Bull Braz Math Soc, New Series (2018) 49:137–157
https://doi.org/10.1007/s00574-017-0045-9

Polynomial Differential Systems in $\mathbb{R}^3$ Having Invariant Weighted Homogeneous Surfaces

Thaís Maria Dalbelo[1] · **Marcelo Messias**[1] · **Alisson C. Reinol**[2]

Received: 10 August 2016 / Accepted: 21 June 2017 / Published online: 27 June 2017

Abstract In this paper we give the normal form of all polynomial differential systems in $\mathbb{R}^3$ having a weighted homogeneous surface $f = 0$ as an invariant algebraic surface and characterize among these systems those having a Darboux invariant constructed uniquely using this invariant surface. Using the obtained results we give some examples of stratified vector fields, when $f = 0$ is a singular surface. We also apply the obtained results to study the Vallis system, which is related to the so-called *El Niño* atmospheric phenomenon, when it has a cone as an invariant algebraic surface, performing a dynamical analysis of the flow of this system restricted to the invariant cone and providing a stratification for this singular surface.

Keywords Polynomial differential systems · Darboux theory of integrability · Invariant algebraic surfaces · Weighted homogeneous surfaces · Singular varieties · Stratified vector fields · Vallis system

✉ Marcelo Messias
marcelo@fct.unesp.br

Thaís Maria Dalbelo
thaisdalbelo@gmail.com

Alisson C. Reinol
alissoncarv@gmail.com

1 Departamento de Matemática e Computação, Faculdade de Ciências e Tecnologia, UNESP-Univ Estadual Paulista, Presidente Prudente, SP, Brazil

2 Departamento de Matemática, Intituto de Biociências, Letras e Ciências Exatas, UNESP-Univ Estadual Paulista, São José do Rio Preto, SP, Brazil

1 Introduction and Statement of the Main Results

Vector fields or differential systems are largely studied in several branches of mathematics due to their theoretical importance and because there are many natural phenomena, arising in Physics, Biology, Engineering, Chemistry and other sciences, which can be mathematically modeled and studied through them. In this way many books and papers have been published aiming to describe the dynamical behavior of vector fields, which is in general a hard task, treated by the qualitative theory and bifurcations of dynamical systems, see for instance Guckenheimer and Holmes (2002), Lorenz (1963), Strogatz (2001), Vallis (1988) and Wiggins (1988, 2003). For polynomial vector fields defined in $\mathbb{R}^3$, which in general present complex dynamical behavior, one of the tools used to study their dynamics is the determination of two-dimensional surfaces, regular or singular ones, embedded in $\mathbb{R}^3$ which are invariant under their flow. This technique is in the context of the Darboux theory of integrability, described in Dumortier et al. (2006), Llibre and Valls (2011) and Llibre and Zhang (2009, 2010, 2012). Invariant algebraic surfaces appear frequently in mathematical models of applied problems and when the model is given by a differential system which has an invariant algebraic surface, as for instance in the Lorenz, Rikitake, Rabinovich, Chen among other systems, this helps strongly to understand its global dynamics, as shown in Llibre and Messias (2009), Llibre et al. (2008, 2010, 2011, 2012) and Messias and Reinol (2015).

An important class of surfaces contained in the Euclidean space $\mathbb{R}^3$ is the well-known weighted homogeneous surfaces, which are defined as $(x, y, z) \in \mathbb{R}^3$ such that $f(x, y, z) = 0$, where $f : \mathbb{R}^3 \to \mathbb{R}$ is a weighted homogeneous polynomial, that is, for all $\lambda \in \mathbb{R}\backslash\{0\}$, f is a polynomial satisfying the condition

$$f(\lambda^{w_1}x, \lambda^{w_2}y, \lambda^{w_3}z) = \lambda^{\alpha} f(x, y, z),$$

where $w_1, w_2, w_3, \alpha \in \mathbb{Z}_+$. In this case, we say that w_1, w_2 and w_3 are the weights of the variables x, y and z, respectively, α is the filtration of f related to the weights (w_1, w_2, w_3) and that f is a weighted homogeneous polynomial of type $(w_1, w_2, w_3 : \alpha)$.

In the context above, consider the polynomial differential system in $\mathbb{R}^3$ given by

$$\dot{x} = P(x, y, z), \quad \dot{y} = Q(x, y, z), \quad \dot{z} = R(x, y, z), \tag{1}$$

where P, Q, R are relatively prime polynomials on the ring $\mathbb{R}[x, y, z]$ of the polynomials in the variables x, y, z with coefficients in $\mathbb{R}$ and the dot denotes derivative with respect to the independent variable t, usually called the time. We can naturally associate to system (1) the vector field

$$X = P(x, y, z)\frac{\partial}{\partial x} + Q(x, y, z)\frac{\partial}{\partial y} + R(x, y, z)\frac{\partial}{\partial z}. \tag{2}$$

We say that $m = max\,\{\deg(P), \deg(Q), \deg(R)\}$ is the degree of the polynomial differential system (1) or the polynomial vector field (2).

An invariant algebraic surface for system (1) or for the vector field (2) is an algebraic surface $f^{-1}(0)$ with $f \in \mathbb{R}[x, y, z]$ such that for some polynomial $K \in \mathbb{R}[x, y, z]$ we have $Xf = Kf$. This implies that if a solution curve of the system (1) has a point on the algebraic surface $f^{-1}(0)$, then the whole solution curve passing by this point is contained in $f^{-1}(0)$. See also Sect. 2 for more details.

In this paper we give the normal form of all polynomial differential systems in $\mathbb{R}^3$ having a weighted homogeneous surface $f = 0$ as an invariant algebraic surface. More precisely, the following result holds.

Theorem 1 *Let $f : \mathbb{R}^3 \rightarrow \mathbb{R}$ be a weighted homogeneous polynomial of type $(w_1, w_2, w_3 : \alpha)$, which is irreducible in $\mathbb{R}[x, y, z]$. Assume that the polynomial differential system* (1) *has $V = f^{-1}(0)$ as an invariant algebraic surface. Then system* (1) *can be written as*

$$\begin{aligned} \dot{x} &= f(x, y, z)A - \frac{\partial f}{\partial y} D + \frac{\partial f}{\partial z} E + w_1\, x\, G, \\ \dot{y} &= f(x, y, z)B + \frac{\partial f}{\partial x} D - \frac{\partial f}{\partial z} F + w_2\, y\, G, \\ \dot{z} &= f(x, y, z)C - \frac{\partial f}{\partial x} E + \frac{\partial f}{\partial y} F + w_3\, z\, G, \end{aligned} \tag{3}$$

where A, B, C, D, E, F and G are arbitrary polynomials belonging to $\mathbb{R}[x, y, z]$, and $\dfrac{\partial f}{\partial v} = \dfrac{\partial f}{\partial v}(x, y, z)$ with v running over the variables x, y and z.

Theorem 1 is proved in Sect. 3. It is interesting to observe that the weights w_1, w_2, w_3 appear explicitly in the equations of system (3) as well as the partial derivatives of the polynomial f. Consequently, the weights of the polynomial f will play an important role in the dynamics of this system. The hypothesis of irreducibility of f is important in order to obtain system (3) as the most general normal form of a polynomial differential system having $V = f^{-1}(0)$ as an invariant algebraic surface. One of the most important properties of weighted homogeneous surfaces is the information given by the weights, which can contribute to solve some topological or geometrical problems related to them, which are difficult to solve for surfaces in general, see for instance Milnor and Orlik (1970). The class of weighted homogeneous surfaces has been largely investigated in algebraic geometry as well as in other branches of mathematics. In singularity theory this object has been deeply investigated by many authors as Grulha (2011), Milnor and Orlik (1970) and Miranda et al. (2013), among others.

Weighted homogeneous surfaces can be a regular surface, that is, a surface of class C^k, with $k \in \mathbb{Z}_+$, or a singular surface (or singular variety) and in both cases it is important to study vector fields defined on these surfaces. Indeed it is common to find problems in mathematics and other sciences in which we need consider vector fields defined on singular varieties, including those given by weighted homogeneous surfaces, see for instance Brasselet et al. (2000, 2009) and Seade (1987). One of the techniques used to study vector fields on a singular variety V contained in some differentiable manifold M is the stratification theory, which appears in several works in the literature, for an overview about this theory see Gibson (1976), Thom (1969) and Whitney (1965). Intuitively, a stratification is a decomposition of a singular variety V in submanifolds of M, called strata, with some type of control on how these strata can be separated and glued again. Such control is called regularity condition. The knowledge of stratifications of singular varieties are related to some invariants of these kind of surfaces, as can be seen in Brasselet et al. (2000, 2009) and Nuño-Ballesteros et al. (2011). In this way, the good property for a vector field defined on the singular variety is to be tangent to each stratum, that is called stratified vector field. It is obvious from the definition of invariant algebraic surface that, when the weighted homogeneous surface $f = 0$ is regular, then the flow of system (3) restricted to this surface generates a vector field tangent to it. In Sect. 5, using the normal form (3), we give some examples of stratified vector fields when $f = 0$ is a singular surface. In such examples the stratification considered will be of Whitney type.

