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Trabalho de Conclusão de Curso

Curso de Graduação em Física

EXOTIC MANIFESTATIONS OF MATTER, LOW TEMPERATURE PHYSICS AND PHASE TRANSITIONS

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Trabalho de Conclusão de Curso apresentado ao Instituto de Geociências e Ciências Exatas - Câmpus de Rio Claro, da Universidade Estadual Paulista Júlio de Mesquita Filho, para obtenção do grau de Bacharel em Física.

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Resumo

Formas espetaculares de manifestação da matéria ocorrem sob condições extremas, ou seja, em baixas temperaturas, altos campos magnéticos ou sob altas pressões. Exemplos incluem supercondutividade, vários tipos de ordem magnética de longo alcance, a fase isolante de Mott [5], a fase super-resfriada da água [6] e a condensação de Bose-Einstein. A terceira lei da Termodinâmica estabelece que a entropia do sistema deve ir a zero quando o sistema é resfriado próximo do zero absoluto, ou seja, o sistema tende a se ordenar. Neste contexto, é importante entender o conceito de transições de fase. Segundo Landau [7], a energia livre do sistema pode ser escrita como uma expansão em termos do parâmetro de ordem. Este último está associado a uma observável cujo valor é nulo acima da temperatura crítica de transição e assume valor finito abaixo da referida temperatura. Como exemplo, vale destacar a magnetização e a polarização elétrica. Ocorre que nas vizinhanças da temperatura crítica, devido à competição entre as fases, há uma flutuação expressiva do parâmetro de ordem a qual se manifesta, por exemplo, na resposta termodinâmica. Por esta razão, medidas de expansão térmica de alta resolução constituem um método importante para se explorar transições de fases. Tal técnica experimental permite ter acesso à variação da energia livre próximo da transição de fase em função de um parâmetro de controle. Como exemplo, pode ser citada a transição de Mott [8], onde um metal torna-se um isolante de Mott devido à localização dos elétrons de condução, podendo ser atingida através de uma variação de pressão, tanto química quanto hidrostática aplicada.

Palavras-chave: Matéria condensada; Transições de fase; Landau; Parâmetro de ordem; Supercondutividade; Mecanismo de Higgs.

Abstract

Spectacular means of matter manifestation occur under extreme conditions, i.e., at low temperatures, high magnetic fields or high pressures. Examples include superconductivity, many forms of long-range magnetic order, the Mott insulator phase [5], the supercooled water phase [6] and the Bose-Einstein condensate. The third law of thermodynamics stablishes that the entropy of a system must go to zero when cooled near absolute zero, i.e., the system tends to get ordered. In this context, it is important to understand the concept of phase transitions. According to Landau [7], the free energy of the system can be described as an expansion in terms of the order parameter. The latter is associated with an observable whose value is zero above the critical temperature and assumes a finite value below said temperature. As an example, magnetisation and electrical polarization can be mentioned. It happens that, at the vicinity of a critical temperature, due to the competition between phases, there is an expressive fluctuation of the order parameter which manifests itself, for example, on the thermodynamic response. For this reason, high resolution measurements of the thermal expansion constitute an important method to explore phase transitions. Such experimental technique gives access to the free energy variation near the phase transition as a function of a control parameter such as pressure or temperature. As an example, the Mott transition can be mentioned [8], where a metal turns into a Mott insulator due to the conduction electrons localyzing, which can be achieved through a variation of both chemical or hydrostatic applied pressure.

Keywords: Condensed matter; Phase transitions; Landau; Order parameter; Superconductivity; Higgs mechanism.

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1 Introduction

Condensed matter Physics is an ever advancing field of study since the beginning of the 20th century [10]. The groundbreaking discoveries by Kammerlingh Onnes of liquid ^{4}He in 1909 [11] and superconductivity in 1911 [12] pushed forward an increasing desire to understand low temperatures phenomena and its properties. In his laboratory, located in Leiden, Netherlands, Onnes began experimenting with low temperatures with the goal of testing the ideas proposed by van der Waals about the existence of critical condensation temperatures for every gas, specially ^{4}He [13]. At the laboratory, the temperature of 1.5 K was achieved enabling the ^{4}He to liquefy. Not stricted to this goal, Onnes also wanted to verify and study how some materials would respond when exposed to such temperatures, which led to the discovery of superconductivity when exposing Au and Hg resistors to liquid ^{4}He while measuring resistivity with respect to temperature. The experiment resulted in a close to zero resistivity for Hg, showcasing the achievement of superconductivity.

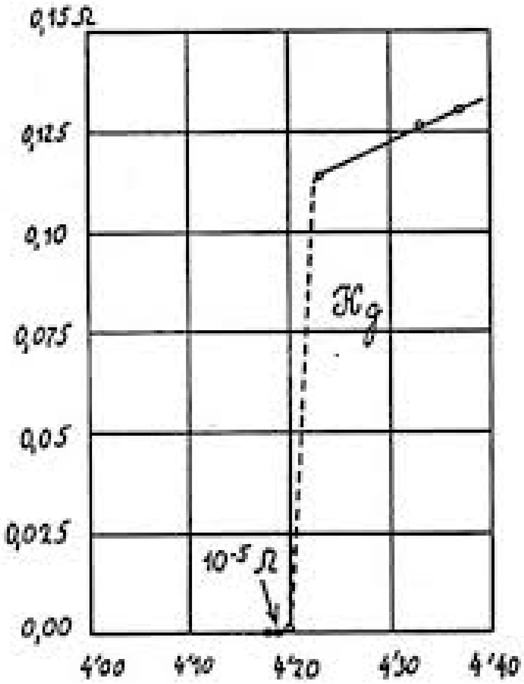

Figure 1 – Resistivity as a function of temperature as made by Onnes in the experiment where the resistivity of Hg is seen to go to zero beyond a certain temperature. This result led to the discovery of a superconducting state, where current flows with little resistance [12].

in 1950, Vitaly Ginzburg and Lev Landau proposed a theory that could explain the macroscopic properties of a superconductor [14], such as the supercurrent and the Meißner effect, through which the magnetic field is expelled from the inside of a superconductor [15]. The superconducting state, as proposed by Anderson [16], is responsible for the spontaneous break of a gauge symmetry, giving rise to a massive photon. Peter Higgs then generalized this mechanism to particle Physics [17], hence the mechanism through which a photon, upon a symmetry break, becomes massive is known as the Anderson-Higgs mechanism. Recently, Mariano de Souza et al. proposed a mechanism through which a massive-like behaviour is observed in the dielectric response [18], which is what motivated my interest in doing the present work. The main goal of this work is to understand mathematically and physically how a photon, in the context of superconductivity, becomes massive. In order to do so, concepts of the experimental and theoretical aspects of low temperatures and phase transitions will be introduced. This work serves as a personal study on the themes above mentioned, so that it can be used in future projects, such as the Master degree. This B. Sc. work is divided into the following chapters:

- **Chapter 2**: In this chapter it is presented an introduction of low temperature Physics as well as experimental aspects of phase transitions, being mostly based on Refs. [19, 20]. Fisrt, a brief discussion of ^{4}He properties is presented. The deduction of the Debye theory of the specific heat is showed. The importance of the Solid State Physics Laboratory in the study of phase transitions and, from that, fundamental aspects of nature is also present in this chapter.

- **Chapter 3**: In this chapter, theoretical aspects of phase transitions are shown through the study of Landau and Ginzburg-Landau theories. The concept of an order parameter is explored and its importance to the mentioned theories of phase transitions. Specially, the case of a complex order parameter is shown, where a phase related to the order parameter appears. In the context of Ginzburg-Landau theory, the gradient of the order parameter is presented.

- **Chapter 4**: In this chapter, aspects of the Higgs mechanism such as the Goldstone theorem are first discussed. Then, a brief section on the gauge invariance of the order parameter is followed by the revisiting of the Higgs mechanism in the superconducting transition. Lastly, the dispersion relation for a photon is computed. This whole chapter was based on Refs. [21, 22].

2 Experimental aspects of low temperatures and phase transitions

In order to explore exotic phases of matter the study of concepts of low temperature Physics will be introduced in this chapter. Properties such as the specific heat of insulators and superconductors, as well as the cooling power of ^{4}He will be explored.

2.1 ^{4}He as a cryoliquid, its latent heat and vapour pressure

This whole section is based on Ref. [19]. According to Pobell, ^{4}He is a very useful *tool* when it comes to achieving low tempratures, such as those below 10 K. In this section, properties of ^{4}He in low temperatures will be explored.

Two isotopes of ^{4}He are used in low temperature experiments, those being ^{4}He and ^{3}He, the latter being rarer and of harder access. It is noticeable that, due to the different quantities of neutrons in each isotope, their spins and thus their properties are different; ^{4}He being a boson and having protons and neutrons in antiparallel orientation, giving its total nuclear spin $I = 0$, while ^{3}He, a fermion, has a total nuclear spin $I = 1/2$.

By analysing the phase diagrams (Figs. 2 and 3) of both isotopes some interesting properties can be seen, such as the low boiling and critical temperatures. It is also possible to see that both isotopes do not become solid under their own vapour pressure, which means that there are no triple points with the vapour, solid and liquid states coexisting.

