# On Computing Discriminants of Subfields of $\mathbb{Q}(\zeta_{p^r})$

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The conductor–discriminant formula, namely, the Hasse Theorem, states that if a number field $K$ is fixed by a subgroup $H$ of $\mathrm{Gal}(\mathbb{Q}(\zeta_n)/\mathbb{Q})$, the discriminant of $K$ can be obtained from $H$ by computing the product of the conductors of all characters defined modulo $n$ which are associated to $K$. By calculating these conductors explicitly, we derive a formula to compute the discriminant of any subfield of $\mathbb{Q}(\zeta_{p^r})$, where $p$ is an odd prime and $r$ is a positive integer. © 2002 Elsevier Science (USA)

*Key Words:* characters; conductors; cyclotomic fields; discriminants of number fields; Hasse Theorem.

## 1. INTRODUCTION

One of the most important invariants of an algebraic number field $K$ is its discriminant, denoted by $\mathrm{Disc}(K)$. In this paper, we are concerned with discriminants of subfields of $\mathbb{Q}(\zeta_{p^r})$ where $p$ is an odd prime and $r$ is a positive integer. For any number field $K \subset \mathbb{Q}(\zeta_{p^r})$, we obtain a closed-form formula to compute $\mathrm{Disc}(K)$ as a function of $p$ and $[K : \mathbb{Q}]$ only. For each divisor $d$ of $\phi(p^r)$, there is a unique subfield $K$ of $\mathbb{Q}(\zeta_{p^r})$ of degree $d$. Such field $K$ is fixed by exactly one subgroup $H$ of $\mathrm{Gal}(\mathbb{Q}(\zeta_{p^r})/\mathbb{Q})$. Our first task will be to characterize the conductors of Dirichlet characters (Lemma 3.2) and then to calculate them explicitly (Lemma 3.3). From these results, the formula for $\mathrm{Disc}(K)$ is obtained using the Hasse Theorem, which states that

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if $K$ is fixed by a subgroup $H$ of $\mathrm{Gal}(\mathbb{Q}(\zeta_n)/\mathbb{Q})$, the discriminant of $K$ is equal to the product of the conductors of all characters associated to $K$ [7].

## 2.  CHARACTERS AND CONDUCTORS

A character of a finite Abelian group $G$ is a homomorphism from $G$ into $\mathbb{C}^*$ [1, 7]. A Dirichlet character defined modulo $m$ is a multiplicative homomorphism $\chi : (\mathbb{Z}/m\mathbb{Z})^* \to \mathbb{C}^*$. If $m$ divides $n$, $\chi$ induces a character modulo $n$ by composition with the natural map $(\mathbb{Z}/n\mathbb{Z})^* \to (\mathbb{Z}/m\mathbb{Z})^*$. Therefore, we can regard the same character defined as either modulo $m$ or modulo $n$ since they are essentially the same map. It is convenient to choose $n$ minimal and call it the conductor of $\chi$, denoted by $f_\chi$. It is sometimes advantageous to think of Dirichlet characters as being characters of Galois groups of cyclotomic fields. If we identify $\mathrm{Gal}(\mathbb{Q}(\zeta_n)/\mathbb{Q})$ with $(\mathbb{Z}/n\mathbb{Z})^*$, then a Dirichlet character modulo $n$ is a Galois character [7]. In general, if $\chi$ is a character modulo $n$, then $\chi$ is a character of $\mathrm{Gal}(\mathbb{Q}(\zeta_n)/\mathbb{Q})$. If $K$ is the fixed field of the kernel of $\chi$, then $K \subseteq \mathbb{Q}(\zeta_n)$, and if $n$ is minimal, then $n = f_\chi$. The field $K$ is exclusively dependent on $\chi$ and is called the field belonging to $\chi$. More generally, if $X$ is a finite group of Dirichlet characters and $n$ is the least common multiple of the conductors of the characters in $X$, then $X$ is a subgroup of the characters of $\mathrm{Gal}(\mathbb{Q}(\zeta_n)/\mathbb{Q})$. If $H$ is the intersection of the kernel of these characters and $K$ is the fixed field of $H$, then $X$ is precisely the set of all homomorphisms from $\mathrm{Gal}(K/\mathbb{Q})$ to $\mathbb{C}^*$. The field $K$ is called the field belonging to $X$, and we have $[K : \mathbb{Q}] = $ order of $X$. If $X$ is cyclic and generated by $\chi$, then $K$ is precisely the same field belonging to $\chi$.

