A 1−Laplacian equation with critical and discontinuous nonlinearities

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Abstract

In this work, a quasilinear elliptic problem involving the 1−Laplacian operator and a discontinuous nonlinearity with a critical Sobolev growth is studied. Through an argument based on an approximation method, an existence result is proved. In order to overcome the lack of compactness, we apply the well known concentration-compactness technique.

Keywords: 1−Laplacian operator, discontinuous nonlinearities, critical growth

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1 Introduction

The study of nonlinear partial differential equations with discontinuous nonlinearities has attracted the attention of several researchers in the last years. Among the applications, we could cite free boundary problems arising in mathematical physics, the obstacle problem, the seepage surface problem, and the Elenbaas equation, see [21, 23, 22]. Several techniques have been applied to study this kind of problem, such as variational methods for nondifferentiable functionals, lower and upper solutions, global branching, etc. See for instance [1, 3, 4, 6, 7, 14, 16, 20, 21, 22, 23, 30].

As far as problems involving the 1−Laplacian operator are concerned, among the first works we could cite [11, 10, 12, 13], as well as [29] and [34]. Up to our knowledge, there are two main approaches one can follow. The first one is based on the study of the energy functional associated to the problem, which is defined in $BV(\Omega)$, whenever one can write it as the difference of a convex and locally Lipschitz functional and a C^1 one. Also, there is another approach which is based on the study of the problem involving the 1−Laplacian operator, through the approximation of it by substituting the 1−Laplacian by the p-Laplacian operator, for $p > 1$. Then, one can use standard arguments to solve the associated problem and then study the family of such solutions as $p \to 1^+$.

In this work, we study the problem

$$\begin{cases} -\Delta_1 u = \lambda H(u-\beta)|u|^{q-2}u + |u|^{1^*-2}u \text{ in } \Omega \\ u = 0 \text{ on } \partial\Omega, \end{cases} \tag{1.1}$$

where $\Delta_1 u = \operatorname{div}\left(\frac{Du}{|Du|}\right)$, $\Omega \subset \mathbb{R}^N$ is a smooth bounded domain, $\lambda, \beta > 0$, $1 < q < \frac{N}{N-1} = 1^*$ and H is the Heaviside function given by

$$H(t) = \begin{cases} 1, & \text{if } t \geq 0 \\ 0, & \text{if } t < 0. \end{cases}$$

We obtain a nontrivial solution of (1.1) when the parameter λ is sufficiently large and β is sufficiently small. We look for solutions in the space of functions of bounded variation $BV(\Omega)$, defined as

$$BV(\Omega) = \{u \in L^1(\Omega) : Du \in \mathcal{M}(\Omega, \mathbb{R}^N)\}.$$

Our main result is the following.

Theorem 1.1 *There exists $\lambda_0, \beta_0 > 0$ such that* (1.1) *admits a nonnegative nontrivial solution $u_{\beta,\lambda} \in BV(\Omega) \cap L^\infty(\Omega)$, for all $\lambda > \lambda_0$ and $0 < \beta < \beta_0$. Furthermore, for each $\lambda > \lambda_0$, there exists $\beta_\lambda > 0$ such that*

$$|\{u_{\beta,\lambda} > \beta\}| > 0 \text{ for all } 0 < \beta < \beta_\lambda, \tag{1.2}$$

where $|A|$ denotes the Lebesgue measure of the measurable set $A \subset \mathbb{R}^N$.

It is worth to highlight the recent paper [39], in which the authors work with a similar problem, but without the critical term.

Inequality (1.2) guarantees that the set $\{u_{\beta,\lambda} > \beta\}$ has positive measure when β is sufficiently small. This information ensures that we are obtaining a solution of a problem with a truly discontinuous nonlinearity.

Theorem 1.1 should be compared with the results of [2], [5], [19] and [31]. In [5] and [19] the problem

$$\begin{cases} -\Delta u = \lambda u^{q-1} + u^{2^*-1} \text{ in } \Omega \\ u > 0 \text{ in } \Omega \\ u = 0 \text{ on } \partial\Omega. \end{cases} \tag{1.3}$$

was studied for different values of $q > 1$. Here, $2^* = \frac{2N}{N-2}$ and $-\Delta$ denotes the usual laplacian. In [5] the authors showed that if $1 < q < 2$, then problem (1.3) admits solutions only for small values of λ. In [19], the authors proved, among other results, that problem (1.3) with $N \geq 4$ and $2 < q < 2^*$ admits a solution for all $\lambda > 0$. In [31] the authors considered the problem

$$\begin{cases} -\Delta_p u = \lambda |u|^{q-2}u + |u|^{p^*-2}u \text{ in } \Omega \\ u = 0 \text{ on } \partial\Omega, \end{cases} \tag{1.4}$$

where $-\Delta_p$ denotes the p-laplacian and $p^* = \frac{Np}{N-p}$. The authors showed that if $p < q < p^*$, then there exists $\lambda_0 > 0$ such that problem (1.4) admits a nontrivial solution for all $\lambda > \lambda_0$. In recent years, problems involving the 1-laplacian operator have been studied, and is natural to ask whether the results of [31] can be extended to the 1-laplacian context or not. Recently, in [2], the authors showed that the problem

$$\begin{cases} -\Delta_1 u = \lambda |u|^{q-2}u + |u|^{1^*-2}u \text{ in } \Omega \\ u = 0 \text{ on } \partial\Omega, \end{cases} \tag{1.5}$$

with $1 < q < \frac{N}{N-1}$ admits many solutions for large values of λ. They tackled the problem directly in the space $BV(\Omega)$. In this work, we study problem (1.1) and apply a different method: our solution is obtained as the limit of a certain sequence of solutions $(u_p)_{p>1}$ in $W_0^{1,p}(\Omega)$ of approximated problems. To do that, the concentration-compactness principle due to Lions (see [36]) takes an important role. More precisely, we prove a version of this principle in the context of $BV(\Omega)$. This version is stated without a full proof in [29, Proposition 3]. It is also worth mentioning that our solution is nonnegative and that due to the presence of the Heavyside function in (1.1), we use techniques from nonsmooth analysis.

Our first step is to study the auxiliary problem

$$\begin{cases} -\Delta_p u = \lambda H(u-\beta)|u|^{q-2}u + |u|^{1^*-2}u \text{ in } \Omega \\ u = 0 \text{ on } \partial\Omega, \end{cases} \tag{1.6}$$

with $p > 1$. We say that $u_{p,\beta,\lambda} \in W_0^{1,p}(\Omega)$ is a solution of (1.6) if there exists $\rho_{p,\beta,\lambda} \in L^{\frac{q}{q-1}}(\Omega)$ such that, for all $\varphi \in W_0^{1,p}(\Omega)$,

$$\int_\Omega |\nabla u_{p,\beta,\lambda}|^{p-2}\nabla u_{p,\beta,\lambda}\nabla\varphi dx = \lambda\int_\Omega \rho_{p,\beta,\lambda}\varphi\, dx + \int_\Omega |u_{p,\beta,\lambda}|^{1^*-2}u_{p,\beta,\lambda}\varphi\, dx, \tag{1.7}$$

where, for a.e. $x \in \Omega$, $\rho_{p,\beta,\lambda}$ satisfies

$$\rho_{p,\beta,\lambda}(x) \in \begin{cases} \{0\} & \text{if } u_{p,\beta,\lambda}(x) < \beta, \\ [0, \beta^{q-1}] & \text{if } u_{p,\beta,\lambda}(x) = \beta, \\ \{u_{p,\beta}(x)^{q-1}\} & \text{if } u_{p,\beta,\lambda}(x) > \beta. \end{cases} \tag{1.8}$$

We proceed in the following way. We first use a non-smooth version of the Mountain Pass Theorem to obtain $\lambda_0, \beta_0 > 0$ which do not depend on p, such that (1.6) is solvable for $\lambda > \lambda_0$ and $0 < \beta < \beta_0$. Next, we show that the sequence of solutions $\{u_{p,\beta,\lambda}\}_{p>1}$ of (1.6) converges to a nontrivial solution of (1.1) as $p \to 1^+$.

Throughout this work, in order to find nonnegative solutions of (1.6), we consider the functional $J_{p,\beta,\lambda} : W_0^{1,p}(\Omega) \to \mathbb{R}$ given by

$$J_{p,\beta,\lambda}(u) = Q_p(u) - \lambda \mathcal{F}_\beta(u) - \frac{1}{1^*} \int_\Omega (u^+)^{1^*}\, dx, \tag{1.9}$$

where $u^\pm = \max\{\pm u, 0\}$,

$$Q_p(u) = \frac{1}{p} \int_\Omega |\nabla u|^p\, dx$$

and

$$\mathcal{F}_\beta(u) = \int_\Omega F_\beta(u)\, dx, \text{ with } F_\beta(t) = \int_0^t f_\beta(s)\, ds \text{ and } f_\beta(s) = H(s-\beta)|s|^{q-2}s.$$

Due to the Heavyside function H, the functional $J_{p,\beta,\lambda}$ is not Fréchet differentiable, but locally Lipschitz continuous. So the nonsmooth critical point theory developed by Chang applies. This work is oganized as follows:

In Section 2 we give some notions and we prove preliminary results.

In Section 3, we state and prove a version of the Concentration-compactness Theorem.

In Section 4 we study the approximated problem (1.6).

In Section 5 we show that the solutions obtained in Section 4 converge in $L^{1^*}(\Omega)$.

In Section 6 we prove Theorem 1.1.

2 Preliminaries

Inspired by the Anzelotti theory, see [15, 24], we give some notations and definitions. Let us denote

$$\mathcal{X}_N(\Omega) = \{\mathbf{z} \in L^\infty(\Omega; \mathbb{R}^N) : \|\mathbf{z}\|_\infty \le 1, \operatorname{div} \mathbf{z} \in L^N(\Omega)\}.$$

For $\mathbf{z} \in \mathcal{X}_N(\Omega)$ and $w \in BV(\Omega)$, we define the distribution $(\mathbf{z}, Dw) \in \mathcal{D}'(\Omega)$ as

$$\langle (\mathbf{z}, Dw), \varphi \rangle = -\int_\Omega w\varphi \mathrm{div}(\mathbf{z})\, dx - \int_\Omega w\mathbf{z} \cdot \nabla\varphi\, dx, \tag{2.1}$$

for every $\varphi \in C_0^\infty(\Omega)$. Furthermore, the following identity, known as the Green's formula, holds

$$\int_\Omega w\, \mathrm{div}(\mathbf{z})\, dx + \int_\Omega (\mathbf{z}, Dw) = \int_{\partial\Omega} [\mathbf{z}, \nu] w\, d\mathcal{H}^{N-1} \text{ for all } w \in BV(\Omega), z \in \mathcal{X}_N(\Omega), \tag{2.2}$$

where $[\cdot, \nu] : \mathcal{X}_N(\Omega) \to L^\infty(\partial\Omega)$ denotes the trace operator and is such that

$$\| [\mathbf{z}, \nu] \|_{L^\infty(\partial\Omega)} \leq \|\mathbf{z}\|_\infty \tag{2.3}$$

and, if $\mathbf{z} \in C^1(\overline{\Omega}_\delta, \mathbb{R}^N)$,

$$[\mathbf{z}, \nu](x) = \mathbf{z}(x) \cdot \nu(x) \quad \text{on } \Omega_\delta,$$

where by Ω_δ we denote a δ-neighborhood of $\partial\Omega$.

