



# Existence and symmetries of solutions in Besov–Morrey spaces for a semilinear heat-wave type equation



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## ABSTRACT

This paper considers a semilinear integro-differential equation of Volterra type which interpolates semilinear heat and wave equations. Global existence of solutions is showed in spaces of Besov type based in Morrey spaces, namely Besov–Morrey spaces. Our initial data is larger than the previous works and our results provide a maximal existence class for semilinear interpolated heat-wave equation. Some symmetries, self-similarity and asymptotic behavior of solutions are also investigated in the framework of Besov–Morrey spaces.

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## 1. Introduction and results

This paper concerns with a semilinear time-fractional partial differential equation (FPDE for short) which describes diverse physical phenomena and mathematical models (see e.g. [13,14,26]). More precisely, in this paper we consider the semilinear integro-partial differential equation in  $\mathbb{R}^n$ , which reads as

$$\begin{cases} u_t = \int_0^t r_\alpha(t-s)[P(D)u(s) + f(u(s))]ds, & (x \in \mathbb{R}^n \text{ and } t > 0) \\ u(0, x) = u_0(x), & x \in \mathbb{R}^n \end{cases} \quad (1.1)$$

where  $u(t) = u(t, x) = (u_1(t, x), \dots, u_n(t, x))$  with  $n \geq 1$ ,  $r_\alpha(t) = \nu t^{\alpha-1}/\Gamma(\alpha)$ ,  $\Gamma(\alpha)$  denotes the gamma function,  $P(D) = \Delta_x$  is the Laplacian operator on  $x$ -variable,  $\nu$  denotes the Newtonian viscosity and  $f : \mathbb{R} \rightarrow \mathbb{R}$  is a function satisfying

$$f(0) = 0 \text{ and } |f(a) - f(b)| \leq C|a - b| (|a|^{\rho-1} + |b|^{\rho-1}). \quad (1.2)$$

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Above  $\rho > 1$  and  $C$  is a positive constant independent of  $a, b \in \mathbb{R}$ . Typical examples of  $f(u)$  are given by  $\gamma|u|^{\rho-1}u$  and  $\gamma|u|^\rho$  for  $\gamma \in \{+, -\}$ . These nonlinearities yield a scaling for (1.1) which is fundamental for our approach in Besov–Morrey spaces  $\mathcal{N}_{p,\mu,\infty}^\sigma$  (see (2.7), for the definition). These spaces are a type of Besov spaces based on Morrey spaces and have been introduced by H. Kozono and M. Yamazaki [16] for analysis of the Navier–Stokes equations. A number of authors have studied PDEs (see [4,5,20,28,37]) and harmonic analysis (see [27,35]) in this framework; for further details, see [30,31] and references therein. As far as we know, the existence problem for (1.1) in Besov–Morrey spaces is new for the fractional case  $\alpha \neq 1$ . Formally, the problem (1.1) is equivalent to (FPDE)

$$\partial_t^\alpha u = \nu P(D)u + f(u) \quad \text{in } (0, \infty) \times \mathbb{R}^n \quad (1.3)$$

$$u_t(0) = 0 \text{ and } u(0) = u_0 \quad \text{in } \mathbb{R}^n, \quad (1.4)$$

where  $\partial_t^\alpha u = D_{0|t}^{\alpha-1} u_t$ ,  $u_t = \frac{\partial u}{\partial t}$  and  $D_{0|t}^{\alpha-1}$  stands for the Riemann–Liouville derivative of order  $\alpha-1$ , namely

$$D_{0|t}^{\alpha-1} u = \frac{1}{\Gamma(2-\alpha)} \frac{\partial}{\partial t} \int_0^t \frac{u(s)}{(t-s)^{\alpha-1}} ds, \quad \text{for } t > 0.$$

Employing a Duhamel-type formula (see [11, Proposition 2.1]) in (1.3)–(1.4) (or (1.1)), formally we obtain the integral equation

$$u(t) = L_\alpha(t)u_0 + B_\alpha(u)(t), \quad (1.5)$$

where

$$B_\alpha(u)(t) = \int_0^t L_\alpha(t-s) \left( \int_0^s r_{\alpha-1}(s-\tau) f(u(\tau)) d\tau \right) ds \quad (1.6)$$

and  $\{L_\alpha(t)\}_{t \geq 0}$  stands for the family of convolution operators (or diffusion-wave operators) defined by

$$\widehat{L_\alpha(t)\varphi}(\xi) = E_\alpha(-t^\alpha|\xi|^2) \widehat{\varphi}(\xi). \quad (1.7)$$

Throughout this paper a *mild solution* for (1.3)–(1.4) (or (1.1)) is a function  $u(t, x)$  satisfying (1.5) and  $u(t, x) \rightharpoonup u_0$  in  $\mathcal{S}'(\mathbb{R}^n)$  as  $t \rightarrow 0^+$ . Actually, using Proposition 3.2 below and Sobolev embedding (2.11), we are going to show this weak convergence in homogeneous Besov space  $\dot{B}_{\infty,\infty}^{2/(\rho-1)}$ , see details in Lemmas 3.4 and 3.5. Here  $E_\alpha(-t^\alpha|\xi|^2)$  stands for Mittag-Leffler function (see (2.15)) and  $\widehat{\cdot} = \mathcal{F}$  stands for the Fourier transform. For  $\alpha = 1$ , the operator  $L_1(t) = S(t)$  is the heat semigroup, because  $E_1(-t|\xi|^2) = e^{-t|\xi|^2}$ . The kernel  $k_\alpha$  of  $L_\alpha(t)$  is the fundamental solution of (1.3) with  $f \equiv 0$ , namely

$$k_\alpha(t, x) = \int_{\mathbb{R}^n} e^{ix \cdot \xi} E_\alpha(-t^\alpha|\xi|^2) d\xi, \quad (1.8)$$

which, in one-dimensional case, reads as (see [8])

$$k_\alpha(t, x) = \frac{1}{\alpha} \int_{\mathbb{R}} \exp\{i\xi x - t|\xi|^{\frac{2}{\alpha}} e^{-i\frac{\gamma\pi}{2} \operatorname{sgn}(\xi)}\} d\xi, \quad \left(\gamma = 2 - \frac{2}{\alpha}\right).$$

The FPDE (1.3)–(1.4) interpolates two PDEs (see e.g. [9]), namely semilinear wave ( $\alpha = 2$ ) and heat ( $\alpha = 1$ ) equations, which have been widely investigated in the last years. These PDEs present many

differences in the theory of existence and asymptotic behavior of solutions in scaling invariant spaces (critical spaces). In the case  $\alpha = 1$ , the FPDE (1.3) is well documented in *critical spaces*, see e.g. [16]. Without making a complete list, we mention  $L^p$ -spaces, weak- $L^p$  spaces, Besov spaces  $\dot{B}_{p,\infty}^s$ , Morrey spaces  $\mathcal{M}_{p,\mu}$ , Besov–Morrey spaces  $\mathcal{N}_{p,\mu,\infty}^s$  and among others. However, there are few papers dealing with FPDEs in those spaces when  $1 < \alpha < 2$ . In [11], the authors used Mihlin–Hörmander’s theorem in order to establish  $L^p$ – $L^r$  estimates for Mittag-Leffler’s family (1.7) and obtained local well-posedness in an  $L^r(\mathbb{R}^n)$ -framework. Using the estimates of [11] and employing techniques of [2,22], the authors of [21] showed the existence of self-similar global solution with initial data  $u_0 \in \dot{B}_{p,\infty}^{n/p-2/(\rho-1)} \cap \mathcal{E}_{q(r,p_0),r}$  (for  $\mathcal{E}_{q(r,p_0),r}$ , see Remark 1.2-(ii)). In [3], the authors studied qualitative properties, like self-similarity, antisymmetry and positivity, of global solutions for small initial data in Morrey space  $\mathcal{M}_{p,\lambda}$ ,  $\lambda = n - \frac{2p}{\rho-1}$ . Now, let  $P(D)f$  be the Riesz potential  $(-\Delta_x)^{\beta/2}f = \mathcal{F}^{-1}|\xi|^\beta \mathcal{F}f$  and  $f(u(t)) = h(x,t)|u(t)|^{\rho-1}u(t)$ . It is worth to mention the works [10,15,32] where the authors, motivated by works of Fujita [7,36], established conditions for either blow up or global existence of weak nonnegative solutions. It is not known if solutions of the Navier–Stokes equations are smooth for all  $t > 0$ , however Lions [18] showed a priori estimate

$$\int_0^T \|D_{0|t}^\gamma u(t)\|_{L^2(\mathbb{R}^n)} dt \leq \text{const.}(J + J^{\frac{3}{2}}), \quad J = \|u_0\|_{L^2(\mathbb{R}^n)} \quad (1.9)$$

where  $0 \leq \gamma < 1/4$  and  $u$  is a weak solution in  $L^2((0,T); L^2(\mathbb{R}^n))$  associated to the data  $u_0 \in L^2(\mathbb{R}^n)$ , for  $n \leq 4$ . In [29, Theorem 5.3], Shinbrot gave a step ahead showing (1.9) for all dimensions  $n$  and  $0 \leq \gamma < 1/2$ . This shows that solutions of the Navier–Stokes have more smoothness in  $t$  than at first appears. It seems that our initial data class (see Theorem 1.1) is larger than the previous works and contains strongly singular functions (see Remark 1.2-(iii)). For  $\mu = n - \frac{2}{\rho-1}$  and  $\lambda = n - \frac{2p}{\rho-1}$ , one has the continuous inclusions

$$L^q \subset \text{weak-}L^q \subset \mathcal{M}_{p,\lambda} \subset \mathcal{N}_{p,\mu,\infty}^\sigma \quad \text{and} \quad \dot{B}_{r,\infty}^k \subset \mathcal{N}_{p,\mu,\infty}^\sigma \quad (1.10)$$

provided that  $\frac{n}{q} = \frac{n-\lambda}{p} = -\sigma + \frac{n-\mu}{p} = -k + \frac{n}{r}$ , where  $\sigma = \frac{n-\mu}{p} - \frac{2}{\rho-1}$ ,  $k = \frac{n}{r} - \frac{2}{\rho-1}$  and  $1 \leq p < r < q \leq \frac{n(\rho-1)}{2}$  (all spaces in (1.10) are invariant by scaling (1.12)). Moreover, there is no known existence of solutions for (1.3)–(1.4) in a class such that  $X \supsetneq \mathcal{N}_{p,\mu,\infty}^\sigma$ . In this sense, we provide a maximal existence class for (1.3)–(1.4) also we improve the well-posedness result in [3,21].

