Gabriel Alexis Rondón Vielma

Sistemas holomorfos por partes e regularização de campos de Filippov no entorno de singularidades degeneradas e de policiclos tangenciais regulares

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Supervisor: Prof. Dr. Paulo Ricardo da Silva

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Gabriel Alexis Rondón Vielma
Bolsista

Prof. Dr. Paulo Ricardo da Silva
Supervisor

Sumário

1 Resumo do Projeto

O presente projeto de pesquisa pretende desenvolver resultados no contexto da teoria de Filippov para compreender o comportamento das órbitas do sistema regularizado no entorno de uma singularidade degenerada tangencial e de policiclos tangenciais regulares. Além disso, estudaremos as singularidades típicas de sistemas lento-rápidos que surgem de regularizações (lineares ou não lineares). Vamos a desenvolver um algoritmo para construir funções de transição adequadas e aplicaremos essas ideias para criar singularidades lento-rápidas a partir de formas normais de campos vetoriais suaves por partes (PSVF). Apresentaremos exemplos de funções de transição que, após regularização de uma forma normal de PSVF, geram singularidades normalmente hiperbólicas, dobras, transcríticas e pitchfork. Também, estudaremos a natureza dos pontos criticos das funções de transição quando removemos a condição de monotonicidade.

Vamos estudar sistemas suaves por partes do tipo $f^+ = (u^+, v^+)$ e $f^- = (u^-, v^-)$ onde u^+, v^+ e u^-, v^- são funções contínuas, possuem derivadas parciais de primeira ordem e satisfazem as equações de Cauchy-Riemann no plano. Tais condições, conforme o Teorema de Looman–Menchoff são suficientes para garantir a analiticidade de $f^+ = u^+ + iv^+$ e $f^- = u^- + iv^-$.

Sistemas do tipo $\dot{z} = f(z)$ são conhecidos na literatura como sistemas holomorfos. Formas normais de tais sistemas foram estudadas por Gasull e seus colaboradores em "Garijo, A., Gasull, A. and Jarque, X. Normal forms for singularities of one dimensional holomorphic vector fields. *Electronic Journal of Differential Equations* **122** (2004), 1–7."Uma das propriedades mais interessantes dos sistemas holomorfos é a não existência de ciclos limites. No entanto, em [6] o bolsista junto com seus colaboradores provaram que os sistemas holomorfos por partes possuem ciclos limite e foi demonstrado que os sistemas holomorfos lineares por partes cujos pontos de equilíbrio estão na variedade de descontinuidade Σ têm no máximo um ciclo limite, mas em geral a cota maximal de ciclos limite de um sistema holomorfo por partes ainda encontra-se em aberto.

Finalmente, vamos investigar os sistemas dinâmicos complexos

$$\varepsilon \dot{z} = f(z, w) \quad \dot{w} = g(z, w),$$

onde $f, g : D \subset \mathbb{C}^2 \to \mathbb{C}$ são funções holomorfas das variáveis complexas $(z, w) \in D$. Estamos interessados em estudar a dinâmica desta classe de sistemas para quando ε é um parâmetro positivo suficientemente pequeno. A ideia é obter resultados semelhantes à Teoria de Fenichel para estes sistemas. O estudo de sistemas lento-rápidos em $\mathbb{C}^2$ pode ser útil no processo de regularização de campos descontínuos no $\mathbb{R}^4$ com superfície de descontinuidade com codimensão 2. Para mais detalhes ver [15].

1.1 Detalhamento

1.2 Regularização de sistemas de Filippov e singularidades

Considere o campo vetorial suave por partes (PSVF)

$$Z(p) = (f, g)_\Sigma = \begin{cases} f(p), & \text{se} \quad h(p) > 0, \\ g(p), & \text{se} \quad h(p) < 0, \end{cases} \qquad \text{para} \quad p \in D. \tag{1}$$

onde $h : D \to \mathbb{R}$, é função tendo 0 como um valor regular, de modo que $\Sigma = h^{-1}(0)$. As trajetórias sobre Σ são definidas seguindo a convenção de Filippov (ver [3]).

Grosso modo, um processo de regularização de um campo vetorial suave por partes Z consiste em obter uma família de um parâmetro de campos vetoriais contínuos Z_ε convergindo para Z quando $\varepsilon \to 0$. Definimos a função de regularização como segue.

Definição 1.1. A função de regularização $\phi : \mathbb{R} \to [-1, 1]$ é uma função suave C^n a qual é estritamente crescente $\phi'(s) > 0$ para todo s tal que $\phi(s) \in (-1, 1)$ e $\phi(s) \to \pm 1$ para $s \to \pm\infty$.

O conjunto de funções da Definição 1.1 inclui funções de transição analíticas como por exemplo $\tanh(x)$ e $\frac{2}{\pi}\arctan(x)$. No entanto, esta classe de funções de transição também inclui as funções de transição não analíticas de Sotomayor-Teixeira, as quais foram introduzidas em [21]. Para mais detalhes, ver [1, 2, 8, 9, 10, 11, 12].

Agora, definimos a regularização para campos vetoriais suaves por partes.

Definição 1.2. A regularização linear de $Z = (f, g)$ dado em (1) é o campo vetorial suave dado por

$$Z_\varepsilon^\phi(p) = \frac{1 + \phi\,(h(p)/\varepsilon)}{2} f(p) + \frac{1 - \phi\,(h(p)/\varepsilon)}{2} g(p), \tag{2}$$

para $0 < \varepsilon << 1$, onde ϕ satisfaz a Definição 1.1 mas não é necessariamente monotônica.

Em [16, 20] os autores consideraram outra forma de generalizar as noções de região deslizante e campo vetorial deslizante por meio de *regularizações não lineares*.

Definição 1.3. Uma regularização Z_ε entre f e g é chamada não linear se existir uma família de 1 parâmetro de campos vetoriais suaves $\widetilde{Z}(\lambda, .)$, com $\lambda \in [-1, 1]$, tal que $\widetilde{Z}(-1, p) = g(p)$, $\widetilde{Z}(1, p) = f(p)$ e $Z_\varepsilon(p) \in \{\widetilde{Z}(\lambda, p), \lambda \in [-1, 1]\}$, $\forall p \in U$. Mais ainda, uma $\varphi-$regularização não linear de f e g é a família de 1 parâmetro dada por $Z_\varepsilon(p) = \widetilde{Z}(\varphi(\frac{F}{\varepsilon}), p)$, onde φ é uma função de transição monotônica.

Um sistema da forma

$$\varepsilon\dot{x} = f(x, y, \varepsilon);\ \dot{y} = g(x, y, \varepsilon); \tag{3}$$

é chamado *sistema lento-rápido*, onde $x \in \mathbb{R}$, $y \in \mathbb{R}$, $0 < \varepsilon << 1$ e f, g são suficientemente suaves.

A seguir, lembramos brevemente algumas formas normais de sistemas lento-rápidos. Uma visão geral sobre este assunto pode ser encontrada no Capítulo 4 de [14]. As formas normais de singularidades SF-transcríticas genéricas planas e singularidades SF-pitchfork genéricas planas foram dadas em [13].

Dizemos que a variedade crítica $C_0 = \{(x, y) : f(x, y, 0) = 0\}$ tem uma *SF-dobra genérica plana* na origem se $f_x(0, 0, 0) = 0, f_{xx}(0, 0, 0) \neq 0, f_y(0, 0, 0) \neq 0$ e $g(0, 0, 0) \neq 0$.

Para obter uma *SF-singularidade transcrítica genérica* na origem, o sistema planar lento-rápido (3) deve satisfazer as seguintes condições: $f(0, 0, 0) = f_x(0, 0, 0) = f_y(0, 0, 0) = 0$, $\det \mathrm{Hes}(f) < 0, f_{xx}(0, 0, 0) \neq 0 \neq g(0, 0, 0)$.

Por outro lado, para obter uma *SF-singularidade pitchfork genérica* na origem, devemos exigir as seguintes condições: $f(0, 0, 0) = f_x(0, 0, 0) = f_{xx}(0, 0, 0) = f_y(0, 0, 0) = 0, f_{xxx}(0, 0, 0) \neq 0, f_{xy}(0, 0, 0) \neq 0, g(0, 0, 0) \neq 0$.

Proposta 1

- Estudar as singularidades típicas de sistemas lento-rápidos que surgem de regularizações (lineares ou não lineares). ¿Quais singularidades típicas dos sistemas lento-rápidos podem ser geradas por regularizações lineares e não lineares? ¿Quais condições devem satisfazer as funções de transição e os campos vetoriais suaves por partes para gerar ditas singularidades?

- Desenvolver um algoritmo para construir funções de transição adequadas e aplicar essas ideias para criar singularidades lento-rápidas a partir de formas normais de campos vetoriais suaves por partes.

- Apresentar exemplos de funções de transição que, após regularização de uma forma normal de PSVF, geram singularidades normalmente hiperbólicas, dobras, transcríticas e pitchfork.

- Estudar a natureza dos pontos criticos das funções de transição quando removemos a condição de monotonicidade. ¿Estes pontos criticos são hiperbólicos?

Além disso, embora já tenhamos publicado artigos em relação as propostas 1 e 2 do projeto original, vamos continuar trabalhando no aprofundamento dos resultados já obtidos em [6, 17].

1.3 Sistemas Holomorfos por Partes

Seja $\mathcal{V} \subseteq \mathbb{C}$ um domínio. Uma função holomorfa $f : \mathcal{V} \to \mathbb{C}$ é uma aplicação que satisfaz

- $u = \mathrm{Re}(f)$ e $v = \mathrm{Im}(f)$ são contínuas;
- as derivadas parciais u_x, u_y, v_x, v_y existem em $\mathcal{V}$ e
- as derivadas parciais satisfazem as equações de Cauchy–Riemann

$$u_x = v_y, \quad u_y = -v_x, \quad \forall z = x + iy \in \mathcal{V}.$$

De acordo com o Teorema de Looman–Menchoff as condições acima são suficientes para garantir a analiticidade de f.

Os retratos de fase locais de sistemas holomorfos podem ser classificados amenos de conjugação conforme.

Sejam f e g funções holomorfas definidas em $0 \in \mathbb{C}$. Dizemos que f e g são 0–*conformemente conjugadas* se existem $R > 0$ e uma aplicação conforme $\phi : D(0,R) \to D(0,R)$ tal que $\phi(0) = 0$ e $\phi(\varphi_f(t,z)) = \varphi_g(t, \phi(z))$, para $z \in D(0,R)$ e t tais que as expressões acima estão bem definidas em $D(0,R)$.

Proposição 1.1. *Seja f uma função holomorfa definida numa vizinhança de $z_0 \in \mathbb{C}$.*

(a) Se $f(z_0) \neq 0$ então f e $g(z) \equiv 1$ são $z_0 0$–conformemente conjugadas.

(b) Se $f(z_0) = 0$ e $f'(z_0) \neq 0$ então f e $g(z) \equiv f'(z_0)z$ são $z_0 0$–conformemente conjugadas.

(c) Se $f(z_0) = 0$, z_0 é um zero de F de ordem $n > 1$ e $\mathrm{Res}(1/F, z_0) = 1/\gamma$ então f e $g(z) \equiv \gamma z^n/(1 + z^{n-1})$ são $z_0 0$–conformemente conjugadas.

(d) Se $f(z_0) = 0$, z_0 é um zero de f de ordem $n > 1$ e $\mathrm{Res}(1/f, z_0) = 0$ então f e $g(z) \equiv \gamma z^n$ são $z_0 0$–conformemente conjugadas.

(e) Se z_0 é um polo de f da ordem n então f e $g(z) \equiv \frac{1}{z^n}$ são $z_0 0$–conformemente conjugados.

Em [6], o bolsista juntamente com o supervisor e o Dr. Luiz Gouveia, deram várias famílias de sistemas holomorfos por partes que apresentam no máximo um ciclo limite, como pode ser visto no teorema abaixo.

Teorema 1.4. *Os sistemas holomorfos lineares por partes cujos pontos de equilíbrio estão na variedade Σ têm no máximo um ciclo limite.*

Ressaltamos que este fato é interessante, já que os sistemas holomorfos não possuem ciclos limite. Nossa proposta então é aprofundar o estudo sobre ciclos limite de sistemas holomorfos por partes. Nossos objetivos incluem estudar se sistemas holomorfos por partes podem apresentar mais de um ciclo limite. Mais ainda, queremos responder também se é possível determinar a cota maximal de ciclos limite de um sistema holomorfo por partes. Os objetivos da proposta são:

Proposta 2:

- Estudar se os sistemas holomorfos por partes podem apresentar mais de um ciclo limite. Neste caso, apresentar alguns exemplos.
- Determinar se as cotas de ciclos limite encontradas são maximais.

- Encontrar as condições para a existência de ciclos limites do PWHS que são formados por transformações Moebius $T(z) = \frac{Az+B}{Cz+D}$, com $AD - BC \neq 0$ e as formas normais dadas na Proposição 1.1.
- Investigar se os conjuntos invariantes de sistemas holomorfos por partes são preservados por conjugação conforme. Em particular, verificar se os ciclos limite de sistemas holomorfos por partes são preservados por conjugação conforme.

Por outro lado, a partir de reuniões com o supervisor do bolsista, surgiram novos problemas interessantes nesta linha de pesquisa a partir dos resultados já obtidos em [5, 6], os quais detalharemos logo abaixo.

Considere o sistema dinâmico complexo

$$\varepsilon \dot{z} = f(z, w) = u_1 + iv_1, \quad \dot{w} = g(z, w) = u_2 + iv_2, \tag{4}$$

onde $f, g : D \to \mathbb{C}$ são funções holomorfas das variáveis complexas $(z, w) \in D$ e $D \subset \mathbb{C}^2$ é um domínio simplesmente conexo. Observe que $C_0 = \{(z, w) \in D : f(z, w) = 0\}$ é a variedade crítica unidimensional do sistema (4). Seja $q_0 = (z_0, w_0) \in C_0$ um ponto de equilíbrio do sistema (4) para $\varepsilon = 0$. Como f, g são funções holomorfas, então as equações de Cauchy-Riemann são válidas. Assim, a matriz Jacobiana do sistema (4) em q_0 implica que os autovalores de $J(q_0)$ são $\lambda_\pm = (u_1)_{x_1} \pm i(v_1)_{x_1}$. Assim, podemos enunciar a seguinte definição.

Definição 1.5. Um subconjunto $S \subset C_0$ é chamado normalmente hiperbólico desde que $\frac{\partial}{\partial x_1} \Re(f(q)) = (u_1)_{x1}(q) \neq 0$ para todos $q \in S$.

Proposta 3:

Suponha que f seja uma função holomorfa linear e g seja uma forma normal dada na Proposição 1.1, assim o sistema reduzido associado ao sistema (4) é dado por

$$0 = f(z, w), \quad \dot{w} = g(z, w). \tag{5}$$

Ressaltamos que sobre o plano $f(z, w) = 0$ a dinâmica do sistema reduzido corresponde aos sistemas estudados em [5, 6]. Estamos interessados em estudar a dinâmica do sistema (4) para quando ε é um parâmetro positivo suficientemente pequeno. A ideia é obter resultados semelhantes à Teoria de Fenichel para esta classe de sistemas. O estudo de sistemas lento-rápidos em $\mathbb{C}^2$ pode ser útil no processo de regularização de campos descontínuos no $\mathbb{R}^4$ com superfície de descontinuidade com codimensão 2. Para mais detalhes ver [15].

2 Trabalhos Realizados no Período

2.1 Trabalhos realizados referentes à Proposta 1

Aqui apresentamos 3 dos trabalhos que foram realizados nesta área de estudo.

2.1.1 Artigo 1

Estudamos sistemas diferenciais suaves por partes no plano da forma

$$\dot{z} = Z(z) = \frac{1 + \operatorname{sgn}(h)}{2} X(z) + \frac{1 - \operatorname{sgn}(h)}{2} Y(z), \tag{6}$$

onde $h : \mathbb{R}^2 \to \mathbb{R}$ é uma função suave com 0 como valor regular. Consideramos regularizações lineares Z_ε^φ de Z substituindo $\operatorname{sgn}(F)$ por $\varphi(h/\varepsilon)$ na última equação, para $\varepsilon > 0$ pequeno e φ sendo uma função de transição (não necessariamente monótona). Também são consideradas regularizações não lineares do campo vetorial Z cuja função de transição é monótona. É um fato bem conhecido que o sistema regularizado é um sistema lento-rápido. Nesta área de pesquisa, estudamos as singularidades típicas dos sistemas lento-rápido que surgem das (linear ou não linear) regularizações, nomeadamente as singularidades de dobra, transcrítica e pitchfork. Além disso, a dependência do sistema lento-rápido das propriedades gráficas da função de transição também é investigada.

Os principais resultados obtidos neste trabalho são os seguintes:

Considere o campo vetorial de deslizamento de Filippov Z^{Σ} associado ao PSVF (6). Embora na literatura seja considerada apenas a dinâmica de Z^{Σ} nas regiões de deslizamento ou fuga de Filippov, o domínio $D\Big(Z^{\Sigma}\Big) \subset \Sigma$ de Z^{Σ} pode ser maior que Σ^s. Nesse sentido, para nossos propósitos, o domínio $D\Big(Z^{\Sigma}\Big)$ de Z^{Σ} é o subconjunto de Σ no qual Z^{Σ} está bem definido, e não apenas a região de deslizamento de Filippov Σ^s.

Teorema A. *Considere o PSVF* (6) *e denote seu campo vetorial de deslizamento de Filippov por* Z^{Σ}*, cujo domínio é o conjunto* $D\Big(Z^{\Sigma}\Big) \subset \Sigma$*. Considere uma regularização linear de* Z *e seja* φ *uma função de transição, não necessariamente monótona. Seja* $\Pi : \mathbb{R}^2 \to \Sigma$ *a projeção canônica e* $x_0 \in (-1, 1)$*. Então, os seguintes resultados se aplicam:*

(a) *Se* $\varphi'(x_0) = 0$*, então o conjunto de pontos* (x_0, y) *tal que*

$$f_1(x_0, y) + g_1(x_0, y) + \varphi(x_0)\Big(f_1(x_0, y) - g_1(x_0, y)\Big) = 0$$

está contido em $C_0 \backslash \mathcal{NH}\Big(C_0\Big)$*. Em outras palavras, os pontos críticos de* φ *geram pontos não normalmente hiperbólicos do conjunto crítico* C_0 *de* (2).

(b) *Suponha que* $(x_0, y_0) \in \mathcal{NH}(C_0)$ *e* $|\varphi(x_0)| > 1$*. Então* $\Pi\Big(\mathcal{NH}(C_0)\Big) \cap \Sigma^w \neq \emptyset$*. Além disso,* $\Sigma^s \subsetneq \Sigma^s_r$*. Em outras palavras, se* $|\varphi(x_0)| > 1$ *então a região de deslizamento r é maior que a região clássica de deslizamento de Filippov.*

(c) *Se* $(0, y_0) \in \Sigma^s_r$ *satisfaz* $f_1(0, y_0) \neq g_1(0, y_0)$*, perto de tal ponto a dinâmica na região de deslizamento* Σ^s_r *é dada pelo campo vetorial clássico de deslizamento de Filippov* Z^{Σ}*. Em outras palavras, mesmo que extendamos* Σ^s *para* Σ^s_r*, a dinâmica em tal conjunto é dada por* Z^{Σ}*. Em particular,* (x_0, y_0) *é um ponto de equilíbrio SF de* (2) *se, e somente se,* $(0, y_0)$ *é um ponto de equilíbrio de* Z^{Σ}*.*

(d) *Se*

$$\Pi\Big(C_0\Big) \cap \left(\Sigma \backslash D\Big(Z^{\Sigma}\Big)\right) \neq \emptyset,$$

então $(0, y_0) \in \Pi\Big(C_0\Big) \cap \left(\Sigma \backslash D\Big(Z^{\Sigma}\Big)\right)$ *é um ponto de tangência para ambos os campos vetoriais* X *e* Y *simultaneamente, e a linha* $\{y = y_0\}$ *é uma componente de* C_0*.*

Agora, estamos preocupados em estabelecer as condições que tanto o campo vetorial por partes quanto a função de transição devem satisfazer para gerar singularidades SF.

Teorema B. *Considere o PSVF* (6) *e seja* φ *uma função de transição, não necessariamente monótona. Após a regularização linear e o blow-up direcional, é possível gerar pontos normalmente hiperbólicos, singularidades SF-dobra e singularidades SF-transcríticas. No entanto, não é possível gerar singularidades SF-pitchfork.*

O Teorema B assegura que não existe uma função de transição que gere uma singularidade SF-pitchfork. Isso nos leva a considerar regularizações não lineares.

Teorema C. *Considere o PSVF* (6) *e seja* φ *uma função de transição monótona. Após a regularização não linear de* φ $\widetilde{Z}(\varphi(\frac{x}{\varepsilon}), x, y)$ *e o blow-up direcional, é possível gerar pontos normalmente hiperbólicos, singularidades SF-dobra, singularidades SF-transcríticas e singularidades SF-pitchfork.*

Este trabalho foi realizado em conjunto com o Prof. Paulo Ricardo da Silva e o Dr. Otávio Perez. Ao final deste relatório, as primeiras páginas do artigo escrito por nós serão anexadas. Você pode encontrar todos os resultados obtidos no artigo com mais detalhes no link https://link.springer.com/article/10.1007/s10883-023-09657-x.

Relembramos que os resultados obtidos nesta linha de pesquisa são interessantes, pois nos permitiram obter algumas respostas para as questões levantadas no projeto de pesquisa. Embora já tenhamos publicado artigos sobre a proposta 1, ainda estamos trabalhando no aprofundamento dos resultados já obtidos em [17, 18, 19] e acreditamos que isso nos permitirá melhorar ainda mais nossos resultados.

2.1.2 Artigo 2

Aqui, nosso objetivo é investigar o *Problema de Shilnikov em Sistemas de Filippov e a Robustez do Caos sob Suavização*. Este trabalho ainda está em desenvolvimento e está sendo realizado em colaboração com o professor Douglas Novaes da Unicamp. O trabalho teve início durante a minha visita técnica na UNICAMP no mês de junho de 2024. A ideia central deste estudo é primeiramente obter uma versão do artigo [17], que foi desenvolvido por nós, mas desta vez focando no $\mathbb{R}^3$, especificamente sobre a regularização de curvas de singularidades tangenciais do tipo dobra-regular em sistemas de Filippov no $\mathbb{R}^3$. Este estudo nos proporcionará informações locais sobre o comportamento do sistema nas proximidades dessas singularidades tangenciais, o que será fundamental para o estudo da dinâmica global do sistema.

A seguir apresentamos a teoria básica dos sistemas dinâmicos não suaves, com o objetivo de definir as órbitas de Shilnikov deslizantes e enunciar os nossos resultados principais.

Seja U um subconjunto aberto e limitado de $\mathbb{R}^3$. Denotamos por $\mathcal{C}^r(K, \mathbb{R}^3)$, com $K = \overline{U}$, o conjunto de todos os campos vetoriais $\mathcal{C}^r$ $X : K \to \mathbb{R}^3$, dotados da topologia induzida pela norma $||X||_r = \sup\{||D^i X(x)|| : x \in K, i \in \{0, 1, \ldots, r\}\}$. Aqui, D^r é o operador identidade para $r = 0$, e a r-ésima derivada para $r > 0$. Para manter a propriedade de unicidade das trajetórias dos campos vetoriais em $\mathcal{C}^0(K, \mathbb{R}^3)$, assumimos, adicionalmente, que esses campos vetoriais são Lipschitz.

Seja $h : K \to \mathbb{R}$ uma função diferenciável com 0 como valor regular. Denotamos por $\Omega_h^r(K, \mathbb{R}^3)$ o espaço dos campos vetoriais por partes definidos por

$$Z(p) = \begin{cases} X(p), & \text{se} \quad h(x) > 0, \\ Y(p), & \text{se} \quad h(x) < 0, \end{cases} \tag{7}$$

com $X, Y \in \mathcal{C}^r(K, \mathbb{R}^3)$. Como de costume, o sistema (7) é denotado por $Z = (X, Y)$ e a superficie $\Sigma = h^{-1}(0)$. Assim, tomamos $\Omega_h^r(K, \mathbb{R}^3) = \mathcal{C}^r(K, \mathbb{R}^3) \times \mathcal{C}^r(K, \mathbb{R}^3)$ dotado da topologia produto.

Os pontos em Σ onde ambos os campos vetoriais X e Y apontam simultaneamente para fora ou para dentro de Σ definem, respectivamente, as regiões *de fuga* Σ^e ou *deslizante* Σ^s, e o interior de seu complemento em Σ define a *região de cruzamento* Σ^c. O complementar da união dessas regiões é constituído pelos *pontos de tangência* entre X ou Y com Σ.

Um ponto de tangência p é chamado de *tangência visível* de X (resp. Y) se existir uma órbita de X (resp. Y), na região $h(x) > 0$ (resp. $h(x) < 0$), que atinja p em um tempo finito. Caso contrário, p é chamada de *tangência invisível* de X (resp. Y). Além disso, um ponto de tangência é chamado de *singular* se for uma tangência invisível para ambos os campos vetoriais X e Y, e é chamado de *regular* se não for singular.

Os pontos em Σ^c satisfazem $Xh(\xi) \cdot Yh(\xi) > 0$, onde Xh denota a derivada da função h na direção do vetor X, ou seja, $Xh(\xi) = \langle \nabla h(\xi), X(\xi) \rangle$. Os pontos em Σ^s (resp. Σ^e) satisfazem $Xh(\xi) < 0$ e $Yh(\xi) > 0$ (resp. $Xh(\xi) > 0$ e $Yh(\xi) < 0$). Finalmente, os pontos de tangência de X (resp. Y) satisfazem $Xh(\xi) = 0$ (resp. $Yh(\xi) = 0$).

Agora definimos o *campo vetorial deslizante*

$$\widetilde{Z}(\xi) = \frac{Yh(\xi)X(\xi) - Xh(\xi)Y(\xi)}{Yh(\xi) - Xh(\xi)}. \tag{8}$$

A trajetória local $\varphi_Z(t,p)$ do sistema diferencial descontínuo por partes $\dot{x} = Z(x)$ que passa por um ponto $p \in \mathbb{R}^3$ é dada pela convenção de Filippov (ver [7]). Aqui, $0 \in I_p \subset \mathbb{R}$ denota o intervalo máximo de definição de $\varphi_Z(t,p)$, e φ_W denota o fluxo de um campo vetorial W. A convenção de Filippov é resumida da seguinte forma:

(*i*) Para $p \in \mathbb{R}^3$ tal que $h(p) > 0$ (resp. $h(p) < 0$) e tomando a origem do tempo em p, a trajetória é definida como $\varphi_Z(t,p) = \varphi_X(t,p)$ (resp. $\varphi_Z(t,p) = \varphi_Y(t,p)$) para $t \in I_p$;

(*ii*) Para $p \in \Sigma^c$ tal que $(Xh)(p), (Yh)(p) > 0$ e tomando a origem do tempo em p, a trajetória é definida como $\varphi_Z(t,p) = \varphi_Y(t,p)$ para $t \in I_p \cap t < 0$ e $\varphi_Z(t,p) = \varphi_X(t,p)$ para $t \in I_p \cap t > 0$. Para o caso $(Xh)(p), (Yh)(p) < 0$ a definição é a mesma, invertendo o tempo;

(*iii*) Para $p \in \Sigma^s$ e tomando a origem do tempo em p, a trajetória é definida como $\varphi_Z(t,p) = \varphi_{\widetilde{Z}}(t,p)$ para $t \in I_p \cap t \geq 0$ e $\varphi_Z(t,p)$ é seja $\varphi_X(t,p)$ ou $\varphi_Y(t,p)$ ou $\varphi_{\widetilde{Z}}(t,p)$ para $t \in I_p \cap t \leq 0$. Para o caso $p \in \Sigma^e$ a definição é a mesma, invertendo o tempo;

(*iv*) Para $p \in \partial\Sigma^c \cup \partial\Sigma^s \cup \partial\Sigma^e$ tal que as definições das trajetórias para pontos em Σ em ambos os lados de p possam ser estendidas para p e coincidam, a órbita que passa por p é essa órbita limite. Chamaremos esses pontos de pontos de *tangência regular*;

(*v*) Para qualquer outro ponto (pontos de *tangência singular*) $\varphi_Z(t,p) = p$ para todo $t \in \mathbb{R}$.

Observação 2.1. Um ponto de tangência $\xi \in \Sigma$ é chamado de *dobras visível* de X (resp. Y) se $(X)^2 h(\xi) > 0$ (resp. $(Y)^2 h(\xi) < 0$). Analogamente, invertendo as desigualdades, definimos uma *dobra invisível.* Suponha que p seja uma dobra visível de X tal que $Yh(p) > 0$, então p é um exemplo de ponto de tangência regular. Nesse caso, tomando a origem do tempo em p, a trajetória que passa por p é definida como $\varphi_Z(t,p) = \varphi_1(t,p)$ para $t \in I_p \cap t \leq 0$ e $\varphi_Z(t,p) = \varphi_2(t,p)$ para $t \in I_p \cap t \geq 0$, onde cada φ_1, φ_2 é φ_X, φ_Y ou $\varphi_{\widetilde{Z}}$.

Um *pseudo-equilíbrio* é um ponto crítico $\xi \in \Sigma^{s,e}$ do campo vetorial deslizante, ou seja, $\widetilde{Z}(\xi) = 0$. Quando ξ é um ponto crítico hiperbólico de $\widetilde{Z}$, é chamado de *pseudo-equilíbrio hiperbólico.* Particularmente, se $\xi \in \Sigma^s$ (resp. $\xi \in \Sigma^e$) é um foco hiperbólico instável (resp. estável) de $\widetilde{Z}$, chamamos ξ de *pseudo-equilíbrio de sela-foco hiperbólico* ou apenas *pseudo-sela-foco hiperbólico.*

Para estudar as órbitas do campo vetorial deslizante, é conveniente definir o campo vetorial deslizante normalizado ($\mathcal{C}^r$)

$$\widehat{Z}(\xi) = (Yh(\xi) - Xh(\xi))\widetilde{Z}(\xi) = Yh(\xi)X(\xi) - Xh(\xi)Y(\xi), \tag{9}$$

que tem o mesmo retrato de fase de $\widetilde{Z}$ invertendo a direção do fluxo na região de fuga.

Definição 2.1. Seja $Z = (X,Y)$ um campo vetorial descontínuo por partes com um pseudo-sela foco hiperbólico $p \in \Sigma^s$ (resp. $p \in \Sigma^e$), e seja $q \in \partial\Sigma^s$ (resp. $q \in \partial\Sigma^e$) um ponto de dobra visível do campo vetorial X tal que

(i) a órbita que passa por q seguindo o campo vetorial deslizante $\widetilde{Z}$ converge para p para trás no tempo (resp. para frente no tempo);

(ii) a órbita que começa em q e segue o campo vetorial X passa um tempo $t_0 > 0$ (resp. $t_0 < 0$) para alcançar p.

Assim, a partir de p e q, uma órbita deslizante Γ é facilmente caracterizada. Chamamos Γ de uma *órbita de Shilnikov deslizante.*

Agora, vamos introduzir o espaço afim das funções de transição. Seja $\mathcal{T}$ o conjunto de todas as funções $\phi : [-1,1] \to [-1,1]$ que são C^1 no intervalo aberto $(-1,1)$ e que satisfazem $\phi(\pm 1) = \pm 1$. Este conjunto é composto por funções de transição monótonas.

Seja $\mathcal{T}_0$ o conjunto de funções $\phi \in \mathcal{T}$ tais que

$$\phi'(1) := \lim_{s \to 1^-} \frac{\phi(s) - 1}{s} \neq 0 \quad \text{e} \quad \phi'(-1) := \lim_{s \to -1^+} \frac{\phi(s) + 1}{s} \neq 0.$$

Considere $\mathcal{V}$ o conjunto de funções $\xi : [-1,1] \to \mathbb{R}$ que são C^1 no intervalo aberto $(-1,1)$ e satisfazem $\xi(\pm 1) = 0$ e denotemos por $\mathcal{U}$ o conjunto de funções $\xi \in \mathcal{V}$ tais que $-s-1 < \xi(s) < 1-s$ e $\xi'(s) > -1$, para todo $s \in (-1,1)$. Finalmente, seja $\mathcal{U}_0$ o conjunto de funções $\xi \in \mathcal{U}$ tais que

$$\xi'(1) := \lim_{s \to 1^-} \frac{\xi(s)}{s} \neq -1 \quad \text{e} \quad \xi'(-1) := \lim_{s \to -1^+} \frac{\xi(s)}{s} \neq -1.$$

Proposição 2.1. *Considere os conjuntos de funções definidos anteriormente e seja $\phi_1 \in \mathcal{T}_0$ dada por $\phi_1(s) = s$ para $s \in [-1,1]$. Então, as seguintes afirmações são verdadeiras:*

(a) $\mathcal{V}$ é um espaço de Banach e $\mathcal{U}_0$ é um subconjunto aberto de $\mathcal{V}$;

(b) $\mathcal{T} = \mathcal{V} + \phi_1$ (espaço afim de Banach) e $\mathcal{T}_0 = \mathcal{U}_0 + \phi_1$.

Finalmente, apresentamos o primeiro resultado principal deste tópico.

Teorema D. *Suponha que $\langle v^\perp, L\rangle \neq 0$ e que $\left|\dfrac{C}{\langle v^\perp, L\rangle}\right| < 1$, onde*

$$C = \langle \partial_z \varphi^X(t_0, q_0), (X_1(0)X_3(0), X_2(0)X_3(0), X_1(0)^2 + X_2(0)^2 \rangle.$$

Então, existe uma curva $\phi_\delta \in \mathcal{T}_0$ tal que o campo vetorial $Z_\delta(p, \phi_\delta)$ tem uma Conexão de Shilnikov.

Este resultado já foi provado e omitimos os detalhes aqui. Estamos aprofundando nesta teoria e suas aplicações, mas ainda estamos na fase inicial do estudo.

2.1.3 Artigo 3

Neste linha de pesquisa, estamos preocupados com a regularização de sistemas de Filippov em torno de conexões do tipo homoclínicas a singularidades tangenciais regulares. Damos condições para garantir a existência de ciclos limites que se bifurcam a partir de tais conexões. Condições adicionais também são fornecidas para garantir a estabilidade e unicidade de tais ciclos limites. Todas as provas são baseadas na construção do primeiro mapa de retorno do sistema de Filippov regularizado em torno de conexões do tipo homoclínicas. Tal mapa é obtido usando a caracterização do comportamento local do sistema de Filippov regularizado em torno de singularidades tangenciais regulares dada em [17]. Teoremas de ponto fixo e argumentos de Poincaré-Bendixson também são usados.

Alguns dos principais resultados obtidos neste trabalho são os seguintes:

Estamos interessados em realizar uma análise qualitativa de $\Sigma-$policiclos com várias singularidades tangenciais regulares visíveis. Considere o campo vetorial não suave $Z = (X^+, X^-)$ que admite um $\Sigma-$policiclo tendo l singularidades tangenciais regulares, $p_i \in \Sigma$, de ordem $n_i = 2k_i$, para $i = 1, \cdots, l$ satisfazendo uma e apenas uma das seguintes propriedades:

(i) para cada $i = 1, \cdots, l$, existe uma curva γ_i conectando p_i e p_{i+1}, orientada de p_i a p_{i+1}, tal que $\gamma_i \backslash \{p_i, p_{i+1}\}$ é uma órbita regular de Z, γ_i é tangente a Σ em p_i e p_{i+1}, onde $p_{l+1} = p_1$. Localmente em torno de cada p_i, Z satisfaz as mesmas condições de policiclos do tipo (a).

(ii) para cada $i = 1, \cdots, l$, existe uma curva γ_i conectando p_i e p_{i+1}, orientada de p_i a p_{i+1}, tal que $\gamma_i \backslash \{p_i, p_{i+1}\}$ é uma órbita regular de Z, γ_i é tangente a Σ em p_i e transversal a Σ em p_{i+1}, onde $p_{l+1} = p_1$. Localmente em torno de cada p_i, Z satisfaz as mesmas condições de policiclos do tipo (b).

Teorema 2.2. *Considere um sistema de Filippov $Z = (X^+, X^-)_\Sigma$ e suponha que Z tenha um policiclo Γ, que é do tipo (i). Para $n \geq \max\limits_{1 \leq i \leq l}\{n_i\} - 1$, seja Φ uma função de regularização de Sotomayor-Teixeira, K, S_l como acima, $\lambda_i^* = \frac{n}{1+n_i(n-1)}$, e considere o sistema regularizado Z_ε^Φ (2). Se $K_l + S_l - 1 \neq 0$, então as seguintes afirmações são válidas.*

(a) Dado $0 < \lambda < \min\left\{\frac{n_l \lambda_l^*}{n_1}, \lambda_l^*\right\}$, if $K_l + S_l - 1 > 0$, então existe $\rho > 0$ tal que o sistema regularizado Z_ε^Φ não admite ciclos limite passando pela seção $\widehat{H}_{\rho,\lambda}^{\varepsilon,1} = [-\rho, -\varepsilon^\lambda] \times \{\varepsilon\}$, para $\varepsilon > 0$ suficientemente pequeno.

(b) Dado $\frac{1}{n_1} < \lambda < \lambda_l^*$, se $K_l + S_l - 1 < 0$, então existe $\rho > 0$ tal que o sistema regularizado Z_ε^Φ admite um único ciclo limite Γ_ε passando pela seção $\widehat{H}_{\rho,\lambda}^{\varepsilon,1} = [-\rho, -\varepsilon^\lambda] \times \{\varepsilon\}$, para $\varepsilon > 0$ suficientemente pequeno. Além disso, Γ_ε é assintoticamente estável e ε-próximo a Γ.

Teorema 2.3. *Considere um sistema de Filippov $Z = (X^+, X^-)_\Sigma$ e suponha que Z tenha um $\Sigma-$polycycle Γ do tipo (ii). Para $n \geq \max\limits_{1 \leq i \leq l}\{n_i\} - 1$, seja Φ uma função de regularização de Sotomayor-Teixeira, $\lambda_i^* = \frac{n}{1+n_i(n-1)}$, e considere o sistema regularizado Z_ε^Φ (2). Então Z_ε^Φ admite pelo menos um ciclo limite Γ_ε, para $\varepsilon > 0$ suficientemente pequeno. Além disso, Γ_ε converge para Γ.*

Este trabalho foi realizado juntamente com o Dr. Douglas Novaes. No final deste relatório, estará anexado o artigo feito por nós, onde é possível encontrar todos os resultados obtidos detalhadamente. Além disso, o paper pode ser encontrado no link:

https://www.sciencedirect.com/science/article/pii/S0167278922002305?via%3Dihub.

2.2 Trabalhos realizados referentes à Proposta 2

Aqui apresentamos 4 dos trabalhos que foram realizados nesta área de estudo.

2.2.1 Artigo 1

Neste linha de pesquisa, estamos preocupados em determinar cotas inferiores do número de ciclos limites para sistemas holomorfos polinomiais por partes dados por

$$\begin{cases} \dot{z} = F^+(z), \text{ when } \operatorname{Im}(z) > 0, \\ \dot{z} = F^-(z), \text{ when } \operatorname{Im}(z) < 0, \end{cases} \tag{10}$$

onde $z = x + iy$ e $F^\pm(z)$ são funções polinomiais holomorfas com $\deg(F^\pm) = n^\pm$. Abordamos este problema com diferentes pontos de vista: estudo do número de zeros das funções averaging de primeira e segunda ordem, ou com o controle dos ciclos limites que aparecem a partir de um ponto de equilíbrio monodrômico através de uma bifurcação degenerada do tipo Andronov-Hopf, adicionando no final, os efeitos deslizantes. Também utilizamos o teorema de Poincaré-Miranda para obter sistemas holomorfos lineares por partes explícitos com 3 ciclos limites, resultado que melhora os exemplos conhecidos na literatura que tinham um único ciclo limite.

Vamos expor nossos principais resultados. Nosso primeiro teorema trata da análise dos ciclos limites obtidos pelo cálculo da função averaging de segunda ordem de uma perturbação do centro global $\dot{z} = iz$ dentro do sistema (10). Como de costume, no próximo resultado, $[.]$ denota a parte inteira de um número real, ou seja, dado $n \leq x < n+1$ então a parte inteira de x é $[x] = n$.

Teorema E. *Seja $\mathcal{L}_{n^+,n^-} \in \mathbb{N} \cup \{\infty\}$ o número máximo de ciclos limite que* (10) *pode ter. Considere o sistema* (10) *com $F^{\pm}(z) = iz + \epsilon h^{\pm}(z)$ e $\deg(h^{\pm}) = n^{\pm}$. Então, usando a teoria de averaging até a ordem dois, para ϵ suficiente pequeno, o número máximo de ciclos limites que se bifurcam das órbitas periódicas do centro é $[(3\max\{n^+, n^-\} - 1))/2]$, esta cota superior é alcançada e, neste caso, todos esses ciclos limites são hiperbólicos. Como consequência, $\mathcal{L}_{n^+,n^-} \geq [(3\max\{n^+, n^-\} - 1))/2]$.*

Nosso segundo resultado estuda com mais detalhes os casos $n^+ + n^- \leq 4$ e trata dos ciclos limites obtidos de uma bifurcação Andronov-Hopf degenerada juntamente com um ciclo limite extra obtido pela adição de deslizamento. É claro que este resultado é baseado no cálculo das quantidades de Lyapunov associadas ao caso foco fraco-foco fraco, que são desenvolvidas para os sistemas holomorfos por partes. Chegamos até a 5^a constante, veja o resultado a seguir.

Teorema F. *Considere o sistema* (10) *onde $F^{\pm}(z) = \sum_{k=1}^{\infty} F_k^{\pm}(z)$. Suponha que $F_1^{\pm}(z) = (i + \lambda^{\pm})z$ e escreva $F_2^{\pm}(z) = A^{\pm}z^2$, $F_3^{\pm}(z) = B^{\pm}z^3$, $F_4^{\pm}(z) = C^{\pm}z^4$, e $F_5^{\pm}(z) = D^{\pm}z^5$. Então suas primeiras cinco quantidades de Lyapunov são:*

(i) $V_1 = e^{(\lambda^+ + \lambda^-)\pi} - 1$;

(ii) $V_2 = \omega_2^+(\pi) + \omega_2^-(\pi)e^{3\lambda^+\pi}$;

(iii) $V_3 = e^{\lambda^+\pi}\omega_3^+(\pi) - 2(\omega_2^+(\pi))^2 + \omega_3^-(\pi)e^{5\lambda^+\pi}$;

(iv) $V_4 = e^{2\lambda^+\pi}\omega_4^+(\pi) - 5e^{\lambda^+\pi}\omega_2^+(\pi)\omega_3^+(\pi) + 5(\omega_2^+(\pi))^3 + \omega_4^-(\pi)e^{7\lambda^+\pi}$;

(v)
$$\begin{aligned} V_5 =& e^{3\lambda^+\pi}\omega_5^+(\pi) + 21e^{\lambda^+\pi}(\omega_2^+(\pi))^2\omega_3^+(\pi) - 14(\omega_2^+(\pi))^4 \\ & -3e^{2\lambda^+\pi}(\omega_3^+(\pi))^2 - 6e^{2\lambda^+\pi}\omega_2^+(\pi)\omega_4^+(\pi) + \omega_5^-(\pi)e^{9\lambda^+\pi}. \end{aligned}$$

onde $\omega_i^{\pm}(\pi)$ for $i = 1, \cdots, 5$ são certas constantes. Além disso, a estabilidade de origem do sistema (10) *é determinada pelo sinal da primeira quantidade de Lyapunov diferente de zero.*

Teorema G. *Seja $\mathcal{L}^0_{n^+,n^-} \leq \mathcal{L}_{n^+,n^-}$ o número de ciclos limites hiperbólicos que se bifurcam a partir da origem do sistema* (10) *Na Tabela 1 damos algumas cotas inferiores de $\mathcal{L}^0_{n^+,n^-}$ para $n^+ + n^- \leq 4$,*

n^+/n^-	1	2	3
1	1	3	5
2	3	4	-
3	5	-	-

Tabela 1: Limites inferiores para $\mathcal{L}^0_{n^+,n^-}$.

Destacamos que no caso linear $n^+ = n^- = 1$, Teoremas E e G apenas mostram que $\mathcal{L}_{1,1} \geq 1$, esse era um resultado conhecido ([6]). Nosso teorema final melhora esse cota inferior.

Teorema H. *Existem sistemas holomorfos lineares por partes com pelo menos 3 ciclos limites, ou seja,* $\mathcal{L}_{1,1} \geq 3$.

Este trabalho foi realizado em conjunto com os professores Armengol Gassul e Paulo Ricardo da Silva. No final deste relatório, estará anexado o artigo feito por nós, onde é possível encontrar todos os resultados obtidos detalhadamente. Mais ainda, este trabalho pode ser encontrado na revista *SIAM Journal On Applied Dynamical Systems* em https://doi.org/10.1137/23M1620922.

Embora já tenhamos feito alguns artigos com relação à proposta 2, ainda estamos trabalhando no aprofundamento dos resultados já obtidos nesta linha de pesquisa. Mais ainda, nesta proposta surgiram novos problemas interessantes a partir dos resultados já obtidos, nos quais estamos trabalhando.

2.2.2 Artigo 2

Seja $\dot{z} = f(z)$ uma equação diferencial holomorfa com centro em p. Aqui, nos queremos estudar os sistemas de perturbação por partes

$$\dot{z} = f(z) + \begin{cases} \epsilon R_m^+(z, \overline{z}), & \text{when } \operatorname{Im}(z) > 0, \\ \epsilon R_m^-(z, \overline{z}), & \text{when } \operatorname{Im}(z) < 0, \end{cases} \tag{11}$$

onde $R^{\pm}(z, \overline{z})$ são polinômios complexos definidos para $\pm \operatorname{Im}(z) > 0$. Fornecemos uma expressão integral, semelhante a uma integral abeliana, para o anel de períodos de p. Os zeros dessa integral controlam os ciclos limites que bifurcam das órbitas periódicas dessa região anular. Esta expressão é dada em termos da conjugação conformal entre $\dot{z} = f(z)$ e sua linearização $\dot{z} = f'(p)z$ em p. Usamos este resultado para controlar a bifurcação simultânea de ciclos limites dos dois períodos anulares de $\dot{z} = \mathrm{i}(z^2 - 1)/2$, após as perturbações por partes complexas e holomorfas polinomiais. Em particular, até onde sabemos, apresentamos o primeiro exemplo de ciclos limites não aninhados para sistemas holomorfos por partes.

A seguir, apresentamos nossos resultado principais nesta linha de pesquisa:

Teorema I. *Considere o sistema complexo por partes* (11). *Suponha que* ϕ *é a linearização de* $\dot{z} = f(z)$ *em* p *tal que* $\phi(\Sigma) \subset \Sigma$. *Então, sua função de Melnikov de primeira ordem é* $M_1(r) = M_1^+(r) - M_1^-(r)$, *onde*

$$M_1^{\pm}(r) = -\operatorname{Im}\left(\int_0^{\pm\pi} \overline{\phi'(\phi^{-1}(re^{\mathrm{i}\theta})) R_m^{\pm}\left(\phi^{-1}(re^{\mathrm{i}\theta}), \overline{\phi^{-1}(re^{\mathrm{i}\theta})}\right)}\, \mathrm{i}e^{\mathrm{i}\theta} d\theta \right). \tag{12}$$

Em particular, cada zero simples $r = r_0$ *de* M_1 *fornece, para* ϵ *suficientemente pequeno, um ciclo limite hiperbólico de* (11) *que tende a* $r = r_0$ *quando* ϵ *tende a 0.*

O resultado acima é uma extensão para o caso descontínuo do obtido em [4] na situação suave.

Utilizamos o Teorema I para estudar os sistemas PWCS (11) com $f(z) = \mathrm{i}(z^2 - 1)/2$ e

$$R_m^{\pm}(z, \overline{z}) = \sum_{l=0}^{m} \sum_{k=0}^{l} \overline{a}_{k,l} z^{l-k} \overline{z}^k, \ a_{k,l} \in \mathbb{C} \text{ e } m = 0, 1, 2, 3. \tag{13}$$

De fato, este sistema não perturbado também foi considerado em [4] no contexto de perturbações suaves.

Ressaltamos que $\dot{z} = f(z) = \mathrm{i}(z^2 - 1)/2$ tem 2 centros em -1 e 1, separados pela linha reta invariável $\operatorname{Re}(z) = 0$. Cada um dos semiplanos perfurados $\{z : \pm \operatorname{Re}(z) > 0\} \setminus \{\pm 1\}$ esta repleto de órbitas periódicas do sistema. Para realizar este estudo, usamos explicitamente a linearização $w = \phi(z) = (1 + z)/(1 - z)$ de $\dot{z} = f(z)$ em -1, que usamos para encontrar a função de bifurcação em $z = -1$. Assim, ao trocar as variáveis e o tempo, obteremos a função de bifurcação em $z = 1$. Vale a pena notar que esta linearização é especialmente simples e também possui um inverso simples, mas, infelizmente, para a maioria dos campos vetoriais holomórficos f, os cálculos podem ser muito mais complicados.

Antes de apresentar nosso segundo resultado principal, introduzimos algumas notações. Denotamos por M_1 e N_1 as funções de primeira ordem médias em -1 e 1, respectivamente. Além disso, dizemos que o sistema (11) apresenta a *configuração de ciclos limites* $[i, j]$ se M_1 e N_1 tiverem simultaneamente i e j zeros simples no intervalo $(0, 1)$, respectivamente. Nossa definição é motivada pela teoria de averaging de primeira ordem, porque, para ε suficiente pequeno, o sistema diferencial tem i ciclos limites ao redor de -1 e j ciclos limites ao redor de 1. Obviamente, pela simetria do problema, se a configuração $[i, j]$ é válida, a configuração $[j, i]$ também o é.

Para $m \leq 3$, temos que ambas as funções, multiplicadas por r, pertencem ao espaço vetorial $\mathcal{F}$ gerado pelas funções

$$\mathcal{F} = [r, r^2, r^3, r^4, (r^2 - 1)^2 \operatorname{arctanh}(r), (r^4 - 1) \operatorname{arctanh}(r), r^2 \operatorname{arctanh}(r)],$$

onde lembramos que

$$\operatorname{arctanh}(r) = \tanh^{-1}(r) = \frac{1}{2} \ln\left(\frac{1 + r}{1 - r}\right).$$

Nós temos provado que o conjunto ordenado acima de funções forma um *sistema de Chebyshev completo e expandido (ECT-system) em* $(0, 1)$. Essa propriedade permitirá controlar o número exato de zeros de cada uma das funções, separadamente.

Para valores maiores de m, o número de monômios r^k, assim como o número de funções da forma $S_l(r) \operatorname{arctanh}(r)$ para alguns polinômios de grau crescente fixo S_l em M_1 e N_1, crescerá, mas nossa abordagem também se aplicará. Em resumo, para um valor fixo de m, o controle do número máximo de zeros de M_1 e N_1 parece poder ser completamente entendido, mas exigiria muito mais esforço computacional. Por essa razão, restringimos nossa atenção ao caso $m \leq 3$. Por outro lado, o conhecimento do número máximo que ambas as funções podem ter simultaneamente é um problema difícil e desafiador. Todos os resultados que obtivemos nesta direção estão resumidos nos próximos teoremas.

Teorema J. *Considere os sistemas PWCS* (11) *com* $f(z) = \mathrm{i}(z^2 - 1)/2$. *Se* $m = 0$, *as únicas configurações realizáveis são* $[[1, 0]]$ *e* $[[0, 1]]$. *Quando* $1 \leq m \leq 3$, *se* $[[i, j]]$ *é uma configuração realizável, então* $i, j \leq m + 3$. *Além disso, as seguintes configurações, satisfazendo* $i, j \leq m + 3$, *são realizáveis:*

(a) $[[i, j]]$ *com* $0 \leq i + j \leq 4$ *quando* $m = 1$.

(b) $[[i, j]]$ *com* $0 \leq i + j \leq 6$, *quando* $m = 2$.

(c) $[[i, j]]$ *com* $0 \leq i + j \leq 8$ *quando* $m = 3$.

Como um caso muito particular na prova do Teorema J, aparece a situação em que tanto M_1 quanto N_1 são polinômios. Nesse caso, a questão do número simultâneo de zeros no intervalo $(0, 1)$ de ambas as funções pode ser abordada com muito mais detalhes. Acreditamos que este é um problema que é interessante por si só.

Neste ponto, outra pergunta natural surge: O que acontece se a função de perturbação complexa $R_m^{\pm}$ em PWCS (11) for holomorfa? De forma abreviada, chamaremos esses sistemas de PWHS. Para eles, $R_m^{\pm}$ dependem apenas de z. Enfatizamos que o fato de a perturbação ser holomorfa simplifica muito os cálculos, e não é surpreendente que o número de ciclos que surgem seja menor do que na situação mais geral acima.

Neste caso e para $m \geq 3$, as funções $M_1(r)$ e $N_1(r)$, multiplicadas por $(r^2 - 1)^{m-3}$, pertencem ao espaço vetorial $\mathcal{G}$ gerado pelas funções

$$\mathcal{G} = [1, r, r^2, \ldots, r^{2(m-2)}, r(r^2 - 1)^{m-3} \operatorname{arctanh}(r)],$$

que também é um ECT-system em $(0,1)$. Apenas abordamos o problema das bifurcações simultâneas quando $m \leq 3$. Nosso principal resultado para PWHS é:

Teorema K. *Considere sistemas holomorfos por partes da forma* (11), *com* $f(z) = \mathrm{i}(z^2-1)/2$, *onde* $R_m^{\pm}$ *dependem apenas de* z. *O seguinte se verifica:*

(a) Quando $m = 0$ *as únicas configurações realizáveis são* $[[1,0]]$ *e* $[[0,1]]$ *e quando* $m \in \{1,2\}$ *as únicas configurações realizáveis são* $[[i,j]]$ *com* $i+j \leq 2$.

(b) Quando $m = 3$, *se* $[[i,j]]$ *é uma configuração realizável, então* $i,j \leq 3$. *Mais ainda, as seguintes configurações, satisfazendo* $i,j \leq 3$, *são realizáveis:* $[[i,j]]$ *com* $0 \leq i+j \leq 4$.

(c) Quando $m > 3$, *se* $[[i,j]]$ *é uma configuração realizável, então* $i,j \leq 2m-3$. *Além disso, existem exemplos onde* i *ou* j *é* $2m-3$.

Este prepint foi realizado em conjunto com os professores Armengol Gassul e Paulo Ricardo da Silva. Ao final deste relatório, o trabalho feito por nós será anexado, onde é possível encontrar todos os resultados obtidos em detalhes.

2.2.3 Artigo 3

Sistemas holomorfos planares $\dot{x} = u(x,y)$, $\dot{y} = v(x,y)$ são aqueles que $u = \mathrm{Re}(f)$ e $v = \mathrm{Im}(f)$ para alguma função holomorfa $f(z)$. Possuem propriedades dinâmicas importantes, destacando-se, por exemplo, o fato de não possuírem ciclos limite e que o problema centro-foco é trivial. Em particular, a hipótese de que um sistema polinomial é holomorfo reduz o número de parâmetros do sistema. Embora um sistema polinomial de grau n dependa dos parâmetros n^2+3n+2, um polinômio holomorfo depende apenas dos parâmetros $2n+2$. Neste trabalho, além de fazer uma visão geral da teoria dos sistemas holomorfos, classificamos todos os possíveis retratos de fase global, no disco de Poincaré, dos sistemas $\dot{z} = f(z)$ e $\dot{z} = 1/f(z)$, onde $f(z)$ é um polinômio de grau 2, 3 e 4 na variável $z \in \mathbb{C}$. Também classificamos todos os possíveis retratos de fase global dos sistemas Moebius $\dot{z} = \frac{Az+B}{Cz+D}$, onde $A,B,C,D \in \mathbb{C}, AD-BC \neq 0$. Finalmente, obtemos expressões explícitas de primeiras integrais de sistemas holomorfos e de sistemas holomorfos conjugados, que têm importantes aplicações no estudo da dinâmica dos fluidos.

Um dos principais resultados obtidos neste trabalho é o seguinte;

Teorema 2.4. *Seja* $\dot{z} = f(z)$ *um sistema holomorfo definido no conjunto aberto* $A \subseteq \mathbb{C}$. *Assim sua as trajetórias estão contidas nas curvas de nível de* $H(x,y) = \Im G(z)$ *onde* $G'(z) = \dfrac{1}{f(z)}$.

Este trabalho foi realizado juntamente com o supervisor Paulo Ricardo e com o Dr. Luiz Fernando Gouveia. Mais detalhes poderão ser encontrados no paper que estará anexado ao final deste relatório. Além disso, o trabalho pode ser encontrado no link:

https://link.springer.com/article/10.1007/s12346-022-00734-3.

2.2.4 Artigo 4

Um difícil problema clássico na teoria qualitativa dos sistemas diferenciais no plano $\mathbb{R}^2$ é o problema do centro-foco, ou seja, distinguir entre um foco e um centro. Outro problema difícil é distinguir dentro de uma família de centros aqueles que são globais. Um centro global é um centro p tal que $\mathbb{R}^2 \setminus \{p\}$ esta repleto de órbitas periódicas.

Neste trabalho, classificamos os centros globais da família das equações complexas

$$i\dot{w} = w - A_3\overline{w}^2 - A_4 w^3 - A_5 w^2 \overline{w} - A_6 w \overline{w}^2, \tag{14}$$

onde $w = x+iy$ e $A_k \in \mathbb{C}$ para $k = 3,4,5,6$.

Ressaltamos que ter informações sobre centros globais de (14) nos permite saber quando essa equação complexa não possui ciclo limite.

O principal resultado obtido neste trabalho é o seguinte:

Teorema L. *As equações complexas polinomiais* (14) *têm um centro global na origem das coordenadas se, e somente se, uma das seguintes condições for satisfeita:*

(a) $a_1 = a_2 = b_2 = c_2 = d_2 = 0$, $d_1 = 3b_1$, $b_1 = -c_1/4$, $c_1 < 0$*;*

(b) $a_1 = a_2 = b_2 = c_1 = c_2 = d_2 = 0$, $d_1 = 3b_1$, $b_1 < 0$*;*

(c) $a_2 = b_1 = b_2 = c_2 = d_1 = d_2 = 0$, $a_1^2 + c_1 < 0$*;*

(d) $a_1 = a_2 = b_1 = b_2 = c_1 = c_2 = d_1 = d_2 = 0$*;*

(e) $a_1 = a_2 = b_2 = c_2 = d_2 = 0$, $d_1 = -b_1 - c_1 \neq 3b_1$, $2b_1 + c_1 \leq 0$, $b_1 > 0$*;*

(f) $a_2 = b_2 = c_1 = c_2 = d_2 = 0$, $b_1 \neq 0$, $a_1^2 + 3b_1 - d_1 < 0$, $a_1^2 + 4(b_1 + d_1) < 0$*;*

(g) $a_1 = a_2 = b_1 = b_2 = c_2 = d_2 = 0$, $d_1 = -c_1 > 0$.

As técnicas utilizadas neste artigo são o método de blow-up vertical e a compactificação de Poincaré.

Este trabalho foi realizado em conjunto com o prof. Jaume Llibre. Ao final deste relatório, o paper feito por nós será anexado, onde é possível encontrar todos os resultados obtidos em detalhes. Além disso, o artigo pode ser encontrado no *Rendiconti del Circolo Matematico di Palermo* em https://doi.org/10.1007/s12215-024-01034-2.

Observe que as técnicas utilizadas e aprendidas no desenvolvimento deste trabalho permitirão ao pesquisador dispor de um grande número de métodos a serem utilizados em futuros projetos.

2.3 Trabalhos realizados referentes à Propostas 3

Aqui, estamos interessados em estudar a existência de variedades complexas invariantes de sistemas holomorfos bidimensionais. A partir da teoria geométrica da perturbação singular sabemos que se um sistema lento-rápido tem associada uma variedade crítica compacta normalmente hiperbólica, então existe uma variedade suave localmente invariante. No entanto, esta variedade suave não possui necessariamente uma estrutura complexa. Aqui, fornecemos condições para garantir a existência de variedades complexas invariantes unidimensionais. Consequentemente, isso nos permite estabelecer que os centros, focos do problema reduzido são persistentes por perturbação singular. As ferramentas utilizadas por nós são as técnicas usuais das Teorias de Fenichel e Briot-Bouquet.

Os principais resultados obtidos neste trabalho são os seguintes:

Teorema M. *Considere o sistema* (4) *com* $f(z, w) = \alpha z + \tilde{f}(z, w)$, $g(z, w) = \beta w + \tilde{g}(z, w)$ *e* $\alpha, \beta \in \mathbb{C}$*. Suponha que* C_0 *seja uma variedade crítica normalmente hiperbólica associada ao sistema complexo* (4) *e considere* $\varepsilon > 0$ *um parâmetro suficiente pequeno. Então, o sistema* (4) *tem uma variedade complexa unidimensional invariante única* C_ε *passando pelo ponto de equilíbrio* q_ε*, que é dado por*

$$C_\varepsilon = \{(z, w) : w = h_\varepsilon(z), \quad h_\varepsilon(0) = 0, \quad h'_\varepsilon(0) = 0\},$$

onde h_ε *é holomorfa em* $B = B(0, r)$.

No próximo resultado, temos usado o Teorema M para provar que os centros, focos e nós do problema reduzido associado ao sistema holomorfo lento-rápido (4), cujo jacobiano de f e g é diagonalizável na origem, são preservados pela perturbação singular.

Teorema N. *Suponha para* $\varepsilon > 0$ *suficiente pequeno que os autovalores de* $J_{\mathbb{C}}(f, g)|_{q_\varepsilon}$ *são* $\alpha, \beta \in \mathbb{C} \setminus \{0\}$ *e que* α, β *são do mesmo tipo. Então, centros, focos e nodos do problema reduzido associados a* (4) *são persistentes por perturbação singular.*

Além disso, estamos interessados em determinar a existência de variedades invariantes complexas para certas famílias de sistemas holomorfos bidimensionais, cujo jacobiano de f e g não é necessariamente diagonalizável na origem e este ponto não precisa ser um ponto de equilíbrio do sistema. A resposta a estas perguntas é dada nos próximos teoremas.