## 3.  PRELIMINARIES

In this section, we present the supporting lemmas to obtain the main result.

Given an odd prime $p$ and a positive integer $r$, $(\mathbb{Z}/p^r\mathbb{Z})^*$ is a cyclic multiplicative group [3]. By Galois Theory, given a divisor $d$ of $\phi(p^r) = (p - 1)p^{r-1}$, there is a unique subfield $K$ of $\mathbb{Q}(\zeta_{p^r})$ fixed by a subgroup of $(\mathbb{Z}/p^r\mathbb{Z})^*$ of order $d$ [5].

LEMMA 3.1.  *Let $p$ be an odd prime number, $r$ a positive integer, and $g$ an integer such that $\bar{g} = g \bmod p^r$ is a generator of $(\mathbb{Z}/p^r\mathbb{Z})^*$. Then for $0 < j \leqslant r$, $g^k \equiv 1 \pmod{p^j}$ if and only if $k \equiv 0 \pmod{(p - 1)p^{j-1}}$.*

*Proof.* If $g^k = 1 + p^j t$, then $g^{kp} = 1 + p^{j+1}t_1$ where $t$ and $t_1$ are integers. Repeating this reasoning, we have $g^{kp^{r-j}} = 1 + p^r t_{r-j}$ where $t_{r-j}$ is an integer. With this, if $g^k \equiv 1 \pmod{p^j}$, then $g^{kp^{r-j}} \equiv 1 \pmod{p^r}$ and therefore

$(p-1)p^{r-1}$ divides $kp^{r-j}$, that is, $k \equiv 0 \,(\mathrm{mod}\,(p-1)p^{j-1})$. Conversely, suppose $k \equiv 0 \,(\mathrm{mod}\,(p-1)p^{j-1})$. Then $(\mathbb{Z}/p^j\mathbb{Z})^*$ has order $(p-1)p^{j-1}$; since $g$ and $p$ are relatively prime, $g^k \equiv 1 \,(\mathrm{mod}\,p^j)$, concluding the proof.  ∎

LEMMA 3.2.  *Let n and m be positive integers and $\chi$ a Dirichlet character defined modulo n. The conductor of $\chi$ is m if and only if m is the least integer dividing n and the following condition holds true: $\forall a \in \mathbb{Z}$ with $(a,n)=1$, if $a \equiv 1 \,(\mathrm{mod}\,m)$, then $\chi(\bar{a}) = 1$.*

*Proof.*  Suppose $m$ is the conductor of $\chi$ and $a \in \mathbb{Z}$ is such that $(a,n)=1$. Since $a \equiv 1 \,(\mathrm{mod}\,m), (a,n) = 1$, and $\chi$ can be defined modulo $m$, then $(a,m) = 1$ and $\chi(\bar{a}) = 1$. Conversely, let $a$ and $b$ be relatively prime to $n$, and consequently, relatively prime to $m$. If $a \equiv b \,(\mathrm{mod}\,m)$, then $ab^{-1} \equiv 1 \,(\mathrm{mod}\,m)$ and thus $\chi(\overline{ab^{-1}}) = 1$, that is, $\chi(\bar{a}) = \chi(\bar{b})$. Since $m$ is the least divisor of $n$ with the above property, then by definition it is the conductor of $\chi$.  ∎