One of the aims of this paper is to establish the existence of solutions for (1.3)–(1.4) in the framework of Besov–Morrey spaces. For that matter, we obtain estimates in Sobolev–Morrey and Besov–Morrey spaces for the diffusion-wave operator  $L_\alpha(t)$  (see Lemma 3.1) which could have an interest of its own. Furthermore, some symmetries properties, self-similarity and asymptotic behavior of solutions are also investigated. We perform a scaling analysis in order to choose the correct indexes of spaces such that their norms are invariant under the scaling (1.11). Indeed, it is well known that if  $u$  solves (1.3) with  $f(u) = \gamma|u|^{\rho-1}u$  then, for each  $\lambda > 0$ , the rescaled function  $u_\lambda(t, x) = \lambda^{\frac{2}{\rho-1}}u(\lambda^{\frac{2}{\alpha}}t, \lambda x)$  is also a solution. This leads us to define a scaling map for (1.3) as

$$u(t, x) \mapsto u_\lambda(t, x). \quad (1.11)$$

Making  $t \rightarrow 0^+$  in (1.11), this map induces the following scaling for initial data

$$u_0(x) \mapsto u_{0\lambda}(x) = \lambda^{\frac{2}{\rho-1}}u_0(\lambda x). \quad (1.12)$$

Solutions invariant by (1.11), namely  $u(t, x) = u_\lambda(t, x)$ , are called *forward self-similar solutions*.

Let  $BC((0, \infty), X)$  be the class of bounded functions from  $(0, \infty)$  to a Banach space  $X$ . For  $1 < p \leq q < \infty$ , we define our ambient space based on Besov–Morrey type spaces (see (2.7)) as

$$X_q^p = \{u \in BC((0, \infty); \mathcal{N}_{p, \mu, \infty}^\sigma) : t^\eta u \in BC((0, \infty); \mathcal{M}_{q, \mu})\}, \quad (1.13)$$

which is a Banach space endowed with the norm

$$\|u\|_{X_q^p} := \sup_{t>0} \|u(t, \cdot)\|_{\mathcal{N}_{p, \mu, \infty}^\sigma} + \sup_{t>0} t^\eta \|u(t, \cdot)\|_{\mathcal{M}_{q, \mu}}. \quad (1.14)$$

Here  $\eta \in \mathbb{R}$  and  $\sigma < 0$  are given by

$$\eta = \frac{\alpha}{2} \left( \frac{2}{\rho-1} - \frac{n-\mu}{q} \right) \text{ and } \sigma = \frac{n-\mu}{p} - \frac{2}{\rho-1}, \quad (1.15)$$

where these values have been chosen in such a way that the norm (1.14) is invariant under the scaling map (1.11).

### 1.1. Main results

In what follows, we state our main results.

**Theorem 1.1** (Well-posedness). *Let  $n \geq 1$ ,  $1 \leq \alpha < 2$ ,  $1 < \{\rho, p\} \leq q < \infty$ , and  $0 \leq \mu < n$  be such that*

$$\frac{2}{\rho-1} - \frac{2}{\alpha\rho} < \frac{n-\mu}{q} < \frac{2}{\alpha(\rho-1)} \text{ and } \frac{n-\mu}{p} < \frac{2}{\rho-1}. \quad (1.16)$$

- (i) (Existence and uniqueness) *There are  $\varepsilon > 0$  and  $\delta = \delta(\varepsilon)$  such that if  $\|u_0\|_{\mathcal{N}_{p, \mu, \infty}^\sigma} \leq \delta$  then the problem (1.1) has a mild solution  $u \in X_q^p$  which is unique in the closed ball  $\{u \in X_q^p; \|u\|_{X_q^p} \leq 2\varepsilon\}$ . Also,  $u(t) \rightharpoonup u_0$  in the weak- $*$  topology of  $\dot{B}_{\infty, \infty}^{2/(\rho-1)}$  as  $t \rightarrow 0^+$ .*
- (ii) (Continuous dependence on data) *Consider the ball*

$$\mathcal{D}_\delta = \{u_0 \in \mathcal{N}_{p, \mu, \infty}^\sigma; \|u_0\|_{\mathcal{N}_{p, \mu, \infty}^\sigma} \leq \delta\}$$

*in the space  $\mathcal{N}_{p, \mu, \infty}^\sigma$ . The data-solution map  $u_0 \in \mathcal{D}_\delta \mapsto u \in X_q^p$  is Lipschitz continuous.*

### Remark 1.2.

- (i) Let  $l > 0$  be such that  $\{p, \rho\} \leq q \leq l$  and  $(n-\mu)/q = n/l$ . By (1.16) it follows that  $\alpha n(\rho-1) < 2l < \alpha n(\rho-1)\rho$  for  $1 \leq \alpha < 2$ . For every  $a \in \mathcal{N}_{p, \mu, \infty}^\sigma$  satisfying the assumptions of Theorem 1.1, there exists a unique solution  $u(t, x)$  of (1.3) in  $L^\infty((0, \infty); \mathcal{N}_{p, \mu, \infty}^\sigma)$  such that  $\|u(t, \cdot)\|_{\mathcal{M}_{q, \mu}} \leq C t^{-\alpha/(\rho-1) + \alpha(n-\mu)/2q}$ . In particular, we recover Theorem 1 of [16].
- (ii) Under the assumptions of Theorem 1.1, for  $\mu = 0$  and  $q \leq r \leq p$ , we reobtain the result in [21]. Indeed, in view of  $\mathcal{N}_{r, 0, \infty}^k = \dot{B}_{r, \infty}^k$  and proceeding as in Lemma 3.4 with  $(p, q) = (r, q)$ , one has

$$\begin{aligned} \|u_0\|_{\mathcal{E}_{q(r, p_0), r}} &:= \sup_{t>0} t^{\frac{1}{q(p_0, r)}} \|L_\alpha(t)u_0\|_r \\ &= \sup_{t>0} t^{\frac{\alpha}{\rho-1} - \frac{\alpha n}{2r}} \|L_\alpha(t)u_0\|_r \leq C \|u_0\|_{\dot{B}_{r, \infty}^{\frac{n}{r} - \frac{2}{\rho-1}}}, \end{aligned}$$

where  $\frac{1}{q(p_0, r)} = \frac{n\alpha}{2} \left( \frac{1}{p_0} - \frac{1}{r} \right)$  and  $p_0 = \frac{n(\rho-1)}{2}$ . Now, using the assumption  $\|u_0\|_{\dot{B}_{r, \infty}^{n/r - 2/(\rho-1)}} \leq \delta$  in Theorem 1.1(i), one obtains Theorem 1.1 of [21].

(iii) Let  $\rho > 1 + 2/n$  and  $\lambda = n - 2p/(\rho - 1)$  for  $p > 1$ . It follows that

$$\mathcal{M}_{p,\lambda} \subset \mathcal{M}_{1,n-\frac{n-\lambda}{p}} \subset \mathcal{N}_{1,\mu,\infty}^0 \subset \mathcal{N}_{p,\mu,\infty}^\sigma, \quad \mu = n - \frac{2}{\rho-1}$$

in view of (2.9) and (2.11). Our initial data can be taken strictly larger than those in [3], see [16, Example 0.10].

(iv) Suppose that  $n = 1$  and let  $P(D) = D_\theta^\beta$  be the Riesz–Feller operator which is given by  $\widehat{D_\theta^\beta \varphi}(\xi) = \psi_\beta^\theta(\xi) \widehat{\varphi}(\xi)$ , where  $\psi_\beta^\theta(\xi) = -|\xi|^\beta e^{i(\operatorname{sgn} \xi) \frac{\pi\theta}{2}}$  with  $0 < \beta \leq 2$  and  $|\theta| \leq \min\{\beta, 2 - \beta\}$ ,  $\xi \in \mathbb{R}$ . Hence (see e.g. [19]), the diffusion-wave operator  $L_\alpha(t)$  reads as

$$\widehat{L_{\beta,\alpha}^\theta(t) \varphi}(\xi) = E_\alpha[-t^\alpha |\xi|^\beta e^{i(\operatorname{sgn} \xi) \frac{\pi\theta}{2}}] \widehat{\varphi}(\xi)$$

which has kernel

$$k_{\beta,\alpha}^\theta(t, x) = \begin{cases} \int_{\mathbb{R}} \exp\{i\xi x - t|\xi|^\beta e^{-i\frac{\theta\pi}{2}\operatorname{sgn}(\xi)}\} d\xi, & \alpha = 1 \\ \int_{\mathbb{R}} e^{i\xi x} E_\alpha[-t^\alpha \psi_\beta^\theta(\xi)] d\xi, & 1 < \alpha < 2. \end{cases}$$

If  $(1 \leq \alpha < 2)$  and  $(\beta = 2)$ , Theorem 1.1 gives us an insight on how to proceed on the study of SFPDEs (stochastic fractional partial differential equations)

$$\frac{\partial^\alpha u}{\partial t}(t, x) = D_\theta^\beta u(t, x) + g(t, x, u(t, x)) + \sum_{k=1}^n \frac{\partial^k h_k}{\partial x^k} + f(t, x, u(t, x)) \frac{\partial^2 W(t, x)}{\partial t \partial x}$$

with datum  $u_0$  in spaces more singular than  $L^p(\mathbb{R})$  spaces. Here, the functions  $f, g, h_k$  satisfy Lipschitz and certain growth conditions (see e.g. [1] and [24]).

