^2 \tan\rho\tan\rho'\Omega'_i \tag{269}$$

$$\partial_{\Omega'_i} u = -R^2 \tan\rho\tan\rho'\Omega_i \tag{270}$$

And so:

$$\partial_\tau\partial_{\tau'} u = R^2 \frac{\cos(\tau - \tau')}{\cos\rho\cos\rho'} \tag{271}$$

$$\partial_{\Omega_j}\partial_{\Omega'_i} u = -R^2 \tan\rho\tan\rho'\delta_{ij} \tag{272}$$

$$\partial_{\Omega_i}\partial_{\tau'} u = 0 \tag{273}$$

$$\partial_\tau\partial_{\Omega'_i} u = 0 \tag{274}$$

So we find the only non-vanishing contribution to the basis tensor:

$$T^{\tau'\Omega'_j\Omega'_i}_{\tau\Omega_i\Omega_j} = R^6 \frac{\cos(\tau - \tau')}{\cos\rho\cos\rho'}(\tan\rho\tan\rho')^2\delta_{ij} \tag{275}$$

To determine $G_{B\partial}$ is not only taking $\rho' \to \pi/2$, we must also determine the asymptotic behavior of $S_{\tau\Omega_i\Omega_j}$ in these coordinates. So the propagator has the form: $G = T^1(g + 3h)$. Then taking the limit $\rho' \to \pi/2$ we find:

$$S_{\tau\Omega_i\Omega_j}(r) \sim \int d\Sigma \frac{m+3}{m} \frac{\cos(\tau - \tau')(\cos\rho)^m(\sin\rho)^2}{R^{2m}(\cos(\tau - \tau') - \sin\rho\Omega\cdot\Omega')^{m+3}} S_{\tau'\Omega'_i\Omega'_j} \tag{276}$$

Now if we want to compare the results in both coordinate systems, we need the mapping between them. We find:

$$z = \frac{R\cos\rho}{(\cos\tau + \sin\rho\Omega_n)} \tag{277}$$

$$x_i = \frac{R\sin\rho\Omega_i}{(\cos\tau + \sin\rho\Omega_n)} \tag{278}$$

$$t = \frac{R\sin\tau}{(\cos\tau + \sin\rho\Omega_n)} \tag{279}$$

We consider the results for the scalar case found in previous chapters:

$$\left(\frac{x_0}{x_0^2 + (\vec{x} - \vec{y})^2}\right)^{\Delta} \xrightarrow{?} \left(\frac{\cos\rho}{\cos(\tau - \tau') - \sin\rho\Omega \cdot \Omega'}\right)^{\Delta} \tag{280}$$

By using the mapping we check that they only coincide at $\tau = \tau' = 0$ and $\Omega_n = \Omega'_n = 0$ up to a factor of 2:

$$\left(\frac{x_0}{x_0^2 + (\vec{x} - \vec{y})^2}\right)^{\Delta} \longrightarrow \left(\frac{1}{2} \times \frac{\cos\rho}{1 - \sin\rho\Omega_i \cdot \Omega'_i}\right)^{\Delta} \tag{281}$$

That is what we mean when we say holography in different coordinates are not easily related. Even for the simplest case, the bulk-to-boundary propagator for scalar fields, the mapping does not give a direct relation. Somehow we expected that result, global coordinates system covers the entire AdS and Poincaré system does not.

# 8 Conclusions

In this dissertation we showed the non-trivial relation of holography in global coordinates and Poincaré coordinates in minkowskian and euclidean signature. Starting from a succint study of Anti-de Sitter space and conformal field theory we introduced the main idea of the AdS/ CFT Correspondence in order to present the calculations of bulk-to-boundary propagators and observables such as $n$-point functions for scalar fields coming from a simple supergravity action and propagators and 2-point function for gauge and spinor fields, all of them living in Euclidean Poincaré AdS. Then, by using the example of the scalar field, we have analyzed AdS holography for minkowskian signature and did the same working in global coordinates. Here we have shown the new features(extra modes) appearing in these cases. As mentioned before, the idea was to show how holography looks so different when working in different coordinates, and even if we know the CFT's are conformally related this relation is not easily found. Few people working with AdS/CFT and its applications use other coordinate system than euclidean Poincaré coordinates and we do not know what is the role of the rest of AdS space. Then some future work must be directed to clarify this issue.