Teorema O. *Considere o sistema lento-rápido* (4) *e suponha que existem funções holomorfas* η, κ *definidas em* Ω *tais que* $\frac{g(z,w)}{f(z,w)} = \eta(z)\kappa(w)$, *para todo* $z, w \in \Omega$. *Então, existe uma variedade complexa invariante unidimensional* C_ε *associado ao sistema diferencial* (4), *tal que*

(a) o fluxo em C_ε *converge para o fluxo lento como* $\varepsilon \to 0$.

(b) a dinâmica sobre C_ε *é dada por* $\dot{z} = g(z, w)$.

Teorema P. *Considere o sistema* (4) *com* $f(z, w) = \alpha z + \beta G(w)$, $g(z, w) = g(w)$, $\alpha \in \mathbb{C}$ *e* $G' \equiv 1/g$. *Então, existe uma variedade complexa invariante unidimensional* C_ε *associada ao sistema diferencial* (4), *tal que*

(a) O fluxo em C_ε *converge para o fluxo lento como* $\varepsilon \to 0$.

(b) A dinâmica sobre C_ε *é dada por* $\dot{w} = g(w)$.

(c) Se $\alpha = i\alpha_2$, *com* $\alpha_2 \in \mathbb{R} \setminus \{0\}$, *então a órbita do sistema* (4) *com condição inicial* $z(w_0) = z_0$ *permanece exponencialmente próximo da variedade invariante* C_ε *na região* $\mathcal{R} = \{(z, w) : \alpha_2 \Im(G(w) - G(w_0)) \geq \eta > 0\}$, *para algum* $\eta > 0$.

Este trabalho foi realizado juntamente com o supervisor Paulo Ricardo e com o doutor Luiz Fernando Gouveia. Mais detalhes poderão ser encontrados no preprint que estará anexado ao final deste relatório. Mais ainda, o preprint pode ser encontrado no ArXiv no link https://arxiv.org/abs/2304.00142.

3 Plano de gestão de dados

A FAPESP publicou em setembro de 2020 normas que exigem a vinculação de um Plano de Gestão de Dados no momento da submissão de um projeto de pesquisa. Planos de Gestão de Dados (PGD's) são considerados hoje em dia parte integral das boas práticas em pesquisa, pois planejam como e onde os dados serão preservados após terminar a pesquisa, e sua disponibilização para outros projetos.

Esta bolsa não possui um PGD vinculado ao projeto inicial, visto que ela foi solicitada ainda em 14/08/2020. Apesar da não obrigatoriedade em apresentá-lo, prezamos pelas boas práticas em pesquisa e elaboramos um PGD para este projeto, que pode ser visualizado ao fim deste projeto de prorrogação.

4 Execução do Plano de trabalho

Durante o período vigente da bolsa, as reuniões com o supervisor Paulo Ricardo da Silva foram feitas de forma presencial. Reuniões semanais eram feitas, nas quais discutíamos os problemas propostos no plano de trabalho do bolsista, assim como questões sugeridas pelo supervisor. Mais ainda, com o intuito de aprender novas técnicas da área de pesquisa, o bolsista participava semanalmente, do ciclo de seminários dos grupos de sistemas dinâmicos da UNESP, que discutiam problemas relacionados ao projeto de pesquisa. As reuniões semanais continuarão até o fim da vigência da bolsa.

5 Participação em eventos científicos

No período ao qual se refere este relatório, o bolsista participou e apresentou alguns trabalhos em eventos científicos nacionais e internacionais de sua área de pesquisa. Os trabalhos intitulados *Lower bounds for the number of limit cycles for piecewise polynomial holomorphic systems* e *Sistemas Dinâmicos Descontínuos: Simplificando a Análise com Variáveis Complexas* foram apresentados na modalidade *apresentação oral* nos seguintes congressos:

- MAT80 XIII Workshop on Dynamical Systems, ocorrido de 25 a 29 de novembro de 2024. `https://www.ime.unicamp.br/~mat80/`;
- IV Colóquio de Matemática da Região Sudeste/XXXVI Semana da Matemática, ocorrido de 21 a 25 de outubro de 2024. `https://www.ibilce.unesp.br/#!/departamentos/matematica/eventos/4coloquio-xxxvisemat/`.

Além disso, o trabalho intitulado *Smoothing of nonsmooth differential systems near regular-tangential singularities* foi apresentado na forma de pôster nas seguintes conferências:

- XLIII Dynamical Days Europe, realizado de 3 a 8 de setembro de 2023. https://sites.google.com/view/dynamicsdayseurope2023;
- Ddays 2023, realizado de 3 a 6 de outubro de 2023. https://www.dance-net.org/dday/34.

Também, o trabalho intitulado *On the number of limit cycles for piecewise holomorphic systems* foi apresentado na forma de pôster nas seguintes conferências:

- 20ª Winter School "Recent Trends in Nonlinear Science", realizada de 29 de janeiro a 2 de fevereiro de 2024. https://www.dance-net.org/rtns/35;
- Workshop on Periodic Orbits, realizado de 7 a 9 de fevereiro de 2024. https://www.crm.cat/op_2024/.

Finalmente, o bolsista participou no ciclo de palestras do grupo de sistema dinâmicos da Unesp.
`https://www.ibilce.unesp.br/#!/departamentos/matematica/grupos-de-pesquisa/sistemas-dinamicos/`.

6 Lista de Trabalhos Publicados e Preprints

6.1 Trabalhos publicados: desde o início da bolsa

(1) D.D. Novaes and G. Rondón. Smoothing of nonsmooth differential systems near regular tangential singularities and boundary limit cycles. **Nonlinearity** 34(6):4202–4263, 2021.
Doi: https://doi.org/10.1088/1361-6544/ac04be.

(2) L.F.S. Gouveia; G. Rondón and P.R. da Silva. Piecewise holomorphic systems. **Journal of Differential Equations** 332:440–472, 2022.
Doi: https://doi.org/10.1016/j.jde.2022.05.027.

(3) D.D. Novaes and G. Rondón. On limit cycles in regularized Filippov systems bifurcating from homoclinic-like connections to regular-tangential singularities. **Physica D-Nonlinear Phenomena**, v.1, p.133526, 2022.
Doi: https://www.sciencedirect.com/science/article/pii/S0167278922002305?via%3Dihub.

(4) L.F.S. Gouveia; G. Rondón and P.R. da Silva. Global Phase Portrait and Local Integrability of Holomorphic Systems. **Qualitative Theory of Dynamical Systems**, v.22, p.35, 2023.
Doi: https://link.springer.com/article/10.1007/s12346-022-00734-3.

(5) G. Rondón; P.R. Silva and O.H. Perez. Slow-Fast Normal Forms Arising from Piecewise Smooth Vector Fields. **Journal of Dynamical and Control Systems**, v.1, p.1-18, 2023.
Doi: https://link.springer.com/article/10.1007/s10883-023-09657-x.

(6) J. Llibre and G. Rondón. Global centers of a class of cubic polynomial differential systems. **Rendiconti del Circolo Matematico di Palermo (Testo Stampato)**, v.1, p.1, 2024.
Doi: https://doi.org/10.1007/s12215-024-01034-2.

(7) A. Gasull; G. Rondón and P.R. Silva. On the number of limit cycles for piecewise polynomial holomorphic systems. **SIAM Journal On Applied Dynamical Systems**, v.23, p.2593 - 2622, 2024.
Doi: https://doi.org/10.1137/23M1620922.

6.2 Trabalhos publicados: durante o período atual

(1) D.D. Novaes and G. Rondón. On limit cycles in regularized Filippov systems bifurcating from homoclinic-like connections to regular-tangential singularities. **Physica D-Nonlinear Phenomena**, v.1, p.133526, 2022.
Doi: https://www.sciencedirect.com/science/article/pii/S0167278922002305?via%3Dihub.

(2) L.F.S. Gouveia; G. Rondón and P.R. da Silva. Global Phase Portrait and Local Integrability of Holomorphic Systems. **Qualitative Theory of Dynamical Systems**, v.22, p.35, 2023.
Doi: https://link.springer.com/article/10.1007/s12346-022-00734-3.

(3) G. Rondón; P.R. Silva and O.H. Perez. Slow-Fast Normal Forms Arising from Piecewise Smooth Vector Fields. **Journal of Dynamical and Control Systems**, v.1, p.1-18, 2023.
Doi: https://link.springer.com/article/10.1007/s10883-023-09657-x.

(4) J. Llibre and G. Rondón. Global centers of a class of cubic polynomial differential systems. **Rendiconti del Circolo Matematico di Palermo (Testo Stampato)**, v.1, p.1, 2024.
Doi: https://doi.org/10.1007/s12215-024-01034-2.

(5) A. Gasull; G. Rondón and P.R. Silva. On the number of limit cycles for piecewise polynomial holomorphic systems. **SIAM Journal On Applied Dynamical Systems**, v.23, p.2593 - 2622, 2024.
Doi: https://doi.org/10.1137/23M1620922.

6.3 Trabalhos em Formato Preprint

(1) G. Rondón, P. R. da Silva, L. F. S. Gouveia. Holomorphic slow-fast systems.
https://arxiv.org/abs/2304.00142.

(2) B. Grajales, G. Rondón, J. Saavedra. Riemannian Geometry of G2-type Real Flag Manifolds.
https://arxiv.org/abs/2401.02805.

6.4 Trabalhos em Andamento

(1) Simultaneous bifurcation of limit cycles for Piecewise Holomorphic systems
Gasull A.; Rondón G.; da Silva P.R.

(2) Global dynamics of the Ehrhard-Muller system with invariant algebraic surfaces
Llibre J.; Rondón G.

(3) Limit cycles of discontinuous piecewise complex systems separated by $\mathbb{S}^1$.
Rondón G.; da Silva P.R.

(4) Regularization of nonsmooth differential system near regular-tangential singularities in $\mathbb{R}^3$
D.D. Novaes; Rondón G.

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On the Number of Limit Cycles for Piecewise Polynomial Holomorphic Systems*

Armengol Gasull†, Gabriel Rondón‡, and Paulo Ricardo da Silva‡

Abstract. In this paper, we are concerned with determining lower bounds of the number of limit cycles for piecewise polynomial holomorphic systems with a straight line of discontinuity. We approach this problem with different points of view. Initially, we study the number of zeros of the first- and second-order averaging functions. We also use the Lyapunov quantities to produce limit cycles appearing from a monodromic equilibrium point via a degenerated Andronov–Hopf type bifurcation, adding at the very end the sliding effects. Finally, we use the Poincaré–Miranda theorem for obtaining an explicit piecewise linear holomorphic system with 3 limit cycles, a result that improves the known examples in the literature that had a single limit cycle.

Key words. piecewise polynomial holomorphic system, limit cycles, averaging method, Lyapunov quantities, Poincaré–Miranda theorem

MSC codes. 32A10, 34A36, 34C07, 37G15

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1. Introduction.

The models in nonsmooth dynamics of differential equations have attracted the attention of many researchers due to the accuracy of the obtained results comparing with the real observations; see more details in the three books [1, 5, 26] and their references. Several of these models are given by piecewise smooth systems with some switching manifold. Moreover, on many of them, the smooth systems are linear, and for planar cases, the switching manifold is a straight line.

Holomorphic functions have a wide range of applications in several areas of applied science such as the study of fluid dynamics; for more information, see, for instance, [3, 7, 11]. One of the most remarkable dynamical properties of holomorphic systems $\dot{z} = f(z)$ is the fact that

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†Departament de Matemàtiques, Edifici C, Universitat Autònoma de Barcelona, 08193 Bellaterra, Barcelona, Spain, and Centre de Recerca Matemàtica, Campus de Bellaterra, 08193 Bellaterra, Barcelona, Spain (armengol.gasull@uab.cat).

‡Institute of Biosciences, Humanities and Exact Sciences, São Paulo State University (Unesp), 2265, CEP 15054–000, S. J. Rio Preto, São Paulo, Brazil (garv202020@gmail.com, paulo.r.silva@unesp.br).

these systems do not have limit cycles. The study and the properties of these systems make them interesting and beautiful, but precisely this absence of limit cycles makes them dynamically poor. However, in [22], the authors proved that there are piecewise linear holomorphic systems that have one limit cycle. Moreover, they proved that, if the equilibrium points are on the straight line of discontinuity, this limit cycle is unique.

In this paper, we are interested in piecewise polynomial holomorphic systems (PWHSs). On one hand, each of the smooth systems has the beautiful properties of the holomorphic systems, but on the other hand, considered as a piecewise defined system, they exhibit all the interesting features of the piecewise linear systems and much more. The complex analysis is useful to determine the phase portrait of holomorphic and piecewise holomorphic systems; see [15, 16, 22]. In particular, in [22], the authors have used the intrinsic properties of holomorphic functions, such as their integrability, to construct limit cycles in the piecewise case. Also, in [17], Garijo et al. use the Cauchy integral formula to calculate the Abelian integrals that arise in their study.

Essentially, the techniques we employed to prove the results of this paper could be applied to piecewise smooth vector fields without necessarily being piecewise holomorphic. However, the holomorphic character endows the vector field with important properties that simplify calculations. In fact, holomorphic systems, apart from the absence of limit cycles, have other surprising and interesting properties: reversibility, integrability, all their centers being isochronous, and complete knowledge of the phase portraits around their nonessential singularities with simple local normal forms. Moreover, when we add the hypothesis of being holomorphic to a system, we reduce the number of involved parameters. More concretely, while a polynomial system of degree n depends on n^2+3n+2 parameters, a polynomial holomorphic system depends only on $2n+2$ parameters. In particular, while planar polynomial systems of degree n that do not have continua of equilibrium points have at most n^2 of them, holomorphic polynomial systems of degree n have at most n equilibrium points provided that f is not identically null. Moreover, as we will see, their Lyapunov quantites in the piecewise case have simpler and more compact expressions than in the general piecewise case.

The aim of this work is the study of the limit cycles of PWHSs:

$$\begin{cases} \dot{z} = F^{+}(z) \text{ when } \operatorname{Im}(z) > 0, \\ \dot{z} = F^{-}(z) \text{ when } \operatorname{Im}(z) < 0, \end{cases} \tag{1}$$

where $z = x + iy$ and $F^{\pm}(z)$ are holomorphic polynomial functions with $\deg(F^{\pm}) = n^{\pm}$. To carry out this analysis, we will start from system (1) when the vector fields $F^{\pm}$ share a monodromic equilibrium point at origin, and then, we will investigate the effects of considering sliding segments in the system.

Notice that the straight line $\Sigma = \{z \in \mathbb{C} : \operatorname{Im}(z) = 0\}$ divides the plane into two half-planes $\Sigma^{\pm}$ given by $\{z \in \mathbb{C} : \operatorname{Im}(z) > 0\}$ and $\{z \in \mathbb{C} : \operatorname{Im}(z) < 0\}$, respectively. The orbits on Σ are defined following the Filippov convention; see [12] for more details.

Let $\mathcal{L}_{n^+,n^-} \in \mathbb{N} \cup \{\infty\}$ be the maximum number of limit cycles that (1) can have. The main goal of this article is to determine lower bounds for $\mathcal{L}_{n^+,n^-}$. Notice that trivially, $\mathcal{L}_{n^+,n^-} = \mathcal{L}_{n^-,n^+}$. The main tools that we will use are the following:

- The averaging theory up to second order to compute the so-called averaged functions of orders 1 and 2. We will base our computations on the general results given in

[24, Thm. 1]. For the sake of completeness, we recall them in Proposition 1. With this approach, we obtain lower bounds of $\mathcal{L}_{n^-,n^+}$ (nonoptimal) for any n^-, n^+.

- The computation and use of the Lyapunov quantities to produce limit cycle via degenerated Andronov–Hopf type bifurcations when $n^- + n^+ \leq 4$. We develop the method introduced in [9, 32] and apply it in the particular holomorphic context to obtain explicit expression of the first five Lyapunov quantities.
- The effect of the sliding for increasing by one the number of limit cycles obtained in the previous item. In this point, we follow the ideas of [14, Prop. 7.3] to perform a suitable final perturbation inside the holomorphic world.
- The use of the conformal map $w = 1/z$ to transform a neighborhood of infinity of our system with $n^+ = n^- = 1$ into a neighborhood of the origin of a new system (1) but with $n^+ = n^- = 2$. Then, by applying the results obtained by using the Lyapunov quantities in this latter case, we prove that $\mathcal{L}_{1,1} \geq 3$. This approach is very related with the one of Freire et al. [13], which proves the same result and with similar ideas but for piecewise linear systems.
- The Poincaré–Miranda theorem to prove the existence of 3 limit cycles for an explicit piecewise linear holomorphic sytem, providing a second proof that $\mathcal{L}_{1,1} \geq 3$. This result adapts to the holomorphic setting the same approach used in [19] to obtain a piecewise linear system with at least 3 limit cycles.

Let us state our main results. Our first theorem deals with the analysis of the limit cycles obtained by computing the second-order averaged function of a perturbation of the global center $\dot{z} = iz$ inside system (1). As usual, in the next result, $\lfloor . \rfloor$ denotes the integer part of a real number; i.e., given $n \leq x < n+1$, then the integer part of x is $\lfloor x \rfloor = n$.

Theorem A. *Consider system (1) with $F^{\pm}(z) = iz + \epsilon h^{\pm}(z)$ and $\deg(h^{\pm}) = n^{\pm}$. Then, by using the averaging theory up to order two, for ϵ small enough, the maximum number of limit cycles that bifurcate from the periodic orbits of the center is $\lfloor (3\max\{n^+, n^-\} - 1))/2 \rfloor$; this upper bound is attained, and, in this case, all these limit cycles are hyperbolic. As a consequence, $\mathcal{L}_{n^+,n^-} \geq \lfloor (3\max\{n^+, n^-\} - 1))/2 \rfloor$.*

Our second result studies with more detail the cases $n^+ + n^- \leq 4$ and deals with the limit cycles obtained from a degenerated Andronov–Hopf bifurcation together with an extra limit cycle obtained by adding sliding (see Proposition 4). Of course, this result is based on the computation of the Lyapunov quantities associated with the weak focus-weak focus case, which are developed and particularized for the piecewise holomorphic systems. We calculate up to the 5th constant; see Theorem D for more details. The main difficulty in calculating these quantities arises from the length of the expressions involved, which was expected since, in the discontinuous case, we only consider half-return maps around the monodromic equilibrium points, and some cancellations that happen in the smooth case disappear. For this reason, we have deferred some of the computations of our proof of Theorem D to an appendix.

Theorem B. *Let $\mathcal{L}^0_{n^+,n^-} \leq \mathcal{L}_{n^+,n^-}$ be the number of hyperbolic limit cycles bifurcating from the origin of system (1). In Table 1, we give some lower bounds of $\mathcal{L}^0_{n^+,n^-}$ for $n^+ + n^- \leq 4$,*

We highlight that, in the linear case $n^+ = n^- = 1$, Theorems A and B only show that $\mathcal{L}_{1,1} \geq 1$, which was a known result ([22]). Our final theorem improves this lower bound.

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Global centers of a class of cubic polynomial differential systems

Jaume Llibre[1] **· Gabriel Rondón**[2]

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Abstract
A difficult classical problem in the qualitative theory of differential systems in the plane $\mathbb{R}^2$ is the center-focus problem, i.e. to distinguish between a focus and a center. Another difficult problem is to distinguish inside a family of centers the ones which are global. A global center is a center p such that $\mathbb{R}^2 \setminus \{p\}$ is filled with periodic orbits. In this paper we classify the global centers of the family of real polynomial differential systems of degree 3 that in complex notation write

$$i\dot{w} = w - A_3\overline{w}^2 - A_4w^3 - A_5w^2\overline{w} - A_6w\overline{w}^2,$$

where $w = x + iy$ and $A_k \in \mathbb{C}$ for $k = 3, 4, 5, 6$.

Keywords Global centers · Vertical blow-up · Polynomial differential equations

Mathematics Subject Classification 34C05

1 Introduction

The general cubic differential equations in complex notation having either a weak focus or a center at the origin of coordinates are

$$i\dot{w} = w - A_1w^2 - A_2w\overline{w} - A_3\overline{w}^2 - A_4w^3 - A_5w^2\overline{w} - A_6w\overline{w}^2 - A_7\overline{w}^3, \qquad (1)$$

where $w = x+iy$ and $A_k \in \mathbb{C}$, $k = 1, \ldots, 7$. These differential systems have been considered in several articles, thus in the papers [2, 12, 13, 15] the authors provide for some subclasses of these differential systems necessary and sufficient conditions in order that the equilibrium point at the origin of coordinates be a center. In this paper we restrict our attention to the subclass $A_1 = A_2 = A_7 = 0$. Then the complex differential equation (1) can be written as

✉ Gabriel Rondón
garv202020@gmail.com

Jaume Llibre
jaume.llibre@uab.cat

1 Departament de Matemàtiques, Edifici Cc, Universitat Autònoma de Barcelona, 08193 Bellaterra, Barcelona, Catalonia, Spain

2 Institute of Biosciences, Humanities and Exact Sciences, São Paulo State University (Unesp), Rua C. Colombo, 2265, São José do Rio Preto, São Paulo CEP 15054–000, Brazil

the following real cubic polynomial differential system

$$
\begin{aligned}
\dot{x} &= y + 2a_1xy - a_2(x^2 - y^2) - (b_2 + c_2 + d_2)x^3 - (3b_1 + c_1 - d_1)x^2y \\
&\quad +(3b_2 - c_2 - d_2)xy^2, \\
\dot{y} &= -x + a_1(x^2 - y^2) + 2a_2xy + (b_1 + c_1 + d_1)x^3 - (3b_2 + c_2 - d_2)x^2y \\
&\quad +(-3b_1 + c_1 + d_1)xy^2 + (b_2 - c_2 + d_2)y^3,
\end{aligned}
\tag{2}
$$

in the plane $\mathbb{R}^2$, where $A_3 = a_1 + ia_2$, $A_4 = b_1 + ib_2$, $A_5 = c_1 + ic_2$ and $A_6 = d_1 + id_2$.

The main goal of this paper is to classify the global centers of system (2). We recall that a *center* of a differential equation in $\mathbb{R}^2$ is an equilibrium point p having a neighbourhood $\mathcal{U}$ such that $\mathcal{U} \setminus \{p\}$ is filled with periodic orbits. In particular, a *global center* is a center p such that $\mathbb{R}^2 \setminus \{p\}$ is filled with periodic orbits.

The classical problem of distinguishing between a focus and a center is a difficult one and it is one of the challenges of the theory of nonlinear differential systems in the plane $\mathbb{R}^2$. The rigorous notion of center appeared in the literature with the works of Poincaré [14] and Dulac [5].

The classification of all centers of the cubic polynomial differential systems is a problem unsolved now. It has been solved only for some subclasses of cubic polynomial differential systems, thus the authors of [13] provided necessary and sufficient conditions to determine when the origin of the differential system (2) is a center. Their result is the following.

Theorem 1 *The cubic polynomial differential systems* (2) *have a center at the origin if, and only if, one of the following four sets of conditions is satisfied:*

(i) $c_2 = 0$, $b_2d_1 + d_2b_1 = 0$, $3b_1 - d_1 = 0$;
(ii) $c_2 = 0$, $b_2d_1 + d_2b_1 = 0$, $F = a_2^2d_2^3 - 3a_2^2d_2d_1^2 + 6a_2a_1d_2^2d_1 - 2a_2a_1d_1^3 - a_1^2d_2^3 + 3a_1^2d_2d_1^2 = 0$;
(iii) $c_1 = c_2 = 0$, $b_2 - d_2 = b_1 + d_1 = 0$;
(iv) $c_2 = 0 = d_1 = d_2 = 0$, $G = -a_2^2b_2^3 + 3a_2^2b_2b_1^2 + 6a_2a_1b_2^2b_1 - 2a_2a_1b_1^3 + a_1^2b_2^3 - 3a_1^2b_2b_1^2$.

There are several works studying the global centers of different classes of polynomial differential systems, for example in [7] and [10] the authors proved that polynomial differential systems of even degree cannot have global centers because such systems always have orbits coming from and going to infinity. However to classify all polynomial differential systems of odd degree having a global center is a very difficult problem. This last problem was proposed by Conti in [4], and in fact up to now only few partial results exist for certain families of polynomial differential systems of odd degree, for more details see, for instance [8, 9, 11].

Roughly speaking the Poincaré compactification, first, consists in identifying the plane $\mathbb{R}^2$ with the interior of unit closed disc $\mathbb{D}^2$ centered at the origin of coordinates, and the boundary of this disc, the circle $\mathbb{S}^1$, with the infinity of $\mathbb{R}^2$. After the polynomial differential system defined in $\mathbb{R}^2$ is extended analytically to the whole closed disc $\mathbb{D}^2$. In this way we can study the dynamics of the polynomial differential systems in a neighbourhood of the infinity. All the details on the Poincaré compactification can be found in [6, Chapter 5].

We emphasize that in order that the origin of a cubic polynomial differential system (2) be a global center, that system either does not have equilibrium points at infinity in the Poincaré compactification, or the local phase portraits of all its infinite equilibrium points are formed by two hyperbolic sectors having their two separatrices at infinity, see Fig. 1. This implies that the Jacobian matrix at any infinite equilibrium point of system (2) must be identically zero, otherwise the infinite equilibrium point would be either hyperbolic, or semi-hyperbolic, or nilpotent, and it is known that the local phase portraits of such kind of equilibrium points

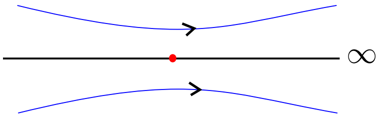

Fig. 1 An equilibrium point on the infinity line whose local phase portrait is formed by two hyperbolic sectors whose two separatrices are contained on the infinity line

are not formed by two hyperbolic sectors having their two separatrices contained at infinity, for details see Theorems 2.15, 2.19 and 3.5 of [6].

Using Theorem 1 we prove the main result of this article, stated in the next theorem.

Theorem 2 *The polynomial differential systems* (2) *have a global center at the origin of coordinates if, and only if, one of the following sets of conditions holds:*

(a) $a_1 = a_2 = b_2 = c_2 = d_2 = 0$, $d_1 = 3b_1$, $b_1 = -c_1/4$, $c_1 < 0$;
(b) $a_1 = a_2 = b_2 = c_1 = c_2 = d_2 = 0$, $d_1 = 3b_1$, $b_1 < 0$;
(c) $a_2 = b_1 = b_2 = c_2 = d_1 = d_2 = 0$, $a_1^2 + c_1 < 0$;
(d) $a_1 = a_2 = b_1 = b_2 = c_1 = c_2 = d_1 = d_2 = 0$;
(e) $a_1 = a_2 = b_2 = c_2 = d_2 = 0$, $d_1 = -b_1 - c_1 \neq 3b_1$, $2b_1 + c_1 \leq 0$, $b_1 > 0$;
(f) $a_2 = b_2 = c_1 = c_2 = d_2 = 0$, $b_1 \neq 0$, $a_1^2 + 3b_1 - d_1 < 0$, $a_1^2 + 4(b_1 + d_1) < 0$;
(g) $a_1 = a_2 = b_1 = b_2 = c_2 = d_2 = 0$, $d_1 = -c_1 > 0$.

The paper is organized as follows. In Sect. 2 we present some basic results on the Poincaré compactification for polynomial vector fields in the plane $\mathbb{R}^2$, on the characterization of global centers, and on the vertical blow up's. In Sect. 3 we prove our main result (Theorem 2) about the global centers of system (2).

2 Preliminaries

This section is devoted to establishing some basic results that will be used for proving Theorem 2.

2.1 Poincaré compactification for vector fields in the plane

To study the global dynamics of a planar polynomial differential system $X = (P, Q)$, we must classify the local phase portraits of its finite and infinite equilibrium points in the Poincaré disc.

Let $\mathbb{S}^2 = \{\mathbf{z} \in \mathbb{R}^3 : ||\mathbf{z}|| = 1\}$ be the unit sphere of $\mathbb{R}^3$. We know that the polynomial vector field X induces an analytic vector field in $\mathbb{S}^2$, that we denote by $p(X)$, see for instance [6, Chaper 5] or [3].

The vector field $p(X)$ allows to study the dynamics of the vector field X in the neighbourhood of infinity, that is, in the neighbourhood of the equator $\mathbb{S}^1 = \{\mathbf{z} \in \mathbb{S}^2 : z_3 = 0\}$.

To obtain the analytical expression for $p(X)$ we shall consider the sphere as a smooth manifold. We choose the six local charts given by $U_i = \{\mathbf{z} \in \mathbb{S}^2 : z_i > 0\}$ and $V_i = \{\mathbf{z} \in \mathbb{S}^2 : z_i < 0\}$, for $i = 1, 2, 3$, with the corresponding coordinate maps $\varphi_i : U_i \to \mathbb{R}^2$ and $\psi_i : V_i \to \mathbb{R}^2$, defined by $\varphi_k(\mathbf{z}) = \psi_k(\mathbf{z}) = (z_m/z_k, z_n/z_k)$ for $m < n$ and $m, n \neq k$. Denote by (u, v) the local coordinates on U_i and V_i for $i = 1, 2, 3$. From [6, Chaper 5] the

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Slow-Fast Normal Forms Arising from Piecewise Smooth Vector Fields

Otavio Henrique Perez[1] · **Gabriel Rondón**[2] · **Paulo Ricardo da Silva**[2]

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Abstract
We study planar piecewise smooth differential systems of the form

$$\dot{z} = Z(z) = \frac{1 + \operatorname{sgn}(F)}{2} X(z) + \frac{1 - \operatorname{sgn}(F)}{2} Y(z),$$

where $F : \mathbb{R}^2 \to \mathbb{R}$ is a smooth map having 0 as a regular value. We consider linear regularizations Z_ε^φ of Z by replacing $\operatorname{sgn}(F)$ by $\varphi(F/\varepsilon)$ in the last equation, with $\varepsilon > 0$ small and φ being a transition function (not necessarily monotonic). Nonlinear regularizations of the vector field Z whose transition function is monotonic are considered too. It is a well-known fact that the regularized system is a slow-fast system. In this paper, we study typical singularities of slow-fast systems that arise from (linear or nonlinear) regularizations, namely, fold, transcritical and pitchfork singularities. Furthermore, the dependence of the slow-fast system on the graphical properties of the transition function is investigated.

Keywords Piecewise smooth vector fields · Geometric singular perturbation theory · Regularization of piecewise smooth vector fields · Transition function

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Dedicated to the memory of Jorge Sotomayor Tello

✉ Gabriel Rondón
gabriel.rondon@unesp.br

Otavio Henrique Perez
otavio.perez@icmc.usp.br

Paulo Ricardo da Silva
paulo.r.silva@unesp.br

[1] Institute of Mathematics and Computer Science, University of São Paulo (USP), Avenida Trabalhador São Carlense, 400, CEP 13566-590 São Carlos, São Paulo, Brazil

[2] Institute of Biosciences, Humanities and Exact Sciences, São Paulo State University (UNESP), Rua C. Colombo, 2265, CEP 15054–000 S. J. Rio Preto, São Paulo, Brazil

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1 Introduction

In real life, there are phenomena whose mathematical models are expressed by piecewise smooth vector fields, which have been studied at least since 1937. These systems are used in many branches of applied sciences, for example, Physics, Control Theory, Economics, Cell Mitosis, etc. For more details see, for instance, [3, 5].

A piecewise smooth vector field (or PSVF for short) is defined as follows: let Σ be a closed subset with empty interior of the ambient space (for example, a manifold embedded in $\mathbb{R}^n$). Such subset is called *discontinuity locus*, and it divides the ambient space in finitely many open subsets $\{U_i\}_{i=1}^k$. In each open subset, U_i is defined a smooth vector field. This paper deals with the case where a smooth curve divides a neighbourhood of $0 \in \mathbb{R}^2$ in two open regions. See Section 2 for a precise definition.

One of the most important question concerning PSVF's is how to define the dynamics in Σ? In other words, how to define the transition between the dynamics defined in two different open sets?

Filippov [5] gave an answer defining the dynamics in Σ as the convex combination of two vector fields. This defines the so called *sliding vector field*. We say that this vector field defined according to Filippov's ideas follows the *Filippov's convention*.

However, for some models, the Filippov's convention is not sufficient to describe the dynamics. For example, in [15], a model involving friction between an object and a flat surface was studied. The author gave an example that Filippov's convention takes into account only kinetic friction, while it is possible to consider static friction as well.

Another way to define the dynamics in the discontinuity locus Σ is combining two powerful tools: regularizations of PSVF's and blow-ups. A regularization process that is compatible with the Filippov's convention is the *Sotomayor-Teixeira regularization* [22], which consists in obtaining a one-parameter family of smooth vector fields Z_ε converging to Z when $\varepsilon \to 0$ (see Section 2.2). By using blow-up techniques, the regularized system $\dot{z} = Z_\varepsilon(z)$ becomes a slow-fast system, and therefore, we are able to apply classical results on geometric singular perturbation theory (see Section 2.4) in the study of PSVF's. Such a link between regularization processes and geometric singular perturbation theory is a recent approach in mathematics, and we refer to [1, 16–20] for further details. A similar approach can also be seen in [10].

Different regularization processes lead to different slow-fast systems, which gives rise to different sliding or sewing regions (see [19–21]). In this paper, we consider *linear* regularizations and *nonlinear* regularizations. See Sections 2.2 and 2.3 for precise definitions.

The dynamics of the *linearly* regularized system depends on the so called *transition function* φ, which can be monotonic or non monotonic. In this paper, we highlighted the relation between the properties of the graph of φ, the properties of the slow-fast system, and the sliding regions of the PSVF's. See Theorem A below.

The main goal of this paper is to study typical singularities of slow-fast systems that arise from (linear or nonlinear) regularizations. For both linear and nonlinear regularizations are presented examples of PSVF's such that, after (linear or nonlinear) regularization and directional blow-up, the slow-fast system presents normally hyperbolic, fold, transcritical or pitchfork singularities.

At some point, the reader may think that, after *linear* regularization and blow-up, it is possible to generate any slow-fast singularity, since it is just a matter of a suitable choice of the transition function. In general, this is not true. Indeed, we show that it does not exist a transition function that generate a pitchfork singularity. However, if we consider *nonlinear* *regularizations*, it is possible to generate such a singularity (see Example 13). This shows that nonlinear regularizations are more general than the linear ones (see also [18, 20]).

Our main results, Theorems A, B and C, are stated and proved in Section 3. In what follows, we briefly describe them.

Firstly, consider linear regularizations. Suppose that we drop the monotonicity condition of the transition function φ. In this context, we will prove that the critical points of φ give rise to non normally hyperbolic points of the critical set C_0 of $\dot{z} = Z_\varepsilon(z)$. For more details, see Item (a) of Theorem A.

In addition, item (b) of Theorem A assures that we extend the classical Filippov sliding region when the transition function satisfy $|\varphi(x_0)| > 1$ for some x_0 in the open interval $(-1, 1)$. According to item (c) of the same Theorem, the dynamics in this extended sliding region is naturally defined using the classical Filippov sliding vector field. It is important to emphasize that item (c) of Theorem A was already proved in [21, Theorem 3]. For completness sake, we incorporated it in the statement of Theorem A and proved it as well.

Finally, item (d) of Theorem A says that there are cases in which it is not possible to apply geometric singular perturbation theory in order to define the sliding dynamics in some points of Σ (see Fig. 1).

Slow-fast normal forms are well known in the literature (see Section 2.5 and the references therein). In Theorem B, we state conditions that both PSVF and transition function must satisfy in order to generate classical slow-fast normal forms, such as fold and transcritical singularities. Moreover, we prove that there are slow-fast normal forms that can not be generated by *linear* regularization processes. This is the case of the pitchfork singularity (see Fig. 2).

In order to generate pitchfork singularities, we must consider nonlinear regularization. Theorem C gives the conditions that must satisfy both monotonic transition function and vector field associated with the nonlinearly regularized system to generate this type of singularity.

Fold, transcritical and pitchfork singularities have very interesting dynamical properties. For example, M. Krupa, P. Szmolyan in [11, 12] studied the dynamics of the slow-fast system around this type of singularities for $\varepsilon > 0$ and built a map of transition between transversal sections. By applying these results together with Theorems B and C, one can determine the local dynamics of the system regularized around these singularities, and thus, it is possible to make a global study of the dynamics of these systems.

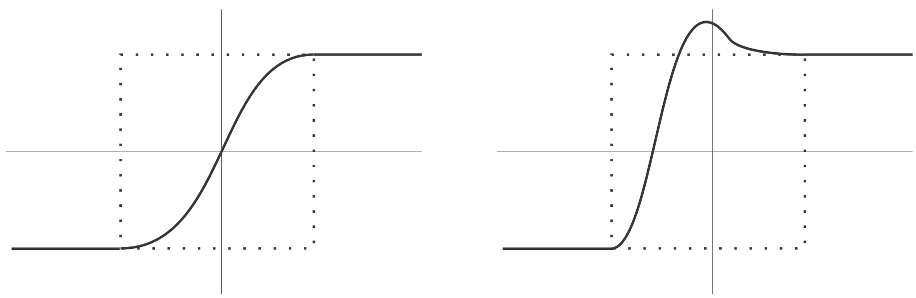

Fig. 1 Monotonic transition function (left) and non monotonic transition function (right). The monotonic one generates only normally hyperbolic critical sets, and the sliding region coincides with the one proposed by Filippov. The non monotonic one has a critical point, which generates a non normally hyperbolic point of the critical manifold. Moreover, in this example, such a transition function extends the classical notion of sliding region

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Global Phase Portrait and Local Integrability of Holomorphic Systems

Luiz F. S. Gouveia[1] · Paulo R. da Silva[1] · Gabriel Rondón[1]

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Abstract

Planar holomorphic systems $\dot{x} = u(x, y)$, $\dot{y} = v(x, y)$ are those that $u = \mathrm{Re}(f)$ and $v = \mathrm{Im}(f)$ for some holomorphic function $f(z)$. They have important dynamical properties, highlighting, for example, the fact that they do not have limit cycles and that center-focus problem is trivial. In particular, the hypothesis that a polynomial system is holomorphic reduces the number of parameters of the system. Although a polynomial system of degree n depends on $n^2 + 3n + 2$ parameters, a polynomial holomorphic depends only on $2n+2$ parameters. In this work, in addition to prove that holomorphic systems are locally integrable, we classify all the possible global phase portraits, on the Poincaré disk, of systems $\dot{z} = f(z)$ and $\dot{z} = 1/f(z)$, where $f(z)$ is a polynomial of degree 2, 3 and 4 in the variable $z \in \mathbb{C}$. We also classify all the possible global phase portraits of Moebius systems $\dot{z} = \frac{Az+B}{Cz+D}$, where $A, B, C, D \in \mathbb{C}$, $AD - BC \neq 0$.

Mathematics Subject Classification 32A10 · 34A34 · 34C20 · 37C10

1 Introduction

The understanding of the phase portrait of planar differential systems involves some questions:

- How is the local behavior around the equilibria points? A serious problem consists in distinguishing between a focus and a center.

✉ Luiz F. S. Gouveia
paulo.r.silva@unesp.br

Paulo R. da Silva
fernando.gouveia@unesp.br

Gabriel Rondón
gabriel.rondon@unesp.br

[1] Departamento de Matemática - Instituto de Biociências, Letras e Ciências Exatas, UNESP - Universidade Estadual Paulista, Rua C. Colombo, 2265, Sao José do Rio Preto, São Paulo CEP 15054-000, Brazil

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- After equilibria points the main subjects are limit cycles, i.e., periodic orbits that are isolated in the set of all periodic orbits of a differential system. How many limit cycles are there in the phase portrait?
- A first integral completely determines its phase portrait. Given a vector field on $\mathbb{R}^2$, how can one determine if this vector field has a first integral?

These problems are unsolved in general. However for a specific class of systems we get very satisfactory answers. The class we are referring to is the one formed by holomorphic systems.

Planar holomorphic systems $\dot{x} = u(x, y)$, $\dot{y} = v(x, y)$ are those that $u = \text{Re}(f)$ and $v = \text{Im}(f)$ for some holomorphic function $f(z)$.

Holomorphic systems have surprising properties:

- the equilibria are isolated and the topological classification of the local phase portraits is fully known,
- they do not have limit cycles,
- the center–focus problem is totally solved,
- there is a first integral $H(x, y)$ defined in a subset of total measure on $\mathbb{R}^2$.

A *holomorphic function* f is a complex-valued function defined in a domain $\mathcal{V} \subseteq \mathbb{C}$ and satisfying that

- $u = \text{Re}(f)$ and $v = \text{Im}(f)$ are continuous,
- there exist the partial derivatives u_x, u_y, v_x, v_y in $\mathcal{V}$ and,
- the partial derivatives satisfy the Cauchy–Riemann equations, see [13],

$$u_x = v_y, \quad u_y = -v_x, \quad \forall z = x + iy \in \mathcal{V}.$$

We remark that *Looman–Menchoff's Theorem* states that the above conditions are sufficient to guarantee the analyticity of f. It means that for any $z_0 \in \mathcal{V}$

$$f(z) = A_0 + A_1(z - z_0) + A_2(z - z_0)^2 + \ldots, \quad A_k = a_k + ib_k = \frac{f^{(k)}(z_0)}{k!} \tag{1}$$

for $z \in D(z_0, R_{z_0}) \subseteq \mathcal{V}$ where $D(z_0, R_{z_0})$ is the largest possible z_0-centered disk contained in $\mathcal{V}$. Unless a translation we can always assume that $z_0 = 0$.

If f is holomorphic in a punctured disc $D(0, R)\backslash\{0\}$ and it is not derivable at 0 we say that 0 is a singularity of f. In this case $f(z)$ is equal to its Laurent's series in $D(0, R)\backslash\{0\}$

$$f(z) = \sum_{k=1}^{\infty} \frac{B_k}{z^k} + \sum_{k=0}^{\infty} A_k z^k, \tag{2}$$

where $B_k = \dfrac{1}{2\pi i} \int_{C_\varepsilon} f(z) z^{k-1} dz, \quad A_k = \dfrac{1}{2\pi i} \int_{C_\varepsilon} \dfrac{f(z)}{z^{k+1}} dz$ with C_ε parameterized by $z(t) = \varepsilon e^{it}$, $\varepsilon \sim 0$, $t \in [0, 2\pi]$.

If $B_k \neq 0$ for an infinite set of indices k we say that 0 is an *essential singularity* and if there exists $n \geq 1$ such that $B_n \neq 0$ and $B_k = 0$ for every $k > n$ then we say that 0 is a *pole of order* n. Moreover B_1 is called *residue* of f at 0 and it is denoted by $B_1 = \operatorname{res}(f, 0)$.

Let $f : D(0, R)\backslash\{0\} \to \mathbb{C}$ be a holomorphic function as (2). Consider the ordinary differential equation

$$\dot{z}(t) = f(z(t)), \quad t \in \mathbb{R}. \tag{3}$$

The solution of (3) passing through $z \in D(0, R)\backslash\{0\}$ at $t = 0$ is denoted by $\varphi_f(t, z) = x(t) + iy(t)$.

The real and imaginary parts of $\varphi_f(t, z)$ must satisfy the following system

$$\begin{cases} \dot{x} = \displaystyle\sum_{k=1}^{\infty}\left(c_k \frac{p_k}{(x^2+y^2)^k} + d_k \frac{q_k}{(x^2+y^2)^k}\right) + a_0 + \sum_{k=1}^{\infty}(a_k p_k - b_k q_k) \\ \dot{y} = \displaystyle\sum_{k=1}^{\infty}\left(d_k \frac{p_k}{(x^2+y^2)^k} - c_k \frac{q_k}{(x^2+y^2)^k}\right) + b_0 + \sum_{k=1}^{\infty}(b_k p_k + a_k q_k) \end{cases} \tag{4}$$

with $p_k = \operatorname{Re}(x + iy)^k$, $q_k = \operatorname{Im}(x + iy)^k$.

Indeed. Taking $f(z) = \sum_{k=1}^{\infty} \frac{c_k + id_k}{z^k} + \sum_{k=0}^{\infty}(a_k + ib_k)z^k$. A direct calculation using Newton's binomial formula gives us

$$z^k = (x + iy)^k = p_k + iq_k.$$

Thus

$$(a_k + ib_k)z^k = (a_k p_k - b_k q_k) + i(b_k p_k + a_k q_k)$$

and

$$\begin{aligned} \frac{c_k + id_k}{z^k} &= (c_k + id_k)\frac{\bar{z}^k}{|z|^{2k}} = \frac{(c_k + id_k)}{(x^2+y^2)^k}(p_k - iq_k) \\ &= \frac{1}{(x^2+y^2)^k}\left((c_k p_k + d_k q_k) + i(d_k p_k - c_k q_k)\right). \end{aligned}$$

Since $\dot{x} = \operatorname{Re}(f(z))$ and $\dot{y} = \operatorname{Im}(f(z))$ we prove that the real and imaginary parts of $\varphi_f(t, z)$ satisfy system (4).

Proposition 1 *If* $A_k, B_k \in \mathbb{R}$ *then system* (4) *is reversible with involution* $R(x, y) = (x, -y)$.

Proof As $A_k, B_k \in \mathbb{R}$, it follows that b_k and d_k are equal zero. Then, from equation (4), $\dot{x}$ only has polynomials of p_k type which is even in the variable y and $\dot{y}$ only has polynomials of q_k type which is odd in variable y. In other words, the system is reversible with involution $R(x, y) = (x, -y)$. ◻

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On limit cycles in regularized Filippov systems bifurcating from homoclinic-like connections to regular-tangential singularities

Douglas D. Novaes [a], Gabriel Rondón [b,*]

[a] Departamento de Matemática, Instituto de Matemática, Estatística e Computação Científica (IMECC), Universidade Estadual de Campinas (UNICAMP), Rua Sérgio Buarque de Holanda, 651, Cidade Universitária Zeferino Vaz, 13083–859, Campinas, SP, Brazil
[b] UNESP - Universidade Estadual Paulista, São José do Rio Preto, São Paulo, Brazil

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In this paper, we are concerned about smoothing of Filippov systems around homoclinic-like connections to regular-tangential singularities. We provide conditions to guarantee the existence of limit cycles bifurcating from such connections. Additional conditions are also provided to ensure the stability and uniqueness of such limit cycles. All the proofs are based on the construction of the first return map of the regularized Filippov system around homoclinic-like connections. Such a map is obtained by using a recent characterization of the local behavior of the regularized Filippov system around regular-tangential singularities. Fixed point theorems and Poincaré–Bendixson arguments are also employed.

1. Introduction

The increasing interest in the study of piecewise smooth vector fields arises naturally due to the number of applications in different areas of knowledge, such as engineering, physics, economics, and biology. For more details see the books [1,2] and the references therein.

In this work, we are interested in planar piecewise smooth vector fields. More precisely, let $\mathcal{D}$ be an open set in $\mathbb{R}^2$ and $h:\mathcal{D}\to\mathbb{R}$ be a C^∞ function having 0 as a regular value. We assume for simplicity that $\Sigma = h^{-1}(0)$ has a single connected component so that $M\backslash\mathcal{D}$ has two connected components, denoted by $\Sigma^+ = h^{-1}(0,\infty)$ and $\Sigma^- = h^{-1}(-\infty,0)$. Accordingly, the piecewise smooth vector field in $\mathbb{R}^2$ is defined by

$$Z(p) = (X^+, X^-)_\Sigma = \begin{cases} X^+(p), & \text{if} \quad p\in\Sigma^+, \\ X^-(p), & \text{if} \quad p\in\Sigma^-, \end{cases} \quad \text{for} \quad p\in\mathcal{D}. \quad (1)$$

We suppose that the vector fields X^+ and X^- have an extension to Σ which is, at least C^2.

Throughout this paper we will denote the components of $X^\pm$ by $X_i^\pm$ for each $i\in\{1,2\}$, i.e. $X^\pm = (X_1^\pm, X_2^\pm)$.

1.1. Filippov's convention

In [3], the local trajectories of piecewise smooth vector fields (1) were defined as solutions of a differential inclusion $\dot p \in \mathcal{F}_Z(p)$. This approach is called Filippov's convention. The piecewise smooth vector field (1) is called a Filippov system when its dynamics is provided by the Filippov's convention.

In order to illustrate the Filippov's convention we define the following open regions on Σ :

(i) $\Sigma^s = \{p\in\Sigma : X^+h(p)\cdot X^-h(p) < 0\}$,
(ii) $\Sigma^c = \{p\in\Sigma : X^+h(p)\cdot X^-h(p) > 0\}$,

where $X^\pm h(p) = \langle \nabla h(p), X^\pm(p)\rangle$ denotes the Lie derivative of h in the direction of the vector fields $X^\pm$. In addition, $(X^\pm)^i h(p) = X^\pm((X^\pm)^{i-1}h)(p)$ for $i>1$.

The set Σ^c is called *crossing region*. For $p\in\Sigma^c$, the trajectories from either side of the discontinuity Σ, reaching p, can be joined continuously, forming a trajectory that crosses Σ^c.

The set Σ^s is called *sliding region* and both vectors fields X^+ and X^- simultaneously point inward to or outward from Σ. We consider the following *sliding vector field* defined on Σ^s :

$$Z^s(p) = \frac{X^-h(p)X^+(p) - X^+h(p)X^-(p)}{X^-h(p) - X^+h(p)}, \quad \text{for } p\in\Sigma^s.$$

Thus, the trajectories from either side of the discontinuity Σ reaching $p\in\Sigma^s$ can be joined continuously to trajectories that slide on Σ^s following $Z^s(p)$. In addition, a singularity of the sliding vector field Z^s is called pseudo-equilibrium.

* Corresponding author.
E-mail addresses: ddnovaes@unicamp.br (D.D. Novaes), gabriel.rondon@unesp.br (G. Rondón).

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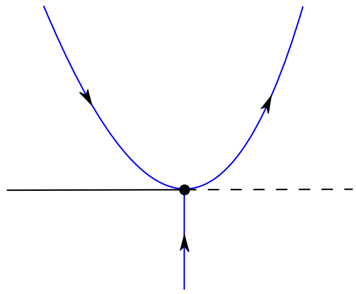

Fig. 1. Visible regular-tangential singularity.

In the Filippov context, the notion of *Σ-singular points* also contains the tangential points Σ^t constituted by the contact points between X^+ and X^- with Σ, i.e. $\Sigma^t = \{p \in \Sigma : X^+h(p) \cdot X^-h(p) = 0\}$. In this paper, we are interested in contact points of finite degeneracy. Recall that p is a *contact point of order* $k-1$ (or multiplicity k) between $X^\pm$ and Σ if 0 is a root of multiplicity k of $f(t) := h \circ \varphi_{X^\pm}(t,p)$, where $t \mapsto \varphi_{X^\pm}(t,p)$ is the trajectory of $X^\pm$ starting at p. Equivalently,

$$X^\pm h(p) = (X^\pm)^2 h(p) = \cdots = (X^\pm)^{k-1} h(p) = 0, \text{ and}$$
$$(X^\pm)^k h(p) \neq 0.$$

In addition, an even multiplicity contact, say $2k$, is called *visible* for X^+ (resp. X^-) when $(X^+)^{2k}h(p) > 0$ (resp. $(X^-)^{2k}h(p) < 0$). Otherwise, it is called *invisible*.

In this work, we shall focus our attention in *visible regular-tangential singularities of multiplicity* $2k$, which are formed by a visible even multiplicity tangential point of X^+ and a regular point of X^-, or vice versa (see Fig. 1).

1.2. Sotomayor–Teixeira regularization

Frequently, non-smooth mathematical models are discontinuous idealizations of regular phenomena. Thus, in order to investigate a regular phenomenon with non-smooth mathematical model, it seems natural to inquire how the solutions of smooth systems, which converge to the non-smooth one, behave in the limit. Therefore, it is usual to consider a non-smooth system as a singular limit of a 1-parameter family of smooth vector fields.

The concept of smoothing of a piecewise smooth vector field Z was firstly introduced by Sotomayor and Teixeira in [4] and consists in obtaining a one-parameter family of smooth vector fields Z_ε converging to Z when ε converges to 0. More precisely, consider a C^∞ function $\phi : \mathbb{R} \to \mathbb{R}$ satisfying $\phi(\pm 1) = \pm 1$, $\phi^{(i)}(\pm 1) = 0$ for $i = 1, 2, \dots, n$, and $\phi'(s) > 0$ for $s \in (-1, 1)$. Then, a C^n*-monotonic transition function* is defined as follows:

$$\Phi(s) = \begin{cases} \phi(s) & \text{if} \quad |s| \leqslant 1, \\ \operatorname{sign}(s) & \text{if} \quad |s| \geqslant 1. \end{cases} \tag{2}$$

The set of C^n-monotonic transition functions Φ which are not C^{n+1} at ± 1 is denoted by C^n_{ST}. Hence, the Sotomayor–Teixeira C^n-regularization of Z is the following one-parameter family of smooth vector fields Z^Φ_ε

$$Z^\Phi_\varepsilon(p) = \frac{1 + \Phi(h/\varepsilon)}{2} X^+(p) + \frac{1 - \Phi(h/\varepsilon)}{2} X^-(p), \text{ for } \varepsilon > 0. \tag{3}$$

Notice that the regularized vector field $Z^\Phi_\varepsilon(p)$ coincides with $X^+(p)$ or $X^-(p)$ whether $h(p) \geqslant \varepsilon$ or $h(p) \leqslant -\varepsilon$, respectively. In the region $|h(p)| \leqslant \varepsilon$, the vector $Z^\Phi_\varepsilon(p)$ is a linear combination of $X^+(p)$ and $X^-(p)$.

The Sotomayor–Teixeira regularization posses an intrinsic relation with the Filippov's convention. Indeed, in [5] the authors proved that the Sotomayor–Teixeira regularization of Filippov systems gives rise to *Singular Perturbation Problems*, for which the corresponding reduced dynamics is conjugated to the sliding dynamics (3). Afterwards, this relation has been proven to hold for more general classes of transition functions, which include analytic [6,7] and non-monotonic [8,9] transition functions. In [10], the authors studied how visible regular-tangential singularities evolve under Sotomayor–Teixeira regularization. This represented a first step towards the complete understanding of how tangential singularities behave under more general regularization processes. In the present paper, we shall use the local results provided by [10] in order to understand the Sotomayor–Teixeira regularization of Σ-Polycycles, which are simple closed curves composed by a collection of singularities and regular orbits, inducing a first return map.

1.3. Σ-polycycles in Filippov systems

In these last decades, the study of Σ-polycycles in Filippov systems have been considered in several papers. For instance, in [11] the authors introduced the critical crossing cycle bifurcation, which is defined as a one-parameter family Z_α of Filippov systems, where Z_0 has a homoclinic-like connection to a fold-regular singularity. In [12], Freire et al. proved that the unfolding of a critical crossing cycle bifurcation provided in [11] holds in a generic scenario. More degenerate homoclinic-like connections to Σ-singularities has also been considered. In [13], the authors studied a codimension-two homoclinic-like connection to a visible–visible fold–fold singularity. In [14] its unfolding under non-autonomous periodic perturbation has been addressed. Recently, in [15], Andrade et al. developed a rather general method to investigate the unfolding of Σ-polycycles in Filippov systems. This method was applied to describe bifurcation diagrams of Filippov systems around several Σ-polycycles. The readers are referred to [16–20] for more studies on Σ-polycycles. The interest in studying Σ-polycycles is due to the fact that they are non-local invariant sets that provide information on the dynamics of the system.

In this paper, we consider Σ-polycycles having visible regular-tangential singularities of multiplicity $2k$, which are formed by a visible even multiplicity tangential point of X^+ and a regular point of X^-. In the process of regularization, we consider C^n- monotonic transition functions. We are concerned to establish conditions for the existence of limit cycles of the regularized system Z^Φ_ε when we consider a particular class of Filippov system, which possesses Σ-polycycles with homoclinic-like connections to visible regular-tangential singularities. In addition, we want to know if it is possible to determine the number of limit cycles that can appear for the regularized system Z^Φ_ε.

Throughout this article we will use the results obtained in [10] on the dynamics of the regularized system around visible regular-tangential singularities of multiplicity $2k$. More specifically, the authors obtained upper and lower transition maps near tangential-regular singularities defined by the flow of the regularized system (see Fig. 2), which we will use to construct the first return map of the regularized system Z^Φ_ε. We will employ this map to prove the existence of limit cycles of Z^Φ_ε.

Our two main results, Theorems A and B provide sufficient conditions for the existence of an asymptotically stable limit cycle of the regularized system bifurcating from a Σ-polycycle of a Filippov system with degenerated contact with the switching manifold. Additional conditions on Theorem A ensuring the uniqueness of such a limit cycle will be provided in Section 2.2.

We recall that the approach on Σ-polycycles passing through visible regular-tangencies of even multiplicity is the main novelty of the present study and this extends the results obtained

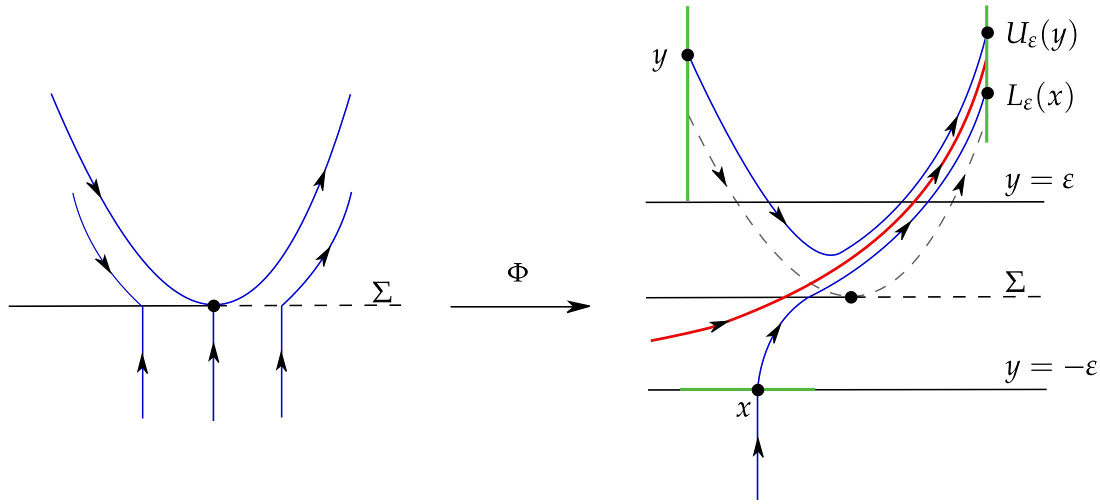

Fig. 2. Upper Transition Map $U_\varepsilon(y)$ and Lower Transition Map $L_\varepsilon(y)$ defined for C^n-regularizations of Filippov systems around visible regular-tangential singularities of multiplicity $2k$.

by [6,21] for the simplest class of Σ-polycycles, namely boundary limit cycle with fold contact with the switching manifold. Specifically, Theorem A generalizes the problem addressed in [21], which was also considered in [6] to a more general class of regularizations. It is worth mentioning that in [10, Theorem C] the authors considered the boundary limit cycles with visible even multiplicity contact points with Σ and we highlight that Theorem A generalizes that result. However, Theorem B has not appeared in previous articles.

In what follows, we shall introduce some basic concepts for this article. First, we define the local separatrix at a point $p \in \Sigma$.

Definition 1. If $p \in \Sigma$, the **asymptotically stable (resp. unstable) separatrix** $W^s_{t,\overline{m}}(p)$, (resp. $W^u_{t,\overline{m}}(p)$) of $Z = (X^+, X^-)$ at a visible regular-tangential singularity p in $\Sigma^{\pm}$ is defined as

$$W^{s,u}_t(p) = \{q = \varphi_{X^+}(t(q), p) : \varphi_{X^+}(I(q), p) \subset \Sigma^+ \text{ and } \delta_{s,u} t(q) > 0\},$$
$$W^{s,u}_{\overline{m}}(p) = \{q = \varphi_{X^-}(t(q), p) : \varphi_{X^-}(I(q), p) \subset \Sigma^- \text{ and } \delta_{s,u} t(q) > 0\},$$

where, $\delta_u = 1$, $\delta_s = -1$, and $I(q)$ is the open interval with extrema 0 and $t(q)$.

Now, we introduce the concepts of regular orbit and Σ-polycycle for planar Filippov systems.

Definition 2. Consider the Filippov system $Z = (X^+, X^-)$.

(i) We say that γ is a **regular orbit** of Z if it is a piecewise smooth curve such that $\gamma \cap \Sigma^+$ and $\gamma \cap \Sigma^-$ are unions of regular orbits of X^+ and X^-, respectively, and $\gamma \cap \Sigma \subset \Sigma^c$.

(ii) A closed curve Γ is said to be a Σ-**polycycle** of Z if it is composed by a finite number of Σ-singularities, $p_1, p_2, \dots, p_n \in \Sigma$, and a finite number of regular orbits of Z, $\gamma_1, \gamma_2, \dots, \gamma_n$, such that for each $1 \leqslant i \leqslant n$, γ_i has ending points p_i and p_{i+1}, and satisfies that

   (a) Γ is a S^1-immersion and it is oriented by increasing time along the regular orbits;
   (b) there exists a non-constant first return map π_Γ defined, at least, in one side of Γ.

Remark 1. Condition (a) in Definition 2 gives the minimality of Σ-polycycles of Z, i.e. a Σ-polycycle Γ cannot be written as union of two or more Σ-polycycles. This condition also establishes the notion of sides of Γ, which is used in Condition (b).

