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The Q-D algorithm for transforming series expansions into a corresponding continued fraction: an extension to cope with zero coefficients ☆

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Abstract

An algorithm for deriving a continued fraction that corresponds to two series expansions simultaneously, when there are zero coefficients in one or both series, is given. It is based on using the Q-D algorithm to derive the corresponding fraction for two related series, and then transforming it into the required continued fraction. Two examples are given.

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1. Introduction

Given two series expansions, one of each of the forms

$$\frac{\mu_0}{z} + \frac{\mu_1}{z^2} + \frac{\mu_2}{z^3} + \cdots + \frac{\mu_k}{z^{k+1}} + \cdots \tag{1}$$

and

$$-\mu_{-1} - \mu_{-2}z - \mu_{-3}z^2 - \cdots - \mu_{-k}z^{k-1} - \cdots , \tag{2}$$

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where each of the $\mu_k,\ k=0,\pm 1,\pm 2,\dots$, are real numbers, there are various techniques for transforming each series, or both series simultaneously, into a corresponding continued fraction. In particular it is well known that if the coefficients $\mu_k,\ k=0,\pm 1,\pm 2,\dots$, are such that the Hankel determinants $H_n^{(-n)}$ and $H_n^{(-n+1)}$ are different from zero for all $n \geqslant 1$, where

$$H_n^{(m)} = \begin{vmatrix} \mu_m & \mu_{m+1} & \cdots & \mu_{m+n-1} \\ \mu_{m+1} & \mu_{m+2} & \cdots & \mu_{m+n} \\ \vdots & \vdots & \cdots & \vdots \\ \mu_{m+n-1} & \mu_{m+n} & \cdots & \mu_{m+2n-2} \end{vmatrix} \tag{3}$$

for $n=1,2,3,\dots$ and $m=0,\pm 1,\pm 2,\dots$, then the two series can be transformed into the corresponding continued fraction

$$\frac{\mu_0}{z-\beta_1^{(0)}} - \frac{\alpha_2^{(0)} z}{z-\beta_2^{(0)}} - \frac{\alpha_3^{(0)} z}{z-\beta_3^{(0)}} - \frac{\alpha_4^{(0)} z}{z-\beta_4^{(0)}} - \cdots . \tag{4}$$

The correspondence is such that the nth convergent

$$\frac{A_n(z)}{B_n(z)} = \frac{\mu_0}{z-\beta_1^{(0)}} - \frac{\alpha_2^{(0)} z}{z-\beta_2^{(0)}} - \frac{\alpha_3^{(0)} z}{z-\beta_3^{(0)}} - \cdots - \frac{\alpha_n^{(0)} z}{z-\beta_n^{(0)}}, \tag{5}$$

is a ratio of polynomials of degree $n-1$ and n, respectively, and *fits* n terms of each series when expanded accordingly. That is

$$\frac{A_n(z)}{B_n(z)} = \begin{cases} \dfrac{\mu_0}{z} + \dfrac{\mu_1}{z^2} + \dfrac{\mu_2}{z^3} + \cdots + \dfrac{\mu_{n-1}}{z^n} + \text{lower order terms} \\ -\mu_{-1} - \mu_{-2} z - \mu_{-3} z^2 - \cdots - \mu_{-n} z^{n-1} + \text{higher order terms.} \end{cases} \tag{6}$$

The coefficients $\alpha_k^{(0)}$ and $\beta_k^{(0)},\ k=1,2,3,\dots$, can be expressed as ratios of the Hankel determinants $H_n^{(m)}$ defined by (3) and hence calculated from the series coefficients. Alternatively, provided that $\mu_k \neq 0$ for $k=0,\pm 1,\pm 2,\dots$, the continued fraction can be derived by means of the recurrence relations

$$\alpha_n^{(r)} = \beta_{n-1}^{(r+1)} + \alpha_{n-1}^{(r+1)} - \beta_{n-1}^{(r)},$$