Another interesting problem in the study of the polynomial differential system (1) is to recognize when it has a Darboux invariant (Llibre et al. 2014; Llibre and Valls 2011), which is a special type of invariant of this differential system (for details, see Sect. 2), since the knowledge of this kind of invariant provides information about the α- and ω-limit sets of all orbits of the differential system, helping us to determine the asymptotic behavior of these orbits. However, the determination of Darboux invariants is in general a very difficult work. Here we characterize among systems (3) those having a Darboux invariant constructed uniquely using the invariant algebraic surface $f = 0$. In fact, in Sect. 4 we prove the following result.

Theorem 2 *Systems* (3) *having a Darboux invariant I constructed uniquely using the invariant weighted homogeneous surface $V = f^{-1}(0)$ can be written as*

$$\begin{aligned}
\dot{x} &= A\left(f - \frac{xw_1}{\alpha}\frac{\partial f}{\partial x}\right) - \frac{\partial f}{\partial y}\left(D + \frac{xw_1}{\alpha}B\right) + \frac{\partial f}{\partial z}\left(E - \frac{xw_1}{\alpha}C\right) - \frac{xw_1}{\alpha}a,\\
\dot{y} &= B\left(f - \frac{yw_2}{\alpha}\frac{\partial f}{\partial y}\right) + \frac{\partial f}{\partial x}\left(D - \frac{yw_2}{\alpha}A\right) - \frac{\partial f}{\partial z}\left(F + \frac{yw_2}{\alpha}C\right) - \frac{yw_2}{\alpha}a, \qquad (4)\\
\dot{z} &= C\left(f - \frac{zw_3}{\alpha}\frac{\partial f}{\partial z}\right) - \frac{\partial f}{\partial x}\left(E + \frac{zw_3}{\alpha}A\right) + \frac{\partial f}{\partial y}\left(F - \frac{zw_3}{\alpha}B\right) - \frac{zw_3}{\alpha}a,
\end{aligned}$$

where A, B, C, D, E and F are arbitrary polynomials in $\mathbb{R}[x, y, z]$, $a \in \mathbb{R}\backslash\{0\}$, $f = f(x, y, z)$ and $\frac{\partial f}{\partial v} = \frac{\partial f}{\partial v}(x, y, z)$ with v running over the variables x, y and z. Moreover the Darboux invariant of system (4) *is given by $I(x, y, z, t) = f(x, y, z)^{\frac{1}{a}} e^t$.*

Once again we can observe the role of the weights w_1, w_2, w_3, of filtration α and of the partial derivatives of the polynomial f in the construction of system (4). Theorem 2 is proved in Sect. 4.

As an application of the study considered here, in Sect. 6 we analyze a differential system defined in $\mathbb{R}^3$, known as the Vallis system, which for some parameter values has a weighted homogeneous surface as an invariant algebraic surface. This system was introduced by Vallis (1988) and it consists of the periodic nonautonomous three-dimensional differential system given by

$$\dot{x} = -ax + \mu y + a\nu(t), \quad \dot{y} = -y + xz, \quad \dot{z} = -z - xy + 1, \tag{5}$$

where the function $\nu(t)$ is some C^1 T-periodic function and a, μ are positive real parameters. System (5) models the atmosphere dynamics in the tropics over the Pacific Ocean, related to the yearly oscillations of precipitation, temperature and wind force and it provides a description of the processes and recovers many of the observed properties of the so-called *El Niño* phenomenon (Strozzi 1999; Vallis 1988). In Strozzi (1999) it was proved that there exists a chaotic attractor for system (5), showing the complex dynamics of this system. The Vallis system has been deeply investigated by many authors, see for instance Euzébio and Llibre (2014), Kanatnikov and Krishchenko (2009) and Krishchenko and Starkov (2008). For $\nu \equiv 0$ and $a = 1$, system (5) reduces to a polynomial differential system. In Sect. 6 we show that, in this case, the weighted homogeneous surface $x^2 - \mu y^2 - \mu(z-1)^2 = 0$ is an invariant algebraic surface of system (5), which also has a Darboux invariant constructed uniquely using this invariant surface, which is a cone.

This paper is organized as follows. In Sect. 2 we present some background material concerning the Darboux theory of integrability, which will be used in the following sections. In Sect. 3 we provide the normal form of all polynomial differential systems in $\mathbb{R}^3$ having a weighted homogeneous surface $f = 0$ as an invariant algebraic surface, proving Theorem 1. In Sect. 4 we characterize among systems (3) those having a Darboux invariant constructed uniquely using the invariant algebraic surface $f = 0$, proving Theorem 2. In Sect. 5 we present some basic concepts about stratification theory and, using the normal form (3), we give some examples of stratified vector fields when $f = 0$ is a singular surface. In Sect. 6 we consider the Vallis system when it has a cone as an invariant algebraic surface, performing a dynamical analysis of the flow of this system restricted to the invariant cone and providing a stratification for this singular surface.

2 Darboux Theory of Integrability

In this subsection we introduce some definitions and results about the Darboux theory of integrability for polynomial differential systems. This kind of integrability provides a link between the integrability of polynomial differential systems and their invariant algebraic surfaces. More details about this theory for planar polynomial vector fields can be found in Llibre (2004) and in Chapter 8 of Dumortier et al. (2006). It can be extended in a natural way for polynomial vector fields in $\mathbb{R}^n$ or $\mathbb{C}^n$, see for instance Jouanolou (1979) and Llibre and Zhang (2009, 2010, 2012).

Consider the polynomial differential system in $\mathbb{R}^3$ given by (1), that is,

$$\dot{x} = P(x, y, z), \quad \dot{y} = Q(x, y, z), \quad \dot{z} = R(x, y, z),$$

and its associated vector field (2), given by,

$$X = P(x, y, z)\frac{\partial}{\partial x} + Q(x, y, z)\frac{\partial}{\partial y} + R(x, y, z)\frac{\partial}{\partial z},$$

which is also denoted by $X = (P, Q, R)$. We have the following definitions (Dumortier et al. 2006; Jouanolou 1979; Llibre 2004; Llibre and Zhang 2009).

Definition 3 Let U be an open subset of $\mathbb{R}^3$. If there exists a nonlocally constant analytic function $H : U \to \mathbb{R}$, which is constant on all solution curves $(x(t), y(t), z(t))$ of system (1) contained in U, then H is called a *first integral* of X in U. Clearly H is a first integral of system (1) if and only if $X(H) \equiv 0$ on U, i.e.

$$X(H) = \frac{\partial H}{\partial x}P + \frac{\partial H}{\partial y}Q + \frac{\partial H}{\partial z}R = \frac{dH}{dt} = 0$$

on the orbits of X contained in U.

Definition 4 An invariant of system (1) on an open subset $U \subset \mathbb{R}^3$ is a nonlocally constant analytic function I in the variables x, y, z and t such that I is constant on all solution curves $(x(t), y(t), z(t))$ of system (1) contained in U, i.e.

$$\frac{dI}{dt} = \frac{\partial I}{\partial x}P + \frac{\partial I}{\partial y}Q + \frac{\partial I}{\partial z}R + \frac{\partial I}{\partial t} = 0$$

on the orbits of X contained in U.

In some sense an invariant I is a first integral of system (1) which depends on the time t.

Definition 5 Let $f \in \mathbb{K}[x, y, z]$ nonlocally constant, where $\mathbb{K}$ is either $\mathbb{R}$ or $\mathbb{C}$. The surface $f(x, y, z) = 0$ is an invariant algebraic surface of system (1) if for some polynomial $K \in \mathbb{K}[x, y, z]$ we have

$$X(f) = \frac{\partial f}{\partial x}P + \frac{\partial f}{\partial y}Q + \frac{\partial f}{\partial z}R = Kf.$$

The polynomial K is called the cofactor of the invariant algebraic surface $f = 0$.

Note that, if system (1) has degree m, then the degree of K is at most $m - 1$. Moreover, when $K = 0$, then f is a polynomial first integral of system (1).

