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UNIVERSIDADE ESTADUAL PAULISTA “JÚLIO DE MESQUITA FILHO”
CÂMPUS DE ILHA SOLTEIRA

AFONSO WILLIAN NUNES

SYMMETRY AND EQUIVALENCE ANALYSIS OF WAVE DYNAMICS IN CONTINUOUS MEDIA: ELASTIC RODS, PLATES, AND NONLINEAR SOLITONIC STRUCTURES

Ilha Solteira
2025

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SYMMETRY AND EQUIVALENCE ANALYSIS OF WAVE DYNAMICS IN CONTINUOUS MEDIA: ELASTIC RODS, PLATES, AND NONLINEAR SOLITONIC STRUCTURES

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Prof. Dr. Samuel da Silva
Supervisor

Prof. Dr. Stylianos Dimas
Co-supervisor

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ORIENTADOR: SAMUEL DA SILVA
COORIENTADOR: STYLIANOS DIMAS

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Prof. Dr. SAMUEL DA SILVA (Participaçao Virtual)
Departamento de Engenharia Mecanica / Faculdade de Engenharia de Ilha Solteira - UNESP

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Prof. Dr. ADRIANO TODOROVIC FABRO (Participaçao Virtual)
Departamento de Engenharia Mecânica / Universidade de Brasília - UnB

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Prof. Dr. AMERICO BARBOSA DA CUNHA JUNIOR (Participaçao Virtual)
Instituto de Matemática e Estatística / Universidade do Estado do Rio de Janeiro - UERJ

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Prof. Dr. IGOR LEITE FREIRE (Participaçao Virtual)
Centro de Ciências Exatas e de Tecnologia / Universidade Federal de São Carlos - UFSCar

UNIVERSIDADE ESTADUAL PAULISTA

Câmpus de Ilha Solteira

Prof. Dr. PETER BERND HAGEDORN (Participaçao Virtual)
Departamento de Engenharia Mecânica / Universidade Técnica de Darmstadt - TUDa

Ilha Solteira, 04 de fevereiro de 2025

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"What is to give light must endure burning."

Viktor E. Frankl

ABSTRACT

Wave analysis supports numerous contemporary engineering systems' optimal design, performance, and safety. Yet, several mathematical models essential for describing the physical phenomena these systems undergo are either difficult to tackle or have not yet been analytically investigated. As a result, such challenges are frequently bypassed in favor of standard numerical studies, overlooking valuable insights that analytical methods could provide when predicting the dynamics exhibited by these systems. This thesis focuses on employing analytical symmetry analysis to examine differential equations that arise from wave dynamics models in continuous media. Simpler, equivalent problems are derived, allowing for the computation of original exact mode shapes and traveling-wave solutions for non-uniform, functionally graded rods described by the elementary rod model. In addition, new complete sets of exact invariant solutions are found for the mode shape equation of rectangular Kirchhoff plates and the generalized Calogero–Bogoyavlenskii–Schiff equation for solitonic structures through the proposed symmetry-based perspective.

Keywords: symmetry analysis; exact solutions; elementary rods; Kirchhoff plates; generalized Calogero–Bogoyavlenskii–Schiff equation.

RESUMO

O estudo da dinâmica de ondas tem papel fundamental no design, desempenho e segurança de diversos sistemas de engenharia contemporâneos. No entanto, muitos modelos matemáticos úteis na descrição dos fenômenos físicos desses sistemas apresentam dificuldades de resolução ou ainda não foram investigados de maneira analítica. Como resultado, tais desafios frequentemente são deixados de lado em favor de estudos numéricos, o que pode levar à perda de noções valiosas fornecidas por métodos analíticos na predição de comportamentos dinâmicos. Esta tese apresenta investigações analíticas por meio de análises de simetria para examinar equações diferenciais provenientes da dinâmica de ondas em meios contínuos. Como resultado, surgem problemas equivalentes mais simples, permitindo a obtenção exata de modos de vibrar e soluções em termos de ondas viajantes para estruturas do tipo barra elementar com geometrias e propriedades materiais não constantes. Além disso, soluções invariantes exatas são determinadas para a equação dos modos de vibrar de placas de Kirchhoff retangulares e para a equação generalizada de Calogero–Bogoyavlenskii–Schiff da teoria de sólitons, analisando-as sob a perspectiva dos métodos de simetria.