$$\beta_n^{(r)} = \frac{\alpha_n^{(r)} \beta_{n-1}^{(r-1)}}{\alpha_n^{(r-1)}} \tag{7}$$

for $r=\dots,-2,-1,0,1,2,\dots$ and $n=2,3,4,\dots$, and with the starting values $\alpha_1^{(r)}=0$ and $\beta_1^{(r)}=\mu_r/\mu_{r-1}$ for $r=\dots,-2,-1,0,1,2,\dots$. This is the Q-D algorithm and the table of coefficients generated can provide many continued fractions that correspond to the two series, either separately or simultaneously, see [2] for instance. The convergents of all the continued fractions are two point Padé approximants for the series (1) and (2).

Clearly the Q-D algorithm cannot be used directly if one or more of the series coefficients are zero, since the associated values of $\beta_1^{(r)} = \mu_r/\mu_{r-1}$ will not be defined. This is not a problem if the determinant method is used of course. In this paper an algorithm is provided that, when used in conjunction with the Q-D algorithm, can provide the continued fraction (4) even when there are coefficients equal to zero in one or both of the series (1) and (2).

2. Deriving one continued fraction from another

Let the two series (1) and (2) be formally represented by $F(z)$ and $F^\star(z)$, respectively.

The series expansions of the function $K/(1+z)$ at $z=\infty$ and $z=0$ are given by

$$K\left(\frac{1}{z}-\frac{1}{z^2}+\frac{1}{z^3}-\cdots\right) \tag{8}$$

and

$$K(1-z+z^2-z^3+\cdots). \tag{9}$$

Let the two series expansions at $z=\infty$ and 0, formally represented by $G(z)$ and $G^\star(z)$, be

$$G(z)=F(z)-\frac{K}{1+z}=\frac{\gamma_0}{z}+\frac{\gamma_1}{z^2}+\frac{\gamma_2}{z^3}+\frac{\gamma_3}{z^4}+\cdots \tag{10}$$

and

$$G^\star(z)=F^\star(z)-\frac{K}{1+z}=-\gamma_{-1}-\gamma_{-2}z-\gamma_{-3}z^2-\gamma_{-4}z^3-\cdots\ . \tag{11}$$

The coefficients in these two later series are easily calculated by subtracting (8) and (9) from (1) and (2), respectively.

Let the continued fraction of the form (4) which corresponds to (10) and (11), in the same way as (4) does for (1) and (2), be

$$\frac{\gamma_0}{z-\delta_1^{(0)}}-\frac{\lambda_2^{(0)}z}{z-\delta_2^{(0)}}-\frac{\lambda_3^{(0)}z}{z-\delta_3^{(0)}}-\frac{\lambda_4^{(0)}z}{z-\delta_4^{(0)}}-\cdots \tag{12}$$

and the nth convergent of this continued fraction be

$$\frac{C_n(z)}{D_n(z)}=\frac{\gamma_0}{z-\delta_1^{(0)}}-\frac{\lambda_2^{(0)}z}{z-\delta_2^{(0)}}-\frac{\lambda_3^{(0)}z}{z-\delta_3^{(0)}}-\cdots-\frac{\lambda_n^{(0)}z}{z-\delta_n^{(0)}}. \tag{13}$$

The denominators of the rational functions (5) and (13) are monic polynomials of degree n. So let them be, respectively,

$$B_n(z)=z^n+b_{n,n-1}z^{n-1}+\cdots+b_{n,1}z+b_{n,0}$$

and

$$D_n(z)=z^n+d_{n,n-1}z^{n-1}+\cdots+d_{n,1}z+d_{n,0}.$$

It is easily seen that

$$b_{n,n-1}=b_{n-1,n-2}-(\beta_n^{(0)}+\alpha_n^{(0)}),\quad b_{n,0}=-\beta_n^{(0)}b_{n-1,0} \tag{14}$$

and

$$d_{n,n-1}=d_{n-1,n-2}-(\delta_n^{(0)}+\lambda_n^{(0)}),\quad d_{n,0}=-\delta_n^{(0)}d_{n-1,0}. \tag{15}$$

Now let us assume that there are zero coefficients in the series (1) and/or (2), but $\mu_0\mu_{-1} \neq 0$, and that the number K in the function $K/(1+z)$ is chosen so that there are no zero coefficients in the series (10) and (11).

