

# Solitons in atomic condensates, with optical lattices and field-induced dipole moments

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**Abstract.** We report some investigations on the existence of matter-wave solitons when considering cross-combined optical lattices (OL) in three dimensions (3D), where we have a nonlinear OL in one of the directions, which is perpendicular to a two-dimensional (2D) plane with linear OLs in either one or both directions. This study can be useful to manage 3D solitons through spatial modulations of the scattering length in one of the OL directions. Another independent study is reported, by considering bright solitons manifested in a bosonic condensate gas carrying collinear dipole moments, which induces an external polarizing field with strength periodically modulated along one of the coordinates. This leads to an effective nonlocal nonlinear lattice, with solitonic solutions. Their dynamics and mobility can be investigated by an effective one-dimensional (1D) model. Interactions between solitons are also reported within this 1D model. In all the cases, we consider full numerical and variational approaches.

## 1. Introduction

In the present contribution we review some recent works on nonlinear optical lattices that we have been investigating in Refs. [1, 2]. By considering Ref. [1], we show results obtained on the existence of three-dimensional (3D) matter-wave solitons when considering cross-combined optical lattices (OL), with a nonlinear optical lattice (NOL) in one of the directions, perpendicular to a plane where we have linear OL in either one or both directions. Our section II is dedicated to this case. First, we noticed that one cannot obtain stable 3D solitons when in one of the perpendicular directions there is no optical lattice or trapping interaction. However, in the full 3D OL cross-combined case, we can obtain families of 3D solitons, which can be stable for both attractive and repulsive interactions. Sample results are presented in this case, analyzing the existence of soliton solutions, as well as their time evolution and stability.

Next, in section III, by considering a bosonic gas of particles carrying collinear dipole moments, following [2], it is shown that one can obtain an effective nonlocal nonlinear lattice in a condensate, where bright solitons can be manifested. Such dipole moments are induced by an external polarizing field with the strength periodically modulated along the coordinate. Our aim, in this case, is to investigate the existence and stability of bright solitons in a system controlled by an effective induced dipole-dipole interaction (DDI) in nonlocal nonlinear lattice. Some results on the soliton stability and dynamics are also presented. The model we consider



is a quasi-1D dipolar Bose-Einstein condensate (BEC) with electric dipole moments of particles induced by the dc field with the local strength periodically varying in space, while its direction is uniform, being oriented along the system's axis, thus giving rise to the attractive DDI.

In both sections the reported results were obtained from exact numerical simulations, as well as from variational procedures, as detailed in [1, 2]. In section III we also add sample results on the dynamics and collisions of solitons. In section IV we have some final considerations.

## 2. Solitons in cross-combined linear and nonlinear optical lattices

When we have cross-combined OL, with linear ones in the  $x$ - and  $z$ -direction, and a nonlinear one in the remaining  $y$ -direction, it is shown in [1] that one can generate 3D matter-wave solitons for some parametric regions. In this section we report some of the results obtained in this work. In case of a 2D BEC, it was shown in [3] that, for conservative systems, a NOL in one direction by itself cannot give stable localized solutions in the case of attractive interactions. Experimentally, one can generate a NOL by a periodic spatial modulation of the scattering length on an optically induced Feshbach resonance [4]. In [5] it was verified that a crossed linear and nonlinear OL allows to stabilize 2D solitons against collapse or decay with either attractive or repulsive mean interaction.

The existence of 3D matter-wave solitons were demonstrated, analytically by using a multi-Gaussian variation approach (VA); and, numerically, by the relaxation method detailed in [6], as well as by direct time integrations of the Gross-Pitaevskii equation (GPE), which also allows us to study the dynamics of the corresponding matter waves. In order to investigate the stability of the solutions, it was considered the Vakhitov-Kolokolov (VK) [7] necessary criterion.

Due to the lattice anisotropy, the solitons display elliptical cross sections, which in our VA are accounted by a multi-Gaussian *ansatz* with different parameters for each one of the perpendicular spatial directions. The VA results are shown in terms of chemical potentials ( $\mu$ ) as functions of the number of particles ( $N$ ), with the VK criterion being used to investigate the stability of the solutions. Such results are compared with direct numerical integrations of the full GPE.

The basic mean-field formalism for the 3D GPE, in dimensionless units ( $\hbar = m = 1$ ), follows below. The GPE with cross-combined linear and nonlinear OLs, can be given by

$$i\frac{\partial u}{\partial t} = -\nabla^2 u - Vu - \Gamma|u|^2 u, \quad (1)$$

where the linear  $V$  and nonlinear  $\Gamma$  interactions are given, respectively, by

$$V \equiv V(x, z) = \varepsilon_x \cos(2x) + \varepsilon_z \cos(2z) \quad (2)$$

and

$$\Gamma \equiv \Gamma(y) = \chi + \gamma \cos(\lambda y). \quad (3)$$

The parameter  $\chi$  is linearly proportional to the two-body scattering length, with the optical lattices parametrized by  $\varepsilon_x$  and  $\varepsilon_z$ , in the linear case; and by  $\gamma$  and  $\lambda$  in the nonlinear case. The variational *ansatz* is a combination of three Gaussian functions, with parameters  $A$ ,  $a$ ,  $b$  and  $c$ :

$$U(x, y, z) = A \exp \left[ -\frac{(ax^2 + by^2 + cz^2)}{2} \right]. \quad (4)$$