Let $p$ be an odd number, $r$ a positive integer, $g$ an integer such that $\bar{g} = g \bmod p^r$ is a generator of $(\mathbb{Z}/p^r\mathbb{Z})^*$, and $\chi : (\mathbb{Z}/p^r\mathbb{Z})^* \to \mathbb{C}^*$ a Dirichlet character. Given that there are exactly $n$ characters over an Abelian group of order $n$, there are $(p-1)p^{r-1}$ Dirichlet characters $\chi$ defined over $(\mathbb{Z}/p^r\mathbb{Z})^*$, and each character is completely determined by its image in $\bar{g}$ [7]. On the other hand, $1 = \chi(\bar{1}) = \chi(\bar{g}^{(p-1)p^{r-1}}) = \chi(\bar{g})^{(p-1)p^{r-1}}$, that is, $\chi(\bar{g})$ is a $(p-1)p^{r-1}$th root of unity [3]. With this, given a character $\chi$ defined modulo $p^r$, there is an integer $i$ where $0 \leqslant i < (p-1)p^{r-1}$ such that $\chi(\bar{g}) = \zeta_{(p-1)p^{r-1}}^i$. Considering the number of characters and the possibilities for the integer $i$, we can conclude that all Dirichlet characters defined modulo $p^r$ are of the form $\chi_i(\bar{g}) = \zeta_{(p-1)p^{r-1}}^i$, $i = 0, \ldots, (p-1)p^{r-1} - 1$.

LEMMA 3.3.  *With the above notation, let i be an integer such that $0 \leqslant i < (p-1)p^{r-1}$. Then $p^j = (i, p^r)$ if and only if the conductor $f_{\chi_i}$ of $\chi_i$ is $p^{r-j}$.*

*Proof.*  For $i = 0$, the result is straightforward. Suppose then that $i \neq 0$ and $p^j = (i, p^r)$. So, $i = p^j t$ for some positive integer $t$. Let $\mathscr{H} = \{\bar{g}^a \in (\mathbb{Z}/p^r\mathbb{Z})^*; g^a \equiv 1 \,(\mathrm{mod}\,p^{r-j})\}$. From Lemma 3.2, $\chi_i$ can then be defined modulo $p^{r-j}$ if and only if $\chi_i(x) = 1, \forall x \in \mathscr{H}$. By Lemma 3.1, $\mathscr{H} = \langle \bar{g}^{(p-1)p^{r-j-1}} \rangle$, and because $\chi_i(\bar{g}^a) = \zeta_{(p-1)p^{r-1}}^{ai}$, one has $\chi_i(\bar{g}^{(p-1)p^{r-j-1}}) = 1$. For the converse, suppose $\chi_i$ can be defined modulo $p^{r-j}$. Then $\chi_i(x) = 1, \forall x \in \mathscr{H}$, and, in particular, $\chi_i(\bar{g}^{(p-1)p^{r-j-1}}) = 1$. Since $\chi_i(\bar{g}^{(p-1)p^{r-j-1}}) = \zeta_{(p-1)p^{r-1}}^{i(p-1)p^{r-j-1}}$, there is an integer $t$ such that $i(p-1)p^{r-j-1} = (p-1)p^{r-1}t$, which implies that $i = p^j t$. In summary, $i = p^j t$ if and only if $\chi_i$ can be defined modulo $p^{r-j}$, which is equivalent to saying that the conductor of $\chi_i$ is $p^{r-j}$ if $p^j$ is the greatest power of $p$ which divides $i$, that is, $p^j = (i, p^r)$.  ∎

Given an integer $n$ and the field $L = \mathbb{Q}(\zeta_n)$, $L$ is a Galois extension of the field of rational numbers, with the Galois group isomorphic to $(\mathbb{Z}/n\mathbb{Z})^*$. From the Fundamental Theorem of Galois Theory [5], it follows that there exists a one-to-one correspondence between the subgroups of $(\mathbb{Z}/n\mathbb{Z})^*$ and the subfields of $L$. The isomorphism between the Galois group and $(\mathbb{Z}/n\mathbb{Z})^*$ is determined by associating the automorphism $\sigma_i$ of $L$ (taking $\zeta_n$ in $\zeta_n^i$) to each integer $i$ in $(\mathbb{Z}/n\mathbb{Z})^*$. Then we can define a character $\chi$ acting on the automorphisms of $L$, where $\chi(\sigma_i)$ means $\chi(i)$.