Let  $O(n)$  be the orthogonal matrix group in  $\mathbb{R}^n$  and let  $\mathcal{G}$  be a subset of  $O(n)$ . If  $h(x) = h(Mx)$  and  $h(x) = -h(Mx)$ , for every  $M \in \mathcal{G}$ , then  $h$  is said *even* (or symmetric) and *odd* (or antisymmetric) under the action of  $\mathcal{G}$ , respectively.

**Theorem 1.3.** Assume the hypotheses of Theorem 1.1. Let  $f(u) = \gamma|u|^{\rho-1}u$ .

- (i) (Symmetry and antisymmetry) The solution  $u(x, t)$  is antisymmetric (resp. symmetric) for  $t > 0$ , when  $u_0$  is antisymmetric (resp. symmetric) under the action of  $\mathcal{G}$ .
- (ii) (Self-similarity) Let  $u_0$  be a homogeneous function of degree  $-\frac{2}{\rho-1}$ , then the mild solution given in Theorem 1.1 is self-similar.

**Remark 1.4.** If  $\mathcal{G} = O(n)$  we have radial symmetry. Indeed, it follows from Theorem 1.3(i) that if  $u_0$  is radially symmetric then  $u(x, t)$  is **radially symmetric** for all  $t > 0$ .

Also, we prove an asymptotic behavior result of the solutions obtained in Theorem 1.1 as  $t \rightarrow \infty$ .

**Theorem 1.5.** Assume the hypotheses of Theorem 1.1. Let  $u$  and  $v$  be two global mild solutions for (1.1) given by Theorem 1.1, with respective data  $u_0$  and  $v_0$ . We have that

$$\lim_{t \rightarrow +\infty} \|u(\cdot, t) - v(\cdot, t)\|_{\mathcal{N}_{p,\mu,\infty}^\sigma} = \lim_{t \rightarrow +\infty} t^\eta \|u(\cdot, t) - v(\cdot, t)\|_{\mathcal{M}_{q,\mu}} = 0 \quad (1.17)$$

if and only if

$$\lim_{t \rightarrow +\infty} \|L_\alpha(t)(u_0 - v_0)\|_{\mathcal{N}_{p,\mu,\infty}^\sigma} + t^\eta \|L_\alpha(t)(u_0 - v_0)\|_{\mathcal{M}_{q,\mu}} = 0. \quad (1.18)$$

The manuscript is organized as follows. In Section 2, basic properties of Sobolev–Morrey, Besov–Morrey spaces and Mittag-Leffler functions are reviewed. Section 3 is devoted to estimates for operators coming from (1.5). Proofs of the theorems are performed in Section 4.

## 2. Preliminaries

In this section we collect some well-known properties about Sobolev–Morrey and Besov–Morrey spaces. Also, we recall properties of the Mittag-Leffler functions.

### 2.1. Besov–Morrey space

The basic properties of Morrey and Besov–Morrey spaces is reviewed in the present subsection for the reader convenience, more details can be found in [12,16,17,25,33].

Let  $Q_r(x_0)$  be the open ball in  $\mathbb{R}^n$  centered at  $x_0$  and with radius  $r > 0$ . Given two parameters  $1 \leq p < \infty$  and  $0 \leq \mu < n$ , the Morrey spaces  $\mathcal{M}_{p,\mu} = \mathcal{M}_{p,\mu}(\mathbb{R}^n)$  is defined to be the set of functions  $f \in L^p(Q_r(x_0))$  such that

$$\|f\|_{p,\mu} := \sup_{x_0 \in \mathbb{R}^n} \sup_{r>0} r^{-\frac{\mu}{p}} \|f\|_{L^p(Q_r(x_0))} < \infty \quad (2.1)$$

which is a Banach space endowed with norm (2.1). For  $s \in \mathbb{R}$  and  $1 \leq p < \infty$ , the homogeneous Sobolev–Morrey space  $\mathcal{M}_{p,\mu}^s = (-\Delta)^{-s/2} \mathcal{M}_{p,\mu}$  is the Banach space with norm

$$\|f\|_{\mathcal{M}_{p,\mu}^s} = \left\| (-\Delta)^{s/2} f \right\|_{p,\mu}. \quad (2.2)$$

Taking  $p = 1$ , we have  $\|f\|_{L^1(Q_r(x_0))}$  denotes the total variation of  $f$  on open ball  $Q_r(x_0)$  and  $\mathcal{M}_{1,\mu}$  stands for space of signed measures. In particular,  $\mathcal{M}_{1,0} = \mathcal{M}$  is the space of finite measures. For  $p > 1$ , we have  $\mathcal{M}_{p,0} = L^p$  and  $\mathcal{M}_{p,0}^s = \dot{H}_p^s$  is the well known Sobolev space. The space  $L^\infty$  corresponds to  $\mathcal{M}_{\infty,\mu}$ . Morrey and Sobolev–Morrey spaces presents the following scaling

$$\|f(\lambda \cdot)\|_{p,\mu} = \lambda^{-\frac{n-\mu}{p}} \|f\|_{p,\mu} \quad (2.3)$$

and

$$\|f(\lambda \cdot)\|_{\mathcal{M}_{p,\mu}^s} = \lambda^{s-\frac{n-\mu}{p}} \|f\|_{\mathcal{M}_{p,\mu}^s}, \quad (2.4)$$

where the exponent  $s - \frac{n-\mu}{p}$  is called scaling index and  $s$  is called regularity index. We have that

$$(-\Delta)^{l/2} \mathcal{M}_{p,\mu}^s = \mathcal{M}_{p,\mu}^{s-l}. \quad (2.5)$$

Morrey spaces contain Lebesgue and weak- $L^p$ , with the same scaling indexes. Precisely, we have the continuous proper inclusions

$$L^p(\mathbb{R}^n) \subsetneq \text{weak-}L^p(\mathbb{R}^n) \subsetneq \mathcal{M}_{r,\mu}(\mathbb{R}^n) \quad (2.6)$$

where  $r < p$  and  $\mu = n(1 - r/p)$  (see e.g. [23]). Let  $\mathcal{S}(\mathbb{R}^n)$  and  $\mathcal{S}'(\mathbb{R}^n)$  be the Schwartz space and the tempered distributions, respectively. Let  $\varphi \in \mathcal{S}(\mathbb{R}^n)$  be nonnegative radial function such that

$$\text{supp}(\varphi) \subset \{\xi \in \mathbb{R}^n; \frac{1}{2} < |\xi| < 2\} \quad \text{and} \quad \sum_{j=-\infty}^{\infty} \varphi_j(\xi) = 1, \text{ for all } \xi \neq 0$$

where  $\varphi_j(\xi) = \varphi(2^{-j}\xi)$ . Let  $\phi(x) = \mathcal{F}^{-1}(\varphi)(x)$  and  $\phi_j(x) = \mathcal{F}^{-1}(\varphi_j)(x) = 2^{jn}\phi(2^jx)$  where  $\mathcal{F}^{-1}$  stands for inverse Fourier transform. For  $1 \leq q < \infty$ ,  $0 \leq \mu < n$  and  $s \in \mathbb{R}$ , the homogeneous Besov–Morrey space  $\mathcal{N}_{q,\mu,r}^s(\mathbb{R}^n)$  ( $\mathcal{N}_{q,\mu,r}^s$  for short) is defined to be the set of  $u \in \mathcal{S}'(\mathbb{R}^n)$ , modulo polynomials  $\mathcal{P}$ , such that  $\mathcal{F}^{-1}\varphi_j(\xi)\mathcal{F}u \in \mathcal{M}_{q,\mu}$  for all  $j \in \mathbb{Z}$  and

$$\|u\|_{\mathcal{N}_{q,\mu,r}^s} = \begin{cases} \left( \sum_{j \in \mathbb{Z}} (2^{js} \|\phi_j * u\|_{q,\mu})^r \right)^{\frac{1}{r}} < \infty, & 1 \leq r < \infty \\ \sup_{j \in \mathbb{Z}} 2^{js} \|\phi_j * u\|_{q,\mu} < \infty, & r = \infty. \end{cases} \quad (2.7)$$

The space  $\mathcal{N}_{q,\mu,r}^s$  is a Banach space and, in particular,  $\mathcal{N}_{q,0,r}^s = \dot{B}_{q,r}^s$  (case  $\mu = 0$ ) corresponds to the homogeneous Besov space. We have the real-interpolation properties

$$\mathcal{N}_{q,\mu,r}^s = (\mathcal{M}_{q,\mu}^{s_1}, \mathcal{M}_{q,\mu}^{s_2})_{\theta,r}$$

and

$$\mathcal{N}_{q,\mu,r}^s = (\mathcal{N}_{q,\mu,r_1}^{s_1}, \mathcal{N}_{q,\mu,r_2}^{s_2})_{\theta,r}, \quad (2.8)$$

for all  $s_1 \neq s_2$ ,  $0 < \theta < 1$  and  $s = (1 - \theta)s_1 + \theta s_2$ . Here  $(X, Y)_{\theta,r}$  stands for the real interpolation space between  $X$  and  $Y$  constructed via the  $K_{\theta,q}$ -method. Recall that  $(\cdot, \cdot)_{\theta,r}$  is an exact interpolation functor of exponent  $\theta$  on the category of normed spaces.

In the next lemmas, we collect basic facts about Morrey spaces and Besov–Morrey spaces (see e.g. [12, 16, 33]).