Palavras-chave: análise de simetria; soluções exatas; barras elementares; placas de Kirchhoff; equação generalizada de Calogero–Bogoyavlenskii–Schiff.

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LIST OF SYMBOLS

A	cross-section area / polar area moment
A_i	new area parameter
C	constant
C_{ij}^k	algebra structure constant
D	flexural stiffness
$\mathcal{E}$	Jacobi epsilon function
F	first kind elliptic integral
${}_2\mathrm{F}_1$	ordinary hypergeometric function
F, G, F_i	arbitrary functions
$\mathcal{F}$, $\mathcal{G}$	differential equations
H_i	smooth function
$\mathcal{H}$	enhanced system restrictions
HeunC	Heun confluent function
HeunG	Heun general function
I_j	first kind Bessel function of order j
J_j	second kind Bessel function of order j
K	constant
$\mathcal{L}$	Lie algebra
M	Young's modulus / shear modulus
M_i	new modulus parameter
P_j^i	associated first kind Legendre functions of degree j and order i
$\mathcal{P}_i$	differentiable transformation function
Q	gCBS dependent variable
Q_j^i	associated second kind Legendre functions of degree j and order i
$\mathcal{Q}$	symmetry characteristic
$\mathcal{R}_i$, $\mathcal{U}$	differentiable transformation functions
$\mathcal{X}$, $\mathcal{X}_i$	symmetry generators
$\mathcal{X}^{(k)}$, $\mathcal{X}_i^{(k)}$	kth-order prolonged symmetry generators
$\hat{\mathcal{X}}$, $\tilde{\mathcal{X}}$	general symmetry generators
Y	mode shape equation dependent variable
$\mathcal{Y}$, $\mathcal{Y}_i$	equivalence generators

a_i, b_i	constants
am	Jacobi's amplitude function
c	constant
c_i	constants of integration
f, g	initial conditions
f_2, g_2	equivalent initial conditions
h	thickness
i	unit imaginary number
i, j	indexes
k	index or constant
l	number of arbitrary parameters
ℓ, ℓ_i	lengths
m	number of independent variables
n	differential equation order
p_i	arbitrary parameter
$\mathbf{p}$	vector of arbitrary parameters
r, r_i	independent variables
$\mathbf{r}$	vector of independent variables
s	invariant
sn	Jacobi elliptic function sn
t	time coordinate
v	differential invariant
u	dependent variable
v, w, w_i	differential invariants
$\mathbf{w}$	vector of differential invariants
x, x_i	spatial coordinates
y	elastodynamic equation dependent variable
y_i	equivalent elastodynamic dependent variable
z	equivalent spatial coordinate
z_i	equivalent length
Γ	Gamma function
Θ_i	ith-dimensional optimal system of subalgebras
Φ	smooth function

Ψ	Lie group of scaling transformations
α, β, δ, γ	constants
δ_i	non-zero real parameters
ε, ε_i	continuous parameters
ζ, ζ_i	new independent variables
η, η_i	infinitesimal functions
$\eta^{(k)}_{r_{i_1}\ldots r_{i_k}}$	kth-order prolonged infinitesimal function
θ	transformation function
ϑ_i	constant
κ_i	optimal system constant
μ_i, ν_i	constants
ν	Poisson's ratio
λ	wavenumber
ξ, ξ_i	infinitesimal functions
ρ	specific mass
σ	dimension of the Lie algebra
τ	dummy or independent variable
ϕ	torsion angle
φ	smooth function
ψ	transformation function
ω	natural angular frequency
∇^2	Laplacian operator
$\square'$	ordinary derivative of $\square$ with respect to its independent variable
$\square_{,\boxtimes}$	partial derivative of $\square$ with respect to $\boxtimes$
$\partial^k\square$	set of kth-order partial derivatives of $\square$
$D_\square$	total derivative operator with respect to $\square$
$\mathrm{e}^\square$	exponentiation of the vector field $\square$
$\bar{\square}$	transformed variable $\square$
$[\square, \boxtimes]$	commutator between the operators $\square$ and $\boxtimes$
$\mathrm{Ad}(\square)$	adjoint representation of $\square$