We now state what it means to be a solution of problem (1.1).

Definition 2.1 *We say that $u \in BV(\Omega)$ is a solution of (1.1) if there exists $\rho \in L^{\frac{q}{q-1}}(\Omega)$ and $\mathbf{z} \in \mathcal{X}_N(\Omega)$ with $\|\mathbf{z}\|_\infty \leq 1$ such that*

$$\begin{cases} \int_\Omega \mathbf{z} \cdot \nabla\varphi dx = \lambda \int_\Omega \rho\varphi dx + \int_\Omega |u|^{1^*-2} u\varphi dx \textit{ for all } \varphi \in C_0^\infty(\Omega), \\ (\mathbf{z}, Du) = |Du| \textit{ in } \mathcal{M}(\Omega), \\ [z, \nu] \in sgn(-u) \quad \mathcal{H}^{N-1} - a.e \textit{ on } \partial\Omega, \end{cases}$$

and

$$\rho(x) \in \begin{cases} \{0\} \textit{ if } u(x) < \beta, \\ [0, \beta^{q-1}] \textit{ if } u(x) = \beta, \\ \{u(x)^{q-1}\} \textit{ if } u(x) > \beta, \end{cases}$$

for almost every $x \in \Omega$.

We remark that in a first sight, Definition 2.1 only allows us to take test functions in $C_0^\infty(\Omega)$. However, as a consequence of Anzelotti's theory, we may choose any $w \in BV(\Omega)$ as a test function. More precisely, if $u \in BV(\Omega)$ is a solution in the sense of Definition 2.1, then (from Green's formula)

$$\int_\Omega (\mathbf{z}, Dw) = \lambda \int_\Omega \rho w dx + \int_\Omega |u|^{1^*-2} uw dx + \int_{\partial\Omega} w[\mathbf{z}, \nu]\, d\mathcal{H}^{N-1}, \text{ for all } w \in BV(\Omega). \tag{2.4}$$

We now give some elements of non-smooth analysis. Let E be a real Banach space. We say that a functional $I : E \to \mathbb{R}$ is locally Lipschitz continuous if for each $u \in E$ there exists a open neighborhood V of u and a constant $M > 0$ (which may depend on u) such that

$$|I(v) - I(w)| \leq M\|v - w\|_E \text{ for all } v, w \in V.$$

We now introduce the notion of non-smooth Palais-Smale condition.

Definition 2.2 *Let E be a Banach space and $I : E \to \mathbb{R}$ be a locally Lipschitz continuous function. We say that I satisfies the non-smooth Palais-Smale condition if every sequence (u_n) in E such that $I(u_n) \to c$ and $\Lambda_I(u_n) \to 0$ possesses a strongly convergent subsequence. Here $\Lambda_I(u)$ denotes the number*

$$\Lambda_I(u) = \min\{\|\mu\|_{E^*} : \mu \in \partial I(u)\},$$

where

$$\partial I(u) = \{\mu \in E^* : \langle \mu, v\rangle \le I^0(u; v), v \in E\},$$

with $\langle \cdot, \cdot\rangle$ denoting de duality pairing between E^* and E, and

$$I^0(u; v) = \lim_{h\to 0} \lim_{\sigma\to 0^+} \frac{I(u + h + \sigma v) - I(u + h)}{\sigma}.$$

We further have (see [25, 26, 33].)

Proposition 2.3 *Let E be a Banach space and $I_1, I_2 : E \to \mathbb{R}$ be locally Lipschitz functions. The following assertions hold.*

(i) $I_1 + I_2 \in Lip_{loc}(E; \mathbb{R})$ and $\partial(I_1 + I_2)(u) \subset \partial I_1(u) + \partial I_2(u)$ for all $u \in E$;

(ii) $\partial(\lambda I_1)(u) = \lambda \partial I_1(u)$ for all $\lambda \in \mathbb{R}$ and $u \in E$;

(iii) Suppose that for each v in a neighborhood of u in E, I_1 admits a Gateux derivative $I_1'(v)$ and that $I_1' : E \to E^*$ is continuous. Then $\partial I_1(u) = \{I_1'(u)\}$.

We say that $u \in E$ is a critical point of I if $0 \in \partial I(u)$, and we say that $c \in \mathbb{R}$ is a critical value of I if there exists a critical point $u \in E$ of I such that $I(u) = c$. We now give some properties of the functional $J_{p,\beta,\lambda} : W_0^{1,p}(\Omega) \to \mathbb{R}$ given by (1.9). This functional is locally Lipschitz continuous and Proposition 2.3 implies that

$$\partial J_{p,\beta,\lambda}(u) \subset \{Q_p'(u)\} - \lambda \partial \mathcal{F}_\beta(u) - \{Q_*'(u)\} \text{ for all } u \in W_0^{1,p}(\Omega),$$

where

$$Q_*(u) = \frac{1}{1^*} \int_\Omega (u^+)^{1^*}\, dx.$$

Furthermore,

$$\partial \mathcal{F}_\beta(u) = \begin{cases} \{0\} \text{ if } u < \beta, \\ [0, \beta^{q-1}] \text{ if } u = \beta, \\ \{u^{q-1}\} \text{ if } u > \beta. \end{cases} \tag{2.5}$$

Our first result is the following. Its proof follows from standard arguments, but we present it here for the sake of completeness.

Lemma 2.4 *The functional $J_{p,\beta,\lambda}$ satisfies the non-smooth Palais-Smale condition.*

Proof. For simplicity, we denote the functional $J_{p,\beta,\lambda}$ merely by $J_{p,\beta}$. Let $(u_n) \subset W_0^{1,p}(\Omega)$ be a Palais-Smale sequence for $J_{p,\beta}$, that is, $J_{p,\beta}(u_n) \to c$ and $\Lambda_{J_{p,\beta}}(u_n) \to 0$. Hence, we know that there exists $\mu_n \in \partial J_{p,\beta}(u_n)$ such that $\|\mu_n\|_{(W_0^{1,p})^*} \to 0$. Consequently, there exists $\rho_n \in \partial \mathcal{F}_\beta(u_n)$ such that

$$\|Q'_p(u_n) - \lambda\rho_n - Q'_*(u_n)\|_{(W_0^{1,p})^*} \to 0.$$

From (2.5) we get

$$\rho_n \in \begin{cases} \{0\} \text{ in } u_n < \beta \\ [0, \beta^{q-1}] \text{ in } u_n = \beta \\ \{u_n^{q-1}\} \text{ in } u_n > \beta. \end{cases}$$

Consequently, $\rho_n \in L^{q/q-1}(\Omega)$ and

$$\sup_{\|v\|_{W_0^{1,p}(\Omega)} \le 1} \left| \int_\Omega |\nabla u_n|^{p-2} \nabla u_n \nabla v \, dx - \lambda \int_\Omega \rho_n v - \int_\Omega (u_n^+)^{1^*-1} v \, dx \right| \to 0.$$

That is, there exists a sequence (τ_n) in $(0,1)$ with $\tau_n \to 0$ such that

$$\left| \int_\Omega |\nabla u_n|^{p-2} \nabla u_n \nabla v \, dx - \lambda \int_\Omega \rho_n v - \int_\Omega (u_n^+)^{1^*-1} v \, dx \right| \le \tau_n \|v\|_{W_0^{1,p}(\Omega)} \text{ for all } v \in W_0^{1,p}(\Omega).$$

Taking $v = u_n$, we get

$$\left| \int_\Omega |\nabla u_n|^p \, dx - \lambda \int_\Omega \rho_n u_n - \int_\Omega (u_n^+)^{1^*} \, dx \right| \le \tau_n \|u_n\|_{W_0^{1,p}(\Omega)}.$$

Consequently,

$$\frac{\tau_n}{q} \|u_n\|_{W_0^{1,p}(\Omega)} \ge \frac{\lambda}{q} \int_\Omega \rho_n u_n + \frac{1}{q} \int_\Omega (u_n^+)^{1^*} \, dx - \frac{1}{q} \int_\Omega |\nabla u_n|^p \, dx.$$

Furthermore, $J_{p,\beta}(u_n) \to c$ and then

$$1 + c + \frac{\tau_n}{q} \|u_n\|_{W_0^{1,p}(\Omega)} \ge J_{p,\beta}(u_n) + \frac{\lambda}{q} \int_\Omega \rho_n u_n + \frac{1}{q} \int_\Omega (u_n^+)^{1^*} \, dx - \frac{1}{q} \int_\Omega |\nabla u_n|^p \, dx.$$

Hence,

$$1 + c + \frac{\tau_n}{q} \|u_n\|_{W_0^{1,p}(\Omega)} \ge \left(\frac{1}{p} - \frac{1}{q}\right) \int_\Omega |\nabla u_n|^p \, dx + \lambda \left(\frac{1}{q} \int_\Omega \rho_n u_n - \int_\Omega F_\beta(u_n) \, dx \right) + \left(\frac{1}{q} - \frac{1}{1^*}\right) \int_\Omega (u_n^+)^{1^*} \, dx.$$

We claim that

$$\left(\frac{1}{q} \int_\Omega \rho_n u_n - \int_\Omega F_\beta(u_n) \, dx \right) \ge 0. \tag{2.6}$$

Indeed,

$$\left(\frac{1}{q} \int_\Omega \rho_n u_n - \int_\Omega F_\beta(u_n) \, dx \right) = \frac{\beta}{q} \int_{\{u_n = \beta\}} \rho_n dx + \frac{\beta^q}{q} |\{u_n > \beta\}| \ge 0.$$

Hence,

$$1 + c + \frac{\tau_n}{q}\|u_n\|_{W_0^{1,p}(\Omega)} \geq \left(\frac{1}{p} - \frac{1}{q}\right)\int_\Omega |\nabla u_n|^p \, dx = \left(\frac{1}{p} - \frac{1}{q}\right)\|u_n\|^p_{W_0^{1,p}(\Omega)}$$

and (u_n) is bounded in $W_0^{1,p}(\Omega)$. Using a similar argument than [16, (proof of) Lemma 3.1], we conclude that $J_{p,\beta}$ satisfies the nonsmooth Palais-Smale condition. □

We recall that $u \in W_0^{1,p}(\Omega)$ is a critical point of the functional $J_{p,\beta,\lambda}$ if $0 \in \partial J_{p,\beta,\lambda}(u)$. We have

Lemma 2.5 *Critical points of the functional $J_{p,\beta,\lambda}$ are nonnegative weak solutions of problem* (1.6).

Proof. Indeed, if u is a critical point of the functional $J_{p,\beta,\lambda}$, then there must exist $\rho_{p,\beta} \in \partial \mathcal{F}_\beta(u)$ such that

$$\|Q'_p(u) - \lambda \rho_{p,\beta} - Q'_*(u)\|_{(W_0^{1,p})^*} = 0.$$

Consequently,

$$\int_\Omega |\nabla u|^{p-2}\nabla u \nabla v - \lambda \int_\Omega \rho_{p,\beta} v - \int_\Omega (u^+)^{1^*-1} v = 0 \text{ for all } v \in W_0^{1,p}(\Omega).$$

Choosing $v = u^-$, we get

$$-\int_\Omega |\nabla u^-|^p \, dx = 0.$$

Consequently, $u^- = 0$ in Ω and hence $u \geq 0$. Hence,

$$\int_\Omega |\nabla u|^{p-2}\nabla u \nabla v - \lambda \int_\Omega \rho_{p,\beta} v - \int_\Omega u^{1^*-1} v = 0 \text{ for all } v \in W_0^{1,p}(\Omega),$$

so that u satisfies (1.7). □

We now state a version of the nonsmoth Mountain-Pass Theorem. See [25, 26, 33].