Our main interest in this study consists in generalizing the results obtained in [10,21] for a type of Σ-polycycles, which we call homoclinic-like Σ-polycycles through a unique Σ-singularity of regular-tangential type of planar Filippov systems (see Fig. 3). In what follows, we divide this special type of Σ-polycycles into 2 classes. In order to define them, consider the Filippov system $Z = (X^+, X^-)$ and suppose that Z has a Σ-polycycle Γ having a unique visible regular-tangential singularity at p of multiplicity $2k$, for $k \geqslant 1$. Through a local change of coordinates, we can assume that $p = (0, 0)$ and $h(x, y) = y$. Now, we shall locally characterize Γ around p. For this, it is enough to divide the homoclinic-like Σ-polycycles at p of Z in the following 2 classes:

(a) $W^s_t(p) \cup W^u_t(p) \subset \Gamma$ and $X^-h(p) > 0$ or $X^-h(p) < 0$;
(b) $W^s_{\overline{m}}(p) \cup W^u_t(p) \subset \Gamma$ or $W^u_{\overline{m}}(p) \cup W^s_t(p) \subset \Gamma$.

Here, Σ-polycycles satisfying condition (a) or (b) will be called Σ-polycycles of type (a) or type (b), respectively (see Fig. 3).

1.4. Main results and structure of the paper

The first main result of this paper (Theorem A) is concerned with regularization of Filippov systems having a Σ-polycycle of type (a). It establishes sufficient conditions for the existence of limit cycles of the regularized system Z^Φ_ε passing through of a certain compact set with nonempty interior. When the limit cycle exists, its stability is characterized and its convergence to the Σ-polycycle is ensured. Theorem A is stated and proven in Section 2. In Section 2.2, we study cases of uniqueness and nonexistence of limit cycles for the regularization of Σ-polycycles of type (a). More specifically, in Proposition 1 we provide a class of piecewise smooth vector fields having a Σ-polycycle of type (a) for which its regularization either does not admit limit cycle or admit a unique limit cycle converging to the Σ-polycycle.

The second main result of this paper (Theorem B) is concerned with regularization of Filippov systems having a Σ-polycycle of type (b). It establishes sufficient conditions for the existence of limit cycles of the regularized system Z^Φ_ε converging to the Σ-polycycle. Theorem B is stated and proven in Section 3.

The proofs of Theorems A and B are mainly based on the characterization of the upper and lower transition maps around regular-tangential singularities, fixed point theorems, and *Poincaré–Bendixson Theorem*.

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Piecewise holomorphic systems

Luiz F.S. Gouveia, Gabriel Rondón *, Paulo R. da Silva

São Paulo State University (Unesp), Institute of Biosciences, Humanities and Exact Sciences, Rua C. Colombo, 2265, CEP 15054-000, S. J. Rio Preto, São Paulo, Brazil

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Abstract

Holomorphic systems are well known in the literature due to their important properties: reversibility, integrability, non-existence of limit cycles and complete knowledge of the phase portraits around their non-essential singularities. In this paper we are concerned about studying the piecewise holomorphic systems (PWHS). Specifically, we study the sewing, sliding, and tangential regions of the PWHS and we classify the typical singularities of these systems. Also, we are interested in understanding how the trajectories of the regularized system associated with the PWHS transit through the region of regularization. In addition, we know that holomorphic systems have no limit cycles, but piecewise holomorphic systems do, so we provide conditions to ensure the existence of limit cycles of these systems. Additional conditions are provided to guarantee the stability and uniqueness of such limit cycles.

MSC: 32A10; 34C20; 34A34; 34A36; 34C05

Keywords: Piecewise holomorphic systems; Limit cycles; Regularization

1. Introduction

The holomorphic systems $\dot{z} = f(z)$ have interesting dynamical properties, for example, the fact that these systems have no limit cycles and that they have a finite number of equilibrium points, which are isolated provided that f is not identically null. In general, when we add the

* Corresponding author.
E-mail addresses: fernando.gouveia@unesp.br (L.F.S. Gouveia), gabriel.rondon@unesp.br (G. Rondón), paulo.r.silva@unesp.br (P.R. da Silva).

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hypothesis of being holomorphic to a system, we reduce the number of parameters in the system. Although a polynomial system of degree n depends on $n^2 + 3n + 2$ parameters, a polynomial holomorphic system depends only on $2n+2$ parameters. Furthermore, the holomorphic functions have their interest in several areas of applied science, for example, in the study of fluid dynamics. In this context, it is possible to verify that the complex potential of the conjugate holomorphic system $\dot{z} = \overline{f(z)}$ is a primitive of $f(z)$. For more information see, for instance, [1,7,8].

In this paper, we are interested in the study of piecewise holomorphic systems (PWHS),

$$\begin{cases} \dot{z} = f^+(z) = u_1 + iv_1, \text{ when } \Im(z) > 0, \\ \dot{z} = f^-(z) = u_2 + iv_2, \text{ when } \Im(z) < 0, \end{cases} \tag{1}$$

where $z = x + iy$ and $f^{\pm}(z)$ are holomorphic functions defined in a domain $\mathcal{V} \subseteq \mathbb{C}$, that is,

(i) $u_{1,2} = \mathrm{Re}(f^{\pm})$ and $v_{1,2} = \mathrm{Im}(f^{\pm})$ are continuous;
(ii) there exist the partial derivatives $(u_{1,2})_x, (u_{1,2})_y, (v_{1,2})_x, (v_{1,2})_y$ in $\mathcal{V}$, and
(iii) the partial derivatives satisfy the Cauchy–Riemann equations

$$\begin{array}{lll} (u_1)_x = (v_1)_y, & (u_1)_y = -(v_1)_x, & \forall z = x + iy \in \mathcal{V}, \\ (u_2)_x = (v_2)_y, & (u_2)_y = -(v_2)_x, & \forall z = x + iy \in \mathcal{V}. \end{array}$$

We remark that the straight line $\Sigma = \{z \in \mathbb{C} : \Im(z) = 0\}$ divides the plane in two half-planes $\Sigma^{\pm}$ given by $\{z \in \mathbb{C} : \Im(z) > 0\}$ and $\{z \in \mathbb{C} : \Im(z) < 0\}$, respectively. The trajectories on Σ are defined following the Filippov convention. For more information on Filippov convention see, for instance, [9].

Throughout this article we use the normal forms associated with the holomorphic functions, which are given in [5] and [11], namely: 1, $(a + ib)z$, z^n, $\frac{\gamma z^n}{1+z^{n-1}}$, and $\frac{1}{z^n}$. For more details see Proposition 1. A priori these normal forms depend on the notion of conformal conjugation.

One of the properties of the PWHS that we will prove here is that the sliding, sewing and tangential regions are preserved by conformal conjugation, see Theorem 4. In particular, Lemma 8 establishes that regular-fold singularities are preserved by conformal conjugation. We will use this last result to characterize the order of tangential contact of the holomorphic functions with Σ, which are conformally conjugated to some of the normal forms. For more information see Theorem 10.

An interesting property is that the regularized vector field associated to (1) loses the property of being holomorphic, see Theorems 12 and 14. For that, we will use *the principle of identity of the holomorphic functions*, which states: given functions f and g holomorphic on a domain D (open and connected subset of $\mathbb{C}$), if $f = g$ on some $S \subseteq D$, where S has an accumulation point of D, then $f = g$ on D.

Also, we are interested in regularizations of PWHS around visible regular-fold singularities. More specifically, using Theorem 1 of [20] and the normal forms associated with the holomorphic functions, we analyze how the trajectories of the regularized system transit through the region of regularization, see Theorem 16.

In addition, we are concerned about studying the existence of limit cycles for the PWHS. One of the reasons for this study is the fact that the holomorphic systems have no limit cycles, for more information see, for instance, [3,6,11,13,14,19,22]. For that we will use the normal forms

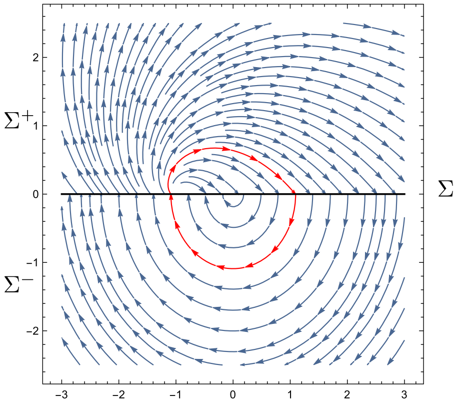

Fig. 1. Phase portrait of PWHS (2). The red trajectory is the limit cycle of (2). (For interpretation of the colors in the figures, the reader is referred to the web version of this article.)

mentioned above and we will establish conditions for the existence of limit cycles, see Theorems 17, 19, and 27. Furthermore, additional conditions are provided to guarantee the stability and uniqueness of such limit cycles. In particular, Theorem 17 establishes that the piecewise linear holomorphic systems whose equilibrium points are on manifold Σ have at most one limit cycle. Also, Corollary 18 establishes necessary and sufficient conditions for the existence of limit cycles. For example, consider the PWHS

$$\begin{cases} \dot{z} = (1-i)(z+1), \text{ when } \Im(z) > 0, \\ \dot{z} = -iz, \text{ when } \Im(z) < 0, \end{cases} \tag{2}$$

notice that it has a unique unstable limit cycle (see Fig. 1).

In the context of piecewise linear systems in the real plane, depending on the number of zones generated by the discontinuity manifold Σ, the maximum number of limit cycles varies. For example, in [10], Freire et al. considered 2 zones divided by a straight line and proved that piecewise linear systems in the real plane have at most one limit cycle. However, when considering 3 zones (for example, the discontinuity manifold Σ could be 2 parallel straight lines) it is possible to prove the existence of more than one limit cycle, for more details see, for instance, [2,18,21].

We emphasize that the focus on the existence of limit cycles in PWHS is one of the main novelties of the present study. Some of the main challenges when working in this context is that building the first return map is a bit complicated, however, if we use the normal forms associated with the holomorphic functions in their polar forms, it is much easier to work with. For the construction of the limit cycles we will use the symmetry of the polar equation of the orbits of z^n and $\frac{1}{z^n}$ and the invariance of the rays of such normal forms.

The paper is organized in the following form. In Section 2, we present some basic results on the normal forms of holomorphic functions and the way that the PWHS is defined in the straight line Σ. In Section 3, we prove that the sliding, sewing and tangential regions are preserved by conformal conjugation. For the tangential region, we study the order of tangential singularities existing in PWHS. In Section 4, we perform an analysis of the regularization of PWHS. Finally, in Section 5, we establish conditions for the existence of limit cycles in PWHS.

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Smoothing of nonsmooth differential systems near regular-tangential singularities and boundary limit cycles

Douglas D Novaes[1,*] and **Gabriel Rondón**[2,*]

[1] Departamento de Matemática, Instituto de Matemática, Estatística e Computação Científica (IMECC), Universidade Estadual de Campinas (UNICAMP), Rua Sérgio Buarque de Holanda, 651, Cidade Universitária Zeferino Vaz, 13083-859, Campinas, SP, Brazil

[2] UNESP—Universidade Estadual Paulista, São José do Rio Preto, São Paulo, Brazil

E-mail: ddnovaes@unicamp.br and garv202020@gmail.com

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Abstract

Understanding how tangential singularities evolve under smoothing processes was one of the first problem concerning regularization of Filippov systems. In this paper, we are interested in C^n-regularizations of Filippov systems around visible regular-tangential singularities of even multiplicity. More specifically, using Fenichel theory and blow-up methods, we aim to understand how the trajectories of the regularized system transits through the region of regularization. We apply our results to investigate C^n-regularizations of boundary limit cycles with even multiplicity contact with the switching manifold.

Keywords: nonsmooth differential systems, regularization, blow-up method, Fenichel theory, tangential singularities, boundary limit cycles

Mathematics Subject Classification numbers: 34A26, 34A36, 34C23, 37G15.

(Some figures may appear in colour only in the online journal)

1. Introduction

The analysis of differential equations with discontinuous right-hand side dates back to the work of Andronov *et al* [1] in 1937. Recently, the interest in such systems has increased significantly, mainly motivated by its wide range of applications in several areas of applied sciences. Piecewise smooth differential systems are used for modeling phenomena presenting abrupt behavior changes such as impact and friction in mechanical systems [4], refuge and switching

*Authors to whom any correspondence should be addressed.
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feeding preference in biological systems [18, 24], gap junctions in neural networks [7], and many others.

In this paper, we are interested in planar piecewise smooth systems. Formally, let M be an open subset of $\mathbb{R}^2$ and let $N \subset M$ be a codimension 1 submanifold of M. Denote by C_i, $i = 1, 2, \ldots, k$, the connected components of $M\backslash N$ and let $X_i : M \to \mathbb{R}^2$, for $i = 1, 2, \ldots, k$, be vector fields defined on M. A piecewise smooth vector field Z on M is defined by

$$Z(p) = X_i(p) \quad \text{if } p \in C_i, \ \text{for } i = 1, 2, \ldots, k. \tag{1}$$

Since N is a codimension 1 submanifold of M, for each $p \in N$ there exists a neighborhood $D \subset M$ of p and a function $h : D \to \mathbb{R}$, having 0 as a regular value, such that $\Sigma = N \cap D = h^{-1}(0)$. Moreover, the neighborhood D can be taken sufficiently small in order that $D\backslash\Sigma$ is composed by two disjoint regions Σ^+ and Σ^- such that $X^+ = Z|_{\Sigma^+}$ and $X^- = Z|_{\Sigma^-}$ are smooth vector fields. Accordingly, the piecewise smooth vector field (1) can be locally described as follows:

$$Z(p) = (X^+, X^-)_\Sigma = \begin{cases} X^+(p), & \text{if } h(p) > 0, \\ X^-(p), & \text{if } h(p) < 0, \end{cases} \qquad \text{for } p \in D.$$

1.1. Filippov systems

The notion of local trajectories of piecewise smooth vector fields (1) was stated by Filippov [13] as solutions of the following differential inclusion

$$\dot{p} \in \mathcal{F}_Z(p) = \frac{X^+(p) + X^-(p)}{2} + \text{sign}(h(p))\frac{X^+(p) - X^-(p)}{2}, \tag{2}$$

where

$$\text{sign}(s) = \begin{cases} -1 & \text{if } s < 0, \\ [-1, 1] & \text{if } s = 0, \\ 1 & \text{if } s > 0. \end{cases}$$

This approach is called Filippov's convention. The piecewise smooth vector field (1) is called a Filippov system when it is ruled by the Filippov's convention. For more information on differential inclusions see, for instance, [25].

The solutions of the differential inclusion (2) are well described in the literature (see, for instance, [13]) and have a simple geometrical interpretation. In order to illustrate this convention we define the following open regions on Σ,

$$\begin{aligned} \Sigma^{\mathrm{c}} &= \{p \in \Sigma : X^+h(p) \cdot X^-h(p) > 0\}, \\ \Sigma^{\mathrm{s}} &= \{p \in \Sigma : X^+h(p) < 0, X^-h(p) > 0\}, \\ \Sigma^{\mathrm{e}} &= \{p \in \Sigma : X^+h(p) > 0, X^-h(p) < 0\}. \end{aligned}$$

Here, $X^\pm h(p) = \langle \nabla h(p), X^\pm(p) \rangle$ denotes the Lie derivative of h in the direction of the vector fields $X^\pm$. Usually, they are called *crossing*, *sliding*, and *escaping* region, respectively. Notice that the points on Σ where both vectors fields X^+ and X^- simultaneously point outward or inward from Σ constitute, respectively, the *escaping* Σ^{e} and *sliding* Σ^{s} regions, and the complement of its closure in Σ constitutes the *crossing region*, Σ^{c}. The complement of the union of those regions in Σ constitutes the *tangency* points between X^+ or X^- with Σ, Σ^{t}.

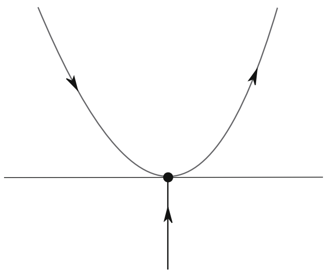


Figure 1. Visible regular-tangential singularity.

For $p \in \Sigma^c$ the trajectories either side of the discontinuity Σ, reaching p, can be joined continuously, forming a trajectory that crosses Σ^c. Alternatively, for $p \in \Sigma^{s,e} = \Sigma^s \cup \Sigma^e$ the trajectories either side of the discontinuity Σ, reaching p, can be joined continuously to trajectories that slide on $\Sigma^{s,e}$ following the sliding vector field,

$$Z^s(p) = \frac{X^-h(p)X^+(p) - X^+h(p)X^-(p)}{X^-h(p) - X^+h(p)}, \quad \text{for } p \in \Sigma^{s,e}. \tag{3}$$

In addition, a singularity of the sliding vector field Z^s is called pseudo-equilibrium.

In the Filippov context, the notion of Σ-*singular points* also contains the tangential points Σ^t constituted by the contact points between X^+ and X^- with Σ, that is, $\Sigma^t = \{p \in \Sigma : X^+h(p) \cdot X^-h(p) = 0\}$. In this paper, we are interested in contact points of finite degeneracy. Recall that p is a *contact of order* $k-1$ (or multiplicity k) between $X^\pm$ and Σ if 0 is a root of multiplicity k of $f(t) := h \circ \varphi_{X^\pm}(t, p)$, where $t \mapsto \varphi_{X^\pm}(t, p)$ is the trajectory of $X^\pm$ starting at p. Equivalently,

$$X^\pm h(p) = (X^\pm)^2 h(p) = \cdots = (X^\pm)^{k-1} h(p) = 0, \quad \text{and} \quad (X^\pm)^k h(p) \neq 0.$$

In addition, an even multiplicity contact, say $2k$, is called *visible* for X^+ (resp. X^-) when $(X^+)^{2k}h(p) > 0$ (resp. $(X^-)^{2k}h(p) < 0$). Otherwise, it is called *invisible*. In the above definitions, the higher order Lie derivatives X^ih are defined, inductively, by $Xh(p) = \langle \nabla h(p), X(p) \rangle$ and $X^ih(p) = X(X^{i-1}h)(p)$ for $i > 1$.

In this paper, we shall focus our attention on *visible regular-tangential singularities of multiplicity* $2k$ which are formed by a visible even multiplicity of X^+ and a regular point of X^-, or vice versa (see figure 1).

1.2. Sotomayor–Teixeira regularization

Roughly speaking, a smoothing process of a piecewise smooth vector field Z consists in obtaining a one-parameter family of continuous vector fields Z_ε converging to Z when $\varepsilon \to 0$.

A well known smoothing process is the Sotomayor–Teixeira regularization, which was introduced in [26]. Let $\phi : \mathbb{R} \to \mathbb{R}$ be a C^∞ function satisfying $\phi(\pm 1) = \pm 1$, $\phi^{(i)}(\pm 1) = 0$ for $i = 1, 2, \ldots, n$, and $\phi'(s) > 0$ for $s \in (-1, 1)$. Then, a C^n-Sotomayor–Teixeira regularization (or just C^n-regularization for short) takes

$$Z_\varepsilon^\Phi(p) = \frac{1 + \Phi_\varepsilon(h(p))}{2} X^+(p) + \frac{1 - \Phi_\varepsilon(h(p))}{2} X^-(p), \quad \Phi_\varepsilon(h) = \Phi(h/\varepsilon), \tag{4}$$

where $\Phi : \mathbb{R} \to \mathbb{R}$ is defined as the following C^n function

$$\Phi(s) = \begin{cases} \phi(s) & \text{if } |s| \leqslant 1, \\ \text{sign}(s) & \text{if } |s| \geqslant 1. \end{cases} \tag{5}$$

SIMULTANEOUS BIFURCATION OF LIMIT CYCLES FOR PIECEWISE HOLOMORPHIC SYSTEMS

ARMENGOL GASULL[1], GABRIEL RONDÓN[2] AND PAULO R. DA SILVA[2]

Abstract. Let $\dot{z} = f(z)$ be a holomorphic differential equation with center at p. In this paper we are concerned about studying the piecewise perturbation systems $\dot{z} = f(z) + \epsilon R^{\pm}(z,\overline{z})$, where $R^{+}(z,\overline{z})$ and $R^{-}(z,\overline{z})$ are complex polynomials defined for $\mathrm{Im}(z) > 0$ and $\mathrm{Im}(z) < 0$, respectively. We provide an integral expression, similar to an Abelian integral, whose zeros control the bifurcating limit cycles of the periodic orbits of the annular period of p. This expression is given in terms of the conformal conjugation between $\dot{z} = f(z)$ and its linearization $\dot{z} = f'(p)z$ at p. We use this result to control the simultaneous bifurcation of limit cycles of the two annular periods of $\dot{z} = \frac{i}{2}(z^2 - 1)$, after both complex and holomorphic polynomial perturbation.

1. Introduction

There are several important aspects to understand the dynamics of differential systems such as knowledge of the existence of limit cycles. In fact, the famous Hilbert's 16th problem is one of the main open problems in the qualitative theory of planar polynomial vector fields. Finding an upper bound for the maximum number of limit cycles of such systems in terms of their degrees is a very difficult problem.

In recent years, great interest has arisen in the study of limit cycles of piecewise holomorphic systems, which is a subfamily of piecewise smooth systems. This is because holomorphic functions have many applications in various areas of applied science, such as the study of fluid dynamics [1, 4, 5]. Furthermore, the study and properties of holomorphic systems $\dot{z} = f(z)$ make them interesting and beautiful but the absence of limit cycles makes them dynamically poor. However, in [12] and [13] the authors showed that there are piecewise holomorphic systems that have limit cycles. More precisely, in [13] the authors have used the intrinsic properties of holomorphic functions, such as their integrability, to construct limit cycles, whereas in [12] Gasull et al. approach this problem with different points of view: study of the number of zeros of the first and second order averaging functions, and with the control of the limit cycles appearing from a monodromic equilibrium point via a degenerated Andronov–Hoph type bifurcation.

Consider the piecewise polynomial complex systems (PWCS),

$$\dot{z} = f(z) + \begin{cases} \epsilon R^{+}(z,\overline{z}), & \text{when } \mathrm{Im}(z) > 0, \\ \epsilon R^{-}(z,\overline{z}), & \text{when } \mathrm{Im}(z) < 0, \end{cases} \tag{1}$$

where $\dot{z} = f(z)$ has a center at p, $0 < \epsilon \ll 1$, $z = x + iy \in \mathbb{C}$ and $R^{\pm}(z,\overline{z})$ are complex polynomial functions with $\deg(R^{\pm}_m) = m$.

We emphasize that the straight line $\Sigma = \{z \in \mathbb{C} : \mathrm{Im}(z) = 0\}$ divides the plane in two halfplanes $\Sigma^{+} = \{z \in \mathbb{C} : \mathrm{Im}(z) > 0\}$ and $\Sigma^{-} = \{z \in \mathbb{C} : \mathrm{Im}(z) < 0\}$. In addition, we assume that the orbits on Σ are defined following the Filippov convention, see [7] for more details.

2020 *Mathematics Subject Classification.* 32A10, 34A36, 34C07, 37G15.

Key words and phrases. Piecewise polynomial complex systems, limit cycles, averaging method, bifurcation.

In the $(r,\theta)-$coordinates $z = re^{i\theta}$, (1) is converted into

$$\frac{dr}{d\theta} = \begin{cases} F^+(\theta, r, \epsilon) \text{ if } 0 \leq \theta \leq \pi, \\ F^-(\theta, r, \epsilon) \text{ if } \pi \leq \theta \leq 2\pi, \end{cases} \tag{2}$$

where $F^{\pm}(\theta, r, \epsilon) = \epsilon F_1^{\pm}(\theta, r) + \mathcal{O}(\epsilon^2)$, with $\theta \in \mathbb{S}^1$, $r > 0$ and $\epsilon > 0$ is a sufficiently small parameter. As usual, $\mathcal{O}$ represents higher order terms of $F^{\pm}$.

From [14], each simple zero $r = r_0$ of

$$M_1^{\pm}(r) = \int_0^{\pm\pi} F_1^{\pm}(\theta, r) d\theta \tag{3}$$

provides, for ϵ small enough, a hyperbolic limit cycle of the piecewise smooth system (2) that tends to $r = r_0$ when ϵ tends to 0.

Here we are interested in finding an equivalent expression of (3) when the unperturbed system is a holomorphic system, $\dot{z} = f(z)$. Specifically, suppose that p is a center of an holomorphic equation $\dot{z} = f(z)$. From [2, 9], we know that there exists a conformal map $w = \phi(z)$ such that this differential equation can be written as $\dot{w} = -iw$. The map ϕ is called the linearizing change of $\dot{z} = f(z)$ at p. In what follows, we state our first main result.

Theorem A. *Consider the piecewise complex system* (1). *Suppose that ϕ is the linearizing change of $\dot{z} = f(z)$ at p such that $\phi(\Sigma^{\pm}) \subset \Sigma^{\pm}$ and $\phi(\Sigma) \subset \Sigma$. Then, each simple zero $r = r_0$ of $M_1(r) = M_r^+(r) - M_1^-(r)$, where*

$$M_1^{\pm}(r) = -\operatorname{Im}\left(\int_0^{\pm\pi} \overline{\phi'(\phi^{-1}(re^{i\theta}))R^{\pm}\left(\phi^{-1}(re^{i\theta}), \overline{\phi^{-1}(re^{i\theta})}\right)} ie^{i\theta} d\theta\right) \tag{4}$$

provides, for ϵ sufficiently small, a hyperbolic limit cycle of (1) *that tends to $r = r_0$ when ϵ tends to 0.*

We employ Theorem A to study PWCS (1) for $f(z) = \frac{i}{2}(z^2 - 1)$ and $R^{\pm}(z, \overline{z}) = R_m^{\pm}(z, \overline{z})$, where

$$R_m^{\pm}(z, \overline{z}) = \sum_{l=0}^{m} \sum_{k=0}^{l} \overline{a_{k,l}} z^{l-k} \overline{z}^k, \ a_{k,l} \in \mathbb{C} \text{ and } m = 0, 1, 2, 3. \tag{5}$$

We emphasize that $f(z) = \frac{i}{2}(z^2 - 1)$ has 2 centers at -1 and 1, respectively (see Figure 1). To carry out this study, we explicitly calculate the linearization change ϕ of $\dot{z} = f(z)$ at -1, which we employ to find the bifurcation function at $z = -1$. Thus, by changing variables and time we will obtain the bifurcation function at $z = 1$. It is worth noting that depending on the holomorphic vector field f, the calculation of ϕ can be complicated.

This type of problem has been addressed in several papers. Specifically, in [10] Garijo et al. study the smooth case, that is, $R = R^+ = R^-$, providing an integral expression for the differential equation $\dot{z} = f(z) + \epsilon R(z, \overline{z})$ and use this formula to control the simultaneous bifurcation of limit cycles of the two annular periods of $\dot{z} = iz + z^2$, after a polynomial perturbation. Also, in [6] the authors investigate the number of bifurcating periodic orbits of a cubic polynomial vector field having two period rings using piecewise perturbations. They study, up to first-order averaging analysis, the bifurcation of periodic orbits of the two annular periods, the first separately and the second simultaneously. There are other works that consider the problem of simultaneous bifurcation such as [3, 11, 16] although in the context of piecewise systems the type of bifurcation that we consider in this paper is a complete novelty.

Before introducing our second main result, we introduce some notations. We denote by M_1 and N_1 the first Melnikov functions on -1 and 1, respectively. In addition, we say that system

(1) presents a configuration with exactly (z_{M_1}, z_{N_1}) limit cycles when it has z_{M_1} and z_{N_1} limit cycles that bifurcate from -1 and 1, respectively.

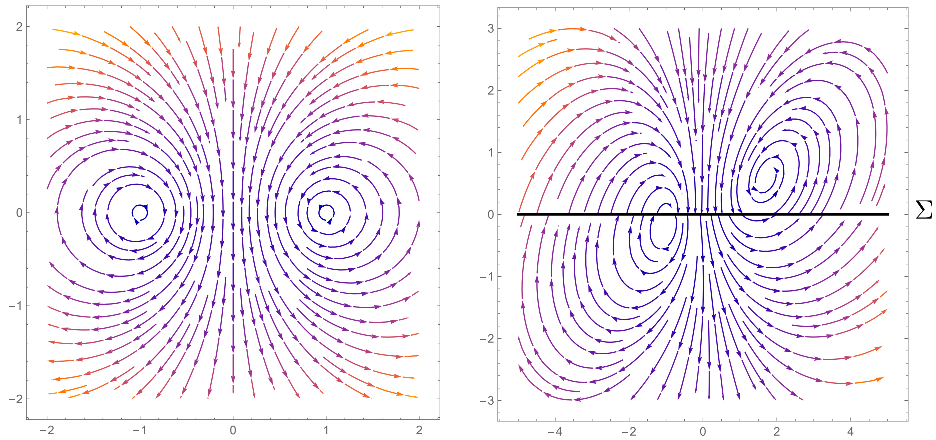


FIGURE 1. The left figure shows the phase portrait of PWCS (1) at $\epsilon = 0$ (unperturbed system). The right figure depicts a phase portrait of PWCS (1) with $0 < \epsilon \ll 1$ (perturbed system).

Theorem B. *Let z_{M_1} (resp. z_{N_1}) be the number of zeros counting its multiplicity of M_1 (resp. N_1) on the interval $(0,1)$. For PWCS (1), the following configurations of limit cycles are realizable:*

(a) (z_{M_1}, z_{N_1}) *with* $z_{M_1} = z_{N_1}$ *and* $0 \leq z_{M_1} + z_{N_1} \leq 2$ *provided that* $m = 0$*;*
(b) (z_{M_1}, z_{N_1}) *with* $0 \leq z_{M_1} + z_{N_1} \leq 4$ *provided that* $m = 1$*;*
(c) (z_{M_1}, z_{N_1}) *with* $0 \leq z_{M_1} + z_{N_1} \leq 4$ *provided that* $m = 2$*. Even more, it is possible to obtain 6 limit cycles with the configuration* $(z_{M_1}, z_{N_1}) = (4,2)$, $(z_{M_1}, z_{N_1}) = (2,4)$, $(z_{M_1}, z_{N_1}) = (5,1)$ *or* $(z_{M_1}, z_{N_1}) = (1,5)$.
(d) (z_{M_1}, z_{N_1}) *with* $0 \leq z_{M_1} + z_{N_1} \leq 5$ *provided that* $m = 3$*. Even more, it is possible to obtain 8 limit cycles with the configuration* $(z_{M_1}, z_{N_1}) = (6,2)$, $(z_{M_1}, z_{N_1}) = (2,6)$.

There are several methods that can be used to find the zeros of a polynomial in a certain interval I, such as the characterization of the extended complete Chebyshev systems. However, finding simultaneous zeros of polynomials can be quite a challenge, since there is no method to carry out such a study. However, here we provide a method for this purpose. Specifically, we use the notion of polynomials whose coefficients depend continuously on certain real parameters, for more details see Lemma 4.

At this point a natural question arises, what happens if the complex perturbation function $R^{\pm}_m$ is holomorphic? In this sense, we want to know the possible configurations (z_{M_1}, z_{N_1}) that PWCS (1) can have when $R^{\pm}_m$ depends only on z. As we will see, the fact that the perturbation is holomorphic greatly simplifies the calculations and it is not surprising that the number of cycles that arise is less than the general case. The following result answers the previous question.

Theorem C. *Let z_{M_1} (resp. z_{N_1}) be the number of zeros counting its multiplicity of M_1 (resp. N_1) on the interval $(0,1)$. If $R^{\pm}$ is a holomorphic polynomial function, then for PWCS (1) the following configurations of limit cycles are realizable:*

(a) (z_{M_1}, z_{N_1}) *with* $z_{M_1} = z_{N_1}$ *and* $0 \leq z_{M_1} + z_{N_1} \leq 2$ *provided that* $m = 0$*;*

(b) (z_{M_1}, z_{N_1}) *with* $0 \le z_{M_1} + z_{N_1} \le 2$ *provided that* $m = 1, 2$;
(c) (z_{M_1}, z_{N_1}) *with* $0 \le z_{M_1} + z_{N_1} \le 3$ *provided that* $m = 3$. *Even more, it is possible to obtain 4 limit cycles with the configuration* $(z_{M_1}, z_{N_1}) = (3, 1)$ *or* $(z_{M_1}, z_{N_1}) = (1, 3)$.

The paper is organized as follows. In Section 2, we present some basic results on the averaging theory and some more technical results. Then we dedicate next four sections to prove Theorems A, B, and C.

2. Preliminaries

This section is devoted to establishing some results that will be used throughout the paper. We divide it in two subsections.

2.1. **The averaging method.** We briefly recall some basic results of the averaging theory for piecewise smooth systems written in polar coordinates. An overview on this subject can be found in [14], and the reader can see the details of the proofs there. Consider the piecewise smooth systems of the form

$$\frac{dr}{d\theta} = \begin{cases} F^{+}(\theta, r, \epsilon) \text{ if } 0 \le \theta \le \pi, \\ F^{-}(\theta, r, \epsilon) \text{ if } \pi \le \theta \le 2\pi, \end{cases} \tag{6}$$

where $F^{\pm}(\theta, r, \epsilon) = \sum_{i=1}^{k} \epsilon^i F_i^{\pm}(\theta, r) + \epsilon^{k+1} R^{\pm}(\theta, r, \epsilon)$, with $\theta \in S^1$, $r > 0$ and $\epsilon > 0$ is a sufficiently small parameter.

From [14] we introduce the following functions $M_1^{\pm}(r)$ related to the above system (6):

$$M_1^{\pm}(r) = \int_0^{\pm\pi} F_1^{\pm}(\theta, r) d\theta.$$

Now, we define the function $M_1(r) = M_1^{+}(r) - M_1^{-}(r)$, which is called *the averaged function of order* 1. The following result can be found in [14, Theorem 1]:

Proposition 1. *Let* M_1 *be the first non identically zero averaged function. Then, each simple zero* $r = r_0$ *of* M_1 *provides, for* ϵ *small enough, a hyperbolic limit cycle of the piecewise smooth system* (6) *that tends to* $r = r_0$ *when* ϵ *tends to 0.*

2.2. **A miscellany of results.** A very useful and well-kown characterization of extended complete Chebyshev system (ECT-systems) is the following:

Lemma 2. $\{f_0, \cdots, f_n\}$ *is an ECT-system on* I *if and only if for all* $k = 0, 1, \cdots, n$, $W_k(x) \neq 0$ *for all* $x \in I$, *where*

$$W_k(x) = W[f_0, ..., f_k](x) = det\left(f_j^{(i)}(x)\right)_{0 \le i,j \le k}$$

is the Wronskian of $(f_0, \cdots, f_k)$ *at* $x \in I$.

This result allows us to estimate the number of real zeroes of any non-zero function $F \in Span\{f_0, \cdots, f_n\}$, where $Span(\mathcal{F})$ denotes the set of all functions given by linear combinations of the functions of $\mathcal{F}$. In what follows, we state a classical result related to the ECT-system, whose proof can be found in [15].

Theorem 3. *Let* $\mathcal{F} = [f_0, ..., f_n]$ *be an ECT-system on* I. *Then, the number of isolated zeros for every element of* $Span(\mathcal{F})$ *does not exceed* n. *Moreover, for each configuration of* $m \le n$ *zeros, taking into account their multiplicity, there exists* $F \in Span(\mathcal{F})$ *with this configuration of zeros.*

In what follows, we provide a simple result for finding the zeros of k-parameter families of polynomials in one variable.

Let $F_\lambda(x)$ be a k-parametric family of polynomials. We denote the discriminant of a polynomial $p(x) = a_n x^n + \cdots + a_1 x + a_0$ as $\Delta_x(p)$, i.e.

$$\Delta_x(p) = (-1)^{\frac{n(n-1)}{2}} \frac{1}{a_n} Res(p(x), p'(x)),$$

where $Res(p, p')$ is the resultant of p and p_0.

Using the same ideas as in [8, Lemma 8.1], it is easy to prove the following result, which will be used throughout the paper.

Lemma 4. *Let $F_\lambda(x) = f_n(\lambda)x^n + f_{n-1}(\lambda)x^{n-1} + \cdots + f_1(\lambda)x + f_0(\lambda)$, $n > 1$, be a family of real polynomials depending continuously on a parameter $\lambda \in \mathbb{R}^k$ and set $\Omega_\lambda = (a(\lambda), b(\lambda))$, for some continuous functions $a(\lambda)$ and $b(\lambda)$. Assume that there exists an connected open set $\mathcal{U} \subset \mathbb{R}^k$ such that:*

(i) *For some $\lambda_0 \in \mathcal{U}$, F_{λ_0} has exactly m zeros in Ω_{λ_0} and all of them are simple.*
(ii) *For all $\lambda \in \mathcal{U}$, $F_\lambda(a(\lambda)) \cdot F_\lambda(b(\lambda)) \neq 0$.*
(iii) *For all $\lambda \in \mathcal{U}$, $\Delta_x(F_\lambda) \neq 0$.*

Then for all $\lambda \in \mathcal{U}$, F_λ has also exactly m zeros in Ω_λ and all of them are simple.

3. Proof of Theorem A

Since p is a center of $\dot{z} = f(z)$, then from [2, 9] we know that there exists ϕ conformal map such that $\phi'(z)f(z) = -i\phi(z)$. Using this on $\dot{z} = f(z) + \epsilon R^\pm(z, \overline{z})$, we get

$$\dot{w} = -iw + \epsilon L^\pm(w, \overline{w}), \tag{7}$$

where $L^\pm(w, \overline{w}) = \phi'(\phi^{-1}(w))R^\pm(\phi^{-1}(w), \overline{\phi^{-1}(w)})$. In the $(r, \theta)-$coordinates $w = re^{i\theta}$, (7) is converted into

$$\frac{dr}{d\theta} = \frac{r\epsilon\Big(\operatorname{Re}(L^\pm(re^{i\theta}, re^{-i\theta}))\cos(\theta) + \operatorname{Im}(L^\pm(re^{i\theta}, re^{-i\theta}))\sin(\theta)\Big)}{-r + \epsilon\Big(\operatorname{Im}(L^\pm(re^{i\theta}, re^{-i\theta}))\cos(\theta) - \operatorname{Re}(L^\pm(re^{i\theta}, re^{-i\theta}))\sin(\theta)\Big)} = F^\pm(r, \theta, \epsilon). \tag{8}$$

Hence, expanding $F^\pm(r, \theta, \epsilon)$ around $\epsilon = 0$, (8) is written as

$$\frac{dr}{d\theta} = \epsilon F_1^\pm(\theta, r) + \mathcal{O}(\epsilon^2),$$

where $F_1^\pm(\theta, r) = -\operatorname{Re}(L^\pm(re^{i\theta}, re^{-i\theta}))\cos(\theta) - \operatorname{Im}(L^\pm(re^{i\theta}, re^{-i\theta}))\sin(\theta)$. Computing the first averaged function

$$\begin{aligned}
M_1^\pm(r) = & \int_0^{\pm\pi} F_1^\pm(\theta, r)d\theta \\
= & -\int_0^{\pm\pi} \operatorname{Im}(i\overline{L^\pm(re^{i\theta}, re^{-i\theta})})\cos(\theta) - \operatorname{Im}(\overline{L^\pm(re^{i\theta}, re^{-i\theta})})\sin(\theta)d\theta \\
= & -\int_0^{\pm\pi} \operatorname{Im}\left(\overline{L^\pm(re^{i\theta}, re^{-i\theta})}(i\cos(\theta) - \sin(\theta))\right)d\theta \\
= & -\operatorname{Im}\left(\int_0^{\pm\pi} \overline{L^\pm(re^{i\theta}, re^{-i\theta})}ie^{i\theta}d\theta\right) \\
= & -\operatorname{Im}\left(\int_0^{\pm\pi} \overline{\phi'(\phi^{-1}(re^{i\theta}))R^\pm\left(\phi^{-1}(re^{i\theta}), \overline{\phi^{-1}(re^{i\theta})}\right)}ie^{i\theta}d\theta\right).
\end{aligned}$$

From Proposition 1 the result follows.

4. The bifurcation function at $z=-1$ and $z=1$

We will do a detailed study around $z=-1$. The analysis around $z=1$ will be reduced from the previous one.

4.1. The bifurcation at $z=-1$. To employ Theorem A to (1) at $z=-1$ we must first linearize $\dot{z}=\frac{i}{2}(z^2-1)$. It is easy to verify that

$$\phi(z)=\frac{1+z}{1-z},\quad \phi'(z)=\frac{2}{(z-1)^2}\quad\text{and}\quad \phi^{-1}(w)=\frac{w-1}{w+1}. \tag{9}$$

In addition, $\phi(\Sigma)=\Sigma$ and $\phi(\mathbb{S}^1)=\{(0,y):y\in\mathbb{R}\}$. From Theorem A with $R^{\pm}(z,\overline{z})=$

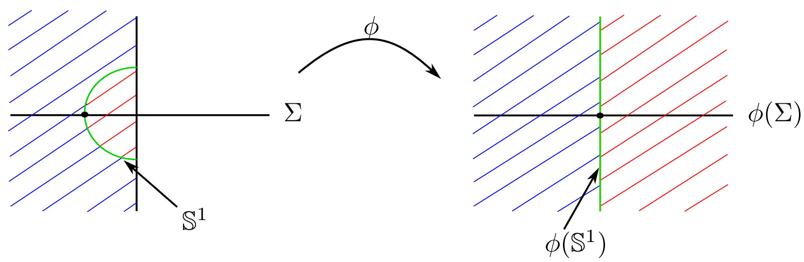


Figure 2. Conformal map $\phi(z)=\frac{1+z}{1-z}$.

$R^{\pm}_m(z,\overline{z})$ given in (5) and $f(z)=\frac{i}{2}(z^2-1)$ we get that $I^{\pm}_m(r):=M^{\pm}_1(r)$ is

$$I^{\pm}_m(r)=\sum_{l=0}^{m}J^{\pm}_l(r), \tag{10}$$

where

$$J^{\pm}_l(r)=\sum_{k=0}^{l}I^{\pm}_{k,l}(r)\quad\text{and}\quad I^{\pm}_{k,l}=-\operatorname{Im}\left(a^{\pm}_{k,l}\int_0^{\pm\pi}\overline{\phi'(\phi^{-1}(re^{i\theta}))}\left(\phi^{-1}(re^{i\theta})\right)^k\left(\overline{\phi^{-1}(re^{i\theta})}\right)^{l-k}ie^{i\theta}d\theta\right).$$

In addition, the following recursive formula holds:

$$I^{\pm}_{m+1}(r)=I^{\pm}_m(r)+J^{\pm}_{m+1}(r). \tag{11}$$

Lemma 5. *Fix $l\geq 0$ and $0\leq k\leq l$. Then*

$$I^{\pm}_{k,l}=-\frac{1}{2}\operatorname{Im}\left(a^{\pm}_{k,l}\int_0^{\pm\pi}\frac{(re^{-i\theta}+1)^{k-l+2}(re^{i\theta}-1)^k(re^{-i\theta}-1)^{l-k}ie^{i\theta}}{(re^{i\theta}+1)^k}d\theta\right).$$

Proof. From (9), we know that

$$\phi'\left(\phi^{-1}(w)\right)=\frac{1}{2}(w+1)^2\quad\text{and}\quad\overline{\phi^{-1}(w)}=\frac{\overline{w}-1}{\overline{w}+1}.$$

Thus, using (10) we get

$$\begin{aligned}I^{\pm}_{k,l}&=-\operatorname{Im}\left(a^{\pm}_{k,l}\int_0^{\pm\pi}\frac{1}{2}(re^{-i\theta}+1)^2\left(\frac{re^{i\theta}-1}{re^{i\theta}+1}\right)^k\left(\frac{re^{-i\theta}-1}{re^{-i\theta}+1}\right)^{l-k}ie^{-i\theta}d\theta\right)\\&=-\frac{1}{2}\operatorname{Im}\left(a^{\pm}_{k,l}\int_0^{\pm\pi}\frac{(re^{-i\theta}+1)^{k-l+2}(re^{i\theta}-1)^k(re^{-i\theta}-1)^{l-k}ie^{i\theta}}{(re^{i\theta}+1)^k}d\theta\right).\end{aligned}$$

$\square$

Lemma 6. *The following hold:*

$$
\begin{array}{llcl}
(a) & J_0^{\pm}(r) & = & -\operatorname{Im}\left(a_{0,0}^{\pm}\right)(-1+r^2) \mp \pi \operatorname{Re}\left(a_{0,0}^{\pm}\right) r, \\
(b) & J_1^{\pm}(r) & = & \dfrac{1}{r}\Big(-\operatorname{Im}\left(a_{0,1}^{\pm}\right) r(1+r^2) - \operatorname{Im}\left(a_{1,1}^{\pm}\right)\Big(-r(1+r^2)+2(-1+r^2)^2\tanh^{-1}(r)\Big) \\
& & & \mp\pi \operatorname{Re}\left(a_{1,1}^{\pm}\right) r^2(-1+r^2)\Big), \\
(c) & J_2^{\pm}(r) & = & \dfrac{1}{r}\Big(-\operatorname{Im}\left(a_{0,2}^{\pm}\right) r(-1+r^2) - \operatorname{Im}\left(a_{1,2}^{\pm}\right)\Big(r-r^3+2(-1+r^4)\tanh^{-1}(r)\Big) \\
& & & -\operatorname{Im}\left(a_{2,2}^{\pm}\right)\Big(5r(-1+r^2)-4(-1+r^4)\tanh^{-1}(r)\Big) \pm \pi \operatorname{Re}\left(a_{0,2}^{\pm}\right) r^2 \\
& & & \mp\pi \operatorname{Re}\left(a_{1,2}^{\pm}\right) r^4 \mp \pi \operatorname{Re}\left(a_{2,2}^{\pm}\right) r^2(1-2r^2)\Big), \\
(d) & J_3^{\pm}(r) & = & \dfrac{1}{r}\Big(-\operatorname{Im}(a_{0,3}^{\pm})\Big(r(1+r^2)-8r^2\tanh^{-1}(r)\Big) \mp 2\pi \operatorname{Re}(a_{0,3}^{\pm}) r^2 \\
& & & -\operatorname{Im}\left(a_{1,3}^{\pm}\right)\Big(-r(1+r^2)+2(1+r^2)^2\tanh^{-1}(r)\Big) \mp \pi \operatorname{Re}\left(a_{1,3}^{\pm}\right) r^2(1+r^2) \\
& & & -\operatorname{Im}\left(a_{2,3}^{\pm}\right)\Big(5r(1+r^2)-4(1+r^4)\tanh^{-1}(r)\Big) \pm 2\pi \operatorname{Re}\left(a_{2,3}^{\pm}\right) r^4 \\
& & & -\operatorname{Im}\left(a_{3,3}^{\pm}\right)\Big(-5r(1+r^2)+(6-4r^2+6r^4)\tanh^{-1}(r)\Big) \\
& & & \mp\pi \operatorname{Re}\left(a_{3,3}^{\pm}\right) r^2(-1+3r^2)\Big).
\end{array}
$$

Proof. From Lemma 5 for $l = 0$, we get

$$
\begin{aligned}
J_0^{\pm}(r) &= I_{0,0}^{\pm}(r) \\
&= -\frac{1}{2}\operatorname{Im}\left(a_{0,0}^{\pm}\int_0^{\pm\pi}(re^{-i\theta}+1)^2 ie^{i\theta}d\theta\right) \\
&= -\operatorname{Im}\left(a_{0,0}^{\pm}(-1 \pm i\pi r + r^2)\right) \\
&= -\operatorname{Im}\left(a_{0,0}^{\pm}\right)(-1+r^2) \mp \pi \operatorname{Re}\left(a_{0,0}^{\pm}\right) r.
\end{aligned}
$$

Thus, we obtain item (a). To prove item (b) we employ Lemma 5 for $l = 1$ and $k = 0, 1$. Indeed,

$$
\begin{aligned}
I_{0,1}^{\pm}(r) &= -\frac{1}{2}\operatorname{Im}\left(a_{0,1}^{\pm}\int_0^{\pm\pi}(re^{-i\theta}+1)(re^{-i\theta}-1)ie^{i\theta}d\theta\right) \\
&= -\operatorname{Im}\left(a_{0,1}^{\pm}\right)(1+r^2), \\
I_{1,1}^{\pm}(r) &= -\frac{1}{2}\operatorname{Im}\left(a_{1,1}^{\pm}\int_0^{\pm\pi}\frac{(re^{-i\theta}+1)^2(re^{i\theta}-1)ie^{i\theta}}{re^{i\theta}+1}d\theta\right) \\
&= -\frac{1}{r}\operatorname{Im}\left(a_{1,1}^{\pm}\Big(-r(1+r^2)+2(-1+r^2)^2\tanh^{-1}(r) \pm i\pi r^2(-1+r^2)\Big)\right) \\
&= -\frac{\operatorname{Im}\left(a_{1,1}^{\pm}\right)}{r}\Big(-r(1+r^2)+2(-1+r^2)^2\tanh^{-1}(r)\Big) \mp \pi \operatorname{Re}\left(a_{1,1}^{\pm}\right) r(-1+r^2).
\end{aligned}
$$

Since $J_1^{\pm}(r) = I_{0,1}^{\pm}(r) + I_{1,1}^{\pm}(r)$, then we get item (b). To verify item (c), we employ Lemma 5 for $l = 2$ and $k = 0, 1, 2$. Indeed,

$$\begin{aligned}
I_{0,2}^{\pm}(r) &= -\frac{1}{2}\operatorname{Im}\left(a_{0,2}^{\pm}\int_0^{\pm\pi}(re^{-i\theta}-1)^2 ie^{i\theta}d\theta\right)\\
&= -\operatorname{Im}\left(a_{0,2}^{\pm}\Big((-1+r^2)\mp i\pi r\Big)\right),\\
&= -\operatorname{Im}\left(a_{0,2}^{\pm}\right)(-1+r^2)\pm\pi\operatorname{Re}\left(a_{0,2}^{\pm}\right)r,\\
I_{1,2}^{\pm}(r) &= -\frac{1}{2}\operatorname{Im}\left(a_{1,2}^{\pm}\int_0^{\pm\pi}\frac{(re^{-i\theta}+1)(re^{i\theta}-1)(re^{-i\theta}-1)ie^{i\theta}}{re^{i\theta}+1}d\theta\right)\\
&= -\frac{1}{r}\operatorname{Im}\left(a_{1,2}^{\pm}\Big(r-r^3+2(-1+r^4)\tanh^{-1}(r)\pm i\pi r^4\Big)\right)\\
&= -\frac{\operatorname{Im}\left(a_{1,2}^{\pm}\right)}{r}\Big(r-r^3+2(-1+r^4)\tanh^{-1}(r)\Big)\mp\pi\operatorname{Re}\left(a_{1,2}^{\pm}\right)r^3,\\
I_{2,2}^{\pm}(r) &= -\frac{1}{2}\operatorname{Im}\left(a_{2,2}^{\pm}\int_0^{\pm\pi}\frac{(re^{-i\theta}+1)^2(re^{i\theta}-1)^2ie^{i\theta}}{(re^{i\theta}+1)^2}d\theta\right)\\
&= -\frac{1}{r}\operatorname{Im}\left(a_{2,2}^{\pm}\Big(5r(-1+r^2)-4(-1+r^4)\tanh^{-1}(r)\pm i\pi r^2(1-2r^2)\Big)\right)\\
&= -\frac{\operatorname{Im}\left(a_{2,2}^{\pm}\right)}{r}\Big(5r(-1+r^2)-4(-1+r^4)\tanh^{-1}(r)\Big)\mp\pi\operatorname{Re}\left(a_{2,2}^{\pm}\right)r(1-2r^2).
\end{aligned}$$

Since $J_2^{\pm}(r) = I_{0,2}^{\pm}(r) + I_{1,2}^{\pm}(r) + I_{2,2}^{\pm}(r)$, then we obtain item (c). Finally, to check item (d), we employ Lemma 5 for $l = 3$ and $k = 0, 1, 2, 3$. Indeed,

$$\begin{aligned}
I_{0,3}^{\pm}(r) &= -\frac{1}{2}\operatorname{Im}\left(a_{0,3}^{\pm}\int_0^{\pm\pi}\frac{(re^{-i\theta}-1)^3}{re^{-i\theta}+1}ie^{i\theta}d\theta\right)\\
&= -\operatorname{Im}\left(a_{0,3}^{\pm}\Big((1+r^2)-8r\tanh^{-1}(r)\pm i2\pi r\Big)\right),\\
&= -\operatorname{Im}(a_{0,3}^{\pm})\Big((1+r^2)-8r\tanh^{-1}(r)\Big)\mp 2\pi\operatorname{Re}(a_{0,3}^{\pm})r,\\
I_{1,3}^{\pm}(r) &= -\frac{1}{2}\operatorname{Im}\left(a_{1,3}^{\pm}\int_0^{\pm\pi}\frac{(re^{i\theta}-1)(re^{-i\theta}-1)^2ie^{i\theta}}{re^{i\theta}+1}d\theta\right)\\
&= -\frac{1}{r}\operatorname{Im}\left(a_{1,3}^{\pm}\Big(-r(1+r^2)+2(1+r^2)^2\tanh^{-1}(r)\pm i\pi r^2(1+r^2)\Big)\right)\\
&= -\frac{\operatorname{Im}\left(a_{1,3}^{\pm}\right)}{r}\Big(-r(1+r^2)+2(1+r^2)^2\tanh^{-1}(r)\Big)\mp\pi\operatorname{Re}\left(a_{1,3}^{\pm}\right)r^2(1+r^2),\\
I_{2,3}^{\pm}(r) &= -\frac{1}{2}\operatorname{Im}\left(a_{2,3}^{\pm}\int_0^{\pm\pi}\frac{(re^{-i\theta}+1)(re^{i\theta}-1)^2(re^{-i\theta}-1)ie^{i\theta}}{(re^{i\theta}+1)^2}d\theta\right)\\
&= -\frac{1}{r}\operatorname{Im}\left(a_{2,3}^{\pm}\Big(5r(1+r^2)-4(1+r^4)\tanh^{-1}(r)\mp i2\pi r^4\Big)\right)\\
&= -\frac{\operatorname{Im}\left(a_{2,3}^{\pm}\right)}{r}\Big(5r(1+r^2)-4(1+r^4)\tanh^{-1}(r)\Big)\pm 2\pi\operatorname{Re}\left(a_{2,3}^{\pm}\right)r^4,
\end{aligned}$$

$$
\begin{aligned}
I^{\pm}_{3,3}(r) &= -\frac{1}{2}\operatorname{Im}\left(a^{\pm}_{3,3}\int_0^{\pm\pi}\frac{(re^{-i\theta}+1)^2(re^{i\theta}-1)^3 ie^{i\theta}}{(re^{i\theta}+1)^3}d\theta\right)\\
&= -\frac{1}{r}\operatorname{Im}\left(a^{\pm}_{3,3}\Big(-5r(1+r^2)+(6-4r^2+6r^4)\tanh^{-1}(r)\pm i\pi r^2(-1+3r^2)\Big)\right)\\
&= -\frac{\operatorname{Im}\left(a^{\pm}_{3,3}\right)}{r}\Big(-5r(1+r^2)+(6-4r^2+6r^4)\tanh^{-1}(r)\Big)\mp\pi\operatorname{Re}\left(a^{\pm}_{3,3}\right)r^2(-1+3r^2).
\end{aligned}
$$

Since $J^{\pm}_3(r)=I^{\pm}_{0,3}(r)+I^{\pm}_{1,3}(r)+I^{\pm}_{2,3}(r)+I^{\pm}_{3,3}(r)$, then we obtain item (d). □

Lemma 7. *The following hold:*

(a) $I_0(r)=(\operatorname{Im}(a^-_{00})-\operatorname{Im}(a^+_{00}))(-1+r^2)-\pi(\operatorname{Re}(a^-_{00})+\operatorname{Re}(a^+_{00}))r.$

(b) $I_1(r)=\frac{1}{r}\Big(ar+br^2+cr^3+dr^4+\alpha(-1+r^2)^2\tanh^{-1}(r)\Big)$, *where a,b,c,d and α depend of the coefficients $a^{\pm}_{k,l}$ for all $k=0,\dots,l$ and $l=0,1$.*

(c) $I_2(r)=\frac{1}{r}\Big(ar+br^2+cr^3+dr^4+\alpha(-1+r^2)^2\tanh^{-1}(r)+\beta(-1+r^4)\tanh^{-1}(r)\Big)$, *where a,b,c,d,α and β depend of the coefficients $a^{\pm}_{k,l}$ for all $k=0,\dots,l$ and $l=0,1,2$.*

(d) $I_3(r)=\frac{1}{r}\Big(ar+br^2+cr^3+dr^4+\alpha(-1+r^2)^2\tanh^{-1}(r)+\beta(-1+r^4)\tanh^{-1}(r)+\gamma r^2\tanh^{-1}(r)\Big)$, *where a,b,c,d,α,β and γ depend of the coefficients $a^{\pm}_{k,l}$ for all $k=0,\dots,l$ and $l=0,1,3$.*

Proof. In lemma 6 we have proven the expression of $J_l(r)$ for all $l=0,1,2$. From equations (10), (11) and $I_m(r)=I^+_m(r)-I^-_m(r)$, we get that

$$
\begin{aligned}
I_0(r) &= J^+_0(r)-J^-_0(r)\\
&= (\operatorname{Im}(a^-_{00})-\operatorname{Im}(a^+_{00}))(-1+r^2)-\pi(\operatorname{Re}(a^-_{00})+\operatorname{Re}(a^+_{00}))r,
\end{aligned}
$$

$$
\begin{aligned}
I_1(r) &= I_0(r)+J^+_1(r)-J^-_1(r)\\
&= ar+br^2+cr^3+dr^4+\alpha(-1+r^2)^2\tanh^{-1}(r),
\end{aligned}
$$

where

- $a=\operatorname{Im}(a^+_{00})-\operatorname{Im}(a^-_{00})-\operatorname{Im}(a^+_{01})+\operatorname{Im}(a^-_{01})+\operatorname{Im}(a^+_{11})-\operatorname{Im}(a^-_{11})$,
- $b=-\pi\operatorname{Re}(a^+_{00})-\pi\operatorname{Re}(a^-_{00})+\pi\operatorname{Re}(a^+_{11})+\pi\operatorname{Re}(a^-_{11})$,
- $c=-\operatorname{Im}(a^+_{00})+\operatorname{Im}(a^-_{00})-\operatorname{Im}(a^+_{01})+\operatorname{Im}(a^-_{01})+\operatorname{Im}(a^+_{11})-\operatorname{Im}(a^-_{11})$,
- $d=-\pi\operatorname{Re}(a^+_{11})-\pi\operatorname{Re}(a^-_{11})$,
- $\alpha=-2\operatorname{Im}(a^+_{11})+2\operatorname{Im}(a^-_{11})$.

$$
\begin{aligned}
I_2(r) &= I_1(r)+J^+_2(r)-J^-_2(r)\\
&= ar+br^2+cr^3+dr^4+\alpha(-1+r^2)^2\tanh^{-1}(r)+\beta(-1+r^4)\tanh^{-1}(r),
\end{aligned}
$$

where

- $a=\operatorname{Im}(a^+_{00})-\operatorname{Im}(a^-_{00})-\operatorname{Im}(a^+_{01})+\operatorname{Im}(a^-_{01})+\operatorname{Im}(a^+_{11})-\operatorname{Im}(a^-_{11})+\operatorname{Im}(a^+_{02})-\operatorname{Im}(a^-_{02})-\operatorname{Im}(a^+_{12})+\operatorname{Im}(a^-_{12})+5\operatorname{Im}(a^+_{22})-5\operatorname{Im}(a^-_{22})$,
- $b=-\pi\operatorname{Re}(a^+_{00})-\pi\operatorname{Re}(a^-_{00})+\pi\operatorname{Re}(a^+_{11})+\pi\operatorname{Re}(a^-_{11})+\pi\operatorname{Re}(a^-_{02})+\pi\operatorname{Re}(a^+_{02})-\pi\operatorname{Re}(a^+_{22})-\pi\operatorname{Re}(a^-_{22})$,
- $c=-\operatorname{Im}(a^+_{00})+\operatorname{Im}(a^-_{00})-\operatorname{Im}(a^+_{01})+\operatorname{Im}(a^-_{01})+\operatorname{Im}(a^+_{11})-\operatorname{Im}(a^-_{11})-\operatorname{Im}(a^+_{02})+\operatorname{Im}(a^-_{02})+\operatorname{Im}(a^+_{12})-\operatorname{Im}(a^-_{12})-5\operatorname{Im}(a^+_{22})+5\operatorname{Im}(a^-_{22})$,
- $d=-\pi\operatorname{Re}(a^+_{11})-\pi\operatorname{Re}(a^-_{11})-\pi\operatorname{Re}(a^+_{12})-\pi\operatorname{Re}(a^-_{12})+2\pi\operatorname{Re}(a^+_{22})+2\pi\operatorname{Re}(a^-_{22})$,
- $\alpha=-2\operatorname{Im}(a^+_{11})+2\operatorname{Im}(a^-_{11})$,
- $\beta=-2\operatorname{Im}(a^+_{12})+2\operatorname{Im}(a^-_{12})+4\operatorname{Im}(a^+_{22})-4\operatorname{Im}(a^-_{22})$.

$$\begin{aligned} I_3(r) &= I_2(r) + J_3^+(r) - J_3^-(r) \\ &= ar + br^2 + cr^3 + dr^4 + \alpha(-1+r^2)^2 \tanh^{-1}(r) + \beta(-1+r^4)\tanh^{-1}(r) \\ &\quad + \gamma r^2 \tanh^{-1}(r), \end{aligned}$$

where

- $a = \mathrm{Im}(a_{00}^+) - \mathrm{Im}(a_{00}^-) - \mathrm{Im}(a_{01}^+) + \mathrm{Im}(a_{01}^-) + \mathrm{Im}(a_{11}^+) - \mathrm{Im}(a_{11}^-) + \mathrm{Im}(a_{02}^+) - \mathrm{Im}(a_{02}^-) - \mathrm{Im}(a_{12}^+) + \mathrm{Im}(a_{12}^-) + 5\,\mathrm{Im}(a_{22}^+) - 5\,\mathrm{Im}(a_{22}^-) - \mathrm{Im}(a_{03}^+) + \mathrm{Im}(a_{03}^-) + \mathrm{Im}(a_{13}^+) - \mathrm{Im}(a_{13}^-) - 5\,\mathrm{Im}(a_{23}^+) + 5\,\mathrm{Im}(a_{23}^-) + 5\,\mathrm{Im}(a_{33}^+) - 5\,\mathrm{Im}(a_{33}^-)$,
- $b = -\pi\,\mathrm{Re}(a_{00}^+) - \pi\,\mathrm{Re}(a_{00}^-) + \pi\,\mathrm{Re}(a_{11}^+) + \pi\,\mathrm{Re}(a_{11}^-) + \pi\,\mathrm{Re}(a_{02}^-) + \pi\,\mathrm{Re}(a_{02}^+) - \pi\,\mathrm{Re}(a_{22}^+) - \pi\,\mathrm{Re}(a_{22}^-) - 2\pi\,\mathrm{Re}(a_{03}^-) - 2\pi\,\mathrm{Re}(a_{03}^+) - \pi\,\mathrm{Re}(a_{13}^+) - \pi\,\mathrm{Re}(a_{13}^-) + \pi\,\mathrm{Re}(a_{33}^+) + \pi\,\mathrm{Re}(a_{33}^-)$,
- $c = -\mathrm{Im}(a_{00}^+) + \mathrm{Im}(a_{00}^-) - \mathrm{Im}(a_{01}^+) + \mathrm{Im}(a_{01}^-) + \mathrm{Im}(a_{11}^+) - \mathrm{Im}(a_{11}^-) - \mathrm{Im}(a_{02}^+) + \mathrm{Im}(a_{02}^-) + \mathrm{Im}(a_{12}^+) - \mathrm{Im}(a_{12}^-) - 5\,\mathrm{Im}(a_{22}^+) + 5\,\mathrm{Im}(a_{22}^-) + \mathrm{Im}(a_{13}^+) - \mathrm{Im}(a_{13}^-) - 5\,\mathrm{Im}(a_{23}^+) + 5\,\mathrm{Im}(a_{23}^-) + 5\,\mathrm{Im}(a_{33}^+) - 5\,\mathrm{Im}(a_{33}^-)$,
- $d = -\pi\,\mathrm{Re}(a_{11}^+) - \pi\,\mathrm{Re}(a_{11}^-) - \pi\,\mathrm{Re}(a_{12}^+) - \pi\,\mathrm{Re}(a_{12}^-) + 2\pi\,\mathrm{Re}(a_{22}^+) + 2\pi\,\mathrm{Re}(a_{22}^-) - \pi\,\mathrm{Re}(a_{13}^+) - \pi\,\mathrm{Re}(a_{13}^-) + 2\pi\,\mathrm{Re}(a_{23}^+) + 2\pi\,\mathrm{Re}(a_{23}^-) - 3\pi\,\mathrm{Re}(a_{33}^+) - 3\pi\,\mathrm{Re}(a_{33}^-)$,
- $\alpha = -2\,\mathrm{Im}(a_{11}^+) + 2\,\mathrm{Im}(a_{11}^-) - 2\,\mathrm{Im}(a_{13}^+) + 2\,\mathrm{Im}(a_{13}^-) + 4\,\mathrm{Im}(a_{23}^+) - 4\,\mathrm{Im}(a_{23}^-) - 6\,\mathrm{Im}(a_{33}^+) + 6\,\mathrm{Im}(a_{33}^-)$,
- $\beta = -2\,\mathrm{Im}(a_{12}^+) + 2\,\mathrm{Im}(a_{12}^-) + 4\,\mathrm{Im}(a_{22}^+) - 4\,\mathrm{Im}(a_{22}^-)$,
- $\gamma = -\mathrm{Im}(a_{03}^+) + \mathrm{Im}(a_{03}^-) + \mathrm{Im}(a_{13}^+) - \mathrm{Im}(a_{13}^-) - \mathrm{Im}(a_{23}^+) + \mathrm{Im}(a_{23}^-) + \mathrm{Im}(a_{33}^+) - \mathrm{Im}(a_{33}^-)$.