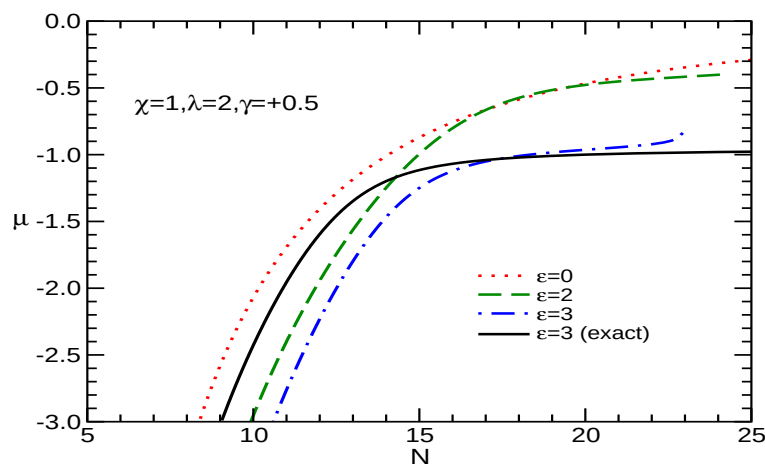
By defining  $\mu$  as the chemical potential, we look for variational solutions of the form

$$u(x, y, z, t) = U(x, y, z) \exp(-i\mu t). \quad (5)$$

### 2.1. Results - 3D Solitons with quasi-2D OLs

We start this 3D investigation by considering a cross-combined quasi-2D OLs, where we have a nonlinear OL in the  $y$ -direction, but no trap or OL in the  $z$ -direction, such that  $\varepsilon_z = 0$  and  $\varepsilon_x \equiv \varepsilon$ . The corresponding variational results are being exemplified by some curves given in Fig. 1, in the  $\mu - N$  plane, obtained for different values of the parameter  $\varepsilon$ , when we fix the parameters of the nonlinear interaction to  $\chi = 1$ ,  $\gamma = 0.5$  and  $\lambda = 2$ . As verified, no stable 3D solitons are predicted to exist, according to the VK criterion, which predicts stability for  $d\mu/dN < 0$  in the attractive case.

The variational approach gives in general a good agreement with full numerical PDE simulations. It is also verified that PDE simulations confirm the prediction of the VK criterion about the instability of 3D solitons. The agreement between VA and PDE numerical simulations is exemplified by the case with  $\varepsilon_x = 3$ , as shown in Fig. 1.



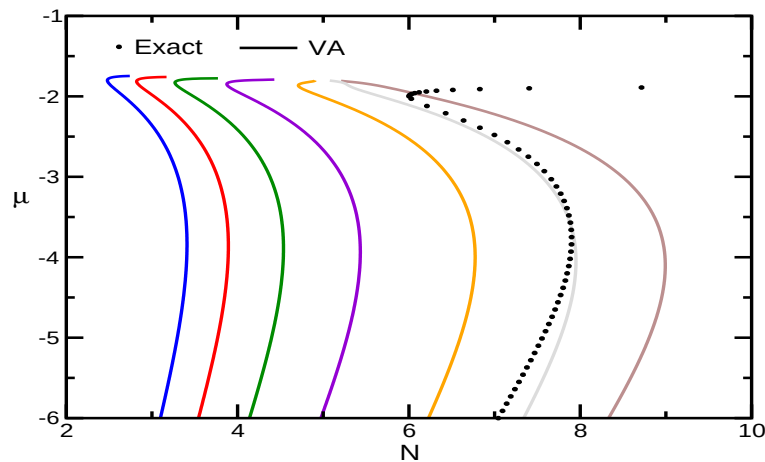
**Figure 1.** Variational results obtained for  $\mu$  in terms of  $N$ , are shown in case of 3D solitons, for crossed 1D LOL ( $x$ -direction) with 1D NLOL ( $y$ -direction), without any constraint in the third  $z$ -direction ( $\varepsilon_z = 0$ ), considering attractive mean two-body interactions ( $\chi = 1$ ). The results are presented for  $\varepsilon_x \equiv \varepsilon$ , with other parameters fixed as shown inside the frame. For  $\varepsilon = 3$ , we compare the VA (dot-dashed line) with direct numerical solutions (solid line).