Theorem 2.6 *Let E be a Banach space and $I \in Lip_{loc}(E; \mathbb{R})$ with $I(0) = 0$. Suppose that there are numbers $\alpha, r > 0$ and a function $e \in E$ such that*

(*i*) *$I(u) \geq \alpha$ for all $u \in E$ with $\|u\|_E = r$;*

(*ii*) *$I(e) < 0$ and $\|e\|_E > r$.*

Let

$$c = \inf_{\gamma \in \Gamma} \max_{t \in [0,1]} I(\gamma(t)) \text{ and } \Gamma = \{\gamma \in C([0,1]; E) : \gamma(0) = 0, \gamma(1) = e\}.$$

Then, $c \geq \alpha$ and there exists a sequence $(u_n) \subset E$ satisfying

$$I(u_n) \to c \text{ and } \Lambda_I(u_n) \to 0.$$

If, in addition, I satisfies the nonsmooth Palais-Smale condition, then c is a critical value of I.

We finish by stating a classical result found in [31, Lemma 2.1].

Lemma 2.7 *Suppose that there exists a constant $C > 0$ such that*

$$\left(\int_\Omega |\varphi|^{1^*} \, d\nu \right)^{\frac{1}{1^*}} \leq C \int_\Omega |\varphi| \, d\mu \text{ for all } \varphi \in C_0^\infty(\mathbb{R}^N).$$

Then, there exist $\{x_i\}_{i \in \mathbb{N}}$ in $\overline{\Omega}$ and and numbers $\nu_i \geq 0$ such that

$$\nu = \sum_{i \in \mathbb{N}} \nu_i \delta_{x_i}, \text{ and } \mu \geq C^{-1} \sum_{i \in \mathbb{N}} \nu_i^{\frac{1}{1^*}} \delta_{x_i}.$$

3 A concentration-compactness theorem

This section is devoted to the following result, which is a concentration-compactness principle in the context of $BV(\Omega)$. In [29, Proposition 3], the proof is given but without some important details.

Theorem 3.1 *Let Ω be a bounded open set in $\mathbb{R}^N$ and assume that $(u_p)_{p>1}$ is a sequence such that $u_p \in W_0^{1,p}(\Omega)$ for all $p > 1$ and that there exists a constant $C > 0$ such that*

$$\int_\Omega |\nabla u_p|^p \, dx \leq C \text{ for all } p > 1. \tag{3.1}$$

The following assertions hold.

(i) There exists $u \in BV(\Omega)$ such that $u_p \rightharpoonup u$ weakly in $BV(\Omega)$ as $p \to 1^+$;

(ii) There exist measures $\mu, \nu \in \mathcal{M}(\overline{\Omega})$ such that $|\nabla u_p|^p \rightharpoonup \mu$ and $|u_p|^{1^*} \rightharpoonup \nu$ weakly in $\mathcal{M}(\overline{\Omega})$ as $p \to 1^+$;

(iii) There exist a countable set $(x_i)_{i \in \mathbb{N}}$ of points in Ω and numbers $\nu_i \geq 0$ and $\mu_i \geq 0$ such that

$$\nu = |u|^{1^*} + \sum_i \nu_i \delta_{x_i},$$

and

$$\mu \geq |Du| + \sum_i \mu_i \delta_{x_i}.$$

Moreover,

$$\nu_i^{\frac{1}{1^*}} \leq \frac{\mu_i}{S} \text{ for all } n \in \mathbb{N},$$

where
$$S = \inf_{u \in W_0^{1,1}(\Omega), u \neq 0} \left\{ \frac{\int_\Omega |\nabla u|}{\left(\int_\Omega |u|^{1^*}\right)^{\frac{1}{1^*}}} \right\}. \tag{3.2}$$

Proof. Note that

$$\begin{aligned} \int_\Omega |\nabla u_p|\, dx &\leq \left(\int_\Omega |\nabla u_p|^p\, dx \right)^{\frac{1}{p}} |\Omega|^{1-\frac{1}{p}} \\ &\leq (1+C)(1+|\Omega|), \end{aligned}$$

for all $p > 1$. We conclude from the Sobolev embedding that there exists $u \in L^1(\Omega)$ such that $u_p \to u$ in $L^1(\Omega)$ as $p \to 1^+$. Moreover, from [17, Proposition 10.1.1], we obtain that $u \in BV(\Omega)$ and $u_p \rightharpoonup u$ weakly in $BV(\Omega)$. This proves (i). For (ii), observe that by (3.2), for all $p > 1$,

$$\left(\int_\Omega |u_p|^{1^*} \right)^{\frac{1}{1^*}} \leq \frac{1}{S} \int_\Omega |\nabla u_p|\, dx \leq \frac{C_2}{S}, \tag{3.3}$$

where
$$C_2 = (1+C)(1+|\Omega|).$$

Item (ii) then follows from (3.1), (3.3) and [18, Comments on Chapter 4]

Before proving (iii), we state and prove the following technical result.

Lemma 3.2 *Let $a \geq 0$, $(a_n)_{n\in\mathbb{N}}$ and $(b_n)_{n\in\mathbb{N}}$ be sequences of positive real numbers such that $a_n \to a$ and $b_n \to 1$ as $n \to \infty$. Then $a_n^{b_n} \to a$ as $n \to \infty$.*

Proof. It follows from the continuity of $f : (0, +\infty) \times (0, +\infty) \to \mathbb{R}$, given by $f(x, y) = e^{y \log x}$.

□

We proceed now to prove (iii). For all $p > 1$, let $v_p = u_p - u$. We know that $v_p \rightharpoonup 0$ weakly in $BV(\Omega)$. Furthermore,

$$\int_\Omega |v_p|^{1^*}\, dx \leq \int_\Omega |u_p|^{1^*}\, dx + \int_\Omega |u|^{1^*}\, dx \leq C \text{ for all } p > 1.$$

Hence, there exists $\nu' \in \mathcal{M}(\Omega)$ such that

$$|v_p|^{1^*} \rightharpoonup \nu' \text{ weakly in } \mathcal{M}(\overline{\Omega}).$$

We also know that the total variation of ∇v_p is bounded in $\mathcal{M}(\overline{\Omega})$. We thus get $\mu' \in \mathcal{M}(\overline{\Omega})$ such that

$$|\nabla v_p| \rightharpoonup \mu' \text{ weakly in } \mathcal{M}(\overline{\Omega}).$$

The embedding $BV(\Omega) \hookrightarrow L^{1^*}(\Omega)$ yields that

$$\left(\int_\Omega |\varphi|^{1^*}|v_p|^{1^*}\,dx\right)^{\frac{1}{1^*}} S \leq \int_\Omega |\nabla(\varphi v_p)|\,dx + \int_\Omega |\varphi v_p| dx \text{ for all } \varphi \in C_0^\infty(\mathbb{R}^N).$$

Consequently,

$$\left(\int_\Omega |\varphi|^{1^*}|v_p|^{1^*}\,dx\right)^{\frac{1}{1^*}} S \leq \int_\Omega |v_p||\nabla\varphi|\,dx + \int_W |\varphi||\nabla v_p|\,dx + \int_\Omega |\varphi||v_p| dx \text{ for all } \varphi \in C_0^\infty(\mathbb{R}^N).$$

Letting $p \to 1^+$, we get (since $v_p \to 0$ in $L^r(\Omega)$ for $1 \leq r < 1^*$)

$$\left(\int_\Omega |\varphi|^{1^*}\,d\nu'\right)^{\frac{1}{1^*}} \leq \frac{1}{S}\int_\Omega |\varphi|\,d\mu' \text{ for all } \varphi \in C_0^\infty(\mathbb{R}^N).$$

From Lemma 2.7, we obtain a sequence of points $(x_i) \subset \overline{\Omega}$ such that

$$\nu' = \sum_{i\in\mathbb{N}} \nu_i \delta_{x_i}.$$

To proceed, we apply Brezis-Lieb to get

$$\int_\Omega |\varphi||u_p|^{1^*} dx - \int_\Omega |\varphi||v_p|^{1^*} dx \to \int_\Omega |\varphi||u|^{1^*} dx \text{ as } p \to 1, \text{ for all } \varphi \in C_0^\infty(\mathbb{R}^N).$$

On the other hand,

$$\int_\Omega |\varphi||u_p|^{1^*} dx - \int_\Omega |\varphi||v_p|^{1^*} dx \to \int_\Omega |\varphi|\,d\nu - \int_\Omega |\varphi|\,d\nu' \text{ as } p \to 1 \text{ for all } \varphi \in C_0^\infty(\mathbb{R}^N).$$

Consequently,

$$|u|^{1^*} = \nu - \nu'.$$

We thus get

$$\nu = \sum_{i\in\mathbb{N}} \nu_i \delta_{x_i} + |u|^{1^*}.$$

We proved the first part of the result. To continue, we will use the embedding $W_0^{1,p}(\Omega) \hookrightarrow L^{1^*}(\Omega)$. Indeed,

$$\left(\int_\Omega |\varphi u_p|^{1^*} dx\right)^{\frac{1}{1^*}} S_p \leq \left(\int_\Omega |\nabla(u_p\varphi)|^p\,dx\right)^{\frac{1}{p}}, \text{ for all } \varphi \in C_0^\infty(\mathbb{R}^N), \tag{3.4}$$

where

$$S_p = \inf_{v\in W_0^{1,p}(\Omega), u\neq 0} \frac{\left(\int_\Omega |\nabla v|^p\,dx\right)^{\frac{1}{p}}}{\left(\int_\Omega |v|^{1^*}\,dx\right)^{\frac{1}{1^*}}}.$$

Observe that

$$\frac{\int_\Omega |\nabla v|\,dx}{\left(\int_\Omega |v|^{1^*}\,dx\right)^{\frac{1}{1^*}}} \leq \frac{\left(\int_\Omega |\nabla v|^p\,dx\right)^{1/p}}{\left(\int_\Omega |v|^{1^*}\,dx\right)^{\frac{1}{1^*}}} |\Omega|^{1-1/p} \text{ for all } v \in W_0^{1,p}(\Omega).$$

We conclude that

$$S \leq S_p |\Omega|^{1-1/p} \text{ for all } p > 1,$$

where S is given by (3.2). Substituting in (3.4), we get

$$\left(\int_\Omega |\varphi u_p|^{1^*} dx \right)^{\frac{1}{1^*}} S \leq |\Omega|^{1-1/p} \left(\int_\Omega |\nabla (u_p \varphi)|^p \, dx \right)^{\frac{1}{p}} \text{ for all } \varphi \in C_0^\infty(\mathbb{R}^N).$$