□

4.2. **The bifurcation at $z = 1$.** In what follows, we establish a connection between the coefficients of the bifurcation functions M_1 and N_1 of -1 and 1, respectively. This relation will allow us to study the simultaneous zeros of these functions. Before stating the following result, let us remember that in terms of the previous section, we have denoted the Melnikov function $M_1(r)$ as $I_m(r)$, where m is the degree of the complex function $R_m^\pm$.

Proposition 8. *Suppose that $M_1(r)$ and $N_1(r)$ are the bifurcation functions of PWCS (1) with $f(z) = \frac{i}{2}(z^2-1)$ associated to $z=-1$ and $z=1$, respectively. Then the expression of $N_1(r)$ coincides with the expression of $M_1(r)$ given in Lemma 7 where each $a_{k,l}^\mp$ satisfies $b_{k,l}^\pm = (-1)^l a_{k,l}^\mp$, for all $0 \le k \le l$ and $0 \le l \le m$.*

Proof. Using the change of variables and time $w(t) = -z(-t)$, we transform PWCS (1) with $f(z) = \frac{i}{2}(z^2-1)$ into

$$\dot{w} = \frac{i}{2}(w^2-1) + \begin{cases} \epsilon R_m^-(-w,-\overline{w}), & \text{when } \mathrm{Im}(w) > 0, \\ \epsilon R_m^+(-w,-\overline{w}), & \text{when } \mathrm{Im}(w) < 0. \end{cases} \tag{12}$$

Hence, the zeroes of the bifurcation function $N_1(r)$ of PWCS (1) are the zeroes of the bifurcation function associated to $z=-1$ of (12). Recall that

$$R_m^\mp(-w,-\overline{w}) = \sum_{l=0}^{m}\sum_{k=0}^{l} \overline{a_{k,l}^\mp}(-w)^{l-k}(-\overline{w})^k := \sum_{l=0}^{m}\sum_{k=0}^{l} \overline{b_{k,l}^\pm} w^{l-k}(\overline{w})^k, \tag{13}$$

where $\overline{b_{k,l}^\pm} = (-1)^l \overline{a_{k,l}^\mp}$, for all $0 \le k \le l$ and $0 \le l \le m$. Conjugating both sides of the equality, we can conclude that $b_{k,l}^\pm = (-1)^l a_{k,l}^\mp$, for all $0 \le k \le l$ and $0 \le l \le m$. □

We recall that from the proof of Proposition 8 it also follows that if a configuration (u,v) of limit cycles is realizable for PWCS (1), then the configuration (v,u) is also realizable.

5. Proof of Theorem B

The following proposition gives us the expressions of the Melnikov functions M_1 and N_1 as well as the maximum number of zeros that these functions separately can have.

Proposition 9. *Let $R_m^\pm$ be a complex polynomial of degree m in (1). Then, the Melnikov functions M_1 and N_1 on -1 and 1 associated to (1) are given by*

(a) *If $m=0$, then $M_1(r)=a(-1+r^2)+br$ and $N_1(r)=-M_1(r)$, where a and b depend of the coefficients $a_{0,0}^\pm$. Moreover, the maximum number of zeros of each M_1 and N_1 is* 1.

(b) *If $m=1$, then*

$$M_1(r)=\frac{1}{r}\Big(ar+br^2+cr^3+dr^4+\alpha(-1+r^2)^2\tanh^{-1}(r)\Big)$$

and

$$N_1(r)=\frac{1}{r}\Big(cr+(b+2d)r^2+ar^3-dr^4+\alpha(-1+r^2)^2\tanh^{-1}(r)\Big),$$

where a,b,c,d and α depend of the coefficients $a_{k,l}^\pm$ for $k\in\{0,\dots,l\}$ and $l\in\{0,1\}$. In addition, the maximum number of zeros of each M_1 and N_1 is 4.

(c) *If $m=2$, then*

$$M_1(r)=\frac{1}{r}\Big(ar+br^2+cr^3+dr^4+\alpha(-1+r^2)^2\tanh^{-1}(r)+\beta(-1+r^4)\tanh^{-1}(r)\Big)$$

and

$$\begin{aligned}N_1(r) &= \tfrac{1}{r}\Big(cr+(b+2d-\kappa)r^2+ar^3+(-d+\kappa)r^4+\alpha(-1+r^2)^2\tanh^{-1}(r)\\ &\quad -\beta(-1+r^4)\tanh^{-1}(r)\Big),\end{aligned}$$

where a,b,c,d,α,β and κ depend of the coefficients $a_{k,l}^\pm$ for $k\in\{0,\dots,l\}$ and $l\in\{0,1,2\}$. Furthermore, the maximum number of zeros of each M_1 and N_1 is 5.

(d) *If $m=3$, then*

$$\begin{aligned}M_1(r) &= \tfrac{1}{r}\Big(ar+br^2+cr^3+dr^4+\alpha(-1+r^2)^2\tanh^{-1}(x)+\beta(-1+r^4)\tanh^{-1}(r)\\ &\quad +\gamma r^2\tanh^{-1}(r)\Big)\end{aligned}$$

and

$$\begin{aligned}N_1(r) &= \tfrac{1}{r}\Big(cr+(b+2d-\kappa+\rho)r^2+ar^3+(-d+\kappa)r^4+\alpha(-1+r^2)^2\tanh^{-1}(r)\\ &\quad -\beta(-1+r^4)\tanh^{-1}(r)+\gamma r^2\tanh^{-1}(r)\Big),\end{aligned}$$

where $a,b,c,d,\alpha,\beta,\gamma,\kappa$ and ρ depend of the coefficients $a_{k,l}^\pm$ for $k\in\{0,\dots,l\}$ and $l\in\{0,1,2,3\}$. Furthermore, the maximum number of zeros of each M_1 and N_1 is 6.

Proof. Let us start analyzing item (a). Suppose that $m=0$. Using Lemma 7 we get that

$$M_1(r)=(\mathrm{Im}(a_{00}^-)-\mathrm{Im}(a_{00}^+))(-1+r^2)-\pi(\mathrm{Re}(a_{00}^-)+\mathrm{Re}(a_{00}^+))r.$$

Thus, $M_1(r)=a(-1+r^2)+br$, where $a=\mathrm{Im}(a_{00}^-)-\mathrm{Im}(a_{00}^+)$ and $b=-\pi(\mathrm{Re}(a_{00}^-)+\mathrm{Re}(a_{00}^+))$. In addition, Proposition 8 implies that $b_{00}^\pm=a_{00}^\mp$. Thus the first Melnikov function associated to $z=1$ is $N_1(r)=-M_1(r)$, for all $r\in(0,1)$.

Now, consider the ordered set of functions $\mathcal{G}=\{1-r^2,r\}$. Recall that the Wronskians, defined in Lemma 2, are $W_0(r)=1-r^2$, $W_1(r)=-(1+r^2)$. Hence, $W_j(r)\neq 0$ at $(0,1)$, for all $j\in\{0,1\}$. According to Lemma 2 $\mathcal{G}$ is an ECT-system. Moreover, Theorem 3 implies that one is the maximum number of positive roots for any element of $Span(\mathcal{G})$ and there is a choice for a_0 and a_1 such that any element of $Span(\mathcal{G})$ has exactly a unique zero.

Now, we prove item (b). Suppose that $m = 1$. From Lemma 7 we know that

$$M_1(r) = \frac{1}{r}\Big(ar + br^2 + cr^3 + dr^4 + \alpha(-1+r^2)^2 \tanh^{-1}(r)\Big),$$

where

- $a = \operatorname{Im}(a_{00}^+) - \operatorname{Im}(a_{00}^-) - \operatorname{Im}(a_{01}^+) + \operatorname{Im}(a_{01}^-) + \operatorname{Im}(a_{11}^+) - \operatorname{Im}(a_{11}^-)$,
- $b = -\pi \operatorname{Re}(a_{00}^+) - \pi \operatorname{Re}(a_{00}^-) + \pi \operatorname{Re}(a_{11}^+) + \pi \operatorname{Re}(a_{11}^-)$,
- $c = -\operatorname{Im}(a_{00}^+) + \operatorname{Im}(a_{00}^-) - \operatorname{Im}(a_{01}^+) + \operatorname{Im}(a_{01}^-) + \operatorname{Im}(a_{11}^+) - \operatorname{Im}(a_{11}^-)$,
- $d = -\pi \operatorname{Re}(a_{11}^+) - \pi \operatorname{Re}(a_{11}^-)$,
- $\alpha = -2\operatorname{Im}(a_{11}^+) + 2\operatorname{Im}(a_{11}^-)$.

To obtain the expression of N_1 in terms of the coefficients of M_1, it is enough to use Lemma 8, which guarantees that $b_{k,l}^{\pm} = (-1)^l a_{k,l}^{\mp}$, for all $0 \le k \le l$ and $0 \le l \le 1$. Then

- $\tilde{a} = \operatorname{Im}(a_{00}^-) - \operatorname{Im}(a_{00}^+) + \operatorname{Im}(a_{01}^-) - \operatorname{Im}(a_{01}^+) - \operatorname{Im}(a_{11}^-) + \operatorname{Im}(a_{11}^+) = c$,
- $\tilde{b} = -\pi \operatorname{Re}(a_{00}^-) - \pi \operatorname{Re}(a_{00}^+) - \pi \operatorname{Re}(a_{11}^-) - \pi \operatorname{Re}(a_{11}^+) = b + 2d$,
- $\tilde{c} = -\operatorname{Im}(a_{00}^-) + \operatorname{Im}(a_{00}^+) + \operatorname{Im}(a_{01}^-) - \operatorname{Im}(a_{01}^+) - \operatorname{Im}(a_{11}^-) + \operatorname{Im}(a_{11}^+) = a$,
- $\tilde{d} = \pi \operatorname{Re}(a_{11}^-) + \pi \operatorname{Re}(a_{11}^+) = -d$,
- $\tilde{\alpha} = 2\operatorname{Im}(a_{11}^-) - 2\operatorname{Im}(a_{11}^+) = \alpha$.

Therefore,

$$\begin{aligned} N_1(r) &= \tfrac{1}{r}\Big(\tilde{a}r + \tilde{b}r^2 + \tilde{c}r^3 + \tilde{d}r^4 + \tilde{\alpha}(-1+r^2)^2\tanh^{-1}(r)\Big) \\ &= \tfrac{1}{r}\Big(cr + (b+2d)r^2 + ar^3 - dr^4 + \alpha(-1+r^2)^2\tanh^{-1}(r)\Big). \end{aligned}$$

Let $\mathcal{F}$ be the ordered set of functions $\{r, r^2, r^3, r^4, (-1+r^2)^2\tanh^{-1}(r)\}$. Emphasize that the Wronskians, defined in Lemma 2, are $W_0(r) = r$, $W_1(r) = r^2$, $W_2(r) = 2r^3$, $W_3(r) = 12r^4$, and

$$W_4(r) = 96\left(\frac{r(-3+5r^2)}{(-1+r^2)^2} + 3\tanh^{-1}(r)\right).$$

The Wronskians $W_j(r) \neq 0$ at $(0,1)$, for all $j \in \{0,1,2,3\}$. Even more, $W_4(r) \neq 0$, at $(0,1)$. Indeed, computing the first derivative of the Wronskian we obtain

$$\frac{dW_4}{dr} = \frac{768r^4}{(1-r^2)^3} > 0,$$

for all $r \in (0,1)$. Since $W_4(0) = 0$ and W_4 is increasing at $(0,1)$, we can conclude that $W_4(r) > 0$ for all $r \in (0,1)$. According to Lemma 2 $\mathcal{F}$ is an ECT-system. In addition, Theorem 3 implies that four is the maximum number of positive roots for any element of $Span(\mathcal{F})$ and there is a choice for a, b, c, d and α such that any element of $Span(\mathcal{F})$ has exactly four zeros.

In what follows, we prove item (c). Suppose that $m = 2$. From Lemma 7 we know that

$$M_1(r) = \frac{1}{r}\Big(ar + br^2 + cr^3 + dr^4 + \alpha(-1+r^2)^2\tanh^{-1}(r) + \beta(-1+r^4)\tanh^{-1}(r)\Big),$$

where

- $a = \operatorname{Im}(a_{00}^+) - \operatorname{Im}(a_{00}^-) - \operatorname{Im}(a_{01}^+) + \operatorname{Im}(a_{01}^-) + \operatorname{Im}(a_{11}^+) - \operatorname{Im}(a_{11}^-) + \operatorname{Im}(a_{02}^+) - \operatorname{Im}(a_{02}^-) - \operatorname{Im}(a_{12}^+) + \operatorname{Im}(a_{12}^-) + 5\operatorname{Im}(a_{22}^+) - 5\operatorname{Im}(a_{22}^-)$,
- $b = -\pi \operatorname{Re}(a_{00}^+) - \pi \operatorname{Re}(a_{00}^-) + \pi \operatorname{Re}(a_{11}^+) + \pi \operatorname{Re}(a_{11}^-) + \pi \operatorname{Re}(a_{02}^-) + \pi \operatorname{Re}(a_{02}^+) - \pi \operatorname{Re}(a_{22}^+) - \pi \operatorname{Re}(a_{22}^-)$,
- $c = -\operatorname{Im}(a_{00}^+) + \operatorname{Im}(a_{00}^-) - \operatorname{Im}(a_{01}^+) + \operatorname{Im}(a_{01}^-) + \operatorname{Im}(a_{11}^+) - \operatorname{Im}(a_{11}^-) - \operatorname{Im}(a_{02}^+) + \operatorname{Im}(a_{02}^-) + \operatorname{Im}(a_{12}^+) - \operatorname{Im}(a_{12}^-) - 5\operatorname{Im}(a_{22}^+) + 5\operatorname{Im}(a_{22}^-)$,
- $d = -\pi \operatorname{Re}(a_{11}^+) - \pi \operatorname{Re}(a_{11}^-) - \pi \operatorname{Re}(a_{12}^+) - \pi \operatorname{Re}(a_{12}^-) + 2\pi \operatorname{Re}(a_{22}^+) + 2\pi \operatorname{Re}(a_{22}^-)$,

- $\alpha = -2\,\mathrm{Im}(a_{11}^{+}) + 2\,\mathrm{Im}(a_{11}^{-})$,
- $\beta = -2\,\mathrm{Im}(a_{12}^{+}) + 2\,\mathrm{Im}(a_{12}^{-}) + 4\,\mathrm{Im}(a_{22}^{+}) - 4\,\mathrm{Im}(a_{22}^{-})$.

To get the expression of N_1 in terms of the coefficients of M_1, it is enough to use Lemma 8, which guarantees that $b_{k,l}^{\pm} = (-1)^l a_{k,l}^{\mp}$, for all $0 \le k \le l$ and $0 \le l \le 1$. Then

- $\tilde{a} = -\,\mathrm{Im}(a_{00}^{+}) + \mathrm{Im}(a_{00}^{-}) - \mathrm{Im}(a_{01}^{+}) + \mathrm{Im}(a_{01}^{-}) + \mathrm{Im}(a_{11}^{+}) - \mathrm{Im}(a_{11}^{-}) - \mathrm{Im}(a_{02}^{+}) + \mathrm{Im}(a_{02}^{-}) + \mathrm{Im}(a_{12}^{+}) - \mathrm{Im}(a_{12}^{-}) - 5\,\mathrm{Im}(a_{22}^{+}) + 5\,\mathrm{Im}(a_{22}^{-}) = c$,
- $\tilde{b} = -\pi\,\mathrm{Re}(a_{00}^{-}) - \pi\,\mathrm{Re}(a_{00}^{+}) - \pi\,\mathrm{Re}(a_{11}^{-}) - \pi\,\mathrm{Re}(a_{11}^{+}) + \pi\,\mathrm{Re}(a_{02}^{+}) + \pi\,\mathrm{Re}(a_{02}^{-}) - \pi\,\mathrm{Re}(a_{22}^{-}) - \pi\,\mathrm{Re}(a_{22}^{+}) = b + 2d - \kappa$,
- $\tilde{c} = \mathrm{Im}(a_{00}^{+}) - \mathrm{Im}(a_{00}^{-}) - \mathrm{Im}(a_{01}^{+}) + \mathrm{Im}(a_{01}^{-}) + \mathrm{Im}(a_{11}^{+}) - \mathrm{Im}(a_{11}^{-}) + \mathrm{Im}(a_{02}^{+}) - \mathrm{Im}(a_{02}^{-}) - \mathrm{Im}(a_{12}^{+}) + \mathrm{Im}(a_{12}^{-}) + 5\,\mathrm{Im}(a_{22}^{+}) - 5\,\mathrm{Im}(a_{22}^{-}) = a$,
- $\tilde{d} = \pi\,\mathrm{Re}(a_{11}^{-}) + \pi\,\mathrm{Re}(a_{11}^{+}) - \pi\,\mathrm{Re}(a_{12}^{-}) - \pi\,\mathrm{Re}(a_{12}^{+}) + 2\pi\,\mathrm{Re}(a_{22}^{-}) + 2\pi\,\mathrm{Re}(a_{22}^{+}) = -d + \kappa$,
- $\tilde{\alpha} = 2\,\mathrm{Im}(a_{11}^{-}) - 2\,\mathrm{Im}(a_{11}^{+}) = \alpha$,
- $\tilde{\beta} = -2\,\mathrm{Im}(a_{12}^{-}) + 2\,\mathrm{Im}(a_{12}^{+}) + 4\,\mathrm{Im}(a_{22}^{-}) - 4\,\mathrm{Im}(a_{22}^{+}) = -\beta$.

where $\kappa = -2\pi\,\mathrm{Re}(a_{12}^{-}) - 2\pi\,\mathrm{Re}(a_{12}^{+}) + 4\pi\,\mathrm{Re}(a_{22}^{-}) + 4\pi\,\mathrm{Re}(a_{22}^{+})$. Therefore,

$$\begin{aligned} N_1(r) &= \tfrac{1}{r}\Big(\tilde{a}r + \tilde{b}r^2 + \tilde{c}r^3 + \tilde{d}r^4 + \tilde{\alpha}(-1+r^2)^2\tanh^{-1}(r) + \tilde{\beta}(-1+r^4)\tanh^{-1}(r)\Big), \\ &= \tfrac{1}{r}\Big(cr + (b+2d-\kappa)r^2 + ar^3 + (-d+\kappa)r^4 + \alpha(-1+r^2)^2\tanh^{-1}(r) \\ &\quad -\beta(-1+r^4)\tanh^{-1}(r)\Big). \end{aligned}$$

Consider the ordered set of functions $\mathcal{F} = \{r, r^2, r^3, r^4, (-1+r^2)^2\tanh^{-1}(r), (-1+r^4)\tanh^{-1}(r)\}$. From case (b), we know that the Wronskians $W_j(r) \neq 0$ at $(0,1)$, for all $j \in \{0,1,2,3,4\}$. We claim that $W_5(r) \neq 0$ for all $r \in (0,1)$, which is given by

$$W_5(r) = \frac{3072(-15r + 22r^3 - 3r^5 + 3(-1+r^2)^2(5+r^2)\tanh^{-1}(r))}{(-1+r^2)^6}.$$

Computing the first derivative of this Wronskian we obtain

$$\begin{aligned} \frac{dW_5(r)}{dr} &= \frac{18432r(-21r + 32r^3 - 3r^5 + 3(-1+r^2)^2(7+r^2)\tanh^{-1}(r))}{(1-r^2)^7} \\ &> \frac{18432r(-21r + 32r^3 - 3r^5 + 3(-1+r^2)^2(7+r^2)(r + \frac{r^3}{3} + \frac{r^5}{5}))}{(1-r^2)^7} \\ &= \frac{18432r^6(16 + r^2 + 20r^4 + 3r^6)}{5(1-r^2)^7} \\ &> 0, \end{aligned}$$

where we have used that $\tanh^{-1}(r) > r + \frac{r^3}{3} + \frac{r^5}{5}$, for all $r \in (0,1)$, which can be proven using the Mean Value Theorem. Since $W_5(r)(0) = 0$ and $W_5(r)$ is increasing, we can conclude that $W_5(r) > 0$ for all $r \in (0,1)$. According to Lemma 2, $\mathcal{F}$ is an ECT-system. In addition, Theorem 3 implies that $\mathcal{F}$ has at least 5 simple zeros in $(0,1)$.

Finally, we prove item (d). Suppose that $m = 3$. From Lemma 7 we know that

$$M_1(r) = \frac{1}{r}\Big(ar + br^2 + cr^3 + dr^4 + \alpha(-1+r^2)^2\tanh^{-1}(r) + \beta(-1+r^4)\tanh^{-1}(r) + \gamma r^2\tanh^{-1}(r)\Big),$$

where

- $a = \mathrm{Im}(a_{00}^{+}) - \mathrm{Im}(a_{00}^{-}) - \mathrm{Im}(a_{01}^{+}) + \mathrm{Im}(a_{01}^{-}) + \mathrm{Im}(a_{11}^{+}) - \mathrm{Im}(a_{11}^{-}) + \mathrm{Im}(a_{02}^{+}) - \mathrm{Im}(a_{02}^{-}) - \mathrm{Im}(a_{12}^{+}) + \mathrm{Im}(a_{12}^{-}) + 5\,\mathrm{Im}(a_{22}^{+}) - 5\,\mathrm{Im}(a_{22}^{-}) - \mathrm{Im}(a_{03}^{+}) + \mathrm{Im}(a_{03}^{-}) + \mathrm{Im}(a_{13}^{+}) - \mathrm{Im}(a_{13}^{-}) - 5\,\mathrm{Im}(a_{23}^{+}) + 5\,\mathrm{Im}(a_{23}^{-}) + 5\,\mathrm{Im}(a_{33}^{+}) - 5\,\mathrm{Im}(a_{33}^{-})$,
- $b = -\pi\,\mathrm{Re}(a_{00}^{+}) - \pi\,\mathrm{Re}(a_{00}^{-}) + \pi\,\mathrm{Re}(a_{11}^{+}) + \pi\,\mathrm{Re}(a_{11}^{-}) + \pi\,\mathrm{Re}(a_{02}^{-}) + \pi\,\mathrm{Re}(a_{02}^{+}) - \pi\,\mathrm{Re}(a_{22}^{+}) - \pi\,\mathrm{Re}(a_{22}^{-}) - 2\pi\,\mathrm{Re}(a_{03}^{-}) - 2\pi\,\mathrm{Re}(a_{03}^{+}) - \pi\,\mathrm{Re}(a_{13}^{+}) - \pi\,\mathrm{Re}(a_{13}^{-}) + \pi\,\mathrm{Re}(a_{33}^{+}) + \pi\,\mathrm{Re}(a_{33}^{-})$,

- $c = -\operatorname{Im}(a_{00}^+) + \operatorname{Im}(a_{00}^-) - \operatorname{Im}(a_{01}^+) + \operatorname{Im}(a_{01}^-) + \operatorname{Im}(a_{11}^+) - \operatorname{Im}(a_{11}^-) - \operatorname{Im}(a_{02}^+) + \operatorname{Im}(a_{02}^-) + \operatorname{Im}(a_{12}^+) - \operatorname{Im}(a_{12}^-) - 5\operatorname{Im}(a_{22}^+) + 5\operatorname{Im}(a_{22}^-) + \operatorname{Im}(a_{13}^+) - \operatorname{Im}(a_{13}^-) - 5\operatorname{Im}(a_{23}^+) + 5\operatorname{Im}(a_{23}^-) + 5\operatorname{Im}(a_{33}^+) - 5\operatorname{Im}(a_{33}^-)$,
- $d = -\pi\operatorname{Re}(a_{11}^+) - \pi\operatorname{Re}(a_{11}^-) - \pi\operatorname{Re}(a_{12}^+) - \pi\operatorname{Re}(a_{12}^-) + 2\pi\operatorname{Re}(a_{22}^+) + 2\pi\operatorname{Re}(a_{22}^-) - \pi\operatorname{Re}(a_{13}^+) - \pi\operatorname{Re}(a_{13}^-) + 2\pi\operatorname{Re}(a_{23}^+) + 2\pi\operatorname{Re}(a_{23}^-) - 3\pi\operatorname{Re}(a_{33}^+) - 3\pi\operatorname{Re}(a_{33}^-)$,
- $\alpha = -2\operatorname{Im}(a_{11}^+) + 2\operatorname{Im}(a_{11}^-) - 2\operatorname{Im}(a_{13}^+) + 2\operatorname{Im}(a_{13}^-) + 4\operatorname{Im}(a_{23}^+) - 4\operatorname{Im}(a_{23}^-) - 6\operatorname{Im}(a_{33}^+) + 6\operatorname{Im}(a_{33}^-)$,
- $\beta = -2\operatorname{Im}(a_{12}^+) + 2\operatorname{Im}(a_{12}^-) + 4\operatorname{Im}(a_{22}^+) - 4\operatorname{Im}(a_{22}^-)$,
- $\gamma = -\operatorname{Im}(a_{03}^+) + \operatorname{Im}(a_{03}^-) + \operatorname{Im}(a_{13}^+) - \operatorname{Im}(a_{13}^-) - \operatorname{Im}(a_{23}^+) + \operatorname{Im}(a_{23}^-) + \operatorname{Im}(a_{33}^+) - \operatorname{Im}(a_{33}^-)$.

To get the expression of N_1 in terms of the coefficients of M_1, it is enough to use Lemma 8, which guarantees that $b_{k,l}^{\pm} = (-1)^l a_{k,l}^{\mp}$, for all $0 \le k \le l$ and $0 \le l \le 3$. Then

- $\tilde{a} = -\operatorname{Im}(a_{00}^+) + \operatorname{Im}(a_{00}^-) - \operatorname{Im}(a_{01}^+) + \operatorname{Im}(a_{01}^-) + \operatorname{Im}(a_{11}^+) - \operatorname{Im}(a_{11}^-) - \operatorname{Im}(a_{02}^+) + \operatorname{Im}(a_{02}^-) + \operatorname{Im}(a_{12}^+) - \operatorname{Im}(a_{12}^-) - 5\operatorname{Im}(a_{22}^+) + 5\operatorname{Im}(a_{22}^-) + \operatorname{Im}(a_{13}^+) - \operatorname{Im}(a_{13}^-) - 5\operatorname{Im}(a_{23}^+) + 5\operatorname{Im}(a_{23}^-) + 5\operatorname{Im}(a_{33}^+) - 5\operatorname{Im}(a_{33}^-) = c$,
- $\tilde{b} = -\pi\operatorname{Re}(a_{00}^-) - \pi\operatorname{Re}(a_{00}^+) - \pi\operatorname{Re}(a_{11}^-) - \pi\operatorname{Re}(a_{11}^+) + \pi\operatorname{Re}(a_{02}^+) + \pi\operatorname{Re}(a_{02}^-) - \pi\operatorname{Re}(a_{22}^-) - \pi\operatorname{Re}(a_{22}^+) + 2\pi\operatorname{Re}(a_{03}^+) + 2\pi\operatorname{Re}(a_{03}^-) + \pi\operatorname{Re}(a_{13}^-) + \pi\operatorname{Re}(a_{13}^+) - \pi\operatorname{Re}(a_{33}^-) - \pi\operatorname{Re}(a_{33}^+) = b + 2d - \kappa + \rho$,
- $\tilde{c} = \operatorname{Im}(a_{00}^+) - \operatorname{Im}(a_{00}^-) - \operatorname{Im}(a_{01}^+) + \operatorname{Im}(a_{01}^-) + \operatorname{Im}(a_{11}^+) - \operatorname{Im}(a_{11}^-) + \operatorname{Im}(a_{02}^+) - \operatorname{Im}(a_{02}^-) - \operatorname{Im}(a_{12}^+) + \operatorname{Im}(a_{12}^-) + 5\operatorname{Im}(a_{22}^+) - 5\operatorname{Im}(a_{22}^-) - \operatorname{Im}(a_{03}^+) + \operatorname{Im}(a_{03}^-) + \operatorname{Im}(a_{13}^+) - \operatorname{Im}(a_{13}^-) - 5\operatorname{Im}(a_{23}^+) + 5\operatorname{Im}(a_{23}^-) + 5\operatorname{Im}(a_{33}^+) - 5\operatorname{Im}(a_{33}^-) = a$,
- $\tilde{d} = \pi\operatorname{Re}(a_{11}^-) + \pi\operatorname{Re}(a_{11}^+) - \pi\operatorname{Re}(a_{12}^-) - \pi\operatorname{Re}(a_{12}^+) + 2\pi\operatorname{Re}(a_{22}^-) + 2\pi\operatorname{Re}(a_{22}^+) + \pi\operatorname{Re}(a_{13}^-) + \pi\operatorname{Re}(a_{13}^+) - 2\pi\operatorname{Re}(a_{23}^-) - 2\pi\operatorname{Re}(a_{23}^+) + 3\pi\operatorname{Re}(a_{33}^-) + 3\pi\operatorname{Re}(a_{33}^+) = -d + \kappa$,
- $\tilde{\alpha} = 2\operatorname{Im}(a_{11}^-) - 2\operatorname{Im}(a_{11}^+) + 2\operatorname{Im}(a_{13}^-) - 2\operatorname{Im}(a_{13}^+) - 4\operatorname{Im}(a_{23}^-) + 4\operatorname{Im}(a_{23}^+) + 6\operatorname{Im}(a_{33}^-) - 6\operatorname{Im}(a_{33}^+) = \alpha$,
- $\tilde{\beta} = -2\operatorname{Im}(a_{12}^-) + 2\operatorname{Im}(a_{12}^+) + 4\operatorname{Im}(a_{22}^-) - 4\operatorname{Im}(a_{22}^+) = -\beta$,
- $\tilde{\gamma} = \operatorname{Im}(a_{03}^-) - \operatorname{Im}(a_{03}^+) - \operatorname{Im}(a_{13}^-) + \operatorname{Im}(a_{13}^+) + \operatorname{Im}(a_{23}^-) - \operatorname{Im}(a_{23}^+) - \operatorname{Im}(a_{33}^-) + \operatorname{Im}(a_{33}^+) = \gamma$.

where $\rho = 4\pi\Big(\operatorname{Re}(a_{03}^+ + \operatorname{Re}(a_{03}^- + \operatorname{Re}(a_{13}^+ + \operatorname{Re}(a_{13}^- - \operatorname{Re}(a_{23}^+ - \operatorname{Re}(a_{23}^- + \operatorname{Re}(a_{33}^+) + \operatorname{Re}(a_{33}^-)\Big)$. Therefore,

$$\begin{aligned} N_1(r) &= \tfrac{1}{r}\Big(\tilde{a}r + \tilde{b}r^2 + \tilde{c}r^3 + \tilde{d}r^4 + \tilde{\alpha}(-1+r^2)^2\tanh^{-1}(r) + \tilde{\beta}(-1+r^4)\tanh^{-1}(r) \\ &\quad + \tilde{\gamma}r^2\tanh^{-1}(r)\Big), \\ &= \tfrac{1}{r}\Big(cr + (b + 2d - \kappa + \rho)r^2 + ar^3 + (-d+\kappa)r^4 + \alpha(-1+r^2)^2\tanh^{-1}(r) \\ &\quad -\beta(-1+r^4)\tanh^{-1}(r) + \gamma r^2\tanh^{-1}(r)\Big). \end{aligned}$$

Consider the ordered set of functions

$$\mathcal{F} = \{r, r^2, r^3, r^4, (-1+r^2)^2\tanh^{-1}(r), (-1+r^4)\tanh^{-1}(r), r^2\tanh^{-1}(r)\}.$$

From cases (b) and (c), we know that the Wronskians $W_j(r) \neq 0$ at $(0,1)$, for all $j \in \{0,1,2,3,4,5\}$. We claim that $W_6(r) \neq 0$ for all $r \in (0,1)$, which is given by

$$W_6(r) = \frac{294912r(-r(105 - 145r^2 + 15r^4 + 9r^6) + 3(-1+r^2)^2(35 + 10r^2 + 3r^4)\tanh^{-1}(r))}{(-1+r^2)^{12}}.$$

Computing the first derivative of this Wronskian we obtain

$$\begin{array}{rcl}\dfrac{dW_6(r)}{dr} & = & \dfrac{294912\Big(-r(105+1910r^2-2864r^4+330r^6+135r^8)+15(-1+r^2)^2(7+139r^2+37r^4+9r^6)\tanh^{-1}(r)\Big)}{(1-r^2)^{13}} \\ & > & \dfrac{294912\Big(-r(105+1910r^2-2864r^4+330r^6+135r^8)+15(-1+r^2)^2(7+139r^2+37r^4+9r^6)(r+\frac{r^3}{3}+\frac{r^5}{5}+\frac{r^7}{7})\Big)}{(1-r^2)^{13}} \\ & = & \frac{294912r^9(601-346r^2+1824r^4+474r^6+135r^8)}{7(1-r^2)^{13}} \\ & > & \frac{294912r^9(255+1824r^4+474r^6+135r^8)}{7(1-r^2)^{13}} \\ & > & 0,\end{array}$$

where we have used that $\tanh^{-1}(r) > r + \frac{r^3}{3} + \frac{r^5}{5} + \frac{r^7}{7}$, for all $r \in (0,1)$, which can be proven using the Mean Value Theorem. Since $W_6(r)(0) = 0$ and $W_6(r)$ is increasing, we can conclude that $W_6(r) > 0$ for all $r \in (0,1)$. According to Lemma 2, $\mathcal{F}$ is an ECT-system. In addition, Theorem 3 implies that $\mathcal{F}$ has at least 6 simple zeros in $(0,1)$. □

5.1. **Proof of Theorem B(a).** According to Proposition 9, the bifurcation function of PWHS (1) associated to $z=-1$ and $z=1$ are given respectively by

$$M_1(r) = a(-1+r^2) + br \quad \text{and} \quad N_1(r) = -M_1(r),$$

where a and b are real coefficients, which depend of the real and imaginary parts of the complex coefficients $a^{\pm}_{0,0}$. Even more, the maximum number of zeros that these functions separately can have is 1 and since $N_1(r) = -M_1(r)$ for all $r \in (0,1)$, then these polynomials have simultaneously associated at most a unique positive real root on $(0,1)$.

5.2. **Proof of Theorem B(b).** From Proposition 9, the bifurcation function of PWHS (1) associated to $z=-1$ and $z=1$ are given respectively by $M_1(r) = \frac{1}{r}f(r)$ and $N_1(r) = \frac{1}{r}g(r)$, where

$$f(r) = ar + br^2 + cr^3 + dr^4 + \alpha(-1+r^2)^2\tanh^{-1}(r),$$

and

$$g(r) = cr + (b+2d)r^2 + ar^3 - dr^4 + \alpha(-1+r^2)^2\tanh^{-1}(r),$$

with a,b,c,d and α real coefficients, which depend of the real and imaginary parts of the complex coefficients $a^{\pm}_{k,l}$, for all $0 \le k \le l$ and $0 \le l \le 1$. Furthermore, the maximum number of zeros that these functions separately can have is 4. Let us study the number of simultaneous zeros that M_1 and N_1 possess.

5.2.1. *Case:* $\alpha = 0$. We can assume that $d = 1$ and

$$\begin{array}{rcl} M_1(r) & = & (r-r_1)(r-r_2)(r-r_3). \\ & = & r^3 - (r_1+r_2+r_3)r^2 + (r_1r_2+r_1r_3+r_2r_3)r - r_1r_2r_3,\end{array}$$

where $r_1, r_2, r_3 \in (0,1)$ and $\lambda = (r_1,r_2,r_3) \in (0,1)^3 \subset \mathbb{R}^3$. This implies that $N_1(r) = q_\lambda(r)$ such that

$$q_\lambda(r) = -r^3 - (r_1r_2r_3)r^2 + (r_1r_2+r_1r_3+r_2r_3+2)r - (r_1+r_2+r_3).$$

Thus, $q_\lambda(0) = -(r_1+r_2+r_3) < 0$, $q_\lambda(1) = -(-1+r_1)(-1+r_2)(-1+r_3) > 0$ and $\Delta(r_1,r_2,r_3) := \Delta_r(q_\lambda)$, where

$$\begin{array}{rcl}\Delta_r(q_\lambda) & = & 32 - 27r_1^2 - 6r_1r_2 - 27r_2^2 + 24r_1^2r_2^2 + 4r_1^3r_2^3 - 6r_1r_3 - 6r_2r_3 + 12r_1^2r_2r_3 \\ & & +12r_1r_2^2r_3 - 6r_1^3r_2^2r_3 - 6r_1^2r_2^3r_3 - 27r_3^2 + 24r_1^2r_3^2 + 12r_1r_2r_3^2 - 6r_1^3r_2r_3^2 \\ & & +24r_2^2r_3^2 - 26r_1^2r_2^2r_3^2 - 6r_1r_2^3r_3^2 + 4r_1^3r_2^3r_3^2 + r_1^4r_2^4r_3^2 + 4r_1^3r_3^3 - 6r_1^2r_2r_3^3 \\ & & -6r_1r_2^2r_3^3 + 4r_1^3r_2^2r_3^3 + 4r_2^3r_3^3 + 4r_1^2r_2^3r_3^3 - 2r_1^4r_2^3r_3^3 - 2r_1^3r_2^4r_3^3 + r_1^4r_2^2r_3^4 \\ & & -2r_1^3r_2^3r_3^4 + r_1^2r_2^4r_3^4.\end{array}$$

It is easy to verify that there is no minimum inside the box $[0,1]^3$, the minimum of the function Δ on the box $[0,1]^3$ is 0 and it is reached on the boundary at the point $(0,1,1)$. This tells us that $\Delta_r(q_\lambda) > 0$ for all λ. In addition, taking $\lambda_0 = (1/3, 1/2, 2/3)$ we get

$$q_{\lambda_0}(r) = \frac{1}{18}(-27 + 49r - 2r^2 - 18r^3)$$

and from the Sturm's sequences of q_{λ_0}, we obtain that it has a unique real root on $(0,1)$.

From Lemma 4, we get that q_λ also has a unique real root in that interval for all $\lambda \in (0,1)^3$. Therefore, $z_{M_1} = 3$ and $z_{N_1} = 1$.

Now, if $z_{M_1} = 2$, then all configurations for z_{N_1} are possible, i.e. $0 \le z_{N_1} \le 2$. Indeed, taking $r_1 = 13/50$, $r_2 = 6/25$ and $r_3 \in \mathbb{R} \setminus (0,1)$ as the roots of M_1, we get that

- If $r_3 = 2$ is root of M_1. Then
$$q_\lambda(r) = -\frac{5}{2} + \frac{1914}{625}r - \frac{78}{625}r^2 - r^3.$$
From the Sturm's sequences of g, we get that it has no real root on $(0,1)$.
- If $r_3 = -1/100$ is root of M_1. Then
$$q_\lambda(r) = -\frac{49}{100} + \frac{10287}{5000}r + \frac{39}{62500}r^2 - r^3.$$
By the Sturm's sequences of g, we obtain that it has a unique real root on $(0,1)$.
- If $r_3 = 101/100$ is root of M_1. Then
$$q_\lambda(r) = -\frac{151}{100} + \frac{12837}{5000}r - \frac{3939}{62500}r^2 - r^3.$$
From the Sturm's sequences of g, we get that it has two simple real roots on $(0,1)$.

5.2.2. *Case:* $\alpha \neq 0$. Assume that $\alpha = 1$. We claim that in this case it is possible to obtain the configuration $(z_{M_1}, z_{N_1}) = (4,0)$. Indeed, suppose that $r_1 = 1/4, r_2 = 1/3, r_3 = 1/2$ and $r_4 = 2/3$ are roots of M_1 in $(0,1)$, this implies that:

- $a = \frac{1}{6}(-3645\tanh^{-1}(1/4) + 4352\tanh^{-1}(1/3) - 1215\tanh^{-1}(1/2) + 130\tanh^{-1}(2/3))$,
- $b = \frac{1}{2}(2205\tanh^{-1}(1/4) - 2688\tanh^{-1}(1/3) + 801\tanh^{-1}(1/2) - 95\tanh^{-1}(2/3))$,
- $c = -\frac{1}{6}(270\tanh^{-1}(1/4) - 256\tanh^{-1}(1/3) + 54\tanh^{-1}(1/2) - 5\tanh^{-1}(2/3))$,
- $d = -405\tanh^{-1}(1/4) + 512\tanh^{-1}(1/3) + 162\tanh^{-1}(1/2) - 20\tanh^{-1}(2/3)$.

By using that $\tanh^{-1}(r) > r$ for all $r \in (0,1)$, we obtain

$$\begin{array}{rcl} g(r) & = & cr + (b+2d)r^2 + ar^3 - dr^4 + (-1+r^2)^2\tanh^{-1}(r) \\ & > & h(r), \end{array}$$

for all $r \in (0,1)$, where $h(r) = cr + (b+2d)r^2 + ar^3 - dr^4 + r(-1+r^2)^2$. From the Sturm's sequences of h, we get that it has no real roots on $(0,1)$ and therefore N_1 does not have them either. Consequently, $z_{M_1} = 4$ and $z_{N_1} = 0$.

5.3. **Proof of Theorem B(c).** According Proposition 9, the bifurcation function of PWHS (1) associated to $z = -1$ and $z = 1$ are given respectively by $M_1(r) = \frac{1}{r}f(r)$ and $N_1(r) = \frac{1}{r}g(r)$, where

$$f(r) = ar + br^2 + cr^3 + dr^4 + \alpha(-1+r^2)^2\tanh^{-1}(r) + \beta(-1+r^4)\tanh^{-1}(r)$$

and

$$g(r) = cr + (b+2d-\kappa)r^2 + ar^3 + (-d+\kappa)r^4 + \alpha(-1+r^2)^2\tanh^{-1}(r) - \beta(-1+r^4)\tanh^{-1}(r),$$

with $a, b, c, d, \alpha, \beta$ and κ real coefficients, which depend of the real and imaginary parts of the complex coefficients $a^{\pm}_{k,l}$, for all $0 \le k \le l$ and $0 \le l \le 2$. Furthermore, the maximum number of zeros that these functions separately can have is 5. Let us study the number of simultaneous zeros that M_1 and N_1 possess.

5.3.1. *Case:* $\alpha = \beta = 0$. If $\kappa = 0$, then by the proof of item (b), we have that $0 \leq z_{M_1} + z_{N_1} \leq 4$. Let us analyze the case $\kappa \neq 0$. Then we can assume that $d = 1$ and

$$\begin{array}{rcl} M_1(r) & = & (r - r_1)(r - r_2)(r - r_3). \\ & = & r^3 - (r_1 + r_2 + r_3)r^2 + (r_1 r_2 + r_1 r_3 + r_2 r_3)r - r_1 r_2 r_3, \end{array}$$

where $r_1, r_2, r_3 \in (0,1)$ and $\lambda = (r_1, r_2, r_3, \kappa) \in (0,1)^3 \times \mathbb{R} \setminus \{0\} \subset \mathbb{R}^4$. This implies that $N_1(r) = q_\lambda(r)$ such that

$$q_\lambda(r) = (-1 + \kappa)r^3 - (r_1 r_2 r_3)r^2 + (r_1 r_2 + r_1 r_3 + r_2 r_3 + 2 - \kappa)r - (r_1 + r_2 + r_3).$$

Subcase: $\kappa = 1$. In this case,

$$q_\lambda(r) = -(r_1 r_2 r_3)r^2 + (1 + r_2 r_3 + r_1 r_2 + r_1 r_3)r - (r_1 + r_2 + r_3).$$

Due to $q_\lambda(0) = -(r_1 + r_2 + r_3) < 0$ and $q_\lambda(1) = -((-1 + r_1)(-1 + r_2)(-1 + r_3)) > 0$, then there exist a root $r^* \in (0,1)$ of q_λ in (0,1), which is given by

$$r^* = \frac{1 + r_1 r_2 + r_1 r_3 + r_2 r_3 - \sqrt{D^*}}{2 r_1 r_2 r_3},$$

where $D^* = 1 + r_1^2(r_2^2 + r_3^2) + r_2^2 r_3^2 + 2r_2 r_3(1 - r_1^2) + 2r_1 r_3(1 - r_2^2) + 2r_1 r_2(1 - r_3^2) > 0$, for all $r_1, r_2, r_3 \in (0,1)^3$. To verify that r^* is the unique root in $(0,1)$, just note that the other root r^{**} of q_λ is greater than 1. Indeed,

$$\begin{array}{rcl} r^{**} - 1 & = & \frac{1 + r_1 r_2 + r_1 r_3 + r_2 r_3 + \sqrt{D^*}}{2 r_1 r_2 r_3} - 1 \\ & = & \frac{1 + r_1 r_2 (1 - r_3) + r_1 r_3 + r_2 r_3 + \sqrt{D^*}}{2 r_1 r_2 r_3} \\ & > & 0, \end{array}$$

for all $r_1, r_2, r_3 \in (0,1)^3$. Thus, $z_{N_1} = 1$.
Subcase: $\kappa \neq 1$. Here, $q_\lambda(0) = -(r_1 + r_2 + r_3) < 0$, $q_\lambda(1) = -(-1 + r_1)(-1 + r_2)(-1 + r_3) > 0$, which implies that there is at least one root in the interval $(0,1)$. In fact this will be the unique root of q_λ in that interval. For that, emphasize that $\Delta_r(q_\lambda) = 4\kappa^4 + \eta_3\kappa^3 + \eta_2\kappa^2 + \eta_1\kappa + \eta_0$, where the coefficients $\eta_j = \eta_j(r_1, r_2, r_3)$, $j = 0, 1, 2, 3$, are given by

$$\begin{array}{rcl} \eta_0 & = & 32 - 27r_1^2 - 27r_2^2 + 24r_1^2 r_2^2 - 6r_2 r_3 + 12_1^2 r_2 r_3 - 6r_1^2 r_2^3 r_3 - 27r_3^2 + 24r_1^2 r_3^2 \\ & & +24r_2^2 r_3^2 - 26r_1^2 r_2^2 r_3^2 + r_1^4 r_2^2 (r_2 - r_3)^2 r_3^2 - 6r_1^2 r_2 r_3^3 + 4r_2^3 r_3^3 + 4r_1^2 r_2^3 r_3^3 + r_1^2 r_2^4 r_3^4 \\ & & -6r_1(r_2 + r_3) + 4r_1^3 r_2^2 (r_2 + r_3) + 12r_1 r_2 r_3 (r_2 + r_3) - 10r_1^3 r_2 r_3 (r_2 + r_3) \\ & & +4r_1^3 r_3^2 (r_2 + r_3) - 6r_1 r_2^2 r_3^2 (r_2 + r_3) + 4r_1^3 r_2^2 r_3^2 (r_2 + r_3) - 2r_1^3 r_2^3 r_3^3 (r_2 + r_3), \\ \eta_1 & = & -80 + 54r_1^2 + 54r_2^2 - 36r_1^2 r_2^2 + 12r_2 r_3 - 18r_1^2 r_2 r_3 + 6r_1^2 r_2^3 r_3 + 54r_3^2 - 36r_1^2 r_3^2 \\ & & -36r_2^2 r_3^2 + 26r_1^2 r_2^2 r_3^2 + 6r_1^2 r_2 r_3^3 - 4r_2^3 r_3^3 - 2r_1^2 r_2^3 r_3^3 + 12r_1(r_2 + r_3) \\ & & -4r_1^3 r_2^2 (r_2 + r_3) - 18r_1 r_2 r_3 (r_2 + r_3) + 10r_1^3 r_2 r_3 (r_2 + r_3) - 4r_1^3 r_3^2 (r_2 + r_3) \\ & & +6r_1 r_2^2 r_3^2 (r_2 + r_3) - 2r_1^3 r_2^2 r_3^2 (r_2 + r_3), \\ \eta_2 & = & 72 - 27r_1^2 - 27r_2^2 + 12r_1^2 r_2^2 + 6r_2 r_3 + 6r_1^2 r_2 r_3 - 27r_3^2 + 12r_1^2 r_3^2 + 12r_2^2 r_3^2 + r_1^2 r_2^2 r_3^2 \\ & & +6r_1(r_2 + r_3) + 6r_1 r_2 r_3 (r_2 + r_3), \\ \eta_3 & = & -28 - 12(r_1 r_2 + r_1 r_3 + r_2 r_3). \end{array}$$

Notice that the equation $\Delta_r(q_\lambda) = 0$ has two real roots κ_1, κ_2 and two complex conjugate roots, because the discriminant D associated with that equation is given by

$$D(r_1, r_2, r_3) = -256(-1 + r_1^2)(-1 + r_2^2)(-1 + r_3^2)(r_1 + r_2 + r_3)(E(r_1, r_2, r_3))^3$$

with

$$\begin{aligned} E(r_1,r_2,r_3) \quad = \quad & -27(r_2+r_3)(1+r_2r_3)^2+27r_1(-1+r_2^3r_3-2r_3^2+r_2r_3(-6+r_3^2) \\ & +r_2^2(-2+r_3^2))+r_1^3(27r_2r_3-27r_3^2+r_2^3r_3^3+9r_2^2(-3+2r_3^2)) \\ & +9r_1^2(3r_2(-2+r_3^2)-3r_3(2+r_3^2)+r_2^3(-3+2r_3^2)+r_2^2r_3(3+2r_3^2)), \end{aligned}$$

which is negative for all $(r_1,r_2,r_3)\in(0,1)^3$ since the maximum of the function E on the box $[0,1]^3$ is 0 and it is reached on the boundary at the point $(0,0,0)$. Furthermore, it is easy to see that the roots of $\Delta_r(q_\lambda)=0$, κ_1 and κ_2, are positive. Even more, $1<\kappa_1<\kappa_2$, see Figure 3.

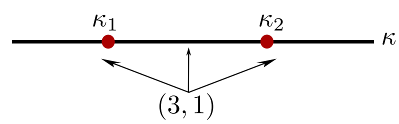


FIGURE 3. Possible configurations of simultaneous zeros of M_1 and N_1.

Depending on the value of κ it is possible to have different configurations of simultaneous zeros of M_1 and N_1. Indeed,

- If $\kappa=\kappa_1$ or $\kappa=\kappa_2$, then the discriminant of q_λ is zero if and only if at least two roots are equal. Then, $z_{N_1}=0$ or $z_{N_1}=1$.

Taking $r_1=1/2, r_2=1/3$ and $r_3=2/3$, we get that

$$\Delta_r(q_\lambda)=4\kappa^4-\frac{110}{3}\kappa^3+\frac{18745}{324}\kappa^2-\frac{1999}{54}\kappa-\frac{155609}{13122}.$$

Since $\Delta_r(q_\lambda)\Big|_{\kappa=1/2}>0$, $\Delta_r(q_\lambda)\Big|_{\kappa=3/2}<0$, $\Delta_r(q_\lambda)\Big|_{\kappa=7}<0$ and $\Delta_r(q_\lambda)\Big|_{\kappa=8}>0$, then there is one root on $(1/2,3/2)$ and another root in $(7,8)$. Thus, we can apply the method of bisections on both of the intervals $[1/2,3/2]$ and $[7,8]$ and we get $\kappa_1=1.00185$ and $\kappa_2=7.3663$ are, therefore, roots of our function.

- If $\kappa<\kappa_1$, then we can take $\lambda_0=(1/2,1/3,2/3,-2)$ we get

$$q_{\lambda_0}(r)=\frac{1}{18}(-27+85r-2r^2-54r^3).$$

  From the Sturm's sequences of q_{λ_0}, we get that it has a unique real root on $(0,1)$. By Lemma 4, we get that q_λ also has a unique real root in that interval for all $\lambda\in(0,1)^3\times(-\infty,\kappa_1)$. Hence, $z_{M_1}=3$ and $z_{N_1}=1$.
- If $\kappa\in(\kappa_1,\kappa_2)$, then taking $\lambda_0=(1/2,1/3,2/3,4)$ we obtain

$$q_{\lambda_0}(r)=\frac{1}{18}(-27+121r-2r^2-90r^3).$$

  From the Sturm's sequences of q_{λ_0}, we get that it has a unique real root on $(0,1)$. According Lemma 4, we get that q_λ also has a unique real root in that interval for all $\lambda\in(0,1)^3\times(\kappa_1,\kappa_2)$. Consequently, $z_{M_1}=3$ and $z_{N_1}=1$.
- If $\kappa>\kappa_3$, then we can take $\lambda_0=(1/2,1/3,2/3,9)$ we get

$$q_{\lambda_0}(r)=-\frac{3}{2}-\frac{113}{18}r-\frac{1}{9}r^2+8r^3.$$

  From the Sturm's sequences of q_{λ_0}, we get that it has a unique real root on $(0,1)$. By Lemma 4, we get that q_λ also has a unique real root in that interval for all $\lambda\in(0,1)^3\times(\kappa_2,\infty)$. Therefore, $z_{M_1}=3$ and $z_{N_1}=1$.

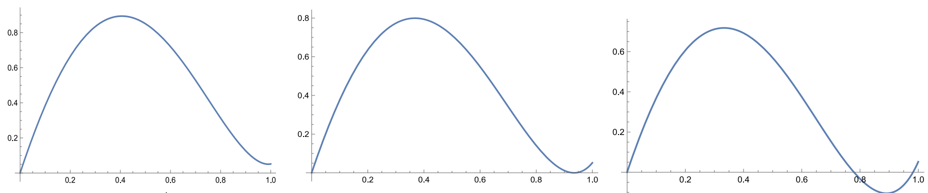
5.3.2. *Case:* $\alpha \neq 0, \beta = 0$. Suppose that $\alpha = 1$. If $\kappa = 0$, we have already proven in item (b) that the configuration $(z_{M_1}, z_{N_1}) = (4, 0)$ is feasible. We claim that in this case it is possible to obtain the configuration $(z_{M_1}, z_{N_1}) = (4, 2)$, for $\kappa \neq 0$. Indeed, consider $\epsilon > 0$ a real number small enough and suppose that $R, R+\epsilon$, $R-\epsilon$ and $1-\epsilon$ are roots of M_1 in $(0, 1)$, i.e. $M_1(R) = 0, M_1(R+\epsilon) = 0$, $M_1(R-\epsilon) = 0$ and $M_1(1-\epsilon) = 0$, this implies, respectively, that:

$$
\begin{aligned}
a &= \tfrac{1}{R}\Big(-bR^2 - cR^3 - dR^4 - \tanh^{-1}(R) + 2R^2\tanh^{-1}(R) - R^4\tanh^{-1}(R)\Big),\\
b &= \frac{R^3(c+dR)(R+\epsilon) - cR(R+\epsilon)^3 - dR(R+\epsilon)^4 + (-1+R^2)^2(R+\epsilon)\tanh^{-1}(R) - R(-1+(R+\epsilon)^2)^2\tanh^{-1}(R+\epsilon)}{-R^2(R+\epsilon)+(R+\epsilon)^2},\\
c &= \frac{-6dR^2(R^2-\epsilon^2)\epsilon^2 + 2(-1+R^2)^2(R^2-\epsilon^2)\tanh^{-1}(R) - R(R+\epsilon)(-1+R^2-2R\epsilon+\epsilon^2)^2\tanh^{-1}(R-\epsilon)}{2R(R^2-\epsilon^2)\epsilon^2}\\
&\quad - \frac{(R-\epsilon)(-1+R^2+2R\epsilon+\epsilon^2)^2\tanh^{-1}(R+\epsilon)}{2(R^2-\epsilon^2)\epsilon^2},\\
d &= (R+\epsilon)\Big(\frac{(-1+R)^3(1+R)^2(-1+\epsilon)(R^3+\epsilon^2-R\epsilon^2-2\epsilon^3+R^2(-1+2\epsilon))\tanh^{-1}(R)}{R(R-\epsilon)\epsilon^2(R+\epsilon)(-1+R)(-1+\epsilon)(-1+R+\epsilon)(R^2+\epsilon(-1+2\epsilon)+R(-1+3\epsilon))}\\
&\quad - \frac{(-2+\epsilon)^2\epsilon^2(R^2-\epsilon^2)\tanh^{-1}(1-\epsilon)}{(R-\epsilon)(R+\epsilon)(-1+R)(-1+\epsilon)(-1+R+\epsilon)(R^2+\epsilon(-1+2\epsilon)+R(-1+3\epsilon))}\\
&\quad - \frac{(-1+\epsilon)(-1+R+\epsilon)(-1+R^2-2R\epsilon+\epsilon^2)^2(R^2+\epsilon(-1+2\epsilon)+R(-1+3\epsilon))\tanh^{-1}(R-\epsilon)}{2(R-\epsilon)\epsilon^2(R+\epsilon)(-1+R)(-1+\epsilon)(-1+R+\epsilon)(R^2+\epsilon(-1+2\epsilon)+R(-1+3\epsilon))}\\
&\quad - \frac{(-1+R)(-1+\epsilon)(-1+R+\epsilon)(-1+R^2+2R\epsilon+\epsilon^2)^2\tanh^{-1}(R+\epsilon)}{2\epsilon^2(R+\epsilon)(-1+R)(-1+\epsilon)(-1+R+\epsilon)(R^2+\epsilon(-1+2\epsilon)+R(-1+3\epsilon))}\Big).
\end{aligned}
$$

Even more, g depends on the parameters R, ϵ and κ. It is easy to see that $g(8/10) < 0 < g(7/10)$ and $g(95/100) < 0 < g(99/100)$ when $(R, \epsilon, \kappa) = (1/2, 1/5, 2)$. From the intermediate value theorem, there exists $r^* \in (7/10, 8/10)$ and $r^{**} \in (95/100, 99/100)$ such that $g(r^*) = 0 = g(r^{**})$. To verify that the roots r^* and r^{**} are simples, note that by calculating the first derivative of g and by using that $r < \tanh^{-1}(r) < r + r^2 + r^3$, for all $r \in (7/10, 99/100)$, we obtain $\mu_2(r) < \frac{dg}{dr} < \mu_1(r)$, for all $r \in (7/10, 99/100)$ where

$$
\begin{aligned}
\frac{dg}{dr} &= 1 - \tfrac{8281}{60}\tanh^{-1}(\tfrac{3}{10}) + \tfrac{675}{4}\tanh^{-1}(\tfrac{1}{2}) - \tfrac{2601}{35}\tanh^{-1}(\tfrac{7}{10}) + \tfrac{81}{5}\tanh^{-1}(\tfrac{4}{5})\\
&\quad + r\Big(-4 + \tfrac{2741011}{6000}\tanh^{-1}(\tfrac{3}{10}) - \tfrac{4515}{8}\tanh^{-1}(\tfrac{1}{2}) + \tfrac{725679}{2800}\tanh^{-1}(\tfrac{7}{10}) - \tfrac{7317}{125}\tanh^{-1}(\tfrac{4}{5})\Big)\\
&\quad - r^2\Big(1 + \tfrac{405769}{7000}\tanh^{-1}(\tfrac{3}{10}) - \tfrac{330750}{7000}\tanh^{-1}(\tfrac{1}{2}) + \tfrac{117045}{7000}\tanh^{-1}(\tfrac{7}{10}) - \tfrac{23814}{7000}\tanh^{-1}(\tfrac{4}{5}))\Big)\\
&\quad + r^3\Big(8 - \tfrac{8281}{30}\tanh^{-1}(\tfrac{3}{10}) + 375\tanh^{-1}(\tfrac{1}{2}) - \tfrac{2601}{14}\tanh^{-1}(\tfrac{7}{10}) + \tfrac{216}{5}\tanh^{-1}(\tfrac{4}{5})\Big)\\
&\quad + 4r(-1+r^2)\tanh^{-1}(r),\\
\mu_1 &= 1 - \tfrac{8281}{60}\tanh^{-1}(\tfrac{3}{10}) + \tfrac{675}{4}\tanh^{-1}(\tfrac{1}{2}) - \tfrac{2601}{35}\tanh^{-1}(\tfrac{7}{10}) + \tfrac{81}{5}\tanh^{-1}(\tfrac{4}{5})\\
&\quad + r\Big(-4 + \tfrac{2741011}{6000}\tanh^{-1}(\tfrac{3}{10}) - \tfrac{4515}{8}\tanh^{-1}(\tfrac{1}{2}) + \tfrac{725679}{2800}\tanh^{-1}(\tfrac{7}{10}) - \tfrac{7317}{125}\tanh^{-1}(\tfrac{4}{5})\Big)\\
&\quad - r^2\Big(5 + \tfrac{405769}{7000}\tanh^{-1}(\tfrac{3}{10}) - \tfrac{330750}{7000}\tanh^{-1}(\tfrac{1}{2}) + \tfrac{117045}{7000}\tanh^{-1}(\tfrac{7}{10}) - \tfrac{23814}{7000}\tanh^{-1}(\tfrac{4}{5}))\Big)\\
&\quad + r^3\Big(8 - \tfrac{8281}{30}\tanh^{-1}(\tfrac{3}{10}) + 375\tanh^{-1}(\tfrac{1}{2}) - \tfrac{2601}{14}\tanh^{-1}(\tfrac{7}{10}) + \tfrac{216}{5}\tanh^{-1}(\tfrac{4}{5})\Big)\\
&\quad + 4r^4,\\
\mu_2 &= 1 - \tfrac{8281}{60}\tanh^{-1}(\tfrac{3}{10}) + \tfrac{675}{4}\tanh^{-1}(\tfrac{1}{2}) - \tfrac{2601}{35}\tanh^{-1}(\tfrac{7}{10}) + \tfrac{81}{5}\tanh^{-1}(\tfrac{4}{5})\\
&\quad + r\Big(-4 + \tfrac{2741011}{6000}\tanh^{-1}(\tfrac{3}{10}) - \tfrac{4515}{8}\tanh^{-1}(\tfrac{1}{2}) + \tfrac{725679}{2800}\tanh^{-1}(\tfrac{7}{10}) - \tfrac{7317}{125}\tanh^{-1}(\tfrac{4}{5})\Big)\\
&\quad - r^2\Big(5 + \tfrac{405769}{7000}\tanh^{-1}(\tfrac{3}{10}) - \tfrac{330750}{7000}\tanh^{-1}(\tfrac{1}{2}) + \tfrac{117045}{7000}\tanh^{-1}(\tfrac{7}{10}) - \tfrac{23814}{7000}\tanh^{-1}(\tfrac{4}{5}))\Big)\\
&\quad + r^3\Big(4 - \tfrac{8281}{30}\tanh^{-1}(\tfrac{3}{10}) + 375\tanh^{-1}(\tfrac{1}{2}) - \tfrac{2601}{14}\tanh^{-1}(\tfrac{7}{10}) + \tfrac{216}{5}\tanh^{-1}(\tfrac{4}{5})\Big)\\
&\quad + 4r^5 + 4r^6,
\end{aligned}
$$

for all $r \in (0,1)$. From the Sturm's sequences of the polynomial $\mu_1(r)$ (resp. $\mu_2(r)$), we obtain that it has no real roots on $(7/10, 8/10)$ (resp. $(95/100, 99/100)$). Even more, $\mu_1(7/10) < 0$ (resp. $\mu_1(95/100) > 0$) implies that $\frac{dg}{dr} < \mu_1(r) < 0$ (resp. $\frac{dg}{dr} > \mu_2(r) > 0$), for all $r \in (95/100, 99/100)$, that means that g is increasing on $(7/10, 8/10)$ (resp. decreasing on $(95/100, 99/100)$). Consequently, r^* (resp. r^{**}) is the unique root of g at $(7/10, 8/10)$ (resp. $(95/100, 99/100)$). We can conclude that $z_{M_1} = 4$ and $z_{N_1} = 2$ (see Figure 4).