With the above notation, if $K$ is the subfield of $L$ fixed by $H$, we say that the character $\chi$ is associated to $K$ if $\chi(\sigma) = 1$ for all $\sigma$ in $H$. As an example, if $K = L$, the subgroup $H$ that fixes $K$ is the identity $\sigma_1$ of $L$. It is easy to see that $\chi(\sigma_1) = 1$ for all $\chi$ defined modulo $n$, and therefore all the characters are associated to $L$. Also, if $K$ is the field of rational numbers, then the only character associated to $K$ is the trivial one.

THEOREM 3.1 (Washington [7]).   *Let $n$ be a positive integer, $L = \mathbb{Q}(\zeta_n)$, $H$ a subgroup of the group of the automorphisms of $L$, and $K$ the subfield of $L$ fixed by $H$. Then the discriminant of the field $K$ is, up to sign, the product of the conductors of the characters defined modulo $n$ which are associated to $K$.*

## 4.   MAIN RESULT

Let $p$ be an odd prime number, $r$ a positive integer, and $L = \mathbb{Q}(\zeta_{p^r})$. Because $L$ is a Galois extension of the rationals whose Galois group is isomorphic to $(\mathbb{Z}/p^r\mathbb{Z})^*$, a cyclic group, there is a one-to-one correspondence between the subfields of $L$ and the divisors of $(p-1)p^{r-1}$, the degree of $L$. In the next theorem, we calculate the discriminant of any subfield $K$ of $L$ as a function of $p$ and its degree only. Since the degree of $K$ is a divisor of $(p-1)p^{r-1}$, we write $[K : \mathbb{Q}] = up^j$, where $u$ is a divisor of $p-1$, and $j \leqslant r-1$.

THEOREM 4.1.   *Let $K$ be a subfield of $\mathbb{Q}(\zeta_{p^r})$ with $[K : \mathbb{Q}] = up^j$ where $p$ does not divide $u$. Then $\mathrm{Disc}(K) = p^{u((j+2)p^j - \frac{p^{j+1}-1}{p-1})-1}$.*

*Proof.*   Observe that $G = \mathrm{Gal}(\mathbb{Q}(\zeta_{p^r})/\mathbb{Q})$ is a cyclic group of order $(p-1)p^{r-1}$. Let $g \in \mathbb{Z}$ be such that its class $\bar{g}$ is a generator of $(\mathbb{Z}/p^r\mathbb{Z})^*$. So if $[K : \mathbb{Q}] = up^j$, then the subgroup $H$ of $G$ that fixes $K$ is cyclic of order $(p-1)p^{r-j-1}/u$. If $\sigma_a$ is a generator of $H$, we conclude that a character $\chi$ defined modulo $p^r$ is associated to $K$ if and only if $\chi(\sigma_a) = 1$. Since the order of $a$ modulo $p^r$ is equal to the order of $H$, that is, $(p-1)p^{r-j-1}/u$, we can assume $a \equiv g^d \pmod{p^r}$ with $d = up^j$ without loss of generality. Therefore, given a character $\chi_i$ defined modulo $p^r$, $\chi_i(\bar{a}) = 1$, namely, $\chi_i$ is associated to $K$ if and only if $di \equiv 0 \pmod{(p-1)p^{r-1}}$ or, equivalently, if and only if

## TABLE I

Number of $\chi_i$ Associated to $K$ Having Conductor $f_{\chi_i}$

| $\ell$ | Number of $\chi_i$ | $f_{\chi_i} = p^{j+1-\ell}$ |
|---|---|---|
| 0 | $up^{j-1}(p-1)$ | $p^{j+1}$ |
| 1 | $up^{j-2}(p-1)$ | $p^j$ |
| $\vdots$ | $\vdots$ | $\vdots$ |
| $j-1$ | $up^0(p-1)$ | $p^2$ |
| $j$ | $u-1$ | $p$ |

$i = (p-1)p^{r-1}t/d$, $t = 0,\ldots,d-1$ (recall that $\chi_i(\bar{g}) = \zeta_{(p-1)p^{r-1}}^i$; thus $\chi_i(\bar{a}) = \zeta_{(p-1)p^{r-1}}^{di}$). Since $d = up^j$, $\chi_i(\bar{a}) = 1$ if and only if $i = \frac{p-1}{u}p^{r-j-1}t$ where $t = 0,\ldots,up^j-1$.