We thus get from Minkowski's inequality that

$$\left(\int_\Omega |\varphi u_p|^{1^*} dx \right)^{\frac{1}{1^*}} S \leq |\Omega|^{1-1/p} \left(\int_\Omega |\varphi|^p |\nabla u_p|^p \, dx \right)^{\frac{1}{p}} + |\Omega|^{1-1/p} \left(\int_\Omega |\nabla \varphi|^p |u_p|^p \, dx \right)^{\frac{1}{p}} \text{ for all } \varphi \in C_0^\infty(\mathbb{R}^N). \tag{3.5}$$

Let $\psi \in C_0^\infty(\mathbb{R}^N)$ be chosen such that $\psi(0) = 1$, $0 \leq \psi \leq 1$ and $supp\, \psi = B_1(0)$.
For all $\epsilon > 0$ and $i \in \mathbb{N}$, we define

$$\psi_\epsilon(x) = \psi \left(\frac{x - x_i}{\epsilon} \right).$$

Substituting ψ_ϵ in (3.5), we get

$$\left(\int_\Omega |\psi_\epsilon|^{1^*} |u_p|^{1^*} \, dx \right)^{\frac{1}{1^*}} S \leq |\Omega|^{1-1/p} \left(\int_\Omega |u_p|^p |\nabla \psi_\epsilon|^p \, dx \right)^{\frac{1}{p}} + |\Omega|^{1-1/p} \left(\int_\Omega |\psi_\epsilon|^p |\nabla u_p|^p \, dx \right)^{\frac{1}{p}}.$$

Letting $p \to 1^+$, we get

$$\left(\int_\Omega |\psi_\epsilon|^{1^*} \, d\nu \right)^{\frac{1}{1^*}} S \leq \lim_{p \to 1^+} \left(\int_\Omega |u_p|^p |\nabla \psi_\epsilon|^p \, dx \right)^{\frac{1}{p}} + \lim_{p \to 1^+} \left(\int_\Omega |\psi_\epsilon|^p |\nabla u_p|^p \, dx \right)^{\frac{1}{p}}.$$

Since $\psi_\epsilon \leq 1$, we get from Lemma 3.2,

$$\lim_{p \to 1^+} \left(\int_\Omega |\psi_\epsilon|^p |\nabla u_p|^p \, dx \right)^{\frac{1}{p}} \leq \lim_{p \to 1^+} \left(\int_\Omega |\psi_\epsilon| |\nabla u_p|^p \, dx \right)^{\frac{1}{p}} = \int_\Omega |\psi_\epsilon| \, d\mu.$$

We further have

$$\lim_{p \to 1} \left(\int_\Omega |u_p|^p |\nabla \psi_\epsilon|^p \, dx \right)^{\frac{1}{p}} = \int_\Omega |u| |\nabla \psi_\epsilon| \, dx.$$

Hence,

$$\left(\int_\Omega |\psi_\epsilon|^{1^*} \, d\nu \right)^{\frac{1}{1^*}} S \leq \int_\Omega |u| |\nabla \psi_\epsilon| \, dx + \int_\Omega |\psi_\epsilon| \, d\mu.$$

Then

$$\nu_i^{\frac{1}{1^*}} S \leq \frac{1}{\epsilon} \int_\Omega \left(\left| \nabla \varphi \left(\frac{x - x_i}{\epsilon} \right) \right| |u| \right) \, dx + \mu(B_\epsilon(x_i))$$

Letting $\epsilon \to 0$, we conclude that

$$\nu_i^{\frac{1}{1^*}} S \leq \mu\{x_i\} \text{ for all } i \in \mathbb{N}.$$

We have proven that
$$\mu \geq S \sum_{i\in\mathbb{N}} \nu_i^{\frac{1}{1^*}} \delta_{x_i}.$$
On the other hand, from weak convergence we get
$$\begin{aligned}\int_\Omega \varphi |Du| &\leq \liminf_{p\to 1} \int_\Omega \varphi |\nabla u_p| \\ &\leq \liminf_{p\to 1} \int_\Omega \varphi |\nabla u_p|^p + \liminf_{p\to 1} \int_\Omega \frac{(p-1)\varphi}{p} \\ &= \int_\Omega \varphi\, d\mu \text{ for all } \varphi \in C_0^\infty(\Omega), \varphi \geq 0,\end{aligned}$$
which means that
$$|Du| \leq \mu.$$
According to [29, Lemma 1], $|Du|(A) = 0$ if the $(N-1)-$Hausdorff measure of A is zero. Consequently,
$$\mu \geq |\nabla u| + S \sum_{i\in\mathbb{N}} \nu_i^{\frac{1}{1^*}} \delta_{x_i}. \tag{3.6}$$
This proves the result □

4 Existence and properties of solutions for the approximated problems

In this section, we show that the functional $J_{p,\beta,\lambda}$ given by (1.9) admits a critical point. When there is no confusion, we will denote $J_{p,\beta,\lambda}$ merely by $J_{p,\beta}$. First, we show that it satisfies the hypothesis of the Mountain-Pass Theorem. We have

Lemma 4.1 *There exist $\overline{p} > 1$, $\alpha > 0$ and $r > 0$ that do not depend on p nor on β such that*
$$J_{p,\beta,\lambda}(u) \geq \alpha \textit{ for all } u \in W_0^{1,p}(\Omega) \textit{ with } \|u\|_{W_0^{1,p}(\Omega)} = r,$$
and for all $1 < p < \overline{p}$. The constants α and r depend on λ.

Proof. We first observe that
$$|\mathcal{F}_\beta(u)| \leq \int_\Omega |F_\beta(u)|\, dx \leq \int_\Omega \int_0^u |f_\beta(s)|\, ds\, dx \leq \int_\Omega \int_0^u |s|^{q-1}\, ds\, dx.$$
Consequently,
$$|\mathcal{F}_\beta(u)| \leq \frac{1}{q} \int_\Omega |u|^q\, dx.$$

Since $1 < q < 1^* < p^* = \frac{Np}{N-p}$, we obtain from Hölder's inequality that

$$|\mathcal{F}_\beta(u)| \leq \frac{1}{q}\left(\int_\Omega |u|^{p^*}\,dx\right)^{\frac{q}{p^*}} |\Omega|^{1-\frac{q}{p^*}}.$$

Similarly, we get

$$\int_\Omega (u^+)^{1^*}\,dx \leq \int_\Omega |u|^{1^*}\,dx \leq \left(\int_\Omega |u|^{p^*}\,dx\right)^{\frac{1^*}{p^*}} |\Omega|^{1-\frac{1^*}{p^*}}.$$

Consequently,

$$J_{p,\beta}(u) \geq \frac{1}{p}\|\nabla u\|^p_{L^p(\Omega)} - \frac{\lambda|\Omega|^{1-\frac{q}{p^*}}}{q}\|u\|^q_{L^{p^*}(\Omega)} - \frac{|\Omega|^{1-\frac{1^*}{p^*}}}{1^*}\|u\|^{1^*}_{L^{p^*}(\Omega)}.$$

From the Sobolev embedding (see [32, proof of Theorem 7.10]), we know that

$$\|u\|_{L^{p^*}(\Omega)} \leq \frac{\theta}{\sqrt{N}}\|\nabla u\|_{L^p(\Omega)} \text{ for all } u \in W^{1,p}_0(\Omega),$$

where

$$\theta = \frac{p(N-1)}{N-p}.$$

Consequently,

$$\begin{aligned} J_{p,\beta}(u) \quad &\geq \quad \frac{1}{p}\|\nabla u\|^p_{L^p(\Omega)} - \frac{\lambda|\Omega|^{1-\frac{q}{p^*}}}{q}\left(\frac{p(N-1)}{\sqrt{N}(N-p)}\right)^q \|\nabla u\|^q_{L^p(\Omega)} \\ &\quad - \frac{|\Omega|^{1-\frac{1^*}{p^*}}}{1^*}\left(\frac{p(N-1)}{\sqrt{N}(N-p)}\right)^{1^*} \|\nabla u\|^{1^*}_{L^p(\Omega)}. \end{aligned}$$

We thus get

$$\begin{aligned} J_{p,\beta}(u) \quad &\geq \quad \frac{1}{p}\|\nabla u\|^p_{L^p(\Omega)} - \frac{\lambda}{q}\max\{1,|\Omega|\}\left(\frac{p(N-1)}{\sqrt{N}(N-p)}\right)^q \|\nabla u\|^q_{L^p(\Omega)} \\ &\quad - \frac{\max\{1,|\Omega|\}}{1^*}\left(\frac{p(N-1)}{\sqrt{N}(N-p)}\right)^{1^*} \|\nabla u\|^{1^*}_{L^p(\Omega)}. \end{aligned}$$

Now, choosing $\overline{p} > 1$ such that $1 < \overline{p} < q$, we get, for all $1 < p < \overline{p}$,

$$J_{p,\beta}(u) \geq \frac{1}{p}\|\nabla u\|^p_{L^p(\Omega)} - C_q\|\nabla u\|^q_{L^p(\Omega)} - C_{1^*}\|\nabla u\|^{1^*}_{L^p(\Omega)},$$

where

$$C_q = \frac{\lambda}{q}\max\{1,|\Omega|\}\left(\frac{\overline{p}(N-1)}{\sqrt{N}(N-\overline{p})}\right)^q \text{ and } C_{1^*} = \frac{\max\{1,|\Omega|\}}{1^*}\left(\frac{\overline{p}(N-1)}{\sqrt{N}(N-\overline{p})}\right)^{1^*}.$$

Observe that C_q and C_{1^*} do not depend on p. Since $1 < q < 1^*$ and $1 < p < \overline{p}$, we get

$$J_{p,\beta}(u) \geq \frac{1}{\overline{p}} \|\nabla u\|^{\overline{p}}_{L^p(\Omega)} - C\|\nabla u\|^q_{L^p(\Omega)} \text{ for all } u \in W^{1,p}_0(\Omega) \text{ with } \|\nabla u\|_{L^p} \leq 1,$$

where $C = C_q + C_{1^*}$. Observe that $\frac{1}{\overline{p}} s^{\overline{p}} - Cs^q \geq \frac{s^q}{\overline{p}}$ if and only if $s^{\overline{p}} \geq (C\overline{p}+1)s^q$. Consequently,

$$J_{p,\beta}(u) \geq \frac{1}{\overline{p}} \|\nabla u\|^q_{L^p(\Omega)} \text{ for all } u \in W^{1,p}_0(\Omega) \text{ with } \|\nabla u\|_{L^p} \leq \left(\frac{1}{1 + C\overline{p}}\right)^{\frac{1}{q-\overline{p}}},$$

which means that if

$$\|\nabla u\|_{L^p} = \left(\frac{1}{1 + C\overline{p}}\right)^{\frac{1}{q-\overline{p}}},$$

then

$$J_{p,\beta}(u) \geq \frac{1}{\overline{p}} \left(\frac{1}{1 + C\overline{p}}\right)^{\frac{1}{q-\overline{p}}}.$$

□

Now we proceed to obtain an element $e \in W^{1,p}_0(\Omega)$ such that $J_{p,\beta}(e) < 0$. The proof of the following result relies on usual arguments, and we thus omit it. See for example [39, Lemma 3.1].