$\kappa = 1/2$ $\kappa \approx 1.246$ $\kappa = 2$

FIGURE 4. Graphs of $g(r)$ for $r \in (0,1)$ for different values of κ.

5.3.3. *Case:* $\alpha, \beta \neq 0$. Following these same ideas, it is possible to obtain the configuration $(z_{M_1}, z_{N_1}) = (5,1)$, when $\alpha \neq 0$ and $\beta = 1$. For that, it is enough to assume that $R, R+\epsilon, R-\epsilon, 1-\epsilon$ and ϵ are roots of M_1 with $(R, \epsilon, \kappa) = (1/2, 1/5, 20)$. We omit details here.

5.4. **Proof of Theorem B(d).** According Proposition 9, the bifurcation function of PWHS (1) associated to $z = -1$ and $z = 1$ are given respectively by $M_1(r) = \frac{1}{r} f(r)$ and $N_1(r) = \frac{1}{r} g(r)$, where

$$f(r) = ar + br^2 + cr^3 + dr^4 + \alpha(-1+r^2)^2 \tanh^{-1}(x) + \beta(-1+r^4)\tanh^{-1}(r) + \gamma r^2 \tanh^{-1}(r),$$

and

$$\begin{aligned} g(r) &= cr + (b + 2d - \kappa + \rho) r^2 + ar^3 + (-d+\kappa) r^4 + \alpha(-1+r^2)^2 \tanh^{-1}(r) \\ &\quad -\beta(-1+r^4)\tanh^{-1}(r) + \gamma r^2 \tanh^{-1}(r). \end{aligned}$$

with $a, b, c, d, \alpha, \beta, \kappa$ and ρ real coefficients, which depend of the real and imaginary parts of the complex coefficients $a^{\pm}_{k,l}$, for all $0 \leq k \leq l$ and $0 \leq l \leq 3$. Moreover, the maximum number of zeros that these functions separately can have is 6. Let us study the number of simultaneous zeros that M_1 and N_1 possess.

5.4.1. *Case:* $\alpha = \beta = \gamma = 0$. If $\rho = 0$, then by the proof of item (c), we have that $0 \leq z_{M_1} + z_{N_1} \leq 4$. Let us analyze the case $\rho \neq 0$. Then we can assume that $d = 1$ and

$$\begin{aligned} M_1(r) &= (r - r_1)(r - r_2)(r - r_3). \\ &= r^3 - (r_1 + r_2 + r_3) r^2 + (r_1 r_2 + r_1 r_3 + r_2 r_3) r - r_1 r_2 r_3, \end{aligned}$$

where $r_1, r_2, r_3 \in (0,1)$ and $\lambda = (r_1, r_2, r_3, \kappa, \rho) \in (0,1)^3 \times \mathbb{R} \times \mathbb{R} \setminus \{0\} \subset \mathbb{R}^5$. This implies that $N_1(r) = q_\lambda(r)$ such that

$$q_\lambda(r) = (-1+\kappa) r^3 - (r_1 r_2 r_3) r^2 + (r_1 r_2 + r_1 r_3 + r_2 r_3 + 2 - \kappa + \rho) r - (r_1 + r_2 + r_3).$$

Just as in case (c) we know that the configurations $(z_{M_1}, z_{N_1}) = (3, j)$ with $j = 0, 1$ are possible, it is enough to take $\rho = 0$. In addition, the configuration $(3,2)$ is also possible. For that it is enough to take $\kappa = -3/5$, $\rho = -1$, $r_1 = 1/10$, $r_2 = 1/5$, $r_3 = 1/20$. Then, the Sturm's sequences of q_λ guarantees that

$$q_\lambda(r) = \frac{1}{1000}(-350 + 1635r - r^2 - 1600r^3),$$

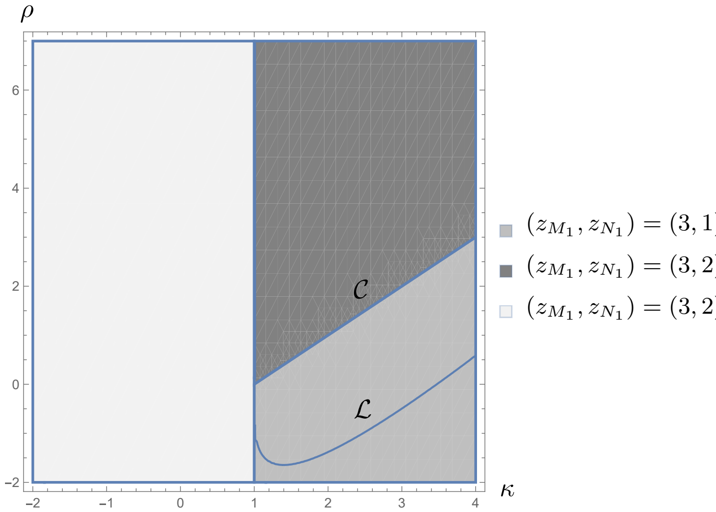


FIGURE 5. Bifurcation diagram of the maximum number of simultaneous zeros in the case $m = 3$ when $\alpha = \beta = \gamma = 0$.

has two real root on $(0, 1)$. However, the configuration $(z_{M_1}, z_{N_1}) = (3, 3)$ is not realizable. Indeed, we can assume that $\kappa \neq 1$, because otherwise q_λ is a polynomial of degree 2 so it has at most 2 real roots. Therefore, we divide this study into 2 subcases:
Subcase: $\kappa < 1$. Since $q_\lambda(0) = -(r_1 + r_2 + r_3) < 0$ and $\lim_{r \to -\infty} q_\lambda(r) = +\infty$ then there exists at least one negative root. Thus, in $(0, 1)$ there can be at most 2 real roots of q_λ.
Subcase: $\kappa > 1$. Consider the derivative function

$$q'_\lambda(r) = 3(-1 + \kappa)r^2 - 2r_1r_2r_3r + 2 + r_1r_2 + r_2r_3 + r_1r_3 - \kappa + \rho,$$

which has two critical points given by

$$r^* = \frac{2r_1r_2r_3 + \sqrt{D}}{6(\kappa - 1)} \quad \text{and} \quad r^{**} = \frac{2r_1r_2r_3 - \sqrt{D}}{6(\kappa - 1)} = \frac{2q'_\lambda(0)}{2r_1r_2r_3 + \sqrt{D}},$$

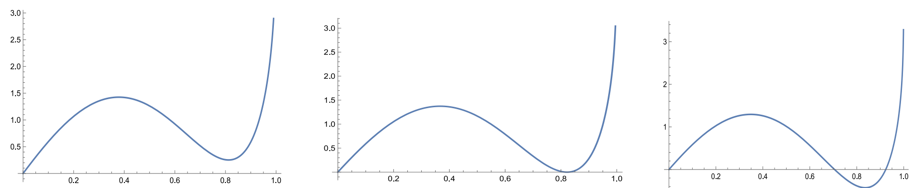
where $D = 4r_1^2r_2^2r_3^2 - 12(\kappa - 1)q'_\lambda(0)$. Now, let $\mathcal{C}$ the set $\{(\kappa, \rho) : q'_\lambda(0) = 0 \,\text{and}\, \kappa > 1\}$, which represents a curve in the parameter space. Notice that it divides the half-plane $\kappa > 1$ into 2 parts (see Figure 5). If $q'_\lambda(0) \leq 0$, then $r^{**} \leq 0 < r^*$. Therefore, for positive values of r the monotocity of q_λ changes from decreasing to increasing only once. This means that in $(0, 1)$ there is at most 1 root. Otherwise, $q'_\lambda(0) > 0$ implies that

- If $q_\lambda(1) < 0$, then there exists a root greater than 1 because $\lim_{r \to +\infty} q_\lambda(r) = +\infty$. Thus, in $(0, 1)$ there can be at most 2 real roots of q_λ.
- If $q_\lambda(1) = 0$, then in $(0, 1)$ there can be at most 2 real roots of q_λ.
- If $q_\lambda(1) > 0$, then $q_\lambda(0)q_\lambda(1) < 0$. In addition, it is easy to see that if $\mathcal{L} = \{(\kappa, \rho) : \Delta_r(q_\lambda) = 0 \,\text{and}\, \kappa > 1\}$ represents a curve in the parameter space, then $\rho_1 > \rho_2$, for all $(\kappa, \rho_1) \in \mathcal{C}$ and $(\kappa, \rho_2) \in \mathcal{L}$. Consequently, $q'_\lambda(0) > 0$ implies $\Delta_r(q_\lambda) > 0$ for all λ. In addition, taking $\lambda_0 = (1/2, 1/3, 1/4, 3, 4)$ we get

$$q_{\lambda_0}(r) = \frac{1}{24}(-26 + 81r - r^2 + 48r^3)$$

and from the Sturm's sequences of q_{λ_0}, we obtain that it has a unique real root on $(0, 1)$. By Lemma 4, we get that q_λ also has a unique real root in that interval for all $\lambda \in (0, 1)^3 \times (1, \infty) \times \mathbb{R} \setminus \{0\}$. Therefore, the result follows.

5.4.2. *Case:* $\alpha, \beta, \gamma \neq 0$. Following these same ideas of item (c), it is possible to obtain the configuration $(z_{M_1}, z_{N_1}) = (6, 2)$, when $\alpha, \beta \neq 0$ and $\gamma = 1$. For that, it is enough to assume that $1/10, 1/8, 1/4, 1/3, 1/2$ and $9/10$ are roots of M_1 with $(\kappa, \rho) = (5, 3)$ (see Figure 6). We omit details here.



$\rho = 2$ $\rho \approx 2.6225$ $\rho = 3$

FIGURE 6. Graphs of $g(r)$ for $r \in (0, 1)$ for different values of ρ.

Other configurations that are possible to obtain when $\alpha, \beta \neq 0$ and $\gamma = 1$ are:

- $(z_{M_1}, z_{N_1}) = (4, 1)$. Here, it is enough to suppose that $1/10, 1/8, 1/4$ and $1/3$ are roots of M_1 with $(\alpha, \beta, \kappa, \rho) = (-9, -13, -12, 25)$. We omit details here.
- $(z_{M_1}, z_{N_1}) = (5, 0)$. For that, it is enough to assume that $1/10, 1/8, 1/4, 1/3$ and $9/10$ are roots of M_1 with $(\beta, \kappa, \rho) = (-2, 2, 5/2)$. We omit details here.
- $(z_{M_1}, z_{N_1}) = (5, 2)$. In this case, it is enough to suppose that $1/10, 1/8, 1/4, 1/3$ and $9/10$ are roots of M_1 with $(\beta, \kappa, \rho) = (-2, 3, 5/2)$. We omit details here.

6. Proof of Theorem C

Since the functions $R_m^{\pm}$ are holomorphic, then $\overline{a_{k,l}} = 0$ for all $1 \leq k \leq l$ and $0 \leq l \leq m$. Thus, from Lemma 5 we know that for each $0 \leq l \leq m$

$$I_{k,l}^{\pm} = \begin{cases} 0, & \text{if} \quad 1 \leq k \leq l, \\ -\dfrac{1}{2} \operatorname{Im}\left(a_{0,l}^{\pm} \displaystyle\int_0^{\pm\pi} (re^{-i\theta} + 1)^{-l+2} (re^{-i\theta} - 1)^l i e^{i\theta} d\theta \right), & \text{if} \quad k = 0. \end{cases}$$

Therefore, $M_1^{\pm}(r)$ are given by

$$M_1^{\pm}(r) = I_m^{\pm}(r) = \sum_{l=0}^{m} J_l^{\pm}(r) = \sum_{l=0}^{m} \left[\sum_{k=0}^{l} I_{k,l}^{\pm}(r) \right] = \sum_{l=0}^{m} I_{0,l}^{\pm}(r). \tag{14}$$

Straightforward calculations allows us to get the following result.

Lemma 10. *The following hold:*

(a) $I_{0,0}^{\pm}(r) = -\operatorname{Im}(a_{00}^{\pm})(-1 + r^2) \mp \pi \operatorname{Re}(a_{00}^{\pm}) r$,
(b) $I_{0,1}^{\pm}(r) = -\operatorname{Im}(a_{01}^{\pm})(1 + r^2)$,
(c) $I_{0,2}^{\pm}(r) = -\operatorname{Im}(a_{02}^{\pm})(-1 + r^2) \pm \operatorname{Re}(a_{02}^{\pm}) \pi r$,
(d) $I_{0,l}^{\pm}(r) = -\operatorname{Im}(a_{0,l}^{\pm}) \left(\frac{P_{2(l-2)}(r)}{(-1+r^2)^{l-3}} - \eta(l)(l-1) 4r \tanh^{-1}(r) \right) \pm (-1)^l \operatorname{Re}(a_{0,l}^{\pm})(l-1)\pi r$,

where $l \geq 3$, $P_{2(l-2)}$ is a polynomial function of degree $2(l-1)$ with rational coefficients and $\eta(l) = 0$ (resp. $\eta(l) = 1$) provided that l is even (resp. l is odd).

The next result provides us with the expressions of the Melnikov functions M_1 and N_1 as well as the maximum number of zeros that these functions separately can have.

Proposition 11. *Let $R_m^{\pm}$ be a holomorphic polynomial of degree m in (1). Then, the Melnikov functions M_1 and N_1 on -1 and 1 associated to (1) are given by*

(a) *If* $m = 0$, *then* $M_1(r) = a_0(-1+r^2) + a_1 r$ *and* $N_1(r) = -M_1(r)$, *where* a_0 *and* a_1 *depend of the coefficients* $a^{\pm}_{0,0}$. *Moreover, the maximum number of zeros of each* M_1 *and* N_1 *is* 1.

(b) *If* $m = 1, 2$, *then* $M_1(r) = a_0 + a_1 r + a_2 r^2$ *and* $N_1(r) = a_2 + a_1 r + a_0 r^2$, *where* a_0, a_1 *and* a_2 *depend of the coefficients* $a^{\pm}_{0,l}$ *for* $l = \{0, \ldots, m\}$. *In addition, the maximum number of zeros of each* M_1 *and* N_1 *is* 2.

(c) *If* $m \geq 3$, *then* $M_1(r) = \dfrac{1}{(-1+r^2)^{m-3}}\left[\displaystyle\sum_{n=0}^{2(m-2)} a_n r^n + \alpha r(-1+r^2)^{m-3}\tanh^{-1}(r)\right]$ *and* $N_1(r) = \dfrac{1}{(-1+r^2)^{m-3}}\left[\displaystyle\sum_{n=0}^{2(m-2)} b_n r^n + \alpha r(-1+r^2)^{m-3}\tanh^{-1}(r)\right]$, *where* a_n, b_n *and* α *depend of the coefficients* $a^{\pm}_{0,l}$ *for* $l = \{0, \ldots, m\}$ *and*
- $b_{2n} = (-1)^{m+1} a_{2m-(4+2n)}$, $n = 0, \ldots, m-2$,
- $b_{2n+1} = a_{2n+1} + \beta_n \kappa$, *such that* $\beta_n, \kappa \in \mathbb{R}$, *for* $n = 0, \ldots, m-3$,
- $\eta(m) = 0$ *(resp.* $\eta(m) = 1$*) provided that* m *even (resp.* m *is odd).*

Furthermore, the maximum number of zeros of each M_1 *and* N_1 *is* $2m-3$.

Proof. Item (a) was proved in Theorem B. Let us start analyzing item (b). Suppose that $m = 2$, the case $m = 1$ follows from this by assuming that $a^{\pm}_{0,2} = 0$. Using Lemma 10 we get that

$$
\begin{aligned}
M_1^{\pm}(r) &= \sum_{l=0}^{2} I^{\pm}_{0,l}(r) \\
&= -(\operatorname{Im}(a^{\pm}_{00}) + \operatorname{Im}(a^{\pm}_{02}))(-1+r^2) \mp \pi(\operatorname{Re}(a^{\pm}_{00}) - \operatorname{Re}(a^{\pm}_{02}))r - \operatorname{Im}(a^{\pm}_{01})(1+r^2) \\
&= \operatorname{Im}(a^{\pm}_{00}) - \operatorname{Im}(a^{\pm}_{01}) + \operatorname{Im}(a^{\pm}_{02}) \mp \pi(\operatorname{Re}(a^{\pm}_{00}) - \operatorname{Re}(a^{\pm}_{02}))r - (\operatorname{Im}(a^{\pm}_{00}) + \operatorname{Im}(a^{\pm}_{01}) \\
&\quad + \operatorname{Im}(a^{\pm}_{02}))r^2.
\end{aligned}
$$

Thus, $M_1(r) = M_1^+(r) - M_1^-(r) = a_0 + a_1 r + a_2 r^2$, where

- $a_0 = \operatorname{Im}(a^+_{00}) - \operatorname{Im}(a^-_{00}) - \operatorname{Im}(a^+_{01}) + \operatorname{Im}(a^-_{01}) + \operatorname{Im}(a^+_{02}) - \operatorname{Im}(a^-_{02})$,
- $a_1 = -\pi\operatorname{Re}(a^+_{00}) - \pi\operatorname{Re}(a^-_{00}) + \pi\operatorname{Re}(a^+_{02}) + \pi\operatorname{Re}(a^-_{02})$,
- $a_2 = -\operatorname{Im}(a^+_{00}) + \operatorname{Im}(a^-_{00}) - \operatorname{Im}(a^+_{01}) + \operatorname{Im}(a^-_{01}) - \operatorname{Im}(a^+_{02}) + \operatorname{Im}(a^-_{02})$.

To obtain the expression of N_1 in terms of the coefficients of M_1, it is enough to use Proposition 8, which guarantees that $b^{\pm}_{0,l} = (-1)^l a^{\mp}_{0,l}$, for all $0 \leq l \leq m$. Then

- $b_0 = -\operatorname{Im}(a^+_{00}) + \operatorname{Im}(a^-_{00}) - \operatorname{Im}(a^+_{01}) + \operatorname{Im}(a^-_{01}) - \operatorname{Im}(a^+_{02}) + \operatorname{Im}(a^-_{02}) = a_2$,
- $b_1 = -\pi\operatorname{Re}(a^+_{00}) - \pi\operatorname{Re}(a^-_{00}) + \pi\operatorname{Re}(a^+_{02}) + \pi\operatorname{Re}(a^-_{02}) = a_1$,
- $b_2 = \operatorname{Im}(a^+_{00}) - \operatorname{Im}(a^-_{00}) - \operatorname{Im}(a^+_{01}) + \operatorname{Im}(a^-_{01}) + \operatorname{Im}(a^+_{02}) - \operatorname{Im}(a^-_{02}) = a_0$.

Therefore, $N_1(r) = b_0 + b_1 r + b_2 r^2 = a_2 + a_1 r + a_0 r^2$.

Now, let $\mathcal{G}$ be the ordered set of functions $\{1, r, r^2\}$. Recall that the Wronskians, defined in Lemma 2, are $W_0(r) = 1$, $W_1(r) = 1$, $W_2(r) = 2$. Hence, $W_j(r) \neq 0$ at $(0,1)$, for all $j \in \{0, 1, 2\}$. According to Lemma 2 $\mathcal{G}$ is an ECT-system. Moreover, Theorem 3 implies that two is the maximum number of positive roots for any element of $Span(\mathcal{G})$ and there is a choice for a_0, a_1 and a_2 such that any element of $Span(\mathcal{G})$ has exactly two zeros.

In what follows, we prove item (c). Assume that $m \geq 3$, then Lemma 10 implies that

$$
\begin{aligned}
M_1^{\pm}(r) &= \sum_{l=0}^{2} I_{0,l}^{\pm}(r) + \sum_{l=3}^{m} I_{0,l}^{\pm}(r) \\
&= -\sum_{l=0}^{2} \operatorname{Im}(a_{0l}^{\pm})((-1)^{l+1} + r^2) \pm \sum_{l=0}^{m} (-1)^l \operatorname{Re}(a_{0,l}^{\pm})(l-1)\pi r \\
&\quad -\sum_{l=3}^{m} \operatorname{Im}(a_{0,l}^{\pm}) \left(\frac{P_{2(l-2)}(r)}{(-1+r^2)^{l-3}} - \eta(l)(l-1)4r\tanh^{-1}(r) \right) \\
&= \frac{1}{(-1+r^2)^{m-3}} \left(-\sum_{l=0}^{2} \operatorname{Im}(a_{0l}^{\pm})((-1)^{l+1} + r^2)(-1+r^2)^{m-3} \right. \\
&\quad \pm \sum_{l=0}^{m} (-1)^l \operatorname{Re}(a_{0,l}^{\pm})(l-1)\pi r(-1+r^2)^{m-3} \\
&\quad \left. -\sum_{l=3}^{m} \operatorname{Im}(a_{0,l}^{\pm}) \left(P_{2(l-2)}(r)(-1+r^2)^{m-l} - \eta(l)(l-1)4r(-1+r^2)^{m-3}\tanh^{-1}(r) \right) \right).
\end{aligned}
$$

Thus,

$$
\begin{aligned}
M_1(r) &= M_1^{+}(r) - M_1^{-}(r) \\
&= \frac{1}{(-1+r^2)^{m-3}} \left[\sum_{n=0}^{2(m-2)} a_n r^n + \alpha r(-1+r^2)^{m-3}\tanh^{-1}(r) \right].
\end{aligned}
$$

where $\alpha = -4\sum_{l=3}^{m}(\operatorname{Im}(a_{0,l}^{+}) - \operatorname{Im}(a_{0,l}^{-}))\eta(l)(l-1)$ and a_n depends of the coefficients $a_{0,l}^{\pm}$ for $l = \{0, \dots, m\}$.

To obtain the expression of N_1 in terms of the coefficients of M_1, it is enough to use Proposition 8 as was done in item (b), i.e, $N_1(r) = \dfrac{r}{(-1+r^2)^{m-3}} \left[\sum_{n=0}^{2(m-2)} b_n r^n + \alpha r(-1+r^2)^{m-3}\tanh^{-1}(r) \right]$, where a_n, b_n and α depend of the coefficients $a_{0,l}^{\pm}$ for $l = \{0, \dots, m\}$ and

- $b_{2n} = (-1)^{m+1} a_{2m-(4+2n)}$, $n = 0, \dots, m-2$,
- $b_{2n+1} = a_{2n+1} + \beta_n \kappa$, $n = 0, \dots, m-3$, such that
  - $\kappa = -2\pi \sum_{l=1}^{\lfloor (m-1)/2 \rfloor} (\operatorname{Re}(a_{0,2l+1}^{+}) + \operatorname{Re}(a_{0,2l+1}^{-}))(2l)$, where $\lfloor . \rfloor$ denotes the integer part of a real number, i.e. given $n \leq x < n+1$ then the integer part of x is $\lfloor x \rfloor = n$.
  - If m is even, then
    - $\beta_0 = 1$,
    - $\beta_1 = -(m-3)$,
    - $\beta_n = (-1)^n (m-3)\left(\frac{m}{2} + 2(n-3)\right)$ $n = 2, \dots, \frac{m}{2} - 2$,
    - $\beta_{m-3-n} = -\beta_n$, $n = 0, \dots, \frac{m}{2} - 2$.
  - If m is odd, then
    - $\beta_0 = -1$,
    - $\beta_1 = m-3$,
    - $\beta_n = (-1)^{n+1}(m-4)\left(\frac{m-3}{2}\right)(n-1)$, $n = 2, \dots, \frac{m-5}{2}$,
    - $\beta_{m-3-n} = \beta_n$, $n = 0, \dots, \frac{m-5}{2}$,
    - $\beta_{\frac{m-3}{2}} = (-1)^{\frac{m-1}{2}}(m-4)(m-5)$, $m = 7, 9$ and $\beta_{\frac{m-3}{2}} = (-2)^{\frac{m-9}{2}}(m-4)(m-6)$, $m \geq 11$.

- $\eta(m) = 0$ (resp. $\eta(m) = 1$) provided that m even (resp. m is odd),
- $\alpha = -4\sum_{l=3}^{m}(\mathrm{Im}(a_{0,l}^{+}) - \mathrm{Im}(a_{0,l}^{-}))\eta(l)(l-1)$.

Now, for each $m \in \mathbb{N}_{\geq 3}$, consider the ordered set of functions $\mathcal{F} = \{1, r, \ldots, r^{2(m-2)}, r(-1+r^2)^{m-3}\tanh^{-1}(r)\}$. Emphasize that the Wronskians, defined in Lemma 2, are $W_j(r) = \prod_{k=0}^{j} k! \neq 0$ at $(0,1)$, for all $j \in \{0,1,2,\ldots,2(m-2)\}$ and $W_{2(m-2)+1}(r) = \frac{(-1)^m \xi_m r}{(-1+r^2)^m} \neq 0$ at $(0,1)$, where ξ_m is a increasing sequence of positive real numbers. According to Lemma 2 $\mathcal{F}$ is an ECT-system. In addition, Theorem 3 implies that $2m-3$ is the maximum number of positive roots for any element of $Span(\mathcal{F})$ and there is a choice for a_n, b_n and α, $n \in \{0,\ldots,2(m-2)+1\}$ such that any element of $Span(\mathcal{F})$ has exactly $2m-3$ zeros. $\square$

6.1. **Proof of Theorem C(a).** This proof is the same as that given in Theorem B(a).

6.2. **Proof of Theorem C(b).** From Proposition 11, the bifurcation function of PWHS (1) associated to $z=-1$ and $z=1$ are given respectively by

$$M_1(r) = a_0 + a_1 r + a_2 r^2$$

and

$$N_1(r) = a_2 + a_1 r + a_0 r^2,$$

where a_0, a_1 and a_3 are real coefficients, which depend of the real and imaginary parts of the complex coefficients $a_{0,l}^{\pm}$, for all $l = 0, \ldots, m$. Furthermore, the maximum number of zeros that these functions separately can have is 2. Let us study the number of simultaneous zeros that M_1 and N_1 possess.

Recall that the discriminant of M_1 and N_1 is $\Delta = a_1^2 - 4a_0a_2$. This means that both polynomials have the same type of roots. If these roots are complex then $z_{M_1} = z_{N_1} = 0$. If they are real roots, we can assume that $a_2 = 1$ and

$$\begin{aligned} M_1(r) &= (r-r_1)(r-r_2). \\ &= r^2 - (r_1+r_2)r + r_1r_2, \end{aligned}$$

where $r_1, r_2 \in \mathbb{R}$. This implies that

$$N_1(r) = r_1r_2r^2 - (r_1+r_2)r + 1 = r_1r_2\left(r - \frac{1}{r_1}\right)\left(r - \frac{1}{r_2}\right).$$

Therefore, $0 \leq z_{M_1} + z_{N_1} \leq 2$.

6.3. **Proof of Theorem C(c).** According Proposition 11, the bifurcation function of PWHS (1) associated to $z=-1$ and $z=1$ are given respectively by

$$M_1(r) = a_0 + a_1 r + a_2 r^2 + \alpha r \tanh^{-1}(r)$$

and

$$N_1(r) = a_2 + (a_1 - \kappa)r + a_0 r^2 + \alpha r \tanh^{-1}(r),$$

where a_0, a_1, a_2, κ and α are real coefficients, which depend of the real and imaginary parts of the complex coefficients $a_{0,l}^{\pm}$, for all $l = 0,1,2,3$. Even more, the maximum number of zeros that these functions separately can have is 3. Let us study the number of simultaneous zeros that M_1 and N_1 possess.

6.3.1. *Case:* $\alpha = 0$. Note that if $\kappa = 0$, then by the proof of item (b), we have that $0 \leq z_{M_1} + z_{N_1} \leq 2$. Let us analyze the case $\kappa \neq 0$. Then we can assume that $a_2 = 1$ and

$$\begin{array}{rcl} M_1(r) & = & (r-r_1)(r-r_2). \\ & = & r^2 - (r_1+r_2)r + r_1 r_2, \end{array}$$

where $r_1, r_2 \in (0,1)$ and $\lambda = (r_1, r_2, \kappa) \in (0,1)^2 \times \mathbb{R}\setminus\{0\} \subset \mathbb{R}^3$. This implies that $N_1(r) = q_\lambda(r)$ such that

$$q_\lambda(r) = r_1 r_2 r^2 - (r_1 + r_2 + \kappa)r + 1.$$

Thus, $q_\lambda(0) = 1 > 0$, $q_\lambda(1) = (-1+r_1)(-1+r_2) - \kappa$ and $\Delta(r_1, r_2) := \Delta_r(q_\lambda)$, where

$$\Delta_r(q_\lambda) = -\frac{1}{r_1 r_2}\Big((-1+r_1^2)(-1+r_1 r_2) - \kappa r_1\Big)\Big((-1+r_2^2)(-1+r_1 r_2) - \kappa r_2\Big).$$

Notice that:

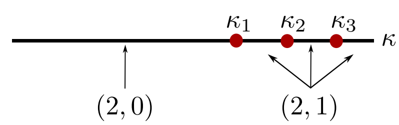


FIGURE 7. Possible configurations of simultaneous zeros of M_1 and N_1. If $\kappa \notin (-\infty, \kappa_1)$ or $\kappa \neq \kappa_i$ for $i = 1,2,3$, then $(z_{M_1}, z_{N_1}) = (2,1)$. If $\kappa \in (-\infty, \kappa_1)$, then $(z_{M_1}, z_{N_1}) = (2,0)$.

- $q_\lambda(1) = 0$ if and only if, $\kappa = \kappa_1 := (-1+r_1)(-1+r_2) > 0$.
- $\Delta_r(q_\lambda) = 0$ if, and only if,

$$\kappa = \kappa_2 := \frac{1}{r_1}(-1+r_1^2)(-1+r_1 r_2) > 0 \quad \text{or} \quad \kappa = \kappa_3 := \frac{1}{r_2}(-1+r_2^2)(-1+r_1 r_2) > 0.$$

Assume without loss of generality that $r_1 > r_2$, then it is easy to see that $\kappa_1 < \kappa_2 < \kappa_3$, see Figure 7. Depending on the value of κ it is possible to have different configurations of simultaneous zeros of M_1 and N_1. Indeed,

- If $\kappa = \kappa_1$, then $r = 1$ and $r = \frac{1}{r_1 r_2}$ are the roots of N_1. Then, $z_{N_1} = 0$.
- If $\kappa = \kappa_2$, then $r = r_1$ and $r = \frac{1}{r_1^2 r_2}$ are the roots of N_1. Then, $z_{N_1} = 1$.
- If $\kappa = \kappa_3$, then $r = r_2$ and $r = \frac{1}{r_1 r_2^2}$ are the roots of N_1. Then, $z_{N_1} = 1$.
- If $\kappa < \kappa_1$, then taking $\lambda_0 = (1/2, 1/3, \kappa_1/2)$ we get

$$q_{\lambda_0}(r) = \frac{1}{6}(6 - 6r + r^2)$$

  which has $3 - \sqrt{3}$ and $3 + \sqrt{3}$ as real root, i.e q_{λ_0} has no real roots between $(0,1)$. From Lemma 4, we get that q_λ also has no real root in that interval for all $\lambda \in (0,1)^2 \times (-\infty, \kappa_1)$. Therefore, $z_{M_1} = 2$ and $z_{N_1} = 0$.
- If $\kappa \in (\kappa_1, \kappa_2)$, then taking $\lambda_0 = (1/2, 1/3, (\kappa_1+\kappa_2)/2)$ we get

$$q_{\lambda_0}(r) = 1 - \frac{13}{6}r + \frac{1}{6}r^2.$$

  which has $\frac{1}{8}(39 - \sqrt{1137})$ and $\frac{1}{8}(39 + \sqrt{1137})$ as real roots, i.e q_{λ_0} has a unique real root between $(0,1)$. From Lemma 4, we get that q_λ also has a unique real root in that interval for all $\lambda \in (0,1)^2 \times (\kappa_1, \kappa_2)$. Therefore, $z_{M_1} = 2$ and $z_{N_1} = 1$.
- If $\kappa \in (\kappa_2, \kappa_3)$, then taking $\lambda_0 = (1/2, 1/3, (\kappa_2+\kappa_3)/2)$ we get

$$q_{\lambda_0}(r) = 1 - \frac{185}{72}r + \frac{1}{6}r^2.$$

which has $\frac{1}{24}(185-\sqrt{30769})$ and $\frac{1}{24}(185+\sqrt{30769})$ as real roots, i.e q_{λ_0} has a unique real root between $(0,1)$. From Lemma 4, we get that q_λ also has a unique real root in that interval for all $\lambda \in (0,1)^2 \times (\kappa_2, \kappa_3)$. Therefore, $z_{M_1} = 2$ and $z_{N_1} = 1$.

- If $\kappa > \kappa_3$, then taking $\lambda_0 = (1/2, 1/3, 2\kappa_3)$ we get
$$q_{\lambda_0}(r) = 1 - \frac{55}{18}r + \frac{1}{6}r^2.$$
which has $\frac{1}{6}(95-\sqrt{8809})$ and $\frac{1}{6}(95+\sqrt{8809})$ as real roots, i.e q_{λ_0} has a unique real root between $(0,1)$. From Lemma 4, we get that q_λ also has a unique real root in that interval for all $\lambda \in (0,1)^2 \times (\kappa_3, \infty)$. Therefore, $z_{M_1} = 2$ and $z_{N_1} = 1$.

6.3.2. *Case:* $\alpha \neq 0$. Assume that $\alpha = 1$. We claim that in this case it is possible to obtain at least 4 limit cycles. Indeed, consider $\epsilon > 0$ a real number small enough and suppose that $R, R+\epsilon$ and $R-\epsilon$ are roots of M_1 in $(0,1)$, i.e. $M_1(R) = 0, M_1(R+\epsilon) = 0$ and $M_1(R-\epsilon) = 0$, this implies that:

- $a_0 = -bR - cR^2 - R\tanh^{-1}(R)$,
- $a_1 = \dfrac{1}{\epsilon}\Big(-2cR\epsilon - c\epsilon^2 + R\tanh^{-1}(R) - R\tanh^{-1}(R+\epsilon) - \epsilon\tanh^{-1}(R+\epsilon)\Big)$,
- $a_2 = \dfrac{1}{2\epsilon^2}\Big(2R\tanh^{-1}(R) - (R-\epsilon)\tanh^{-1}(R-\epsilon) - (R+\epsilon)\tanh^{-1}(R+\epsilon)\Big)$.

Even more, N_1 depends on the parameters R, ϵ and κ. It is easy to see that $N_1(1/10) < 0 < N_1(9/10)$ when $(R, \epsilon, \kappa) = (1/2, 1/10, 35/100)$. From the intermediate value theorem, there exists $r^* \in (1/10, 9/10)$ such that $N_1(r*) = 0$. To verify that this root r^* is simple, note that by calculating the first derivative of N_1 and by using that $\tanh^{-1}(r) > r$ for all $r \in (0,1)$, we obtain

$$\begin{aligned}
\frac{dN_1}{dr} &= \tfrac{7}{20} + 22\tanh^{-1}(\tfrac{2}{5}) - 50\tanh^{-1}(\tfrac{1}{2}) + 27\tanh^{-1}(\tfrac{3}{5}) + \tanh^{-1}(r)\\
&\quad + r\left(\tfrac{1}{1-r^2} - 12\tanh^{-1}(\tfrac{2}{5}) + 24\tanh^{-1}(\tfrac{1}{2}) - 12\tanh^{-1}(\tfrac{3}{5})\right)\\
&> \tfrac{7}{20} + 22\tanh^{-1}(\tfrac{2}{5}) - 50\tanh^{-1}(\tfrac{1}{2}) + 27\tanh^{-1}(\tfrac{3}{5})\\
&\quad + r\left(1 + \tfrac{1}{1-r^2} - 12\tanh^{-1}(\tfrac{2}{5}) + 24\tanh^{-1}(\tfrac{1}{2}) - 12\tanh^{-1}(\tfrac{3}{5})\right) > 0,
\end{aligned}$$

for all $r \in (0,1)$. Thus N_1 is increasing at $(0,1)$, which implies that r^* is the unique root of g at $(0,1)$. Consequently, $z_{M_1} = 3$ and $z_{N_1} = 1$.

Finally, it is possible to obtain the configuration $(z_{M_1}, z_{N_1}) = (3,0)$ by taking $(R, \epsilon, \kappa) = (2/3, 1/4, -11/2)$. From which the result follows.

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[1]Departament de Matemàtiques, Edifici Cc, Universitat Autònoma de Barcelona, 08193 Bellaterra, Barcelona, Spain; and Centre de Recerca Matemàtica, Campus de Bellaterra, 08193 Bellaterra, Barcelona, Spain

[2]São Paulo State University (Unesp), Institute of Biosciences, Humanities and Exact Sciences. Rua C. Colombo, 2265, CEP 15054–000. S. J. Rio Preto, São Paulo, Brazil.

Email address: armengol.gasull@uab.cat

Email address: garv202020@gmail.com

Email address: paulo.r.silva@unesp.br

HOLOMORPHIC SLOW-FAST SYSTEMS

GABRIEL RONDÓN, PAULO R. DA SILVA AND LUIZ F. S. GOUVEIA

Abstract. In this paper, we are concerned with studying the existence of invariant complex manifolds of two-dimensional holomorphic systems. From the geometric singular perturbation theory we know that if a slow-fast system has associated a normally hyperbolic compact critical manifold, then there exists a smooth locally invariant manifold. However, this smooth manifold does not necessarily have a complex structure. Here, we provide conditions to guarantee the existence of one-dimensional invariant complex manifolds. Consequently, this allows us to establish that the centers, foci, and nodes of the reduced problem are persistent by singular perturbation. The tools used by us are the usual techniques of Fenichel and Briot-Bouquet Theories.

1. Introduction

In the qualitative theory of dynamical systems, there are several studies on singular perturbation problems (see, for instance, [3, 9, 10, 17]). In this paper, we want to study singular perturbation problems in $\mathbb{R}^4$, which can be written as two-dimensional holomorphic systems. It is important to highlight that although there are applications that can be modeled by two-dimensional holomorphic systems, see [11, 12], there are few articles on the subject such as [13, 14].

The basic systems we consider are of the form

$$\text{(SF1)} \qquad \begin{cases} \varepsilon \dot{z} & = & f(z,w), \\ \dot{w} & = & g(z,w), \end{cases}$$

where $f, g : D \to \mathbb{C}$ are holomorphic functions, $D \subseteq \mathbb{C}^2$ is a simply connected domain, $\varepsilon > 0$ is a small enough parameter and the dot $\cdot$ represents the derivative of the functions $z(\tau)$ and $w(\tau)$ with respect to the real variable τ.

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.

Emphasize that $C_0 = \{(z,w) \in D : f(z,w) = 0\}$ is the one-dimensional critical manifold associated with the system (SF1) provided that $\frac{\partial f}{\partial z}(z,w) \neq 0$ for all $(z,w) \in C_0$. Recall that (SF1) can be written as a real four-dimensional autonomous system in an appropriate domain of $\mathbb{R}^4$.

Suppose that $S_0 \subset C_0$ is a normally hyperbolic compact critical manifold. Thus, we can assume that S_0 is given as the graph of a function of z in terms of w. Indeed, since $\frac{\partial f}{\partial z}(z,w) \neq 0$ for all $(z,w) \in S_0$, then from *Implicit Function Theorem* (see [15]), there exist an open set U and a holomorphic function $z = h(w)$ such that $f(h(w),w) = 0$, for all $w \in U$. Hence, without loss of generality we can assume that $S_0 = \{(z,w) : z = h(w), w \in U\}$.

One of the main goals of this paper is to study the existence of invariant complex manifolds of two-dimensional holomorphic system (SF1). At some point, the reader may think that it is enough to apply *Fenichel's theorem* (see, for instance, [9]) to obtain a one-dimensional invariant complex manifold. In general, this is not true, due to the fact that there are smooth manifolds that have no complex structure. An obvious impediment is that the dimension of the smooth manifold must be even. Another obstacle is orientability, since every complex manifold is orientable.

Although the *geometric singular perturbation theory* does not allow us to guarantee the existence of invariant complex manifolds, it is possible to use this theory and the Laurent series to construct examples of smooth manifolds without complex structure. Indeed, if we apply the *Fenichel's Theorem* to the equivalent C^∞ four-dimensional real system associated with system (SF1), then there exists a smooth locally invariant manifold $S_\varepsilon = \{(z,w) : z = h_\varepsilon(w)\}$ of slow-fast system (SF1), which is diffeomorphic to S_0. In general, this function $h_\varepsilon(z)$ does not have to be holomorphic, as we illustrate in the following example. Consider the complex dynamical system

$$(1) \qquad \begin{cases} \varepsilon \dot{z} = z + w, \\ \dot{w} = w^2, \end{cases}$$

notice that $C_0 = \{(z,w) : z = -w\}$ is a normally hyperbolic critical manifold of (1). In addition, if $S_0 \subset C_0$ is a compact set, then the equivalent system to (1) in $\mathbb{R}^4$ has a smooth locally invariant manifold S_ε. Nevertheless, this manifold has no complex structure. Indeed, suppose that there exists a holomorphic function h_ε such that $S_\varepsilon = \{(z,w) : z = h_\varepsilon(w)\}$. Then h_ε can be written as a power series. In

Section 5, we will prove that this series is given by

$$h_\varepsilon(w) = -\sum_{k=0}^{\infty} \varepsilon^k k! w^{k+1}.$$

Notice that this series diverges for $w \neq 0$, which contradicts the fact that h_ε is holomorphic.

At this point, it is natural to ask: under what conditions can we ensure the existence of invariant manifolds with complex structure of system (SF1)? In Section 3, we use the *Briot-Bouquet Theory* and provide conditions to guarantee the existence of one-dimensional invariant complex manifolds. Specifically, in Theorem A we prove that, for $\varepsilon > 0$ sufficiently enough, the system

$$\text{(SF2)} \qquad \begin{cases} \varepsilon \dot{z} = \alpha z + \tilde{f}(z,w), \\ \dot{w} = \beta w + \tilde{g}(z,w), \end{cases} \text{where } \tilde{f}, \tilde{g} = \mathcal{O}_2(z,w) \text{ and } \alpha, \beta \in \mathbb{C},$$

has a unique one-dimensional invariant complex manifold C_ε passing through the equilibrium point $q_\varepsilon = (0,0)$ provided that the critical manifold is normally hyperbolic.

We employ Theorem A to prove that the centers, foci and nodes of the reduced problem associated with the holomorphic slow-fast system (SF1), whose Jacobian matrix of f and g is diagonalizable at the origin, are preserved by singular perturbation, for more details see Theorem B and Figure 1.

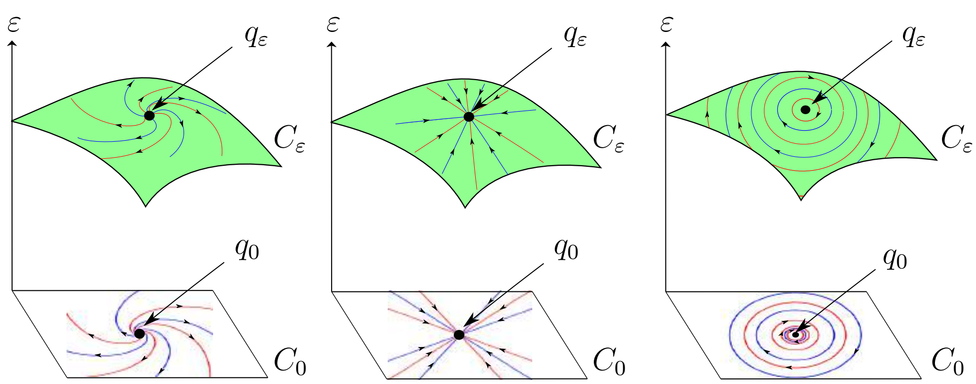

FIGURE 1. Persistence of a focus, node and center of the reduced problem by singular perturbation, respectively.

Also, we are concerned with determining the existence of complex invariant manifolds for certain families of two-dimensional holomorphic systems, whose jacobian matrix of f and g is not necessarily digonalizable at the origin and this point does not need to be an equilibrium point of the system, see Theorems C and D. Specifically, Theorem C stablishes the existence of one-dimensional invariant complex manifolds associated with slow-fast system (SF1) whose equation of orbits is separable, that is, in the case that there exist holomorphic functions η, κ such that $\frac{g(z,w)}{f(z,w)} = \eta(z)\kappa(w)$ in a suitable domain. Furthermore, in Theorem D we state conditions to ensure when the system

$$\text{(SF3)} \qquad \begin{cases} \varepsilon\dot{z} = \alpha z + \beta G(w), \\ \dot{w} = g(w), \end{cases} \text{where } G' \equiv 1/g \text{ and } \alpha, \beta \in \mathbb{C} \setminus \{0\},$$

has associated a one-dimensional exponentially attractive invariant complex manifold C_ε.

Finally, we will present a method to study slow-fast systems of the form

$$\text{(SF4)} \qquad \begin{cases} \varepsilon\dot{z} = \alpha z + \beta w, \\ \dot{w} = g(w), \end{cases}, \text{ where } \alpha \in \mathbb{C} \setminus \{0\} \text{ and } \beta \in \mathbb{C},$$

where g is one of the normal forms given in [4], namely: $\dot{w} = 1$, $\dot{w} = (a + ib)w$, $\dot{w} = w^n$, $\dot{w} = \frac{1}{w^n}$ and $\dot{w} = \frac{\gamma w^n}{1+w^{n-1}}$. We will determine which of these families could have associated manifolds with complex structure and in that case we will investigate what happens when a singularity of the reduced problem is perturbed for $\varepsilon > 0$. Specifically, in Propositions 10 and 11 we prove that the centers and poles of order n of the reduced problem associated with system (SF4) are persistent by singular perturbation.

The paper is organized in the following form. In Section 2, we present some basic results on the complex manifolds, geometric singular perturbation theory and the Briot-Bouquet Theory. In Section 3, we employ Briot-Bouquet theory to prove Theorem A and B. In Section 4, we state and prove Theorems C and D. Lastly, in Section 5 we use the Laurent series to give complex systems that have no associated complex manifolds.

2. Preliminaries

This section is devoted to establishing some basic results that will be used throughout the paper.

2.1. **Complex manifolds.** A *holomorphic function* f is a complex-valued function defined in a domain $\mathcal{V} \subseteq \mathbb{C}$ and satisfying that

- $u = \text{Re}(f)$ and $v = \text{Im}(f)$ are continuous,
- there exist the partial derivatives u_x, u_y, v_x, v_y in $\mathcal{V}$ and,

- the partial derivatives satisfy the Cauchy–Riemann equations, see [5],

$$u_x = v_y, \quad u_y = -v_x, \quad \forall z = x + iy \in \mathcal{V}.$$

We recall that *Looman–Menchoff's Theorem* establishes that the above conditions are sufficient to guarantee the analyticity of f. This result implies that for any $z_0 \in \mathcal{V}$

$$f(z) = A_0 + A_1(z - z_0) + A_2(z - z_0)^2 + \ldots, \quad A_k = a_k + ib_k = \frac{f^{(k)}(z_0)}{k!}$$

for $z \in D(z_0, R_{z_0}) \subseteq \mathcal{V}$ where $D(z_0, R_{z_0})$ is the largest possible z_0–centered disk contained in $\mathcal{V}$.

If f is holomorphic in a punctured disc $D(0, R) \setminus \{0\}$ and it is not derivable at 0 we say that 0 is a singularity of f. In this case f is equal to its Laurent's series in $D(0, R) \setminus \{0\}$

$$f(z) = \sum_{k=1}^{\infty} \frac{B_k}{z^k} + \sum_{k=0}^{\infty} A_k z^k,$$

where $B_k = \frac{1}{2\pi i} \int_{C_\varepsilon} f(z) z^{k-1} dz, \quad A_k = \frac{1}{2\pi i} \int_{C_\varepsilon} \frac{f(z)}{z^{k+1}} dz$ with C_ε parameterized by $z(t) = \varepsilon e^{it}, \varepsilon \sim 0$, $t \in [0, 2\pi]$.

If $B_k \neq 0$ for an infinite set of indices k we say that 0 is an *essential singularity* and if there exists $n \geq 1$ such that $B_n \neq 0$ and $B_k = 0$ for every $k > n$ then we say that 0 is a *pole of order n*.

The notion of holomorphic function can be extended as follow.

Consider $\mathcal{W} \subset \mathbb{C}^2$ an open subset and let $f : \mathcal{W} \to \mathbb{C}$ be a continuously differentiable function. Then f is said *holomorphic* if the Cauchy-Riemann equations hold for all coordinates $z_j = x_j + iy_j$, with $j = 1, 2$, i.e.

$$u_{x_j} = v_{y_j}, \quad u_{y_j} = -v_{x_j}, \quad \forall z_j = x_j + iy_j \in \mathcal{W}.$$

In addition, if f is holomorphic in $\mathcal{W}$, then in each point $\zeta = (\zeta_1, \zeta_2) \in \mathcal{W}$ has an open neighborhood $D(\zeta, R_\zeta)$, such that the function f can be expanded into a power series

$$f(z, w) = \sum_{n=1}^{\infty} \sum_{k_1 + k_2 = n} c_{k_1, k_2} (z - \zeta_1)^{k_1} (w - \zeta_2)^{k_2},$$

which converges for all $(z, w) \in D(\zeta, R_\zeta)$.

The definition of a complex manifold is analogous to the smooth manifold, but we require that the charts take on values in $\mathbb{C}^2$ and that the transition functions be holomorphic (see Figure 2).

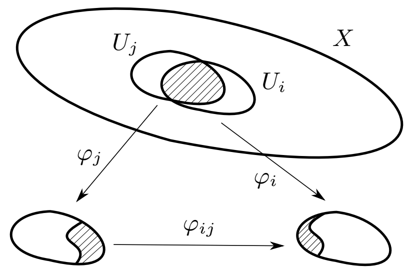

FIGURE 2. Complex manifolds require that the transition functions φ_{ij} be holomorphic.

A *holomorphic atlas* in a topological space $X \subseteq \mathbb{C}^2$ is a collection of pairs (U_i, φ_i) called holomorphic charts where

- Each U_i is an open in X and $X = \displaystyle\bigcup_i U_i$;
- Each $\varphi_i : U_i \to \mathbb{C}^n$ is a homomorphism over an open of $\mathbb{C}^n$, with $n \leq 2$;
- Whenever $U_i \cap U_j \neq \emptyset$ the transition

$$\varphi_{ij} = \varphi_i \circ \varphi_j^{-1} : \varphi_j(U_i \cap U_j) \to \varphi_i(U_i \cap U_j)$$

  is a holomorphic map.

A *complex manifold* is a topological space X, Hausdorff and with enumerable base, equipped with a maximal holomorphic atlas. The number n is called the complex dimension of X.

The following proposition will give us a way to obtain complex submanifolds. For more details see, for instance, [7, 18].

Proposition 1. *Let $\phi : X \subseteq \mathbb{C}^2 \to \mathbb{C}$ be a holomorphic map between complex manifolds. Consider $b \in \phi(X) \subseteq \mathbb{C}$ such that the rank of ϕ is maximal ($rank(\phi) = 1$), for all $a \in \phi^{-1}(b)$. Then $\phi^{-1}(b)$ is a complex submanifold of X of dimension 1.*

2.2. **Geometric singular perturbation theory.** We consider singularly perturbed systems of differential equations

$$\begin{cases} \varepsilon \dot{\mathbf{x}} = f(\mathbf{x}, \mathbf{y}, \varepsilon), \\ \dot{\mathbf{y}} = g(\mathbf{x}, \mathbf{y}, \varepsilon), \end{cases} \tag{2}$$

which is called *slow-fast system*, where $\varepsilon \in (0, \varepsilon_0)$, ε_0 small, $(\mathbf{x}, \mathbf{y}) \in \mathcal{M}$, an open set $\mathcal{M} \subset \mathbb{R}^4$ and $f, g : (0, \varepsilon_0) \times \mathcal{M} \to \mathbb{R}^4$ are C^r. The dot $\cdot$ represents the derivative of the functions $\mathbf{x}(\tau)$ and $\mathbf{y}(\tau)$ with respect to the variable τ.

If we write $t = \dfrac{\tau}{\varepsilon}$, then system (2) becomes

$$\begin{cases} \mathbf{x}' = f(\mathbf{x}, \mathbf{y}, \varepsilon), \\ \mathbf{y}' = \varepsilon g(\mathbf{x}, \mathbf{y}, \varepsilon), \end{cases} \tag{3}$$

in which the apostrophe ' denotes the derivative of the functions $\mathbf{x}(t)$ and $\mathbf{y}(t)$ with respect to the variable t. Notice that the parameter $\varepsilon = \dfrac{\tau}{t}$ represents the ratio of the time scales.

Consider equation (2) and set $\varepsilon = 0$. We obtain the so called *reduced problem* given by

$$0 = f(\mathbf{x}, \mathbf{y}, 0), \quad \dot{\mathbf{y}} = g(\mathbf{x}, \mathbf{y}, 0). \tag{4}$$

Observe that (4) is not an ordinary differential equation, but it is an *algebraic differential equation.*

Solutions of (4) are contained in the set

$$C_0 = \Big\{(\mathbf{x}, \mathbf{y}) \in \mathcal{M} : \ f(\mathbf{x}, \mathbf{y}, 0) = 0\Big\}.$$

The set C_0 is called *critical set.* In the case where C_0 is a manifold, C_0 is called critical manifold.

On the other hand, setting $\varepsilon = 0$ in equation (3) we obtain the so called *layer problem*

$$\mathbf{x}' = f(\mathbf{x}, \mathbf{y}, 0), \quad \mathbf{y}' = 0. \tag{5}$$

Moreover, the system (5) can be seen as a system of ordinary differential equations, where $\mathbf{y} \in \mathbb{R}^2$ is a parameter and the critical set C_0 is a set of equilibrium points of (5).

The main goal of *geometric singular perturbation theory* is to study systems (4) and (5) in order to obtain information of the full system (2). Observe that the systems (2) and (3) are equivalent when $\varepsilon > 0$, since they only differ by time scale.

We next introduce the notion of normally hyperbolic points.

Let $\mathbf{x}_0 \in S$, for any set $S \subset \mathcal{M}$. We say that $\mathbf{x}_0$ is *normally hyperbolic* if the 2×2 matrix $Df_{\mathbf{x}}(\mathbf{x}_0)$ does not have eigenvalues with zero real part.

In what follows we present a version of *Fenichel's Theorem* whose proof can be found in [16].

Theorem 2. *Consider $\mathcal{M}$ a C^{r+1} manifold, $2 \leq r \leq \infty$. Let X_ε, $\varepsilon \in (-\varepsilon_0, \varepsilon_0)$ be a C^r family of vector fields on $\mathcal{M}$, and assume C_0 a C^r submanifold of $\mathcal{M}$ consisting entirely of equilibrium points of X_0. If $S \subset C_0$ is a $j-$dimensional compact normally hyperbolic invariant manifold of the reduced vector field X_R with a $j + j^s-$dimensional local* *stable manifold $\mathcal{W}^s$ and a $j + j^u-$dimensional local unstable manifold $\mathcal{W}^u$, then there exists $\varepsilon_1 > 0$ such that*

(i) *There exists a C^{r-1} family of manifolds $\{S_\varepsilon : \varepsilon \in (-\varepsilon_1, \varepsilon_1)\}$ with $S_0 = S$ and S_ε is a normally hyperbolic invariant manifold of X_ε.*

(ii) *There are C^{r-1} families of $(j + j^s + k^s)-$dimensional and $(j + j^u + k^u)-$dimensional manifolds $\{S^s_\varepsilon : \varepsilon \in (-\varepsilon_1, \varepsilon_1)\}$ and $\{S^u_\varepsilon : \varepsilon \in (-\varepsilon_1, \varepsilon_1)\}$ such that for $\varepsilon > 0$ the manifolds S^s_ε and S^u_ε are local stable and unstable manifolds of S_ε.*

Now, suppose that S is given as in Theorem 2, then $Df_{\mathbf{x}}(\mathbf{x}, \mathbf{y}, 0)$ is invertible for all $(\mathbf{x}, \mathbf{y}) \in S$. From *Implicit Function Theorem* there exists a smooth function h_0 defined on a compact domain $K \subset \mathbb{R}^2$ such that

$$S = \{(\mathbf{x}, \mathbf{y})) : \mathbf{x} = h_0(\mathbf{y})\}.$$

That is, S is locally the graph of a function of $\mathbf{x}$ in terms of $\mathbf{y}$

Theorem 3. *Assume the same hypotheses of Theorem 2. If $\varepsilon > 0$ is sufficiently small, there exists a function $\mathbf{x} = h_\varepsilon(\mathbf{y})$, defined on K, so that the graph*

$$S_\varepsilon = \{(\mathbf{x}, \mathbf{y})) : \mathbf{x} = h_\varepsilon(\mathbf{y})\}$$

is locally invariant under (3). Moreover h_ε is C^r, for any $r < \infty$, jointly in $\mathbf{y}$ and ε.

2.3. **Two-dimensional holomorphic slow-fast systems.** Consider the two-dimensional holomorphic slow-fast system (SF1), where $f(z, w) = u_1 + iv_1$ and $g(z, w) = u_2 + iv_2$ are holomorphic functions of the complex variables $(z, w) = (x_1 + iy_1, x_2 + iy_2) \in D$.

Recall that the fast system associated to system (SF1) is given by

$$\begin{cases} z' = f(z, w) = u_1 + iv_1, \\ w' = \varepsilon g(z, w) = \varepsilon(u_2 + iv_2), \end{cases} \tag{6}$$

where the apostrophe ' denotes the derivative of the functions $z(t)$ and $w(t)$ with respect to the variable t, with $t = \tau/\varepsilon$. Notice that the critical set associated to system (6) is defined as $C_0 = \{(z, w) \in D : f(z, w) = 0\}$.