**Lemma 2.1.** *Suppose that  $s_1, s_2 \in \mathbb{R}$ ,  $1 \leq p_1, p_2, p_3 < \infty$  and  $0 \leq \mu_i < n$ ,  $i = 1, 2, 3$ .*

(i) (Inclusion) *If  $\frac{n-\mu_1}{p_1} = \frac{n-\mu_2}{p_2}$  and  $p_2 \leq p_1$ ,*

$$\mathcal{M}_{p_1,\mu_1} \subset \mathcal{M}_{p_2,\mu_2} \quad \text{and} \quad \mathcal{N}_{p_1,\mu_1,1}^0 \subset \mathcal{M}_{p_1,\mu_1} \subset \mathcal{N}_{p_1,\mu_1,\infty}^0. \quad (2.9)$$

(ii) (Sobolev-type embedding) *Let  $j = 1, 2$  and  $p_j, s_j$  be  $p_2 \leq p_1$ ,  $s_1 \leq s_2$  such that  $s_2 - \frac{n-\mu_2}{p_2} = s_1 - \frac{n-\mu_1}{p_1}$ , we obtain*

$$\mathcal{M}_{p_2,\mu}^{s_2} \subset \mathcal{M}_{p_1,\mu}^{s_1}, (\mu = \mu_1 = \mu_2) \quad (2.10)$$

*and for every  $1 \leq r_2 \leq r_1 \leq \infty$ , we have*

$$\mathcal{N}_{p_2,\mu_2,r_2}^{s_2} \subset \mathcal{N}_{p_1,\mu_1,r_1}^{s_1} \quad \text{and} \quad \mathcal{N}_{p_2,\mu_2,r_2}^{s_2} \subset \dot{B}_{\infty,r_2}^{s_2 - \frac{n-\mu_2}{p_2}}. \quad (2.11)$$

(iii) (Hölder inequality) *Let  $\frac{1}{p_3} = \frac{1}{p_2} + \frac{1}{p_1}$  and  $\frac{\mu_3}{p_3} = \frac{\mu_2}{p_2} + \frac{\mu_1}{p_1}$ . If  $f_j \in \mathcal{M}_{p_j,\mu_j}$  with  $j = 1, 2$ , then  $f_1 f_2 \in \mathcal{M}_{p_3,\mu_3}$  and*

$$\|f_1 f_2\|_{p_3,\mu_3} \leq \|f_1\|_{p_1,\mu_1} \|f_2\|_{p_2,\mu_2}. \quad (2.12)$$

We finish this subsection recalling an estimate for certain multiplier operators on  $\mathcal{M}_{q,\mu}^s$  (see e.g. [17]).

**Lemma 2.2.** *Let  $m, s \in \mathbb{R}$  and  $0 \leq \mu < n$  and  $P(\xi) \in C^{[n/2]+1}(\mathbb{R}^n \setminus \{0\})$ . Assume that there is  $A > 0$  such that*

$$\left| \frac{\partial^k P}{\partial \xi^k}(\xi) \right| \leq A |\xi|^{m-|k|}, \quad (2.13)$$

for all  $k \in (\mathbb{N} \cup \{0\})^n$  with  $|k| \leq [n/2] + 1$  and for all  $\xi \neq 0$ . Then, the multiplier operator  $P(D)f = \mathcal{F}^{-1}P(\xi)\mathcal{F}f$  on  $\mathcal{S}'/\mathcal{P}$  satisfies the estimate

$$\|P(D)f\|_{\mathcal{M}_{q,\mu}^{s-m}} \leq CA \|f\|_{\mathcal{M}_{q,\mu}^s}, \quad (1 < q < \infty) \quad (2.14)$$

where  $C > 0$  is a constant independent of  $f$ , and the set  $\mathcal{S}'/\mathcal{P}$  consists in equivalence classes in  $\mathcal{S}'$  modulo polynomials with  $n$  variables.

## 2.2. Mittag-Leffler function

In this part we collect some basic properties for Mittag-Leffler functions  $E_\alpha(-t^\alpha|\xi|^2)$  as well as the fundamental solution  $k_\alpha$  (see (1.8)), further details can be obtained in [3,8,11] and references therein.

Recall that Mittag-Leffler's function  $E_\alpha(-t^\alpha|\xi|^2)$  can be defined via complex integral as

$$E_\alpha(-t^\alpha|\xi|^2) = \frac{1}{2\pi i} \int_{\zeta} \frac{e^z z^{\alpha-1}}{z^\alpha + t^\alpha|\xi|^2} dz, \quad (\alpha > 0) \quad (2.15)$$

where  $\zeta$  is any Hankel's path on complex plan  $\mathbb{C}$ . The integrand in (2.15) has simple poles given by

$$a_\alpha(\xi) = |\xi|^{\frac{2}{\alpha}} e^{\frac{i\pi}{\alpha}}, \quad b_\alpha(\xi) = |\xi|^{\frac{2}{\alpha}} e^{-\frac{i\pi}{\alpha}}, \quad \text{for } \xi \in \mathbb{R}^n.$$

**Lemma 2.3.** *Let  $1 < \alpha < 2$  and  $k_\alpha$  be as in (1.8). We have that*

$$L^1(\mathbb{R}^n) \ni E_\alpha(-|\xi|^2) = \frac{1}{\alpha} (\exp(a_\alpha(\xi)) + \exp(b_\alpha(\xi))) + l_\alpha(\xi) \quad (n \geq 1) \quad (2.16)$$

where

$$l_\alpha(\xi) = \begin{cases} \frac{\sin(\alpha\pi)}{\pi} \int_0^\infty \frac{|\xi|^2 s^{\alpha-1} e^{-s}}{s^{2\alpha} + 2|\xi|^2 s^\alpha \cos(\alpha\pi) + |\xi|^4} ds & \text{if } \xi \neq 0 \\ 1 - \frac{2}{\alpha}, & \text{if } \xi = 0. \end{cases}$$

Moreover,

$$\frac{\partial^k k_\alpha}{\partial x_i^k}(t, x) = t^{-\frac{\alpha}{2}(k+n)} \frac{\partial^k}{\partial x_i^k} k_\alpha(1, t^{-\frac{\alpha}{2}} x), \quad (t > 0) \quad (2.17)$$

$k_\alpha(t, x) \geq 0$ ,  $P_\alpha(1, |x|) = \alpha k_\alpha(1, x)$  is a probability measure.

**Lemma 2.4.** *Let  $1 \leq \alpha < 2$  and  $0 \leq \delta < 2$ . There exists  $A > 0$  such that*

$$\left| \frac{\partial^k}{\partial \xi^k} \left[ |\xi|^\delta E_\alpha(-|\xi|^2) \right] \right| \leq A |\xi|^{-|k|}, \quad \xi \neq 0, \quad (2.18)$$

for all  $k \in (\mathbb{N} \cup \{0\})^n$  with  $|k| \leq [n/2] + 1$ .



We finish this section by recalling that  $E_\alpha(-|\xi|^2)$  coincides with  $\sum_{k=0}^{\infty} \frac{(-|\xi|^2)^k}{\Gamma(\alpha k + 1)}$ , and then  $E_\alpha(-|\xi|^2)$  is finite, for all  $\xi \in \mathbb{R}^n$ . Indeed, since Gamma function  $\Gamma(x)$  can be expressed as  $\frac{1}{\Gamma(x)} = \frac{1}{2\pi i} \int_{\zeta} e^z z^{-x} dz$ , for  $x \in \mathbb{C}$ , dominated convergence theorem yields

$$\begin{aligned} E_\alpha(-|\xi|^2) &= \frac{1}{2\pi i} \int_{\zeta} e^z z^{-1} \left[ \lim_{n \rightarrow \infty} \frac{1 - (-z^{-\alpha} |\xi|^2)^{n+1}}{1 + z^\alpha |\xi|^2} \right] dz \\ &= \frac{1}{2\pi i} \int_{\zeta} e^z z^{-1} \left[ \lim_{n \rightarrow \infty} \sum_{k=0}^n (-z^{-\alpha} |\xi|^2)^k \right] dz \\ &= \lim_{n \rightarrow \infty} \sum_{k=0}^n (-|\xi|^2)^k \frac{1}{2\pi i} \int_{\zeta} e^z z^{-(\alpha k + 1)} dz = \sum_{k=0}^{\infty} \frac{(-|\xi|^2)^k}{\Gamma(\alpha k + 1)}, \end{aligned} \quad (2.19)$$

for all  $\alpha \geq 1$ .

### 3. Key estimates

The goal of this section is to derive estimates for Mittag-Leffler convolution operators  $\{L_\alpha(t)\}_{t \geq 0}$  on Sobolev–Morrey spaces and Besov–Morrey spaces. Here and below the letter  $C$  will denote constants which can change from line to line.

**Lemma 3.1.** *Let  $s, \beta \in \mathbb{R}$ ,  $1 \leq \alpha < 2$ ,  $1 < p \leq q < \infty$ ,  $0 \leq \mu < n$ , and  $(\beta - s) + \frac{n-\mu}{p} - \frac{n-\mu}{q} < 2$  where  $\beta \geq s$ .*

(i) *There exists  $C > 0$  such that*

$$\|L_\alpha(t)f\|_{\mathcal{M}_{q,\mu}^\beta} \leq Ct^{-\frac{\alpha}{2}(\beta-s)-\frac{\alpha}{2}(\frac{n-\mu}{p}-\frac{n-\mu}{q})} \|f\|_{\mathcal{M}_{p,\mu}^s}, \quad (3.1)$$

*for every  $t > 0$  and  $f \in \mathcal{M}_{p,\mu}^s$ .*

(ii) *Let  $r \in [1, \infty]$ , there exists  $C > 0$  such that*

$$\|L_\alpha(t)f\|_{\mathcal{N}_{q,\mu,r}^\beta} \leq Ct^{-\frac{\alpha}{2}(\beta-s)-\frac{\alpha}{2}(\frac{n-\mu}{p}-\frac{n-\mu}{q})} \|f\|_{\mathcal{N}_{p,\mu,r}^s}, \quad (3.2)$$

*for every  $f \in \mathcal{S}'/\mathcal{P}$  and  $t > 0$ .*

(iii) *Let  $r \in [1, \infty]$  and  $\beta > s$ , there exists  $C > 0$  such that*

$$\|L_\alpha(t)f\|_{\mathcal{N}_{q,\mu,1}^\beta} \leq Ct^{-\frac{\alpha}{2}(\beta-s)-\frac{\alpha}{2}(\frac{n-\mu}{p}-\frac{n-\mu}{q})} \|f\|_{\mathcal{N}_{p,\mu,r}^s}, \quad (3.3)$$