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1 INTRODUCTION

Wave dynamics analysis is essential for ensuring numerous engineering systems' safety, efficiency, and reliability, thereby promoting their advancement toward the optimal performance demanded by contemporary society (Sony; Laventure; Sadhu, 2019; Chen; Chiu; Chen, 2016; Ghasemloonia; Rideout; Butt, 2015; Hussein; Leamy; Ruzzene, 2014). These advancements have led to the development of unconventional systems and sophisticated mathematical models with intricate differential equations for accurately capturing their underlying physical phenomena (Lim; Zhang; Reddy, 2015; Salehipour; Shahidi; Nahvi, 2015; Li; Hu, 2016; Şimşek, 2016; Li; Li; Hu, 2016; Wei *et al.*, 1995; Onorato *et al.*, 2013). Numerical and experimental methods are valuable in assessing the dynamics of such problems, and the challenges imposed by these systems' modeling complexity have led to a tendency among engineers to overlook analytical methods. This tendency became a routine practice and was eventually seen as the only feasible approach, thereby circumventing the inherent insights that analytical studies of the associated physical phenomena could provide (Zhang *et al.*, 2024; Ma *et al.*, 2024; Ma; Zhang; Kennedy, 2015).

Within the realm of unconventional engineering structures, advancements in fabrication have facilitated the development of optimized structures with unusual designs and materials of complex compositions, such as phononic crystals and metamaterials (Fabro *et al.*, 2021; Fan *et al.*, 2021; Minkov *et al.*, 2020; Maldovan, 2013; Soukoulis; Wegener, 2011). Consistently, many of these require intricate continuous-body models to represent their vibration dynamics. This causes most of their inherent properties of interest, such as amplification bands or gaps, vibration control, cloaking, frequency filtering, and topological effects, not to be fully known in early project stages as relying on non-immediate numerical and experimental studies, as they would be if they had exact solutions available (Ji; Huber, 2022; Lu *et al.*, 2021; Pelat *et al.*, 2020; Beli; Arruda; Ruzzene, 2018; Tang *et al.*, 2016; Pai; Peng; Jiang, 2014).

Exploring wave phenomena in continuous media usually leads to many different types of waves of interest. Regarding models incorporating nonlinear effects, stable localized solitonic traveling and dispersive waves follow interesting patterns for vibration dynamics research (Silva; Freire, 2019; Hsiao *et al.*, 2005; Ames; Harrell; Herod, 1993; Ames, 1992). Studying these wave structures provides valuable insights into processes such as wave amplification, energy localization, and the formation of persistent wave patterns. This knowledge is essential for applications across diverse fields, including material science, optics, fluid mechanics, and acoustics, supporting the design of innovative structures (Miao; Wang, 2024; Wu; Hu; Wang, 2023; Nikan; Avazzadeh; Rasoulizadeh, 2022; Lü *et al.*, 2017; Serkin; Hasegawa, 2002).