### 2.2. Results - 3D Solitons for two linear with one nonlinear OLs

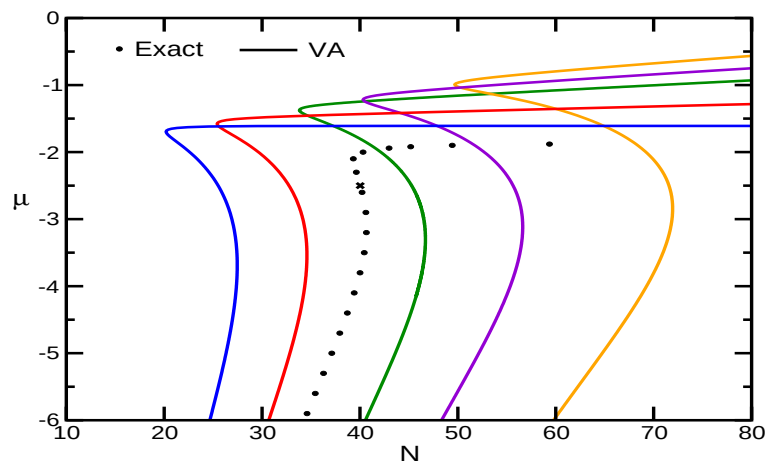
By considering OL in three perpendicular directions, with a nonlinear OL in one of them, some sample results are displayed in Figs. 2 and 3, for attractive and repulsive cases, respectively. As we observe, in both the cases there are branches of the curves for which one can predict stable 3D solitons, by considering the VK criterion ( $d\mu/dN < 0$ ). For comparison with specific VA solutions (solid lines), in each of the figures we add one dotted line with direct PDE calculations.

### 2.3. Results - Time evolution and stability

In Fig. 4, we present some results related to time evolution and stability of the solitons, considering the evolution of a profile obtained from direct numerical time integrations of the GP equation. The profile is for one specific point of the dotted curve shown in Fig. 3 for the repulsive case, with  $\mu = -2.5$ . This point is close to the limit that defines the stable or unstable regions according to the VK criterion. As shown in Fig. 4, by reducing  $\varepsilon_z$ , the instability occurs in a very short time interval.