*for every  $f \in \mathcal{S}'/\mathcal{P}$ .*

**Proof.** Let  $\delta = (\beta - s) + l$ , where  $l = \frac{n-\mu}{p} - \frac{n-\mu}{q}$ . Recalling that  $\widehat{f_\lambda}(\xi) = \lambda^{-n} \widehat{f}(\xi/\lambda)$  for  $f_\lambda(x) := f(\lambda x)$ , we let  $(-\Delta)^{\frac{\delta}{2}} L_\alpha(t)$  be the Fourier multiplier defined as follows

$$\begin{aligned} [(-\Delta)^{\frac{\delta}{2}} L_\alpha(t)f]^\wedge(\xi) &= \widehat{h}_\alpha(\xi, t) \widehat{f}(\xi) \\ &= t^{-\delta \frac{\alpha}{2}} \widehat{h}_\alpha(t^{\frac{\alpha}{2}} \xi, 1) \widehat{f}(\xi) \\ &= t^{-\delta \frac{\alpha}{2}} (h_\alpha(\cdot, 1) * f_{t^{\alpha/2}}(\cdot))^\wedge_{t^{-\alpha/2}}(\xi) \\ &:= t^{-\delta \frac{\alpha}{2}} (P(D)(f_{t^{\alpha/2}}))^\wedge_{t^{-\alpha/2}}(\xi), \end{aligned} \quad (3.4)$$

where the symbol of  $P(D)$  is  $\widehat{h}_\alpha(\xi, 1) = |\xi|^\delta E_\alpha(-|\xi|^2)$ . Noticing that  $0 \leq \delta < 2$ , it follows from Lemma 2.4 that  $P(\xi)$  shall satisfy (2.13) with  $m = 0$ . Using (2.4) and (2.14) we obtain

$$\begin{aligned} \|(P(D)(f_{t^{\alpha/2}}))_{t^{-\alpha/2}}\|_{\mathcal{M}_{p,\mu}^s} &= t^{-\frac{\alpha}{2}(s-\frac{n-\mu}{p})} \|P(D)(f_{t^{\alpha/2}})\|_{\mathcal{M}_{p,\mu}^s} \\ &\leq CA t^{-\frac{\alpha}{2}(s-\frac{n-\mu}{p})} \|f_{t^{\alpha/2}}\|_{\mathcal{M}_{p,\mu}^s} \\ &= CA t^{-\frac{\alpha}{2}(s-\frac{n-\mu}{p})} t^{\frac{\alpha}{2}(s-\frac{n-\mu}{p})} \|f\|_{\mathcal{M}_{p,\mu}^s} \\ &= CA \|f\|_{\mathcal{M}_{p,\mu}^s}. \end{aligned} \quad (3.5)$$

Using (2.10) with  $(s_1, s_2) = (0, l)$  and (3.4), we obtain

$$\begin{aligned} \|L_\alpha(t)f\|_{\mathcal{M}_{q,\mu}^\beta} &= \left\| (-\Delta)^{\frac{\beta}{2}} L_\alpha(t)f \right\|_{\mathcal{M}_{q,\mu}} \\ &\leq \left\| (-\Delta)^{\frac{\beta}{2}} L_\alpha(t)f \right\|_{\mathcal{M}_{p,\mu}^l}, \text{ where } l = \frac{n-\mu}{p} - \frac{n-\mu}{q} \\ &= \left\| (-\Delta)^{\frac{\beta-s}{2} + \frac{l}{2}} L_\alpha(t)f \right\|_{\mathcal{M}_{p,\mu}^s} = \left\| (-\Delta)^{\frac{\delta}{2}} L_\alpha(t)f \right\|_{\mathcal{M}_{p,\mu}^s} \\ &= t^{-\delta\frac{\alpha}{2}} \|(P(D)(f_{t^{\alpha/2}}))_{t^{-\alpha/2}}\|_{\mathcal{M}_{p,\mu}^s} \end{aligned} \quad (3.6)$$

$$\leq CA t^{-\frac{\alpha}{2}(\beta-s) - \frac{\alpha}{2}(\frac{n-\mu}{p} - \frac{n-\mu}{q})} \|f\|_{\mathcal{M}_{p,\mu}^s}, \quad (3.7)$$

where (3.7) is obtained of (3.6) via inequality (3.5). In order to obtain (3.2) we recall the real interpolation  $\mathcal{N}_{q,\mu,r}^\beta = (\mathcal{M}_{q,\mu}^{\beta_1}, \mathcal{M}_{q,\mu}^{\beta_2})_{\theta,r}$ ,  $\mathcal{N}_{p,\mu,r}^s = (\mathcal{M}_{p,\mu}^{s_1}, \mathcal{M}_{p,\mu}^{s_2})_{\theta,r}$ , where  $\beta = (1-\theta)\beta_1 + \theta\beta_2$ ,  $\beta_1 \neq \beta_2$  and  $s = (1-\theta)s_1 + \theta s_2$ ,  $s_1 \neq s_2$ . Then, we have

$$\|L_\alpha(t)f\|_{\mathcal{N}_{q,\mu,r}^\beta} \leq m_0^{1-\theta} m_1^\theta \|f\|_{\mathcal{N}_{p,\mu,r}^s}, \quad 0 < \theta < 1, \quad (3.8)$$

where  $m_i = \|L_\alpha(t)f\|_{\mathcal{M}_{p,\mu}^{s_i} \rightarrow \mathcal{M}_{q,\mu}^{\beta_i}}$ . In view of (3.1), we obtain

$$m_i \leq CA t^{-\frac{\alpha}{2}(\beta_i - s_i) - \frac{\alpha}{2}(\frac{n-\mu}{p} - \frac{n-\mu}{q})}$$

and then, by inserting it into (3.8), we get (3.2). Now, using (3.2), it follows that

$$\|L_\alpha(t)f\|_{\mathcal{N}_{q,\mu,\infty}^{2\beta-s}} \leq C t^{-\alpha(\beta-s) - \frac{\alpha}{2}(\frac{n-\mu}{p} - \frac{n-\mu}{q})} \|f\|_{\mathcal{N}_{p,\mu,\infty}^s}$$

and

$$\|L_\alpha(t)f\|_{\mathcal{N}_{q,\mu,\infty}^s} \leq C t^{-\frac{\alpha}{2}(\frac{n-\mu}{p} - \frac{n-\mu}{q})} \|f\|_{\mathcal{N}_{p,\mu,\infty}^s}.$$

In view of (2.8) and  $(2\beta - s)(1 - 1/2) + s(1/2) = \beta$ , we have  $\mathcal{N}_{q,\mu,1}^\beta = (\mathcal{N}_{q,\mu,\infty}^{2\beta-s}, \mathcal{N}_{q,\mu,\infty}^s)_{1/2,1}$  which yields (3.3).  $\square$

**Proposition 3.2.** Let  $\xi \in \mathbb{R}^n$ . If  $1 \leq \alpha < 2$ , we have  $|E_\alpha(-|\xi|^2)| \leq 1$  and  $E_\alpha(-|t^{\alpha/2}\xi|^2) \rightarrow 1$  as  $t \rightarrow 0^+$ .

**Proof.** It is enough to make a proof for  $1 < \alpha < 2$ , because the Lemma holds for  $E_1(-t|\xi|^2) = e^{-t|\xi|^2}$ . To this end, let  $t = |\xi|^{\frac{2}{\alpha}} s^{\frac{1}{\alpha}}$  for  $\xi \neq 0$  and using Lemma 2.3 and  $|\exp(a_\alpha(\xi))| = |\exp(b_\alpha(\xi))| = \exp(|\xi|^{2/\alpha} \cos(\pi/\alpha)) \leq 1$  we obtain

$$\begin{aligned}
|E_\alpha(-|\xi|^2)| &\leq \frac{2}{\alpha} + |l_\alpha(\xi)| \\
&\leq \frac{2}{\alpha} + \frac{\sin(\alpha\pi)}{\pi} \int_0^\infty \frac{e^{-|\xi|^{\frac{2}{\alpha}} s^{\frac{1}{\alpha}}}}{s^2 + 2s \cos(\alpha\pi) + 1} ds \\
&\leq \frac{2}{\alpha} + \frac{\sin(\alpha\pi)}{\pi} \int_0^\infty \frac{1}{s^2 + 2s \cos(\alpha\pi) + 1} ds \\
&= \frac{2}{\alpha} + \left(1 - \frac{2}{\alpha}\right) = 1.
\end{aligned}$$

Let  $\Phi(t) = \frac{e^z z^{\alpha-1}}{z^\alpha + t^\alpha |\xi|^2}$ . Note that  $|\Phi(t)| \in L^1(0, \infty)$  and  $\Phi(t) \rightarrow e^z/z$  as  $t \rightarrow 0^+$ , using dominated converge theorem and residue theorem we have

$$E_\alpha(-t^\alpha |\xi|^2) \rightarrow \frac{1}{2\pi i} \int_\zeta \frac{e^z z^{\alpha-1}}{z^\alpha} dz = \frac{1}{2\pi i} \left\{ 2\pi i \operatorname{res} \left( \frac{e^z}{z}; z=0 \right) \right\} = 1, \quad (3.9)$$

because  $\operatorname{res}(e^z/z; z=0) = 1$ .  $\square$

### 3.1. Linear estimates

We start by recalling an elementary fixed point lemma whose proof can be found in [6].

**Lemma 3.3.** *Let  $(X, \|\cdot\|)$  be a Banach space and  $1 < \rho < \infty$ . Suppose that  $B : X \rightarrow X$  satisfies  $B(0) = 0$  and*

$$\|B(x) - B(z)\| \leq K\|x - z\|(\|x\|^{\rho-1} + \|z\|^{\rho-1}).$$

*Let  $R > 0$  be the unique positive root of  $2^\rho K R^{\rho-1} - 1 = 0$ . Given  $0 < \varepsilon < R$  and  $y \in X$  such that  $\|y\| \leq \varepsilon$ , there exists a solution  $x \in X$  for the equation  $x = y + B(x)$  which is the unique one in the closed ball  $\{z \in X; \|z\| \leq 2\varepsilon\}$ . Moreover, if  $\|\bar{y}\| \leq \varepsilon$  and  $\|\bar{x}\| \leq 2\varepsilon$  satisfies the equation  $\bar{x} = \bar{y} + B(\bar{x})$  then*

$$\|x - \bar{x}\| \leq \frac{1}{1 - 2^\rho K \varepsilon^{\rho-1}} \|y - \bar{y}\|. \quad (3.10)$$

The integral equation (1.5) has the form  $x = y + B(x)$  in the space  $X = X_q^p$  where  $y = L_\alpha(t)u_0$  and  $B(x) = B_\alpha(u)$  is given by (1.7) and (1.6), respectively. We invoke Lemma 3.3 in our proofs, hence the estimates for linear and nonlinear part of (1.5) will be necessary.