Definition 6 Let $g, h \in \mathbb{K}[x, y, z]\backslash\{0\}$ and assume that g and h are relatively prime in the ring $\mathbb{K}[x, y, z]$, or that $h = 1$, where $\mathbb{K}$ is either $\mathbb{R}$ or $\mathbb{C}$. Then the function $F = exp(g/h)$ is called an exponential factor of system (1) if for some polynomial $L \in \mathbb{K}[x, y, z]$ of degree at most $m - 1$ we have that

$$X(F) = \frac{\partial F}{\partial x}P + \frac{\partial F}{\partial y}Q + \frac{\partial F}{\partial z}R = LF.$$

We say that an invariant I of X is of *Darboux type*, or a *Darboux invariant*, if it can be written as

$$I(x, y, z, t) = {f_1}^{\lambda_1} \ldots {f_p}^{\lambda_p} {F_1}^{\mu_1} \ldots {F_q}^{\mu_q} e^{st}, \tag{6}$$

where $f_i = 0$ are invariant algebraic surfaces of X for $i = 1, \ldots, p$; F_j are exponential factors of X for $j = 1, \ldots, q$; $\lambda_i, \mu_j \in \mathbb{C}$ and $s \in \mathbb{R}\backslash\{0\}$.

While the knowledge of a first integral of system (1) in $\mathbb{R}^3$ allows to reduce the study of this system in one dimension, the knowledge of a Darboux invariant provides information about the α- and the ω-limit sets of all orbits of system (1), see for instance Proposition 5 in Llibre and Oliveira (2015).

The next result explains how to find Darboux invariants of system (1). Its proof for polynomial differential systems in $\mathbb{R}^2$ can be found in Dumortier et al. (2006) (see statement (vi) of Theorem 8.7), but it can be trivially extended to $\mathbb{R}^n$.

Proposition 7 *Suppose that a polynomial differential system* (1) *of degree m admits p invariant algebraic surfaces $f_i = 0$ with cofactors K_i for $i = 1, \ldots, p$, and q exponential factors $F_j = exp(g_j/h_j)$ with cofactors L_j for $j = 1, \ldots, q$. There exist $\lambda_i, \eta_j \in \mathbb{C}$ not all zero such that*

$$\sum_{i=1}^{p} \lambda_i K_i + \sum_{j=1}^{q} \eta_j L_j = -s,$$

for some $s \in \mathbb{R}\backslash\{0\}$, if and only if the real (multivalued) function

$$f_1^{\lambda_1} \ldots f_p^{\lambda_p} F_1^{\eta_1} \ldots F_q^{\eta_q} e^{st}$$

is a Darboux invariant of system (1).

3 Proof of Theorem 1

The following Lemma will be essential to prove Theorem 1, which is the main result of this work.

Lemma 8 *Let $f : \mathbb{R}^3 \to \mathbb{R}$ be a weighted homogeneous polynomial of type $(w_1, w_2, w_3 : \alpha)$, then the following equality holds*

$$\alpha f(x, y, z) = w_1 x \frac{\partial f}{\partial x}(x, y, z) + w_2 y \frac{\partial f}{\partial y}(x, y, z) + w_3 z \frac{\partial f}{\partial z}(x, y, z).$$

Proof As f is a weighted homogeneous polynomial of type $(w_1, w_2, w_3 : \alpha)$, then

$$f(x, y, z) = \sum_{(i_1, i_2, i_3)} a_{(i_1, i_2, i_3)} x^{i_1} y^{i_2} z^{i_3},$$

where $(i_1, i_2, i_3) \in \mathbb{N}^3$ and $a_{(i_1,i_2,i_3)} \in \mathbb{R}$, moreover

$$\alpha = w_1 i_1 + w_2 i_2 + w_3 i_3 \tag{7}$$

for all nonzero terms of f. Then,

$$\begin{aligned} \frac{\partial f}{\partial x}(x, y, z) &= \sum_{(i_1,i_2,i_3)} (i_1)(a_{(i_1,i_2,i_3)}) x^{i_1-1} y^{i_2} z^{i_3}, \\ \frac{\partial f}{\partial y}(x, y, z) &= \sum_{(i_1,i_2,i_3)} (i_2)(a_{(i_1,i_2,i_3)}) x^{i_1} y^{i_2-1} z^{i_3}, \\ \frac{\partial f}{\partial z}(x, y, z) &= \sum_{(i_1,i_2,i_3)} (i_3)(a_{(i_1,i_2,i_3)}) x^{i_1} y^{i_2} z^{i_3-1}. \end{aligned} \tag{8}$$

Therefore the result follows from (7) and (8). □

From now on we consider the following setup. Let f_1, f_2 and f_3 be real functions defined in an open subset U of $\mathbb{R}^3$. As usual the Jacobian matrix of the functions f_1, f_2 and f_3 is defined by

$$J_{(f_1,f_2,f_3)} := J_{(f_1,f_2,f_3)}(x, y, z) = \begin{pmatrix} \frac{\partial f_1}{\partial x} & \frac{\partial f_1}{\partial y} & \frac{\partial f_1}{\partial z} \\ \frac{\partial f_2}{\partial x} & \frac{\partial f_2}{\partial y} & \frac{\partial f_2}{\partial z} \\ \frac{\partial f_3}{\partial x} & \frac{\partial f_3}{\partial y} & \frac{\partial f_3}{\partial z} \end{pmatrix}.$$

The Jacobian of $J_{(f_1,f_2,f_3)}$ is the determinant of the matrix J, which we denote by $|J_{(f_1,f_2,f_3)}|$.

Proof of Theorem 1 Let us denote the weighted homogeneous polynomial $f = f(x, y, z)$ in the statement of Theorem 1 by $f_1 = f_1(x, y, z)$. According to Theorem 5 of Llibre et al. (2015) (which is a particular case of Theorem 5 of Llibre et al. 2014), any polynomial differential system in $\mathbb{R}^3$ which admits $f_1 = 0$ as an invariant algebraic surface is given by

$$\dot{v} = \lambda_1 \, |J_{(v,f_2,f_3)}| + \lambda_2 \, |J_{(f_1,v,f_3)}| + \lambda_3 \, |J_{(f_1,f_2,v)}|, \tag{9}$$

where v runs over the variables x, y and z, $\lambda_1 = \varphi f_1$ and φ, λ_2, λ_3 are rational functions, f_2 and f_3 are arbitrary polynomials in $\mathbb{R}[x, y, z]$ which must be chosen in such a way that $|J_{(f_1,f_2,f_3)}| \not\equiv 0$.

From expression (9), performing the involved calculations we obtain that the most general polynomial differential system having the invariant weighted homogeneous surface $f_1 = 0$ can be written as

$$\begin{aligned}\dot{x} &= \varphi f_1 J_1 + \lambda_2(f_{1z} f_{3y} - f_{1y} f_{3z}) + \lambda_3(f_{1y} f_{2z} - f_{1z} f_{2y}),\\ \dot{y} &= \varphi f_1 J_2 + \lambda_2(f_{1x} f_{3z} - f_{1z} f_{3x}) + \lambda_3(f_{1z} f_{2x} - f_{1x} f_{2z}),\\ \dot{z} &= \varphi f_1 J_3 + \lambda_2(f_{1y} f_{3x} - f_{1x} f_{3y}) + \lambda_3(f_{1x} f_{2y} - f_{1y} f_{2x}),\end{aligned} \tag{10}$$

where $f_1 = f_1(x, y, z)$, $f_{iv} = (\partial f_i/\partial v)(x, y, z)$, with v running over the variables x, y and z, for $i = 1, 2, 3$, and

$$J_1 = f_{2y} f_{3z} - f_{2z} f_{3y}, \quad J_2 = f_{3x} f_{2z} - f_{2x} f_{3z}, \quad J_3 = f_{2x} f_{3y} - f_{3x} f_{2y}.$$