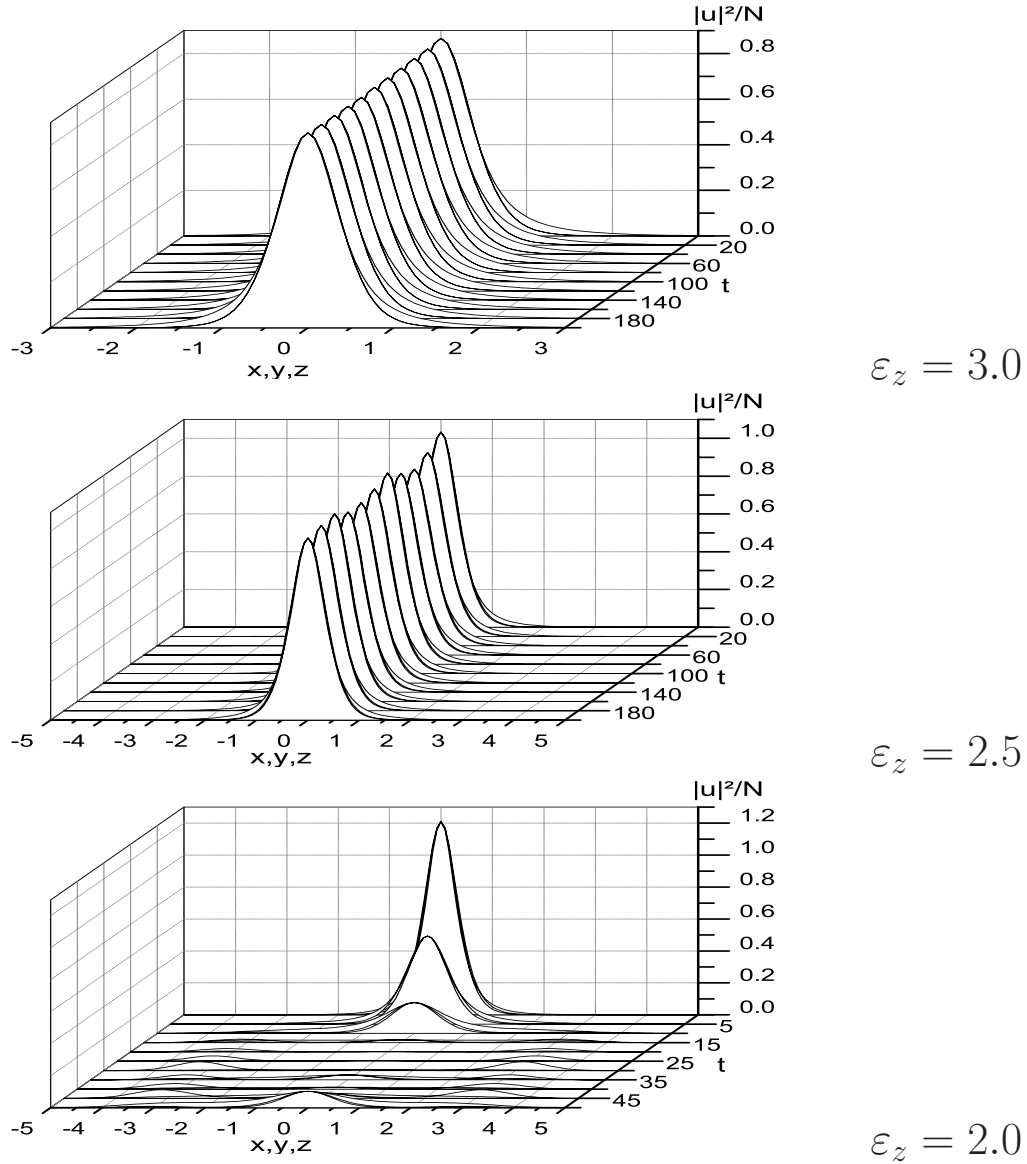


**Figure 2.** In cross-combined 2D LOL with 1D NOL, VA results (solid lines) for 3D solitons are shown for  $\mu$  in terms of  $N$ . We fix  $\varepsilon_x = \varepsilon_z \equiv \varepsilon = 3$ , with attractive interaction  $\chi = 1$ , varying the NOL strength, such that, from left to right,  $\gamma = 3.0, 2.5, 2.0, 1.5, 1.0, 0.7, 0.5$ , respectively. For comparison with PDE simulations, a sample result with  $\gamma = 0.5$  is given by the dotted curve.



**Figure 3.** VA results (solid lines) for  $\mu$  versus  $N$  are shown for fixed  $\varepsilon = 3$  and  $\gamma = 0.5$ , with several repulsive mean nonlinearity interactions,  $\chi = 0.0, -0.1, -0.2, -0.25, -0.3$  (from left to right, respectively). For comparison with VA, it is included PDE results for  $\chi = -0.2$ .

This section is concluded by some remarks evidenced by the results. While in the 2D OL cross-combined case, 3D solitons are always unstable, in the full 3D OL cross-combined case, we can obtain families of 3D solitons, which can be stable for both attractive and repulsive interactions. In perspective of potential applications, we should point out that, besides providing an alternate way to create stable solitons in 3D, with cross-combined linear and nonlinear optical lattices (both in BEC and nonlinear optics), these results can also be useful in practical applications, opening the possibility to manage stable 3D solitons through spatial modulations of the scattering length in one of the optical lattice directions.



**Figure 4.** The soliton dynamics, for the repulsive case given in Fig. 3, for  $\mu = -2.5$ , with  $\varepsilon_x = 3.0$ ,  $\gamma = 0.5$ , and  $\chi = -0.2$ . The loss of stability is verified by the reduction of the linear parameter in one direction, from  $\varepsilon_z = 3.0$  (top panel) to  $\varepsilon_z = 2.5$  (middle panel) and  $\varepsilon_z = 2.0$  (lower panel). The profiles in the  $y$ -direction are slightly wider.

### 3. Nonlocal nonlinear lattice from induced dipole-dipole interactions

In this section we report some results that have been investigated in Ref. [2], considering a condensate elongated along axis  $x$ , with dipole moments of polarizable molecules or atoms induced by an external field directed along  $x$  too. The local strength of the polarizing field also varies along  $x$ . Accordingly, one can use an effectively 1D GPE, with the DDI term derived from the underlying 3D GPE. The necessary spatially modulated dc electric and/or magnetic field can be imposed by ferroelectric or ferromagnetic lattices. Besides the fact that one could consider magnetic dipole moments induced by a solenoid, within our formalism we have considered local dipole moment induced by a polarizing electric field.

In addition to the ferroelectric lattice, the periodic modulation of the strength of the electric

field oriented *perpendicular* to the system's axis ( $x$ ) can be provided by a capacitor with the separation between its electrodes modulated in  $x$  periodically, as discussed in Ref. [8].

### 3.1. 1D GPE formalism with DDI interaction

From the original 3D GPE, as shown in Ref. [2], by considering DDI interaction, we can derive the following reduced 1D equation:

$$i\frac{\partial\phi}{\partial t} = -\frac{1}{2}\frac{\partial^2\phi}{\partial x^2} + \beta f^2(x)\phi + g|\phi|^2\phi - f(x)\phi \int_{-\infty}^{+\infty} f(x')|\phi'|^2 R(x-x')dx', \quad (6)$$

where  $\phi \equiv \phi(x, t)$  and  $\phi' \equiv \phi(x', t)$ , with the effective 1D kernel given by

$$R(x) = \sigma \frac{10}{\pi} \left[ \left(1 + 2x^2\right) \exp\left(x^2\right) \operatorname{erfc}(|x|) - \frac{2}{\sqrt{\pi}}|x| \right]. \quad (7)$$

Here, the parameter  $\sigma$  takes values  $\sigma = +1$  for the attractive DDI, and  $\sigma = -1/2$  for the repulsive DDI between dipoles oriented perpendicular to  $x$ , which makes it possible to consider the latter case too. However, for the perpendicular orientation of the dipoles, we have shown by means of numerical results for different values of  $\mu$ , that the wave-function profiles are delocalized, i.e., they do not build bright solitons. Therefore, in the next we restrict our analysis to the attractive DDI, with  $\sigma = 1$ . Actually, the kernel (7) can be replaced by a simplified expression [9],

$$R(x) = \frac{10}{\pi \sqrt{(\pi x^2 + 1)^3}}. \quad (8)$$

It is very close to (7), but smoother near  $x = 0$ , while (7) has a cusp at this point.