Lemma 4.2 *There exists $e \in C^\infty_0(\Omega)$ such that*

$$J_{p,\beta,\lambda}(e) < -|\Omega| \text{ for all } 1 < p < \overline{p}. \tag{4.1}$$

Furthermore, e does not depend on λ nor on β.

We define the functional $I_{p,\beta,\lambda} : W^{1,p}_0(\Omega) \to \mathbb{R}$ by

$$I_{p,\beta,\lambda}(u) = J_{p,\beta,\lambda}(u) + \frac{p-1}{p}|\Omega|. \tag{4.2}$$

Observe that from Lemma 4.1, there exist $\overline{p} > 1$, $\alpha > 0$ and $r > 0$ such that

$$I_{p,\beta,\lambda}(u) \geq \alpha \text{ for all } \|u\| = r \text{ and } 1 < p < \overline{p}.$$

Furthermore,

$$I_{p,\beta,\lambda}(e) < J_{p,\beta}(e) + |\Omega| < 0 \text{ for all } 1 < p < \overline{p},$$

where $e \in C^\infty_0(\Omega)$ is given by Lemma 4.2. Since $I_{p,\beta,\lambda}$ also satisfies the nonsmooth Palais-Smale condition, Theorem 2.6 implies that $I_{p,\beta,\lambda}$ has a critical point $u_{p,\beta,\lambda} \in W^{1,p}_0(\Omega)$ at the level

$$c_{p,\beta,\lambda} = \inf_{\gamma \in \Gamma_p} \max_{t \in [0,1]} I_{p,\beta,\lambda}(\gamma(t)) \text{ with } \Gamma_p = \{\gamma \in C([0,1]; W^{1,p}_0(\Omega)) : \gamma(0) = 0 \text{ and } \gamma(1) = e\}. \tag{4.3}$$

We finish this section with an uniform estimate for the solutions $u_{p,\beta,\lambda}$ of (1.6). The proof follows from arguments that can be found in [38].

Lemma 4.3 *There exists $C > 0$, such that*

$$\int_\Omega |\nabla u_{p,\beta,\lambda}|^p dx \le C. \tag{4.4}$$

Moreover, the family $(u_{p,\beta,\lambda})_{1<p<\overline{p}}$ is bounded in $BV(\Omega)$.

5 Convergence as $p \to 1^+$

First of all, from Lemma 4.3 and the Sobolev embeddings,

$$u_{p,\beta,\lambda} \to u_{\beta,\lambda} \text{ in } L^r(\Omega) \text{ for all } 1 \le r < 1^*, \text{ and } u_{p,\beta,\lambda} \to u_{\beta,\lambda} \text{ a.e. in } \Omega \text{ as } p \to 1^+. \tag{5.1}$$

Our first goal is to obtain convergence in $L^{1^*}(\Omega)$. To do that, we will require λ to be large. We first state a technical lemma.

Lemma 5.1 *Let $e \in C_0^\infty(\Omega)$ be given as in Lemma 4.2. Then there exist $\overline{p} > 1$, $\beta_0 > 0$ and $\lambda_0 > 0$ such that*

$$\max_{t\in[0,1]} I_{\overline{p},\beta,\lambda}(te) < \frac{1}{N} S^N \text{ for all } 0 < \beta < \beta_0, \lambda > \lambda_0, \tag{5.2}$$

where S is given by (3.2).

Proof. We choose β_1 and $\overline{p}$ such that

$$\frac{\beta_1^{\overline{p}}}{\overline{p}} \int_\Omega |\nabla e|^{\overline{p}} + \frac{\beta_1^{1^*}}{1^*} \int_\Omega |e|^{1^*} dx < \frac{1}{4} \frac{1}{N} S^N$$

and

$$\frac{\overline{p}-1}{\overline{p}} |\Omega| < \frac{1}{4} \frac{1}{N} S^N.$$

We will obtain $\lambda_0 > 0$ such that (5.2) holds for all $0 < \beta < \beta_0 = \beta_1/3$. Indeed, choosing such a β we have

$$\begin{aligned} I_{\overline{p},\beta,\lambda}(te) &= \frac{t^{\overline{p}}}{\overline{p}} \int_\Omega |\nabla e|^{\overline{p}} - \lambda \int_\Omega F_\beta(te)\, dx - \frac{t^{1^*}}{1^*} \int_\Omega |e|^{1^*} dx + \frac{\overline{p}-1}{\overline{p}} |\Omega| \\ &\le \frac{\beta^{\overline{p}}}{\overline{p}} \int_\Omega |\nabla e|^{\overline{p}} + \frac{\beta^{1^*}}{1^*} \int_\Omega |e|^{1^*} dx + \frac{\overline{p}-1}{\overline{p}} |\Omega|, \end{aligned}$$

for all $t \le \beta$. The choices of β_1 and $\overline{p}$ imply that

$$I_{\overline{p},\beta,\lambda}(te) < \frac{1}{2} \frac{1}{N} S^N \text{ for all } t \le \beta.$$

We further have

$$\begin{aligned}
I_{\overline{p},\beta,\lambda}(te) &\leq \frac{\beta_1^{\overline{p}}}{\overline{p}}\int_\Omega |\nabla e|^{\overline{p}} + \frac{\beta_1^{1^*}}{1^*}\int_\Omega |e|^{1^*}\,dx + \frac{\overline{p}-1}{\overline{p}}|\Omega| \\
&< \frac{1}{2}\frac{1}{N}S^N \text{ for all } \beta < t \leq \beta_1.
\end{aligned}$$

On the other hand, we have

$$I_{\overline{p},\beta,\lambda}(te) = \frac{t^{\overline{p}}}{\overline{p}}\int_\Omega |\nabla e|^{\overline{p}} - \lambda\int_{\{te>\beta\}} F_\beta(te)\,dx - \frac{t^{1^*}}{1^*}\int_\Omega |e|^{1^*}\,dx + \frac{\overline{p}-1}{\overline{p}}|\Omega|,$$

for all $t > \beta_1$. Since

$$\left\{e > \frac{1}{2}\right\} \subset \{te > \beta\} \text{ for all } 0 < \beta < \beta_0 \text{ and } t \geq \beta_1,$$

we get

$$I_{\overline{p},\beta,\lambda}(te) \leq \frac{t^{\overline{p}}}{\overline{p}}\int_\Omega |\nabla e|^{\overline{p}} - \lambda\int_{\{e>1/2\}} F_{\beta_0}(te)\,dx - \frac{t^{1^*}}{1^*}\int_\Omega |e|^{1^*}\,dx + \frac{\overline{p}-1}{\overline{p}}|\Omega|,$$

for all $0 < \beta < \beta_0$ and $\beta_1 < t \leq 1$. We must choose $\lambda_0 > 0$ such that

$$\frac{1}{\overline{p}}\int_\Omega |\nabla e|^{\overline{p}} - \lambda_0\int_{\{e>1/2\}} F_{\beta_0}(te)\,dx + \frac{\overline{p}-1}{\overline{p}}|\Omega| < \frac{1}{N}S^N \text{ for all } \beta_1 < t \leq 1.$$

From the choice of $\overline{p}$, we only need to show that

$$\frac{1}{\overline{p}}\int_\Omega |\nabla e|^{\overline{p}} - \lambda_0\int_{\{e>1/2\}} F_{\beta_0}(te)\,dx < \frac{1}{2N}S^N \text{ for all } \beta_1 < t \leq 1.$$

Hence, we may choose λ_0 satisfying

$$\frac{1}{\overline{p}}\int_\Omega |\nabla e|^{\overline{p}} - \lambda_0\int_{\{e>1/2\}} F_{\beta_0}(te)\,dx < 0 \text{ for all } \beta_1 < t \leq 1.$$

We also have

$$\begin{aligned}
\int_{\{e>1/2\}} F_{\beta_0}(te)\,dx &= \int_{\{e>1/2\}}\int_{\beta_0}^{te} |s|^{q-1}\,ds\,dx \\
&= \frac{1}{q}\int_{\{e>1/2\}} \left(|te|^q - (\beta_0)^q\right)\,dx \\
&\geq \frac{1}{q}\int_{\{e>1/2\}} \left(\left(\frac{\beta_1}{2}\right)^q - (\beta_0)^q\right)\,dx \\
&= \frac{1}{q}\left|\left\{e > 1/2\right\}\right|\left(\left(\frac{\beta_1}{2}\right)^q - (\beta_1/3)^q\right) \\
&= C > 0.
\end{aligned}$$

Then, it is enough to choose λ_0 such that

$$\frac{1}{\overline{p}}\int_\Omega |\nabla e|^{\overline{p}} < C\lambda_0.$$

Hence, for

$$\lambda_0 = \frac{2}{C\overline{p}}\int_\Omega |\nabla e|^{\overline{p}},$$

it follows that

$$I_{\overline{p},\beta,\lambda}(te) < \frac{1}{N}S^N \text{ for all } 0<\beta<\beta_0 \text{ and } \beta_1 \le t \le 1.$$

This proves the result. $\square$

We now get

Lemma 5.2 *Let $\lambda_0 > 0$ and $\beta_0 > 0$ be given by Lemma 5.1. Then*

$$u_{p,\beta,\lambda} \to u_{\beta,\lambda} \text{ in } L^{1^*}(\Omega) \text{ as } p \to 1^+, \tag{5.3}$$

provided $\lambda > \lambda_0$ and $0 < \beta < \beta_0$.

Proof. We again denote $I_{p,\beta,\lambda}$, $c_{p,\beta,\lambda}$ and $u_{p,\beta,\lambda}$ merely by $I_{p,\beta}$, $c_{p,\beta}$ and $u_{p,\beta}$ respectively. Using (4.4) and Theorem 3.1, we get measures $\mu, \nu \in \mathcal{M}(\overline{\Omega})$, a countable set I and a set of points $\{x_i : i \in I\} \subset \overline{\Omega}$ such that

$$|\nabla u_{p,\beta}|^p \rightharpoonup \mu \ge |Du_\beta| + \sum_i \delta_{x_i}\mu_i, \quad \text{in } \mathcal{M}(\overline{\Omega}) \tag{5.4}$$

$$|u_{p,\beta}|^{1^*} \rightharpoonup \nu = |u_\beta|^{1^*} + \sum_i \delta_{x_i}\nu_i, \quad \text{in } \mathcal{M}(\overline{\Omega}) \tag{5.5}$$

and

$$\nu_i^{\frac{1}{1^*}} \le \frac{\mu_i}{S}. \tag{5.6}$$

We will show that $I = \emptyset$ as in [31]. Indeed, for every $\sigma > 0$ and $i \in I$, we define

$$\psi_{\sigma,i}(x) = \psi\left(\frac{x - x_i}{\sigma}\right),$$

where $\psi \in C_c^\infty(\mathbb{R}^N)$ is a function such that $0 \le \psi \le 1$ in $B_1(0)$, $\psi \equiv 0$ in $\mathbb{R}^N \setminus B_2(0)$ and $\|\nabla\psi\|_{L^\infty(\mathbb{R}^N)} \le 2$. Taking $\psi_{\sigma,i}u_{p,\beta}$ as test function in (1.6), we have from (5.4) and (5.5) that

$$\lim_{p\to 1^+}\int_\Omega u_{p,\beta}|\nabla u_{p,\beta}|^{p-2}\nabla u_{p,\beta}\nabla\psi_{\sigma,i}\,dx = \int_\Omega \psi_{\sigma,i}\,d\nu - \int_\Omega \psi_{\sigma,i}\,d\mu + \lambda \lim_{p\to 1^+}\int_\Omega \rho_{p,\beta}\psi_{\sigma,i}u_{p,\beta}\,dx.$$