Numerical and experimental methods prevail in the framework of complicated models due to their high practicality and effectiveness in rather comprehensive problems, although costly in time and capital resources (Xin *et al.*, 2020; Valentine *et al.*, 2008). Remarkable benefits that complement such methods arise from exploring wave dynamics problems analytically. These include, for instance, avoiding convergence studies and cumulative integration errors, achieving optimization boosts, reducing computational assessment time, and bypassing the need to design and redesign experimental setups, which are often subject to scale limitations, test variability, and measurement uncertainties, despite the inherent exactness of analytical solutions (Momani; Odibat, 2006; Hassani; Hinton, 1998). Indeed, there are many evident drawbacks when performing analytical investigations of highly complicated models, such as encountering non-integrable equations or requiring excessive assumptions to simplify the model into a solvable version, which may lack the desired physical accuracy. In this sense, analytical, numerical, and experimental approaches complement each other, which justifies the importance of investigating wave dynamic problems using analytical methods (Wang; Pan, 2008; Bender; Orszag, 1999; Dimarogonas, 1996).

Symmetry analysis consolidates several sparsely developed analytical *ad hoc* techniques for solving differential equations, constituting a cornerstone of applied mathematics aimed at such purposes (Olver, 1986; Ovsiannikov, 1982). The mathematician Sophus Lie found a way to compute and exploit coordinate transformations that simplify and systematize the resolution of such differential equations, creating the so-called Lie theory. This theory deals with preserving invariant structures under the said transformations, called symmetries, referring to the idea of a symmetric object that looks the same from different perspectives (Hydon, 2000). Literature shows several symmetry applications to wave propagation problems (Raza *et al.*, 2023; Kumar *et al.*, 2022; Kumar; Rani, 2021; Wazwaz; Kaur, 2018; Tu *et al.*, 2016; Anco; Silva; Freire, 2015; Ames, 1985). On the other hand, scarce works address continuous vibrating structures within the symmetry view (Çelik *et al.*, 2021; Nunes; Silva; Ruiz, 2022).

Symmetry analysis is well-established in the mathematical literature as a powerful tool for solving complicated differential equations analytically. On the other hand, the engineering community has seldom applied such analysis. Given this fact, the limited number of analytical studies addressing intricate wave dynamics problems suggests a promising area for research. This gap highlights the potential of a study applying symmetry-based techniques to engineering wave problems, particularly for complex models. The sections in the sequence establish this thesis's goals and contributions and outline the division of chapters along this text.

1.1. GOALS

This thesis aims to apply symmetry analysis to study relevant wave dynamic problems in engineering, focusing on achieving analytical solutions to unusual vibrating structures and intricate wave propagation models. For that, the specific objectives are:

- Investigating vibrations and wave propagation phenomena in slender rod-like structures, where the associated complexity arises from addressed generic geometry non-uniformities and functionally graded material parameters.
- Addressing a comprehensive vibration analysis to thin vibrating rectangular plates.
- Exploring comprehensive wave propagation phenomena of the nonlinear generalized Calogero–Bogoyavlenskii–Schiff (gCBS) equation model.

1.2. CONTRIBUTIONS

The main contributions of this thesis are:

- Presenting a symmetry chapter with less dense mathematical language, aimed at fostering the interest of non-mathematician readers while offering a comprehensive toolkit for the analytical solution of differential equations.
- Computing exact mode shapes of non-uniform, functionally-graded elementary rods undergoing longitudinal or torsional vibrations assisted by symmetry analysis.
- Finding exact traveling wave solutions to the elastodynamic equation of non-uniform, functionally-graded elementary rods undergoing longitudinal or torsional vibrations facilitated by symmetry analysis.
- Providing the mode shape equation of uniform rectangular Kirchhoff plates with exact invariant solutions using symmetry analysis.

- Computing exact invariant solutions to the nonlinear gCBS equation using symmetry analysis.
- Sharing symmetry and equivalence computation algorithms implemented in Wolfram Mathematica language to the interested reader as a symbolic complement to the theoretical symmetry chapter.

1.3. THESIS OUTLINE

The chapters of this thesis, except for the final one, are self-contained and address the following topics:

Chapter 1 introduces the role of vibration analysis in structures, particularly relevant to recently developed metastructures with unconventional designs, restricting the study goal to analytical investigations driven by symmetry methods.