In this context the reduced problem is given by

$$0 = f(z, w), \ \dot{w} = g(z, w), \tag{7}$$

and the layer problem is defined as

$$z' = f(z, w), \ w' = 0. \tag{8}$$

Observe that the systems (SF1) and (6) are equivalent when $\varepsilon > 0$, since they only differ by time scale. In addition, systems (7) and (8) allow us to obtain information about system (SF1).

Now, we introduce the notion of normal hyperbolicity in the complex context. Let $q_0 = (z_0, w_0) \in C_0$ be an equilibrium point of the system (6) for $\varepsilon = 0$. Since f, g are holomorphic functions, then the Cauchy-Riemann equations hold. Thus, the Jacobian matrix of the system in $\mathbb{R}^4$ associated to (6) at q_0 is given by

$$J_{\mathbb{R}}(f,0)|_{q_0} = \begin{pmatrix} (u_1)_{x_1} & -(v_1)_{x_1} & (u_1)_{x_2} & -(v_1)_{x_2} \\ (v_1)_{x_1} & (u_1)_{x_1} & (v_1)_{x_2} & (u_1)_{x_2} \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}, \tag{9}$$

this implies that the eigenvalues of $J_{\mathbb{R}}(f,0)|_{q_0}$ are $\lambda_{\pm} = (u_1)_{x_1} \pm i(v_1)_{x_1}$. Hence, a subset $S \subset C_0$ is called normally hyperbolic provided that $\operatorname{Re}\left(\frac{\partial f}{\partial z}(q)\right) = (u_1)_{x1}(q) \neq 0$ for all $q \in S$.

2.4. **Briot-Bouquet Systems.** Briot-Bouquet systems have been widely used in the research literature (see, for instance, [6, 13, 14]).

A Briot-Bouquet system is of the following form

$$zw' = F(z,w), \quad F(0,0) = 0, \tag{10}$$

where F is a holomorphic function and the apostrophe ' denotes the derivative of the function $w(z)$ with respect to the complex variable z.

The one-dimensional theory is well known since the original work of Briot and Bouquet (see [1]). Here we will use the following result, which was proved in [8, Proposition 1.1.1].

Proposition 4 (Briot-Bouquet criterion). *If $F(z,w) = \lambda w + \mathcal{O}(z, w^2)$ and $\lambda \notin \mathbb{N}$, then (10) admits a unique holomorphic solution at $z = 0$ satisfying $w(0) = 0$.*

3. Persistence of Invariant Manifold

This section aims to prove that centers, foci and nodes of the reduced problem associated with the holomorphic slow-fast system (SF1), whose Jacobian matrix of f and g is diagonalizable at the origin, are preserved by singular perturbation.

For that, first we are going to construct one-dimensional invariant complex manifolds via the *Briot-Bouquet Theory*. Thus, consider the complex differential system given by (SF1). Without loss of generality, assume that $q_\varepsilon = (0,0)$ is an equilibrium point of (SF1) in D. Suppose that the associated linearized system is such that the eigenvalues of the

Jacobian matrix $J_{\mathbb{C}}(f,g)|_{q_\varepsilon}$ of f, g are given by $\alpha, \beta \in \mathbb{C}$ and $J_{\mathbb{C}}(f,g)|_{q_\varepsilon}$ is diagonalizable. Hence, we can write (SF1) as (SF2) where

$$\tilde{f}(z,w) = \sum_{n=2}^{\infty} \sum_{s+l=n} a_{s,l} z^s w^l \quad \text{and} \quad \tilde{g}(z,w) = \sum_{n=2}^{\infty} \sum_{s+l=n} b_{s,l} z^s w^l, \tag{11}$$

which convergent in some neighbourhood of q_ε. Here s, l are nonnegative integers and $a_{s,l}, b_{s,l}$ are complex constants.

In this context, the critical set associated with complex system (SF2) is given by $C_0 = \{(z,w) : \alpha z + \tilde{f}(z,w) = 0\}$. Emphasize that if C_0 is a normally hyperbolic critical manifold, then $\mathrm{Re}(\alpha) \neq 0$.

To find an invariant manifold of system (SF2), we consider the following initial value problem

$$(\alpha z + \tilde{f}(z,w))\frac{dw}{dz} = \varepsilon(\beta w + \tilde{g}(z,w)), \quad w_\varepsilon(0) = 0, \quad w_\varepsilon'(0) = 0. \tag{12}$$

In what follows we present a fundamental result that will be used to prove the main theorem of this section.

Theorem A. *Suppose that C_0 is a normally hyperbolic critical manifold associated with complex system (SF2) and consider $\varepsilon > 0$ a small enough parameter. Then, system (SF2) has a unique one-dimensional invariant complex manifold C_ε passing through the equilibrium point q_ε, which is given by*

$$C_\varepsilon = \{(z,w) : w = h_\varepsilon(z), \quad h_\varepsilon(0) = 0, \quad h_\varepsilon'(0) = 0\},$$

where h_ε is holomorphic in $B = B(0,r)$.

Proof. First, we prove that the initial value problem (12) has a unique solution $w = h_\varepsilon(z)$, which is holomorphic in B. Indeed, consider the holomorphic transformation $w(z) = z\varphi(z)$, where $z \in B$. Then, the initial value problem (12) can be written as

$$(\alpha z + \tilde{f}(z, z\varphi))[\varphi + z\varphi'] = \varepsilon(\beta z\varphi + \tilde{g}(z, z\varphi)), \quad \varphi_\varepsilon(0) = 0. \tag{13}$$

Since C_0 is a normally hyperbolic critical manifold, then $\mathrm{Re}(\alpha) \neq 0$. Thus, dividing equation (13) by αz, we get

$$\left(1 + \frac{\tilde{f}(z,z\varphi)}{\alpha z}\right)[\varphi + z\varphi'] = \varepsilon\left(\frac{\beta\varphi}{\alpha} + \frac{\tilde{g}(z,z\varphi)}{\alpha z}\right). \tag{14}$$

Now, we define the holomorphic functions $p(z,\varphi) := \frac{\tilde{f}(z,z\varphi)}{\alpha z}$ and $q(z,\varphi) := \frac{\tilde{g}(z,z\varphi)}{\alpha z}$. Then, from (11) and $w = z\varphi$, we obtain

$$\tilde{p}(z,\varphi) = \frac{1}{\alpha}\sum_{n=2}^{\infty} \sum_{s+l=n} a_{s,l}\varphi^l z^{n-1} \quad \text{and} \quad \tilde{q}(z,\varphi) = \frac{1}{\alpha}\sum_{n=2}^{\infty} \sum_{s+l=n} b_{s,l}\varphi^l z^{n-1},$$

which converge in some neighbourhood of q_ε. Rewriting equation (14), we get

$$z\varphi' = -\varphi + \varepsilon \frac{\frac{\beta\varphi}{\alpha} + q(z,\varphi)}{1 + p(z,\varphi)} := F_\varepsilon(z,\varphi).$$

Thus, putting the linear part of F_ε in an explicit way, we obtain that

$$z\varphi' = -\left(1 - \frac{\varepsilon\beta}{\alpha}\right)\varphi + \frac{\varepsilon b_{20}}{\alpha} z + \varepsilon Q(z,\varphi), \tag{15}$$

where

$$Q(z,\varphi) = \left(q(z,\varphi) - \frac{b_{20}}{\alpha} z\right) - \frac{q(z,\varphi)p(z,\varphi)}{1 + p(z,\varphi)} - \frac{\beta\varphi p(z,\varphi)}{\alpha(1 + p(z,\varphi))}.$$

Recall that F_ε is a holomorphic function in a neighborhood of q_ε, $F_\varepsilon(0,0) = 0$ and $Q(z,\varphi) = \mathcal{O}_2(z,\varphi)$. For $\varepsilon > 0$ small enough, we have that $\frac{\varepsilon\beta}{\alpha} - 1 \notin \mathbb{N}$, then from *Briot-Bouquet criterion* (see Proposition 4) we conclude that (15) has a unique solution $\varphi = \Lambda_\varepsilon(z)$, which is holomorphic in B. Using that $w = z\varphi$, then the initial value problem (12) has a unique solution $w = h_\varepsilon(z)$, where $h_\varepsilon(z) = z\Lambda_\varepsilon(z)$ is holomorphic in B.

Now, consider the holomoprhic function $H_\varepsilon(z,w) = w - h_\varepsilon(z)$, with $H_\varepsilon(0,0) = 0$ and $\frac{\partial H_\varepsilon}{\partial z}(0,0) = 0$. Define the set $C_\varepsilon := \{(z,w) : w = h_\varepsilon(z), \quad h_\varepsilon(0) = 0, \quad h'_\varepsilon(0) = 0\}$. Emphasize that $C_\varepsilon = H_\varepsilon^{-1}(0)$. By construction, we get that C_ε is an invariant set.

Notice that the rank of H is the rank of the matrix 1×2, which is given by

$$J_{\mathbb{C}} H_\varepsilon(z,w) = \left(\frac{\partial H}{\partial z}(z,w), \frac{\partial H}{\partial w}(z,w)\right) = \left(-\frac{\partial h_\varepsilon}{\partial z}(z), 1\right).$$

Thus, $J_{\mathbb{C}} H$ has maximal rank (rank 1) in each point of C_ε. From Proposition 1, we conclude that C_ε is a complex manifold of dimension 1. $\square$

Remark 5. *From the proof of Theorem A, we can deduce that this result still holds when we replace the hypothesis in which C_0 is a normally hyperbolic manifold by $\frac{\partial f}{\partial z}(0,0) = \alpha \neq 0$, which is a weaker hypothesis.*

In the sequel, assume that:

- **(H)** the eigenvalues of the $J_{\mathbb{C}}(f,g)|_{q_\varepsilon}$ are $\alpha, \beta \in \mathbb{C} \setminus \{0\}$ and that α, β are of the same type, that is, both α and β are pure imaginary complex numbers, real numbers, or complex numbers with nonzero real and imaginary parts.

At this point, we can assume that C_0 is given as the graph of a function of z in terms of w. Indeed, define $\Theta(z,w) = \alpha z + \tilde{f}(z,w)$ and $q_0 = (0,0)$. Since $\Theta(q_0) = 0$ and $\frac{\partial \Theta}{\partial z}(q_0) = \alpha \neq 0$, then from *Implicit*

Function Theorem, there exist a neighborhood U_{q_0} of q_0 and a holomorphic function $z = L(w)$ such that $L(0) = 0$ and $\Theta(L(w), w) = 0$, for all $w \in U_{q_0}$. Therefore, locally the critical manifold is given by $C_0 = \{(z, w) : z = L(w), \quad L(0) = 0, \quad w \in U\}$.

Recall that the dynamics on C_0 is given by

$$\dot{w} = \beta w + g(L(w), w) =: G(w). \tag{16}$$

From item (b) of [4, Theorem 1.1] we know that G and βw are 0-conformally conjugate.

We are ready to state the main result of this section.

Theorem B. *Suppose that the eigenvalues of $J_{\mathbb{C}}(f, g)|_{q_\varepsilon}$ satisfy hypothesis* **(H)***, where ε is a small enough positive parameter. Then, centers, foci, and nodes of the reduced problem associated to* (SF1) *are persistent by singular perturbation.*

Proof. By Theorem A, we know that, for $\varepsilon > 0$ small enough, system (SF2) has a unique one-dimensional invariant complex manifold C_ε at q_ε, which is given by

$$C_\varepsilon = \{(z, w) : w = h_\varepsilon(z), \quad h_\varepsilon(0) = 0, \quad h'_\varepsilon(0) = 0\},$$

where h_ε is holomorphic in B.

Emphasize that the dynamics about C_ε is given by

$$\dot{z} = \frac{\dot{w}}{h'_\varepsilon(z)} = \frac{\alpha}{\varepsilon} z + \frac{\tilde{f}(z, h_\varepsilon(z))}{\varepsilon} =: \Gamma_\varepsilon(z). \tag{17}$$

By item (b) of [4, Theorem 1.1] we can conclude that Γ_ε and $\frac{\alpha}{\varepsilon} z$ are 0-conformally conjugate. From (16) and (17) and the fact that α and β are of the same type the result follows. $\square$

4. Other families of holomorphic slow-fast systems

In this section, we are concerned with finding one-dimensional invariant complex manifolds of slow-fast systems whose Jacobian matrix is not necessarily diagonalizable at the origin. Furthermore, we do not require that the origin be an equilibrium point of the system.

4.1. Uncoupled differential systems.

Consider the two-dimensional holomorphic slow-fast system (SF1), where f, g are holomorphic functions defined in a punctured disk $D = D(0, R) \setminus \{0\}$.

We assume that $\frac{\partial f}{\partial w}(z, w) \neq 0$, for all $(z, w) \in C_0$. From Proposition 1, we know that C_0 is a complex manifold. Moreover, we can assume that C_0 is given as the graph of a function of w in terms of z. Indeed, since $\frac{\partial f}{\partial w}(z, w) \neq 0$ for all $(z, w) \in C_0$, then from *Implicit Function Theorem*, there exist an open set U and a holomorphic function $w = \lambda(z)$ such that $f(z, \lambda(z)) = 0$, for all $z \in U$.

Let us introduce the equations for the orbits of system (SF1):

$$\frac{dw}{dz} = \varepsilon \frac{g(z,w)}{f(z,w)}. \tag{18}$$

Suppose that there exist holomorphic functions η, κ defined in $\Omega = B \backslash L$ such that $\frac{g(z,w)}{f(z,w)} = \eta(z)\kappa(w)$ for all $z, w \in \Omega$, where L is a ray starting at 0, $B = B(0,r) \setminus \{0\}$ and

- either 0 is the single zero of η (resp. κ) in B;
- or 0 is an isolated singularity of η (resp. κ) and η (resp. κ) has no zeros in B.

From [2, Corollary 6.16], we know that the holomorphic functions $\eta, \kappa : \Omega \to \mathbb{C}$ have a primitive in Ω. Thus $1/\eta, 1/\kappa$ are also holomorphic functions in Ω and consequently both η, κ and $1/\eta, 1/\kappa$ have primitives in Ω. Integrating the equation $\frac{dw}{dz} = \varepsilon\eta(z)\kappa(w)$, we get that there exist holomorphic functions F, G such that $G(w) = \varepsilon F(z)$, $F'(z) = \eta(z)$ and $G'(w) = 1/\kappa(w)$.

In what follows we state an interesting result about the existence of invariant complex manifolds associated with slow-fast systems whose equation of the orbits is separable.

Theorem C. *Consider the slow-fast system* (SF1) *and assume that there exist holomorphic functions η, κ defined in Ω such that $\frac{g(z,w)}{f(z,w)} = \eta(z)\kappa(w)$, for all $z, w \in \Omega$. Then, there exists a one-dimensional invariant complex manifold C_ε associated with the differential system* (SF1)*, such that*

(a) *the flow on C_ε converges to the slow flow as $\varepsilon \to 0$.*
(b) *the dynamics over C_ε is given by $\dot{z} = g(z, w)$.*

Proof. Consider the holomoprhic function $H_\varepsilon(z,w) = G(w) - \varepsilon F(z)$ and the set $C_\varepsilon = \{(z,w) : G(w) = \varepsilon F(z)\}$. Recall that $C_\varepsilon = H_\varepsilon^{-1}(0)$. From (18), we obtain that C_ε is an invariant set.

Emphasize that the rank of H_ε is the rank of the matrix 1×2, which is given by

$$J_{\mathbb{C}} H_\varepsilon(z,w) = \left(\frac{\partial H_\varepsilon}{\partial z}(z,w), \frac{\partial H_\varepsilon}{\partial w}(z,w) \right) = \left(-\varepsilon\eta(z), \frac{1}{\kappa(w)} \right).$$

Hence, $J_{\mathbb{C}} H_\varepsilon$ has maximal rank (rank 1) in each point of C_ε. By Proposition 1, we conclude that C_ε is a complex manifold of dimension 1.

Now, we shall prove item (a). Deriving the equation $G(w) = \varepsilon F(z)$, we get that $1 = \varepsilon k(w)\eta(z)$, which implies that $f(z,w) = \varepsilon g(z,w)$. Therefore, $C_\varepsilon \subset D_\varepsilon = \{(z,w) : f(z,w) = \varepsilon g(z,w)\}$ and $D_0 = C_0$.

Since $\frac{\partial H_\varepsilon}{\partial w}(z,w) \neq 0$ for all $(z,w) \in C_\varepsilon$, then from *Implicit Function Theorem*, there exist an open set V and a holomorphic function $w = h_\varepsilon(z)$ such that $H_\varepsilon(z, h_\varepsilon(z)) = 0$, for all $z \in V$. Then, expanding $h_\varepsilon(z)$ in Taylor series around $\varepsilon = 0$, we get that $h_\varepsilon(z) = \lambda(z) + \varepsilon Q(z,\varepsilon)$, for all $\varepsilon \in [0, \varepsilon_0]$ and $z \in V \cap U$, where Q is a holomorphic function and $C_0 = \{(z,w) : w = \lambda(z),\ z \in U\}$ is locally the critical manifold.

In what follow, we show that the Hausdorff distance between C_0 and C_ε is of order ε, that is $d_H(C_0, C_\varepsilon) = \mathcal{O}(\varepsilon)$. Recall that the Hausdorff distance d_H between nonempty subsets A and B is given by

$$d_H(A,B) = \max\left\{\sup_{x\in A}\inf_{y\in B} d(x,y)\,,\ \sup_{x\in B}\inf_{y\in A} d(x,y)\right\}.$$

Hence, $d_H(C_0, C_\varepsilon) = \mathcal{O}(\varepsilon)$ provided that the following statements hold:

i. for each $p \in C_\varepsilon$ there exists $q \in C_0$ such that $d(p,q) = \mathcal{O}(\varepsilon)$;
ii. for each $q \in C_0$ there exists $p \in C_\varepsilon$ such that $d(p,q) = \mathcal{O}(\varepsilon)$.

In the sequel, we shall verify item (i). Item (ii) can be verified analogously.

Consider the point $p = (z,w) \in C_\varepsilon$ and take $q = (z,\xi) \in C_0$. Then, $w = h_\varepsilon(z) = \lambda(z) + \varepsilon Q(z,\varepsilon)$ and $\xi = \lambda(z)$. Thus,

$$d(p,q) = ||(0, w-\xi)|| = ||(0, \varepsilon Q(z,\varepsilon))|| = |Q(z,\varepsilon)|\,\varepsilon \leq \varepsilon M,$$

where we have used that $|Q(z,\varepsilon)| \leq M$, for all $(z,\varepsilon) \in K_{V\cap U} \times [0,\varepsilon_0]$, with $K_{V\cap U}$ a compact subset of $V \cap U$. This implies that $d(p,q) = \mathcal{O}(\varepsilon)$.

Finally, we prove item (b). Since $H_\varepsilon(z,w) = 0$ for all $(z,w) \in C_\varepsilon$, then $\frac{\partial H_\varepsilon}{\partial z}\dot{z} + \frac{\partial H_\varepsilon}{\partial w}\dot{w} = 0$. Consequently, the dynamics over C_ε is determined by

$$\dot{w} = -\frac{\dfrac{\partial H_\varepsilon}{\partial z}\dot{z}}{\dfrac{\partial H_\varepsilon}{\partial w}} = -\frac{-\varepsilon\eta(z)\dot{z}}{\dfrac{1}{\kappa(w)}} = \varepsilon\,\underbrace{\eta(z)\kappa(w)}\,\dot{z} = \frac{g(z,w)}{f(z,w)}f(z,w) = g(z,w).$$

□

Emphasize that it is possible to adapt the previous result for uncoupled slow-fast systems of the form:

$$\left\{\begin{array}{l} \varepsilon\dot{z} = f(z), \\ \dot{w} = g(w), \end{array}\right. \tag{19}$$

where f, g are holomorphic functions defined in Ω. Thus, the following result is a direct consequence of the above.

Corollary 6. *Equilibrium points and poles of order n of the reduced problem associated to (19) are persistent by singular perturbation.*

Example 7. *Let n be natural number with $n \geq 2$. Consider the following system*

$$\begin{cases} \varepsilon \dot{z} = z^n, \\ \dot{w} = g(w), \end{cases} \tag{20}$$

where g is one of the normal forms given in [4, Theorem 1.1]. *Recall that $C_0 = \{(z,w) \in \mathbb{C}^2 : z = 0\}$ is the critical manifold associated with system (20), which is not normally hyperbolic. From Theorem C, there exists an invariant complex manifold C_ε given by:*

- $C_\varepsilon = \{(z,w) : z^{n-1}\ln(w) = \varepsilon \frac{\eta}{-n+1}\}$, *provided that $g(w) = \eta w$, with $\eta \in \mathbb{C}$.*
- $C_\varepsilon = \{(z,w) : z^{n-1} = \varepsilon \frac{m-1}{n-1} w^{m-1}\}$, *provided that $g(w) = w^m$, with $m \geq 2$.*
- $C_\varepsilon = \{(z,w) : z^{n-1} = \varepsilon \frac{m+1}{-n+1} w^{-(m+1)}\}$, *provided that $g(w) = \frac{1}{w^m}$, with $m \geq 2$.*
- $C_\varepsilon = \left\{(z,w) : z^{n-1}\left(\frac{w^{-m+1}}{-m+1} + \ln(w)\right) = \varepsilon \frac{\gamma}{-n+1}\right\}$, *provided that $g(w) = \frac{\gamma w^m}{1+w^{m-1}}$, with $m \geq 2$ and $\gamma \in \mathbb{C}$.*

Moreover, in either case we have that C_ε converges to C_0 when ε tends to 0.

4.2. **Coupled differential equations.** Consider the system (SF3), where g, G are holomorphic functions defined in Ω such that $G'(w) = 1/g(w)$, $\alpha, \beta \in \mathbb{C} \setminus \{0\}$. In this constext, the critical manifold of (SF3) is $C_0 = \{(z,w) : z = -\frac{\beta G(w)}{\alpha}\}$.

Let us introduce the equations for the orbits of system (SF3):

$$\varepsilon \frac{dz}{dw} = \frac{\alpha z + \beta G(w)}{g(w)}. \tag{21}$$

Since $\dot{w} = g(w)$, then $G(w) = t$. From the first equation of (SF3), we get the linear differential equation given by $\varepsilon \dot{z} = \alpha z + \beta t$. Recall that $z = -\frac{\beta}{\alpha^2}(\alpha t + \varepsilon)$ is a solution of previous equation. Thus, we obtain the equation $z = -\frac{\beta}{\alpha^2}(\alpha G(w) + \varepsilon)$, which satisfies the equation of orbits (21).

In the next result we prove that the set of points that satisfies the above equation is in fact a one-dimensional invariant complex manifold. Moreover, if we assume that the critical manifold is not normally hyperbolic, then the *geometric singular perturbation theory* does not determine the type of stability of the invariant manifold C_ε. However, in the following theorem we are giving conditions to ensure when the invariant manifold C_ε is exponentially attractive (see Figure 3).

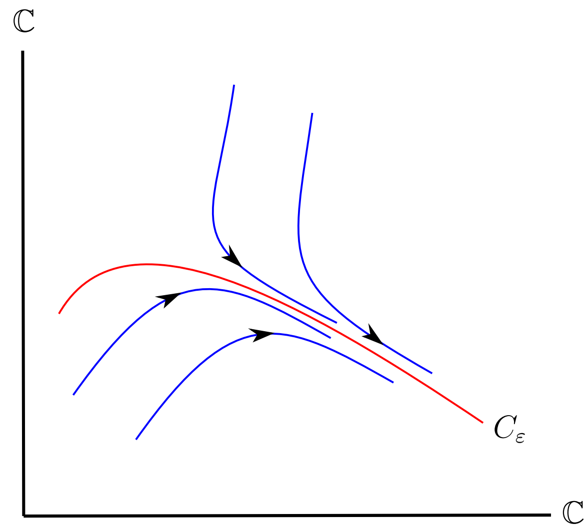

FIGURE 3. C_ε is exponentially attractive in a certain region $\mathcal{R}$, when C_0 is not normally hyperbolic.

Theorem D. *Consider the slow-fast system* (SF3). *Then, there exists a one-dimensional invariant complex manifold C_ε associated with the differential system* (SF3), *such that*

(a) *The flow on C_ε converges to the slow flow as $\varepsilon \to 0$.*
(b) *The dynamics over C_ε is given by $\dot{w} = g(w)$.*
(c) *If $\alpha = i\alpha_2$, with $\alpha_2 \in \mathbb{R} \setminus \{0\}$, then the orbit of system* (21) *with initial condition $z(w_0) = z_0$ stays exponentially close to the invariant manifold C_ε in the region $\mathcal{R} = \{(z, w) : \alpha_2 \Im(G(w) - G(w_0)) \geq \eta > 0\}$, for some $\eta > 0$.*

Proof. Consider the holomoprhic function $H_\varepsilon(z, w) = z + \frac{\beta}{\alpha^2}(\alpha G(w) + \varepsilon)$ and the set $C_\varepsilon = \{(z, w) : z = -\frac{\beta}{\alpha^2}(\alpha G(w) + \varepsilon)\}$. Recall that $C_\varepsilon = H_\varepsilon^{-1}(0)$. By construction C_ε is an invariant set.

Emphasize that the rank of H_ε is the rank of the matrix 1×2, which is given by

$$J_{\mathbb{C}} H_\varepsilon(z, w) = \left(\frac{\partial H_\varepsilon}{\partial z}(z, w), \frac{\partial H_\varepsilon}{\partial w}(z, w) \right) = \left(1, \frac{\beta}{\alpha g(w)} \right).$$

Hence, $J_{\mathbb{C}} H_\varepsilon$ has maximal rank (rank 1) in each point of C_ε. By Proposition 1, we conclude that C_ε is a complex manifold of dimension 1.

Now, we shall prove item (a). For that, it is enough to show that the Hausdorff distance between C_0 and C_ε is of order ε or equivalently that the following statements hold:

i. for each $p \in C_\varepsilon$ there exists $q \in C_0$ such that $d(p,q) = \mathcal{O}(\varepsilon)$;
ii. for each $q \in C_0$ there exists $p \in C_\varepsilon$ such that $d(p,q) = \mathcal{O}(\varepsilon)$.

In the sequel, we shall verify item (i). Item (ii) can be verified analogously.

Consider the point $p = (z,w) \in C_\varepsilon$ and take $q = (\xi, w) \in C_0$. Then, $z = -\frac{\beta}{\alpha^2}(\alpha G(w) + \varepsilon)$ and $\xi = -\frac{\beta}{\alpha}G(w)$. Thus,

$$d(p,q) = ||(z-\xi, 0)|| = \left|\left|\left(-\frac{\beta}{\alpha^2}\varepsilon, 0\right)\right|\right| = \left|\frac{\beta}{\alpha^2}\right|\varepsilon.$$

This implies that $d(p,q) = \mathcal{O}(\varepsilon)$.

Next, we prove item (b). Since $H_\varepsilon(z,w) = 0$ for all $(z,w) \in C_\varepsilon$, then $\frac{\partial H_\varepsilon}{\partial z}\dot{z} + \frac{\partial H_\varepsilon}{\partial w}\dot{w} = 0$. Consequently, the dynamics over C_ε is determined by

$$\dot{w} = -\frac{\dfrac{\partial H_\varepsilon}{\partial z}\dot{z}}{\dfrac{\partial H_\varepsilon}{\partial w}} = -\frac{\dot{z}}{\dfrac{\beta}{\alpha g(w)}} = -\frac{\alpha z + \beta G(w)}{\dfrac{\varepsilon\beta}{\alpha g(w)}} = -\frac{-\dfrac{\varepsilon\beta}{\alpha}}{\dfrac{\varepsilon\beta}{\alpha g(w)}} = g(w).$$

Finally, we prove item (c). Define $h_\varepsilon(w) = -\frac{\beta}{\alpha^2}(\alpha G(w) + \varepsilon)$. Hence, the invariant manifold is given by $C_\varepsilon = \{(z,w) : z = h_\varepsilon(w)\}$.

Now, we perform the change of variables $v = z - h_\varepsilon(w)$ in equation (21) obtaining:

$$\varepsilon\frac{dv}{dw} = \frac{i\alpha_2}{g(w)}v. \tag{22}$$

Recall that the solution of (22) with initial condition $v(w_0) = z(w_0) - h_\varepsilon(w_0)$ can be written as $v(w) = v(w_0)e^{\frac{i\alpha_2(G(w)-G(w_0))}{\varepsilon}}$. Thus,

$$|v(w)| = |v(w_0)|e^{-\dfrac{\alpha_2\Im(G(w)-G(w_0))}{\varepsilon}} \leq |v(w_0)|e^{-\dfrac{\eta}{\varepsilon}}.$$

Since $\eta > 0$, then any solution gets exponentially closer to the invariant manifold C_ε. $\square$

Example 8. *Consider the system defined as*

$$\begin{cases} \varepsilon\dot{z} = iz + w^2, \\ \dot{w} = \frac{1}{2w}, \end{cases} \tag{23}$$

with $(z_0, w_0) = (\varepsilon + i, 1)$. Notice that $G(w) = w^2$ and $\alpha = i$. Since $G'(w) = 2w$, from Theorem D there exists a one-dimensional invariant complex manifold C_ε associated with the differential system (23), which is exponentially attractive on $\mathcal{R} = \{(z,w) \in \mathbb{C}^2 : \mathrm{Re}(w)\,\mathrm{Im}(w) > 0\}$.

5. Fenichel manifold approximation

This section is focused on using Laurent series and *Fenichel's Theorem* to approximate complex manifolds and find smooth manifolds without complex structure of the slow-fast system (SF4). In particular, we are interested in studying the dynamics of the system (SF4).

5.1. **Non-existence of smooth manifolds with complex structure.** Below we present 2 families of slow-fast systems that, despite having associated smooth locally invariant manifolds, have no complex structure.

5.1.1. *Linear-w^n case.* Consider the following system

$$\begin{cases} \varepsilon \dot{z} = \alpha z + \beta w, \\ \dot{w} = w^n, \end{cases} \tag{24}$$

where $\alpha, \beta \in \mathbb{C} \setminus \{0\}$ and $n \geq 2$. Recall that the critical manifold C_0 associated with system (24) is normally hyperbolic provided that $\operatorname{Re}(\alpha) = \alpha_1 \neq 0$. Let S_0 be a compact subset of C_0.

We now apply the *Fenichel's Theorem* to the equivalent $\mathbb{C}^\infty$ four-dimensional real system associated with the system (24), then there exists a smooth locally invariant manifold $S_\varepsilon = \{(z, w) : z = h_\varepsilon(w)\}$ of the slow-fast system (24), which is diffeomorphic to S_0.

Emphasize that this does not guarantee that the complex function $h_\varepsilon(z)$ is a holomorphic function. Indeed, suppose that there exists an invariant manifold $C_\varepsilon = \{(z, w) : z = h_\varepsilon(w)\}$, where

$$h_\varepsilon(w) = a_0 + \cdots + a_n w^n + \cdots + a_{2n-1} w^{2n-1} + \mathcal{O}(w^{2n}). \tag{25}$$

Since $\dot{z} = h'_\varepsilon(w)\dot{w}$ then we obtain the following equation

$$\alpha h_\varepsilon(w) + \beta w = h'_\varepsilon(w) \varepsilon w^n. \tag{26}$$

Substituting (25) in (26), we get that $a_i = 0$, for all $i \neq kn - (k-1)$, $a_1 = -\frac{\beta}{\alpha}$, $a_n = -\frac{\beta\varepsilon}{\alpha^2}$, and $a_{kn-(k-1)} = -\frac{\beta^k \varepsilon^k \prod_{j=2}^{k}[(j-1)n-(j-2)]}{\alpha^{k+1}}$, with $k \geq 2$. Therefore,

$$C_\varepsilon = \left\{ (z, w) : z = -\frac{\beta}{\alpha} w - \frac{\beta\varepsilon}{\alpha^2} w^n + \sum_{k=2}^{\infty} a_{kn-(k-1)} w^{kn-(k-1)} \right\}.$$

Notice that the series $\sum_{k=2}^{\infty} a_{kn-(k-1)} w^{kn-(k-1)}$ diverges for $w \neq 0$, which contradicts the existence of the holomorphic function h_ε.

5.1.2. *Linear-$\frac{\gamma w^n}{1+w^{n-1}}$ case.* Consider the following system

$$\begin{cases} \varepsilon \dot{z} = \alpha z + \beta w, \\ \dot{w} = \frac{\gamma w^n}{1+w^{n-1}}, \end{cases} \tag{27}$$

where $\alpha, \beta \in \mathbb{C} \setminus \{0\}$ and $n \geq 2$. Recall that the critical manifold C_0 associated with system (27) is normally hyperbolic provided that $\operatorname{Re}(\alpha) = \alpha_1 \neq 0$. Take S_0 a compact subset of C_0.

We now apply the *Fenichel's Theorem* to the equivalent $\mathbb{C}^\infty$ four-dimensional real system associated with the system (27), then there exists a smooth locally invariant manifold $S_\varepsilon = \{(z,w) : z = h_\varepsilon(w)\}$ of the slow-fast system (27), which is diffeomorphic to S_0.

Emphasize that this does not guarantee that the complex function $h_\varepsilon(z)$ is a holomorphic function. Indeed, suppose that there exists an invariant manifold $C_\varepsilon = \{(z,w) : z = h_\varepsilon(w)\}$, where

$$h_\varepsilon(w) = a_0 + \cdots + a_n w^n + \cdots + a_{2n-1} w^{2n-1} + \mathcal{O}(w^{2n}). \tag{28}$$

Since $\dot{z} = h'_\varepsilon(w)\dot{w}$, then

$$\alpha h_\varepsilon(w) + \beta w = h'_\varepsilon(w) \varepsilon \frac{\gamma w^n}{1+w^{n-1}}. \tag{29}$$

Substituting (28) in (29), we get that $a_i = 0$, for all $i \neq kn - (k-1)$, $a_1 = -\frac{\beta}{\alpha}$, $a_n = -\frac{\beta\gamma\varepsilon}{\alpha^2}$, and $a_{kn-(k-1)} = -\frac{\beta\gamma\varepsilon \prod_{j=2}^{k}[\alpha - ((j-1)n-(j-2))\gamma\varepsilon]}{\alpha^{k+1}}$, with $k \geq 2$. Therefore,

$$C_\varepsilon = \left\{ (z,w) : z = -\frac{\beta}{\alpha} w - \frac{\beta\gamma\varepsilon}{\alpha^2} w^n + \sum_{k=2}^{\infty} a_{kn-(k-1)} w^{kn-(k-1)} \right\}.$$

Notice that the series $\sum_{k=2}^{\infty} a_{kn-(k-1)} w^{kn-(k-1)}$ diverges for $w \neq 0$, which contradicts the existence of the holomorphic function h_ε.

The following result is a direct consequence of [4, Theorem 1.1] and of cases 5.1.1 and 5.1.2.

Proposition 9. *Consider system* (SF1) *such that g has a zero of order $n > 1$ and f is a linear holomorphic function, then system* (SF1) *has no complex manifolds of form $C_\varepsilon = \{(z,w) : z = h_\varepsilon(w)\}$.*

5.2. **Approximation of complex invariant manifold.** Here we present 2 families of slow-fast systems that, in the case of having associated locally invariant smooth manifolds with complex structure, can be approximated via Laurent series and also the singularity associated with the reduced problem is preserved by singular perturbation.

5.2.1. *Linear-linear case.* Consider the following system

$$\begin{cases} \varepsilon\dot{z} = az + bw, \\ \dot{w} = cz + dw, \end{cases} \tag{30}$$

where $a \in \mathbb{C}\setminus\{0\}$ and $b, c, d \in \mathbb{C}$. Assume that the critical manifold C_0 associated with system (30) is normally hyperbolic, that is, $\mathrm{Re}\,(a) \neq 0$.

Notice that the reduced problem associated with system (30) is given by

$$\begin{cases} 0 = az + bw, \\ \dot{w} = \left(\frac{ad-bc}{a}\right) w =: \widetilde{g}(w). \end{cases} \tag{31}$$

Thus, the equilibrium point is $w_0 = 0$ and $\widetilde{g}'(w_0) = \frac{ad-bc}{a} = \alpha + i\beta$, where $\alpha = \mathrm{Re}\left(\frac{ad-bc}{a}\right)$ and $\beta = \mathrm{Im}\left(\frac{ad-bc}{a}\right)$. In addition, the Jacobian matrix at the equilibrium point of system (31) is

$$J_{\mathbb{R}}\widetilde{g}|_{w_0} = \begin{pmatrix} \alpha & -\beta \\ \beta & \alpha \end{pmatrix}.$$

The determinant D and the trace T are

$$D = \alpha^2 + \beta^2 \quad \text{and} \quad T = 2\alpha.$$

- $\beta \neq 0$. We have that if $\alpha > 0$ then the origin is a repelling focus of the reduced problem (31), if $\alpha < 0$ then the origin is an attracting focus of (31).
- $\beta = 0$. We have that if $\alpha > 0$ then the origin is a repelling node of the reduced problem (31) and if $\alpha < 0$ then the origin is an attracting node of (31).

Therefore, we get the following table:

α	j^u	j^s
$+$	2	0
$-$	0	2

where j^u, j^s have been defined in Theorem 2. In this case, the eigenvalues of the Jacobian matrix (9) are $\lambda_\pm = \mathrm{Re}\,(a) \pm i\,\mathrm{Im}\,(a)$ Thus, we have the following table:

$\mathrm{Re}\,(a)$	k^u	k^s
$+$	2	0
$-$	0	2

where k^u, k^s have been defined in Theorem 2. Hence, if $\alpha, \mathrm{Re}\,(a) > 0$, then $j^u + k^u = 4$ and $j^s + k^s = 0$. if $\alpha, \mathrm{Re}\,(a) < 0$, then $j^u + k^u = 0$ and

$j^s + k^s = 4$. If $sign(\alpha) \neq sign(\mathrm{Re}\,(a))$, then $j^u + k^u = 2$ and $j^s + k^s = 2$. From item (ii) of Theorem 2:

- If $\alpha, \mathrm{Re}\,(a) > 0$, then the equilibrium point $q_\varepsilon = (0,0)$ of system (30) is a global repelling point.
- If $\alpha, \mathrm{Re}\,(a) < 0$, then the equilibrium point $q_\varepsilon = (0,0)$ of system (30) is a global attracting point.
- If $sign(\alpha) \neq sign(\mathrm{Re}\,(a))$, then the equilibrium point $q_\varepsilon = (0,0)$ of system (30) is a saddle point.

Notice that when $\alpha = 0$ we can not use item (ii) of Theorem 2, however, C_0 is normally hyperbolic, thus if we take a compact subset S_0 of C_0, we can apply the *Fenichel's Theorem* to the equivalent $\mathbb{C}^\infty$ four-dimensional real system associated with the system (30), then there exists a smooth locally invariant manifold $S_\varepsilon = \{(z,w) : z = h_\varepsilon(w)\}$ of the slow-fast system (30), which is diffeomorphic to S_0.

Although this does not guarantee that the complex function $h_\varepsilon(z)$ is a holomorphic function, we would like to know the dynamics on the Fenichel manifold if it exists. Hence, suppose that $C_\varepsilon = \{(z,w) : z = h_\varepsilon(w)\}$, where $h_\varepsilon(w) = \lambda_0 + \lambda_1 w + \mathcal{O}(w^2)$. Since $\dot{z} = h'_\varepsilon(w)\dot{w}$, then

$$ah_\varepsilon(w) + bw = h'_\varepsilon(w)\varepsilon((\alpha + i\beta)w).$$

This implies that $\lambda_i = 0$, for all $i \neq 1$ and $\lambda_1 = \frac{-b}{a-\varepsilon(\alpha+i\beta)}$. Therefore,

$$C_\varepsilon = \left\{(z,w) : z = \frac{-b}{a - \varepsilon(\alpha + i\beta)}w\right\}.$$

In particular, if $\alpha = 0$, then

$$C_\varepsilon|_{\alpha=0} = \left\{(z,w) : z = \frac{-b}{a - i\varepsilon\beta}w\right\}.$$

Recall that the dynamics over the manifold $C_\varepsilon|_{\alpha=0}$ is given by the following differential equation

$$\dot{w} = i\beta w.$$

Therefore, if w_0 is a center of reduced problem (31), then w_ε is a center of system (30) such that $w_\varepsilon \to w_0$ when $\varepsilon \to 0$ (see Figure 4).

The following result is a direct consequence of [4, Theorem 1.1] and of case 5.2.1.

Proposition 10. *Consider system* (SF1) *such that g has a zero w_0 of order one and f is a linear holomorphic function. If system* (SF1) *admits an invariant manifold with complex structure, then centers of the reduced problem associated to* (SF1) *are persistent by singular perturbation.*

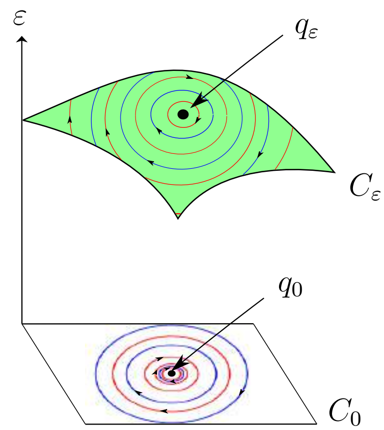

FIGURE 4. Persistence of a center of the reduced problem by singular perturbation.

5.2.2. *Linear-pole case.* Consider the following system

$$\begin{cases} \varepsilon \dot{z} = \alpha z + \beta w, \\ \dot{w} = \frac{1}{w^n}, \end{cases} \tag{32}$$

where $\alpha, \beta \in \mathbb{C} \setminus \{0\}$ and $n \geq 1$. Recall that the critical manifold C_0 associated with system (32) is normally hyperbolic provided that $\mathrm{Re}\,(\alpha) = \alpha_1 \neq 0$. Let S_0 be a compact subset of C_0.

We now apply the *Fenichel's Theorem* to the equivalent $\mathbb{C}^\infty$ four-dimensional real system associated with the system (32), then there exists a smooth locally invariant manifold $S_\varepsilon = \{(z, w) : z = h_\varepsilon(w)\}$ of the slow-fast system (32), which is diffeomorphic to S_0.

Although this does not guarantee that the complex function $h_\varepsilon(z)$ is a holomorphic function, we would like to know the dynamics on the Fenichel manifold if it exists. Hence, suppose that there exists an invariant manifold $C_\varepsilon = \{(z, w) : z = h_\varepsilon(w)\}$, where

$$h_\varepsilon(w) = a_0 + \cdots + a_{n+2} w^{n+2} + \cdots + a_{2n+3} w^{2n+3} + \cdots + \mathcal{O}(w^{3n+4}). \tag{33}$$

Since $\dot{z} = h'_\varepsilon(w)\dot{w}$, then we get the following equation

$$\alpha h_\varepsilon(w) + \beta w = h'_\varepsilon(w) \varepsilon w^{-n}. \tag{34}$$

Substituting (33) in (34), we get that $a_i = 0$, for all $i \neq kn + (k+1)$ and $a_{kn+(k+1)} = \frac{\alpha^{k-1}\beta}{\varepsilon^k \prod_{j=1}^{k} [jn+(j+1)]}$, with $k \geq 1$. Therefore,

$$C_\varepsilon = \left\{ (z, w) : z = \sum_{k=1}^{\infty} a_{kn+(k+1)} w^{kn+(k+1)} \right\}.$$

It is easy to see that the series $\sum_{k=1}^{\infty} a_{kn+(k+1)} w^{kn+(k+1)}$ converges for all $w \in \mathbb{C}$. Recall that the dynamics over the manifold C_ε is given by the following differential equation

$$\dot{w} = \frac{\beta w + \frac{\alpha\beta}{(n+2)\varepsilon} w^{n+2} + \frac{\alpha^2\beta}{(n+2)(2n+3)\varepsilon^2} w^{2n+3} + \mathcal{O}(1/\varepsilon^3, w^{3n+4})}{\frac{\beta}{\varepsilon} w^{n+1} + \frac{\alpha\beta}{(n+2)\varepsilon^2} w^{2n+2} + \mathcal{O}(1/\varepsilon^3, w^{3n+3})} =: F_\varepsilon(w).$$

Thus, expanding F_ε around $w_0 = 0$ we get that

$$F_\varepsilon(w) = \frac{\varepsilon}{w^n} + \mathcal{O}(1/\varepsilon, w^{n+2}).$$

Therefore, w_0 is a pole of order n of F_ε (see Figure 5). From [4, Theorem 1.1], we conclude that F_ε and $G \equiv \frac{1}{w^n}$ are 0-conformally conjugated.

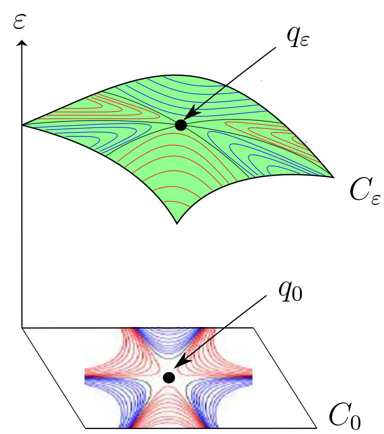

FIGURE 5. Persistence of a pole of the reduced problem by singular perturbation.

The following result is a direct consequence of [4, Theorem 1.1] and of case 5.2.2.

Proposition 11. *Consider system (SF1) such that g has a pole w_0 of order $n \geq 1$ and f is a linear holomorphic function. If system (SF1) admits an invariant manifold with complex structure, then poles of order n of the reduced problem associated to (SF1) are persistent by singular perturbation.*

6. Statements and Declarations

The authors declare that they have no known competing financial interests or personal relationships that could have appeared to influence the work reported in this paper.

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São Paulo State University (Unesp), Institute of Biosciences, Humanities and Exact Sciences. Rua C. Colombo, 2265, CEP 15054–000. S. J. Rio Preto, São Paulo, Brazil.

Email address: gabriel.rondon@unesp.br

Email address: paulo.r.silva@unesp.br

Email address: fernando.gouveia@unesp.br

RIEMANNIAN GEOMETRY OF G_2-TYPE REAL FLAG MANIFOLDS

BRIAN GRAJALES, GABRIEL RONDÓN, AND JULIETH SAAVEDRA

Abstract. In this paper, we investigate homogeneous Riemannian geometry on real flag manifolds of the split real form of $\mathfrak{g}_2$. We characterize the metrics that are invariant under the action of a maximal compact subgroup of G_2. Our exploration encompasses the analysis of g.o. metrics and equigeodesics on the $\mathfrak{g}_2$-type flag manifolds. Additionally, we explore the Ricci flow for the case where the isotropy representation has no equivalent summands, employing techniques from the qualitative theory of dynamical systems.

Contents

1. Introduction

In the context of homogeneous manifolds, the study of real flag manifolds stands out as a fascinating pursuit, engaging in the intricacies of geometric structures and their underlying symmetries. A real flag manifold is a quotient space $\mathbb{F} = G/P$, where G is a connected Lie group with non-compact real simple associated Lie algebra $\mathfrak{g}$, and P is a parabolic subgroup of G. In this paper, we consider the case where $\mathfrak{g}$ is the split real form of $\mathfrak{g}_2$. We present an exploration of the homogeneous Riemannian geometry in these manifolds, focusing particularly on homogeneous geodesics and the Ricci flow of invariant metrics.

For a real flag manifold $\mathbb{F} = G/P$, we have that any maximal compact subgroup $K \subseteq G$ acts transitively on $\mathbb{F}$ with isotropy subgroup $K \cap P$. This leads to an alternative presentation of $\mathbb{F}$, namely, $\mathbb{F} = K/(K \cap P)$. This presentation yields the isotropy representation of $K \cap P$ on the tangent space $T_o\mathbb{F}$ at the point $o := e(K \cap P)$, where e denotes the identity element of K. The compactness of K ensures the complete decomposition of this representation into irreducible subrepresentations. The understanding of these subrepresentations and their relations is key to describing K-invariant tensor fields on $\mathbb{F}$. A detailed study of the isotropy representation of a real flag manifold was developed in [17]. Several authors have contributed to the study of geometry and topology of real flag

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manifolds (see, for instance, [10, 12, 16, 24, 25]). In particular, we point out [12], where the authors described the invariant metrics on flag manifolds of a split real form of a classical Lie algebra (A, B, C, or D) and used this description to characterize those invariant metrics for which every geodesic through the origin o is a homogeneous geodesic. This means that every geodesic is the orbit of a one-parameter subgroup of G. An invariant metric with that property is called a g.o. metric (geodesic orbit metric), and the corresponding Riemannian homogeneous space is called a g.o. space. This class of homogeneous spaces includes simply connected symmetric spaces and naturally reductive homogeneous spaces. While a complete classification of g.o. spaces is far from being accomplished, there exists a substantial body of literature on this matter; we refer to [2, 5, 6, 21] for instance.

The primary objective of this paper is to continue the aforementioned work in [12] by providing a detailed description of the invariant metrics on real flag manifolds associated with the exceptional Lie algebra $\mathfrak{g}_2$ and classifying the g.o. metrics among them (Theorem 3.5). However, our scope extends beyond this; we also aim to characterize homogeneous curves (orbits of a one-parameter subgroup) that are geodesics for every invariant metric (Theorem 3.8). These special curves are known as equigeodesics. Recent works on the classification and properties of these curves include [8, 11, 22]. Here, we mainly use the results provided in [11] to obtain the equigeodesics on real flag manifolds of $\mathfrak{g}_2$.

As a final contribution, we explore the dynamics of the system associated with the homogeneous Ricci flow in the case where all the irreducible subrepresentations of the isotropy representation have multiplicity one. The Ricci flow is currently one of the most studied topics in Differential Geometry. For a differentiable manifold M, it is defined as the nonlinear evolution equation

$$\frac{\partial g}{\partial t} = -2\mathrm{Ric}(g), \tag{1.1}$$

where $t \mapsto g(t)$ is a one-parameter family of Riemannian metrics on M and $\mathrm{Ric}(g)$ is the Ricci tensor associated with g. Hamilton introduced it in [15], gaining significance due to its implications for understanding the geometric and topological structure of Riemannian manifolds. In the case of a homogeneous space, the Ricci tensor is constant on M, and each solution g of (1.1) is a curve on the set of invariant metrics, provided the initial condition $g(0)$ is invariant. Consequently, the equation (1.1) transforms into an ordinary differential equation known as the homogeneous Ricci flow. While this makes it somewhat more manageable, it remains far from straightforward. Exploring the Ricci flow on homogeneous spaces involves tools from the theory of dynamical systems. This approach has been adopted by various works covering different classes of homogeneous spaces, including generalized Wallach spaces [1, 23], Stiefel manifolds [23], and complex flag manifolds [13, 14, 23].

Notably, in [13], the authors employed the Poincaré compactification method [7] for the first time to study the global behavior of the homogeneous Ricci flow on $\frac{\mathrm{SO}(2n+1)}{\mathrm{U}(m)\times\mathrm{SO}(2k+1)}$ and $\frac{\mathrm{Sp}(n)}{\mathrm{U}(m)\times\mathrm{Sp}(k)}$. This tool has proven to be very useful, and we will utilize it Section 4.2 for the analysis of the global dynamics. Our approach for the study of the homogenous Ricci flow consists of a local study and a global one. Concerning the local study, we calculate the invariant algebraic surfaces of degree 2 and since said system is very degenerate we use the Blow-up method [9] to understand the local dynamics of the system around $z-$axis, which consists of changing an equilibrium point, whose Jacobian matrix has eigenvalues with zero real part, for a sphere $\mathbb{S}^2 \subset \mathbb{R}^3$, leaving the dynamics far from this point without changes. For the analysis of global dynamics, as previously mentioned, we employ the classical Poincaré compactification method. This method involves identifying $\mathbb{R}^3$ with the interior of the unit sphere $\mathbb{S}^2$ and $\mathbb{S}^2$ with the infinity of $\mathbb{R}^3$. Subsequently, the polynomial differential system defined in $\mathbb{R}^3$ is analytically extended to the entire sphere. Consequently, we can examine the dynamics of polynomial differential systems in the vicinity of infinity.

The paper is organized in the following form. In Section 2, we present some basic results on compact homogeneous spaces, the non-compact Lie algebra $\mathfrak{g}_2$ and real flag manifolds of $\mathfrak{g}_2$. In particular, we present a description of the isotropy representation for each flag manifold. In Section 3, we describe the g.o. metrics and the equigeodesics on flag manifolds of $\mathfrak{g}_2$. Finally, in Section 4, we perform an analysis on the homogeneous Ricci flow for the flag manifold whose isotropy representation has no equivalent irreducible subrepresentations.

2. Preliminaries

2.1. **Compact homogeneous spaces.** Let G be a compact Lie group, H a closed subgroup of G, and consider the homogeneous space $M = G/H$. There is a natural smooth transitive action of G on G/H given by

$$\phi : G \times G/H \longrightarrow G/H;\ \phi(a, bH) = abH.$$

For each $a \in G$, we can define the map

$$\phi_a := \phi(a, \cdot) : G/H \ni bH \mapsto abH \in G/H.$$

A Riemannian metric g on G/H is called G-invariant (or G-homogeneous) if

$$\{\phi_a : G/H \to G/H \mid a \in G\} \subseteq \mathrm{Iso}(G/H, g),$$

where $\mathrm{Iso}(G/H, g)$ denotes the group of all bijective isometries from G/H to itself.

Let $\mathfrak{g}$ and $\mathfrak{h}$ be the Lie algebras associated with G and H respectively. Consider the adjoint representation $\mathrm{Ad} : G \to \mathrm{GL}(\mathfrak{g})$ of G. Since G is compact, there exists a unique (up to re-scaling) $\mathrm{Ad}(G)$-invariant inner product $(\cdot,\cdot)$ on G. This means that

$$(\mathrm{Ad}(a)X, \mathrm{Ad}(a)Y) = (X, Y),\ a \in G,\ X, Y \in \mathfrak{g}.$$

By fixing this inner product, we obtain a reductive orthogonal decomposition of the Lie algebra $\mathfrak{g}$ as follows: if $\mathfrak{m}$ is the orthogonal complement of $\mathfrak{h}$ in $\mathfrak{g}$ with respect to $(\cdot,\cdot)$, then

$$\mathfrak{g} = \mathfrak{h} \oplus \mathfrak{m},\ \text{and}\ \mathrm{Ad}(h)\mathfrak{m} = \mathfrak{m},\ \forall h \in H.$$

This allows us to define the representation

$$\mathrm{Ad}^H\big|_{\mathfrak{m}} : H \to \mathrm{GL}(\mathfrak{m});\ \mathrm{Ad}^H\big|_{\mathfrak{m}}(h) := \mathrm{Ad}(h)\big|_{\mathfrak{m}},\ \forall h \in H, \tag{2.1}$$

which is equivalent to the isotropy representation of G/H at the left coset eH of the identity element $e \in G$. By compactness of H and the fact that $(\cdot,\cdot)$ is $\mathrm{Ad}(G)$-invariant, we have that this representation is completely reducible into pairwise $(\cdot,\cdot)$-orthogonal irreducible H-submodules, that is,

$$\mathfrak{m} = \mathfrak{m}_1 \oplus \cdots \oplus \mathfrak{m}_s, \tag{2.2}$$

where $\mathrm{Ad}(h)\mathfrak{m}_j = \mathfrak{m}_j,\ \forall h \in H$, and the representation

$$\mathrm{Ad}^H\big|_{\mathfrak{m}_j} : H \to \mathrm{GL}(\mathfrak{m}_j);\ \mathrm{Ad}^H\big|_{\mathfrak{m}_j}(h) := \mathrm{Ad}(h)\big|_{\mathfrak{m}_j}$$

is irreducible for each $j \in \{1, ..., s\}$. The H-submodules $\mathfrak{m}_1, ..., \mathfrak{m}_s$ are called the isotropy summands of the representation (2.1). Two isotropy summands $\mathfrak{m}_i$ and $\mathfrak{m}_j$ are equivalent if the representations $\mathrm{Ad}^H\big|_{\mathfrak{m}_i}$ and $\mathrm{Ad}^H\big|_{\mathfrak{m}_j}$ are equivalent. An inner product $\langle\cdot,\cdot\rangle : \mathfrak{m} \times \mathfrak{m} \to \mathbb{R}$ is called $\mathrm{Ad}(H)$-invariant if it satisfies the equation

$$\langle \mathrm{Ad}(h)X, \mathrm{Ad}(h)Y \rangle = \langle X, Y \rangle,\ h \in H,\ X, Y \in \mathfrak{m}.$$

There exists a bijection between the set of all Riemannian G-invariant metrics on G/H and the set of all $\mathrm{Ad}(H)$-invariant inner products on $\mathfrak{m}$. Since $(\cdot,\cdot)$ is $\mathrm{Ad}(G)$-invariant, then $(\cdot,\cdot)\big|_{\mathfrak{m}\times\mathfrak{m}} : \mathfrak{m} \times \mathfrak{m} \to \mathbb{R}$ is $\mathrm{Ad}(H)$-invariant. Consequently, for any $\mathrm{Ad}(H)$-invariant inner product $\langle\cdot,\cdot\rangle$, there exists a linear operator $A : \mathfrak{m} \to \mathfrak{m}$ such that

$$\langle X, Y \rangle = (AX, Y),\ \forall X, Y \in \mathfrak{m}.$$

The operator A is referred to as the metric operator associated with $\langle\cdot,\cdot\rangle$. As both $\langle\cdot,\cdot\rangle$ and $(\cdot,\cdot)\big|_{\mathfrak{m}\times\mathfrak{m}}$ are $\mathrm{Ad}(H)$-invariant inner products, the operator A is positive definite, self-adjoint (with respect to $(\cdot,\cdot)\big|_{\mathfrak{m}\times\mathfrak{m}}$), and commutes with $\mathrm{Ad}(h)\big|_{\mathfrak{m}}$, for all $h \in H$. Moreover, any linear operator A that satisfies these properties corresponds to the metric operator associated with some $\mathrm{Ad}(H)$-invariant inner product on $\mathfrak{m}$.

Following the notations introduced in [11], let $\{T_i^j : i,j \in \{1,...,s\}\}$ be a family of linear maps $T_i^j : \mathfrak{m}_i \to \mathfrak{m}_j$ that satisfies the following properties:

i) $T_i^i = \mathrm{I}_{\mathfrak{m}_i},\ i = 1,...,s.$

ii) $T_i^j = 0$ whenever $\mathfrak{m}_i$ is not equivalent to $\mathfrak{m}_j$.

iii) If $\mathfrak{m}_i$ is equivalent to $\mathfrak{m}_j$, then $T_i^j : \mathfrak{m}_i \to \mathfrak{m}_j$ is an equivariant map such that

$$(T_i^j(X), T_i^j(Y)) = (X, Y).$$

iv) If $\mathfrak{m}_i$ is equivalent to $\mathfrak{m}_j$, then $(T_i^j)^{-1} = T_j^i$.

There exist $(\cdot,\cdot)\big|_{\mathfrak{m}\times\mathfrak{m}}$-orthonormal sets $\mathcal{B}_1, ..., \mathcal{B}_s$ such that for each $j \in \{1,...,s\}$, $\mathcal{B}_j$ is a basis of $\mathfrak{m}_j$, and $T_i^j(\mathcal{B}_i) = \mathcal{B}_j$ whenever $\mathfrak{m}_i$ is equivalent to $\mathfrak{m}_j$. The set $\mathcal{B} = \mathcal{B}_1 \cup \cdots \cup \mathcal{B}_s$ is then a basis of $\mathfrak{m}$. Any metric operator A associated with a G-invariant metric on G/H can be represented in such a basis by a matrix of the form

$$[A]_{\mathcal{B}} = \begin{pmatrix} \mu_1 \mathrm{I}_{d_1} & A_{21}^T & A_{31}^T & \cdots & A_{s1}^T \\ A_{21} & \mu_2 \mathrm{I}_{d_2} & A_{32}^T & \cdots & A_{s2}^T \\ A_{31} & A_{32} & \mu_3 \mathrm{I}_{d_3} & \cdots & A_{s3} \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ A_{s1} & A_{s2} & A_{s3} & \cdots & \mu_s \mathrm{I}_{d_s} \end{pmatrix}, \tag{2.3}$$

where $\mu_1, ..., \mu_s > 0$, $d_i = \dim \mathfrak{m}_i$, $i = 1, ..., s$, and A_{ij} defines an equivariant map from $\mathfrak{m}_j$ to $\mathfrak{m}_i$ for $1 \leqslant j < i \leqslant s$ (in particular, $A_{ij} = 0$ whenever $\mathfrak{m}_i$ is not equivalent to $\mathfrak{m}_j$). Conversely, the formula (2.3) defines a metric operator corresponding with some G -invariant metric provided that the matrix is positive definite.

2.2. **The non-compact Lie algebra $\mathfrak{g}_2$.** In this section, we present a construction of the split real form of the Lie algebra $\mathfrak{g}_2$. This construction can also be found in [4] or [19, Section 8.4.1].