In order to obtain the normal form (3), taking into account that f_1 is a weighted homogeneous polynomial and using the proof of Theorem 5 of Llibre et al. (2015), which provides a way for determining explicit expressions for the rational functions φ, λ_2 and λ_3, we take

$$\begin{aligned}\varphi &= \frac{f_{1x} A + f_{1y} B + f_{1z} C + \alpha G}{f_{1x} J_1 + f_{1y} J_2 + f_{1z} J_3},\\ \lambda_2 &= \frac{(f_{1x} f_{2y} - f_{1y} f_{2x}) D + (-f_{1x} f_{2z} + f_{1z} f_{2x}) E + (f_{1y} f_{2z} - f_{1z} f_{2y}) F}{f_{1x} J_1 + f_{1y} J_2 + f_{1z} J_3}\\ &\quad + \frac{(f_{2x} A + f_{2y} B + f_{2z} C) f_1 + (w_1 x f_{2x} + w_2 y f_{2y} + w_3 z f_{2z}) G}{f_{1x} J_1 + f_{1y} J_2 + f_{1z} J_3},\\ \lambda_3 &= \frac{(f_{1x} f_{3y} - f_{1y} f_{3x}) D + (-f_{1x} f_{3z} + f_{1z} f_{3x}) E + (f_{1y} f_{3z} - f_{1z} f_{3y}) F}{f_{1x} J_1 + f_{1y} J_2 + f_{1z} J_3}\\ &\quad + \frac{(f_{3x} A + f_{3y} B + f_{3z} C) f_1 + (w_1 x f_{3x} + w_2 y f_{3y} + w_3 z f_{3z}) G}{f_{1x} J_1 + f_{1y} J_2 + f_{1z} J_3},\end{aligned}$$

where A, B, C, D, E, F and G are arbitrary polynomials in $\mathbb{R}[x, y, z]$. Substituting these expressions into system (10), we obtain exactly system (3), which has the weighted homogeneous surface $f_1 = 0$ as an invariant algebraic surface with cofactor $K = f_{1x} A + f_{1y} B + f_{1z} C + \alpha G$. In fact we have

$$\begin{aligned}X(f_1) &= f_{1x} P + f_{1y} Q + f_{1z} R\\ &= f_{1x}(f_1 A - f_{1y} D + f_{1z} E + w_1 x\, G)\\ &\quad + f_{1y}(f_1 B + f_{1x} D - f_{1z} F + w_2 y\, G)\\ &\quad + f_{1z}(f_1 C - f_{1x} E + f_{1y} F + w_3 z\, G),\end{aligned}$$

and using Lemma 8, we get

$$X(f_1) = (f_{1x} A + f_{1y} B + f_{1z} C + \alpha G)\, f_1.$$

This completes the proof of Theorem 1. □

Note that system (3) depends on the weights (w_1, w_2, w_3) of f while the cofactor K depends on the filtration α of f. Besides, the partial derivatives of the polynomial

f appear explicitly in the equations of system (3), which enable us to know easily when a singular point of the surface $f^{-1}(0)$, that is, a point $p_0 = (x_0, y_0, z_0) \in V$ such that

$$\frac{\partial f}{\partial x}(x_0, y_0, z_0) = \frac{\partial f}{\partial y}(x_0, y_0, z_0) = \frac{\partial f}{\partial z}(x_0, y_0, z_0) = 0,$$

is also an *equilibrium point of the differential system* (3), that is,

$$X(p_0) = (P(x_0, y_0, z_0), Q(x_0, y_0, z_0), R(x_0, y_0, z_0)) = (0, 0, 0).$$

4 Darboux Invariants and Weighted Homogeneous Surfaces

Proof of Theorem 2 Suppose that system (1) has an invariant weighted homogeneous surface $V = f^{-1}(0)$. By Theorem 1, we can rewrite system (1) as (3). So $f = 0$ is an invariant algebraic surface of system (3) with cofactor

$$K = \frac{\partial f}{\partial x} A + \frac{\partial f}{\partial y} B + \frac{\partial f}{\partial z} C + \alpha G.$$

If system (3) has a Darboux invariant constructed uniquely using the invariant surface $f = 0$, then by Proposition 7, there exists a nonzero $\eta \in \mathbb{C}$ such that, for some $s \in \mathbb{R}\backslash\{0\}$,

$$\eta K = -s \quad \Leftrightarrow \quad G = -\frac{1}{\alpha}\left(\frac{\partial f}{\partial x} A - \frac{\partial f}{\partial y} B - \frac{\partial f}{\partial z} C - a\right)$$

where $a = s/\eta \in \mathbb{R}\backslash\{0\}$. Substituting G and grouping the polynomials conveniently into system (3) we obtain system (4). Using Proposition 7, it follows that the Darboux invariant of system (4) is $I(x, y, z, t) = f(x, y, z)^{\eta}\, e^{st}$. As $\eta = s/a$ and clearly it is not restrictive to choose $s = 1$, hence $I(x, y, z, t) = f(x, y, z)^{1/a}\, e^{t}$. Theorem 2 is proved. □

In the next two sections we present some nice applications of the obtained results.

5 Stratified Vector Fields

In this section, we use the normal form (3) to give some examples of stratified vector fields when the invariant algebraic surface $f = 0$ is a singular surface, where f is a weighted homogeneous polynomial. We start with the definition of stratification (see Gibson 1976), which, among other things, enable us to study vector fields on singular varieties, called stratified vector fields.

Definition 9 Let M be a manifold and $V \subset M$. A locally finite stratification of V is a partition of V on submanifolds of M, called strata, such that for every point of V there is a neighborhood in M which intersects only a finite number of strata.

The most important notion in stratification theory is a regularity condition between the strata, which indicates the way in which the strata can be separated and glued again. Intuitively, we can think of a regularity condition as a type of control on how these strata are glued. Two of the main regularity conditions are the boundary condition and the conditions (a) and (b) of Whitney (Gibson 1976; Trotman 1986; Whitney 1965). The precise definition of these conditions are given below.

Definition 10 (*Boundary Condition*) A stratification $\{V_\beta\}$ of V satisfies the boundary condition, if for two strata V_β and V_γ such that $V_\beta \cap \overline{V_\gamma} \neq \emptyset$, then $V_\beta \subset \overline{V_\gamma}$.

Note that the boundary condition means that the closure of a stratum is the union of lower dimensional strata.

Definition 11 (*Whitney Conditions*) A stratification $\{V_\beta\}$ satisfies the Whitney conditions if for every pair (V_β, V_γ) of strata, such that $V_\gamma \subset \overline{V_\beta}$ and for every point y of V_γ the following statements hold:

(a) If for all sequence of points x_i of V_β converging to y the limit

$$T := \lim_{i\to\infty} T_{x_i}(V_\beta)$$

exists in the corresponding Grassmannian, then T contains $T_y(V_\gamma)$, where $T_{x_i}(V_\beta)$ denotes the tangent space of V_β at the point x_i;

(b) If moreover, for all sequence y_i of points of V_γ converging to y the directions limit

$$\lambda := \lim_{i\to\infty} \overline{x_i y_i}$$

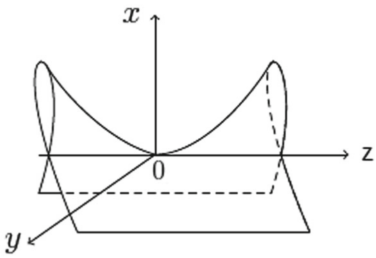
exists in the projective space, then T contain λ, where $\overline{x_i y_i}$ is the straight line passing by the points x_i and y_i, for all $i \in \mathbb{N}$.

A stratification satisfying the boundary condition and the Whitney conditions is called a *Whitney stratification*. In the following we present an example of stratification.

Example 12 Let $h : \mathbb{R}^3 \to \mathbb{R}$ be the function given by $h(x, y, z) = y^2+x^3-x^2z^2$ and V be the variety defined by the zeros of h, that is $V = h^{-1}(0)$, which is known as the Whitney cusp and is drawn in Fig. 1. If we consider V_γ and V_β the submanifolds of $\mathbb{R}^3$ given by $V_\gamma = \{(0, 0, z) \in \mathbb{R}^3\} \subset V$ and $V_\beta = V\backslash V_\gamma$, then V_γ and V_β form the locally finite stratification $\{V_\gamma, V_\beta\}$ of V. Note that this stratification satisfies the condition (a) of Whitney, however it does not satisfies the condition (b). Indeed, consider the sequence $(p_i)_{i\in\mathbb{N}}$ of points of V_β converging to $(0, 0, 0)$ given by $p_i = (x_i, 0, \sqrt{x_i})$, for all $i \in \mathbb{N}$, with $x_i \in \mathbb{R}_+\backslash\{0\}$ and the sequence $(q_i)_{i\in\mathbb{N}}$ of points of V_γ given by $q_i = (0, 0, \sqrt{x_i})$, for all $i \in \mathbb{N}$, with $x_i \in \mathbb{R}_+\backslash\{0\}$, which also converges to $(0, 0, 0)$. Note that the straight lines $\overline{p_i q_i}$ always have the same direction $\lambda = (x_i, 0, 0)$, however $\lambda \not\subset T$, where

$$\lim_{i\to\infty} T_{p_i}(V_\beta) = T,$$

Fig. 1 The Whitney cusp



since T is equal to the plane yz.