An important property when using a cryoliquid is its latent heat of evaporation L, since it dictates the amount of heat needed to evaporate it, and their vapour pressure P_v. The vapour pressure of liquid ^{4}He can be calculated through the Clausius-Clapeyron equation [19]:

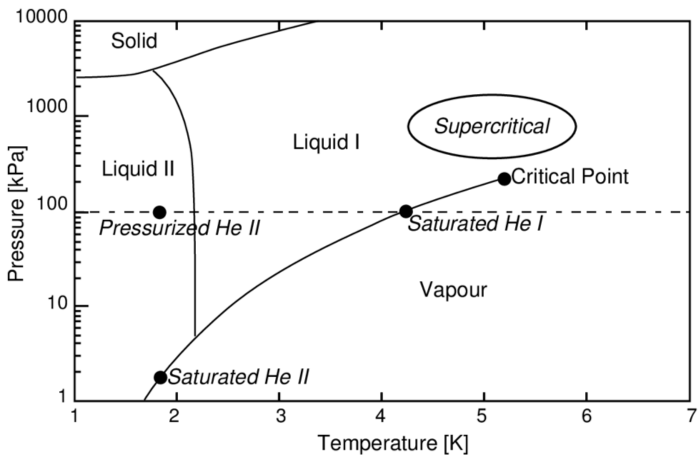

Figure 2 – The phase diagram for ^{4}He [23]. It is seen that the critical temperature, at low pressures, for liquid ^{4}He is a bit less than 2 K, and that vapour and solid phases are far apart.

$$\left(\frac{dP}{dT}\right)_{vap} = \frac{S_{gas} - S_{liq}}{V_{gas} - V_{liq}}, \tag{2.1}$$

where S is the entropy, T the temperature and V the molar volume.

Equation 2.1 can be rewritten by considering that the molar volume in the gas state is significantly larger than in the liquid state. The molar volume of the gas state can be approximated to the ideal gas $V_{gas} \cong \frac{RT}{P}$, where R is the universal gas constant. Also, considering that the difference in the entropies is $\frac{L}{T}$, the equation becomes:

$$\left(\frac{dP}{dT}\right)_{vap} \cong \frac{L(T)P}{RT^2}. \tag{2.2}$$

In order to find the vapour pressure this equation can be solved by separation of variables:

$$\frac{dP}{dT} = \frac{LP}{RT^2} \Rightarrow \frac{1}{P}dP = \frac{L}{RT^2}dT \Rightarrow \int \frac{1}{P}dP = \int \frac{L}{RT^2}dT, \tag{2.3}$$

by approximating L to a constant, the integrals can be solved resulting in:

$$\ln|P| = -\frac{L}{RT} \Rightarrow P_{vap} \propto e^{-\frac{L}{RT}}. \tag{2.4}$$

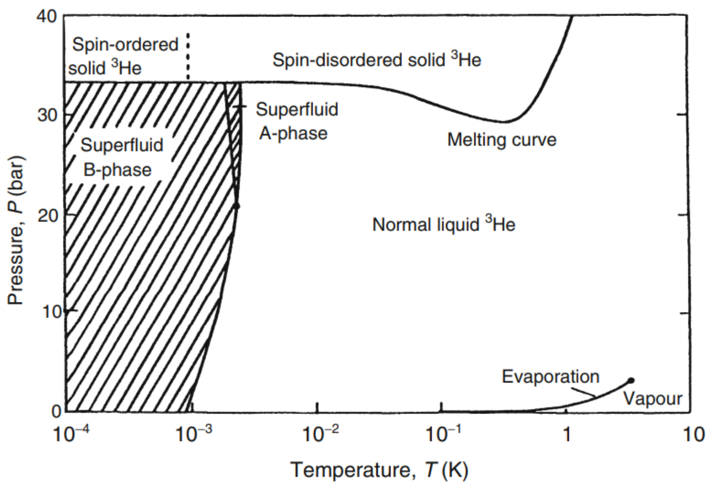

Figure 3 – The phase diagram of ^{3}He [19]. No triple point with vapour, liquid and solid phases exists and the superfluid critical temperature is in the order of mK.

Equation 2.4 shows that the vapour pressure decreases exponentially with temperature. This result can lead to interesting usages of the liquids. The vapour above a liquid can be pumped in order to achieve temperatures below the boiling point: if particles from the hotter phase are pumped away, then the liquid particles with higher energies will escape while only the less energetic ones will remain in the liquid phase, causing the liquid to cool down. It is possible to quantify the cooling power of such method by taking into consideration a mass flow of $\dot{n}$ from the liquid to the vapour phase:

$$\dot{Q} = \dot{n}(H_{liq} - H_{vap}) = \dot{n}\dot{L}, \tag{2.5}$$

where $\dot{Q}$ represents the heat transfer and H the enthalpy associated with each phase. Considering a pump with a constant volume pumping speed, the mass flow $\dot{n}$ will be proportional to the vapour pressure, so that:

$$\dot{Q} \propto LP_{vap} \propto e^{-\frac{1}{T}}, \tag{2.6}$$

which shows that the cooling power will decrease exponentially with temperature, the pump will get more and more inefficient with lowering temperatures. The vapour will eventually be entirely pumped, then the minimum temperature will be achieved. The temperature dependence of the system's vapour pressure is useful, since temperatures can be measured by simply measuring the vapour pressure above a liquid ^{4}He bath.

Having introduced ^{4}He as a useful *tool* for reaching low temperatures, where many phases of matter can be observed, a way of detecting such phase transitions will be explored in the next section via the specific heat.

2.1.1 Revisiting some properties of solids at low temperatures

First, to understand the specific heat of insulators at low temperatures the Debye model will be introduced. The argument present in this section is entirely based on the same discussion from Refs. [19, 20]. The Debye model is used to understand the phononic contribution to the specific heat. The excitations present in crystals are phonons. The total energy at a given temperature with thermal energy given by $\tau = k_BT$ can be seen as a sum of each phonon contribution, with an equilibrium occupancy of phonons $\langle n \rangle$ given by the Planck distribution:

$$\langle n \rangle = \frac{1}{\exp(\frac{\hbar\omega}{k_BT}) - 1}, \tag{2.7}$$

thus, the total thermal energy can be written as:

$$U = \sum_{\vec{k}} D(\omega) \frac{\hbar\omega_{\vec{k}}}{\exp(\hbar\omega_{\vec{k}}/k_BT) - 1}, \tag{2.8}$$

where the indexes $\vec{k}$ refer to the value of the wavevector. $D(\omega)$ is the density of states in the context of phonons, i.e., it refers to the quantity of modes of vibration per frequency value. Now, considering a continuous range of frequencies, the summation over $\vec{k}$ can be turned into an integral over $d\omega$:

$$U = \int d\omega D(\omega) \frac{\hbar\omega}{\exp(\frac{\hbar\omega}{k_B T}) - 1}, \tag{2.9}$$

and, then, by differentiating Eq. 2.9 with respect to the temperature, the specific heat will then be given, while making the substitution $x = \hbar\omega / k_B T$, by:

$$C = k_B \int d\omega D(\omega) \frac{x^2 \exp x}{(\exp x - 1)^2}. \tag{2.10}$$

The frequency can be determined as $\omega = vk$, where v is the sound velocity in the material and is assumed to be constant. The value of k can be determined by the boundary condition of a cube of sides L. The number of modes can be found by multiplying the volume $\left(\frac{L}{2\pi}\right)^3$ (associated with a single possible value of $k = \frac{2N\pi}{L}$) to the volume of a sphere with radius k:

$$N = \left(\frac{L}{2\pi}\right)^3 \left(\frac{4\pi k^3}{3}\right). \tag{2.11}$$

The density of states can then be calculated:

$$D(\omega) = \frac{dN}{d\omega} = \frac{V\omega^2}{2\pi^2 v^3}, \tag{2.12}$$

where $V = L^3$. A maximum frequency for a lattice of N primitive cells can be calculated from Eq. 2.11, resulting in:

$$\begin{aligned} k^3 &= \frac{N}{V} 6\pi^2 \Rightarrow \left(\frac{\omega}{v}\right)^3 = \frac{N}{V} 6\pi^2, \\ \omega_D^3 &= \frac{N}{V} 6\pi^2 v^3. \end{aligned} \tag{2.13}$$

The frequency ω_D is known as the Debye frequency and it represents the highest frequency a phonon can assume. Finally, the thermal energy given by Eq. 2.9 can now be calculated as follows:

$$U = \int d\omega D(\omega) \langle n \rangle \hbar\omega = \int_0^{\omega_D} d\omega \left(\frac{V\omega^2}{2\pi^2 v^3}\right) \left[\frac{\hbar\omega}{\exp(\hbar\omega / k_B T) - 1}\right], \tag{2.14}$$

which results, by differentiating with respect to the temperature, in the specific heat:

$$C(T) = \frac{12}{5}\pi^4 N_0 k_B \left(\frac{T}{\theta_D}\right)^3, \tag{2.15}$$

with θ_D being the Debye temperature, which is the temperature associated with the highest mode of vibration possible in the solid. This model, although being exact for low temperatures, also works well for high temperatures, hence its importance in modeling the specific heat. At high temperatures, i.e., when $T \gg \theta_D$, the specific heat is given by Dulong-Petit value $3N_0k_B$ [20].