Let $\mathfrak{sl}(3)$ represent the Lie algebra of real, traceless 3×3 matrices, and $\mathbb{R}^3$ denote the 3-dimensional Euclidean space. The canonical basis of $\mathbb{R}^3$ is given by $\{e_1, e_2, e_3\}$ and its corresponding dual basis is denoted as $\{\epsilon_1, \epsilon_2, \epsilon_3\}$. We define $\bigwedge^3(\mathbb{R}^3)$ as the vector space consisting of all 3-covectors in $\mathbb{R}^3$. This vector space is isomorphic to $\mathbb{R}$ via the mapping Ψ defined as follows:

$$\Psi : c \in \mathbb{R} \mapsto c\nu \in \bigwedge^3(\mathbb{R}^3),\ \text{ where } \nu := e_1 \wedge e_2 \wedge e_3.$$

Let us introduce the linear isomorphisms

$$T : \bigwedge^2(\mathbb{R}^3) \longrightarrow (\mathbb{R}^3)^*,\ T(u \wedge v)(w) := \Psi^{-1}(u \wedge v \wedge w)$$

and

$$S : \bigwedge^2(\mathbb{R}^3)^* \longrightarrow \mathbb{R}^3,$$

defined implicitly by the formula

$$\alpha \wedge \beta \wedge \gamma = \gamma(S(\alpha \wedge \beta))\nu^*,\ \text{ where } \nu^* := \epsilon_1 \wedge \epsilon_2 \wedge \epsilon_3,\ \text{ and } \alpha, \beta, \gamma \in (\mathbb{R}^3)^*.$$

Consider the vector space

$$\mathfrak{g} := \mathfrak{sl}(3) \oplus \mathbb{R}^3 \oplus \left(\mathbb{R}^3\right)^*$$

endowed with the Lie bracket $[\cdot,\cdot]$ defined by the following relations:

i) The Lie bracket of two matrices $X, Y \in \mathfrak{sl}(3)$ is given by

$$[X, Y] := XY - YX \in \mathfrak{sl}(3).$$

ii) If $X \in \mathfrak{sl}(3)$ and $v \in \mathbb{R}^3$ then
$$[X, v] := Xv \in \mathbb{R}^3.$$

iii) If $X \in \mathfrak{sl}(3)$ and $\alpha \in \left(\mathbb{R}^3\right)^*$ then
$$[X, \alpha] := -\alpha \circ X \in \left(\mathbb{R}^3\right)^*.$$

iv) If $u, v \in \mathbb{R}^3$ then
$$[u, v] := -\frac{4}{3} T(u \wedge v) \in \left(\mathbb{R}^3\right)^*.$$

v)] If $\alpha, \beta \in \left(\mathbb{R}^3\right)^*$ then
$$[\alpha, \beta] := \frac{4}{3} S(\alpha \wedge \beta) \in \mathbb{R}^3.$$

vi) If $v = \sum_{i=1}^{3} v^i e_i \in \mathbb{R}^3$ and $\alpha = \sum_{j=1}^{3} \alpha^j \epsilon_j \in \left(\mathbb{R}^3\right)^*$ then
$$[v, \alpha] := (v^i \alpha^j)_{3\times 3} - \frac{1}{3}\alpha(v) \mathrm{I}_3 \in \mathfrak{sl}(3),$$
where I_3 is the identity matrix of order 3.

vii) For generic elements $X = X_1 + u_1 + \alpha_1,\ Y = X_2 + u_2 + \alpha_2 \in \mathfrak{g},\ X_j \in \mathfrak{sl}(3),\ u_j \in \mathbb{R}^3,\ \alpha_j \in \left(\mathbb{R}^3\right)^*,\ j = 1, 2$; define the Lie bracket $[X, Y]$ by extending the rules $i) - vi)$ so that $[\cdot, \cdot]$ is bilinear and skew-symmetric.

The pair $(\mathfrak{g}, [\cdot, \cdot])$ is a non-compact Lie algebra which is isomorphic to the split real form of the complex simple Lie exceptional Lie algebra $\mathfrak{g}_2$. For simplicity, we shall also denote this Lie algebra as $\mathfrak{g}_2$.

The set $\mathfrak{h}$, consisting of all diagonal and traceless 3×3 matrices, is a Cartan subalgebra $\mathfrak{g}_2$. The corresponding root system is given by
$$\Pi = \{\lambda_i - \lambda_j : 1 \leqslant i \neq j \leqslant 3\} \cup \{\pm\lambda_i : 1 \leqslant i \leqslant 3\},$$
where each λ_i is defined as
$$\lambda_i : \begin{array}{ccc} \mathfrak{h} & \longrightarrow & \mathbb{R} \\ \operatorname{diag}(a_1, a_2, a_3) & \longmapsto & a_i \end{array}, \quad i = 1, 2, 3.$$
A set of positive roots can be chosen as
$$\Pi^+ = \{\lambda_i - \lambda_j : 1 \leqslant i < j \leqslant 3\} \cup \{\lambda_1, \lambda_2, -\lambda_3\}.$$
The corresponding set of simple roots is $\Sigma = \{\alpha_1 := \lambda_1 - \lambda_2, \alpha_2 := \lambda_2\}$. Given $i, j \in \{1, 2, 3\}$, let E_{ij} be the 3×3 square matrix whose (i, j)-entry is equal to 1, and all the other entries are zero. Then, the root spaces associated with Π are
$$\begin{aligned}
(\mathfrak{g}_2)_{\lambda_i - \lambda_j} &= \operatorname{span}\{E_{ij}\},\ 1 \leqslant i \neq j \leqslant 3 \\
(\mathfrak{g}_2)_{\lambda_i} &= \operatorname{span}\{e_i\},\ \text{and} \\
(\mathfrak{g}_2)_{-\lambda_i} &= \operatorname{span}\{\epsilon_i\},\ i = 1, 2, 3.
\end{aligned}$$

2.3. **Real flag manifolds of $\mathfrak{g}_2$.** A real flag manifold of the Lie algebra $\mathfrak{g}_2$ is defined as the homogeneous space $\mathbb{F} = G_2/P$, where G_2 is a connected non-compact simple Lie group with Lie algebra $\mathfrak{g}_2$, and P is a parabolic subgroup of G_2. The non-trivial parabolic subgroups of G_2 correspond bijectively to proper subsets Θ of the set Σ of simple roots associated with the Cartan subalgebra $\mathfrak{h}$. This correspondence is given as follows: for a given $\Theta \subsetneq \Sigma$, we denote by $\langle\Theta\rangle$ the set of all roots that are linear combinations with integer coefficients of elements in Θ. Let $\langle\Theta\rangle^+ = \langle\Theta\rangle \cap \Pi^+$ and $\langle\Theta\rangle^- = \langle\Theta\rangle \cap (-\Pi^+)$. We then define
$$\mathfrak{p}_\Theta := \mathfrak{h} \oplus \bigoplus_{\alpha \in \Pi} (\mathfrak{g}_2)_\alpha \oplus \bigoplus_{\alpha \in \langle\Theta\rangle^-} (\mathfrak{g}_2)_\alpha,$$

and

$$P_\Theta = \{a \in G_2 : \mathrm{Ad}(a)\mathfrak{p}_\Theta = \mathfrak{p}_\Theta\}.$$

Then P_Θ is a parabolic subgroup of G_2 and every non-trivial parabolic subgroup of G_2 is isomorphic to P_Θ for some $\Theta \subsetneq \Sigma$. Consequently, there exist precisely three real flag manifolds of $\mathfrak{g}_2$, corresponding to $\Theta = \varnothing$, $\Theta = \{\alpha_1\}$, and $\Theta = \{\alpha_2\}$.

For each $\Theta \subsetneq \Sigma$, the maximal compact subgroup

$$K \cong \mathrm{SO}(4) \cong (\mathrm{SU}(2) \times \mathrm{SU}(2))/\{\pm(\mathrm{I}, \mathrm{I})\}$$

of G_2 acts transitively on the flag $\mathbb{F}_\Theta = G_2/P_\Theta$. The isotropy subgroup of eP_Θ is $K_\Theta := K \cap P_\Theta$. Therefore, a flag manifold $\mathbb{F}_\Theta = G_2/P_\Theta$ can also be represented as K/K_Θ. The Lie algebra $\mathfrak{k}$ of K is isomorphic to $\mathfrak{so}(4) = \mathfrak{so}(3) \oplus \mathfrak{so}(3)$ (see, for instance, [20]). It is spanned by the vectors

$$X_1 := E_{21} - E_{12},\ X_2 := E_{31} - E_{13},\ X_3 := E_{32} - E_{23},\ Y_i := e_i - \epsilon_i,\ i = 1, 2, 3.$$

which satisfy the following relations:

$$\begin{array}{lll}
[X_1, X_2] = X_3, & [X_2, X_3] = X_1, & [X_3, Y_3] = -Y_2, \\
[X_1, X_3] = -X_2, & [X_2, Y_1] = Y_3, & [Y_1, Y_2] = X_1 + \frac{4}{3}Y_3, \\
[X_1, Y_1] = Y_2, & [X_2, Y_3] = -Y_1, & [Y_1, Y_3] = X_2 - \frac{4}{3}Y_2, \\
[X_1, Y_2] = -Y_1, & [X_3, Y_2] = Y_3, & [Y_2, Y_3] = X_3 + \frac{4}{3}Y_1.
\end{array} \tag{2.4}$$

The Killing form B of $\mathfrak{k} \cong \mathfrak{so}(3) \oplus \mathfrak{so}(3)$ is negative definite. Consequently, $(\cdot, \cdot) := -B$ defines an $\mathrm{Ad}(K)$-invariant inner product on $\mathfrak{k}$. Due to the compactness of K, we have that for each $\Theta \subsetneq \Sigma$, the flag manifold $\mathbb{F}_\Theta = K/K_\Theta$ is reductive. This implies the existence of a decomposition $\mathfrak{k} = \mathfrak{k}_\Theta \oplus \mathfrak{m}_\Theta$, where $\mathfrak{k}_\Theta$ is the Lie algebra of K_Θ and $\mathrm{Ad}(k)\mathfrak{m}_\Theta = \mathfrak{m}_\Theta$ for all $k \in K_\Theta$. The isotropy representation of $\mathbb{F}_\Theta$ is equivalent to the representation

$$\mathrm{Ad}^{K_\Theta}\big|_{\mathfrak{m}_\Theta} : K_\Theta \longrightarrow \mathrm{GL}(\mathfrak{m}_\Theta), \tag{2.5}$$

which is completely reducible, that is, $\mathfrak{m}_\Theta$ can be decomposed as a direct sum of irreducible (possibly equivalent) subrepresentations. The following proposition proved by Patrão and San Martin in [17] gives the description of these subrepresentations and their equivalences.

Proposition 2.1. *Let $\mathbb{F}_\Theta$ be a real flag manifold of $\mathfrak{g}_2$. Then the following statements hold:*

a) If $\Theta = \varnothing$, then $\mathfrak{k}_\varnothing = \{0\}$ and $\mathfrak{m}_\varnothing$ is the direct sum of six one-dimensional $K_\varnothing$-submodules given by

$$\mathrm{span}\{X_i\}, \mathrm{span}\{Y_i\},\ i = 1, 2, 3.$$

All these submodules are irreducible and the equivalence classes among them are

$$\begin{gathered}
\{\mathrm{span}\{X_1\}, \mathrm{span}\{Y_3\}\}, \\
\{\mathrm{span}\{X_2\}, \mathrm{span}\{Y_2\}\}, \\
\{\mathrm{span}\{X_3\}, \mathrm{span}\{Y_1\}\}.
\end{gathered}$$

b) If $\Theta = \{\alpha_1\}$, then $\mathfrak{k}_{\{\alpha_1\}} = \mathrm{span}\{X_1\}$ and $\mathfrak{m}_{\{\alpha_1\}}$ decomposes into the irreducible $K_{\{\alpha_1\}}$-submodules

$$\begin{gathered}
\mathrm{span}\{Y_3\}, \\
\mathrm{span}\{X_2, X_3\}, \\
\mathrm{span}\{Y_2, Y_1\},
\end{gathered}$$

where the two-dimensional submodules are equivalent and the map

$$\begin{array}{rccc}
T: & \mathrm{span}\{X_2, X_3\} & \longrightarrow & \mathrm{span}\{Y_2, Y_1\} \\
& X_2 & \longmapsto & Y_2 \\
& X_3 & \longmapsto & -Y_1
\end{array}$$

is an equivariant isomorphism.

c) *If* $\Theta = \{\alpha_2\}$, *then* $\mathfrak{k}_{\{\alpha_2\}} = \mathrm{span}\{Y_2\}$ *and* $\mathfrak{m}_{\{\alpha_2\}}$ *decomposes into the irreducible inequivalent* $K_{\{\alpha_2\}}$*-submodules*

$$\begin{aligned}&\mathrm{span}\{X_2\},\\ &\mathrm{span}\left\{(\sqrt{13}-2)X_3+3Y_1, (\sqrt{13}-2)X_1+3Y_3\right\},\\ &\mathrm{span}\left\{(\sqrt{13}+2)X_3-3Y_1, (\sqrt{13}+2)X_1-3Y_3\right\}.\end{aligned}$$

Remark 2.2. The irreducible submodules provided in Proposition 2.1 are not unique and are not pairwise orthogonal with respect to the inner product $(\cdot,\cdot)$. This lack of orthogonality complicates the characterization of invariant metrics. Therefore, we will consider different irreducible submodules of the isotropy representation (2.5) beyond those presented in Proposition 2.1 (see Proposition 2.6).

Remark 2.3. The representation

$$\mathrm{Ad}^{K_{\{\alpha_1\}}}\big|_{\mathrm{span}\{X_2,X_3\}} : K_{\{\alpha_1\}} \to \mathrm{GL}(\mathrm{span}\{X_2,X_3\})$$

is orthogonal. This means that the vector space $\mathrm{End}(\mathrm{span}\{X_2,X_3\})$ of all equivariant endomorphism of $\mathrm{span}\{X_2,X_3\}$ is isomorphic to $\mathbb{R}$. As a consequence, any equivariant isomorphism from $\mathrm{span}\{X_2,X_3\}$ to $\mathrm{span}\{Y_2,Y_1\}$ is a scalar multiple of T. Further details can be found in [17] for reference.

Proposition 2.4. *The vectors*

$$\begin{aligned}&W_i := \frac{1}{2}X_i,\ i=1,2,3,\\ &Z_1 := \frac{3}{2\sqrt{13}}Y_1 - \frac{1}{\sqrt{13}}X_3,\quad Z_2 := \frac{3}{2\sqrt{13}}Y_2 + \frac{1}{\sqrt{13}}X_2,\quad Z_3 := \frac{3}{2\sqrt{13}}Y_3 - \frac{1}{\sqrt{13}}X_1,\end{aligned}$$

form an $(\cdot,\cdot)-$*orthonormal basis for* $\mathfrak{k}$, *and satisfy the following bracket relations:*

$$\begin{array}{lll}[W_1,W_2]=\frac{1}{2}W_3, & [W_2,W_3]=\frac{1}{2}W_1, & [W_3,Z_3]=-\frac{1}{2}Z_2,\\ [W_1,W_3]=-\frac{1}{2}W_2, & [W_2,Z_1]=\frac{1}{2}Z_3, & [Z_1,Z_2]=\frac{1}{2}W_1,\\ [W_1,Z_1]=\frac{1}{2}Z_2, & [W_2,Z_3]=-\frac{1}{2}Z_1, & [Z_1,Z_3]=\frac{1}{2}W_2,\\ [W_1,Z_2]=-\frac{1}{2}Z_1, & [W_3,Z_2]=\frac{1}{2}Z_3, & [Z_2,Z_3]=\frac{1}{2}W_3.\end{array} \tag{2.6}$$

Proof. A straightforward calculation reveals that the non-zero inner products among the vectors X_i and Y_i (where $i=1,2,3$) are

$$\begin{aligned}&(X_i,X_i)=4,\quad (Y_i,Y_i)=\frac{68}{9},\quad i=1,2,3,\\ &(X_1,Y_3)=-(X_2,Y_2)=(X_3,Y_1)=\frac{8}{3}.\end{aligned}$$

The $(\cdot,\cdot)$-orthonormal basis $\{W_1,W_2,W_3,Z_1,Z_2,Z_3\}$ is obtained by applying the Gram-Schmidt algorithm to orthonormalize the basis $\{X_1,X_2,X_3,Y_1,Y_2,Y_3\}$ with respect to the inner product $(\cdot,\cdot)$. The relations (4.3) follow from a lengthy calculation using (2.4). □

Lemma 2.5. *Let* $\rho : G \to GL(V)$ *and* $\tau : G \to GL(W)$ *be equivalent representations of a Lie group* G. *Let* $T : V \to W$ *be an equivariant isomorphism. Define the graph of* T *as the set*

$$\mathrm{graph}(T) := \{X + T(X) \in V \oplus W : X \in V\} \subseteq V \oplus W.$$

Then $\mathrm{graph}(T)$ *is a vector space, and the map*

$$\gamma : G \to \mathrm{GL}(\mathrm{graph}(T)),\ \gamma(a)(X+T(X)) := \rho(a)X + \tau(a)T(X),\ a \in G,\ X \in V,$$

is a representation of G *that is equivalent to* ρ.

Proof. Since T is linear, graph(T) is a vector subspace of $V \oplus W$. The fact that γ defines a representation of G comes from the fact that ρ and τ are representations of G. Finally, the map $\tilde{T} : V \to \text{graph}(T)$, defined as $X \mapsto X + T(X)$, is trivially a linear isomorphism, and for every $a \in G$ and $X \in V$, we have

$$\begin{aligned}\tilde{T}\rho(a)(X) &= \rho(a)X + T(\rho(a)X)\\ &= \rho(a)X + \tau(a)T(X), \qquad \text{since } T \text{ is equivariant,}\\ &= \gamma(a)\left(X + T(X)\right)\\ &= \gamma(a)\tilde{T}(X).\end{aligned}$$

That is, $\tilde{T}\rho(a) = \gamma(a)\tilde{T}$, for all $a \in G$. Hence $\tilde{T}$ is an equivariant isomorphism and, consequently, γ and ρ are equivalent. ☐

Proposition 2.6. *Let $\mathbb{F}_\Theta$ be a real flag manifold of $\mathfrak{g}_2$. Then the following statements hold:*

a) *If $\Theta = \varnothing$, the representation (2.5) can be decomposed into six one-dimensional submodules given by*

$$\text{span}\{W_i\}, \text{span}\{Z_i\}, \ i = 1, 2, 3.$$

All these submodules are irreducible and the equivalence classes among them are

$$\begin{aligned}&\{\text{span}\{W_1\}, \text{span}\{Z_3\}\},\\ &\{\text{span}\{W_2\}, \text{span}\{Z_2\}\},\\ &\{\text{span}\{W_3\}, \text{span}\{Z_1\}\}.\end{aligned}$$

b) *If $\Theta = \{\alpha_1\}$, the representation (2.5) can be decomposed into the irreducible subrepresentations*

$$\begin{aligned}&\text{span}\{Z_3\},\\ &\text{span}\{W_2, W_3\},\\ &\text{span}\{Z_2, -Z_1\},\end{aligned}$$

where the two-dimensional submodules are equivalent and the map

$$\begin{array}{cccc}\tilde{T}: & \text{span}\{W_2, W_3\} & \longrightarrow & \text{span}\{Z_2, -Z_1\}\\ & W_2 & \longmapsto & Z_2\\ & W_3 & \longmapsto & -Z_1\end{array}$$

is an equivariant isomorphism.

c) *If $\Theta = \{\alpha_2\}$, the representation (2.5) can be decomposed into the irreducible inequivalent subrepresentations*

$$\begin{aligned}&\text{span}\left\{\frac{\sqrt{13}W_2 + 2Z_2}{\sqrt{17}}\right\},\\ &\text{span}\left\{\frac{W_1 + Z_3}{\sqrt{2}}, \frac{W_3 + Z_1}{\sqrt{2}}\right\},\\ &\text{span}\left\{\frac{W_1 - Z_3}{\sqrt{2}}, \frac{W_3 - Z_1}{\sqrt{2}}\right\}.\end{aligned}$$

Proof. For the proof of a), let us assume that $\Theta = \varnothing$. Then, by Proposition 2.1, the one-dimensional subspaces span$\{X_i\}$, span$\{Y_i\}$, $i = 1, 2, 3$ are $K_\varnothing$-invariant irreducible subspaces of $\mathfrak{m}_\varnothing$. Additionally, span$\{X_1\}$ is equivalent to span$\{Y_3\}$, span$\{X_2\}$ is equivalent to span$\{Y_2\}$ and span$\{X_3\}$ is equivalent to span$\{Y_1\}$. Since these all submodules are one-dimensional, the linear maps

$$\begin{array}{cccc}T_1: & \text{span}\{X_1\} & \longrightarrow & \text{span}\{Y_3\}\\ & X_1 & \longmapsto & -\frac{3}{2}Y_3\end{array},$$

$$\begin{array}{cccc}T_2: & \text{span}\{X_2\} & \longrightarrow & \text{span}\{Y_2\}\\ & X_2 & \longmapsto & \frac{3}{2}Y_2\end{array},$$

$$\begin{array}{rccc} T_3: & \mathrm{span}\{X_3\} & \longrightarrow & \mathrm{span}\{Y_1\} \\ & X_3 & \longmapsto & -\frac{3}{2}Y_1 \end{array}$$

are equivariant isomorphisms. By Lemma 2.5, $\{\lambda X_1 + T_1(\lambda X_1) : \lambda \in \mathbb{R}\}$ is a $K_\varnothing$-invariant subspace of $\mathfrak{m}_\varnothing$ that is equivalent to $\mathrm{span}\{X_1\}$, $\{\lambda X_2 + T_1(\lambda X_2) : \lambda \in \mathbb{R}\}$ is a $K_\varnothing$-invariant subspace of $\mathfrak{m}_\varnothing$ that is equivalent to $\mathrm{span}\{X_2\}$, and $\{\lambda X_3 + T_1(\lambda X_3) : \lambda \in \mathbb{R}\}$ is a $K_\varnothing$-invariant subspace of $\mathfrak{m}_\varnothing$ that is equivalent to $\mathrm{span}\{X_3\}$. The statement $a)$ now follows from the fact that

$$\begin{aligned} \{\lambda X_1 + T_1(\lambda X_1) : \lambda \in \mathbb{R}\} &= \left\{\lambda X_1 - \lambda\frac{3}{2}Y_3 : \lambda \in \mathbb{R}\right\} \\ &= \left\{\lambda\left(X_1 - \frac{3}{2}Y_3\right) : \lambda \in \mathbb{R}\right\} \\ &= \mathrm{span}\left\{X_1 - \frac{3}{2}Y_3\right\} \\ &= \mathrm{span}\left\{-\sqrt{13}Z_3\right\} \\ &= \mathrm{span}\{Z_3\}, \end{aligned}$$

$$\begin{aligned} \{\lambda X_2 + T_2(\lambda X_2) : \lambda \in \mathbb{R}\} &= \left\{\lambda X_2 + \lambda\frac{3}{2}Y_2 : \lambda \in \mathbb{R}\right\} \\ &= \left\{\lambda\left(X_2 + \frac{3}{2}Y_2\right) : \lambda \in \mathbb{R}\right\} \\ &= \mathrm{span}\left\{X_2 + \frac{3}{2}Y_2\right\} \\ &= \mathrm{span}\left\{\sqrt{13}Z_2\right\} \\ &= \mathrm{span}\{Z_2\}, \end{aligned}$$

$$\begin{aligned} \{\lambda X_3 + T_3(\lambda X_3) : \lambda \in \mathbb{R}\} &= \left\{\lambda X_3 - \lambda\frac{3}{2}Y_1 : \lambda \in \mathbb{R}\right\} \\ &= \left\{\lambda\left(X_3 - \frac{3}{2}Y_1\right) : \lambda \in \mathbb{R}\right\} \\ &= \mathrm{span}\left\{X_3 - \frac{3}{2}Y_1\right\} \\ &= \mathrm{span}\left\{-\sqrt{13}Z_1\right\} \\ &= \mathrm{span}\{Z_1\}, \end{aligned}$$

and $\mathrm{span}\{X_i\} = \mathrm{span}\left\{\frac{3}{2}X_i\right\} = \mathrm{span}\{W_i\}$, for $i = 1, 2, 3$.

Let us prove $b)$. Assume that $\Theta = \{\alpha_1\}$. By Proposition 2.1, the sets

$$\mathrm{span}\{Y_3\},\ \mathrm{span}\{X_2, X_3\},\ \mathrm{span}\{Y_2, Y_1\}$$

are $K_{\{\alpha_1\}}$-invariant irreducible subspaces of $\mathfrak{m}_{\{\alpha_1\}}$, and the linear map

$$\begin{array}{rccc} T: & \mathrm{span}\{X_2, X_3\} & \longrightarrow & \mathrm{span}\{Y_2, Y_1\} \\ & X_2 & \longmapsto & Y_2 \\ & X_3 & \longmapsto & -Y_1 \end{array}$$

is an equivariant isomorphism. Then, $\frac{3}{2}T$ is also an equivariant isomorphism. By Lemma 2.5, the set

$$\left\{\lambda_1 X_2 + \lambda_2 X_3 + \frac{3}{2}T\left(\lambda_1 X_2 + \lambda_2 X_3\right) : \lambda_1, \lambda_2 \in \mathbb{R}\right\}$$

is a $K_{\{\alpha_1\}}$-invariant irreducible subspace of $\mathfrak{m}_{\{\alpha_1\}}$ that is equivalent to $\operatorname{span}\{X_2, X_3\}$. But

$$\begin{aligned}
&\left\{\lambda_1 X_2 + \lambda_2 X_3 + \frac{3}{2}T\left(\lambda_1 X_2 + \lambda_2 X_3\right) : \lambda_1, \lambda_2 \in \mathbb{R}\right\} \\
= &\left\{\lambda_1 X_2 + \lambda_2 X_3 + \lambda_1 \frac{3}{2} Y_2 - \lambda_2 \frac{3}{2} Y_1 : \lambda_1, \lambda_2 \in \mathbb{R}\right\} \\
= &\left\{\lambda_1 \left(X_2 + \frac{3}{2} Y_2\right) + \lambda_2 \left(X_3 - \frac{3}{2} Y_1\right) : \lambda_1, \lambda_2 \in \mathbb{R}\right\} \\
= &\left\{\lambda_1 \left(\sqrt{13} Z_2\right) + \lambda_2 \left(-\sqrt{13} Z_1\right) : \lambda_1, \lambda_2 \in \mathbb{R}\right\} \\
= &\operatorname{span}\{Z_2, -Z_1\}.
\end{aligned}$$

Since $\operatorname{span}\{X_2, X_3\} = \operatorname{span}\left\{\frac{1}{2}X_2, \frac{1}{2}X_3\right\} = \operatorname{span}\{W_2, W_3\}$, then $\operatorname{span}\{W_2, W_3\}$ and $\operatorname{span}\{Z_2, -Z_1\}$ are $K_{\{\alpha_1\}}$-invariant irreducible equivalent subspaces of $\mathfrak{m}_{\{\alpha_1\}}$. Moreover, from the proof of Lemma 2.5, we have that the map

$$\operatorname{span}\{W_2, W_3\} \ni X \mapsto X + \frac{3}{2}T(X) \in \operatorname{span}\{Z_2, -Z_1\}$$

is an intertwining isomorphism. Therefore, the map

$$\operatorname{span}\{W_2, W_3\} \ni X \mapsto \frac{2}{\sqrt{13}}\left(X + \frac{3}{2}T(X)\right) \in \operatorname{span}\{Z_2, -Z_1\}$$

is also an interwining isomorphism. This last map is nothing but $\tilde{T}$ since

$$\frac{2}{\sqrt{13}}\left(W_2 + \frac{3}{2}T(W_2)\right) = Z_2 \text{ and } \frac{2}{\sqrt{13}}\left(W_3 + \frac{3}{2}T(W_3)\right) = -Z_1.$$

So far we have shown that the subspaces $\operatorname{span}\{W_2, W_3\}$ and $\operatorname{span}\{Z_2, -Z_1\}$ are $K_{\{\alpha_1\}}$-invariant irreducible and equivalent, and that $\tilde{T}$ is an intertwining isomorphism between them. To prove that $\operatorname{span}\{Z_3\}$ is $K_{\{\alpha_1\}}$-invariant, consider the representation

$$\operatorname{Ad}^{K_{\{\alpha_1\}}}\big|_{\mathfrak{k}_{\{\alpha_1\}} \oplus \operatorname{span}\{Y_3\}} : K_{\{\alpha_1\}} \to \operatorname{GL}\left(\mathfrak{k}_{\{\alpha_1\}} \oplus \operatorname{span}\{Y_3\}\right).$$

The inner product $(\cdot, \cdot)$ is $\operatorname{Ad}(K_{\{\alpha_1\}})$-invariant because it is $\operatorname{Ad}(K)$-invariant. Additionally, $(X_1, Z_1) = 0$, so the vector space $\operatorname{span}\{Z_3\}$ is the orthogonal complement of $\mathfrak{k}_{\{\alpha_1\}} = \operatorname{span}\{X_1\}$ in $\mathfrak{k}_{\{\alpha_1\}} \oplus \operatorname{span}\{Y_3\}$ with respect to $(\cdot, \cdot)$. Since $\mathfrak{k}_{\{\alpha_1\}}$ is the Lie algebra of $K_{\{\alpha_1\}}$, then $\mathfrak{k}_{\{\alpha_1\}}$ is $K_{\{\alpha_1\}}$-invariant, so is its $(\cdot, \cdot)$-orthogonal complement $\operatorname{span}\{Z_3\}$. This completes the proof of *b*).

To prove *c*), let us consider the case where $\Theta = \{\alpha_2\}$. Arguing as before, observe that $(\sqrt{13}W_2 + 2Z_2, Y_2) = 0$, so $\operatorname{span}\{\sqrt{13}W_2 + 2Z_2\}$ is the $(\cdot, \cdot)$-orthogonal complement of $\mathfrak{k}_{\{\alpha_2\}}$ in $\mathfrak{k}_{\{\alpha_2\}} \oplus \operatorname{span}\{\sqrt{13}W_2 + 2Z_2\}$. Since $\mathfrak{k}_{\{\alpha_2\}}$ is $K_{\{\alpha_2\}}$, and $(\cdot, \cdot)$ is $\operatorname{Ad}(K_{\{\alpha_2\}})$-invariant, then $\operatorname{span}\{\sqrt{13}W_2 + 2Z_2\}$ is $K_{\{\alpha_2\}}$-invariant. On the other hand, observe that

$$\frac{W_3 + Z_1}{\sqrt{2}} = \frac{1}{2\sqrt{26}}\left((\sqrt{13} - 2)X_3 + 3Y_1\right), \quad \frac{W_1 + Z_3}{\sqrt{2}} = \frac{1}{2\sqrt{26}}\left((\sqrt{13} - 2)X_1 + 3Y_3\right)$$

$$\frac{W_3 - Z_1}{\sqrt{2}} = \frac{1}{2\sqrt{26}}\left((\sqrt{13} + 2)X_3 - 3Y_1\right), \quad \frac{W_1 + Z_3}{\sqrt{2}} = \frac{1}{2\sqrt{26}}\left((\sqrt{13} + 2)X_1 - 3Y_3\right).$$

Hence

$$\operatorname{span}\left\{(\sqrt{13} - 2)X_3 + 3Y_1, (\sqrt{13} - 2)X_1 + 3Y_3\right\} = \operatorname{span}\left\{\frac{W_3 + Z_1}{\sqrt{2}}, \frac{W_1 + Z_3}{\sqrt{2}}\right\}$$

and

$$\operatorname{span}\left\{(\sqrt{13}+2)X_3 - 3Y_1, (\sqrt{13}+2)X_1 - 3Y_3\right\} = \operatorname{span}\left\{\frac{W_3 - Z_1}{\sqrt{2}}, \frac{W_1 - Z_3}{\sqrt{2}}\right\}.$$

The proof is complete. □

3. Homogeneous geodesics on flag manifolds

3.1. **K-invariant metrics.** In this section we present a useful description of the K-invariant metrics on the real flag manifolds of $\mathfrak{g}_2$. As in Section 2.3, we will fix the $\mathrm{Ad}(K)$-invariant inner product $(\cdot,\cdot)$ defined as the negative of the Killing form on $\mathfrak{k}$.

Theorem 3.1. *Let $\mathbb{F}_\Theta = K/K_\Theta$ be a flag of $\mathfrak{g}_2$.*

a) If $\Theta = \varnothing$, then for each metric operator $A : \mathfrak{m}_\varnothing \to \mathfrak{m}_\varnothing$ associated with an $\mathrm{Ad}(K_\varnothing)$-invariant inner product on $\mathfrak{m}_\varnothing$, there exist positive numbers $\mu_1, ..., \mu_6$, and real numbers $a_1 \in (-\sqrt{\mu_1\mu_2}, \sqrt{\mu_1\mu_2})$, $a_2 \in (-\sqrt{\mu_3\mu_4}, \sqrt{\mu_3\mu_4})$, and $a_3 \in (-\sqrt{\mu_5\mu_6}, \sqrt{\mu_5\mu_6})$ such that A is written the $(\cdot,\cdot)$-orthonormal basis $\mathcal{B}_\varnothing = \{W_1, Z_3, W_2, Z_2, W_3, Z_1\}$ as

$$[A]_{\mathcal{B}_\varnothing} = \begin{pmatrix} \mu_1 & a_1 & 0 & 0 & 0 & 0 \\ a_1 & \mu_2 & 0 & 0 & 0 & 0 \\ 0 & 0 & \mu_3 & a_2 & 0 & 0 \\ 0 & 0 & a_2 & \mu_4 & 0 & 0 \\ 0 & 0 & 0 & 0 & \mu_5 & a_3 \\ 0 & 0 & 0 & 0 & a_3 & \mu_6 \end{pmatrix}. \tag{3.1}$$

This representation covers all metric operators associated with $\mathrm{Ad}(K_\varnothing)$-invariant inner products.

b) If $\Theta = \{\alpha_1\}$, then for each metric operator $A : \mathfrak{m}_{\{\alpha_1\}} \to \mathfrak{m}_{\{\alpha_1\}}$ associated with an $\mathrm{Ad}(K_{\{\alpha_1\}})$-invariant inner product on $\mathfrak{m}_{\{\alpha_1\}}$, there exist positive numbers μ_1, μ_2, μ_3, and a real number $a \in (-\sqrt{\mu_2\mu_3}, \sqrt{\mu_2\mu_3})$ such that A is written in the basis $\mathcal{B}_{\{\alpha_1\}} = \{Z_3, W_2, W_3, Z_2, -Z_1\}$ as

$$[A]_{\mathcal{B}_{\{\alpha_1\}}} = \begin{pmatrix} \mu_1 & 0 & 0 & 0 & 0 \\ 0 & \mu_2 & 0 & a & 0 \\ 0 & 0 & \mu_2 & 0 & a \\ 0 & a & 0 & \mu_3 & 0 \\ 0 & 0 & a & 0 & \mu_3 \end{pmatrix}. \tag{3.2}$$

This representation covers all metric operators associated with $\mathrm{Ad}\left(K_{\{\alpha_1\}}\right)$-invariant inner products.

c) If $\Theta = \{\alpha_2\}$, then there exist positive real numbers μ_i, $i = 1,2,3$ such that the metric operator A associated with an $\mathrm{Ad}\left(K_{\{\alpha_2\}}\right)$-invariant inner product on $\mathfrak{m}_{\{\alpha_2\}}$ is written in the basis $\mathcal{B}_{\{\alpha_2\}} = \left\{\frac{\sqrt{13}W_2+2Z_2}{\sqrt{17}}, \frac{W_1+Z_3}{\sqrt{2}}, \frac{W_3+Z_1}{\sqrt{2}}, \frac{W_1-Z_3}{\sqrt{2}}, \frac{W_3-Z_1}{\sqrt{2}}\right\}$ as

$$[A]_{\mathcal{B}_{\{\alpha_2\}}} = \begin{pmatrix} \mu_1 & 0 & 0 & 0 & 0 \\ 0 & \mu_2 & 0 & 0 & 0 \\ 0 & 0 & \mu_2 & 0 & 0 \\ 0 & 0 & 0 & \mu_3 & 0 \\ 0 & 0 & 0 & 0 & \mu_3 \end{pmatrix}. \tag{3.3}$$

This representation covers all metric operators associated with $\mathrm{Ad}\left(K_{\{\alpha_2\}}\right)$-invariant inner products.

Proof. Assuming $\Theta = \varnothing$, Proposition 2.6 implies that

$$\mathfrak{m}_\varnothing = (\mathfrak{m}_\varnothing)_1 \oplus (\mathfrak{m}_\varnothing)_2 \oplus (\mathfrak{m}_\varnothing)_3 \oplus (\mathfrak{m}_\varnothing)_4 \oplus (\mathfrak{m}_\varnothing)_5 \oplus (\mathfrak{m}_\varnothing)_6,$$

where

$$\begin{aligned} &(\mathfrak{m}_{\varnothing})_1 = \operatorname{span}\{W_1\}, && (\mathfrak{m}_{\varnothing})_2 = \operatorname{span}\{Z_3\},\\ &(\mathfrak{m}_{\varnothing})_3 = \operatorname{span}\{W_2\}, && (\mathfrak{m}_{\varnothing})_4 = \operatorname{span}\{Z_2\},\\ &(\mathfrak{m}_{\varnothing})_5 = \operatorname{span}\{W_3\}, && (\mathfrak{m}_{\varnothing})_6 = \operatorname{span}\{Z_1\}, \end{aligned}$$

with $(\mathfrak{m}_{\varnothing})_1$ equivalent to $(\mathfrak{m}_{\varnothing})_2$, $(\mathfrak{m}_{\varnothing})_3$ equivalent to $(\mathfrak{m}_{\varnothing})_4$, and $(\mathfrak{m}_{\varnothing})_5$ equivalent to $(\mathfrak{m}_{\varnothing})_6$ (as $K_{\varnothing}$-modules). Define the linear maps

$$\begin{aligned} &(T_{\varnothing})_1^2 : (\mathfrak{m}_{\varnothing})_1 \ni W_1 \longmapsto Z_3 \in (\mathfrak{m}_{\varnothing})_2\,, && (T_{\varnothing})_2^1 : (\mathfrak{m}_{\varnothing})_2 \ni Z_3 \longmapsto W_1 \in (\mathfrak{m}_{\varnothing})_1\,,\\ &(T_{\varnothing})_3^4 : (\mathfrak{m}_{\varnothing})_3 \ni W_2 \longmapsto Z_2 \in (\mathfrak{m}_{\varnothing})_4\,, && (T_{\varnothing})_4^3 : (\mathfrak{m}_{\varnothing})_4 \ni Z_2 \longmapsto W_2 \in (\mathfrak{m}_{\varnothing})_3\,,\\ &(T_{\varnothing})_5^6 : (\mathfrak{m}_{\varnothing})_5 \ni W_3 \longmapsto Z_1 \in (\mathfrak{m}_{\varnothing})_6\,, && (T_{\varnothing})_6^5 : (\mathfrak{m}_{\varnothing})_6 \ni Z_1 \longmapsto W_3 \in (\mathfrak{m}_{\varnothing})_5\,,\\ &(T_{\varnothing})_i^i := \mathrm{I}_{(\mathfrak{m}_{\varnothing})_i},\ i = 1,2,3,4,5,6, \end{aligned}$$

and $(T_{\varnothing})_i^j := 0$ for any other i, j, and consider the $(\cdot,\cdot)$-orthonormal sets

$$\mathcal{B}_1^{\varnothing} := \{W_1\},\ \mathcal{B}_2^{\varnothing} := \{Z_3\},\ \mathcal{B}_3^{\varnothing} := \{W_2\},\ \mathcal{B}_4^{\varnothing} := \{Z_2\},\ \mathcal{B}_5^{\varnothing} := \{W_3\},\ \mathcal{B}_6^{\varnothing} := \{Z_1\}.$$

Then, the family $\{(T_{\varnothing})_i^j : i, j \in \{1,2,3,4,5,6\}\}$ satisfies the properties $i) - iv)$ listed in Section 2.1, and $\mathcal{B}_{\varnothing} = \mathcal{B}_1^{\varnothing} \cup \mathcal{B}_2^{\varnothing} \cup \mathcal{B}_3^{\varnothing} \cup \mathcal{B}_4^{\varnothing} \cup \mathcal{B}_5^{\varnothing} \cup \mathcal{B}_6^{\varnothing}$ is an $(\cdot,\cdot)$-orthonormal basis of $\mathfrak{m}_{\varnothing}$ such that $(T_{\varnothing})_i^j(\mathcal{B}_i^{\varnothing}) = \mathcal{B}_j^{\varnothing}$ whenever $(\mathfrak{m}_{\varnothing})_i$ is equivalent to $(\mathfrak{m}_{\varnothing})_j$. Consequently, by formula (2.3), every metric operator $A : \mathfrak{m}_{\varnothing} \to \mathfrak{m}_{\varnothing}$ is determined by a positive numbers $\mu_1, \ldots, \mu_6$ and real number a_1, a_2, a_3 through the relation

$$[A]_{\mathcal{B}_{\varnothing}} = \begin{pmatrix} \mu_1 & a_1 & 0 & 0 & 0 & 0\\ a_1 & \mu_2 & 0 & 0 & 0 & 0\\ 0 & 0 & \mu_3 & a_2 & 0 & 0\\ 0 & 0 & a_2 & \mu_4 & 0 & 0\\ 0 & 0 & 0 & 0 & \mu_5 & a_3\\ 0 & 0 & 0 & 0 & a_3 & \mu_6 \end{pmatrix},$$

where this matrix is positive definite. Observe that the eigenvalues of this matrix are

$$\frac{\mu_1 + \mu_2 \pm \sqrt{(\mu_1 - \mu_2)^2 + 4a_1^2}}{2},$$
$$\frac{\mu_3 + \mu_4 \pm \sqrt{(\mu_3 - \mu_4)^2 + 4a_2^2}}{2},$$
$$\frac{\mu_5 + \mu_6 \pm \sqrt{(\mu_5 - \mu_6)^2 + 4a_3^2}}{2}.$$

Thus, the matrix is positive definite if and only if

$$\begin{aligned} &\mu_1 + \mu_2 \pm \sqrt{(\mu_1 - \mu_2)^2 + 4a_1^2} > 0,\\ &\mu_3 + \mu_4 \pm \sqrt{(\mu_3 - \mu_4)^2 + 4a_2^2} > 0, \text{ and}\\ &\mu_5 + \mu_6 \pm \sqrt{(\mu_5 - \mu_6)^2 + 4a_3^2} > 0, \end{aligned}$$

which is equivalent to

$$|a_1| < \sqrt{\mu_1\mu_2},\ |a_2| < \sqrt{\mu_3\mu_4},\ \text{and } |a_3| < \sqrt{\mu_5\mu_6}.$$

This proves a).

For the proof of b), we can proceed analogously to the proof of a) : let $\Theta = \{\alpha_1\}$, and consider

$$\left(\mathfrak{m}_{\{\alpha_1\}}\right)_1 := \operatorname{span}\{Z_3\},\ \left(\mathfrak{m}_{\{\alpha_1\}}\right)_2 := \operatorname{span}\{W_2, W_3\},\ \text{and } \left(\mathfrak{m}_{\{\alpha_1\}}\right)_3 := \operatorname{span}\{Z_2, -Z_1\}.$$

Then, by Proposition 2.6, we have that

$$\mathfrak{m}_{\{\alpha_1\}} = \left(\mathfrak{m}_{\{\alpha_1\}}\right)_1 \oplus \left(\mathfrak{m}_{\{\alpha_1\}}\right)_2 \oplus \left(\mathfrak{m}_{\{\alpha_1\}}\right)_3,$$

where $\left(\mathfrak{m}_{\{\alpha_1\}}\right)_2$ is equivalent to $\left(\mathfrak{m}_{\{\alpha_1\}}\right)_3$, and

$$\begin{array}{rccc} \tilde{T}: & \left(\mathfrak{m}_{\{\alpha_1\}}\right)_2 & \longrightarrow & \left(\mathfrak{m}_{\{\alpha_1\}}\right)_3 \\ & W_2 & \longmapsto & Z_2 \\ & W_3 & \longmapsto & -Z_1 \end{array}$$

is an intertwining isomorphism. In this case, we define

$$\left(T_{\{\alpha_1\}}\right)_2^3 := \tilde{T}, \qquad \left(T_{\{\alpha_1\}}\right)_3^2 := \tilde{T}^{-1}, \ \left(T_{\{\alpha_1\}}\right)_i^i := \mathrm{I}_{\left(\mathfrak{m}_{\{\alpha_1\}}\right)_i}, \ i = 1, 2, 3,$$

$\left(T_{\{\alpha_1\}}\right)_i^j := 0$ for any other i, j, and

$$\mathcal{B}_1^{\{\alpha_1\}} := \{Z_3\}, \ \mathcal{B}_2^{\{\alpha_1\}} := \{W_2, W_3\}, \ \mathcal{B}_3^{\{\alpha_1\}} := \{Z_2, -Z_1\}.$$

Again, the family $\{\left(T_{\{\alpha_1\}}\right)_i^j : i, j \in \{1, 2, 3\}\}$ satisfies the conditions $i) - iv)$ of Section 2.1, and

$$\mathcal{B}_{\{\alpha_1\}} = \mathcal{B}_1^{\{\alpha_1\}} \cup \mathcal{B}_2^{\{\alpha_1\}} \cup \mathcal{B}_3^{\{\alpha_1\}}$$

is an $(\cdot,\cdot)$-orthonormal basis of $\mathfrak{m}_{\{\alpha_1\}}$ such that is an $(\cdot,\cdot)$-orthonormal basis of $\mathfrak{m}_\varnothing$ such that $(T_\varnothing)_i^j \left(\mathcal{B}_i^{\{\alpha_1\}}\right) = \mathcal{B}_j^{\{\alpha_1\}}$ whenever $(\mathfrak{m}_\varnothing)_i$ is equivalent to $(\mathfrak{m}_\varnothing)_j$. Thus, by formula (2.3), any metric operator $A : \mathfrak{m}_{\{\alpha_1\}} \to \mathfrak{m}_{\{\alpha_1\}}$ is determined by positive numbers μ_1, μ_2, μ_3 and real numbers a, b, c, d through the formula

$$[A]_{\mathcal{B}_{\{\alpha_1\}}} = \begin{pmatrix} \mu_1 & 0 & 0 & 0 & 0 \\ 0 & \mu_2 & 0 & a & c \\ 0 & 0 & \mu_2 & b & d \\ 0 & a & b & \mu_3 & 0 \\ 0 & c & d & 0 & \mu_3 \end{pmatrix},$$

where the matrix is positive definite. The submatrix

$$\begin{pmatrix} a & b \\ c & d \end{pmatrix}$$

defines the map

$$\begin{array}{rccc} A_{32}: & \left(\mathfrak{m}_{\{\alpha_1\}}\right)_2 & \longrightarrow & \left(\mathfrak{m}_{\{\alpha_1\}}\right)_3 \\ & W_2 & \longmapsto & aZ_2 - cZ_1 \\ & W_3 & \longmapsto & bZ_2 - dZ_1, \end{array}$$

and it is an equivariant map. Hence, since $\left(\mathfrak{m}_{\{\alpha_1\}}\right)_2$ is an orthogonal $K_{\{\alpha_1\}}$-module (see Remark 2.3), then A_{32} must be a multiple of $\tilde{T}$. This implies that there exists $\lambda \in \mathbb{R}$ such that

$$A_{32}(W_2) = \lambda \tilde{T}(W_2), \text{ and } A_{32}(W_3) = \lambda \tilde{T}(W_3),$$

or, equivalently,

$$aZ_2 - cZ_1 = \lambda Z_2, \text{ and } bZ_2 - dZ_1 = -\lambda Z_1,$$

that is, $a = d = \lambda$, and $b = c = 0$. Therefore,

$$[A]_{\mathcal{B}_{\{\alpha_1\}}} = \begin{pmatrix} \mu_1 & 0 & 0 & 0 & 0 \\ 0 & \mu_2 & 0 & a & 0 \\ 0 & 0 & \mu_2 & 0 & a \\ 0 & a & 0 & \mu_3 & 0 \\ 0 & 0 & a & 0 & \mu_3 \end{pmatrix}.$$

By computing the eigenvalues of this matrix, we obtain

$$\mu_1, \text{ and } \frac{\mu_2 + \mu_3 \pm \sqrt{(\mu_2 - \mu_3)^2 + 4a^2}}{2}.$$

Thus, it is positive definite if and only if $|a| < \sqrt{\mu_2 \mu_3}$. We have proven $b)$.

For $\Theta = \{\alpha_2\}$, Proposition 2.6 gives us the decomposition

$$\mathfrak{m}_{\{\alpha_2\}} = \left(\mathfrak{m}_{\{\alpha_2\}}\right)_1 \oplus \left(\mathfrak{m}_{\{\alpha_2\}}\right)_2 \oplus \left(\mathfrak{m}_{\{\alpha_2\}}\right)_3,$$

where

$$\left(\mathfrak{m}_{\{\alpha_2\}}\right)_1 = \operatorname{span}\left\{\frac{\sqrt{13}W_2 + 2Z_2}{\sqrt{17}}\right\},$$

$$\left(\mathfrak{m}_{\{\alpha_2\}}\right)_2 = \operatorname{span}\left\{\frac{W_1 + Z_3}{\sqrt{2}}, \frac{W_3 + Z_1}{\sqrt{2}}\right\}$$

$$\left(\mathfrak{m}_{\{\alpha_2\}}\right)_3 = \operatorname{span}\left\{\frac{W_1 - Z_3}{\sqrt{2}}, \frac{W_3 - Z_1}{\sqrt{2}}\right\},$$

and all of them are not equivalent. In this case, since there are not nonzero equivariant maps between these submodules, we have that any metric operator $A : \mathfrak{m}_{\{\alpha_2\}} \to \mathfrak{m}_{\{\alpha_2\}}$ is determined by positive numbers μ_1, μ_2, μ_3 such that

$$A\big|_{\left(\mathfrak{m}_{\{\alpha_2\}}\right)_i} = \mu_i \mathrm{I}_{\left(\mathfrak{m}_{\{\alpha_2\}}\right)_i}, \quad i = 1, 2, 3.$$

The sets

$$\mathcal{B}_1^{\{\alpha_2\}} := \left\{\frac{\sqrt{13}W_2 + 2Z_2}{\sqrt{17}}\right\},$$

$$\mathcal{B}_2^{\{\alpha_2\}} := \left\{\frac{W_1 + Z_3}{\sqrt{2}}, \frac{W_3 + Z_1}{\sqrt{2}}\right\},$$

$$\mathcal{B}_3^{\{\alpha_2\}} := \left\{\frac{W_1 - Z_3}{\sqrt{2}}, \frac{W_3 - Z_1}{\sqrt{2}}\right\}$$

are $(\cdot,\cdot)$-orthonormal bases of $\left(\mathfrak{m}_{\{\alpha_2\}}\right)_1$, $\left(\mathfrak{m}_{\{\alpha_2\}}\right)_2$, $\left(\mathfrak{m}_{\{\alpha_2\}}\right)_3$ respectively. Thus, A can be written in the basis $\mathcal{B}_{\{\alpha_2\}} = \mathcal{B}_1^{\{\alpha_2\}} \cup \mathcal{B}_2^{\{\alpha_2\}} \cup \mathcal{B}_3^{\{\alpha_2\}}$ as

$$[A]_{\mathcal{B}_{\{\alpha_2\}}} = \begin{pmatrix} \mu_1 & 0 & 0 & 0 & 0 \\ 0 & \mu_2 & 0 & 0 & 0 \\ 0 & 0 & \mu_2 & 0 & 0 \\ 0 & 0 & 0 & \mu_3 & 0 \\ 0 & 0 & 0 & 0 & \mu_3 \end{pmatrix}.$$

The proof is complete. □

3.2. **G.o. metrics.** This section is dedicated to classify the K-invariant metrics on flag manifolds of $\mathfrak{g}_2$ that are g.o. metrics.

Definition 3.2. Let G be a compact Lie group, H a closed subgroup of G, and g a G-invariant metric on the homogeneous space G/H. A smooth curve γ on G/H is called *homogeneous* if it is the orbit of a one-parameter subgroup of G, that is, there exists X in the Lie algebra $\mathfrak{g}$ of G such that

$$\gamma(t) = \exp(tX)H,$$

for all t in the domain of γ. If, in addition, the homogeneous curve γ is a geodesic on $(G/H, g)$, then we say that γ is a *homogeneous geodesic* with respect to g. In such a case, the vector $X \in \mathfrak{g}$ is called a *geodesic vector*.

Definition 3.3. A G-invariant metric on a homogeneous space G/H is a *g.o. metric* all geodesics on $(G/H, g)$ starting at eH are homogeneous geodesics.

With the notations of Section 2.1, we recall that the set of G-invariant metrics g on a homogeneous space G/H are in bijective correspondence with the set of $\mathrm{Ad}(H)$-invariant inner products $\langle\cdot,\cdot\rangle$ on $\mathfrak{m}$. So, in what follows we will write g or $\langle\cdot,\cdot\rangle$ interchangeably to refer to either a G-invariant metric on G/H or an $\mathrm{Ad}(H)$-invariant inner product on $\mathfrak{m}$.

The following Proposition was proved by Souris in [21]. It provides a helpful tool to determine whether a G-invariant metric is a g.o. metric.

Proposition 3.4. *Let $\langle\cdot,\cdot\rangle$ be a $\mathrm{Ad}(H)$-invariant inner product on a homogeneous space G/H, and let A be the metric operator associated with $\langle\cdot,\cdot\rangle$. Then $\langle\cdot,\cdot\rangle$ is a g.o. metric if and only if for all $X \in \mathfrak{m}$, there exist a vector $Z \in \mathfrak{h}$ such that*

$$[Z + X, AX] = 0. \tag{3.4}$$

Theorem 3.5. *Let $\mathbb{F}_\Theta$ be a flag of $\mathfrak{g}_2$, $\langle\cdot,\cdot\rangle$ an $\mathrm{Ad}(K_\Theta)$-invariant inner product on $\mathfrak{m}_\Theta$, and A its associated metric operator. Let*

$$\begin{aligned}
\mathcal{B}_\varnothing &= \{W_1, Z_3, W_2, Z_2, W_3, Z_1\}\\
\mathcal{B}_{\{\alpha_1\}} &= \{Z_3, W_2, W_3, Z_2, -Z_1\}, \text{ and}\\
\mathcal{B}_{\{\alpha_2\}} &= \left\{\frac{\sqrt{13}W_2 + 2Z_2}{\sqrt{17}}, \frac{W_1 + Z_3}{\sqrt{2}}, \frac{W_3 + Z_1}{\sqrt{2}}, \frac{W_1 - Z_3}{\sqrt{2}}, \frac{W_3 - Z_1}{\sqrt{2}}\right\}.
\end{aligned}$$

a) If $\Theta = \varnothing$, then $\langle\cdot,\cdot\rangle$ is a g.o. metric if and only if there exist $\mu > 0$, and $a \in (-\mu, \mu)$ such that

$$[A]_{\mathcal{B}_\varnothing} = \begin{pmatrix} \mu & a & 0 & 0 & 0 & 0\\ a & \mu & 0 & 0 & 0 & 0\\ 0 & 0 & \mu & -a & 0 & 0\\ 0 & 0 & -a & \mu & 0 & 0\\ 0 & 0 & 0 & 0 & \mu & a\\ 0 & 0 & 0 & 0 & a & \mu \end{pmatrix}. \tag{3.5}$$

b) If $\Theta = \{\alpha_1\}$, then $\langle\cdot,\cdot\rangle$ is a g.o. metric if and only if there exist $\mu > \mu_1 > 0$ such that

$$[A]_{\mathcal{B}_{\{\alpha_1\}}} = \begin{pmatrix} \mu_1 & 0 & 0 & 0 & 0\\ 0 & \mu & 0 & a & 0\\ 0 & 0 & \mu & 0 & a\\ 0 & a & 0 & \mu & 0\\ 0 & 0 & a & 0 & \mu \end{pmatrix}, \text{ where } a^2 = \mu(\mu - \mu_1). \tag{3.6}$$

c) If $\Theta = \{\alpha_2\}$, then $\langle\cdot,\cdot\rangle$ is a g.o. metric if and only if there exist $\mu_1, \mu_2, \mu_3 > 0$ such that

$$[A]_{\mathcal{B}_{\{\alpha_2\}}} = \begin{pmatrix} \mu_1 & 0 & 0 & 0 & 0\\ 0 & \mu_2 & 0 & 0 & 0\\ 0 & 0 & \mu_2 & 0 & 0\\ 0 & 0 & 0 & \mu_3 & 0\\ 0 & 0 & 0 & 0 & \mu_3 \end{pmatrix},$$

where $\mu_1 = \frac{34\mu_2\mu_3}{(2+\sqrt{13})^2\mu_2 + (2-\sqrt{13})^2\mu_3}$.