**Lemma 3.4.** *Under the assumptions of Theorem 1.1, there exists  $L > 0$  such that*

$$\|L_\alpha(t)u_0\|_{X_q^p} \leq L\|u_0\|_{\mathcal{N}_{p,\mu,\infty}^\sigma}, \quad (3.11)$$

*for all  $u_0 \in \mathcal{N}_{p,\mu,\infty}^\sigma$ . Let  $s = 2/(\rho - 1)$ , if  $u_0 \in \dot{B}_{\infty,\infty}^s$  we obtain  $L_\alpha(t)u_0 \rightharpoonup u_0$  in the weak-\* topology of  $\dot{B}_{\infty,\infty}^s$  as  $t \rightarrow 0^+$ .*

**Proof.** Notice that by (1.15) we obtain  $\eta + \frac{\alpha}{2}\sigma = \frac{\alpha}{2} \left( \frac{n-\mu}{p} - \frac{n-\mu}{q} \right)$ . Using (1.16) one has  $\frac{n-\mu}{p} - \frac{n-\mu}{q} - \sigma = \frac{2}{\rho-1} - \frac{n-\mu}{q} < \frac{2}{\alpha\rho} < 2$  and  $\sigma < 0$  which by (3.2) and afterwards by (2.9) and (3.3), respectively, give us

$$\begin{aligned}
\sup_{t>0} \|L_\alpha(t)u_0\|_{\mathcal{N}_{p,\mu,\infty}^\sigma} + \sup_{t>0} t^\eta \|L_\alpha(t)u_0\|_{\mathcal{M}_{q,\mu}} &\leq C\|a\|_{\mathcal{N}_{p,\mu,\infty}^\sigma} + \sup_{t>0} t^\eta \|L_\alpha(t)u_0\|_{\mathcal{N}_{q,\mu,1}^0} \\
&\leq C\|u_0\|_{\mathcal{N}_{p,\mu,\infty}^\sigma} + C \sup_{t>0} t^{\eta+\frac{\alpha}{2}\sigma-\frac{\alpha}{2}(\frac{n-\mu}{p}-\frac{n-\mu}{q})} \|u_0\|_{\mathcal{N}_{p,\mu,\infty}^\sigma} \\
&\leq L\|u_0\|_{\mathcal{N}_{p,\mu,\infty}^\sigma},
\end{aligned}$$

this yield (3.11). It remains to check the weak- $*$  convergence. To this end, let  $v \in \dot{B}_{1,1}^{-s}$  the predual space of  $\dot{B}_{\infty,\infty}^s$ . Using Proposition 3.2 we have

$$\|L_\alpha(t)v - v\|_{\dot{B}_{1,1}^{-s}} = \sum_{j=-\infty}^{\infty} \{2^{-js} \|\mathcal{F}^{-1} \varphi_j [E_\alpha(-t^\alpha |\xi|^2) - 1] \mathcal{F} v\|_{L^1(\mathbb{R}^n)}\} \rightarrow 0$$

as  $t \rightarrow 0^+$ . Thanks to (2.17) and (2.19) one has  $E_\alpha(-t^\alpha |\xi|^2) = t^{-\frac{\alpha}{2}n} E_\alpha(-|t^{-\frac{\alpha}{2}} \xi|^2) \in \mathbb{R}$  for all  $t > 0$  and  $\xi \in \mathbb{R}^n$ , it follows that

$$|\langle L_\alpha(t)u_0 - u_0, v \rangle| = |\langle u_0, L_\alpha(t)v - v \rangle| \leq \|u_0\|_{\dot{B}_{\infty,\infty}^s} \|L_\alpha(t)v - v\|_{\dot{B}_{1,1}^{-s}} \rightarrow 0, \quad (3.12)$$

as  $t \rightarrow 0^+$ .  $\square$

### 3.2. Nonlinear estimates

Recall the nonlinear term in (1.5)

$$B_\alpha(u)(t) = \int_0^t L_\alpha(t-s) \int_0^s r_{\alpha-1}(s-\tau) f(u(\tau)) d\tau ds. \quad (3.13)$$

**Lemma 3.5** (Nonlinear estimate). Assume the hypotheses of Theorem 1.1. There is a constant  $K > 0$  such that

$$\|B_\alpha(u) - B_\alpha(v)\|_{X_q^p} \leq K\|u - v\|_{X_q^p} (\|u\|_{X_q^p}^{\rho-1} + \|v\|_{X_q^p}^{\rho-1}), \quad (3.14)$$

for all  $u, v \in X_q^p$ . Moreover, we have  $B_\alpha(u)(t) \rightarrow 0$  in the weak- $*$  topology of  $\dot{B}_{\infty,\infty}^{2/(\rho-1)}$  as  $t \rightarrow 0^+$ .

**Proof.** The proof is divided in three steps.

**First step.** Let  $\tilde{s} \in \mathbb{R}$  be such that  $\sigma - \frac{n-\mu}{p} = \tilde{s} - \frac{n-\mu}{q/\rho}$ . In view of (1.16) and  $\alpha \geq 1$  we have  $\frac{n-\mu}{q/\rho} \geq \frac{2}{\rho-1} > \frac{n-\mu}{p}$ , it follows that  $\sigma < 0 \leq \tilde{s}$  and  $p \geq q/\rho$ . Applying (2.10) and (3.2) afterwards (2.9) and (2.12), respectively, we have

$$\begin{aligned}
\|B_\alpha(u)(t) - B_\alpha(v)(t)\|_{\mathcal{N}_{p,\mu,\infty}^\sigma} &\leq \|B_\alpha(u)(t) - B_\alpha(v)(t)\|_{\mathcal{N}_{q/\rho,\mu,\infty}^{\tilde{s}}} \\
&\leq C \int_0^t (t-s)^{\gamma_1} \int_0^s r_{\alpha-1}(s-\tau) \|f(u(\tau)) - f(v(\tau))\|_{\mathcal{N}_{q/\rho,\mu,\infty}^0} d\tau ds \\
&\leq C \int_0^t (t-s)^{\gamma_1} \int_0^s r_{\alpha-1}(s-\tau) \|f(u(\tau)) - f(v(\tau))\|_{\mathcal{M}_{q/\rho,\mu}} d\tau ds \\
&\leq C \int_0^t (t-s)^{\gamma_1} \int_0^s r_{\alpha-1}(s-\tau) \|u - v\|_{\mathcal{M}_{q,\mu}} (\|u\|_{\mathcal{M}_{q,\mu}}^{\rho-1} + \|v\|_{\mathcal{M}_{q,\mu}}^{\rho-1}) d\tau ds \quad (3.15)
\end{aligned}$$

$$:= \psi_1(t) \sup_{t>0} t^\eta \|u(t) - v(t)\|_{\mathcal{M}_{q,\mu}} \sup_{t>0} t^{\eta(\rho-1)} (\|u(t)\|_{\mathcal{M}_{q,\mu}}^{\rho-1} + \|v(t)\|_{\mathcal{M}_{q,\mu}}^{\rho-1}) \quad (3.16)$$

where  $r_\alpha(s) = s^{\alpha-1}/\Gamma(\alpha)$ ,  $f(u(\tau)) = |u(\tau)|^{\rho-1}u(\tau)$ ,  $\gamma_1 = -\frac{\alpha}{2}\tilde{s} = -\frac{\alpha}{2}\sigma - \frac{\alpha}{2}(\frac{n-\mu}{q/\rho} - \frac{n-\mu}{p})$  and

$$\begin{aligned} \psi_1(t) &= C \int_0^t (t-s)^{\gamma_1} \int_0^s r_{\alpha-1}(s-\tau) \tau^{-\eta\rho} d\tau ds \\ &= C\beta(1-\eta\rho, \alpha-1)\beta(\alpha-\eta, \gamma_1+1), \end{aligned} \quad (3.17)$$

for  $\beta(x, y) = \int_0^1 t^{x-1}(1-t)^{y-1}dt$  if  $x, y > 0$ . Indeed, by change of variables  $\tau = zs$  and  $s = t\omega$ , respectively, we get

$$\begin{aligned} \int_0^t (t-s)^{\gamma_1} \int_0^s r_{\alpha-1}(s-\tau) \tau^{-\eta\rho} d\tau ds &= \int_0^t (t-s)^{\gamma_1} s^{\alpha-1-\eta\rho} \left( \int_0^1 (1-z)^{\alpha-2} z^{-\eta\rho} dz \right) ds \\ &= \beta(1-\eta\rho, \alpha-1) t^{\alpha+\gamma_1-\eta\rho} \int_0^1 (1-\omega)^{\gamma_1} \omega^{\alpha-1-\eta\rho} d\omega, \\ &= \beta(1-\eta\rho, \alpha-1)\beta(\alpha-\eta\rho, \gamma_1+1), \end{aligned} \quad (3.18)$$

because by  $\gamma_1 = -\frac{\alpha}{2}\sigma - \frac{\alpha}{2}(\frac{n-\mu}{q/\rho} - \frac{n-\mu}{p})$  and (1.15) we have

$$\begin{aligned} \alpha + \gamma_1 - \eta\rho &= \alpha + \frac{\alpha}{2} \left( \frac{2}{\rho-1} - \frac{n-\mu}{q} \rho \right) - \frac{\alpha}{2} \left( \frac{2}{\rho-1} - \frac{n-\mu}{q} \right) \rho \\ &= \alpha + \frac{\alpha}{\rho-1} - \frac{\alpha\rho}{\rho-1} = 0. \end{aligned}$$