From Eq. 2.15 it is possible to see that, when $T \to 0$, the specific heat of the systems will decrease rapidly. The specific heat, however, does not depend exclusively on the phononic contribution, but also on the electronic contribution as well. The total specific heat can be written as:

$$C(T) = \gamma T + \beta T^3, \tag{2.16}$$

where the linear term is the electronic contribution and the cubic term is given by Eq. 2.15.

Following the discussion from Ref. [19], for a superconductor the phononic contribution for the specific heat is not altered, obeying still a power law proportional to T^3:

$$C = \beta T^3. \tag{2.17}$$

However, the electronic specific heat suffers a sudden discontinuity when the material is cooled down past the critical temperature, which, for a simple superconductor that follows the BCS theory is given by [19]:

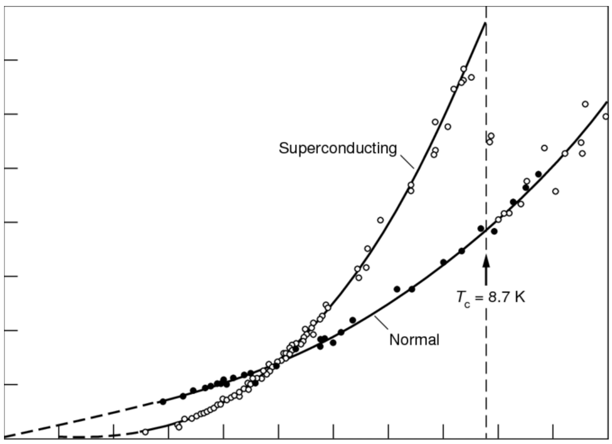

Figure 4 – The behaviour of the specific heat for a niobium superconductor with the decreasing of temperature [24]. The specific heat suffers a discontinuity at the transition point, assuming higher values than expected for the normal state.

$$\Delta C = 1.43\gamma T_C. \tag{2.18}$$

One should note that γT_c is the "expected" specific heat in the normal phase at T_c. After the transition, the specific heat decreases much more rapidly than it would at the "normal" state.

The specific heat can be measured then to identify a superconducting phase transition. As shall be seen in more details in Chapter 3, this discontinuity in the specific heat is characteristic of any second-order phase transition [25].

2.2 Solid State Physics Laboratory

Figure 5 – The Solid State Physics Laboratory. The laboratory has a cryostat consisting on a TeslatronPT cryostat and equipments capable of carrying out measurements of dielectric constant, thermal expansion and electric polarization.

Phase transitions can be experimentally accessed at the Solid State Physics Laboratory. Located in São Paulo State University, Rio Claro, Brazil; the laboratory counts with a TeslatronPT cryostat, made by Oxford Instruments, capable of reaching temperatures down to 1.4 K and magnetic fields up to 12 T, operating with ^{4}He in a closed cycle, i.e., without the need of inserting new ^{4}He into the lines. The laboratory also counts with a high precision capacitance bridge, capable of measuring thermal expansions down to the scale of 0.5 Å. The laboratory was proudly built from scratch by Prof. Dr. Mariano de Souza along with the members of the research gorup at the time, with financial support of FAPESP.

2.2.1 The equipments and cryostat components

Low temperatures can only be achieved via a very complex system of equipments precisely designed to such. The cryostat functions similarly to a thermal bottle: the interior is thermally isolated from the exterior via vacuum. The Outer Vacuum Chamber (OVC) is responsible for maintaining the interior thermally isolated from the exterior, with a better vacuum signifying a better isolation. The OVC is pumped with a turbo pump Pfeiffer Vacuum HiCube 80, granting pressures inside the chamber to be reached down to $\sim 10^{-6}$ mbar. A simple analysis of the ideal gas equation:

$$PV = nRT, \tag{2.19}$$

where P is the pressure inside the chamber, V the volume, R the universal gas constant, T the temperature and n the quantity of moles of an ideal gas, shows that, since P is proportional to n, by lowering the pressure by a factor of 10^9, the quantity of particles decreases proportionally to the same magnitude. The amount of heat exchanged through conduction decreases. Also, to mitigate the heat exchange via irradiation, the interior of the sample chamber is covered in a reflective material.

The equipment responsible for refrigerating the gas is a compressor. Inside the apparatus, the gas is firstly compressed, then, by doing work, it expands thus losing heat. The cold gas is pumped to the so-called cold head where it is able to exchange heat with the circulation gas. The gas is pumped back to the compressor, where the cycle is restarted. This process allows the system to be cooled down to extremely low temperatures without the need of inserting new gas into the system, operating with 16 bar of ^{4}He. Due to the huge amount of heat dissipated by the compressor, both the cold head and the compressor must remain separated in different rooms to prevent the interference in the cooling of the system. The lines extend for 12 m to facilitate this isolation. The system responsible for cooling the compressor consists of a chiller that storages water at temperatures of around 16 °C. Such water is directed to the compressor where it circulates exchanging heat. This system stops the compressor from overheating.

There is also 0.6 mbar of ^{4}He gas that circulates through the cryogenic lines via a circulation pump. The gas, with high purity (99.999%), must not contain moisture and small particles, since these impurities can compromise the temperatures reached during a cooldown. To grant such impurities to not contaminate the gas, a zeloite trap is needed. Zeolites have interesting properties of adsorption, a process through which particles are attached on the surface of the material. The porous structure of the zeolites can trap small impurities such as water particles and dust, not allowing them to circulate along with the gas.

The circulation gas is thermally coupled to the compressor gas through the cold head, where both gases exchange heat. This exchange cools the circulation gas, which in turn is coupled to a Variable Temperature Insert (VTI). The VTI consists in a chamber which is in thermal contact with the sample space, where another gas is present to couple thermally the sample with the rest of the system. All gases are thermally coupled although they do not exchange particles, so the system refrigerates homogenously.

A very important valve in the system is the needle valve, since it controls the "cold breath" pressure of the gas, i.e., it regulates the flow of ^{4}He in the system. When the valve is fully closed, there is no pressure of gas flowing, and by opening it there will be an increase in pressure. The valve also helps coupling the VTI with the circulation gas, so that the tempreature in the sample chamber can be controlled by the opening or closing of the needle valve.

Magnetic fields are applied by a superconductor magnet coil in the interior of the cryostat. Before applying fields, the magnet must have been cooled down below 4.6 K. A heater must then be turned on, since the heating of electrical components inside the cryostat will make contact between the voltage source and the magnet. A superconducting magnet is interesting since a superconductor is able to carry a current almost without energy loss via Joule effect, hence it can be set into a persistent mode where the coil creates a magnetic field persistently without spending much energy nor dissipating heat. The magnet is capable of creating magnetic fields up to 12 T (about 10^5 times more intense than earth's magnetic field) upon being subjected to a 110 A current.

The cooldown is, through these delicate equipments, granted to occur controllably. All these components are necessary so that the intricate Physics of low temperatures and phase transitions can be well explored.

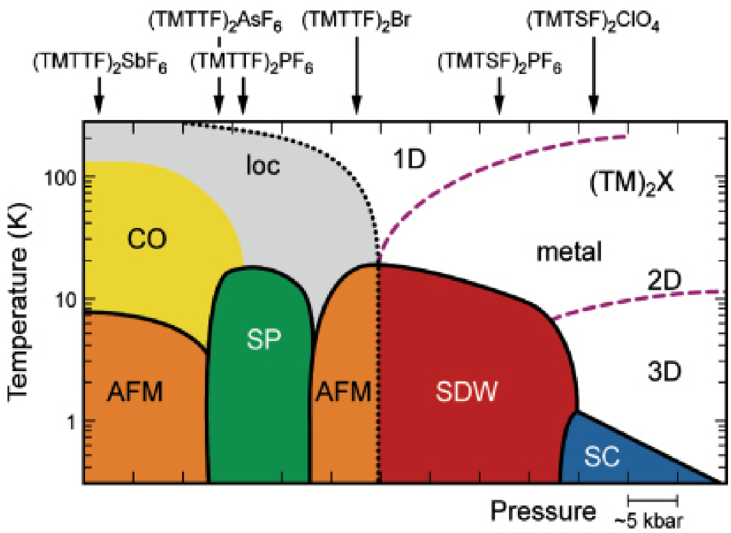

Figure 6 – Phase diagram of the Fabre salts belonging to the families $(TMTTF)_2X$ and $(TMTSF)_2X$, corresponding respectively to a tetramethyltetrathiafulvalene and a tetramethyltetraselenafulvalene molecule with a X monovalent counter-anion. The phase diagram of these materials present a very rich number of phases, such as charge ordering (CO), antiferromagnetic (AFM), spin-Peierls (SP), spin density wave (SDW) and superconductor (SC) [26].