Proof. The proof of *a*) and *b*) can be found in [12, Proposition 4.6]. For the proof of *c*), let $\Theta = \{\alpha_2\}$. Then

$$\mathfrak{k}_{\{\alpha_2\}} = \mathrm{span}\{Y_2\} = \mathrm{span}\{\sqrt{13}Z_2 - 2W_2\}.$$

In this case, Proposition 3.4 says that A is a g.o. metric if and only if for each $X \in \mathfrak{m}_{\{\alpha_2\}}$, there exists $\lambda \in \mathbb{R}$ such that

$$[\lambda\left(\sqrt{13}Z_2 - 2W_2\right) + X, AX]. \tag{3.7}$$

If a metric A defined as

$$[A]_{\mathcal{B}_{\{\alpha_2\}}} = \begin{pmatrix} \mu_1 & 0 & 0 & 0 & 0 \\ 0 & \mu_2 & 0 & 0 & 0 \\ 0 & 0 & \mu_2 & 0 & 0 \\ 0 & 0 & 0 & \mu_3 & 0 \\ 0 & 0 & 0 & 0 & \mu_3 \end{pmatrix},$$

is a g.o. metric, then for $X = \sqrt{13}W_2 + 2Z_2 + W_1 + Z_3 + W_3 - Z_1$ there exists $\lambda \in \mathbb{R}$ such that the equation (3.7) is satisfied. Observe that

$$A(\sqrt{13}W_2 + 2Z_2 + W_1 + Z_3) = \sqrt{13}\mu_1 W_2 + 2\mu_1 Z_2 + \mu_2(W_1 + Z_3) + \mu_3(W_3 - Z_1)$$

so that

$$\begin{aligned}
0 =& [\lambda \left(\sqrt{13}Z_2 - 2W_2\right) + X, AX] \\
=& \lambda[\sqrt{13}Z_2 - 2W_2, AX] + [X, AX] \\
=& \lambda\{\sqrt{13}\mu_2[Z_2, W_1] + \sqrt{13}\mu_2[Z_2, Z_3] + \sqrt{13}\mu_3[Z_2, W_3] - \sqrt{13}\mu_3[Z_2, Z_1] \\
& - 2\mu_2[W_2, W_1] - 2\mu_2[W_2, Z_3] - 2\mu_3[W_2, W_3] + 2\mu_3[W_2, Z_1]\} + \sqrt{13}\mu_2[W_2, W_1] \\
& + \sqrt{13}\mu_2[W_2, Z_3] + \sqrt{13}\mu_3[W_2, W_3] - \sqrt{13}\mu_3[W_2, Z_1] + 2\mu_2[Z_2, W_1] \\
& + 2\mu_2[Z_2, Z_3] + 2\mu_3[Z_2, W_3] - 2\mu_3[Z_2, Z_1] + \sqrt{13}\mu_1[W_1, W_2] + 2\mu_1[W_1, Z_2] \\
& + \mu_3[W_1, W_3] - \mu_3[W_1, Z_1] + \sqrt{13}\mu_1[Z_3, W_2] + 2\mu_1[Z_3, Z_2] + \mu_3[Z_3, W_3] \\
& - \mu_3[Z_3, Z_1] + \sqrt{13}\mu_1[W_3, W_2] + 2\mu_1[W_3, Z_2] + \mu_2[W_3, W_1] + \mu_2[W_3, Z_3] \\
& - \sqrt{13}\mu_1[Z_1, W_2] - 2\mu_1[Z_1, Z_2] - \mu_2[Z_1, W_1] - \mu_2[Z_1, Z_3] \\
=& \frac{\lambda}{2}\{\sqrt{13}\mu_2 Z_1 + \sqrt{13}\mu_2 W_3 - \sqrt{13}\mu_3 Z_3 + \sqrt{13}\mu_3 W_1 + 2\mu_2 W_3 + 2\mu_2 Z_1 - 2\mu_3 W_1 \\
& + 2\mu_3 Z_3\} - \frac{\sqrt{13}\mu_2}{2}W_3 - \frac{\sqrt{13}\mu_2}{2}Z_1 + \frac{\sqrt{13}\mu_3}{2}W_1 - \frac{\sqrt{13}\mu_3}{2}Z_3 + \frac{2\mu_2}{2}Z_1 + \frac{2\mu_2}{2}W_3 \\
& - \frac{2\mu_3}{2}Z_3 + \frac{2\mu_3}{2}W_1 + \frac{\sqrt{13}\mu_1}{2}W_3 - \frac{2\mu_1}{2}Z_1 - \frac{\mu_3}{2}W_2 - \frac{\mu_3}{2}Z_2 + \frac{\sqrt{13}\mu_1}{2}Z_1 - \frac{2\mu_1}{2}W_3 \\
& + \frac{\mu_3}{2}Z_2 + \frac{\mu_3}{2}W_2 - \frac{\sqrt{13}\mu_1}{2}W_1 + \frac{2\mu_1}{2}Z_3 + \frac{\mu_2}{2}W_2 - \frac{\mu_2}{2}Z_2 + \frac{\sqrt{13}\mu_1}{2}Z_3 - \frac{2\mu_1}{2}W_1 \\
& + \frac{\mu_2}{2}Z_2 - \frac{\mu_2}{2}W_2 \\
=& \frac{-\lambda(2 - \sqrt{13})\mu_3 + (2 + \sqrt{13})\mu_3 - (2 + \sqrt{13})\mu_1}{2}W_1 \\
& + \frac{\lambda(2 + \sqrt{13})\mu_2 + (2 - \sqrt{13})\mu_2 - (2 - \sqrt{13})\mu_1}{2}W_3 \\
& + \frac{\lambda(2 + \sqrt{13})\mu_2 + (2 - \sqrt{13})\mu_2 - (2 - \sqrt{13})\mu_1}{2}Z_1 \\
& + \frac{\lambda(2 - \sqrt{13})\mu_3 - (2 + \sqrt{13})\mu_3 + (2 + \sqrt{13})\mu_1}{2}Z_3.
\end{aligned}$$

This implies

$$\begin{aligned}
& \begin{cases} \lambda(2 + \sqrt{13})\mu_2 + (2 - \sqrt{13})\mu_2 - (2 - \sqrt{13})\mu_1 = 0 \\ \lambda(2 - \sqrt{13})\mu_3 - (2 + \sqrt{13})\mu_3 + (2 + \sqrt{13})\mu_1 = 0 \end{cases} \\
\Longrightarrow & \begin{cases} \lambda(2 + \sqrt{13})(2 - \sqrt{13})\mu_2\mu_3 + (2 - \sqrt{13})^2\mu_2\mu_3 - (2 - \sqrt{13})^2\mu_1\mu_3 = 0 \\ \lambda(2 + \sqrt{13})(2 - \sqrt{13})\mu_2\mu_3 - (2 + \sqrt{13})^2\mu_2\mu_3 + (2 + \sqrt{13})^2\mu_1\mu_2 = 0 \end{cases} \\
\Longrightarrow & \left((2 - \sqrt{13})^2 + (2 + \sqrt{13})^2\right)\mu_2\mu_3 - \left((2 + \sqrt{13})^2\mu_2 + (2 - \sqrt{13})^2\mu_3\right)\mu_1 = 0 \\
\Longrightarrow & \mu_1 = \frac{34\mu_2\mu_3}{(2 + \sqrt{13})^2\mu_2 + (2 - \sqrt{13})^2\mu_3}.
\end{aligned}$$

Conversely, if $\mu_1 = \frac{34\mu_2\mu_3}{(2+\sqrt{13})^2\mu_2+(2-\sqrt{13})^2\mu_3}$, then, given

$$X = x_1(\sqrt{13}W_2 + 2Z_2) + x_2(W_1 + Z_3) + x_3(W_3 + Z_1) + x_4(W_1 - Z_3) + x_5(W_3 - Z_1) \in \mathfrak{m}_{\{\alpha_2\}}$$

we have

$$\begin{aligned} A(\sqrt{13}W_2 + 2Z_2 + W_1 + Z_3) =& \sqrt{13}\mu_1 x_1 W_2 + 2\mu_1 x_1 Z_2 + \mu_2 x_2(W_1 + Z_3) \\ &+ \mu_2 x_3(W_3 + Z_1) + \mu_3 x_4(W_1 - Z_3) + \mu_3 x_5(W_3 - Z_1). \end{aligned}$$

For $\lambda \in \mathbb{R}$ we can compute (3.7) as follows:

$$\begin{aligned} &[\lambda\left(\sqrt{13}Z_2 - 2W_2\right) + X, AX] \\ =&\lambda[\sqrt{13}Z_2 - 2W_2, AX] + [X, AX] \\ =&\lambda\{\sqrt{13}\mu_2 x_2[Z_2, W_1] + \sqrt{13}\mu_2 x_2[Z_2, Z_3] + \sqrt{13}\mu_2 x_3[Z_2, W_3] + \sqrt{13}\mu_2 x_3[Z_2, Z_1] \\ &+ \sqrt{13}\mu_3 x_4[Z_2, W_1] - \sqrt{13}\mu_3 x_4[Z_2, Z_3] + \sqrt{13}\mu_3 x_5[Z_2, W_3] - \sqrt{13}\mu_3 x_5[Z_2, Z_1] \\ &- 2\mu_2 x_2[W_2, W_1] - 2\mu_2 x_2[W_2, Z_3] - 2\mu_2 x_3[W_2, W_3] - 2\mu_2 x_3[W_2, Z_1] \\ &- 2\mu_3 x_4[W_2, W_1] + 2\mu_3 x_4[W_2, Z_3] - 2\mu_3 x_5[W_2, W_3] + 2\mu_3 x_5[W_2, Z_1]\} \\ &+ \sqrt{13}\mu_2 x_1 x_2[W_2, W_1] + \sqrt{13}\mu_2 x_1 x_2[W_2, Z_3] + \sqrt{13}\mu_2 x_1 x_3[W_2, W_3] \\ &+ \sqrt{13}\mu_2 x_1 x_3[W_2, Z_1] + \sqrt{13}\mu_3 x_1 x_4[W_2, W_1] - \sqrt{13}\mu_3 x_1 x_4[W_2, Z_3] \\ &+ \sqrt{13}\mu_3 x_1 x_5[W_2, W_3] - \sqrt{13}\mu_3 x_1 x_5[W_2, Z_1] + 2\mu_2 x_1 x_2[Z_2, W_1] \\ &+ 2\mu_2 x_1 x_2[Z_2, Z_3] + 2\mu_2 x_1 x_3[Z_2, W_3] + 2\mu_2 x_1 x_3[Z_2, Z_1] + 2\mu_3 x_1 x_4[Z_2, W_1] \\ &- 2\mu_3 x_1 x_4[Z_2, Z_3] + 2\mu_3 x_1 x_5[Z_2, W_3] - 2\mu_3 x_1 x_5[Z_2, Z_1] + \sqrt{13}\mu_1 x_1 x_2[W_1, W_2] \\ &+ 2\mu_1 x_1 x_2[W_1, Z_2] + \mu_2 x_2 x_3[W_1, W_3] + \mu_2 x_2 x_3[W_1, Z_1] + \mu_3 x_2 x_5[W_1, W_3] \\ &- \mu_3 x_2 x_5[W_1, Z_1] + \sqrt{13}\mu_1 x_1 x_2[Z_3, W_2] + 2\mu_1 x_1 x_2[Z_3, Z_2] + \mu_2 x_2 x_3[Z_3, W_3] \\ &+ \mu_2 x_2 x_3[Z_3, Z_1] + \mu_3 x_2 x_5[Z_3, W_3] - \mu_3 x_2 x_5[Z_3, Z_1] + \sqrt{13}\mu_1 x_1 x_3[W_3, W_2] \\ &+ 2\mu_1 x_1 x_3[W_3, Z_2] + \mu_2 x_2 x_3[W_3, W_1] + \mu_2 x_2 x_3[W_3, Z_3] + \mu_3 x_3 x_4[W_3, W_1] \\ &- \mu_3 x_3 x_4[W_3, Z_3] + \sqrt{13}\mu_1 x_1 x_3[Z_1, W_2] + 2\mu_1 x_1 x_3[Z_1, Z_2] + \mu_2 x_2 x_3[Z_1, W_1] \\ &+ \mu_2 x_2 x_3[Z_1, Z_3] + \mu_3 x_3 x_4[Z_1, W_1] - \mu_3 x_3 x_4[Z_1, Z_3] + \sqrt{13}\mu_1 x_1 x_4[W_1, W_2] \\ &+ 2\mu_1 x_1 x_4[W_1, Z_2] + \mu_2 x_3 x_4[W_1, W_3] + \mu_2 x_3 x_4[W_1, Z_1] + \mu_3 x_4 x_5[W_1, W_3] \\ &- \mu_3 x_4 x_5[W_1, Z_1] - \sqrt{13}\mu_1 x_1 x_4[Z_3, W_2] - 2\mu_1 x_1 x_4[Z_3, Z_2] - \mu_2 x_3 x_4[Z_3, W_3] \\ &- \mu_2 x_3 x_4[Z_3, Z_1] - \mu_3 x_4 x_5[Z_3, W_3] + \mu_3 x_4 x_5[Z_3, Z_1] + \sqrt{13}\mu_1 x_1 x_5[W_3, W_2] \\ &+ 2\mu_1 x_1 x_5[W_3, Z_2] + \mu_2 x_2 x_5[W_3, W_1] + \mu_2 x_2 x_5[W_3, Z_3] + \mu_3 x_4 x_5[W_3, W_1] \\ &- \mu_3 x_4 x_5[W_3, Z_3] - \sqrt{13}\mu_1 x_1 x_5[Z_1, W_2] - 2\mu_1 x_1 x_5[Z_1, Z_2] - \mu_2 x_2 x_5[Z_1, W_1] \\ &- \mu_2 x_2 x_5[Z_1, Z_3] - \mu_3 x_4 x_5[Z_1, W_1] + \mu_3 x_4 x_5[Z_1, Z_3] \\ =&\frac{\lambda}{2}\left\{\left((\sqrt{13} + 2)\mu_2 x_2 + (\sqrt{13} - 2)\mu_3 x_4\right) Z_1\right. \\ &+ \left((\sqrt{13} + 2)\mu_2 x_2 - (\sqrt{13} - 2)\mu_3 x_4\right) W_3 \\ &- \left((\sqrt{13} + 2)\mu_2 x_3 + (\sqrt{13} - 2)\mu_3 x_5\right) Z_3 \\ &\left.- \left((\sqrt{13} + 2)\mu_2 x_3 - (\sqrt{13} - 2)\mu_3 x_5\right) W_1\right\} \\ &+ x_1\frac{(\sqrt{13} - 2)x_2(\mu_1 - \mu_2) - (\sqrt{13} + 2)x_4(\mu_1 - \mu_3)}{2} Z_1 \\ &+ x_1\frac{(\sqrt{13} - 2)x_2(\mu_1 - \mu_2) + (\sqrt{13} + 2)x_4(\mu_1 - \mu_3)}{2} W_3 \end{aligned}$$

$$
\begin{aligned}
&- x_1\frac{(\sqrt{13}-2)x_3(\mu_1-\mu_2)-(\sqrt{13}+2)x_5(\mu_1-\mu_3)}{2}Z_3\\
&- x_1\frac{(\sqrt{13}-2)x_3(\mu_1-\mu_2)+(\sqrt{13}+2)x_5(\mu_1-\mu_3)}{2}W_1.
\end{aligned}
$$

Now, since

$$\mu_1=\frac{34\mu_2\mu_3}{(2+\sqrt{13})^2\mu_2+(2-\sqrt{13})^2\mu_3},$$

then

$$\mu_1-\mu_2=-\frac{(2+\sqrt{13})^2(\mu_2-\mu_3)\mu_2}{(2+\sqrt{13})^2\mu_2+(2-\sqrt{13})^2\mu_3},$$

and

$$\mu_1-\mu_3=\frac{(2-\sqrt{13})^2(\mu_2-\mu_3)\mu_3}{(2+\sqrt{13})^2\mu_2+(2-\sqrt{13})^2\mu_3}.$$

Hence, for $(i,j)\in\{(2,4),(3,5)\}$

$$
\begin{aligned}
&\frac{(\sqrt{13}-2)x_i(\mu_1-\mu_2)\pm(\sqrt{13}+2)x_j(\mu_1-\mu_3)}{2}\\
=&\frac{9(\mu_2-\mu_3)}{2((2+\sqrt{13})^2\mu_2+(2-\sqrt{13})^2\mu_3))}\left(-(\sqrt{13}+2)\mu_2x_i\pm(\sqrt{13}-2)\mu_3x_j\right).
\end{aligned}
$$

This implies that

$$
\begin{aligned}
&[\lambda\left(\sqrt{13}Z_2-2W_2\right)+X,AX]\\
=&\frac{1}{2}\left(\lambda-\frac{9x_1(\mu_2-\mu_3)}{(2+\sqrt{13})^2\mu_2+(2-\sqrt{13})^2\mu_3}\right)\left((\sqrt{13}+2)\mu_2x_2+(\sqrt{13}-2)\mu_3x_4\right)Z_1\\
&+\frac{1}{2}\left(\lambda-\frac{9x_1(\mu_2-\mu_3)}{(2+\sqrt{13})^2\mu_2+(2-\sqrt{13})^2\mu_3}\right)\left((\sqrt{13}+2)\mu_2x_2-(\sqrt{13}-2)\mu_3x_4\right)W_3\\
&-\frac{1}{2}\left(\lambda-\frac{9x_1(\mu_2-\mu_3)}{(2+\sqrt{13})^2\mu_2+(2-\sqrt{13})^2\mu_3}\right)\left((\sqrt{13}+2)\mu_2x_3+(\sqrt{13}-2)\mu_3x_5\right)Z_3\\
&-\frac{1}{2}\left(\lambda-\frac{9x_1(\mu_2-\mu_3)}{(2+\sqrt{13})^2\mu_2+(2-\sqrt{13})^2\mu_3}\right)\left((\sqrt{13}+2)\mu_2x_3-(\sqrt{13}-2)\mu_3x_5\right)W_1.
\end{aligned}
$$

Therefore, $[\lambda\left(\sqrt{13}Z_2-2W_2\right)+X,AX]=0$ for $\lambda=\frac{9x_1(\mu_2-\mu_3)}{(2+\sqrt{13})^2\mu_2+(2-\sqrt{13})^2\mu_3}$. This shows that $\langle\cdot,\cdot\rangle$ is a g.o. metric. The proof is complete. $\square$

3.3. **Equigeodesics.** Given a vector $X\in\mathfrak{g}$ and a linear subspace $\mathfrak{u}\subseteq\mathfrak{g}$, let $X_{\mathfrak{u}}$ denote the orthogonal projection of X onto $\mathfrak{u}$.

Definition 3.6. Let G be a compact Lie group, and H a closed subgroup of G. A vector $X\in\mathfrak{g}$ that is a geodesic vector for any G-invariant metric on G/H is called an *equigeodesic vector.*

The following proposition establishes a characterization of equigeodesic vectors on a homogeneous space under certain conditions.

Proposition 3.7 ([11]). *Let G be a compact Lie group, H a closed subgroup of G, and G/H a homogeneous space such that every irreducible H-submodule of multiplicity greater than one is orthogonal. Then $X\in\mathfrak{m}$ is an equigeodesic vector if and only if*

$$[X,T_i^j(X_{\mathfrak{m}_i})+T_j^i(X_{\mathfrak{m}_j})]_{\mathfrak{m}}=0,\ \ i,j=1,...,s, \tag{3.8}$$

where $\{T_i^j:i,j\in\{1,...,s\}\}$ is a family of linear maps $T_i^j:\mathfrak{m}_i\to\mathfrak{m}_j$ satisfying the conditions $i)-iv)$ in Section 2.1.

We may use Proposition 3.7 to characterize equigeodesic vectors on flags of $\mathfrak{g}_2$.

Theorem 3.8. *Let $\mathbb{F}_\Theta$ be a flag of $\mathfrak{g}_2$.*

a) *A vector* $X \in \mathfrak{m}_\varnothing$ *is equigeodesic if and only if*

$$X \in \operatorname{span}\{W_1, Z_3\} \cup \operatorname{span}\{W_2, Z_2\} \cup \operatorname{span}\{W_3, Z_1\}.$$

b) *A vector* $X \in \mathfrak{m}_{\{\alpha_1\}}$ *is equigeodesic if and only if* $X \in \operatorname{span}\{Z_3\}$ *or*

$$X \in \{w_2W_2 + w_3W_3 + z_1Z_1 + z_2Z_2 : w_2z_1 + w_3z_2 = 0\}$$

c) *A vector* $X \in \mathfrak{m}_{\{\alpha_2\}}$ *is equigeodesic if and only if*

$$X \in \operatorname{span}\{\sqrt{13}Z_2 - 2W_2\} \cup \operatorname{span}\{W_1, W_3, Z_1, Z_3\}.$$

Proof. Let $\Theta = \varnothing$, and consider $(\mathfrak{m}_\varnothing)_i, (T_\varnothing)_i^j$, $i,j = 1,2,3,4,5,6$ as in the proof of Theorem 3.1. For each $i \in \{1,2,3,4,5,6\}$, we have that $(\mathfrak{m}_\varnothing)_i$ is an orthogonal H-module since it is one-dimensional, so we can apply Proposition 3.7. For a given

$$X = \sum_{i=1}^{3} w_iW_i + \sum_{i=1}^{3} z_iZ_i \in \mathfrak{m}_\varnothing$$

we have

$$(T_\varnothing)_1^2\left(X_{(\mathfrak{m}_\varnothing)_1}\right) = w_1Z_3,\ (T_\varnothing)_2^1\left(X_{(\mathfrak{m}_\varnothing)_2}\right) = z_3W_1,$$
$$(T_\varnothing)_3^4\left(X_{(\mathfrak{m}_\varnothing)_3}\right) = w_2Z_2,\ (T_\varnothing)_4^3\left(X_{(\mathfrak{m}_\varnothing)_4}\right) = z_2W_2,$$
$$(T_\varnothing)_5^6\left(X_{(\mathfrak{m}_\varnothing)_5}\right) = w_3Z_1,\ (T_\varnothing)_6^5\left(X_{(\mathfrak{m}_\varnothing)_6}\right) = z_1W_3.$$

Therefore, X is equigeodesic if and only if the following equations hold:

$$\begin{aligned} &2[X, w_iW_i] = 2[X, z_iZ_i] = 0,\ i = 1,2,3, \\ &[X, w_1Z_3 + z_3W_1] = [X, w_2Z_2 + z_2W_2] = [X, w_3Z_1 + z_1W_3] = 0. \end{aligned} \tag{3.9}$$

If X is equigeodesic, then in particular,

$$\begin{aligned}
0 = 2[X, w_1W_1] &= w_1w_3W_2 - w_1w_2W_3 + w_1z_2Z_1 - w_1z_1Z_2, \\
0 = 2[X, w_2W_2] &= -w_2w_3W_1 + w_1w_2W_3 + w_2z_3Z_1 - w_2z_1Z_3, \\
0 = 2[X, w_3W_3] &= w_2w_3W_1 - w_1w_3W_2 + w_3z_3Z_2 - w_3z_2Z_3, \\
0 = 2[X, z_1Z_1] &= -z_1z_2W_1 - z_1z_3W_2 + w_1z_1Z_2 + w_2z_1Z_3, \\
0 = 2[X, z_2Z_2] &= z_1z_2W_1 - z_2z_3W_3 - w_1z_2Z_1 + w_3z_2Z_3, \\
0 = 2[X, z_3Z_3] &= z_1z_3W_2 + z_2z_3W_3 - w_2z_3Z_1 - w_3z_3Z_2.
\end{aligned}$$

which implies that w_i, z_i, $i = 1,2,3$ satisfy the following equations:

$$\begin{cases} w_1w_i = z_3z_i = w_1z_j = z_3w_j = 0, & i \neq 1,\ j \neq 3, \\ w_2w_i = z_2w_i = w_2z_i = z_2z_i = 0, & i \neq 2, \\ w_3w_i = z_1w_i = w_3z_j = z_1z_j = 0, & i \neq 3,\ j \neq 1. \end{cases}$$

From these equations, we can deduce the following statements:

- If $w_1 \neq 0$ or $z_3 \neq 0$ then $w_2 = w_3 = z_1 = z_2 = 0$ which implies $X \in \operatorname{span}\{W_1, Z_3\}$.
- If $w_2 \neq 0$ or $z_2 \neq 0$ then $w_1 = w_3 = z_1 = z_3 = 0$ which implies $X \in \operatorname{span}\{W_2, Z_2\}$.
- If $w_3 \neq 0$ or $z_1 \neq 0$ then $w_1 = w_2 = z_2 = z_3 = 0$ which implies $X \in \operatorname{span}\{W_3, Z_1\}$.

Hence, $X \in \operatorname{span}\{W_1, Z_3\} \cup \operatorname{span}\{W_2, Z_2\} \cup \operatorname{span}\{W_3, Z_1\}$. To show that any $X \in \operatorname{span}\{W_1, Z_3\} \cup \operatorname{span}\{W_2, Z_2\} \cup \operatorname{span}\{W_3, Z_1\}$ satisfies the equations (3.9) is a straightforward calculation. This proves *a*).

For the proof of *b*), consider $\Theta = \{\alpha_1\}$, $\left(\mathfrak{m}_{\{\alpha_1\}}\right)_i$, and $\left(T_{\{\alpha_1\}}\right)_i^j$, $i,j = 1,2,3$ as in the proof of Theorem 3.1. The submodules $\left(\mathfrak{m}_{\{\alpha_1\}}\right)_2$ and $\left(\mathfrak{m}_{\{\alpha_1\}}\right)_3$ are orthogonal (see Remark 2.3). Therefore, due to Proposition 3.7, a vector $X = w_2W_2 + w_3W_3 + \sum_{i=1}^{3} z_iZ_i \in \mathfrak{m}_{\{\alpha_1\}}$ is equigeodesic if and only if

$$2[X, z_3Z_3]_{\mathfrak{m}_{\{\alpha_1\}}} = 2[X, w_2W_2 + w_3W_3]_{\mathfrak{m}_{\{\alpha_1\}}} = 2[X, z_1Z_1 + z_2Z_2]_{\mathfrak{m}_{\{\alpha_1\}}} = 0,$$

$$[X, z_2W_2 - z_1W_3 - w_3Z_1 + w_2Z_2]_{\mathfrak{m}_{\{\alpha_1\}}} = 0.$$

Now, let us evaluate each of these expressions:

$$\begin{aligned}
2[X, z_3Z_3]_{\mathfrak{m}_{\{\alpha_1\}}} &= z_1z_3W_2 + z_2z_3W_3 - w_2z_3Z_1 - w_3z_3Z_2,\\
2[X, w_2W_2 + w_3W_3]_{\mathfrak{m}_{\{\alpha_1\}}} &= w_2z_3Z_1 + w_3z_3Z_2 - (w_2z_1 + w_3z_2)Z_3,\\
2[X, z_1Z_1 + z_2Z_2]_{\mathfrak{m}_{\{\alpha_1\}}} &= -z_1z_2W_2 - z_2z_3W_3 + (w_2z_1 + w_3z_2)Z_3,\\
[X, z_2W_2 - z_1W_3 - w_3Z_1 + w_2Z_2]_{\mathfrak{m}_{\{\alpha_1\}}} &= w_3z_3W_2 - w_2Z_3W_3 + z_2z_3Z_1 - z_1z_3Z_2.
\end{aligned}$$

Therefore, X is equigeodesic if and only if the following system of equations is satisfied

$$\begin{cases} z_3w_i = z_3z_j = 0, \quad i \neq 1,\ j \neq 3,\\ w_2z_1 + w_3z_2 = 0, \end{cases}$$

or, equivalently, $w_2 = w_3 = z_1 = z_2 = 0$, (in which case $X \in \operatorname{span}\{Z_3\}$) or $z_3 = 0,\ w_2z_1 + w_3z_2 = 0$. This completes the proof of b).

For the proof of c), as before we are going to consider $(\mathfrak{m}_{\{\alpha_2\}})_i,\ \left(T_{\{\alpha_2\}}\right)_i^j,\ i,j = 1,2,3$ as in the proof of Theorem 3.1. In this case, the submodules $(\mathfrak{m}_{\{\alpha_2\}})_i$ are all inequivalent, i.e., all of them have multiplicity one, so the hypothesis of Proposition 3.7 hold. Given a vector

$$X = x_1(\sqrt{13}W_2 + 2Z_2) + x_2(W_1 + Z_3) + x_3(W_3 + Z_1) + x_4(W_1 - Z_3) + x_5(W_3 - Z_1) \in \mathfrak{m}_{\{\alpha_2\}},$$

the equations (3.8) are equivalent to

$$\begin{aligned}
2[X, x_1(\sqrt{13}W_2 + 2Z_2)]_{\mathfrak{m}_{\{\alpha_2\}}} &= 0,\\
2[X, x_2(W_1 + Z_3) + x_3(W_3 + Z_1)]_{\mathfrak{m}_{\{\alpha_2\}}} &= 0,\\
2[X, x_4(W_1 - Z_3) + x_5(W_3 - Z_1)]_{\mathfrak{m}_{\{\alpha_2\}}} &= 0.
\end{aligned}$$

By computing these Lie brackets we obtain

$$\begin{aligned}
2[X, x_1(\sqrt{13}W_2 + 2Z_2)]_{\mathfrak{m}_{\{\alpha_2\}}} =& - x_1x_2(2 - \sqrt{13})(W_3 + Z_1)\\
&+ x_1x_3(2 - \sqrt{13})(W_1 + Z_3)\\
&+ x_1x_4(2 + \sqrt{13})(W_3 - Z_1)\\
&- x_1x_5(2 + \sqrt{13})(W_1 - Z_3)\\
2[X, x_2(W_1 + Z_3) + x_3(W_3 + Z_1)]_{\mathfrak{m}_{\{\alpha_2\}}} =& x_1x_2(2 - \sqrt{13})(W_3 + Z_1)\\
&- x_1x_3(2 - \sqrt{13})(W_1 + Z_3)\\
2[X, x_4(W_1 - Z_3) + x_5(W_3 - Z_1)]_{\mathfrak{m}_{\{\alpha_2\}}} =& - x_1x_4(2 + \sqrt{13})(W_3 - Z_1)\\
&+ x_1x_5(2 + \sqrt{13})(W_1 - Z_3).
\end{aligned}$$

Hence, X equigeodesic if and only if $x_1x_i = 0,\ i = 2,3,4,5$; that is, $x_1 = 0$, or $x_i = 0,\ i = 2,3,4,5$. This proves that the space of equigeodesic vectors is

$$\begin{aligned}
&\operatorname{span}\{\sqrt{13}Z_2 - 2W_2\} \cup \operatorname{span}\{W_1 + Z_3, W_3 + Z_1, W_1 - Z_3, W_3 - Z_1\}\\
=&\operatorname{span}\{\sqrt{13}Z_2 - 2W_2\} \cup \operatorname{span}\{W_1, W_3, Z_1, Z_3\}.
\end{aligned}$$

The proof is complete. □

4. The homogeneous Ricci flow

The parametrization of the homogeneous metrics provided by Theorem 3.1 allows us to deal with invariant geometry of real flag manifolds of type $\mathfrak{g}_2$ in a more practical way. In particular, the set of invariant metrics on the flag $\mathbb{F}_{\{\alpha_2\}}$ can be identified with the open subset $(\mathbb{R}^+)^3$ consisting of all the points in the Euclidean three-dimensional space with positive coordinates. In this section, we will make a qualitative analysis of the homogeneous Ricci flow on the flag manifold $\mathbb{F}_{\{\alpha_2\}}$. For this purpose, we recall the following well-known formula for the Ricci curvature on a reductive homogeneous space given in [3, Corollary 7.38].

Theorem 4.1. *Let G be a compact Lie group, H a closed subgroup of G, and $\mathfrak{g} = \mathfrak{h} \oplus \mathfrak{m}$ a reductive decomposition that is orthogonal with respect to the Killing form B of $\mathfrak{g}$. For a $\mathrm{Ad}(H)$-invariant inner product $\langle\cdot,\cdot\rangle$ defined on $\mathfrak{m}$, the Ricci tensor associated with $\langle\cdot,\cdot\rangle$ satisfies the following equation:*

$$\begin{aligned}\mathrm{Ric}(X,Y) =& -\frac{1}{2}\sum_i \langle [X,v_i]_{\mathfrak{m}}, [Y,v_i]_{\mathfrak{m}}\rangle - \frac{1}{2}B(X,Y)\\ &+\frac{1}{4}\sum_{i,j}\langle [v_i,v_j]_{\mathfrak{m}}, X\rangle\langle [v_i,v_j]_{\mathfrak{m}}, Y\rangle - \langle U(X,Y), Z\rangle,\ \ X,Y\in\mathfrak{m},\end{aligned} \tag{4.1}$$

where $\{v_i\}$ is an $\langle\cdot,\cdot\rangle$-orthonormal basis of $\mathfrak{m}$, $Z = \sum_i U(v_i,v_i)$, and $U : \mathfrak{m}\times\mathfrak{m} \to \mathfrak{m}$ is the linear map defined implicitly by the formula

$$2\langle U(u,v), w\rangle = \langle [w,u]_{\mathfrak{m}}, v\rangle + \langle [w,v]_{\mathfrak{m}}, u\rangle,\ \ u,v,w\in\mathfrak{m}. \tag{4.2}$$

Corollary 4.2. *Consider the flag manifold $\mathbb{F}_{\{\alpha_2\}}$. Let $\langle\cdot,\cdot\rangle$ be an $\mathrm{Ad}(K_{\{\alpha_2\}})$-invariant inner product on $\mathfrak{m}_{\{\alpha_2\}}$, and A its associated metric operator defined by positive numbers μ_1,μ_2,μ_3 as in formula (3.3). The components of the Ricci tensor corresponding to $\langle\cdot,\cdot\rangle$ with respect to the basis $\mathcal{B}_{\{\alpha_2\}}$ are given by:*

$$\begin{aligned}\mathrm{Ric}_1 &= \frac{1}{544}\left\{\left(\frac{(\sqrt{13}-2)\mu_1}{\mu_2}\right)^2 + \left(\frac{(\sqrt{13}+2)\mu_1}{\mu_3}\right)^2\right\},\\ \mathrm{Ric}_2 &= -\frac{1}{544}\left(\frac{(\sqrt{13}-2)^2\mu_1}{\mu_2}\right) + \frac{1}{2},\\ \mathrm{Ric}_2 &= -\frac{1}{544}\left(\frac{(\sqrt{13}+2)^2\mu_1}{\mu_3}\right) + \frac{1}{2}.\end{aligned}$$

Proof. For the $\mathrm{Ad}(K_{\{\alpha_2\}})$-invariant metric $\langle\cdot,\cdot\rangle$ determined by $\mu_1,\mu_2,\mu_3>0$, an $\langle\cdot,\cdot\rangle$-orthonormal basis of $\mathfrak{m}_{\{\alpha_2\}}$ is given by the vectors

$$v_1 = \frac{\sqrt{13}W_2 + 2Z_2}{\sqrt{17\mu_1}},\ v_2 = \frac{W_1+Z_3}{\sqrt{2\mu_2}},\ v_3 = \frac{W_3+Z_1}{\sqrt{2\mu_2}},\ v_4 = \frac{W_1-Z_3}{\sqrt{2\mu_3}},\ v_5 = \frac{W_3-Z_1}{\sqrt{2\mu_3}}.$$

Additionally, they satisfy the following relations:

$$\begin{aligned}[v_1,v_2]_{\mathfrak{m}_{\{\alpha_2\}}} &= \tfrac{2-\sqrt{13}}{2\sqrt{17\mu_1}}v_3, & [v_1,v_5]_{\mathfrak{m}_{\{\alpha_2\}}} &= \tfrac{2+\sqrt{13}}{2\sqrt{17\mu_1}}v_4,\\ [v_1,v_3]_{\mathfrak{m}_{\{\alpha_2\}}} &= -\tfrac{2-\sqrt{13}}{2\sqrt{17\mu_1}}v_2, & [v_2,v_3]_{\mathfrak{m}_{\{\alpha_2\}}} &= \tfrac{(2-\sqrt{13})\sqrt{\mu_1}}{4\sqrt{17}\mu_2}v_1,\\ [v_1,v_4]_{\mathfrak{m}_{\{\alpha_2\}}} &= -\tfrac{2+\sqrt{13}}{2\sqrt{17\mu_1}}v_5, & [v_4,v_5]_{\mathfrak{m}_{\{\alpha_2\}}} &= -\tfrac{(2+\sqrt{13})\sqrt{\mu_1}}{4\sqrt{17}\mu_3}v_1.\end{aligned} \tag{4.3}$$

Since $[v_i,v_j]$ is always $\langle\cdot,\cdot\rangle$-orthogonal to v_i and v_j, then, by the formula (4.2), we have $U\equiv 0$. Therefore, when applying the equation (4.1) to v_k, $k=1,2,3,4,5$, it simplifies to

$$\begin{aligned}\mathrm{Ric}(v_k,v_k) &= -\frac{1}{2}\sum_{i=1}^{5}\left|\left|[v_k,v_i]_{\mathfrak{m}_{\{\alpha_2\}}}\right|\right|^2 - \frac{1}{2}B(v_k,v_k) + \frac{1}{2}\sum_{1\leqslant i<j\leqslant 5}\langle [v_i,v_j]_{\mathfrak{m}_{\{\alpha_2\}}}, v_k\rangle^2\\ &= -\frac{1}{2}\sum_{i=1}^{5}\left|\left|[v_k,v_i]_{\mathfrak{m}_{\{\alpha_2\}}}\right|\right|^2 + \frac{1}{2}(v_k,v_k) + \frac{1}{2}\sum_{1\leqslant i<j\leqslant 5}\langle [v_i,v_j]_{\mathfrak{m}_{\{\alpha_2\}}}, v_k\rangle^2,\end{aligned}$$

where the norm $||\cdot||$ is taken with respect to $\langle\cdot,\cdot\rangle$. Computing the formula above we obtain

$$\mathrm{Ric}(v_1,v_1) = -\frac{1}{2}\sum_{i=1}^{5}\left|\left|[v_1,v_i]_{\mathfrak{m}_{\{\alpha_2\}}}\right|\right|^2 + \frac{1}{2}(v_1,v_1) + \frac{1}{2}\sum_{1\leqslant i<j\leqslant 5}\langle [v_i,v_j]_{\mathfrak{m}_{\{\alpha_2\}}}, v_1\rangle^2$$

$$
\begin{aligned}
&= -\left\{\left(\frac{2-\sqrt{13}}{2\sqrt{17\mu_1}}\right)^2 + \left(\frac{2+\sqrt{13}}{2\sqrt{17\mu_1}}\right)^2\right\} + \frac{1}{2\mu_1} \\
&\quad + \frac{1}{2}\left\{\left(\frac{(2-\sqrt{13})\sqrt{\mu_1}}{4\sqrt{17}\mu_2}\right)^2 + \left(-\frac{(2+\sqrt{13})\sqrt{\mu_1}}{4\sqrt{17}\mu_3}\right)^2\right\} \\
&= -\frac{34}{68\mu_1} + \frac{1}{2\mu_1} + \frac{1}{2}\left\{\frac{(2-\sqrt{13})^2\mu_1}{(16)(17)\mu_2^2} + \frac{(2+\sqrt{13})^2\mu_1}{(16)(17)\mu_3^2}\right\} \\
&= \frac{\mu_1}{544}\left\{\left(\frac{2-\sqrt{13}}{\mu_2}\right)^2 + \left(\frac{2+\sqrt{13}}{\mu_3}\right)^2\right\},
\end{aligned}
$$

$$
\begin{aligned}
\mathrm{Ric}(v_2, v_2) &= -\frac{1}{2}\sum_{i=1}^{5}\left\|[v_2, v_i]_{\mathfrak{m}_{\{\alpha_2\}}}\right\|^2 + \frac{1}{2}(v_2, v_2) + \frac{1}{2}\sum_{1\leqslant i<j\leqslant 5}\langle [v_i, v_j]_{\mathfrak{m}_{\{\alpha_2\}}}, v_2\rangle^2 \\
&= -\frac{1}{2}\left\{\left(\frac{2-\sqrt{13}}{2\sqrt{17\mu_1}}\right)^2 + \left(\frac{(2-\sqrt{13})\sqrt{\mu_1}}{4\sqrt{17}\mu_2}\right)^2\right\} + \frac{1}{2\mu_2} \\
&\quad + \frac{1}{2}\left(-\frac{2-\sqrt{13}}{2\sqrt{17\mu_1}}\right)^2 \\
&= -\frac{\mu_1}{544}\left(\frac{2-\sqrt{13}}{\mu_2}\right)^2 + \frac{1}{2\mu_2} = \mathrm{Ric}(v_3, v_3),
\end{aligned}
$$

$$
\begin{aligned}
\mathrm{Ric}(v_4, v_4) &= -\frac{1}{2}\sum_{i=1}^{5}\left\|[v_4, v_i]_{\mathfrak{m}_{\{\alpha_2\}}}\right\|^2 + \frac{1}{2}(v_4, v_4) + \frac{1}{2}\sum_{1\leqslant i<j\leqslant 5}\langle [v_i, v_j]_{\mathfrak{m}_{\{\alpha_2\}}}, v_4\rangle^2 \\
&= -\frac{1}{2}\left\{\left(-\frac{2+\sqrt{13}}{2\sqrt{17\mu_1}}\right)^2 + \left(\frac{(2+\sqrt{13})\sqrt{\mu_1}}{4\sqrt{17}\mu_3}\right)^2\right\} + \frac{1}{2\mu_3} \\
&\quad + \frac{1}{2}\left(\frac{2+\sqrt{13}}{2\sqrt{17\mu_1}}\right)^2 \\
&= -\frac{\mu_1}{544}\left(\frac{2+\sqrt{13}}{\mu_3}\right)^2 + \frac{1}{2\mu_3} = \mathrm{Ric}(v_5, v_5).
\end{aligned}
$$

The result follows from the fact that $\mathcal{B}_{\{\alpha_2\}} = \left\{\sqrt{\mu_1}v_1, \sqrt{\mu_2}v_2, \sqrt{\mu_2}v_3, \sqrt{\mu_3}v_4, \sqrt{\mu_3}v_5\right\}$, so

$$
\begin{aligned}
\mathrm{Ric}_1 &= \mathrm{Ric}(\sqrt{\mu_1}v_1, \sqrt{\mu_1}v_1) = \mu_1\mathrm{Ric}(v_1, v_1) \\
\mathrm{Ric}_2 &= \mathrm{Ric}(\sqrt{\mu_2}v_k, \sqrt{\mu_2}v_k) = \mu_2\mathrm{Ric}(v_k, v_k),\ k = 2, 3, \\
\mathrm{Ric}_3 &= \mathrm{Ric}(\sqrt{\mu_3}v_k, \sqrt{\mu_3}v_k) = \mu_3\mathrm{Ric}(v_k, v_k),\ k = 4, 5.
\end{aligned}
$$

□

Theorem 4.3. *The homogeneous Ricci flow*

$$\frac{d\langle\cdot,\cdot\rangle}{dt} = -2\mathrm{Ric} \tag{4.4}$$

on the flag manifold $\mathbb{F}_{\{\alpha_2\}}$ is equivalent to the autonomous system of ordinary differential equations

$$
\left\{
\begin{array}{rcl}
x' &=& \dfrac{x}{z}\left(-\dfrac{1}{2}x^2 + \dfrac{1}{\sqrt{13}-2}x - \dfrac{1}{4}y^2\right) \\[2ex]
y' &=& \dfrac{y}{z}\left(-\dfrac{1}{4}x^2 + \dfrac{1}{\sqrt{13}+2}y - \dfrac{1}{2}y^2\right) \\[2ex]
z' &=& -\dfrac{1}{4}(x^2 + y^2)
\end{array}
\right. \quad ,\ x, y, z > 0. \tag{4.5}
$$

Proof. By Corollary 4.2, the Ricci flow (4.4) is equivalent to the system

$$\begin{cases} \mu_1' &= -\dfrac{1}{272}\left\{\left(\dfrac{(\sqrt{13}-2)\mu_1}{\mu_2}\right)^2+\left(\dfrac{(\sqrt{13}+2)\mu_1}{\mu_3}\right)^2\right\}, \\ \mu_2' &= \dfrac{1}{272}\left(\dfrac{(\sqrt{13}-2)^2\mu_1}{\mu_2}\right)-1, \\ \mu_3' &= \dfrac{1}{272}\left(\dfrac{(\sqrt{13}+2)^2\mu_1}{\mu_3}\right)-1, \end{cases}$$

with $\mu_1,\mu_2,\mu_3>0$. By making the change of variables $(\mu_1,\mu_2,\mu_3)\mapsto(x,y,z)$ (which maps $(\mathbb{R}^+)^3$ onto itself) defined by

$$x=\frac{(\sqrt{13}-2)\mu_1}{68\mu_2},\; y=\frac{(\sqrt{13}+2)\mu_1}{68\mu_3},\; z=\frac{\mu_1}{68}, \tag{4.6}$$

we obtain the result. □

In what follows, we will do a qualitative analysis of the system (4.5). First, let us set $\alpha:=\sqrt{13}-2$, $\beta:=\sqrt{13}+2$. Performing the time rescaling $t=z\tau$, we obtain the following polynomial system

$$\begin{cases} \dot{x} &= x\left(-\dfrac{1}{2}x^2+\dfrac{1}{\alpha}x-\dfrac{1}{4}y^2\right), \\ \dot{y} &= y\left(-\dfrac{1}{4}x^2+\dfrac{1}{\beta}y-\dfrac{1}{2}y^2\right), \\ \dot{z} &= -\dfrac{z}{4}(x^2+y^2), \end{cases} \tag{4.7}$$

where the dot $\cdot$ represents the derivative of the functions $x(\tau)$, $y(\tau)$ and $z(\tau)$ with respect to the real variable τ. Emphasize that systems (4.5) and (4.7) are topologically equivalent for $z>0$. Furthermore, we will denote by $X=(P^1,P^2,P^3)$ the vector field associated to system (4.7). In addition, if $x,y,z\geqslant 0$, then the equilibrium points of system (4.7) are given by $q_1=(2/\alpha,0,0)$, $q_2=(0,2/\beta,0)$, $q_3=(0.0521831,0.352931,0)$ and $q_4=(0,0,z)$, where $z>0$ and q_4 is a numerical solution of the equation $X\equiv 0$. This leads us to the first result of this section.

Proposition 4.4. *Let $\phi_t:\mathbb{R}^3\to\mathbb{R}^3$ be the flow associated with system (4.7) with $x,y,z>0$. Then, ϕ_t has no equilibrium points in finite time, that is, there is no $q\in\mathbb{R}^3$ such that $\phi_t(q)=q$ for all t.*

Although none of the equilibrium points are strictly in the first octant, the other points will help us understand the local dynamics of the given system in this region.

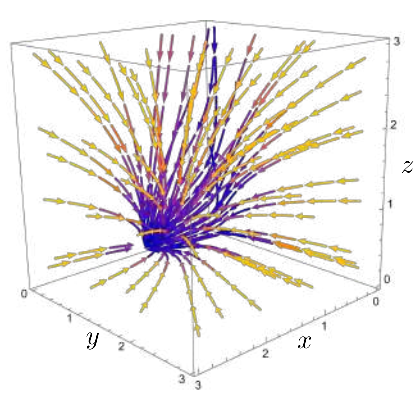

FIGURE 1. Phase portrait of system (4.7).

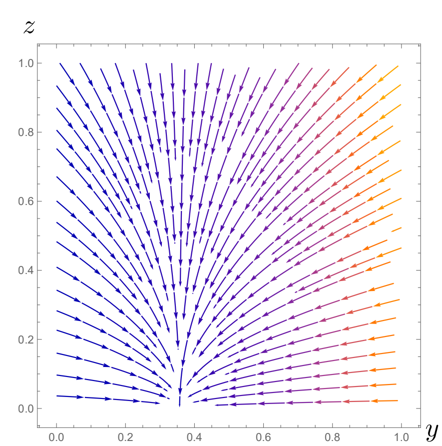

FIGURE 2. Projection of the phase portrait of system (4.7) onto the $yz-$plane.

4.1. **Local dynamic.** Let us start by classifying all degree 2 invariant surfaces of system (4.7). We say that a real polynomial $f = f(x, y, z)$ in the variables x, y and z is a Darboux polynomial of system (4.7) provided that $(\nabla f) \cdot X = kf$, where $k = k(x, y, z)$ is a real polynomial of degree at most 2, called the cofactor of $f(x, y, z)$. If $f(x, y, z)$ is a Darboux polynomial, then the surface $f(x, y, z) = 0$ is an invariant algebraic surface, that is, if a orbit of system (4.7) has a point on this surface, then it is completely contained in it.

Proposition 4.5. *All the invariant algebraic surfaces $f(x, y, z) = 0$ of degree 2 of system (4.7) are given in the following table:*

$f(x,y,z)$	$k(x,y,z)$
z^2	$-\dfrac{x^2}{2} - \dfrac{y^2}{2}$
x^2	$-x^2 - \dfrac{y^2}{2} + \dfrac{2x}{\alpha}$
y^2	$-\dfrac{x^2}{2} - y^2 + \dfrac{2y}{\beta}$
xy	$-\dfrac{3x^2}{4} - \dfrac{3y^2}{4} + \dfrac{x}{\alpha} + \dfrac{y}{\beta}$
xz	$-\dfrac{3x^2}{4} - \dfrac{y^2}{2} + \dfrac{x}{\alpha}$
yz	$-\dfrac{x^2}{2} - \dfrac{3y^2}{4} + \dfrac{y}{\beta}$

TABLE 1. The invariant algebraic surfaces $f(x, y, z) = 0$ of degree 2 of system (4.7).

Proof. Substituting

$$f(z, y, z) = \sum_{i=0}^{2} \sum_{j=0}^{2-i} \sum_{l=0}^{2-i-j} a_{i,j,k} x^i y^j z^l \quad \text{and} \quad k(z, y, z) = \sum_{i=0}^{2} \sum_{j=0}^{2-i} \sum_{l=0}^{2-i-j} k_{i,j,k} x^i y^j z^l,$$

into the equation $(\nabla f) \cdot X = kf$ and using that X is the vector field associated with the system (4.7), we obtain, after some tedious calculations, Table 1. Despite omitting these calculations, the reader can use the functions f and k given in Table 1 and verify that the equation $(\nabla f) \cdot X = kf$ is satisfied. $\square$

From Proposition 4.5 we can conclude that the only invariant algebraic surfaces are $x = 0$, $y = 0$ and $z = 0$. Furthermore, there are no invariant algebraic surfaces of degree 2 for $x, y, z > 0$.

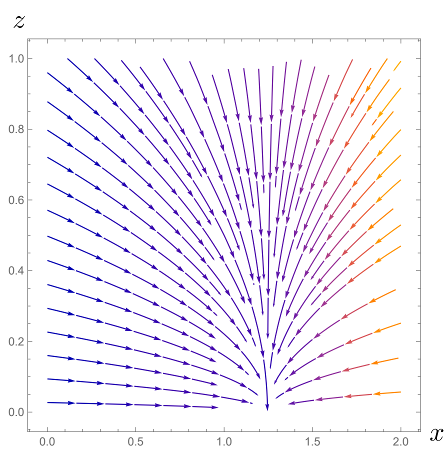

FIGURE 3. Projection of the phase portrait of system (4.7) onto the $xz-$plane.

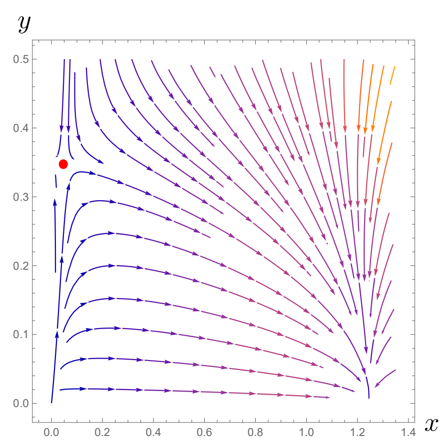

FIGURE 4. Projection of the phase portrait of system (4.7) onto the $xy-$plane.

4.1.1. *Dynamics around q_1.* Here, the Jacobian matrix associated with system (4.7) at the point q_1 has eigenvalues $-2/\alpha^2, -1/\alpha^2$ and $-1/\alpha^2$ with corresponding eigenvectors $(1,0,0), (0,0,1)$ and $(0,1,0)$. Therefore, q_1 is an attractor point, see Figures 3 and 4.

4.1.2. *Dynamics around q_2.* Notice that the eigenvalues associated with system (4.7) at the point q_2 are $-2/\beta^2, -1/\beta^2$ and $-1/\beta^2$ with corresponding eigenvectors $(0,1,0), (0,0,1)$ and $(1,0,0)$. Consequantly, q_2 is an attractor point, see Figure 2.

4.1.3. *Dynamics around q_3.* In this case, the eigenvalues associated with q_3 are $-0.0625182, -0.0318209$ and 0.0306973 with eigenvectors $(0.0992779, 0.99506, 0), (0,0,1)$ and $(0.99506, -0.0992779, 0)$, respectively. This implies that q_3 is a saddle point, see Figure 4.

4.1.4. *Dynamics around q_4.* Recall that the Jacobian matrix of the vector field associated with the system (4.7) at the equilibrium point $q_4 = (0,0,z)$ is given by

$$\begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}.$$

Hence, the equilibrium point q_4 has eigenvalues at with real part zero. These types of equilibrium points are known as *nonhyperbolic equilibrium points.* In order to understand the dynamics of the system (4.7) at $q_4 = (0,0,z)$ we must apply the following Blow-up:

$$x = r\widetilde{x} \quad y = r\widetilde{y} \quad z = \widetilde{z}.$$

where $r \in \overline{\mathbb{R}^+}$ and $(\widetilde{x}, \widetilde{y}, \widetilde{z}) \in \mathbb{S}^2$. Roughly speaking, the geometric idea of the blow-up method is to change the equilibrium point q_4 by a sphere $\mathbb{S}^2 \subset \mathbb{R}^3$, leaving the dynamics away from the q_4 unchanged. This allow us to blow-up the dynamics around q_4.

Now, consider the following chart:

$$\kappa_1 : \widetilde{y} = 1 : x = r_1 x_1 \quad y = r_1 \quad z = z_1.$$

In this chart, the system (4.7) can be written as:

$$\begin{cases} \dot{x_1} &= x_1 \left(\dfrac{r_1}{4}(1 - x_1^2) - \dfrac{1}{\beta} + \dfrac{x_1}{\alpha} \right), \\[2ex] \dot{r_1} &= -r_1 \left(\dfrac{1}{4}(2 + x_1^2) r_1 - \dfrac{1}{\beta} \right), \\[2ex] \dot{z_1} &= -\dfrac{r_1 z_1}{4}(1 + x_1^2), \end{cases} \tag{4.8}$$

after desingularization through the division by r_1 on the right-hand side. In addition, we denote by F the vector field associated to system (4.8). In the new coordinates q_4 is represented by the equilibrium points

$$p^+ = \left(\frac{\alpha}{\beta}, 0, z_1^* \right) \quad \text{and} \quad p^- = (0, 0, z_1^*),$$

where $z_1^* \in \mathbb{R}^+$. Thus, the eigenvalues of the linear part of system (4.8) at the equilibrium point p^+ are $\lambda_1^+ = 0, \lambda_2^+ = 1/\beta$ and $\lambda_3^+ = 1/\beta$, and the ones of the equilibrium point p^- are $\lambda_1^- = 0, \quad \lambda_2^- = 1/\beta$ and $\lambda_3^- = -1/\beta$. This means that $p^\pm$ are nonhyperbolic equilibrium points. It is not difficult to find their corresponding eigenvectors

$$v_1^+ = (0,0,1) \quad \text{and} \quad v_2^+ = (1,0,0)$$

and

$$v_1^- = (0,0,1), \quad v_2^- = \left(0, -\frac{4}{z_1^* \beta}, 1 \right) \quad \text{and} \quad v_3^- = (1,0,0),$$

respectively. Nonzero multiples of these eigenvectors are the only eigenvectors of system (4.8) at p^+ corresponding to $\lambda_1 = 0$ and $\lambda_2 = \lambda_3 = 1/\beta$ respectively. Consequently, we must find one generalized eigenvector corresponding to $\lambda_3^+ = 1/\beta$ and independent of v_2^+. Solving the equation $(F - \lambda_3^+ I)^2 v_3^+ = 0$, we get the following generalized eigenvector

$$v_3^+ = \left(0, 1, -\frac{z_1^*(\alpha^2+\beta^2)}{4\beta}\right).$$

From *center manifold theorem* [18], we know that there exists an one-dimensional center manifold $W^c(p^\pm)$ tangent to the center subspace $E^c : z_1-$axis of (4.8) at $p^\pm$, there exists a one-dimensional (resp. 2-dimensional) unstable manifold $W^u(p^-)$ (resp $W^u(p^+)$) tangent to the unstable subspace

$$E^u = \left\{(0, r_1, z_1) : r_1 = -\frac{4z_1}{z_1^*\beta}\right\} \quad \left(\text{resp. } E^u = span\left\{v_2^+, v_3^+\right\}\right)$$

of (4.8) at p^- (resp. p^+) and there exists a one-dimensional stable manifold $W^s(p^-)$ tangent to the stable subspace $E^s : x_1-$axis of (4.8) at p^-. Even more, $W^c(p^\pm)$, $W^s(p^\pm)$ and $W^u(p^\pm)$ are invariant under the flow of (4.8). The local dynamics of system (4.7) can be seen in Figure 1.

4.2. **Global dynamic.** To investigate the global dynamics of a polynomial differential system in the space X, we need to classify the local phase portraits of its finite and infinite equilibrium points on the Poincaré disk.

Let $\mathbb{S}^3 = \{\mathbf{y} \in \mathbb{R}^4 : ||\mathbf{y}|| = 1\}$ be a sphere in $\mathbb{R}^3$. From [7], we know that X induces a vector field in $\mathbb{S}^3$, which we denote by $p(X)$. The vector field $p(X)$ allows us to study the behavior of X in the neighborhood of infinity, i.e., in the neighborhood of the equator $\mathbb{S}^2 = \{\mathbf{y} \in \mathbb{S}^3 : y_4 = 0\}$. To get the analytical expression for $p(X)$ we shall consider the sphere as a smooth manifold. In this context, it is enough to choose the 3 coordinate neighbourhoods given by $U_i = \{\mathbf{y} \in \mathbb{S}^3 : y_i > 0\}$, for $i = 1, 2, 3$. Denote by (z_1, z_2, z_3) the local coordinates on U_i for $i = 1, 2, 3$. By [7], the vector field $p(X)$ in U_1 becomes

$$\frac{z_3^3}{\Delta(z)^2}(-z_1P^1 + P^2, -z_2P^1 + P^3, -z_3P^1),$$

where $P^i = P^i(1/z_3, z_1/z_3, z_2/z_3)$ and $\Delta(z) = (1 + \sum_{i=1}^3 z_i^2)^{\frac{1}{2}}$. Then, system (4.7) in the chart U_1 is

$$\begin{cases} \dot{z_1} &= z_1\left(\dfrac{1-z_1^2}{4} - \dfrac{z_3}{\alpha} + \dfrac{z_1z_3}{\beta}\right), \\ \dot{z_2} &= z_2\left(\dfrac{1}{4} - \dfrac{z_3}{\alpha}\right), \\ \dot{z_3} &= \dfrac{z_3}{4}\left(2 + z_1^2 - \dfrac{4z_3}{\alpha}\right), \end{cases} \tag{4.9}$$

We have that in chart U_1 system (4.9) has three equilibrium points:

$$p_1 = (1, 0, 0), \quad p_2 = (-1, 0, 0) \quad \text{and} \quad p_3 = (0, 0, 0),$$

The eigenvalues of the linear part of system (4.9) at the equilibrium points p_1 and p_2 are $-1/2, 1/4$ and $3/4$ with corresponding eigenvectors $(1, 0, 0)$, $(0, 1, 0)$ and $(4(\alpha \mp \beta)/5\alpha\beta, 0, 1)$ and the ones of the origin are $1/4, 1/4$ and $1/2$ with corresponding eigenvectors $(1, 0, 0)$, $(0, 1, 0)$ and $(0, 0, 1)$. This implies that the origin is a source and p_1, p_2 are saddle points.

Likewise, we know that the expression for $p(X)$ in U_2 is given by

$$\frac{z_3^3}{\Delta(z)^2}(-z_1P^2 + P^1, -z_2P^2 + P^3, -z_3P^2),$$

where $P^i = P^i(z_1/z_3, 1/z_3, z_2/z_3)$ and $\Delta(z) = (1 + \sum_{i=1}^3 z_i^2)^{\frac{1}{2}}$. Then, system (4.7) in the chart U_2 is

$$\begin{cases} \dot{z_1} &= z_1\left(\dfrac{1-z_1^2}{4} - \dfrac{z_3}{\beta} + \dfrac{z_1 z_3}{\alpha}\right), \\ \dot{z_2} &= z_2\left(\dfrac{1}{4} - \dfrac{z_3}{\beta}\right), \\ \dot{z_3} &= \dfrac{z_3}{4}\left(2 + z_1^2 - \dfrac{4z_3}{\beta}\right), \end{cases} \tag{4.10}$$

We get that in chart U_2 system (4.10) has three equilibrium points:

$$p_1 = (1,0,0)\,, \quad p_2 = (-1,0,0) \quad \text{and} \quad p_3 = (0,0,0),$$

The eigenvalues of the linear part of system (4.10) at the equilibrium points p_1 and p_2 are $-1/2, 1/4$ and $3/4$ with corresponding eigenvectors $(1,0,0)$, $(0,1,0)$ and $(\mp 4(\alpha \mp \beta)/5\alpha\beta, 0, 1)$ and the ones of the equilibrium point p_3 are $1/4, 1/4$ and $1/2$ with corresponding eigenvectors $(1,0,0)$, $(0,1,0)$ and $(0,0,1)$. This implies that the origin is a source and p_1, p_2 are saddle points.

Now, the expression for $p(X)$ in U_3 is given by

$$\frac{z_3^3}{\Delta(z)^2}(-z_1P^3 + P^1, -z_2P^3 + P^2, -z_3P^3),$$

where $P^i = P^i(z_1/z_3, z_2/z_3, 1/z_3)$ and $\Delta(z) = (1 + \sum_{i=1}^3 z_i^2)^{\frac{1}{2}}$, Then, system (4.7) in the chart U_3 is

$$\begin{cases} \dot{z_1} &= z_1^2\left(-\dfrac{z_1}{4} + \dfrac{z_3}{\alpha}\right), \\ \dot{z_2} &= -z_2^2\left(\dfrac{z_2}{4} - \dfrac{z_3}{\beta}\right), \\ \dot{z_3} &= \dfrac{z_3}{4}\left(z_1^2 + z_2^2\right), \end{cases} \tag{4.11}$$

We get that in chart U_3 system (4.11) has the origin as a unique equilibrium point. In addition, the origin is a linearly zero equilibrium (i.e. the Jacobian matrix at $(0,0,0)$ is identically zero). From *center manifold theorem* [18], we know that there exists an 3-dimensional center manifold $W^c(0,0,0)$ tangent to the center subspace $E^c = \mathbb{R}^3$ of (4.11) at the origin. Even more, $W^c(0,0,0)$, is invariant under the flow of (4.11).

Let us denote by $\pi : \mathbb{R}^3 \to \mathbb{R}^3$ the map $\pi(x_1, x_2, x_3) = \frac{1}{\Delta(x)}(x_1, x_2, x_3)$, which shrinks $\mathbb{R}^3$ to its unitary ball and takes the infinity to the sphere $\mathbb{S}^2$.

Now, denote η_i the point in $\pi(\mathbb{R}^3)$ corresponding to p_i for $i = 1,2,3$. Using the information given in the charts U_i for $i = 1,2,3$ we have that there are no η_j that are in the first octant. This completes the qualitative behavior of differential system (4.7).

It is worth pointing out that the previous analysis was conducted for the parameters x, y, z defined in (4.6). However, this is sufficient to understand and deduce the behavior and properties of the Ricci flow by considering the original parametrization (μ_1, μ_2, μ_3) of the invariant metrics. To illustrate this, we will prove the following proposition that concerns the long-time behavior of the solutions in their original parametrization.

Proposition 4.6. *Let $t \mapsto \langle \cdot, \cdot \rangle_t$ be a solution of the homogeneous Ricci flow (4.4), and A_t the metric operator associated with $\langle \cdot, \cdot \rangle_t$. Then $\langle \cdot, \cdot \rangle_t$ collapses over time.*

Proof. Assume that A_t is determined by $\mu_1(t), \mu_2(t), \mu_3(t) > 0$, for all t and notice that $\mu_1(t), \mu_2(t), \mu_3(t)$ satisfy the relations

$$\mu_1(t) = 68z(t), \quad \mu_2(t) = \frac{\alpha z(t)}{x(t)} \quad \text{and} \quad \mu_3(t) = \frac{\beta z(t)}{y(t)}, \tag{4.12}$$

with $\alpha = \sqrt{13} - 2$ and $\beta = \sqrt{13} + 2$. Emphasize that there exist four equilibrium points denoted as $\{q_i\}_{i=1}^4$ of system (4.7). Further, consider $\mathfrak{R} = \{(x, y, z) \in \mathbb{R}^3 : \quad x, y, z > 0\}$ as the domain where this metric is defined. Within this domain, there are distinct neighborhoods U_i containing q_i for each i, such that $U_i \cap U_j = \varnothing$ for all $i \neq j$.

In what follows, we study the behavior of the metric at each equilibrium point to understand its effects. For the attractor points q_1 and q_2 we have that any orbit $\gamma \equiv (\gamma_1, \gamma_2, \gamma_3) \subset U_i$ has as ω-limit set the equilibrium point q_i. Thus, for $q_1 = (2/\alpha, 0, 0)$, the curve $\gamma_1(t) \to 2/\alpha$ and $\gamma_j(t) \to 0$ for $j = 2, 3$ as $t \to \infty$. Using the relations (4.12) we get that

$$\mu_1(t) = 68\gamma_3(t), \quad \mu_2(t) = \alpha\frac{\gamma_3(t)}{\gamma_1(t)} \quad \text{and} \quad \mu_3(t) = \beta\frac{\gamma_3(t)}{\gamma_2(t)} \tag{4.13}$$

concluding that $\mu_1(t), \mu_2(t) \to 0$ as $t \to \infty$, thus the metric is collapsing. For $q_2 = (0, \frac{2}{\beta}, 0)$, the curve $\gamma_2(t) \to 2/\beta$ and $\gamma_j(t) \to 0$ for $j = 1, 3$ as $t \to \infty$. Therefore, using (4.13), we have that $\mu_1(t), \mu_3(t) \to 0$ as $t \to \infty$, so the metric is collapsing.

The point $q_3 = (0.0521831, 0.352931, 0)$ stands as a saddle point. Upon considering equations (4.13), it is evident that this point is associated with a trivial metric and that in the unstable subspace q_3 behaves like an unstable equilibrium point. Moreover, since q_3 is a saddle point there exists a stable subspace in which all orbits converge to q_3, then let $\tilde{\gamma} \equiv (\tilde{\gamma_1}, \tilde{\gamma_2}, \tilde{\gamma_3})$ be a orbit in U_3 such that $\tilde{\gamma}(t) \to q_3$ as $t \to \infty$. Thus, $\tilde{\gamma}_3(t) \to 0$, which implies that $\mu_1(t), \mu_2(t), \mu_3(t) \to 0$ as $t \to \infty$, concluding that $\langle \cdot, \cdot \rangle_t$ is collapsing. □

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Brian Grajales
IMECC-Unicamp, Departamento de Matemática, Rua Sérgio Buarque de Holanda 651, Cidade Universitária Zeferino Vaz. 13083-859, Campinas - SP, Brazil
E-mail address grajales@ime.unicamp.br

Gabriel Rondón
São Paulo State University (Unesp), Institute of Biosciences, Humanities and Exact Sciences. Rua C. Colombo, 2265, CEP 15054–000. S. J. Rio Preto, São Paulo, Brazil
Email address: gabriel.rondon@unesp.br

Julieth Saavedra
Centro de Ciências - Departamento de Matemática, Av. Humberto Monte, s/n, Campus do Pici - Bloco 914. 60455-760, Fortaleza, Ceará, Brazil
Email address: julieth.p.saavedra@gmail.com