Inserting (3.17) into (3.16) yields

$$\|B_\alpha(u)(t) - B_\alpha(v)(t)\|_{\mathcal{N}_{p,\mu,\infty}^\sigma} \leq K_1 \sup_{t>0} t^\eta \|u(t) - v(t)\|_{\mathcal{M}_{q,\mu}} \sup_{t>0} t^{\eta(\rho-1)} (\|u(t)\|_{\mathcal{M}_{q,\mu}}^{\rho-1} + \|v(t)\|_{\mathcal{M}_{q,\mu}}^{\rho-1}). \quad (3.19)$$

**Second step.** Let  $\beta = s = 0$ . By estimate (3.1) and Hölder inequality (2.12) we obtain

$$\|B_\alpha(u)(t) - B_\alpha(v)(t)\|_{\mathcal{M}_{q,\mu}} \leq C \int_0^t (t-s)^{\gamma_2} \theta(s) ds \quad (3.20)$$

where  $\gamma_2 = -\frac{\alpha}{2}(\frac{n-\mu}{q/\rho} - \frac{n-\mu}{q})$  and  $\theta(s)$  is given by

$$\theta(s) = \int_0^s r_{\alpha-1}(s-\tau) \|u(\tau) - v(\tau)\|_{\mathcal{M}_{q,\mu}} (\|u(\tau)\|_{\mathcal{M}_{q,\mu}}^{\rho-1} + \|v(\tau)\|_{\mathcal{M}_{q,\mu}}^{\rho-1}) d\tau.$$

Mimicking the *First step* we get

$$\|B_\alpha(u)(t) - B_\alpha(v)(t)\|_{\mathcal{M}_{q,\mu}} \leq \psi_2(t) \sup_{t>0} t^\eta \|u(t) - v(t)\|_{\mathcal{M}_{q,\mu}} \sup_{t>0} t^{\eta(\rho-1)} (\|u(t)\|_{\mathcal{M}_{q,\mu}}^{\rho-1} + \|v(t)\|_{\mathcal{M}_{q,\mu}}^{\rho-1}) \quad (3.21)$$

where  $\psi_2(t)$  can be estimated as

$$\psi_2(t) \leq C\beta(1-\eta\rho, \alpha-1)\beta(\alpha-\eta\rho, \gamma_2+1)t^{\alpha+\gamma_2-\eta\rho} = K_2 t^{-\eta}, \quad (3.22)$$

because in view of (1.15) we have

$$\alpha + \gamma_2 - \eta\rho = \alpha - \frac{\alpha}{2} \frac{n-\mu}{q} - \frac{\alpha}{2} \frac{2\rho}{\rho-1} = \alpha + \frac{\alpha}{\rho-1} - \frac{\alpha\rho}{\rho-1} - \eta = -\eta.$$

Inserting (3.22) into (3.21) it follows that

$$t^\eta \|B_\alpha(u)(t) - B_\alpha(v)(t)\|_{\mathcal{M}_{q,\mu}} \leq K_2 \sup_{t>0} t^\eta \|u(t) - v(t)\|_{\mathcal{M}_{q,\mu}} \sup_{t>0} t^{\eta(\rho-1)} (\|u(t)\|_{\mathcal{M}_{q,\mu}}^{\rho-1} + \|v(t)\|_{\mathcal{M}_{q,\mu}}^{\rho-1}). \quad (3.23)$$

The convergence of the beta functions appearing in (3.18) and (3.22) is obtained by restrictions (1.16) and  $\alpha \geq 1$ , because this yields in  $\gamma_1, \gamma_2 > -1$  and  $\eta\rho < 1 \leq \alpha$ . It follows that  $(\frac{n-\mu}{q/\rho} - \frac{n-\mu}{q}) < \frac{2}{\alpha} \leq 2$  which we have used in *Second step*. Recalling (1.14) and using (3.19) and (3.23) we obtain (3.14) with  $K = K_1 + K_2$ .

**Third step.** As  $\mathcal{S}(\mathbb{R}^n)$  is dense in  $\dot{B}_{1,1}^{2/(\rho-1)}$  (see [34, p. 48]) the weak- $*$  convergence can be obtained by estimate

$$\begin{aligned} |\langle B_\alpha(u)(t), v \rangle| &\leq |\langle B_\alpha(u)(t), v - \varphi \rangle| + |\langle B_\alpha(u)(t), \varphi \rangle| \\ &\leq \|B_\alpha(u)(t)\|_{\dot{B}_{\infty,\infty}^{\sigma-(n-\mu)/p}} \|v - \varphi\|_{\dot{B}_{1,1}^{2/(\rho-1)}} + |\langle B_\alpha(u)(t), \varphi \rangle| \\ &\leq C \|u\|_{X_q^p} \varepsilon + C \|u\|_{X_q^p}^\rho \|\varphi\|_{\dot{B}_{1,1}^{2/(\rho-1)}} t^\alpha \leq C \|u\|_{X_q^p} \varepsilon \text{ as } t \rightarrow 0^+, \end{aligned} \quad (3.24)$$

because for  $v \in \dot{B}_{1,1}^{\sigma-(n-\mu)/p} = \dot{B}_{1,1}^{2/(\rho-1)}$  one has  $\|v - \varphi\|_{\dot{B}_{1,1}^{2/(\rho-1)}} \leq \varepsilon$ , for all  $\varepsilon > 0$ . Moreover, by embedding  $\mathcal{N}_{l,\mu,\infty}^s \subset \dot{B}_{\infty,\infty}^{s-(n-\mu)/l}$  (see (2.11)) and  $\frac{n-\mu}{p} < \frac{2}{\rho-1} < \frac{n-\mu}{q/\rho}$  we have that

$$\begin{aligned} |\langle B_\alpha(u)(t), \varphi \rangle| &= \left| \int_0^t \int_0^s r_{\alpha-1}(s-\tau) \langle f(u(\tau)), L_\alpha(t-s)\varphi \rangle d\tau ds \right| \\ &\leq C \int_0^t \int_0^s r_{\alpha-1}(s-\tau) \| |u(\tau)|^{\rho-1} u(\tau) \|_{\dot{B}_{\infty,\infty}^{\frac{n-\mu}{q/\rho}}} \|L_\alpha(t-s)\varphi\|_{\dot{B}_{1,1}^{\frac{n-\mu}{q/\rho}}} d\tau ds \\ &\leq C \int_0^t \int_0^s (s-\tau)^{\alpha-2} (t-s)^{-\frac{\alpha}{2}(\frac{n-\mu}{q/\rho} - \frac{2}{\rho-1})} \| |u(\tau)|^{\rho-1} u(\tau) \|_{\mathcal{N}_{q/\rho,\mu,\infty}^0} \|\varphi\|_{\dot{B}_{1,1}^{\frac{2}{\rho-1}}} d\tau ds \\ &\leq C \int_0^t \int_0^s (s-\tau)^{\alpha-2} (t-s)^{-\frac{\alpha}{2}(\frac{n-\mu}{q/\rho} - \frac{2}{\rho-1})} \tau^{-\eta\rho} d\tau ds \|u\|_{X_q^p}^\rho \|\varphi\|_{\dot{B}_{1,1}^{\frac{2}{\rho-1}}} \\ &\leq C t^\alpha \|u\|_{X_q^p}^\rho \|\varphi\|_{\dot{B}_{1,1}^{\frac{2}{\rho-1}}} \rightarrow 0 \text{ as } t \rightarrow 0^+ \end{aligned} \quad (3.25)$$

and

$$\begin{aligned} \|B_\alpha(u)(t)\|_{\dot{B}_{\infty,\infty}^{\sigma-\frac{n-\mu}{p}}} &\leq C \int_0^t \int_0^s r_{\alpha-1}(s-\tau) \|L_\alpha(t-s) |u(\tau)|^{\rho-1} u(\tau)\|_{\dot{B}_{\infty,\infty}^{\sigma-\frac{n-\mu}{p}}} d\tau ds \\ &\leq \int_0^t \int_0^s r_{\alpha-1}(s-\tau) \|L_\alpha(t-s) |u(\tau)|^{\rho-1} u(\tau)\|_{\mathcal{N}_{p,\mu,\infty}^\sigma} d\tau ds \end{aligned}$$

$$\begin{aligned}
&\leq \int_0^t \int_0^s r_{\alpha-1}(s-\tau) \|L_\alpha(t-s) |u(\tau)|^{\rho-1} u(\tau)\|_{\mathcal{N}_{q/\rho, \mu, \infty}^{\bar{s}}} d\tau ds \\
&\leq C \int_0^t \int_0^s r_{\alpha-1}(s-\tau) (t-s)^{\gamma_1} \| |u(\tau)|^{\rho-1} u(\tau)\|_{\mathcal{N}_{q/\rho, \mu, \infty}^0} d\tau ds \\
&\leq C \int_0^t \int_0^s r_{\alpha-1}(s-\tau) (t-s)^{\gamma_1} \tau^{-\eta\rho} d\tau ds \|u\|_{X_q^p}^\rho \\
&\leq C \|u\|_{X_q^p}^\rho,
\end{aligned} \tag{3.26}$$

as required, this finish our proof.  $\square$

#### 4. Proof of theorems

##### 4.1. Proof of Theorem 1.1

Let  $0 < \varepsilon < R = (1/2^\rho K)^{\rho-1}$ , where  $K > 0$  and  $L > 0$  are the constants obtained in Lemma 3.4 and Lemma 3.5, respectively. Let  $\delta = \varepsilon/L$ , Lemma 3.3 with  $X = X_q^p$  and  $y = L_\alpha(t)u_0$  yields the existence of a unique global mild solution  $u \in X_q$  such that  $\|u\|_{X_q} \leq \varepsilon$ . Moreover, Lemmas 3.4 and 3.5 yield  $u(t) \rightharpoonup u_0$  in the weak-\* topology of  $\dot{B}_{\infty, \infty}^{2/(\rho-1)}$  as  $t \rightarrow 0^+$ . The dependence of the initial data can be obtained from Lemma 3.4 and Lemma 3.3. Indeed, let  $\bar{y} = L_\alpha(t)\bar{u}_0$  where  $\bar{u}_0 \in \mathcal{N}_{p, \mu, \infty}^\sigma$ , then

$$\|u(t) - \bar{u}(t)\|_{X_q^p} \leq \frac{1}{1 - 2^\rho K \varepsilon^{\rho-1}} \|L_\alpha(t)(u_0 - \bar{u}_0)\|_{X_q} \leq \frac{1}{1 - 2^\rho K \varepsilon^{\rho-1}} \|u_0 - \bar{u}_0\|_{\mathcal{N}_{p, \mu, \infty}^\sigma}. \quad \square$$

##### 4.2. Proof of Theorem 1.3

The proof follows from analogous argument found in [3, Theorem 3.3]. For the reader convenience, we indicate the main steps of proofs.