Low temperature experiments are carried out in the laboratory in order to explore interesting properties of molecular system samples belonging to the $(TMTTF)_2X$ family. Such materials are of great importance in the study of fundamental Physics, since they present a wide variety of phases that can be achieved by lowering the temperature as shown in Fig. 6.

The work of the research group is fundamentally based on the study of such phases, as well as properties that arise from them. Fundamental research could only be done by the group because of the laboratory, hence its importance and inclusion in this work. Now, having briefly introduced experimental aspects of phase transitions, a dive into the Landau theory shall be done in the next chapter in order to explore them theoretically.

3 Revisiting phase transitions

This chapter will focus on understanding a mathematical description of phase transitions. A classical phase transition can be defined as a change in the state of a crystal, for example, by varying the values of temperature or pressure of the material beyond a critical value, after which its symmetry properties change as well as its thermodynamic properties [25].

3.1 A brief introduction to first- and second-order phase transitions

First-order phase transitions refer to the penomenon where, upon a phase transition, the thermodynamic functions suffer a discontinuity [25]. The paraelectric-ferroelectric transition can be understood to be of a first-order depending on the material [27]. The symmetry properties of crystal changes discontinuously in first-order phase transitions.

Second-order phase transitions can be defined as transitions which uccur with a discontinuity in the derivatives of a thermodynamic function. Such case can be observed, as seen in Chapter 2, in the transition from insulating and superconducting phases, where there is a discontinuity in the specific heat when the sample is cooled beneath the critical temperature [15].

3.2 The basics of Landau theory of phase transitions

The Landau theory of phase transitions [25] is a way to mathematically describe such phenomena through an expansion in terms of a parameter η which defines the phase. This theory shall be revisited in this section in order to shine some light on how phase transitions work.

Firstly, η shall be defined as an order parameter. This physical quantity is seen as a parameter that is not manifested before the transition, while assuming a finite value after the material has undergone it. As an example, the polarization in the ferroelectric transition can be cited [27].

Considering a crystal that undergoes a phase transition, its free energy F can be written as a function of temperature, pressure and the order parameter η, or simply $F = F(P, T, \eta)$. While temperature and pressure can assume arbitrary values, η must be determined by the thermal equilibrium condition, so that F must be a point of minimum at a given T and P [25]. The free energy can be then expanded in terms of η, given the fact that the latter assumes arbitrarily small values near a transition point, which will result in:

$$F(P,T,\eta) = F_0 + \alpha\eta + A\eta^2 + B\eta^3 + C\eta^4 + ... , \tag{3.1}$$

where the coefficients $\alpha, A, B, C, ...$ are functions of P and T.

It is worth mentioning that such expansion should not be continued until arbitrarily high powers of η [7], the degree to which it is to be continued depends on the order of the transition. The linear term coefficient α corresponds to a field associated with the order parameter [21], and for now on will be considered zero for brevity (the transition occurs in the absence of fields). Above the critical temperature T_c, the second order coefficient $A > 0$, given the fact that before the transition the free energy must correspond to a minimum at $\eta = 0$, and below T_c it changes signs for in this phase the minimum occurs at $\eta \neq 0$ (see Fig. 7). Due to its sign depending on the temperature, the coefficient A can be expressed as [25, 28]:

$$A = \frac{a(T - T_c)}{2}, \tag{3.2}$$

where a is a constant depending on the material. This equation shows that the transition points can be calculated at $A = 0$, since it occurs when $T = T_c$. From now on, the phase above T_c will be referred to as phase I, and the one below it is phase II.

Moreover, the third order coefficient must equal zero since a third order equation will not result in a singularity, instead, the point defined by $\frac{\partial F_3}{\partial \eta} = B\frac{d}{d\eta}\eta^3 = 0$, where F_3 is the third order term of the free energy expansion, will correspond to an inflection point. Therefore, the coefficient B can be considered zero as well as all odd terms coefficients. The expansion is now written:

$$F(\eta) = F_0 + \frac{r}{2}\eta^2 + \frac{u}{4}\eta^4 + ... , \tag{3.3}$$

with $r = a(T - T_c)$ and $C = \frac{u}{4} > 0$. Equating C to a new coefficient $\frac{u}{4}$ is done due to convenience and to facilitate the algebra.

In order to inspect first-order phase transitions, where the order parameter presents a discontinuity at the transition point, the free energy is expanded until the sixth power of η as follows [28]:

$$F = \frac{r}{2}\eta^2 + \frac{u}{4}\eta^4 + \frac{b}{6}\eta^6, \tag{3.4}$$

from the condition that the free energy must be minimized in order to explore the behaviour of the order parameter, first derivatives of the free energy must be taken:

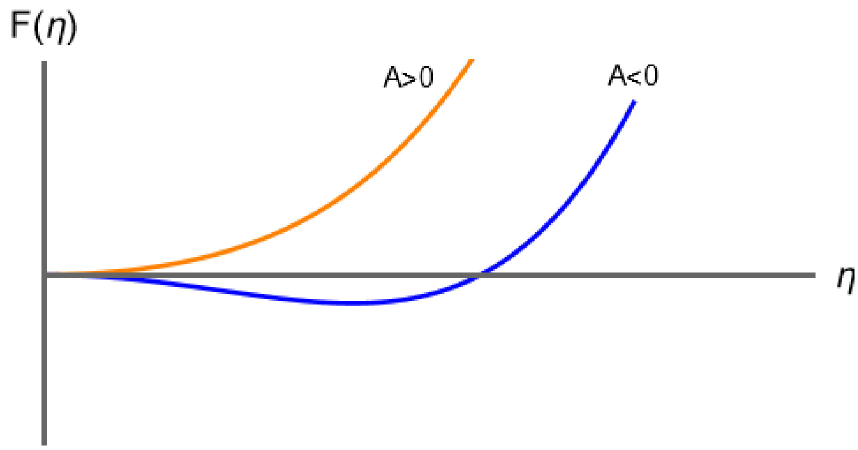

Figure 7 – The positive part of the free energy F as a function of η in both phases with arbitrary parameters. When the system is at phase I, the coefficient A is greater than zero, while upon transitioning it shifts to below zero. The free energy, for $A > 0$, will have its minimum at $\eta = 0$, and when $A < 0$ the minimum accurs at $\eta \neq 0$.

$$\frac{\partial F}{\partial \eta} = r\eta + u\eta^3 + b\eta^5 = 0 \Rightarrow r + u\eta^2 + b\eta^4 = 0. \tag{3.5}$$

The expression for the order parameter can be found by solving Equation 3.5 for η^2 while observing its behaviour at the critical point, i.e., when $T = T_c$, the u-depending terms vanish, resulting in:

$$\eta^2 = \frac{-u \pm u}{2b}. \tag{3.6}$$

The order parameter at the transition point will either be $\eta = 0$ or:

$$\eta = \left[-\frac{u}{b}\right]^{1/2}, \tag{3.7}$$

with $u < 0$ in this case. There is a discontinuity in the order parameter at the transition point, since the order parameter can be either $\eta = 0$ or $\eta \neq 0$, as Eq. 3.7 suggests. This discontinuity in the order parameter characterizes a classical first-order phase transition [21].

Now, for second-order phase transitions, the order parameter as a function of temperature can be obtained by simply equating the derivative of the free energy to zero, while carrying the expansion up to the fourth term [25]:

$$\frac{\partial F}{\partial \eta} = 2A\eta + 4C\eta^3 = 0 \Rightarrow \eta\left(A + 2C\eta^2\right) = 0 \Rightarrow A + 2C\eta^2 = 0, \tag{3.8}$$

therefore:

$$\eta^2 = -\frac{A}{2C} = a\frac{(T_c - T)}{2C}. \tag{3.9}$$

This equation shows how the order parameter behaves at temperatures equal or below T_c. Now, the entropy near a transition point can be determined, while dismissing higher powers of η, by [25]:

$$S = -\frac{\partial F}{\partial T} = S_0 - \left(\frac{\partial A}{\partial T}\right)\eta^2, \tag{3.10}$$

where $S_0 = -\frac{\partial F_0}{\partial T}$, and is understood as the entropy associated with phase I. As previously stated, in phase I the parameter $\eta = 0$, and in this case $S = S_0$, while, in phase II, $\eta^2 = -\frac{A}{2C}$ and its entropy is given by:

$$S = S_0 + \frac{A}{2C}\frac{\partial A}{\partial T} = S_0 + \frac{a^2}{2C}(T - T_c). \tag{3.11}$$

Equation 3.11 shows that $S = S_0$ when the system is at the transition point ($T = T_c$), receiving a contribution $\frac{a^2}{2C}(T - T_c)$ when the system is below the critical temperature, and thus the entropy is continuous throughout the transition.

Now, finally, the specific heat $C_P = T\left(\frac{\partial S}{\partial T}\right)_p$ of both phases at the transition point shall be determined. Then, for phase II C_P can be expressed as follows:

$$C_P = C_0 + a^2\frac{T_c}{2C}. \tag{3.12}$$

For the phase above T_c, $S = S_0$, and therefore $C_P = C_0$, with C_{p0} referring to the specific heat associated with phase I. This leads to the fact that the specific heat is not continous, since a contribution $+a^2\frac{T_c}{2C}$ is given at T_c. It assumes a higher value at phase II than at phase I at the transition point. This behaviour of the thermodynamic functions, as seen in the previous section, corresponds to a second order phase transition: while the thermodynamic functions remain continuous through the transition, their derivatives suffer a discontinuity [25].