Hence, from these coefficients, the continued fraction (12) can be derived using the Q-D algorithm and then the numbers $d_{n,n-1}$ and $d_{n,0}$, $n = 2,3,4,\ldots$, can be calculated.

Now

$$F(z) - \frac{A_n(z)}{B_n(z)} = \mathrm{O}(z^n), \quad F^\star(z) - \frac{A_n(z)}{B_n(z)} = \mathrm{O}\left(\frac{1}{z^{n+1}}\right),$$

$$G(z) - \frac{C_n(z)}{D_n(z)} = \mathrm{O}(z^n), \quad G^\star(z) - \frac{C_n(z)}{D_n(z)} = \mathrm{O}\left(\frac{1}{z^{n+1}}\right).$$

Hence, from these four later equations

$$\frac{C_n(z)}{D_n(z)} - \frac{A_n(z)}{B_n(z)} + \frac{K}{1+z} = \mathrm{O}\left(z^n, \frac{1}{z^{n+1}}\right)$$

or

$$\frac{(1+z)B_n(z)C_n(z) + KB_n(z)D_n(z) - (1+z)A_n(z)D_n(z)}{(1+z)B_n(z)D_n(z)} = \mathrm{O}\left(z^n, \frac{1}{z^{n+1}}\right),$$

where $\mathrm{O}(z^n, 1/(z^{n+1}))$ denotes the series expansion of the left-hand sides, in powers of z and $1/z$, respectively, begin with z^n and $1/z^{n+1}$, respectively.

Since the numerator of the left-hand side is a polynomial of degree $2n$ at most, while the denominator is of degree $2n+1$, it follows that the numerator consists of a single term, namely $R_n z^n$, where R_n is a real number. Hence

$$(1+z)\left\{\frac{C_n(z)}{D_n(z)} - \frac{A_n(z)}{B_n(z)}\right\} + K = \frac{R_n z^n}{B_n(z)D_n(z)}$$

and, similarly

$$(1+z)\left\{\frac{C_{n+1}(z)}{D_{n+1}(z)} - \frac{A_{n+1}(z)}{B_{n+1}(z)}\right\} + K = \frac{R_{n+1} z^{n+1}}{B_{n+1}(z)D_{n+1}(z)}.$$

Subtracting the first of these equations from the second and using the standard result concerning the difference between consecutive convergents of a continued fraction yields

$$\frac{zR_{n+1}}{B_{n+1}(z)D_{n+1}(z)} = \frac{R_n}{B_n(z)D_n(z)} + (1+z)\left\{\frac{\gamma_0\lambda_2^{(0)}\ldots\lambda_{n+1}^{(0)}}{D_{n+1}(z)D_n(z)} - \frac{\mu_0\alpha_2^{(0)}\ldots\alpha_{n+1}^{(0)}}{B_{n+1}(z)B_n(z)}\right\}$$

$$= \frac{R_n}{B_n(z)D_n(z)} + (1+z)\left\{\frac{u_{n+1}}{D_{n+1}(z)D_n(z)} - \frac{w_{n+1}}{B_{n+1}(z)B_n(z)}\right\},$$

where

$$u_{n+1} = \gamma_0\lambda_2^{(0)}\ldots\lambda_{n+1}^{(0)} = u_n\lambda_{n+1}^{(0)} \tag{16}$$

and

$$w_{n+1} = \mu_0 \alpha_2^{(0)} \ldots \alpha_{n+1}^{(0)} = w_n \alpha_{n+1}^{(0)} \tag{17}$$

with $u_1 = \gamma_0$ and $w_1 = \mu_0$. Thus

$$zR_{n+1}D_n(z)B_n(z) = R_nB_{n+1}(z)D_{n+1}(z) + (1+z)\{u_{n+1}B_{n+1}(z)B_n(z) - w_{n+1}D_{n+1}(z)D_n(z)\}. \tag{18}$$