# A Bessel Functions

The modified Bessel functions $I_{\pm\nu(z)}$ and $K_{\nu(z)}$ are solutions of the following differential equation:

$$z^2\frac{d^2w}{dz^2}+z\frac{dw}{dz}-(z^2+\nu^2)=0 \tag{282}$$

And the Bessel functions of first kind $J_{\pm\nu(z)}$ are solutions of the equation:

$$z^2\frac{d^2w}{dz^2}+z\frac{dw}{dz}+(z^2-\nu^2)=0 \tag{283}$$

The asymptotic expansions at large $z$, $\nu$ fixed and $\mu=4\nu^2$, are given by:

$$I_\nu(z)\sim\frac{e^z}{\sqrt{2\pi z}}\left(1-\frac{\mu-1}{8z}+\frac{(\mu-1)(\mu-9)}{2!(8z)^2}+\dots\right) \tag{284}$$

$$K_\nu(z)\sim\sqrt{\frac{\pi}{2z}}e^{-z}\left(1+\frac{\mu-1}{8z}+\frac{(\mu-1)(\mu-9)}{2!(8z)^2}+\dots\right) \tag{285}$$

$$J_{\pm\nu(z)}\sim\sqrt{\frac{2}{\pi z}}\left(\cos(z\mp\frac{\pi}{2}\nu-\frac{\pi}{4})+\dots\right) \tag{286}$$

# B Hypergeometric Functions

Hypergeometric functions are solutions of the differential equation:

$$z(1-z)\frac{d^2u}{dz^2} + [\gamma - (\alpha+\beta+1)z]\frac{du}{dz} - \alpha\beta u = 0 \tag{287}$$

If $\gamma$ not an integer:

$$\begin{aligned} u_1 &= F(\alpha, \beta; \gamma; z) \\ u_2 &= z^{1-\gamma} F(\alpha-\gamma+1, \beta-\gamma+1; 2-\gamma; z) \end{aligned} \tag{288}$$

If $\gamma = m+1$:

$$\begin{aligned} u_1 &= F(\alpha, \beta; m+1; z) \\ u_2 &= F(\alpha, \beta; m+1; z) \ln z + \cdots \end{aligned} \tag{289}$$

# C Bitensors and limits

$$
\begin{aligned}
&\partial_\mu u = \frac{1}{z_0}\left[\frac{(z-w)_\mu}{w_0} - u\delta_{\mu 0}\right] \\
&\partial_{\nu'} u = \frac{-1}{w_0}\left[\frac{(z-w)_{\nu'}}{z_0} + u\delta_{\nu' 0}\right] \\
&\partial_\mu \partial_{\nu'} u = \frac{-1}{z_0 w_0}\left[\delta_{\mu\nu'} + \frac{(z-w)_\mu \delta_{0\nu'}}{w_0} - \frac{(z-w)_{\nu'}\delta_{0\mu}}{w_0} - u\delta_{0\nu'}\delta_{0\mu}\right] \\
&D^\mu \partial_\mu u = (d+1)(u+1) \\
&\partial^\mu u \partial_\mu u = u(u+2) \\
&D_\mu \partial_\nu u = g_{\mu\nu}(u+1) \\
&D_\mu \partial_\nu \partial_{\nu'} u = g_{\mu\nu}\partial_{\nu'} u \\
&(\partial^\mu u)(D_\mu \partial_\nu \partial_{\nu'} u) = \partial_\nu u \partial_{\nu'} u \\
&(\partial^\mu u)(\partial_\mu \partial_{\nu'}) u = (u+1)\partial_{\nu'} u
\end{aligned}
\tag{290}
$$

$$
\lim_{\epsilon\to 0} \frac{\epsilon^{2m}}{(\vec{x}^2+\epsilon^2)^{3+m}} = \frac{1}{m(m+1)(m+2)} Area(S_6)\delta^6(\vec{x}) \tag{291}
$$

$$
\lim_{\epsilon\to 0} \frac{\epsilon^{2m+2}}{(\vec{x}^2+\epsilon^2)^{4+m}} = \frac{1}{(m+1)(m+2)(m+3)} Area(S_6)\delta^6(\vec{x}) \tag{292}
$$

$$
\lim_{\epsilon\to 0} \frac{\epsilon^{2m}\vec{x}^2}{(\vec{x}^2+\epsilon^2)^{4+m}} = \frac{3}{m(m+1)(m+2)(m+3)} Area(S_6)\delta^6(\vec{x}) \tag{293}
$$

$$
\lim_{\epsilon\to 0} \int d^6x \frac{\epsilon^{2m}x^i x^j}{(\vec{x}^2+\epsilon^2)^{4+m}} s_{jkl}(\vec{x}+\vec{y}) = \frac{1}{2m(m+1)(m+2)(m+3)} Area(S_6) s_{ikl}(\vec{y}) \tag{294}
$$

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