Now if we add the zero dimensional strata $V_\tau = \{0\}$, then $\{V_\beta, V_{\gamma'}, V_\tau\}$ is a Whitney stratification of V, where $V_{\gamma'} = V_\gamma \backslash \{0\}$.

Let M be a differentiable manifold, $V \subset M$ an analytic variety and $\{V_\beta\}$ a locally finite stratification of V. Let $\bigcup \mathrm{TV}_\beta$ be the union of the tangent bundles of all strata V_β. We can consider this union as a subset of the tangent bundle $\mathrm{TM}|_\mathrm{V}$.

Definition 13 Let X be a section of TM on a subset A of M. We say that the vector field X is stratified if for each $x \in V_\beta \cap A$, we have that $X(x) \in T_x\mathrm{V}_\beta$, where V_β is the stratum that contains x.

Given a polynomial $f : \mathbb{R}^3 \to \mathbb{R}$, the *singular set* of the algebraic surface $V = f^{-1}(0)$ is the set of all singular points of V and we denote it by S, that is,

$$S := \mathrm{Sing}(V) = \left\{(x, y, z) \in V\,;\ \frac{\partial f}{\partial x}(x, y, z) = \frac{\partial f}{\partial y}(x, y, z) = \frac{\partial f}{\partial z}(x, y, z) = 0\right\}.$$

Let X be the polynomial vector field associated to the polynomial differential system (3). An immediate consequence of Theorem 1 is that, if $S = \emptyset$, i.e., if V is a smooth surface, then $X|_V$ is a vector field tangent to V. Furthermore, when V is a singular surface with an isolated singularity at the origin, the following result holds.

Corollary 14 *If $S = \{(0, 0, 0)\}$ and $X|_V$ has a finite number of equilibrium points, then $X|_V$ is a stratified vector field on V and the stratification given by the flow of $X|_V$ is of Whitney type.*

If $\dim(S) = 1$, then we have yet situations in which we can obtain stratified vector fields. In the following we present two examples where it happens.

Example 15 Consider the polynomial $h : \mathbb{R}^3 \to \mathbb{R}$ given by

$$h = h(x, y, z) = y^2 + x^3 - x^2 z^2,$$

which is a weighted homogeneous polynomial of type (2, 3, 1:6). Then by Theorem 1 the polynomial differential systems having the weighted homogeneous surface $h^{-1}(0)$ (the Whitney cusp) as an invariant algebraic surface have the form

$$
\begin{aligned}
\dot{x} &= hA - \frac{\partial h}{\partial y} D + \frac{\partial h}{\partial z} E + 2\, x\, G,\\
\dot{y} &= hB + \frac{\partial h}{\partial x} D - \frac{\partial h}{\partial z} F + 3\, y\, G,\\
\dot{z} &= hC - \frac{\partial h}{\partial x} E + \frac{\partial h}{\partial y} F + z\, G,
\end{aligned} \tag{11}
$$

where A, B, C, D, E, F and G are arbitrary polynomials in $\mathbb{R}[x, y, z]$.

Taking $A = B = E = D = 0$ and $C := c$, $F := f$, $G := g$ as nonzero constants, system (11) becomes

$$
\begin{aligned}
\dot{x} &= 2\, x\, g,\\
\dot{y} &= 2\, f\, x^2\, z + 3\, y\, g,\\
\dot{z} &= c\,(y^2 + x^3 - x^2 z^2) + 2\, f\, y + z\, g.
\end{aligned} \tag{12}
$$

System (12) has only the origin (0, 0, 0) as an equilibrium point. The vector field X associated to system (12), when restricted to the set $S = \{(0, 0, z) \in \mathbb{R}^3\}$, is the linear transformation given by $X|_S(0, 0, z) = g\, z$. Note that S is exactly the singular set of the Whitney cusp $h^{-1}(0)$, as seen in Example 12. In this way, the restricted vector field $X|_S$ vanishes only in (0, 0, 0) and $X|_S$ is tangent to S. On the other hand, from the definition of invariant algebraic surface we have that the vector field $X|_Z$ is tangent to Z, where $Z = h^{-1}(0)\backslash S$ is the regular part of $h^{-1}(0)$. Therefore, the flow of system (12) determines a stratification of the singular surface $h^{-1}(0)$, given by $\{Z, S\backslash\{(0, 0, 0)\}, \{(0, 0, 0)\}\}$, which is a Whitney stratification of this surface, as seen in Example 12.

We observe that the stratification obtained in Example 15 is not unique. Indeed, it is possible to make other choices for the polynomials $A, \ldots, G$ and to obtain other stratifications of this singular surface.

Example 16 In Trotman (1986), the author presents diagrams showing when Whitney regularity conditions hold for the stratification with two strata given by $\{V\backslash S, S\}$, whenever V is one of the algebraic surfaces

$$\{(x, y, z) \in \mathbb{R}^3;\;\; y^m = x^r + x^p z^n \text{ with } \; m, r, p, n \in \mathbb{Z}_+\}.$$

Consider $V \subset \mathbb{R}^3$ a weighted homogeneous surface given by $f^{-1}(0)$, where $f : \mathbb{R}^3 \to \mathbb{R}$ is the polynomial

$$f(x, y, z) = x^r - y^m + x^p z^n,$$

with m, n, p, r positive integers, such that Whitney's regularity conditions are satisfied (according to Trotman 1986). Then by Theorem 1, the polynomial differential system

$$\dot{x} = w_1\, x\, G,$$

$$\dot{y} = -\frac{\partial f}{\partial z} F + w_2\, y\, G, \tag{13}$$
$$\dot{z} = f C + \frac{\partial f}{\partial y} F + w_3\, z\, G,$$

admits $f = 0$ as an invariant algebraic surface, $X|_V$ is a stratified vector field on V and the stratification given by the flow of $X|_V$ is of Whitney type, where C, F and G are nonzero constants and (w_1, w_2, w_3) are the weights of f. Indeed, system (13) has only the origin $(0, 0, 0)$ as an equilibrium point and $S = \{(0, 0, z) \in \mathbb{R}^3\}$ is the singular set of V, then proceeding in an analogous way of Example 15, we obtain that the flow of system (13) determines a stratification of the singular surface $f^{-1}(0)$, given by $\{Z, S\backslash\{(0, 0, 0)\}, \{(0, 0, 0)\}\}$, which is a Whitney stratification of this surface, since $\{Z, S\}$ is a Whitney stratification, where $Z = f^{-1}(0)\backslash S$ is the regular part of $f^{-1}(0)$.

Note that if $X|_V$ has an infinite number of equilibrium points, then its flow does not determine a locally finite stratification of V.

6 Dynamical Analysis of Vallis System with an Invariant Cone

The Vallis system is the periodic nonautonomous three–dimensional system given by

$$\dot{x} = -ax + \mu y + a\nu(t), \quad \dot{y} = -y + xz, \quad \dot{z} = -z - xy + 1,$$

where a and μ are positive real parameters and $\nu(t)$ is some C^1 T-periodic function. It is easy to check that for $\nu \equiv 0$ and $a = 1$, the cone $x^2 - \mu y^2 - \mu(z - 1)^2 = 0$ is an invariant algebraic surface of the Vallis system.

In order to simplify the next calculations, considering the affine change of coordinates

$$(x, y, z) \to \left(z, \frac{y}{\sqrt{\mu}}, \frac{x}{\sqrt{\mu}} + 1\right)$$

we can write the Vallis system as

$$\dot{x} = -yz - x, \quad \dot{y} = xz + \sqrt{\mu}\, z - y, \quad \dot{z} = \sqrt{\mu}\, y - z. \tag{14}$$

We can show that the (normalized) cone $f(x, y, z) = x^2 + y^2 - z^2 = 0$ is an invariant algebraic surface of system (14) with constant cofactor $K = -2$.

In this section we give a detailed description of the flow of system (14) restricted to the invariant cone V and prove that this flow provides a Whitney stratification of this singular surface. Furthermore we show that system (14) has a Darboux invariant constructed uniquely using the invariant cone and give explicitly the expression of the Darboux invariant.