Chapter 2 describes symmetry analysis for differential equations, focusing on making these mathematically rigorous approaches more accessible and practical for engineering applications.

Chapter 3 brings computed exact mode shape expressions for elementary rods with various non-uniform geometries and functionally graded materials undergoing free longitudinal or torsional vibrations, obtained using an equivalent mode shape equation with fewer arbitrary parameters, then reduced via its symmetries and solved.

Chapter 4 provides exact solutions to the elastodynamic equation of rods employing classification via equivalence and nonlocal symmetry transformations, writing initial and boundary value problems as equivalent constant-coefficient wave equation problems admitting the classical d'Alembert solution, making it suitable for infinite, semi-infinite, and finite rods with diverse geometries and materials.

Chapter 5 studies the mode shape equation of uniform rectangular Kirchhoff plates applying equivalence studies from symmetry analysis to find the simplest model's representative equation and solve it using classical and nonclassical symmetry transformations, deriving exact invariant solutions to the mode shapes of thin plates.

Chapter 6 explores the Lie algebra of the gCBS equation, computing its admitted two-dimensional optimal system of subalgebras for reducing the equation into easier-to-solve ones, namely invariant equations, whose solutions immediately recover as exact wave solutions to the original equation.

Chapter 7 closes the text with its final remarks and discusses open investigations left along the work for future research.

1.4. WRITING TRANSPARENCY AND AUTHOR RESPONSIBILITY

Regarding transparency in English writing, Grammarly was used to identify potentially unclear sentences and improve the overall cohesion of this text.

The opinions, hypotheses, conclusions, or recommendations expressed in this material are the sole responsibility of the author and do not necessarily reflect the views of FAPESP.

7 FINAL REMARKS

This chapter concludes the studies presented herein in Section 7.1 and outlines potential future research in Section 7.2, prompted by questions and ideas that emerged during the development of this thesis.

7.1. CONCLUSIONS

This thesis applied symmetry methods to wave dynamics problems in continuous media and yielded a wide range of original exact solutions for them. The relevance of these solutions extends to qualitative and quantitative aspects of the investigated wave phenomena, such as identifying asymptotic behaviors, establishing benchmarks for and validating numerical methods, and accelerating structural optimization. From this, applying symmetry methods not only offered new insights into the exact behavior of complex wave phenomena but also provided a powerful and systematic framework for analytically tackling previously intractable problems in wave propagation. These contributions are expected to foster efficient wave dynamics applications in engineering and science.

Particularly, the provided symmetry analysis allowed computing several exact mode shapes and traveling-wave solutions for rods with non-uniform geometries and functionally graded materials. These modal and wave solutions complement each other in assessing the corresponding dynamics of such rods, offering them a comprehensive analytical study under free and forced vibrations in transient or steady-state motions. Meanwhile, thin plate structures have been provided with new invariant solutions within the scope of modal analysis, from where exact mode shapes for plates arise to effectively grasp its dynamics. Finally, a set of invariant solutions has been computed to the gCBS model, comprising new solitonic waves found analytically that elucidate the propagation of nonlinear waves in many kinds of continuous media. These results illustrate how the rich dynamics of these complex models can be assessed through analytical investigations.

For such investigations, the symmetry tools from enhanced group analysis helped explore equivalence transformations to yield better representative equations within the family of the given differential equation from the model. This led, as expected, to the study of equivalent, simpler equations with fewer arbitrary coefficients, either functions or constants. Then, the group analysis machinery assisted in identifying symmetry transformations that preserve the essential nature of original and equivalent problems while reducing the order of given ODEs or the number of independent variables of PDEs. These tools provide a systematic framework for uncovering deeper insights into the mathematical and physical nature of the problem. From classical and nonclassical point symmetries to nonlocal ones, exploiting symmetries of differential equations of these wave dynamics problems has led to their aimed analytical resolution.