The left-hand side is a polynomial of degree $2n+1$ with no constant term. The coefficient of z^{2n+2} on the right-hand side must therefore vanish, which means that

$$R_n = w_{n+1} - u_{n+1}. \tag{19}$$

Similarly, the constant term on the right-hand side of (18) vanishes. Substituting from (19) then yields

$$w_{n+1}\left\{1 - \frac{d_{n,0}}{b_{n+1,0}}\right\} = u_{n+1}\left\{1 - \frac{b_{n,0}}{d_{n+1,0}}\right\}. \tag{20}$$

Finally, equate coefficients of z^{2n+1} in (18), remembering that $B_n(z)$ and $D_n(z)$ are monic, and again, using (19) and replacing n by $n-1$ yields

$$w_{n+1} - u_{n+1} = w_n\{b_{n,n-1} - d_{n-1,n-2} - 1\} - u_n\{d_{n,n-1} - b_{n-1,n-2} - 1\}, \quad n \geqslant 1. \tag{21}$$

Eqs. (20) and (21), used alternately for $n = 1, 2, 3, \ldots$, together with the Eqs. (14), (15) and (17) can then generate the sequence $\alpha_2^{(0)}, \beta_2^{(0)}, \alpha_3^{(0)}, \beta_3^{(0)}, \ldots$.

Given the coefficients μ_i, $i = 0, \pm 1, \pm 2, \pm m$, such that $\mu_k = 0$ for some $k \neq 0, -1$, the following algorithm can be used to generate the continued fraction corresponding to the two formal series expansions (1) and (2).

Algorithm

- *Determination of the modified series for a convenient* K:
  For $i = \ldots, -2, -1, 0, 1, 2, \ldots$ do

  $$\gamma_i = \mu_i + (-1)^{i+1}K \neq 0.$$

- *Application of the* Q-D *algorithm*:
  For $r = \ldots, -2, -1, 0, 1, 2, \ldots$ do

  $$\lambda_1^{(r)} = 0 \quad \text{and} \quad \delta_1^{(r)} = \frac{\gamma_r}{\gamma_{r-1}}.$$

  Then, for $i = 2, 3, \ldots$ and for $r = \ldots, -1, 0, 1, \ldots$ do

  $$\lambda_i^{(r)} = \delta_{i-1}^{(r+1)} + \lambda_{i-1}^{(r+1)} - \delta_{i-1}^{(r)} \quad \text{and} \quad \delta_i^{(r)} = \frac{\lambda_i^{(r)}\delta_{i-1}^{(r-1)}}{\lambda_i^{(r-1)}}.$$

- *Determination of the coefficients* $\alpha_i,\ \beta_i,\ i=2,3,\dots$:
  Let

$$\alpha_1 = 0, \quad \beta_1 = \frac{\delta_1^{(0)}\mu_0}{\gamma_0 - K\delta_1^{(0)}},$$

$$w_1 = \mu_0, \quad u_1 = \gamma_0,$$

$$b_{0,-1} = 0, \quad d_{0,-1} = 0.$$

Then, for $i = 1,2,\dots$ do

$$b_{i,i-1} = b_{i-1,i-2} - (\alpha_i + \beta_i),$$

$$d_{i,i-1} = d_{i-1,i-2} - (\lambda_i^{(0)} + \delta_i^{(0)}),$$

$$u_{i+1} = u_i\lambda_{i+1}^{(0)},$$

$$d_{i+1,0} = -d_{i,0}\delta_{i+1}^{(0)},$$

$$w_{i+1} = u_{i+1} + w_i(b_{i,i-1} - d_{i-1,i-2} - 1) - u_i(d_{i,i-1} - b_{i-1,i-2} - 1),$$