The flow of system (14) restricted to the invariant cone V is given by

$$\dot{x} = -y\sqrt{x^2 + y^2} - x, \quad \dot{y} = (\sqrt{\mu} + x)\sqrt{x^2 + y^2} - y, \tag{15}$$

for $z \geq 0$ and by

$$\dot{x} = y\sqrt{x^2 + y^2} - x, \quad \dot{y} = -(\sqrt{\mu} + x)\sqrt{x^2 + y^2} - y, \tag{16}$$

for $z \leq 0$. The equilibrium points of system (15) are

$$p_0 = (0, 0) \quad \text{and} \quad p_1 = \left(\frac{1 - \mu}{\sqrt{\mu}}, \frac{\sqrt{\mu - 1}}{\sqrt{\mu}}\right)$$

and the equilibrium points of system (16) are

$$p_0 = (0, 0) \quad \text{and} \quad p_2 = \left(\frac{1 - \mu}{\sqrt{\mu}}, -\frac{\sqrt{\mu - 1}}{\sqrt{\mu}}\right).$$

The number of equilibrium points of systems (15) and (16) depends on the value of μ. Then we consider two cases: $0 < \mu \leq 1$ and $\mu > 1$

Suppose that $0 < \mu \leq 1$. Then p_0 is the only equilibrium point of systems (15) and (16). In order to study the dynamics of systems (15) and (16) near the origin we consider polar coordinates. Performing the change of coordinates $(x, y) \to (\rho \cos\theta, \rho \sin\theta)$ we can write system (15) as

$$\dot{\rho} = \rho(\sqrt{\mu} \sin\theta - 1), \quad \dot{\theta} = \rho + \sqrt{\mu} \cos\theta, \tag{17}$$

and system (16) as

$$\dot{\rho} = -\rho(\sqrt{\mu} \sin\theta + 1), \quad \dot{\theta} = -(\rho + \sqrt{\mu} \cos\theta). \tag{18}$$

Note that $\dot{\rho} \leq 0$ in systems (17) and (18), since $0 < \mu \leq 1$. Hence the origin is a stable equilibrium point of systems (15) and (16). Now consider the circle

$$\mathcal{C} = \left\{(x, y) \in \mathbb{R}^2; \left(x + \frac{\sqrt{\mu}}{2}\right)^2 + y^2 = \left(\frac{\sqrt{\mu}}{2}\right)^2\right\}.$$

For system (17) we have that $\dot{\theta} < 0$ if (x, y) is inside $\mathcal{C}$, $\dot{\theta} = 0$ if (x, y) is on $\mathcal{C}$ and $\dot{\theta} > 0$ if (x, y) is outside $\mathcal{C}$. Hence when the orbits of system (15) are outside $\mathcal{C}$ they rotate counter-clockwise and when they are inside $\mathcal{C}$ they rotate clockwise. For system (16), by similar arguments, when its orbits are outside $\mathcal{C}$ they rotate clockwise and when they are inside $\mathcal{C}$ they rotate counter-clockwise. The phase portrait of the restricted system (15) when $0 < \mu \leq 1$ is drawn in Fig. 2a and the phase portrait of the restricted system (16) when $0 < \mu \leq 1$ is drawn in Fig. 2b. Note that the circle $\mathcal{C}$ is indicated in both cases.

Now suppose that $\mu > 1$. Then p_0 and p_1 are equilibrium points of system (15) and p_0 and p_2 are equilibrium points of system (16). The eigenvalues of the linear part of system (15) at p_1 are

$$\lambda_{1,2} = -\frac{1}{2} \pm \frac{1}{2}\sqrt{5-4\mu}\,.$$

Note that $0 \leq 5-4\mu < 1$ if $1 < \mu \leq 5/4$ and $5-4\mu < 0$ if $\mu > 5/4$. Hence the equilibrium point p_1 of system (15) is either a stable node if $1 < \mu \leq 5/4$ or a stable focus if $\mu > 5/4$. Analogously we can check that the equilibrium point p_2 of system (16) also is either a stable node if $1 < \mu \leq 5/4$ or a stable focus if $\mu > 5/4$.

In order to study the dynamics of system (15) near the equilibrium point p_0 we use a polar blow-up (for a general reference on this theory see Chapter 3 of Dumortier et al. 2006). System (15) in polar coordinates is given by (17). The zeros of system (17) on $\{\rho = 0\}$ are located at $\theta = \pm\pi/2$ and $\theta = 3\pi/2$. It is easy to verify that these equilibrium points are saddles. Figure 3 illustrates this process.

By a similar study we can conclude that the dynamics of system (16) near of the origin is topologically equivalent to the dynamics of system (15) near the origin. Indeed in Fig. 4 we can see the phase portraits of the restricted systems (15) and (16) in a small neighborhood of the origin. The phase portraits of system (15) when $1 < \mu \leq 5/4$ and when $\mu > 5/4$ are drawn in Fig. 5 and the phase portrait of system (16) when $1 < \mu \leq 5/4$ and when $\mu > 5/4$ are drawn in Fig. 6.

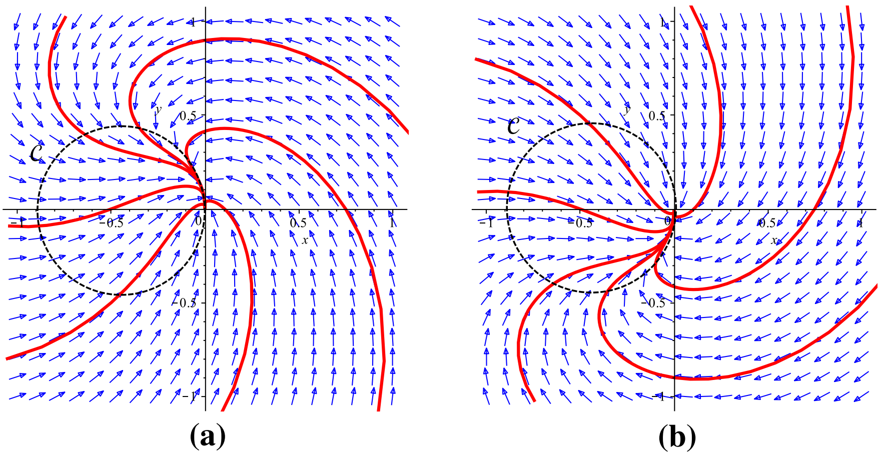


Fig. 2 **a** Phase portrait of the restricted system (15) when $0 < \mu \leq 1$. **b** Phase portrait of the restricted system (16) when $0 < \mu \leq 1$

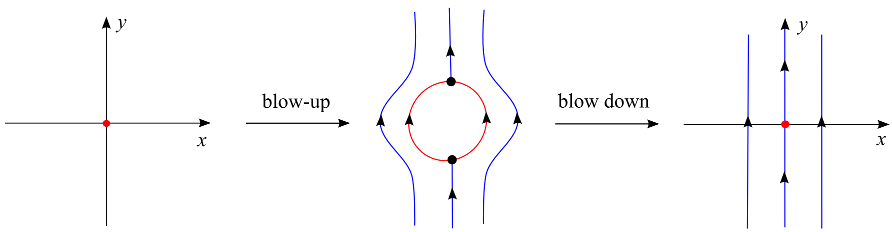


Fig. 3 Polar blow-up of the equilibrium point p_0 of system (15)

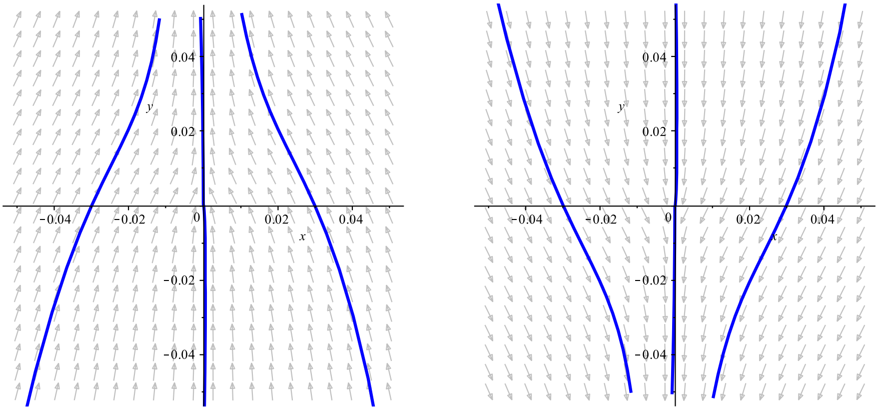


Fig. 4 Phase portrait of systems (15) and (16), respectively, in a small neighborhood of the origin (i.e. equilibrium point p_0)