From the definition of $\rho_{p,\beta}$ and $\psi_{\sigma,i}$

$$\lim_{\sigma\to 0}\lim_{p\to 1^+}\int_\Omega \rho_{p,\beta}\psi_{\sigma,i}u_{p,\beta}\,dx = 0.$$

On the other hand, arguing as in [31], we get

$$\lim_{\sigma\to 0}\lim_{p\to 1}\int_\Omega u_{p,\beta}|\nabla u_{p,\beta}|^{p-2}\nabla u_{p,\beta}\nabla\psi_{\sigma,i}\,dx = 0. \tag{5.7}$$

Indeed, Hölder's inequality implies that

$$\begin{aligned}\int_\Omega u_{p,\beta}|\nabla u_{p,\beta}|^{p-2}\nabla u_{p,\beta}\nabla\psi_{\sigma,i}\,dx &\leq \left(\int_\Omega |\nabla u_{p,\beta}|^p\,dx\right)^{\frac{p-1}{p}}\left(\int_\Omega |u_{p,\beta}|^p|\nabla\psi_{\sigma,i}|^p\,dx\right)^{1/p}\\ &\leq C\left(\int_\Omega |u_{p,\beta}|^p|\nabla\psi_{\sigma,i}|^p\,dx\right)^{1/p}.\end{aligned}$$

On the other hand, the Dominated Convergence Theorem and Lemma 3.2 imply that

$$\lim_{p\to 1}\int_\Omega u_{p,\beta}|\nabla u_{p,\beta}|^{p-2}\nabla u_{p,\beta}\nabla\psi_{\sigma,i}\,dx \leq C\int_\Omega |u_\beta||\nabla\psi_{\sigma,i}|\,dx.$$

Consequently,

$$\lim_{\sigma\to 0}\int_\Omega \psi_{\sigma,i}\,d\mu = \lim_{\sigma\to 0}\int_\Omega \psi_{\sigma,i}\,d\nu.$$

Observe that

$$\lim_{\sigma\to 0}\int_\Omega \psi_{\sigma,i}\,d\mu \geq \lim_{\sigma\to 0}\int_\Omega \psi_{\sigma,i}|\nabla u_\beta|\,dx + \mu_i$$

and

$$\lim_{\sigma\to 0}\int_\Omega \psi_{\sigma,i}\,d\nu = \nu_i.$$

Hence we conclude that $\mu_i \leq \nu_i$. On the other hand, this fact and (5.6) impliy that

$$\nu_i^{\frac{1}{1^*}} \leq \frac{\nu_i}{S},$$

which is equivalent to

$$\nu_i \geq S^N.$$

Now we will show that this will lead us to a contradiction. Indeed, taking $u_{p,\beta}$ as test function in (1.6), we see that

$$0 = \int_\Omega |\nabla u_{p,\beta}|^p\,dx - \int_\Omega |u_{p,\beta}|^{1^*}\,dx - \lambda\int_\Omega \rho_{p,\beta}u_{p,\beta}\,dx.$$

We thus get

$$c_{p,\beta} = \left(\frac{1}{p}-\frac{1}{1^*}\right)\int_\Omega |u_{p,\beta}|^{1^*}\,dx + \lambda\left(\int_\Omega\left(\frac{1}{p}\rho_{p,\beta}u_{p,\beta} - F_\beta(u_{p,\beta})\right)\,dx\right) + \left(\frac{p-1}{p}|\Omega|\right).$$

If $1 < p < q$, we conclude that (see (2.6))

$$c_{p,\beta} \geq \left(\frac{1}{p} - \frac{1}{1^*}\right) \int_\Omega \psi_{\sigma,i} |u_{p,\beta}|^{1^*} \, dx + \left(\frac{p-1}{p}|\Omega|\right),$$

for all $\sigma > 0$ and $i \in I$. Letting $p \to 1^+$, we get

$$\lim_{p \to 1^+} c_{p,\beta} \geq \left(1 - \frac{1}{1^*}\right) \nu_i \geq \frac{1}{N} S^N.$$

This contradicts Lemma 5.1 and then $I = \emptyset$. □

Now we proceed to show that $u_{\beta,\lambda} \in L^\infty(\Omega)$. In order to do it, we follow similar ideas than [38, Section 3.2]. In the remainder of this section, we denote $u_{p,\beta,\lambda}$ and $u_{\beta,\lambda}$ merely by $u_{p,\beta}$ and u_β respectively.

Lemma 5.3 *Assume that $\lambda > \lambda_0$ and $0 < \beta < \beta_0$. For each $\epsilon > 0$ there exist $\eta > 0$ and $\overline{p} > 1$ (independent of p) such that*

$$\int_B (1 + u_{p,\beta}^{\frac{1}{N-1}})^N \, dx \leq \epsilon \text{ for all Borel sets } B \text{ such that } |B| < \eta, \text{ and for all } 1 < p < \overline{p}. \tag{5.8}$$

Proof. Since $u_\beta^{1^*} \in L^1(\Omega)$, we may choose $\eta > 0$ such that

$$\int_B u_\beta^{\frac{N}{N-1}} \, dx \leq \frac{\epsilon}{2} \text{ for all Borel sets } B \text{ such that } |B| < \eta.$$

Since $u_{p,\beta} \to u_\beta$ in $L^{1^*}(\Omega)$, as $p \to 1^+$, there must exists $\overline{p} > 1$ such that

$$\int_\Omega |u_\beta - u_{p,\beta}|^{1^*} \, dx < \frac{\epsilon}{2} \text{ for all } 1 < p < \overline{p}.$$

Consequently, if B is a Borelian subset of Ω such that $|B| < \eta$, we get

$$\int_B |u_{p,\beta}|^{1^*} \, dx \leq \int_B |u_{p,\beta} - u_\beta|^{1^*} \, dx + \int_B |u_\beta|^{1^*} \, dx < \epsilon \text{ for all } 1 < p < \overline{p}.$$

The result then follows from the fact that

$$(1 + u_{p,\beta}^{\frac{1}{N-1}})^N \leq 2^N (1 + u_{p,\beta}^{1^*}),$$

and that $|\Omega|$ is finite. □

Lemma 5.4 *Assume that $\lambda > \lambda_0$ and $0 < \beta < \beta_0$. For every $\epsilon > 0$, there exists $k > 0$ that does not depend on p such that*

$$\int_{\{u_{p,\beta} > k\}} (1 + u_{p,\beta}^{\frac{1}{N-1}})^N \, dx < \epsilon \text{ for all } 1 < p < \overline{p}.$$

Proof. Let $\epsilon > 0$. From Lemma 5.3, we may choose $\eta > 0$ such that

$$\int_B (1 + u_{p,\beta}^{\frac{1}{N-1}})^N \, dx < \frac{\epsilon}{2} \text{ for all Borel sets } B \text{ such that } |B| < \eta, 1 < p < \overline{p}.$$

We then choose $k > 0$ such that

$$|\{u_\beta > k/2\}| < \eta.$$

We then have

$$\int_{\{u_\beta > k/2\}} (1 + u_{p,\beta}^{\frac{1}{N-1}})^N \, dx < \frac{\epsilon}{2} \text{ for all } 1 < p < \overline{p}.$$

On the other hand, the Egorov Theorem imply that there exists a Borel set $A \subset \Omega$ such that $|A| < \eta$ and $u_{p,\beta} \to u_\beta$ uniformly in $\Omega \setminus A$. Consequently,

$$\begin{aligned}
\int_{\{u_p > k\}} (1 + u_{p,\beta}^{\frac{1}{N-1}})^N \, dx &= \int_{A \cap \{u_p > k\}} (1 + u_{p,\beta}^{\frac{1}{N-1}})^N \, dx + \int_{A^c \cap \{u_p > k\}} (1 + u_{p,\beta}^{\frac{1}{N-1}})^N \, dx \\
&\leq \frac{\epsilon}{2} + \int_{A^c \cap \{u_p > k\}} (1 + u_{p,\beta}^{\frac{1}{N-1}})^N \, dx.
\end{aligned}$$

Uniform convergence implies that there exists $\overline{p} > 1$ such that

$$|u_{p,\beta}(x) - u_\beta(x)| < k/2 \text{ for all } x \in A^c, 1 < p < \overline{p}.$$

As a consequence, if $x \in A^c \cap \{u_{p,\beta} > k\}$ then

$$u_\beta(x) = u_{p,\beta}(x) + u_\beta(x) - u_{p,\beta}(x) \geq k - k/2 = k/2.$$

Hence $A^c \cap \{u_{p,\beta} > k\} \subset \{u_\beta > k/2\}$. We conclude that

$$\int_{\{u_p > k\}} (1 + u_{p,\beta}^{\frac{1}{N-1}})^N \, dx < \frac{\epsilon}{2} + \frac{\epsilon}{2} = \epsilon.$$

□

We finally show the main result of this section.

Proposition 5.5 *If $\lambda > \lambda_0$ and $0 < \beta < \beta_0$, then the function u_β belongs to $L^\infty(\Omega)$.*

Proof. For $k > 0$ we define

$$G_k(s) = \begin{cases} s - k, & \text{if } s > k \\ 0, & \text{if } s \leq k. \end{cases}$$

Choosing $G_k(u_{p,\beta})$ as a test function in (1.7), we get

$$\int_{\{u_{p,\beta} > k\}} |\nabla u_{p,\beta}|^p \, dx = \int_\Omega (\lambda \rho_{p,\beta} + u_{p,\beta}^{1^*-1}) G_k(u_{p,\beta}) \, dx.$$

Since $\rho_{p,\beta} \le u_{p,\beta}^{q-1}$, we get

$$\int_{\{u_{p,\beta}>k\}} |\nabla u_{p,\beta}|^p \, dx \le \int_\Omega (\lambda u_{p,\beta}^{q-1} + u_{p,\beta}^{1^*-1}) G_k(u_{p,\beta}) \, dx.$$

Consequently,

$$\int_{\{u_{p,\beta}>k\}} |\nabla u_{p,\beta}|^p \, dx \le (1+\lambda) \int_\Omega (1 + u_{p,\beta}^{1^*-1}) G_k(u_{p,\beta}) \, dx. \tag{5.9}$$

On the other hand, the Sobolev embedding implies that

$$\begin{aligned}
\left(\int_\Omega G_k(u_{p,\beta})^{\frac{N}{N-1}} dx\right)^{\frac{N-1}{N}} &\le S \int_\Omega |\nabla G_k(u_{p,\beta})| dx \\
&\le \frac{S}{p} \int_\Omega |\nabla G_k(u_{p,\beta})|^p \, dx + \frac{S(p-1)}{p} |\Omega| \\
&\le S(1+\lambda) \int_\Omega (1 + u_{p,\beta}^{1^*-1}) G_k(u_{p,\beta}) \, dx + \frac{S(p-1)}{p} |\Omega| \\
&\le S(1+\lambda) \int_{\{u_{p,\beta}>k\}} (1 + u_{p,\beta}^{1^*-1}) G_k(u_{p,\beta}) \, dx + \frac{S(p-1)}{p} |\Omega|.
\end{aligned}$$