$$b_{i+1,0} = \frac{d_{i,0}}{1 - \frac{u_{i+1}}{w_{i+1}}(1 - \frac{b_{i,0}}{d_{i+1,0}})},$$

$$\alpha_{i+1} = \frac{w_{i+1}}{w_i},$$

$$\beta_{i+1} = -\frac{b_{i+1,0}}{b_{i,0}}.$$

Remark. It is just as well possible to choose the modification $K/(1 \pm az)$ instead of $K/(1+z)$, and a more general algorithm can be introduced. However, this will not add any significant benefits to our objective of generating the continued fraction.

A simple illustration, for which alternate coefficients in both series expansions are zero, but for which the continued fraction can easily be derived by other means, illustrates the algorithm. It is the function

$$F(z) = \frac{1}{\sqrt{1+z^2}}.$$

The two series expansions are

$$\frac{1}{\sqrt{1+z^2}} = \frac{1}{z} - \frac{1}{2z^3} + \frac{3}{8z^5} - \frac{5}{16z^7} + \cdots$$

and

$$\frac{1}{\sqrt{1+z^2}} = 1 - \frac{z^2}{2} + \frac{3z^4}{8} - \frac{5z^6}{16} + \cdots$$

which is valid for $|z| < 1$.

Setting $K=2$ and subtracting the series expansions of $2/(1+z)$ from the above yields the required two series expansions, each with non-zero coefficients.

It is easily seen that the coefficients in the two series are symmetric in that

$$\gamma_{-k} = \gamma_{k-1}, \quad k = 1, 2, 3 \ldots .$$

That is the series are of the form

$$\frac{\zeta_0}{z} + \frac{\zeta_1}{z^2} + \frac{\zeta_2}{z^3} + \frac{\zeta_3}{z^4} + \cdots + \frac{\zeta_{k+1}}{z^k} + \cdots$$

and

$$\zeta_0 + \zeta_1 z + \zeta_2 z^2 + \zeta_3 z^3 + \cdots \zeta_k z^{k-1} + \cdots .$$

As a consequence of this symmetry, when using the Q-D algorithm, only half of the elements $\lambda_n^{(r)}$ and $\delta_n^{(r)}$ need to be calculated, namely those with non-negative values of r. This is because

$$\delta_n^{(-r)} = \frac{1}{\delta_n^{(r)}}, \quad \lambda_n^{(-r)} = \frac{\lambda_n^{(r)}}{\delta_n^{(r)}\delta_{n-1}^{(r)}}, \quad r = 0, 1, 2, \ldots .$$

These results can be proved using elementary transformations of continued fractions and the corresponding properties. It is seen in the derivation that $\delta_n^{(0)} = -1$ for all n and thus the corresponding continued fraction (12) has the form

$$\frac{\lambda_1^{(0)}}{z+1} \underset{-}{} \frac{\lambda_2^{(0)} z}{z+1} \underset{-}{} \frac{\lambda_3^{(0)} z}{z+1} \underset{-}{} \cdots .$$

The numerator coefficients $\lambda_k^{(0)}$ can be expressed explicitly in terms of k and they are

$$\lambda_{2k}^{(0)} = \frac{4k-7}{2(4k-3)}, \quad \lambda_{2k+1} = 1 - \lambda_{2k}^{(0)} = \frac{4k+1}{2(4k-3)}$$

with $\lambda_1^{(0)} = -1$. The continued fraction for $F(z) = 1/\sqrt{1+z^2}$ is then obtained by the algorithm above and it is

$$\frac{1}{\sqrt{1+z^2}} = \frac{1}{z+1} \underset{-}{} \frac{z}{z+1} \underset{-}{} \frac{z/2}{z+1} \underset{-}{} \frac{z/2}{z+1} \underset{-}{} \frac{z/2}{z+1} \underset{-}{} \frac{z/2}{z+1} \underset{-}{} \cdots .$$

This continued fraction can be obtained by other means, see [5], where it and some applications are discussed.