3.3 The Landau theory with a complex order parameter

Now, taking the discussion a little further, a many components order parameter can be introduced. Some phase transitions can be described through an n-components order parameter, and in this section the discussion is focused specifically on complex order parameters, which can be expressed as [21]:

$$\hat{\eta} = \eta_1 + i\eta_2, \tag{3.13}$$

where η_1 and η_2 are real values corresponding to the components of the complex order parameter.

The Landau free energy can be formed by introducing $\hat{\eta}$ into the expansion of the free energy of second-order phase transitions. Hence, the expression is given by:

$$F = r\left[\eta^*\eta\right] + \frac{u}{2}\left[\eta^*\eta\right]^2, \tag{3.14}$$

since $|\eta|^2 = \eta^*\eta$. In phase I, as discussed in Sec. 3.2, F is minimum at $|\eta| = 0$.

The free energy remains unchanged at any value for the phase ϕ of the order parameter: $|e^{i\phi}\eta|^2 = |\eta|^2$. Upon a phase transition, the Landau free energy then assumes a "mexican hat" shape with the points of minima represented by its brim, cf. Fig. 8. These points represent all the possible values of η_1 and η_2 for which the amplitude of the order parameter minimizes the free energy. From Eq. 3.9, the points of minima for this system are described by:

$$\eta = \sqrt{\frac{|r|}{u}}e^{i\phi}, \tag{3.15}$$

In this scenario, the order parameter can assume any phase, i.e., a "position" along the brim, as long as its magnitude is maintained constant. However, if the order parameter magnitude oscilates, the phase shift is accompanied by an energy spend, creating a sort of resistance for the phase to shift. Landau theory provides a good way of describing phase transitions in the vicinity of a critical temperature, but a more general way of describing a phase transition lies in the Ginzburg-Landau theory [14], where the energy needed to a space variation of the order parameter is also accounted for.

3.4 Introducing the basics of Ginzburg-Landau theory

The Ginzburg-Landau theory was first introduced to describe superfluids and superconductors near a critical point [14]. In this theory, a free energy is built similarly to the Landau theory discussed above, but it also accounts for an energy cost associated with variations in space of the order parameter, thus being more general. This energy is added to the Landau free energy resulting, for the simplest, one-component case, in [21]:

$$F_{GL} = f_0 + \frac{s}{2}\left|\nabla\eta\right|^2 + \frac{r}{2}\eta^2 + \frac{u}{4}\eta^4 - h\eta, \tag{3.16}$$

where s is a constant associated with the energy cost for varying spacially the order parameter and h refers to an applied field. This expansion is valid near a critical point only, since in this regime the order parameter assumes sufficiently small values.

From this equation, the correlation length, i.e., the length scale associated with the order parameter fluctuations, can be derived:

$$\xi = \sqrt{\frac{s}{|r|}} = \sqrt{\frac{s}{a(T_c - T)}} = \sqrt{\frac{s}{aT_c - aT}} \Rightarrow \sqrt{\frac{s}{aT_c}} \cdot \sqrt{\frac{1}{1 - \frac{T}{T_c}}} = \xi_0 \left(1 - \frac{T}{T_c}\right)^{-1/2}, \tag{3.17}$$

where $\xi_0 = \sqrt{\frac{s}{aT_c}}$ corresponds to the correlation length at $T = 0$. The equation above comes from a dimensional analysis of both coefficients s and r, which have dimensions of $[E]/[L]$ and $[E]/[L^3]$, respectively. The distance to which the gradient and the quadratic terms are comparable is the correlation length, which dictates the scale to which magnitude oscillations are relevant [21].

Now, applying the concept of a complex order parameter into Ginzburg-Landau free energy, more specifically in the context of superconductivity, where the order parameter can be seen as a wavefunction of the Cooper pairs that, upon having them condensed, assumes a macroscopic form in the superconducting phase:

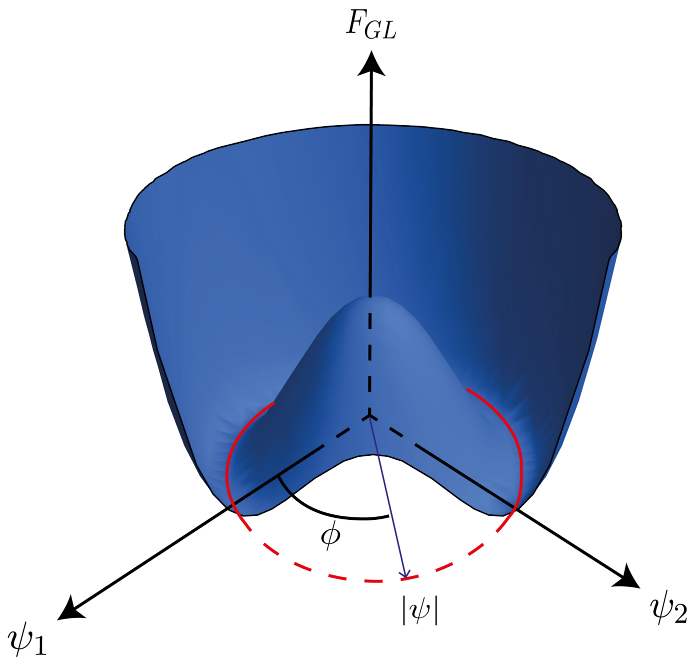

Figure 8 – A representation of the Ginzburg-Landau free energy. The amplitude of the order parameter is kept invariant along the circle of minima (seen as the red circle along the brim) independently of the phase ϕ.

$$\psi(x) = |\psi(x)|e^{i\phi(x)}, \tag{3.18}$$

Then the order parameter can also be described as the density of Cooper pairs n_s formed [21], hence:

$$|\psi(x)|^2 = n_s(x). \tag{3.19}$$

The gradient term in Equation 3.16 behaves similarly to the kinetic energy density in quantum mechanics [29]: $\frac{\hbar^2}{2m}|\nabla\psi|^2$, so the coefficient s can be identified as $s = \frac{h^2}{m}$ and the free energy assumes then the form:

$$F_{GL} = \frac{\hbar^2}{2m}|\nabla\psi|^2 + r|\psi|^4 + \frac{u}{2}|\psi|^4. \tag{3.20}$$

The gradient term is thus the kinetic contribution to the free energy:

$$\frac{\hbar^2}{2m}|\nabla\psi|^2 = \frac{1}{2m}|-i\hbar\nabla\psi|^2, \tag{3.21}$$

where $\hat{p} = -i\hbar\nabla$ is the momentum operator. As for the correlation length, which governs the range of amplitude fluctuations of the order parameter, it can be expressed as [21]:

$$\xi = \sqrt{\frac{s}{r}} = \sqrt{\frac{\hbar^2}{2m|r|}} = \xi_0 \left(1 - \frac{T}{T_c}\right)^{-1/2}, \tag{3.22}$$

with the correlation length at zero temperature corresponding to $\xi_0 = \sqrt{\frac{\hbar^2}{2maT_c}}$. Beyond such length, the amplitude fluctuations are negligible and only phase fluctuations become expressive.

Ginzburg-Landau theory furnishes a more complete way of understanding phase transitions, since it accounts for space fluctuations of the order parameter as well as the expansion of the free energy given by Landau theory. The gradient term is core to the appearence of a massive mode, as the next chapter will show.

4 The Anderson-Higgs mechanism revisited

Now, with the basic understanding of the Ginzburg-Landau theory, this chapter will explore how this theory results in the appearence of a massive photon. Firstly, an introduction to the Goldstone theory will be given to stablish the concept of Goldstone modes.

4.1 The Goldstone theorem applied to the Ginzburg-Landau theory

The Goldstone theorem states that, upon a continuous symmetry breaking, a massless mode of vibration should appear in the direction of the broken symmetry [30]. In order to inspect this phenomenon in the context of superconducting transition, the concepts of symmetry and mass matrix shall be revisited. Understanding the concept of a massless mode is essential to the Anderson-Higgs mechanism, as shall be seen.

The idea of a symmetry consists primarily in the invariance under a transformation. A continuous symmetry can be seen as one which is defined by an invariance upon a continuous transformation of a parameter within the theory [31].

The classic example of a continuous symmetry consists of a rotation symmetry, where the system remains the same under any transformation on the angles that describe the system, those being the continuous parameters describing the symmetry. In this chapter, the interesting case of symmetry to be looked upon is the phase symmetry, since Goldstone modes can be observed through it. In phase transitions, symmetry is spontaneously broken, as described in Chapter 3 where, upon transitioning its phase, the system assumes a finite value for the order parameter. For a complex order parameter a continuous symmetry is broken since the order parameter can assume any phase, which is a continuous parameter, when a phase transition is carried out.