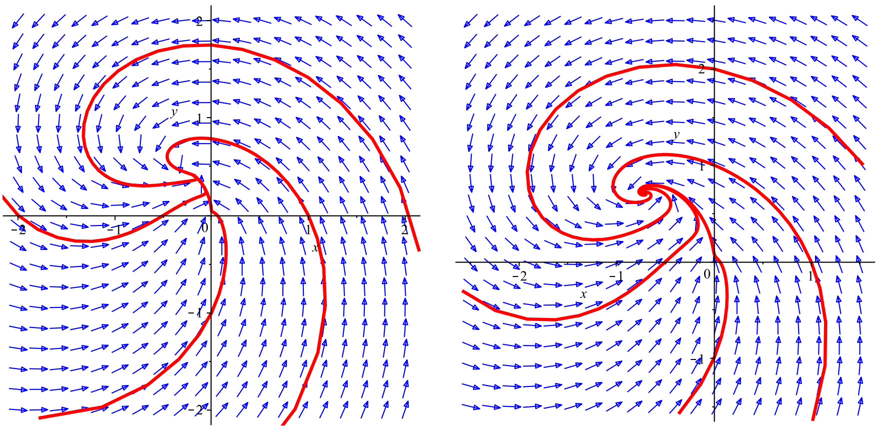


Fig. 5 Phase portrait of system (15) when $1 < \mu \leq 5/4$ (p_1 is a stable node) and when $\mu > 5/4$ (p_1 is a stable focus), respectively

The phase portrait of system (14) on the invariant cone V is obtained by gluing the phase portraits of systems (15) (for $z \geq 0$) and (16) (for $z \leq 0$). In Fig. 7 is drawn the dynamics of system (14) on the invariant cone V when $0 < \mu \leq 1$ and when $\mu > 1$.

Observe that a degenerate Pitchfork bifurcation occurs on the invariant cone V at p_0 when $\mu = 1$, with the emergence of two equilibrium points p_1 and p_2, and the singular set of the invariant cone V is $S = \{0\}$. Then, by Proposition 14 the flow of the vector field X associated to the Vallis system determines a Whitney stratification of V which is given by $\{V \backslash \{p_0\}, \{p_0\}\}$ if $\mu < 1$ and by $\{V \backslash \{p_0, p_1, p_2\}, \{p_0\}, \{p_1\}, \{p_2\}\}$ if $\mu \geq 1$. In this case, the bifurcation that occurs on the invariant cone does not change the property that the flow of X determines a Whitney stratification of the invariant algebraic surface V.

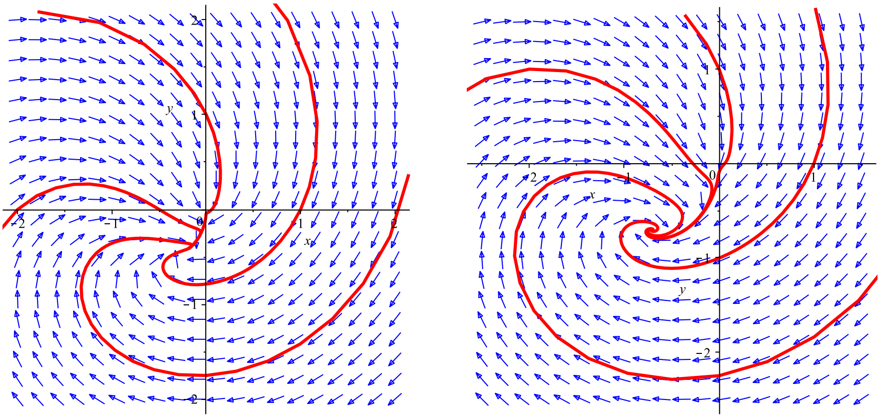


Fig. 6 Phase portrait of system (16) when $1 < \mu \leq 5/4$ (p_2 is a stable node) and when $\mu > 5/4$ (p_2 is a stable focus), respectively

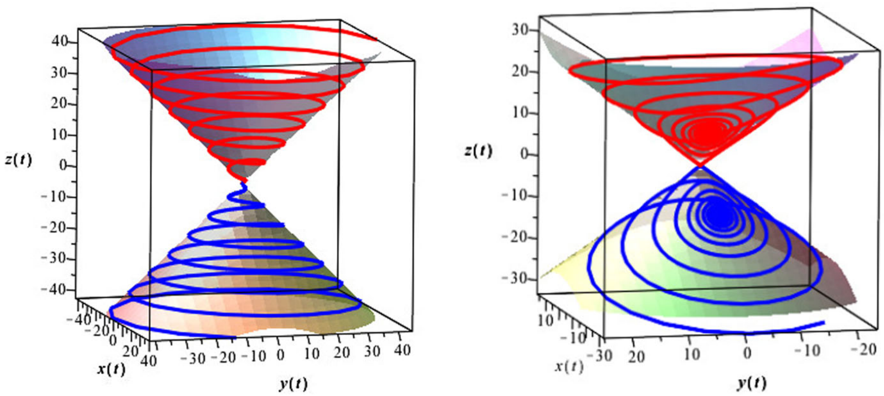


Fig. 7 Dynamics of system (14) on the invariant cone V when $0 < \mu \leq 1$ and when $\mu > 1$, respectively

Finally note that system (14) has a Darboux invariant constructed uniquely using the invariant cone V. Indeed we can write system (14) as system (4), because the cone V is an invariant algebraic surface of system (14) with cofactor $K = -2$ and it is a surface defined by a weighted homogeneous polynomial of type (1, 1, 1:2). Then $\omega_1 = \omega_2 = \omega_3 = 1$ and $a = \alpha = 2$ and we must consider $A = B = C = E = 0$, $D = z/2$, $F = \sqrt{\mu}/2$ and $G = -1$. Therefore the Darboux invariant of system (14) is given by $I(x, y, z, t) = (x^2 + y^2 - z^2)^{1/2} e^t$.

In Figs. 8, 9 and 10 we can observe the behavior of the orbits of the Vallis system near the invariant cone V when $0 < \mu \leq 1$ and when $\mu > 1$. Note that in both cases the orbits inside and outside of the invariant cone V tends to it when the time tends to infinity. It illustrates what is stated in Proposition 5 of Llibre and Oliveira (2015), since the Vallis system, in these cases, has a Darboux invariant constructed using the invariant cone V.

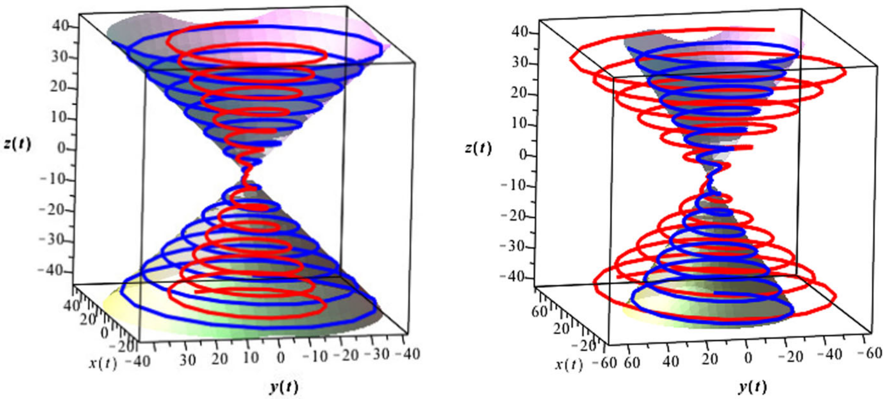
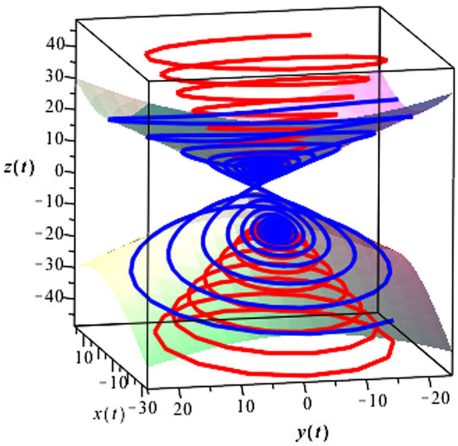


Fig. 8 Phase portrait of system (14) when $0 < \mu \leq 1$. *Blue orbits* are on the invariant cone V and *red orbits* are inside the invariant cone V and outside the invariant cone V, respectively (color figure online)

Fig. 9 Phase portrait of system (14) when $\mu > 1$. *Blue orbits* are on the invariant cone V and *red orbits* are inside the invariant cone V (color figure online)