2.1. *The function* arccot(z)

The function arccot(z), $\cot^{-1}(z)$, has the two series expansions

$$\frac{\pi}{2} - z + \frac{z^3}{3} - \frac{z^5}{5} + \cdots + (-1)^k \frac{z^{2k-1}}{2k-1} + \cdots, \quad |z| < 1 \tag{22}$$

and

$$\frac{1}{z} - \frac{1}{3z^3} - \frac{1}{5z^5} - \frac{1}{7z^7} + \cdots + (-1)^k \frac{1}{(2k+1)z^{2k+1}} + \cdots, \quad |z| > 1. \tag{23}$$

As in the previous example, these series each have zero coefficients and the continued fraction of the form (4) can be obtained by the Q-D algorithm and the additional algorithm. However, in this example too we can exploit the partial symmetry of the coefficients in the two series to our advantage. These series, and the two series expansions of the amended function

$$\operatorname{arccot}(z) - \frac{K}{1+z},$$

are of the form

$$\frac{\mu_0}{z} + \frac{\mu_1}{z^2} + \frac{\mu_2}{z^3} + \frac{\mu_3}{z^4} + \cdots + \frac{\mu_{k+1}}{z^k} + \cdots$$

and

$$-\mu_{-1} - \mu_0 z - \mu_1 z^2 - \mu_2 z^3 - \cdots - \mu_{k-1} z^k - \cdots \ .$$

That is $\mu_k = \mu_{-k-2},\ k = 0, 1, 2, \ldots$.

As a consequence, the elements generated by the Q-D algorithm for such series satisfy

$$\beta_n^{(r)} = \frac{1}{\beta_n^{(-r-1)}}, \quad \alpha_n^{(r)} = \frac{\alpha_n^{(-r-1)}}{\beta_{n-1}^{(-r-1)}\beta_n^{(-r-1)}}, \quad r = 1, 2, \ldots \ .$$

Hence, in the table of coefficients

$$\begin{array}{cccccccc}
\cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots \\
\cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots \\
\beta_1^{(-4)} & \alpha_2^{(-4)} & \beta_2^{(-4)} & \alpha_3^{(-4)} & \beta_3^{(-4)} & \alpha_4^{(-4)} & \beta_4^{(-4)} & \cdots \\
\beta_1^{(-3)} & \alpha_2^{(-3)} & \beta_2^{(-3)} & \alpha_3^{(-3)} & \beta_3^{(-3)} & \alpha_4^{(-3)} & \beta_4^{(-3)} & \cdots \\
\beta_1^{(-2)} & \alpha_2^{(-2)} & \beta_2^{(-2)} & \alpha_3^{(-2)} & \beta_3^{(-2)} & \alpha_4^{(-2)} & \beta_4^{(-2)} & \cdots \\
\beta_1^{(-1)} & \alpha_2^{(-1)} & \beta_2^{(-1)} & \alpha_3^{(-1)} & \beta_3^{(-1)} & \alpha_4^{(-1)} & \beta_4^{(-1)} & \cdots \\
\hline
\beta_1^{(0)} & \alpha_2^{(0)} & \beta_2^{(0)} & \alpha_3^{(0)} & \beta_3^{(0)} & \alpha_4^{(0)} & \beta_4^{(0)} & \cdots \\
\beta_1^{(1)} & \alpha_2^{(1)} & \beta_2^{(1)} & \alpha_3^{(1)} & \beta_3^{(1)} & \alpha_4^{(1)} & \beta_4^{(1)} & \cdots \\
\beta_1^{(2)} & \alpha_2^{(2)} & \beta_2^{(2)} & \alpha_3^{(2)} & \beta_3^{(2)} & \alpha_4^{(2)} & \beta_4^{(2)} & \cdots \\
\beta_1^{(3)} & \alpha_2^{(3)} & \beta_2^{(3)} & \alpha_3^{(3)} & \beta_3^{(3)} & \alpha_4^{(3)} & \beta_4^{(3)} & \cdots \\
\beta_1^{(4)} & \alpha_2^{(4)} & \beta_2^{(4)} & \alpha_3^{(4)} & \beta_3^{(4)} & \alpha_4^{(4)} & \beta_4^{(4)} & \cdots \\
\cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots \\
\cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots
\end{array}$$

those above the line can be expressed in terms of those below the line, and vice versa. These relations are derived as follows.