Note that the DDI can be represented by a pseudopotential that includes a contact-interaction (delta-functional) term [10, 11]. Therefore, the spatially modulation may induce a position-depending part of the contact interactions too. However, in the present setting, we restrict ourselves to a pure nonlinear nonlocal lattice. We assume that the dynamics of the system in the perpendicular directions is completely frozen, i.e., the transverse trapping frequency,  $\omega_{\perp}$ , is much larger than the longitudinal one,  $\omega_{\perp} \gg \omega_{\parallel}$ . On the other hand, if  $\omega_{\perp}$  is not too large, interesting transverse effects may occur, such as the Einstein-de Haas effect [12, 13, 14, 15], which is beyond the scope of our present analysis.

In Eq. (6), the nonlinear term  $g|\phi|^2\phi$  accounts for the usual collisional two-body interaction, with the effective DDI potential being composed by linear and nonlinear terms, as follows:

$$V_{\text{eff}}^{(\text{DDI})}(x; |\phi|^2) = f(x) \left[ \beta f(x) - \int_{-\infty}^{+\infty} f(x')|\phi'|^2 R(x-x')dx' \right], \quad (9)$$

where the modulation function is chosen, as said above, in the form of a periodic one:

$$f(x) = f_0 + f_1 \cos(kx), \quad (10)$$

with the parameters  $f_0$ ,  $f_1$ , and  $k \equiv 2\pi a_{\perp}/\lambda = 2\pi/\Lambda$ .  $\beta$  in Eq.(6) can vary from 1 to 10, under typical physical conditions, if the constant part of the modulation function is fixed as  $f_0 \equiv 1$ .

The Hamiltonian corresponding to Eq. (6) is

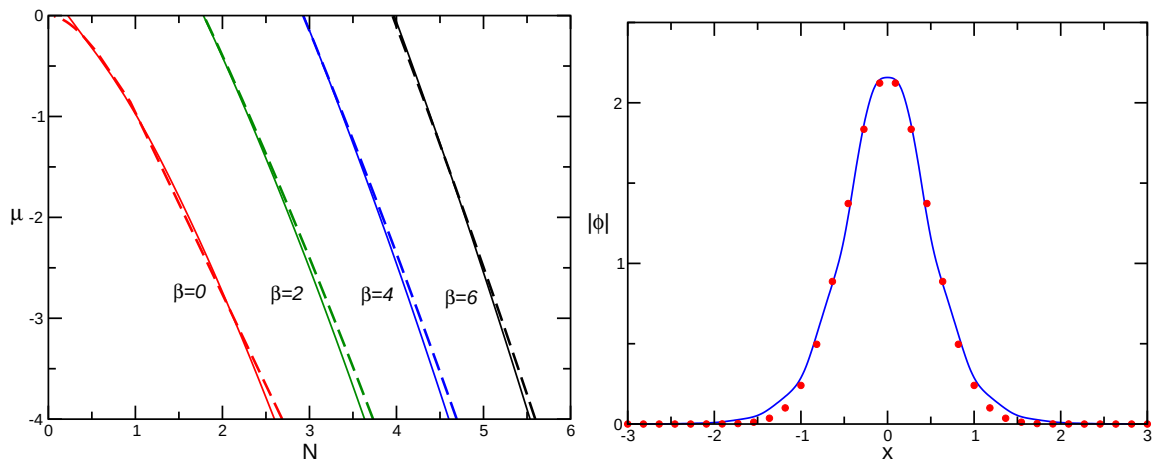
$$\begin{aligned} H = & \int_{-\infty}^{+\infty} dx \left[ \frac{1}{2} \left| \frac{\partial\phi}{\partial x} \right|^2 + \frac{g}{2} |\phi|^4 + \beta f^2(x) |\phi|^2 \right] \\ & - \frac{1}{2} \int_{-\infty}^{+\infty} dx f(x) |\phi|^2 \int_{-\infty}^{+\infty} dx' f(x') |\phi'|^2 R(x-x'). \end{aligned} \quad (11)$$

Note that it contains not only the spatially modulated nonlinear DDI, but also the additional linear potential,  $\beta f^2(x)$ , which is derived from the locally induced dipole moment with the polarizing field [8]. The existence of bright-soliton solutions can be investigated by solving the corresponding eigenvalue problem, obtained from Eqs. (6) and (9), with  $\phi = |\phi|e^{-i\mu t}$ :

$$-\frac{1}{2}\frac{\partial^2\phi}{\partial x^2} + g|\phi|^2\phi + V_{eff}^{DDI}(x;|\phi|^2)\phi = \mu\phi. \quad (12)$$