Meanwhile, the required machinery for many symmetry computations comes at the cost of relying on symbolic manipulation software. The trade-off between exactness and the effort required to compute and solve symmetry equations must be considered depending on the desired application. Advantages such as the instantaneous assessment of an exact solution may be valuable during the early design stages, reducing computational costs and enhancing the reliability of predicted dynamics with valuable insights, whereas the complexity arising from very realistic modeling equations might preclude analytical approaches, given the number of variables and parameters involved.

7.2. FUTURE WORKS

Future work could explore additional vibration models of non-uniform functionally graded structures, such as Mindlin-Herrmann rods, Euler-Bernoulli and Timoshenko beams, and Kirchhoff and Mindlin plates (Hagedorn; DasGupta, 2007; Inman, 2013; Timoshenko; Krieger, 1959; Mei, 2013; Güven, 2013). Once these models contain high-order equations and systems of differential equations, they might benefit from the systematicity of reductions provided by symmetry methods. In addition, the high number of arbitrary parameters that these models for structures with varying geometries and material parameters might have encourages finding equivalence classes of best representative equations via equivalence transformations. Future studies aiming for a more theoretical standpoint could also delve into the algebraic structure of equivalence transformations, enabling successive reductions without the need for repeated equivalence analysis calculations.

Exact solutions, such as those provided in this thesis, might pave the way for comprehensive analytical investigations of metastructures, where variations in geometry and material properties play a fundamental role in shaping the expected, often unusual, dynamic behaviors. In this framework, solutions developed for periodic or continuously varying property structures offer valuable tools to explore, for instance, the design of structures with topological acoustic black holes to passively control vibrations, the optimization of damping-layer materials for mitigating residual vibrations in systems of precision subject to dynamic loading, and the study of hyperbolic metamaterials, known for their ability to localize vibration modes and exhibit robustness in adapting to structural modifications (Pelat *et al.*, 2020; Tang *et al.*, 2016; Ji *et al.*, 2023; Ding *et al.*, 2017; Yan; Cheng; Wang, 2018; Poddubny *et al.*, 2013).

Symmetries of initial and boundary value problems are not as well documented in the literature as expected, even though most bounded unsolved real-world problems would take some advantage of the symmetry approach. Even within traditional group analysis, there is no consensus among researchers. Some authors argue that the initial and boundary-value problem symmetries are confined to smaller subgroups of the Lie symmetry group of the governing differential equation (Bluman; Anco, 2002). Others argue that these symmetries may be entirely distinct from those of the original equation and must be constructed independently (Hydon, 2005). The situation becomes even more intricate when addressing enhanced group analysis, where checking whether the employed equivalence transformations degenerate the initial and boundary conditions falls out of the scope of even the most complete symmetry books. For this reason, one fundamental step in investigating more complex problems, which require more equivalence and symmetry reductions to be solved, is to expand knowledge in this area.

The recent growth in the symmetry community among physicists and engineers reflects the rising interest in applying symmetry methods to practical problems, such as the wave propagation ones brought in this thesis. It's worth emphasizing that the power of symmetry methods is not restricted to persistently seeking exact solutions. Numerical approaches assisted by symmetries follow dimension reductions, stability improvements, coarse meshes, and computational time optimizations (Kobilarov; Marsden; Sukhatme, 2010; Yong; Sun; Gao, 2021; Cheng; Frazzoli; LaValle, 2003; Bensoam; Carré, 2019). Embedding symmetry methods in machine learning approaches is a hot topic nowadays, for which feeding physics-informed neural networks with conservations laws or estimating parameters may restructure the contemporary use and spread the teaching of symmetry methods across new fields of knowledge (Liu *et al.*, 2024; Otto *et al.*, 2023; Zhang; Cai; Zhang, 2023; Alpar *et al.*, 2024; Zhang *et al.*, 2023).

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