The continued fraction

$$\frac{\mu_r}{z-\beta_1^{(r)}}-\frac{\alpha_2^{(r)}z}{z-\beta_2^{(r)}}-\frac{\alpha_3^{(r)}z}{z-\beta_3^{(r)}}-\frac{\alpha_4^{(r)}z}{z-\beta_4^{(r)}}-\cdots$$

corresponds to the two series

$$\frac{\mu_r}{z}+\frac{\mu_{r+1}}{z^2}+\frac{\mu_{r+2}}{z^3}+\frac{\mu_{r+3}}{z^4}+\frac{\mu_{r+4}}{z^5}+\cdots$$

and

$$-\mu_{r-1}-\mu_{r-2}z-\mu_{r-3}z^2-\mu_{r-4}z^3-\cdots\ .$$

If we replace z by $1/z$ in the continued fraction we obtain, after some manipulation,

$$\frac{-(\mu_r z)/\beta_1^{(r)}}{z-1/\beta_1^{(r)}}-\frac{(\alpha_2^{(r)}z)/(\beta_1^{(r)}\beta_2^{(r)})}{z-1/\beta_2^{(r)}}-\frac{(\alpha_3^{(r)}z)/(\beta_2^{(r)}\beta_3^{(r)})}{z-1/\beta_3^{(r)}}-\cdots\ .$$

and since $\beta_1^{(r)}=\mu_r/\mu_{r-1}$ this is

$$\frac{-\mu_{r-1}z}{z-1/\beta_1^{(r)}}-\frac{(\alpha_2^{(r)}z)/(\beta_1^{(r)}\beta_2^{(r)})}{z-1/\beta_2^{(r)}}-\frac{(\alpha_3^{(r)}z)/(\beta_2^{(r)}\beta_3^{(r)})}{z-1/\beta_3^{(r)}}-\cdots \tag{24}$$

which corresponds to the two series

$$-\mu_{r-1}-\frac{\mu_{r-2}}{z}-\frac{\mu_{r-3}}{z^2}-\frac{\mu_{r-4}}{z^3}-\cdots$$

and

$$\mu_r z+\mu_{r+1}z^2+\mu_{r+2}z^3+\mu_{r+3}z^4+\cdots\ .$$

Dividing (24) by $-z$ yields the continued fraction

$$\frac{\mu_{r-1}}{z-1/\beta_1^{(r)}}-\frac{(\alpha_2^{(r)}z)/(\beta_1^{(r)}\beta_2^{(r)})}{z-1/\beta_2^{(r)}}-\frac{(\alpha_3^{(r)}z)/(\beta_2^{(r)}\beta_3^{(r)})}{z-1/\beta_3^{(r)}}-\cdots\ , \tag{25}$$

corresponding to

$$\frac{\mu_{r-1}}{z}+\frac{\mu_{r-2}}{z^2}+\frac{\mu_{r-3}}{z^3}+\frac{\mu_{r-4}}{z^3}+\cdots \tag{26}$$

and

$$-\mu_r-\mu_{r+1}z-\mu_{r+2}z^2-\cdots\ . \tag{27}$$

Similarly, the continued fraction

$$\frac{\mu_{-r-1}}{z-\beta_1^{(-r-1)}}-\frac{\alpha_2^{(-r-1)}z}{z-\beta_2^{(-r-1)}}-\frac{\alpha_3^{(-r-1)}z}{z-\beta_3^{(-r-1)}}-\cdots \tag{28}$$