**Item (i):** Let  $M \in \mathcal{G}$  and  $u_0$  be antisymmetric, then  $\Phi(x, t) = L_\alpha(t)u_0$  and  $B_\alpha(u)$  is antisymmetric. Indeed, in view of the orthogonality of  $M$  and  $u_0(Mx) = -u_0(x)$  we have

$$-\widehat{u_0}(\xi) = [u_0(M \cdot)]^\wedge(\xi) = \widehat{u_0}(M^{-1}\xi), \tag{4.1}$$

it follows that

$$\begin{aligned}
[\Phi(Mx, t)]^\wedge(\xi) &= E_\alpha(-t^\alpha |M^{-1}\xi|^2) \widehat{u_0}(M^{-1}\xi) \\
&= -E_\alpha(-t^\alpha |\xi|^2) \widehat{u_0}(\xi) \\
&= -\widehat{\Phi(x, t)}(\xi),
\end{aligned}$$

this shows us that  $L_\alpha(t)u_0$  is antisymmetric for each fixed  $t > 0$ . Similarly, we can show that  $B_\alpha(u)$  is antisymmetric whether  $u$  is also. So, employing an induction argument, one can prove that each element  $u_k$  of the Picard sequence

$$u_1(x, t) = \Phi(x, t) \tag{4.2}$$

$$u_k(x, t) = \Phi(x, t) + B_\alpha(u_{k-1})(x, t), \quad k = 2, 3, \dots \tag{4.3}$$

is antisymmetric. It follows that  $u(x, t)$  is antisymmetric, for all  $t > 0$ . The symmetric property is analogous.

**Item (ii):** Let  $\Phi(x, t) = L_\alpha(t)u_0$ . In view of  $u_0$  be homogeneous of degree  $-\frac{2}{\rho-1}$  we set  $u_0(\lambda x) = \lambda^{-2/(\rho-1)}u_0(x)$ . It follows that

$$\begin{aligned} [\Phi(\lambda \cdot, t)]^\wedge(\xi) &= E_\alpha(-t^\alpha |\xi/\lambda|^2) \widehat{u_0}(\xi/\lambda) \\ &= \lambda^{-\frac{2}{\rho-1}} \lambda^{-n} E_\alpha(-t^\alpha |\xi/\lambda|^2) \widehat{u_0}(\xi) \\ &= \lambda^{-\frac{2}{\rho-1}} E_\alpha(-t^\alpha |\xi|^2) \widehat{u_0}(\xi) \\ &= \lambda^{-\widehat{\frac{2}{\rho-1}}} \Phi(\cdot, t)(\xi), \end{aligned}$$

that is,  $\Phi(\lambda x, t) = \lambda^{-\frac{2}{\rho-1}} \Phi(x, t)$ . Now proceeding like **Item (i)** we obtain that

$$u(x, t) \equiv u_\lambda(x, t), \text{ for every } \lambda > 0,$$

in other words,  $u$  is forward self-similar solution.  $\square$

#### 4.3. Proof of Theorem 1.5

We only show that (1.18) implies (1.17). The converse is left to the reader. We have that

$$\begin{aligned} t^\eta \|u(\cdot, t) - v(\cdot, t)\|_{\mathcal{M}_{q,\mu}} &\leq t^\eta \|L_\alpha(t)(u_0 - v_0)\|_{\mathcal{M}_{q,\mu}} + t^\eta \|B_\alpha(u) - B_\alpha(v)\|_{\mathcal{M}_{q,\mu}} \\ &:= t^\eta \|L_\alpha(t)(u_0 - v_0)\|_{\mathcal{M}_{q,\mu}} + J_1(t) \end{aligned} \quad (4.4)$$

and

$$\begin{aligned} \|u(\cdot, t) - v(\cdot, t)\|_{\mathcal{N}_{p,\mu,\infty}^\sigma} &\leq \|L_\alpha(t)(u_0 - v_0)\|_{\mathcal{N}_{p,\mu,\infty}^\sigma} + \|B_\alpha(u) - B_\alpha(v)\|_{\mathcal{N}_{p,\mu,\infty}^\sigma} \\ &\leq \|L_\alpha(t)(u_0 - v_0)\|_{\mathcal{N}_{p,\mu,\infty}^\sigma} + J_2(t). \end{aligned} \quad (4.5)$$

Using the inequality (3.20),  $\|u\|_{X_q} \leq 2\varepsilon$  and  $\|v\|_{X_q} \leq 2\varepsilon$ ,  $J_1(t)$  can be estimated as

$$\begin{aligned} J_1(t) &\leq C t^\eta \int_0^t (t-s)^{\gamma_2} \int_0^s r_{\alpha-1}(s-\tau) \|u(\tau) - v(\tau)\|_{\mathcal{M}_{q,\mu}} \left( \|u(\tau)\|_{\mathcal{M}_{q,\mu}}^{\rho-1} + \|v(\tau)\|_{\mathcal{M}_{q,\mu}}^{\rho-1} \right) d\tau ds \\ &\leq t^\eta 2(2\varepsilon)^{\rho-1} C \int_0^t (t-s)^{\gamma_2} \int_0^s r_{\alpha-1}(s-\tau) \tau^{-\eta\rho} \Sigma_1(\tau) d\tau ds, \end{aligned} \quad (4.6)$$

where  $\Sigma_1(\tau) = t^\eta \|u(\tau) - v(\tau)\|_{\mathcal{M}_{q,\mu}}$  and  $\gamma_2 = \eta\rho - \alpha - \eta$ . For  $J_2$ , in view of (3.15), we have that

$$J_2(t) \leq (2^\rho \varepsilon^{\rho-1}) C \int_0^t (t-s)^{\gamma_1} \int_0^s r_{\alpha-1}(s-\tau) \tau^{-\eta\rho} \Sigma_2(\tau) d\tau ds, \quad (4.7)$$

where  $\Sigma_2(\tau) = \|u(\tau) - v(\tau)\|_{\mathcal{N}_{p,\mu,\infty}^\sigma}$  and  $\gamma_1 = \eta\rho - \alpha$ .

Let us define  $\Sigma(\tau) = \Sigma_1(\tau) + \Sigma_2(\tau)$ . After performing a change of variables in (4.6) and (4.7), we get

$$J_1(t) + J_2(t) \leq (2^\rho \varepsilon^{\rho-1}) C \int_0^1 (1-\sigma)^{\gamma_2} \sigma^{\alpha-1-\eta\rho} \int_0^1 r_{\alpha-1}(1-z) z^{-\eta\rho} \Sigma(t\sigma z) dz d\sigma +$$



$$+ (2^\rho \varepsilon^{\rho-1}) C \int_0^1 (1-\sigma)^{\gamma_1} \sigma^{\alpha-1-\eta\rho} d\sigma \int_0^1 r_{\alpha-1}(1-z) z^{-\eta\rho} \Sigma(t\sigma z) dz d\sigma. \quad (4.8)$$

We claim that

$$\Pi := \limsup_{t \rightarrow +\infty} \Sigma(t) = 0, \quad (4.9)$$

which is enough for our purposes. For that, we take  $\limsup_{t \rightarrow +\infty}$  in (4.8) in order to obtain

$$\begin{aligned} \limsup_{t \rightarrow +\infty} [J_1(t) + J_2(t)] &\leq (2^\rho \varepsilon^{\rho-1}) C \int_0^1 (1-\sigma)^{\gamma_2} \sigma^{\alpha-1-\eta\rho} d\sigma \limsup_{t \rightarrow +\infty} \int_0^1 r_{\alpha-1}(1-z) z^{-\eta\rho} \Sigma(t\sigma z) dz \\ &\quad + (2^\rho \varepsilon^{\rho-1}) C \int_0^1 (1-\sigma)^{\gamma_1} \sigma^{\alpha-1-\eta\rho} d\sigma \limsup_{t \rightarrow +\infty} \int_0^1 r_{\alpha-1}(1-z) z^{-\eta\rho} \Sigma(t\sigma z) dz \\ &\leq (2^\rho \varepsilon^{\rho-1}) C \left( \int_0^1 (1-\sigma)^{\gamma_2} \sigma^{\alpha-1-\beta\rho} d\sigma \int_0^1 r_{\alpha-1}(1-z) z^{-\eta\rho} dz \right) \Pi \\ &\quad + (2^\rho \varepsilon^{\rho-1}) C \left( \int_0^1 (1-\sigma)^{\gamma_1} \sigma^{\alpha-1-\eta\rho} d\sigma \int_0^1 r_{\alpha-1}(1-z) z^{-\eta\rho} dz \right) \Pi \\ &= (K_1 + K_2) (2^\rho \varepsilon^{\rho-1}) \Pi. \end{aligned} \quad (4.10)$$

It follows from (4.4), (4.5), (4.10) and (1.18) that

$$\begin{aligned} \Pi &\leq \limsup_{t \rightarrow +\infty} (t^\eta \|L_\alpha(t)(u_0 - v_0)\|_{\mathcal{M}_{q,\mu}} + \|L_\alpha(t)(u_0 - v_0)\|_{\mathcal{N}_{p,\mu,\infty}^\sigma}) + \limsup_{t \rightarrow +\infty} [J_1(t) + J_2(t)] \\ &\leq 0 + (K_1 + K_2) (2^\rho \varepsilon^{\rho-1}) \Pi = (2^\rho \varepsilon^{\rho-1} K) \Pi. \end{aligned} \quad (4.11)$$

So, due to  $2^\rho \varepsilon^{\rho-1} K < 1$ , we get  $\Pi = 0$ , as required.  $\square$

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