4.1.1 Introducing the mass matrix and a generalization of the theorem

In the context of field theory, the mass matrix encapsulates all possible relations between the fields involved in a system. A classical field can be seen as a physical quantity that permeates all of space and to which is assigned a value at each position value, as an example the electric potential can be cited [32]. The order parameter magnitude and phase can both be seen as fields in the classical sense, which will be explored in the next section. For now, a general field theory invariant under rotation of a vector with components ϕ_n will be studied. The potential energy for such theory is written in the form [22]:

$$V(\phi) = \frac{\mathcal{M}}{2}\sum_n \phi_n\phi_n + \frac{g}{4}\left(\sum_n \phi_n\phi_n\right)^2, \tag{4.1}$$

where the coefficients $\mathcal{M}$ and g are constants depending on the system. From this potential, a mass matrix can be formed by simply taking the second derivative with respect to all fields involved, while analysing the case where the potential is minimized:

$$M_{nm} = \left(\frac{\partial^2 V(\phi)}{\partial\phi_n\partial\phi_m}\right)_{\phi_0}. \tag{4.2}$$

where ϕ_0 represents the value for the field that minimizes the potential. Roughly speaking, the mass matrix takes into consideration how the potential curves around its minimum in the direction of each fild component. Understanding this matrix is important to verify the Goldstone theorem, since its eigenvalues correspond to the masses associated with the fields present in the theory [22]. If the symmetry is broken, i.e., if the potential is minimized at a non-zero value of $\sum_n \phi_n\phi_n$, with its respective value being:

$$\sum_n \phi_{0n}\phi_{0n} = \frac{|\mathcal{M}|}{g}, \tag{4.3}$$

then there must be a massless mode of vibration associated with the direction of such broken symmetry according to the theorem [22]. The mass matrix for this situation can be explored:

$$\frac{\partial V}{\partial\phi_n} = \mathcal{M}\delta nm + g\delta_{nm}\sum_n \phi_n\phi_n + 2g\phi_n\phi_m. \tag{4.4}$$

The off-diagonal terms for the matrix will therefore be just $2g\phi_n\phi_m$. This matrix will only have one eigenvector with a non zero eigenvalue, while (N-1) whose associated mass is zero.

4.1.2 The mass matrix in the Ginzburg-Landau free energy

Now, applying this to the example of the Ginzburg-Landau free energy from chapter 3, a massless mode of vibration can be found. Microscopically, the free energy involves a potential energy given by:

$$V(\psi) = r|\psi|^2 + \frac{u}{2}|\psi|^2. \tag{4.5}$$

As discussed earlier, when $r < 0$ the minimum of the order parameter is given by Eq. 3.18. By considering small fluctuations around this minimum such as:

$$\psi' = \left(\sqrt{\frac{|r|}{u}} + \psi\right) e^{i(\phi_0+\phi)}, \tag{4.6}$$

it is easily seen that:

$$|\psi'|^2 = \left(\sqrt{\frac{|r|}{u}} + \psi\right)^2 . \tag{4.7}$$

Now, applying this result to Eq. 4.5 yields the new potential that accounts for oscillations around the state of minimum, given by the form:

$$V = r\left(\sqrt{\frac{|r|}{u}} + \psi\right)^2 + \frac{u}{2}\left(\sqrt{\frac{|r|}{u}} + \psi\right)^4 . \tag{4.8}$$

The mass matrix can be written as the second derivative of the potential with respect to both amplitude and phase of the order parameter (the perturbation $\psi = 0$):

$$M_{\psi\psi} = \left(\frac{\partial^2 V}{\partial \psi^2}\right)_{\psi=0} = \frac{\partial}{\partial \psi}\left[2r\left(\sqrt{\frac{|r|}{u}} + \psi\right) + 2u\left(\sqrt{\frac{|r|}{u}} + \psi\right)^3\right]_{\psi=0} , \tag{4.9}$$

which results in:

$$M_{\psi\psi} = 2r + 2u\left(\sqrt{\frac{|r|}{u}} + \psi\right)^2_{\psi=0} = 2r + 6|r|. \tag{4.10}$$

Since $r < 0$, then $|r| = -r$ which therefore implies that:

$$M_{\psi\psi} = 4|r|. \tag{4.11}$$

Since the potential does not depend explicitly on the phase, all other terms that depend on phase derivatives will be zero, so that:

$$M_{\psi\phi} = M_{\phi\psi} = M_{\phi\phi} = 0, \tag{4.12}$$

resulting, finally in the mass matrix for this potential:

$$M = \begin{bmatrix} 4|r| & 0 \\ 0 & 0 \end{bmatrix} . \tag{4.13}$$

It is possible to conclude therefore that any fluctuation on the direction of the phase will have no mass associated, while amplitude fluctuations will then have a mass given by $4|r|$. This result is in agreement with the Goldstone theorem [22]. It will be investigated in the next sections how this mode, when interacting with a gauge field, will be absorbed and become massive.

4.2 Gauge invariance of a charged particle

For a charged particle such as the electron, the momentum operator in the electromagnetic field is expressed as $\hat{p} = \hat{p} - e\vec{A}$ [29], with e being the fundamental charge and $\vec{A}$ the magnetic vector potential. The Hamiltonian can be then expressed as [33]:

$$\hat{H} = \frac{\hat{p}^2}{2m} + e\varphi \Rightarrow \hat{H} = \frac{\left(\hat{p} - e\vec{A}\right)^2}{2m} + e\varphi. \tag{4.14}$$

Hence, the time dependent Schrödinger equation becomes:

$$\hat{H}\psi = -i\hbar\frac{\partial\psi}{\partial t} \Rightarrow \left[\frac{1}{2m}\left(\hat{p} - e\vec{A}\right)^2 + e\varphi\right]\psi = -i\hbar\frac{\partial\psi}{\partial t}. \tag{4.15}$$

Applying $\vec{p} = -i\hbar\nabla$ into Eq. 4.15 yields the one-body Schrödinger equation for a charged particle:

$$\left[-\frac{\hbar^2}{2m}\left(\nabla - i\frac{e}{\hbar}\vec{A}\right)^2 + e\varphi\right]\psi = -i\hbar\frac{\partial}{\partial t}\psi. \tag{4.16}$$

Considering a shift α on the phase of the wavefunction such as $e^{i\alpha}\psi$, the equation above must be invariant under the gauge transformations [21, 29]:

$$\phi \rightarrow \phi + \alpha; \quad \vec{A} \rightarrow \vec{A} + \frac{\hbar}{e}\nabla\alpha; \quad \varphi \rightarrow \varphi - \frac{\hbar}{e}\frac{\partial\alpha}{\partial t}, \tag{4.17}$$

where φ is the electric potential. Therefore, space derivatives, when dealing with a charged particle in an electromagnetic field, are replaced by the gauge invariant counterpart:

$$\nabla \rightarrow \nabla - i\frac{e}{\hbar}\vec{A}. \tag{4.18}$$

The same gauge transformations must hold for the phase transformations since the order parameter in Ginzburg-Landau theory correspond to the macroscopic effects of a wavefunction corresponding to the condensation of Cooper pairs [21], i.e., the pairs start to occupy the same quantum state, which leads to a same wavefunction occupying the whole space associated with the superconducting, so some properties like the order parameter phase become macroscopic.

4.3 The Anderson-Higgs mechanism

In the context of quantum field theory, the so-called Anderson-Higgs mechanism is responsible for the appearence of a massive boson [17], but not stricted to this field of study, such mechanism can be also observed in condensed matter Physics. In the superconductor phase transition, for example, this phenomenon is observable through Ginzburg-Landau theory where a gradient of the order parameter is accounted for, cf. Eq. 3.13. This section is mostly based on the discussion present in Ref. [21]. The gradient of the order parameter can be separated into two derivative terms:

$$\nabla\psi = \nabla(e^{i\phi}|\psi|) = \nabla|\psi|e^{i\phi} + i|\psi|\nabla\phi e^{i\phi}, \tag{4.19}$$

therefore:

$$|\nabla\psi|^2 = \nabla^2|\psi|^2 + |\psi|^2(\nabla\phi)^2. \tag{4.20}$$

Applying Eq. 4.20 into the free energy yields:

$$F_{GL} = \frac{\hbar^2}{2m}|\psi|^2(\nabla\phi)^2 + \frac{\hbar^2}{2m}(\nabla|\psi|)^2 + r|\psi|^2 + \frac{u}{2}|\psi|^4. \tag{4.21}$$

An energy associated with a phase gradient now explicitly appears. This energy term can be understood as the energy cost for shifting the phase. The remaining terms are associated with amplitude fluctuations. On large length scales, however, the leading contribution will be that of the phase gradient, since the phase assumes a macroscopic configuration in the supercodnucting phase, and amplitude fluctuations are confined to the coherence length. The energy for shifting the phase is known as the phase stiffness. It can be seen as an elastic energy for the shifting phase.