The Hölder inequality implies

$$\begin{aligned}
\left(\int_\Omega G_k(u_{p,\beta})^{\frac{N}{N-1}} dx\right)^{\frac{N-1}{N}} &\le \mathcal{S}(1+\lambda) \left(\int_{\{u_{p,\beta}>k\}} (1 + u_{p,\beta}^{1^*-1})^N \, dx\right)^{\frac{1}{N}} \left(\int_\Omega G_k(u_{p,\beta})^{\frac{N}{N-1}} \, dx\right)^{\frac{N-1}{N}} \\
&+ \frac{S(p-1)}{p} |\Omega|.
\end{aligned}$$

From Lemma 5.4, we may choose $k_0 > 0$ such that

$$\int_{\{u_{p,\beta}>k\}} (1 + u_{p,\beta}^{1^*-1})^N \, dx \le \frac{1}{(2S(1+\lambda))^N} \text{ for all } k \ge k_0,$$

and for all $1 < p < \overline{p}$. We conclude that

$$\left(\int_\Omega G_k(u_{p,\beta})^{\frac{N}{N-1}} dx\right)^{\frac{N-1}{N}} \le \frac{2S(p-1)}{p} |\Omega| \text{ for all } 1 < p < \overline{p}.$$

Rewriting,

$$\int_\Omega G_k(u_{p,\beta})^{\frac{N}{N-1}} \, dx \le \left(\frac{2S(p-1)}{p} |\Omega|\right)^{\frac{N}{N-1}} \quad \text{for all } k \ge k_0, 1 < p < \overline{p}.$$

Consequently, passing to the limit as $p \to 1^+$ and using Fatou's Lemma,

$$\int_\Omega ((u_\beta - k)^+)^{\frac{N}{N-1}} \, dx = 0 \text{ for all } k \ge k_0.$$

Then $u_\beta \le k_0$ a.e. in Ω and this proves the result. □

6 Proof of Theorem 1.1

This section is devoted to show that the function $u_{\beta,\lambda}$ obtained in (5.1) is a solution of problem (1.1). We have shown that there exist $\lambda_0, \beta_0 > 0$ such that if $\lambda > \lambda_0$ and $0 < \beta < \beta_0$, then $u_{\beta,\lambda} \in BV(\Omega) \cap L^\infty(\Omega)$ and, as $p \to 1^+$,

$$u_{p,\beta,\lambda} \to u_{\beta,\lambda} \text{ in } L^r(\Omega), 1 \leq r \leq 1^*, \text{ and } u_{p,\beta,\lambda} \to u_{\beta,\lambda} \text{ a.e. in } \Omega. \tag{6.1}$$

We further observe that the Sobolev embedding and (4.4) imply that there exists $C > 0$ that does not depend on p nor on β such that

$$\|u_{p,\beta,\lambda}\|_{L^{1^*}(\Omega)} < C \text{ for all } 1 < p < \overline{p}. \tag{6.2}$$

We first study the convergence of $\rho_{p,\beta,\lambda}$ as $p \to 1^+$. We have

Lemma 6.1 *Let $\lambda > \lambda_0$ and $0 < \beta < \beta_0$. There exists $\rho_{\beta,\lambda} \in L^{\frac{q}{q-1}}(\Omega)$ such that*

$$\rho_{p,\beta,\lambda} \rightharpoonup \rho_{\beta,\lambda} \text{ weakly in } L^{\frac{q}{q-1}}(\Omega) \text{ as } p \to 1^+, \tag{6.3}$$

and

$$\rho_{\beta,\lambda}(x) \in \begin{cases} \{0\} \text{ if } u_{\beta,\lambda}(x) < \beta, \\ [0, \beta^{q-1}] \text{ if } u_{\beta,\lambda}(x) = \beta, \\ \{u_{\beta,\lambda}(x)^{q-1}\} \text{ if } u_{\beta,\lambda}(x) > \beta. \end{cases} \tag{6.4}$$

Proof. We again denote $\rho_{p,\beta,\lambda}$, $u_{p,\beta,\lambda}$ and $u_{\beta,\lambda}$ merely by $\rho_{p,\beta}$, $u_{p,\beta}$ and u_β respectively. By the definition of $|\rho_{p,\beta}(x)| \leq u_{p,\beta}(x)^{q-1}$ a.e. in Ω, Hölder's inequality and (6.2), we get for all $1 < p < \overline{p}$,

$$\int_\Omega |\rho_{p,\beta}(x)|^{\frac{q}{q-1}}\, dx \leq \left(\int_\Omega |u_{p,\beta}(x)|^{1^*}\, dx \right)^{\frac{q}{1^*}} |\Omega|^{1-\frac{q}{1^*}} \leq C^q |\Omega|^{1-\frac{q}{1^*}}. \tag{6.5}$$

Then $(\rho_{p,\beta})_p$ is uniformly bounded in $L^{q/q-1}(\Omega)$, from which (6.3) follows. We denote $\rho_{\beta,\lambda}$ merely by ρ_β. We now claim that if $E \subset \Omega$ is a measurable set, then

$$\lim_{p\to 1^+} \int_E \rho_{p,\beta} u_{p,\beta}(x)\, dx = \int_E \rho_\beta u_\beta(x)\, dx. \tag{6.6}$$

Indeed, observe that

$$\rho_{p,\beta} u_{p,\beta} - \rho_\beta u_\beta = u_\beta(\rho_{p,\beta} - \rho_\beta) + \rho_{p,\beta}(u_{p,\beta} - u_\beta).$$

Consequently, (6.3) implies that

$$\lim_{p\to 1^+} \int_E u_\beta(\rho_{p,\beta} - \rho_\beta)(x)\, dx = 0. \tag{6.7}$$

On the other hand, Hölder's inequality and (6.5) imply that there exists $C > 0$ such that

$$\begin{aligned}\lim_{p\to 1^+}\int_E \rho_{p,\beta}(u_{p,\beta}-u_\beta)\,dx &\le \left(\int_\Omega \rho_{p,\beta}^{\frac{q}{q-1}}\,dx\right)^{\frac{q-1}{q}}\left(\int_\Omega |u_{p,\beta}(x)-u_\beta(x)|^q\,dx\right)^{\frac{1}{q}}\\ &\le C\left(\int_\Omega |u_{p,\beta}(x)-u_\beta(x)|^q\,dx\right)^{\frac{1}{q}}.\end{aligned}$$

From (6.1), we conclude that

$$\lim_{p\to 1^+}\int_E \rho_{p,\beta}(u_{p,\beta}-u_\beta)\,dx = 0. \tag{6.8}$$

We now prove (6.4). It is clear that $\rho_\beta(x) \ge 0$ in Ω. We will show the following statements:

$$\rho_\beta(x) = 0 \text{ a.e. in } E_1 = \{0 < u_\beta < \beta\}. \tag{6.9}$$

$$\rho_\beta(x) \le \beta^{q-1} \text{ a.e. in } E_2 = \{u_\beta = \beta\}. \tag{6.10}$$

$$\rho_\beta(x) = u_\beta^{q-1} \text{ a.e. in } E_3 = \{u_\beta > \beta\}. \tag{6.11}$$

First we show (6.9). For each $x \in E_1$ there exists $p_x > 1$ such that

$$u_{p,\beta}(x) < \beta \text{ for all } 1 < p < p_x,$$

and consequently,

$$\rho_{p,\beta}(x) = 0 \text{ for all } 1 < p < p_x.$$

Hence, $\rho_{p,\beta}(x)u_{p,\beta}(x) \to 0$ a.e. in E_1 as $p \to 1^+$. Furthermore,

$$|\rho_{p,\beta}(x)u_{p,\beta}(x)| \le u_{p,\beta}(x)^q \to u_\beta(x)^q \text{ in } L^1(\Omega).$$

Consequently, the Dominated Convergence Theorem implies that

$$0 = \lim_{p\to 1^+}\int_{E_1} \rho_{p,\beta}u_{p,\beta}(x)\,dx = \int_{E_1} \rho_\beta u_\beta(x)\,dx,$$

and this implies that (6.9) holds. Now we show (6.10). Indeed, assume that the set $F_1 = \{x \in E_2 : \rho_\beta(x) > \beta^{q-1}\}$ has positive measure. Then

$$\begin{aligned}\beta^q|F_1| < \int_{F_1} \rho_\beta(x)u_\beta(x)\,dx &= \lim_{p\to 1^+}\int_{F_1} \rho_{p,\beta}(x)u_{p,\beta}(x)\,dx \le \lim_{p\to 1^+}\int_{F_1} u_{p,\beta}(x)^q\\ &= \int_{F_1} u_\beta(x)^q\,dx = \beta^q|F_1|,\end{aligned}$$

which is a clear contradiction. It remains to show (6.11). Let $F_2 = \{x \in E_3 : \rho_\beta(x) > u_\beta^{q-1}(x)\}$ and $F_3 = \{x \in E_3 : \rho_\beta(x) < u_\beta^{q-1}(x)\}$. If $|F_2| > 0$, then

$$\begin{aligned} \int_{F_2} u_\beta(x)^q \, dx &< \int_{F_2} \rho_\beta(x) u_\beta(x) \, dx \\ &= \lim_{p\to 1} \int_{F_2} \rho_{p,\beta}(x) u_{p,\beta}(x) \, dx \le \lim_{p\to 1} \int_{F_2} u_{p,\beta}(x)^q \\ &= \int_{F_2} u_\beta(x)^q \, dx, \end{aligned}$$

a clear contradiction. Then, $|F_2| = 0$. Furthermore, if $|F_3| > 0$, then

$$\lim_{p\to 1^+} \int_{F_3} \rho_{p,\beta}(x) u_{p,\beta}(x) \, dx = \int_{F_3} \rho_\beta(x) u_\beta(x) \, dx < \int_{F_3} u_\beta(x)^q \, dx. \tag{6.12}$$

But for almost all $x \in F_3$, there exists $p_x > 1$ such that

$$u_{p,\beta}(x) > \beta \text{ for all } 1 < p < p_x.$$

Then

$$(\rho_{p,\beta} u_{p,\beta})(x) = u_{p,\beta}(x)^q \text{ for all } 1 < p < p_{x_0}.$$

Consequently, $\lim_{p\to 1^+} (\rho_{p,\beta} u_{p,\beta})(x) = \lim_{p\to 1^+} u_{p,\beta}(x)^q = u_\beta(x)^q$ for almost all $x \in F_3$. Since

$$(\rho_{p,\beta} u_{p,\beta})(x) \le u_{p,\beta}^q \to u_\beta^q \text{ in } L^1(\Omega),$$

we conclude from the Dominated Convergence Theorem that

$$\lim_{p\to 1^+} \int_{F_3} \rho_{p,\beta}(x) u_{p,\beta}(x) \, dx = \int_{F_3} u_\beta^q \, dx. \tag{6.13}$$

Hence, (6.12) and (6.13) lead us to a contradiction. Then $|F_3| = 0$.