### 3.2. Results - Bright solitons, existence and stability

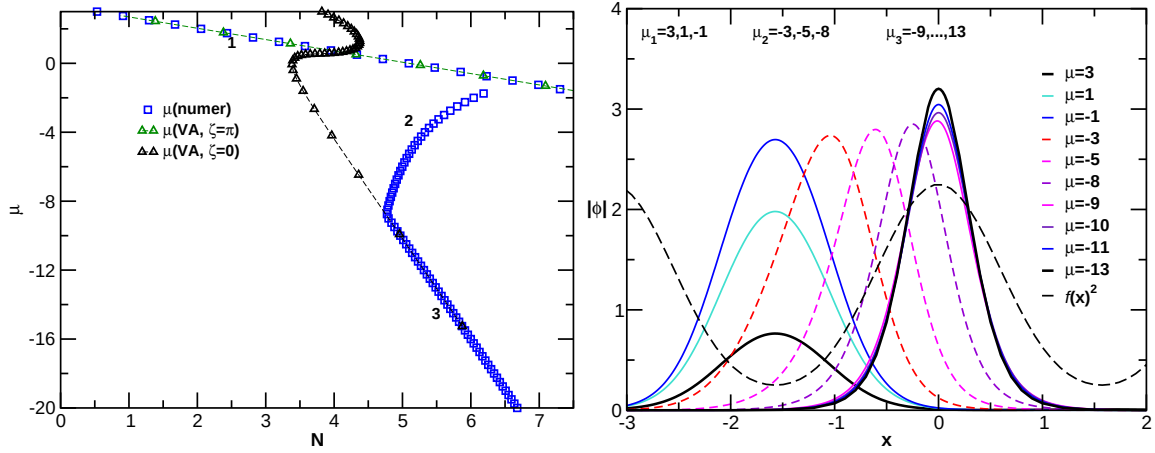
In our following analysis, we consider full numerical solutions of Eq. (12), as well as the corresponding VA results, for two characteristic cases. The solutions produced by the VA are compared with their numerically counterparts in Figs. 5 and 6, for  $g = 0$ ,  $f_0 = 1$ ,  $f_1 = 0.5$ , and  $\Lambda = 0.5$  ( $k = 4\pi$ ). The corresponding numerical solutions of Eq. (12) were obtained by means of a relaxation technique [6]. The effect of the nonlinearity,  $\beta$  (the scaled polarizability), is illustrated in the left panel of Fig. 5. As shown, VA and the numerical results are in good agreement. The variational and numerical results for the soliton profile corresponding to  $\mu = 0$  and  $\beta = 6$  are compared in the right panel of the figure. We also note that for large  $\beta$  the contribution of the linear lattice grows, leading to more accurate results for the variational approach [16].



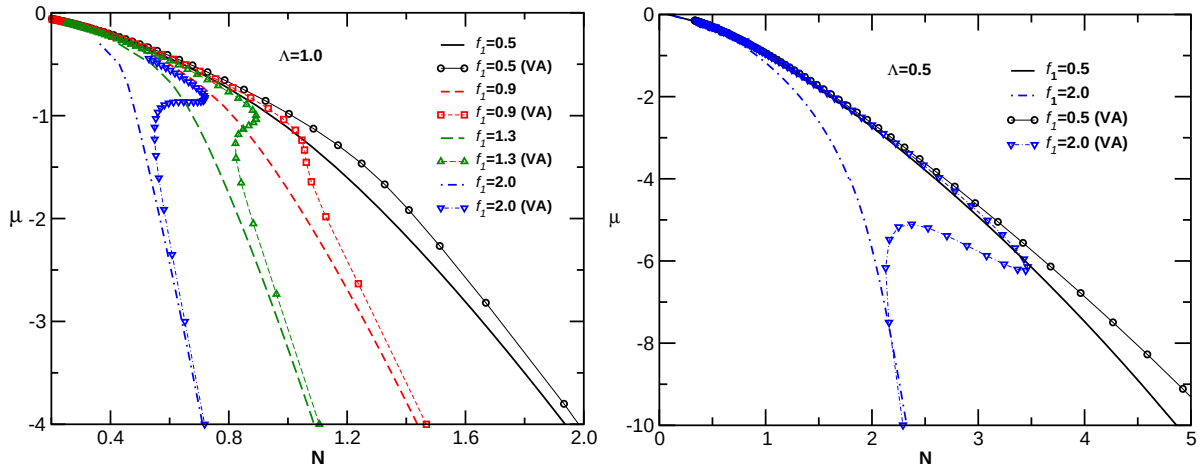
**Figure 5.** In the left panel, we show  $\mu$  versus  $N$  from direct numerical solutions of Eq. (12) (solid lines), and from the corresponding VA (dashed lines). The parameter  $\beta$  is indicated inside the frame, with the other parameters fixed to  $g = 0$ ,  $f_0 = 1$ ,  $f_1 = 0.5$  and  $\Lambda = 0.5$  ( $k = 4\pi$ ). In the right panel, we have the profile of the wave function centered in  $x = 0$ , for  $\beta = 6$  and  $\mu = 0$ . The full-numerical solution (solid line) is compared with VA (dotted line).