In order to derive the equation of motion for the phase gradient term a Lagrangian can be useful. The potential energy is, on large scales beyond the correlation length, $V = \frac{1}{2}\frac{\hbar^2 n_s}{m}(\nabla\phi)^2$ and the kinetic energy can be associated with a time variating phase $T = \frac{1}{2}\frac{\hbar^2 n_s}{m}(\frac{\dot{\phi}}{c^*})^2$ [21]. The Lagrangian density, which refers to the Lagrangian per unit volume and the action, referring to the time integration of each lagrangian contribution, are then:

$$\mathcal{L} = \frac{\rho}{2}\left[\left(\frac{\dot{\phi}}{c^*}\right)^2 - (\nabla\phi)^2\right], \quad S = \frac{\rho}{2}\int dt d^3x \left[\left(\frac{\dot{\phi}}{c^*}\right)^2 - (\nabla\phi)^2\right], \tag{4.22}$$

where $\rho = \dfrac{\hbar^2 n_s}{m}$ is the term associated with the phase stiffness and c^* represents the characteristic velocity of propagation of the system. Since the superconducting state is made of charged particles, the electromagnetic field plays a key role in the action, and gauge invariance must be taken into account. The variation on the phase must be acompanied by gauge transformations of the electromagnetic field such as Eq. 4.17, with their space and time derivatives given by:

$$\dot{\phi} \rightarrow \dot{\phi} + \frac{e^*}{\hbar}\varphi; \quad \nabla\phi \rightarrow \nabla\phi - \frac{e^*}{\hbar}\vec{A}, \tag{4.23}$$

where e^* corresponds to the charge of an electron pair i.e., $e^* = 2e$. Applying these transformations into the action, while also introducing the electromagnetic contribution, yields:

$$S = \frac{1}{2}\int dtd^3x \left\{\rho\left[\frac{1}{c^*}\left(\dot{\phi} + \frac{e^*}{\hbar}\varphi\right)\right]^2 - \left(\nabla\phi - \frac{e^*}{\hbar}\vec{A}\right)^2\right\} + \frac{1}{2\mu_0}\left[\left(\frac{\vec{E}}{c}\right)^2 - \vec{B}^2\right], \tag{4.24}$$

with $\dfrac{1}{2\mu_0}\displaystyle\int dtd^3x\left[\left(\frac{\vec{E}}{c}\right)^2 - \vec{B}^2\right] = S_{EM}$ corresponding to the electromagnetic contribution to the action, and $\vec{E} = -\dfrac{\partial\vec{A}}{\partial t} - \nabla\varphi$ is the gauged electric field.

Assuming that the phase is being absorbed, $\alpha = -\phi$, the gauge fields can be written in terms of the phase derivatives:

$$\vec{A'} = \vec{A} - \frac{\hbar}{e^*}\nabla\phi; \quad \varphi' = \varphi + \frac{\hbar}{e^*}\dot{\phi}, \tag{4.25}$$

thus, the superconducting contribution to the action becomes:

$$\begin{aligned} S_{sc} &= \frac{\rho}{2}\int dtd^3x \left\{\frac{1}{c^{*2}}\left[\dot{\phi} + \frac{e^*}{\hbar}\left(\varphi' - \frac{\hbar}{e^*}\dot{\phi}\right)\right]^2 - \left[\nabla\phi - \frac{e^*}{\hbar}\left(\vec{A'} + \frac{\hbar}{e^*}\nabla\phi\right)\right]\right\}, \\ \Rightarrow S_{sc} &= \frac{\rho}{2}\int dtd^3x\left[\frac{1}{c^{*2}}\left(\frac{e^*}{\hbar}\varphi\right)^2 - \left(\frac{e^*}{\hbar}\vec{A}\right)^2\right]. \end{aligned} \tag{4.26}$$

Given the fact that $\rho = \dfrac{\hbar^2 n_s}{m}$, the action can be rewritten as:

$$S = \frac{1}{2}\frac{\rho e^{*2}}{\hbar^2}\int dtd^3x\left[\left(\frac{\varphi}{c^*}\right)^2 - (\vec{A})^2\right] + S_{EM} = \frac{1}{2}\frac{\hbar^2 n_s}{m}\frac{e^{*2}}{\hbar^2}\int dtd^3x\left[\left(\frac{\varphi}{c^*}\right)^2 - (\vec{A})^2\right] + S_{EM}, \tag{4.27}$$

which implies in the new action where the order parameter phase has been absorbed by the gauge fields:

$$S = \int dtd^3x \left\{ \frac{1}{2}\frac{n_s e^{*2}}{m}\left[\left(\frac{\varphi}{c^*}\right)^2 - (\vec{A})^2\right] + \frac{1}{2\mu_0}\left[\left(\frac{E}{c}\right)^2 - B^2\right]\right\}. \tag{4.28}$$

Now, the action has become completely electromagnetic and independent of the phase gradient. The phase degrees of liberty have been absorbed by the gauge fields, giving rise to a new action where their contribution carries a mass coefficient: the gauge symmetry has been broken. One can understand this absorption as the photon corresponding to the field carrying the stifness associated with the phase gradient: it is slower. The absorption of phase fluctuations by gauge fields, turning them into massive fields is the core of the well-known Anderson-Higgs mechanism [17].

4.3.1 Deriving the dispersion relation of a photon

Now, having assembled the action for the fields in the superconductor, the next step is to derive the dispertion relation of a photon associated with the gauge field in order to observe the appearence of a mass term in the propagation energy for this photon, since the dispersion relation for a massive particle must present a rest mass. The idea is to arrive at a wave equation from the action. Solving it must result in an energy carrying a mass term independent of its momentum.

The superconducting contribution to the Lagrangian density, from Eq. 4.28, is given by:

$$\mathcal{L}_{sc} = \frac{1}{2}\frac{n_s e^{*2}}{m}\left[\left(\frac{\varphi}{c^*}\right)^2 - (\vec{A})^2\right], \tag{4.29}$$

therefore, the variation of the action can be taken:

$$\delta S_{sc} = \frac{1}{2}\frac{n_s e^{*2}}{m}\int dtd^3x \frac{\partial \mathcal{L}_{sc}}{\partial \varphi}\delta\varphi + \frac{\partial \mathcal{L}_{sc}}{\partial \vec{A}}\delta\vec{A} = \frac{n_s e^{*2}}{m}\int \left[\frac{\varphi}{c^{*2}}\delta\varphi - \vec{A}\delta\vec{A}\right]. \tag{4.30}$$

The same can be done for the electromagnetic action:

$$\delta S_{EM} = \int dtd^3x \frac{\partial \mathcal{L}_{EM}}{\partial \varphi}\delta\varphi + \frac{\partial \mathcal{L}_{EM}}{\partial \vec{A}}\delta\vec{A} = \frac{1}{\mu_0}\int dtd^3x \left[\left(-\nabla\times\vec{B} + \frac{1}{c^2}\dot{\vec{E}}\right)\delta\vec{A} + \frac{\nabla\cdot\vec{E}}{c^2}\delta\varphi\right]. \tag{4.31}$$

Hence, the total action variation $\delta S = \delta S_{sc} + \delta S_{EM}$ is expressed as:

$$\delta S = \int dt d^3x \frac{n_s e^{*2}}{m}\left[\frac{\varphi}{c^{*2}}\delta\varphi - \vec{A}\delta\vec{A}\right] + \frac{1}{\mu_0}\left[\left(-\nabla\times\vec{B} + \frac{1}{c^2}\dot{\vec{E}}\right)\delta\vec{A} + \frac{\nabla\cdot\vec{E}}{c^2}\delta\varphi\right]. \quad (4.32)$$

From Eq. 4.20, both Gauss's and Ampère's laws can be derived. First, by taking variations of the action with respect to the electric potential:

$$\frac{\delta S}{\delta\varphi} = \frac{n_s e^{*2}}{mc^{*2}}\varphi + \frac{1}{\mu_0}\frac{\nabla\cdot\vec{E}}{c^2} = 0, \quad (4.33)$$

and by isolating the divergence of the electric field:

$$\nabla\cdot\vec{E} = -\frac{c^2\mu_0 n_s e^{*2}}{mc^{*2}}\varphi = -\frac{n_s e^{*2}}{mc^{*2}\varepsilon_0}\varphi, \quad (4.34)$$

the celebrated Gauss's Law is achieved. A dimensional analysis shows that $\frac{n_s e^{*2}}{mc^{*2}}\varphi$ corresponds to the charge density of the system and thus is relabeled as ρ_c. Now, taking variations with respect to $\vec{A}$ results in:

$$\frac{\delta S}{\delta\vec{A}} = -\frac{n_s e^{*2}}{m}\vec{A} + \frac{1}{\mu_0 c^2}\dot{\vec{E}} - \frac{\nabla\times\vec{B}}{\mu_0} = -\frac{n_s e^{*2}}{m}\vec{A} + \frac{1}{\mu_0}\left(\frac{1}{c^2}\dot{\vec{E}} - \nabla\times\vec{B}\right). \quad (4.35)$$

Again, a dimensional analysis shows that $\frac{n_s e^{*2}}{m}\vec{A}$ corresponds to the current, hence it is labeled $\vec{J}$. Ampère's equation is finally obtained:

$$\vec{J} + \frac{1}{\mu_0}\left(\frac{1}{c^2}\dot{\vec{E}} - \nabla\times\vec{B}\right) = 0. \quad (4.36)$$