If $t = 0$, then $i = 0$; according to Lemma 3.3, $f_{\chi_i} = 1$. If $t \neq 0$, let $t = p^\ell t_k$, where $\ell$ is an integer in $[0,j]$, and $(t_k,p) = 1$. Note that for each $\ell \in [0,j-1]$, there are $up^{j-\ell-1}(p-1)$ elements $t_k$ in these conditions. From Lemma 3.3, the conductors of the corresponding $\chi_i$ are all equal to $p^{j+1-\ell}$. If $\ell = j$, there are $u-1$ elements $t_k$ with $(t_k,p) = 1$, and the conductors of the corresponding $\chi_i$ are all equal to $p$. Table I summarizes these results. The first column displays each possible value for $\ell$; the second, the number of (nontrivial) characters $\chi_i$ for which $i = \frac{p-1}{u}p^{r-j-1+\ell}t_k$ and $(t_k,p) = 1$; the third column lists the common conductor of these characters $\chi_i$. Note that except for the trivial character, all the others that are associated to $K$ are counted in Table I.

From Theorem 3.1, the discriminant of $K$ is, up to sign, equal to the product of the conductors of the characters $\chi_i$ that are associated to $K$. Using this and the results in Table I, we obtain

$$\mathrm{Disc}(K) = \prod_{i;\chi_i \text{ is assoc. to } K} f_{\chi_i} = p^\alpha,$$

where $\alpha$ is computed as the sum of the second column entries times $\log_p$ of the third column entries. Hence,

$$\alpha = u(p-1)((j+1)p^{j-1} + jp^{j-2} + \cdots + 2p^0) + u - 1$$

$$= \frac{u(p-1)}{p}\sum_{i=0}^{j}(i+1)p^i - \frac{u(p-1)}{p} + u - 1$$

$$= \frac{u(p-1)}{p}\frac{d}{dp}\left(\frac{p^{j+2}-1}{p-1}\right) + \frac{u}{p} - 1$$

$$= \frac{u(p-1)}{p}\left(\frac{(j+2)p^{j+1}(p-1) - (p^{j+2}-1)}{(p-1)^2}\right) + \frac{u}{p} - 1.$$

This implies that

$$\alpha = u\left((j+2)p^j - \frac{p^{j+2}-1}{p(p-1)} + \frac{1}{p}\right) - 1,$$

and therefore,

$$\alpha = u\left((j+2)p^j - \frac{p^{j+1}-1}{p-1}\right) - 1. \quad \blacksquare$$

COROLLARY 4.1 (Washington [7]). *Given positive integers p and r with p an odd prime, the discriminant of the cyclotomic field $\mathbb{Q}(\zeta_{p^r})$ is, up to sign, equal to $p^{(p-1)((r+1)p^{r-1}-(p^r-1)/(p-1))-1}$.*

COROLLARY 4.2. *If p is an odd prime and $K \subset \mathbb{Q}(\zeta_p)$, then $\mathrm{Disc}(K) = \pm p^{[K:\mathbb{Q}]-1}$.*

*Remark* 4.1.    Note that the main result in this paper, Theorem 4.1, was possible due to the following: Given an odd prime $p$, a positive integer $r$, and the cyclotomic field $F = \mathbb{Q}(\zeta_{p^r})$, the divisors of $\phi(p^r)$ are in a one-to-one correspondence with the subfields of $F$. Hence, for each $d = up^j$ where $j = 0, \ldots, r-1$, there is exactly one subfield $K$ of $\mathbb{Q}(\zeta_{p^r})$ with degree $d$. In this way, the discriminant of $K$ can be obtained as a function of its degree.

On the other hand, for more general cyclotomic extensions, say of the form $\mathbb{Q}(\zeta_n)$ where $n$ is any positive integer, there may be two or more subfields with a given degree that have different discriminants. In other words, the extension of the results in this paper depends on determining the other properties (besides the degree) which specify a given subfield of $\mathbb{Q}(\zeta_n)$ and therefore, its discriminant.

## ACKNOWLEDGMENTS

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