In the left panel of Fig. 6, for  $k = 2$  ( $\Lambda = \pi$ ) and  $\beta = 6$ , with the other parameters the same as in Fig. 5, the numerical results for the  $\mu$  versus  $N$  are shown in three distinct regions: two stable (1 and 3) and one unstable (region 2). The profiles, displayed in the right panel, clearly show that the wave function profiles are centered at  $x = \pi/2$  in the stable region 1. In the unstable region 2, it is verified a transition from the stable region 1 to the stable region 3, with the center of the profiles moving to  $x = 0$ . As seen in Fig. 6, the VA gives a perfect agreement with the numerical solutions in the region 1 (where the center of the profile is located at  $x = \zeta = \pi/2$ ), when  $\mu > -2$ , and in region 3 (where the center of the profile is at  $x = \zeta = 0$ ), when  $\mu < -9$ . However, the simple Gaussian ansatz we have used cannot follow the behavior presented by the numerical results in region 2.





**Figure 6.** In the left panel,  $\mu$  is given in terms of  $N$ , by considering numerical and variational results, for  $\Lambda = \pi$  (i.e.,  $k = 2$ ). Other parameters are  $g = 0$ ,  $f_0 = 1$ ,  $f_1 = 0.5$  and  $\beta = 6$ . In this panel, we indicate three regions (1, 2 and 3) of the numerical solutions, following the variation of the chemical potential ( $\mu_1$ ,  $\mu_2$  and  $\mu_3$ ), to identify the corresponding profiles in the right panel. The modulation function,  $f^2(x)$ , is shown by the black-dashed line in the right one.



**Figure 7.** Plots for  $\mu$  versus  $N$  are shown for different values of  $f_1$  (as indicated inside the frames), with  $\beta = 0$ ,  $f_0 = 1$ , and  $\Lambda = 1$  (left panel) or  $0.5$  (right panel).

Numerical results, for  $\mu$  versus  $N$  compared with VA findings for  $\beta = 0$ , are shown in Fig. 7, for different values of  $f_1$ , with  $\Lambda = 1$  (left panel) and  $0.5$  (right panel). The VK criterion for stability,  $\partial\mu/\partial N < 0$  [7], is well verified by our numerical results. Simulations of temporal evolutions that were performed validate the VK criterion in the present model. However, the VA results cannot follow the results to the full extension, besides the fact that they present very good agreement for large negative values of  $\mu$ . As seen in the left panel of Fig. 7, for  $\Lambda = 1$  the VA correctly predicts the stability and converges to numerical results for  $\mu < -1.5$  at all values of  $f_1$ . In the case of  $\Lambda = 0.5$  (right panel), the VA results are equally accurate at  $\mu < -6$ . On the other hand, for  $f_1 = 2$  and  $\Lambda = 0.5$ , the VA solutions deviate from the full numerical results in an interval of  $N$ , where one can obtain more than one  $\mu$  for the same  $N$ .

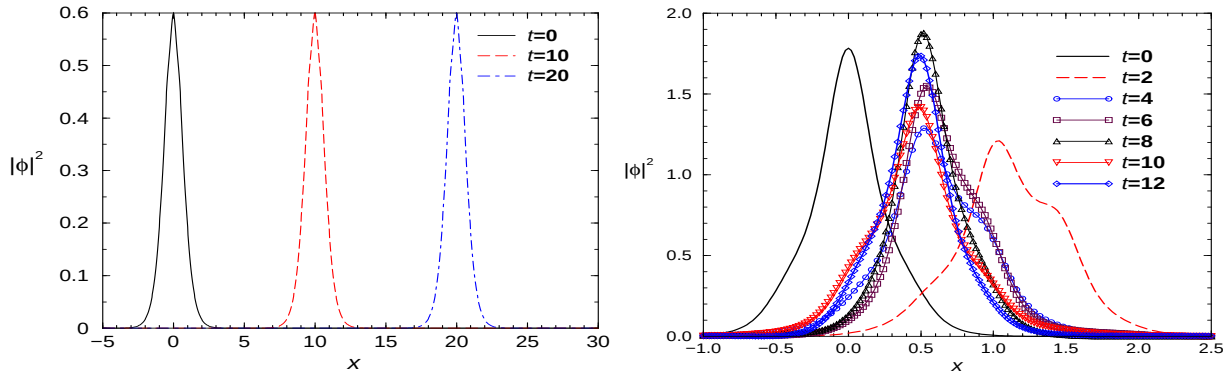


### 3.3. Dynamics of bright solitons

In order to conclude this section, we address some results on the mobility of solitons and their collisions. The soliton motion in nonlinear lattices was previously considered in Refs. [17, 18, 19]. First, we present full numerical solutions of the 1D GPE (6), exploring a parameter region for finding stable bright solitons. This is followed by consideration of a dynamical version of the VA, with some results for frequencies of oscillations of perturbed solitons being compared with full numerical calculations.