Finally, a continuity equation is obtained by taking the divergence of Eq. 4.36:

$$\nabla\cdot\vec{J} + \frac{1}{\mu_0}\nabla\cdot\left(\frac{1}{c^2}\dot{\vec{E}} - \nabla\times\vec{B}\right) = \nabla\cdot\vec{J} + \frac{1}{\mu_0 c^2}\frac{\partial}{\partial t}(\nabla\cdot\vec{E}). \quad (4.37)$$

From Equation 4.34:

$$\nabla\cdot\vec{J} + \varepsilon_0\frac{\partial}{\partial t}\left(\frac{\rho_c}{\varepsilon_0}\right) = \nabla\cdot\vec{J} + \frac{\partial\rho_c}{\partial t} = 0, \quad (4.38)$$

therefore:

$$-\frac{n_s e^{*2}}{m}\left(\nabla\cdot\vec{A} + \frac{1}{c^{*2}}\frac{\partial\varphi}{\partial t}\right) = 0. \quad (4.39)$$

Solving Equation 4.39 for $\frac{\partial \varphi}{\partial t}$ yields:

$$\frac{\partial \varphi}{\partial t} = -c^{*2}\nabla \cdot \vec{A}. \tag{4.40}$$

Now, rewriting Ampère's equation in terms of the vector and scalar potentials:

$$\nabla \times \vec{B} = \nabla \times \left(\nabla \times \vec{A}\right) = \nabla\left(\nabla \cdot \vec{A}\right) - \nabla^2 \vec{A} = \mu_0 \vec{J} + \frac{1}{c^2}\frac{\partial}{\partial t}\left(-\frac{\partial \vec{A}}{\partial t} - \nabla \varphi\right). \tag{4.41}$$

So that:

$$\nabla\left(\nabla \cdot \vec{A}\right) - \nabla^2 \vec{A} = -\frac{\mu_0 n_s e^{*2}}{m}\vec{A} - \frac{1}{c^2}\frac{\partial^2 \vec{A}}{\partial t^2} + \frac{c^{*2}}{c^2}\nabla\left(\nabla \cdot \vec{A}\right), \tag{4.42}$$

which in fact can be simplifyed to a diferential equation corresponding to the wave eqution for a gauge field $\vec{A}$:

$$\left[1 - \left(\frac{c^*}{c}\right)^2\right]\nabla\left(\nabla \cdot \vec{A}\right) = \left(\nabla^2 - \frac{1}{c^2}\frac{\partial^2}{\partial t^2} - \frac{\mu_0 n_s e^{*2}}{m}\right)\vec{A}. \tag{4.43}$$

This wave equation can be solved for $\vec{A}$ by assuming the vector potential as a plane wave: $\vec{A} = A_0 e^{i(\vec{p}\cdot\vec{x}-Et)/\hbar}\hat{e}$, where A_0 is a constant and $\hat{e}$ the propagation direction. Two scenarios are possible: the wave is either longitudinal ($\vec{p} \parallel \hat{e}$) or transverse ($\vec{p} \perp \hat{e}$). The latter implies that $\nabla \cdot \vec{A} = 0$, so that:

$$\begin{aligned}
&\nabla^2\left(A_0 e^{i(\vec{p}\cdot\vec{x}-Et)/\hbar}\hat{e}\right) - \frac{1}{c^2}\frac{\partial^2}{\partial t^2}\left(A_0 e^{i(\vec{p}\cdot\vec{x}-Et)/\hbar}\hat{e}\right) - \frac{\mu_0 n_s e^{*2}}{m}\left(A_0 e^{i(\vec{p}\cdot\vec{x}-Et)/\hbar}\hat{e}\right) = 0.\\
&\Rightarrow -A_0 e^{i(\vec{p}\cdot\vec{x}-Et)/\hbar}\left(\frac{p}{\hbar}\right)^2 + \frac{1}{c^2}A_0 e^{i(\vec{p}\cdot\vec{x}-Et)/\hbar}\left(\frac{E}{\hbar}\right)^2 - \frac{\mu_0 n_s e^{*2}}{m}\left(A_0 e^{i(\vec{p}\cdot\vec{x}-Et)/\hbar}\hat{e}\right) = 0,
\end{aligned} \tag{4.44}$$

which will result in:

$$\frac{1}{c^2}\frac{E^2}{\hbar^2} = \frac{p^2}{\hbar^2} + \frac{\mu_0 n_s e^{*2}}{m} \Rightarrow E^2 = (cp)^2 + \frac{\mu_0 n_s e^{*2}}{m}(c\hbar)^2. \tag{4.45}$$

At last, the dispersion relation for the transverse mode can be expressed as:

$$E_t = \left[(cp)^2 + \frac{\mu_0 n_s e^{*2}}{m}(c\hbar)^2\right]^{1/2}. \tag{4.46}$$

Now, the dispersion relation for the longitudinal case shall be calculated. In this case, $\nabla \cdot \vec{A} \neq 0$ and the right-hand side of Eq. 4.43 thus becomes:

$$\left[1+\frac{c^{*2}}{c^2}\right]\nabla\left(\nabla\cdot\left(A_0 e^{i(\vec{p}\cdot\vec{x}-Et)/\hbar}\hat{e}\right)\right)=-\left[1+\frac{c^{*2}}{c^2}\right]A_0 e^{i(\vec{p}\cdot\vec{x}-Et)/\hbar}\left(\frac{p}{\hbar}\right)^2. \tag{4.47}$$

Hence, substituting Eq. 4.47 into Eq. 4.43 yields:

$$-\left[1-\frac{c^{*2}}{c^2}\right]\left(\frac{p}{\hbar}\right)^2=\frac{1}{c^2}\frac{E^2}{\hbar^2}-\left(\frac{p}{\hbar}\right)^2-\frac{\mu_0 n_s e^{*2}}{m}, \tag{4.48}$$

then:

$$-\left(\frac{p}{\hbar}\right)^2+\frac{c^{*2}}{c^2}\left(\frac{p}{\hbar}\right)^2=\frac{1}{c^2}\frac{E^2}{\hbar^2}-\left(\frac{p}{\hbar}\right)^2-\frac{\mu_0 n_s e^{*2}}{m}, \tag{4.49}$$

which leads to:

$$\frac{1}{c^2}\frac{E^2}{\hbar^2}=\frac{c^{*2}}{c^2}\left(\frac{p}{\hbar}\right)^2+\frac{\mu_0 n_s e^{*2}}{m}\Rightarrow E^2=(c^*p)^2+\frac{\mu_0 n_s e^{*2}}{m}(c\hbar)^2. \tag{4.50}$$

Hence, the dispersion relation has been finally derived for the longitundinal case:

$$E_l=\left[(c^*p)^2+\frac{\mu_0 n_s e^{*2}}{m}(c\hbar)^2\right]^{1/2}. \tag{4.51}$$

Both energies then were acquired and are now compiled:

$$\begin{cases} E_t(p)=\left[(cp)^2+(c^2m_a)^2\right]^{1/2} \\ E_l(p)=\left[(c^*p)^2+(c^2m_a)^2\right]^{1/2}, \end{cases} \tag{4.52}$$

where $m_a^2=\frac{\mu_0 n_s e^{*2}}{mc^2}\hbar^2$, corresponding to the mass acquired by the photon when the order parameter phase is absorbed. the dispersion relation imparts a new energy, associated with a mass m_a. The photon has now acquired a resistance to movement, since more energy must be spent for it to propagate. Two dispersion relations were calculated: a longitudinal and a transverse one; the first propagating with lightspeed c and the latter with the characteristic velocity of the system c^*.

This result where the study of the superconductor phase transition led to the observation of Anderson-Higgs mechanism has been now understood. The absorption of the order parameter phase into the gauge fields, making them behave slower, is in fact a result of the Anderson-Higgs mechanism [17, 21].

5 Conclusions and perspectives

It is possible to conclude, from this revision work, that at low temperatures matter behaves in exotic ways, such as the achievement of superconductivity, where phase transitions take place. Phase transitions can be explored experimentally in the Solid State Physics Laboratory, where the exotic behaviour of phases at low temperatures are explored. Landau theory is responsible for describing phase transitions, where a free energy is expanded in terms of an order parameter. A more complete theory for phase transitions, more specifically for the superconductor transition, is given by Ginzburg-Landau theory, where space fluctuations of the order parameter are accounted for. This theory can be used to observe the appearence of a massive mode of vibration, the so-called Higgs mode, in the superconductor transition. An introduction to the Goldstone theorem was given, while applying it to the Ginzburg-Landau potential, resulting in the appearence of a massless mode of vibration along the phase direction of the order parameter, meaning that the phase oscillations are not originally massive. However, when the electromagnetic gauge field couples to the order parameter phase the appearence of a mass term was seen to occur, granting it a slow and weak behaviour. This phenomenon is known as the Anderson-Higgs mechanism.

Although this work is a revision one, it served as a way to understand the concepts above discussed in order to be able to explore them more deeply in future projects. The main goal which was the understanding of the appearence of a massive mode in the superconducting phase was fulfilled in Chapter 4.

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