corresponds to the two series

$$\frac{\mu_{-r-1}}{z}+\frac{\mu_{-r}}{z^2}+\frac{\mu_{-r+1}}{z^3}+\frac{\mu_{-r+2}}{z^4}+\frac{\mu_{-r+3}}{z^5}+\cdots \tag{29}$$

and

$$-\mu_{-r-2}-\mu_{-r-3}z-\mu_{-r-4}z^2-\mu_{-r-5}z^3-\cdots\ . \tag{30}$$

But the series coefficients satisfy $\mu_r=\mu_{-r-2}$ for $r=0,1,2,\ldots$.
Hence, for $r\geqslant 0$, the series (26) and (27) become identical to (29) and (30), respectively.
It follows that the continued fractions (25) and (28) are identical and so

$$\beta_n^{(r)}=\frac{1}{\beta_n^{(-r-1)}},\quad \alpha_n^{(r)}=\frac{\alpha_n^{(-r-1)}}{\beta_{n-1}^{(-r-1)}\beta_n^{(-r-1)}},\quad r=0,1,2,\ldots\ . \tag{31}$$

The Q-D algorithm, abbreviated by the use of (31), can now be used to obtain the continued fraction for the function $\operatorname{arccot}(z)-K/(1+z)$. The continued fraction for $\operatorname{arccot}(z)$ is then obtained and the coefficients of the first ten convergents are as below.

$$\alpha_1^{(0)}=1,\qquad \beta_1^{(0)}=-0.6366197724,$$
$$\alpha_2^{(0)}=0.6366197724,\quad \beta_2^{(0)}=-1.070461461,$$
$$\alpha_3^{(0)}=0.5468626849,\quad \beta_2^{(0)}=-1.021048760,$$
$$\alpha_4^{(0)}=0.5197242291,\quad \beta_4^{(0)}=-1.007543872,$$
$$\alpha_5^{(0)}=0.5101710861,\quad \beta_5^{(0)}=-1.003285499,$$
$$\alpha_6^{(0)}=0.5061199276,\quad \beta_6^{(0)}=-1.001691558,$$
$$\alpha_7^{(0)}=0.5040866744,\quad \beta_7^{(0)}=-1.000985436,$$
$$\alpha_8^{(0)}=0.5029242687,\quad \beta_8^{(0)}=-1.000623423,$$
$$\alpha_9^{(0)}=0.5021889793,\quad \beta_9^{(0)}=-1.000394789,$$
$$\alpha_{10}^{(0)}=0.5016364377,\quad \beta_{10}^{(0)}=-1.000087202.$$

This continued fraction expansion holds in the complex plane cut from $-i$ to $+i$ along the unit circle and on the whole real and axis the tenth convergent will be accurate to the seventh decimal place. A discussion of the error estimates for the continued fractions corresponding to two series can be found in [4] and, for the expansion of Dawson's integral, in [1]. The related function

$$G(z)=\frac{\pi-4\tan^{-1}(z)}{\pi(1-z)}$$

has series expansions of the form

$$\frac{\mu_0}{z}+\frac{\mu_1}{z^2}+\frac{\mu_2}{z^3}+\frac{\mu_3}{z^4}+\cdots$$

and

$$\mu_0+\mu_1 z+\mu_2 z^2+\mu_3 z^3+\cdots$$

and so the coefficients in the partial denominators of the continued fraction are all equal to unity, as in the first example above. This was first observed by D. Drew, see [3].

The function defined in terms of the error function by

$$F(z) = \frac{\sqrt{\pi}}{2} e^{z^2}(1 - erf(z))$$

is another function that has suitable series expansions at $z = 0$ and at infinity, with zero coefficients in the latter. The continued fraction of form (4) can be obtained by the above technique.

Finally, continued fractions corresponding to one series only, whether it is of the form (1) or (2), can be obtained either as a by-product of the two series case or, if only one series exists, by modifying the techniques described above.

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