The propagation of a soliton is presented in the two panels of Fig. 8. In the left panel, for  $\mu = -1$  and  $N \approx 1.02$ , we show the soliton propagation by considering a time interval from 0 to 20, and velocity equal to 1 (dimensionless units). In the right panel, for  $\mu = -10$  and  $N \approx 4.86$ , with profiles separated by time intervals  $\Delta t = 2$ , we show that the soliton is being trapped at a fixed position. The parameters, in both the cases, are  $\beta = g = 0$ ,  $f_0 = 1$ ,  $f_1 = 0.5$ , and  $\Lambda = 0.5$ .

In Fig. 9 it is given a density plot for the interaction of two solitons, with zero phase difference between them, such that they attract each other. The parameters are the same as in Fig. 8. In such a case, we noticed a transition from a bound state to a breather. These results are extracted from figures 7 and 8 of Ref. [2], where more details are presented.



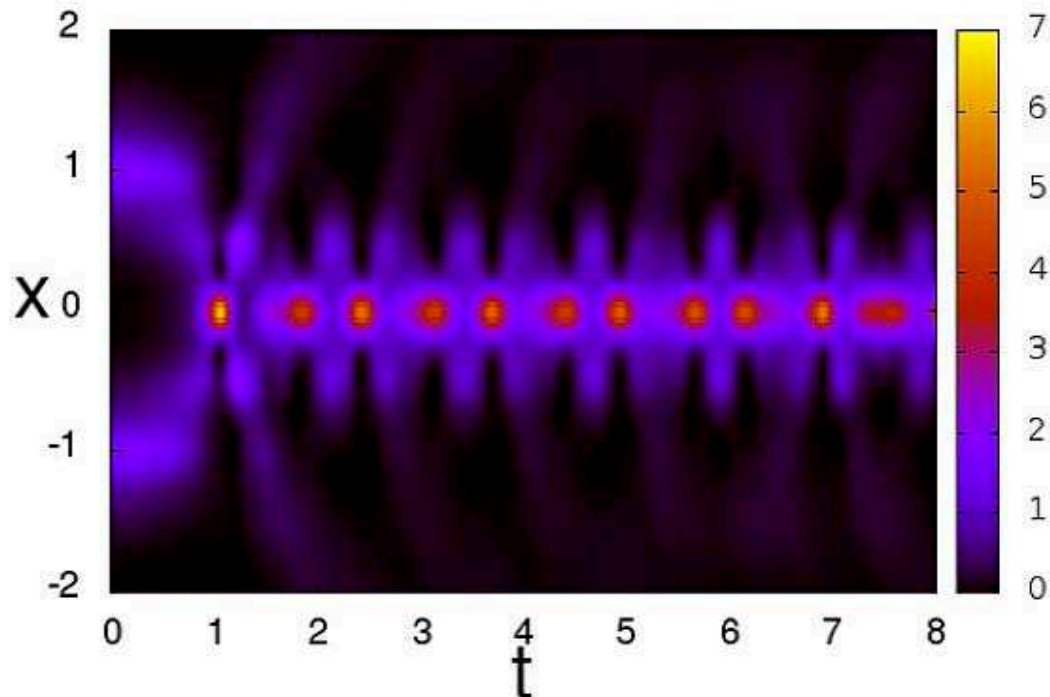
**Figure 8.** Propagation of a soliton. In the left panel, soliton profiles are shown with time intervals  $\Delta t = 10$  (the velocity is 1), for  $\mu = -1$  and  $N \approx 1.02$ . In the right panel,  $\mu = -10$ ,  $N \approx 4.86$ , we show the soliton being trapped at a fixed position, with the profiles given in intervals  $\Delta t = 2$ . In both cases,  $f_0 = 1$ ,  $f_1 = 0.5$  and  $\Lambda = 0.5$  (with  $\beta = 0$ ).

### 4. General conclusions

We report in section II results obtained on the existence of solitons and the corresponding dynamics from the case of cross-combined linear and nonlinear optical lattices, within a full 3D model. As shown, in the full 3D OL cross-combined case, we can obtain families of 3D solitons, which can be stable for both attractive and repulsive interactions. In perspective of potential applications, we should point out that, besides providing an alternate way to create stable solitons in 3D, with cross combined linear and nonlinear optical lattices (both in BEC and nonlinear optics), these results can also be useful in practical applications, opening the possibility to manage stable 3D solitons through spatial modulations of the scattering length in one of the OL directions. In section III, we report investigations related to another model, considering a 1D reduced case. Within this model, we study the conditions for the existence of bright solitons considering induced dipole-dipole interactions. The results were verified by comparison with numerical solutions of the respective 1D GPE. The stability of the soliton solutions is verified by considering the VK criterion. We also report investigations related to the dynamics of solitons and the interactions between them, including merger into breathers.

### Acknowledgments

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**Figure 9.** The interaction of two solitons represented in a density plot (scale in the right of the figure), for  $\mu = -10$  and  $N \approx 4.86$ , with zero average velocity. The soliton position  $x$  is shown as a function of time  $t$ . The parameters are the same as in Fig. 8, with zero phase difference between the solitons. A transition from a bound state to a breather is evidenced.

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