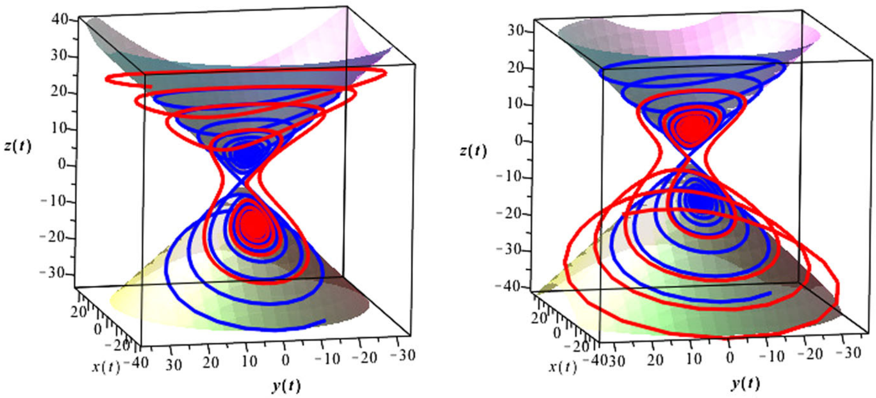


Fig. 10 Phase portrait of system (14) when $\mu > 1$. *Blue orbits* are on the invariant cone V and *red orbits* are outside the invariant cone V (color figure online)

Acknowledgements The first author was supported by a scholarship of PNPD/CAPES, developed in the Graduate Program of Applied and Computational Mathematics at the School of Science and Technology, Campus of UNESP at Presidente Prudente, SP, Brazil, and also partially supported by CAPES-Program PVE Grant number 88881.068165/2014-01. The second author was supported by FAPESP Grant number 2013/24541-0, by CNPq Grant number 308159/2015-2, and by CAPES-Program CSF-PVE Grant number 88881.030454/2013. The third author was supported by FAPESP Grant number 2013/26602-7. The authors are grateful to N. Grulha and D. Trotman for their careful reading and suggestions about the stratification part of the paper. The authors also thank the anonymous referee for his valuable comments and suggestions, which help them to improve the results and their presentation in this paper.

References

Brasselet, J.-P., Lê, D.T., Seade, J.: Euler obstruction and indices of vector fields. Topology **6**, 1193–1208 (2000)

Brasselet, J.-P., Seade, J., Suwa, T.: Vector Fields on Singular Varieties. Lecture Notes in Mathematics. Springer-Verlag, Berlin (2009)

Dumortier, F., Llibre, J., Artés, J.C.: Qualitative Theory of Planar Differential Systems. Springer-Verlag, New York (2006)

Euzébio, R., Llibre, J.: Periodic solutions of El niño model through the Vallis differential system. Discrete Contin. Dyn. Syst. **34**, 3455–3469 (2014)

Gibson, C.G., Wirthmuller, K., du Plessis A.A., Looijenga E.J.N.: Topological Stability of Smooth Mappings. Lecture Notes in Mathematics 552 (1976)

Grulha Jr., N.G.: Stability of the Euler obstruction of a function. Bol. Soc. Mat. Mexicana **17**, 95–103 (2011)

Guckenheimer, J., Holmes, P.: Nonlinear Oscillations, Dynamical Systems and Bifurcations of Vector Fields. Springer-Verlag, New York (2002)

Jouanolou, J.P.: Équations de Pfaff Algébriques. Lecture Notes in Mathematics 708 (1979)

Kanatnikov, A., Krishchenko, A.: Localization of invariant compact sets of nonautonomous systems. Differ. Equ. **45**, 46–52 (2009)

Krishchenko, A.P., Starkov, K.E.: Localization of compact invariant sets of nonlinear time-varying systems. Int. J. Bifurcation Chaos **18**, 1599–1604 (2008)

Llibre, J.: Integrability of Polynomial Differential Systems. Handbook of differential equations, pp. 437–532. Elsevier/North-Holland, Amsterdam (2004)

Llibre, J., Messias, M.: Global dynamics of the Rikitake system. Phys. D **238**, 241–252 (2009)

Llibre, J., Messias, M., Reinol, A.C.: Darboux invariants for planar polynomial differential systems having an invariant conic. Z. Angew. Math. Phys. **65**, 1127–1136 (2014)

Llibre, J., Messias, M., Reinol, A.C.: Normal forms for polynomial differential systems in $\mathbb{R}^3$ having an invariant quadric and a Darboux invariant. Int. J. Bifurcation Chaos **25**, 1550015 (2015)

Llibre, J., Messias, M., Silva, P.R.: Global dynamics in the Poincaré ball of the Chen system having invariant algebraic surfaces. Int. J. Bifurcation Chaos **22**, 1–17 (2012)

Llibre, J., Messias M., da Silva P.R.: Global dynamics of stationary solutions of the extended Fisher–Kolmogorov equation. J. Math. Phys. **52**, 112701, 12 (2011)

Llibre, J., Messias, M., da Silva, P.R.: Global dynamics of the Lorenz system with invariant algebraic surfaces. Int. J. Bifurcation Chaos **20**, 3137–3155 (2010)

Llibre, J., Messias M, da Silva P.R.: On the global dynamics of the Rabinovich system. J. Phys. A Math. Theor. **41**, 275210, 21 (2008)

Llibre, J., Oliveira, R.D.S.: Quadratic systems with invariant straight lines of total multiplicity two having Darboux invariants. Commun. Contemp. Math. **17**, 1450018 (2015)

Llibre, J., Ramirez, R., Sadovskaia, N.: Inverse approach in ordinary differential equations: applications to Lagrangian and Hamiltonian mechanics. J. Dyn. Differ. Equ. **26**, 529–581 (2014)

Llibre, J., Valls, C.: Polynomial, rational and analytic first integrals for a family of 3-dimensional Lotka–Volterra systems. Z. Angew. Math. Phys. **62**, 761–777 (2011)

Llibre, J., Zhang, X.: Darboux theory of integrability in $\mathbb{C}^n$ taking into account the multiplicity. J. Diff. Eqn. **246**, 541–551 (2009)

Llibre, J., Zhang, X.: Rational first integrals in the Darboux theory of integrability in $\mathbb{C}^n$. Bull. Sci. Math. **134**, 189–195 (2010)

Llibre, J., Zhang, X.: On the Darboux integrability of the polynomial differential systems. Qualit. Th. Dyn. Syst. **11**, 129–144 (2012)
Lorenz, E.N.: Deterministic nonperiodic flow. J. Atmos. Sci. **20**, 130–141 (1963)
Messias, M., Reinol, A.C.: Integrability and dynamics of quadratic three-dimensional differential systems having an invariant paraboloid. Int. J. Bifurcation Chaos. 26, 1650134 (2016)
Milnor, J., Orlik, P.: Isolated singularities defined by weighted homogeneous polynomials. Topology **9**, 385–393 (1970)
Miranda, A.J., Rizzioli, E.C., Saia, M.J.: Stable singularities of co-rank one quasi homogeneous map germs from $(C^{n+1}; 0)$ to $(C^n; 0)$, $n = 2$; 3. JP J. Geom. Topol. **13**, 303–337 (2013)
Nuño-Ballesteros, J.J., Oréfice, B., Tomazella, J.N.: The Bruce–Roberts number of a function on a weighted homogeneous hypersurface. Q. J. Math. 269–280 (2011)
Seade J.: The index of a vector field on a complex surface with singularities. In: Verjovsky, A. (ed.) The Lefschetz Centennial Conf., Contemp. Math. **58**, Part III, Amer. Math. Soc., 225–232 (1987)
Strogatz, S.H.: Nonlinear Dynamics and Chaos: with Applications to Physics. Chemistry and Engineering. Westview Press, New York, Biology (2001)
Strozzi, D.: On the Origin of Interannual and Irregular Behaviour in the El Niño Properties. Report of Department of Physics, Princeton University, available at the WEB (1999)
Trotman, D.: Publications Mathématiques de L'Université Paris VII: Séminaire sur la géométrie algébrique réelle. Paris VII, Tome I (1986)
Thom, R.: Ensembles et morphisms stratifiés. Bull. Am. Math. Soc. **75**, 240–284 (1969)
Vallis, G.K.: Conceptual models of El Niño and the southern oscillation. Geophys. Res. **93**, 13979–13991 (1988)
Whitney, H.: Tangents to an analytic variety. Ann. Math. **81**(3), 496–549 (1965)
Wiggins, S.: Global Bifurcation and Chaos. Springer-Verlag, New York (1988)
Wiggins, S.: Introduction to Applied Nonlinear Dynamical Systems and Chaos. Springer-Verlag, New York (2003)