□

By using the same arguments that in [39], one can show that, for $\lambda > \lambda_0$ and $0 < \beta < \beta_0$, there exists a vector field $\mathbf{z}_\beta \in L^\infty(\Omega; \mathbb{R}^N)$ such that $\|\mathbf{z}_\beta\|_\infty \le 1$ and

$$-\operatorname{div} \mathbf{z}_\beta = \lambda \rho_\beta + u_\beta^{1^*-1} \text{ in } \mathcal{D}'(\Omega).$$

Moreover,

$$(\mathbf{z}_\beta, Du_\beta) = |Du_\beta| \quad \text{in } \mathcal{M}(\Omega) \tag{6.14}$$

and also

$$[z_\beta, \nu] \in \operatorname{sgn}(-u_\beta) \quad \text{on } \partial\Omega.$$

We finish by showing that $u_\beta \not\equiv 0$. Indeed, from Lemma 4.1, (4.2) and (4.3), we know that there exists $\alpha > 0$ independent of p such that

$$I_{p,\beta}(u_{p,\beta}) = c_{p,\beta} \ge \alpha > 0 \text{ for all } p > 1.$$

Hence,

$$J_{p,\beta}(u_{p,\beta})+\left(\frac{p-1}{p}\right)|\Omega|\geq\alpha>0 \text{ for all } p>1,$$

which means

$$\frac{1}{p}\int_\Omega|\nabla u_{p,\beta}|^p\,dx,-\lambda\mathcal{F}_\beta(u_{p,\beta})-\frac{1}{1^*}\int_\Omega(u_{p,\beta}^+)^{1^*}\,dx+\left(\frac{p-1}{p}\right)|\Omega|\geq\alpha>0.$$

Consequently,

$$\frac{1}{p}\int_\Omega|\nabla u_{p,\beta}|^p\,dx+\left(\frac{p-1}{p}\right)|\Omega|\geq\alpha>0 \text{ for all } p>1.$$

Letting $p\to 1^+$, we get

$$\lim_{p\to 1^+}\frac{1}{p}\int_\Omega|\nabla u_{p,\beta}|^p\,dx\geq\alpha>0. \tag{6.15}$$

From (1.7), we get

$$\int_\Omega|\nabla u_{p,\beta}|^p\,dx=\lambda\int_\Omega\rho_{p,\beta}u_{p,\beta}\,dx+\int_\Omega u_{p,\beta}^{1^*}\,dx,$$

Using (6.15), Lemmas 5.2 and 6.1 we conclude that

$$0<\alpha\leq\lambda\int_\Omega\rho_\beta u_\beta\,dx+\int_\Omega u_\beta^{1^*}\,dx. \tag{6.16}$$

Consequently, $u_\beta\not\equiv 0$.

Finally, what is left to prove is that there exists $\beta_\lambda>0$ such that

$$|\{u_\beta>\beta\}|>0 \text{ for all } 0<\beta<\beta_\lambda.$$

Indeed, from (6.16) and Lemma 6.1 we get

$$\alpha\leq\lambda\beta^q|\Omega|+\beta^{1^*}|\Omega|+\lambda\int_{\{u_\beta>\beta\}}u_\beta^q\,dx+\int_{\{u_\beta>\beta\}}u_\beta^{1^*}\,dx$$

Consequently, if we choose $\beta_\lambda>0$ such that

$$\alpha-(\lambda\beta^q|\Omega|+\beta^{1^*}|\Omega|)>\alpha/2 \text{ for all } 0<\beta<\beta_\lambda,$$

we get

$$\alpha/2<\lambda\int_{\{u_\beta>\beta\}}u_\beta^q\,dx+\int_{\{u_\beta>\beta\}}u_\beta^{1^*}\,dx \text{ for all } 0<\beta<\beta_\lambda. \tag{6.17}$$

Consequently,

$$|\{u_\beta>\beta\}|>0 \text{ for all } 0<\beta<\beta_\lambda,$$

since otherwise the right-hand-side of (6.17) would be equal to zero. This proves the result.

□

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References

[1] C.O. Alves, Z. Yuan, and L. Huang *Existence and multiplicity of solutions for discontinuous elliptic problems in RN*. Proc. R. Soc. Edinb. Sect. A Math. **151** (2021), 548–572.

[2] C. O. Alves, A. Ourraoui and M. T. O. Pimenta *Multiplicity of solutions for a class of quasilinear problems involving the 1-Laplacian operator with critical growth*. J. Differential Equations. **308** (2022), 545–574.

[3] C.O. Alves, J.A. dos Santos and R.C. Nemer, *Multiple solutions for a problem with discontinuous nonlinearity*. Ann. Mat. Pura Appl. **197** (2018), 883-903.

[4] A. Ambrosetti and M. Badiale, *The dual variational principle and elliptic problems with discontinuous nonlinearities*. J Math Anal Appl. **140** (1989), 363–373.

[5] A. Ambrosetti, H. Brezis and G. Cerami *Combined effects of concave and convex nonlinearities in some elliptic problems*. J. Funct. Anal. **122** (1994) 519–543.

[6] A. Ambrosetti, M. Calahorrano, F. Dobarro *Global branching for discontinuous problems*. Comment. Math. Univ. Carol. **31** (1990), 213 – 222.

[7] A. Ambrosetti, R. E. L. Turner *Some discontinuous variational problems*. Differential Integral Equations. **1** (1988), 341-349.

[8] A. Ambrosetti, H. Brezis and G. Cerami *Combined effects of concave and convex nonlinearities in some elliptic problems*. J. Funct. Anal. **122** (1994), 519-543.

[9] L. Ambrosio, N. Fusco and D. Pallara, *Functions of bounded variation and free discontinuity problems*, Oxford University Press, Oxford (2000).

[10] F. Andreu, C. Ballester, V. Caselles and J. M. Mazón *Minimizing total variation flow.* C. R. Acad. Sci., Paris, Sr. I, Math. **331** (2000), 867 - 872 .

[11] F. Andreu, C. Ballester, V. Caselles and J. M. Mazón *The Dirichlet problem for the total variation flow.* J. Funct. Anal. **180** (2001), 347 - 403

[12] F. Andreu, C. Ballester, V. Caselles and J. M. Mazón, *Minimizing total variation flow.* Differential Integral Equations. **14** (2001), 321 - 360.

[13] F. Andreu, V. Caselles and J. M. Mazón *Parabolic quasilinear equations minimizing linear growth functionals.* Progress in mathematics **233** Birkhäuser Verlag, Basel (2004).

[14] M. Anmin, H. Chang *Kirchhoff type problems in $\mathbb{R}^N$ with radial potentials and locally lipschitz functional.* Appl. Math. Lett. **62** (2016), 49-54.

[15] G. Anzellotti, *Pairings between measures and bounded functions and compensated compactness.* Ann. Mat. Pura Appl. **4** (1983), 293 - 318.

[16] D. Arcoya and M. Calahorrano *Some Discontinuous Problems with a Quasilinear Operator.* J. Math. Anal. Appl.**187** (1994), 1059–1072

[17] H. Attouch, G. Buttazzo and G. Michaille *Variational analysis in Sobolev and BV spaces.* MOS-SIAM Series on Optimization, 2014, ISBN:978-1-61197-347-1

[18] H. Brezis, *Functional Analysis, Sobolev Spaces and Partial Differential Equations.* Springer, New York, Universitext, 2010.

[19] H. Brezis and L. Nirenberg *Positive solutions of nonlinear elliptic equations involving critical Sobolev exponents.* Comm. Pure Appl. Math. **36** (1983), 437–477.

[20] G. Cerami *Soluzioni positive di problemi con parte nonlineare discontinua e applicazioni a un problema di frontiera libera.* Boll. UMI. **6** (1983), 321-338.

[21] K. C. Chang *The obstacle problem and partial differential equations with discontinuous nonlinearities.* Commun Pure Appl Math. **33** (1978), 117-146

[22] K. C. Chang *On the multiple solutions of the elliptic differential equations with discontinuous nonlinear terms.* Sci. Sin. **21** (1978), 139–158.

[23] K.C. Chang *Variational methods for nondifferentiable functionals and their applications to partial differential equations* J. Math. Anal. **80** (1981), 102-129.

[24] G. Chen and H. Frid *Divergence-measure fields and hyperbolic conservation laws* Arch. Ration. Mech. Anal. **147** (1999), 89 - 118.

[25] F. H. Clarke *Optimization and Nonsmooth Analysis*. John Wiley & Sons, N.Y, (1983).

[26] F. H. Clarke *Generalized gradients and applications*. Trans. Amer. Math. Soc. **265** (1975), 247-262.

[27] L.M. de Cave *Nonlinear elliptic equations with singular nonlinearities*. Asymptotic Anal. **84** (2013), 181 - 195.

[28] V. De Cicco, D. Giachetti and S. Segura de León *Elliptic problems involving the* $1-$*Laplacian and a singular lower order term*. J. Lond. Math. Soc. **99** (2019), 349-376.

[29] F. Demengel *On some nonlinear partial differential equations involving the* $1-$*Laplacian and critical Sobolev exponent*. ESAIM, Control Optim. Calc. Var. **4** (1999), 667 - 686.

[30] G.M. Figueiredo and G. G. dos Santos *Solution for a Kirchhoff equation with critical Caffarelli-Kohn-Nirenberg growth and discontinuous nonlinearity*. Z. Angew. Math. Phys. **69** (2018), 13pp.

[31] J. Garcia-Azorero and I. Peral-Alonso *Multiplicity of solutions for elliptic problems with critical exponent or with a nonsymmetric term.* Transactions of the American Mathematical Society **323** (1991), 877–895

[32] D. Gilbarg and N. Trudinger *Elliptic partial differential equations of second order*. Reprint of the 1998 edition, Classics in Mathematics. Springer-Verlag, Berlin 2001.

[33] M. R. Grossinho and S. A. Tersian *An Introduction to Minimax theorems and their applications to Differential Equations* (2001).

[34] B. Kawohl *On a family of torsional creep problems* J. Reine Angew. Math **410** (1990), 1 - 22.

[35] P.L. Lions *On the existence of positive solutions of semilinear elliptic equations*. SIAM Rev. **24** (1982), 441-467.

[36] P. L. Lions *The concentration-compactness principle in the calculus of variations. The limit case*. Rev. Mat. Iberoamericana **1** (1985), 145–201.

[37] A. Mercaldo, S. Segura de León and C. Trombetti *On the solutions to* $1-$*Laplacian equation with* L^1 *data*. J. Funct. Anal. **256** (2009), 2387-2416.

[38] A. Molino Salas and S. Segura de León *Elliptic equations involving the* $1-$*Laplacian and a subcritical source term.* Nonlienar Analysis **168** (2018), 50-66

[39] M.T.O. Pimenta, G. dos Santos, J. Santos Júnior *On a quasilinear elliptic problem involving the* $1-$*Laplacian operator and a discontinuous nonlinearity.* Proc. Roy. Soc. Edinburgh Sect. A (2022), 1-27. doi:10.1017/prm.2022.86.

[40] L. Rudin, S. Osher, E. Fatemi *Nonlinear total variation based noise removal algorithms.* Phys. D **60** (1992), 259 - 268.

[41] Y. Sun, S. Wu and Y. Long *Combined effects of singular and superlinear nonlinearities in some singular boundary value problems.* J. Differential Equations **176** (2001), 511 - 531.