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**Contributions in Interval Optimization and
Interval Optimal Control**

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Post-Graduation in Mathematics

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*Contributions in Interval Optimization and
Interval Optimal Control*

Tese apresentada como parte dos requisitos para obtenção do título de Doutor em Matemática, junto ao Programa de Pós-Graduação em Matemática, do Instituto de Biociências, Letras e Ciências Exatas da Universidade Estadual Paulista “Júlio de Mesquita Filho”, Câmpus de São José do Rio Preto.

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*Dedicated to
my mother, Flor.*

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“Let nothing disturb thee;
Let nothing dismay thee:
All things pass;
God never changes.
Patience attains
All that it strives for.
He who has God
Finds he lacks nothing:
God alone suffices.”

Saint Teresa of Ávila (2002, p. 288), [62].

Neste trabalho, primeiramente, serão apresentados problemas de otimização nos quais a função objetivo é de múltiplas variáveis e de valor intervalar e as restrições de desigualdade são dadas por funcionais clássicos, isto é, de valor real. Serão dadas as condições de otimalidade usando a E -diferenciabilidade e, depois, a gH -diferenciabilidade total das funções com valor intervalar de várias variáveis. As condições necessárias de otimalidade usando a gH -diferenciabilidade total são do tipo KKT e as suficientes são do tipo de convexidade generalizada. Em seguida, serão estabelecidos problemas de controle ótimo nos quais a função objetivo também é com valor intervalar de múltiplas variáveis e as restrições estão na forma de desigualdades e igualdades clássicas. Serão fornecidas as condições de otimalidade usando o conceito de Lipschitz para funções intervalares de várias variáveis e, logo, a gH -diferenciabilidade total das funções com valor intervalar de várias variáveis. As condições necessárias de otimalidade, usando a gH -diferenciabilidade total, estão na forma do célebre Princípio do Máximo de Pontryagin, mas desta vez na versão intervalar.

Palavras Chave: Problemas de otimização intervalar, condições de tipo Karush–Kuhn–Tucker, condições suficientes, problemas de controle ótimo intervalar, Princípio do Máximo de Pontryagin local intervalar.

ABSTRACT

In this work, firstly, it will be presented optimization problems in which the objective function is interval-valued of multiple variables and the inequality constraints are given by classical functionals, that is, real-valued ones. It will be given the optimality conditions using the E -differentiability and then the total gH -differentiability of interval-valued functions of several variables. The necessary optimality conditions using the total gH -differentiability are of KKT -type and the sufficient ones are of generalized convexity type. Next, it will be established optimal control problems in which the objective function is also interval-valued of multiple variables and the constraints are in the form of classical inequalities and equalities. It will be furnished the optimality conditions using the Lipschitz concept for interval-valued functions of several variables and then the total gH -differentiability of interval-valued functions of several variables. The necessary optimality conditions using the total gH -differentiability is in the form of the celebrated local Pontryagin Maximum Principle, but this time in the intervalar version.

Keywords: Interval optimization problems, Karush–Kuhn–Tucker–type conditions, sufficient conditions, interval optimal control problems, interval local Pontryagin Maximum Principle.

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LIST OF SYMBOLS

Description

$\mathbb{R}$	Euclidean vectorial space;
$\mathbb{R}^n$	n –dimensional Euclidean vectorial space;
e_j	j –th canonical direction in $\mathbb{R}^n$;
$\mathbb{R}_+^n$	nonnegative orthant of $\mathbb{R}^n$;
$\mathring{\mathbb{R}}_+^n$	positive orthant of $\mathbb{R}^n$;
$\mathcal{K}_C(\mathbb{R}), \mathcal{K}_C$	interval space, set of all bounded closed intervals in $\mathbb{R}$;
$(\mathcal{K}_C)^n$	n –cartesian interval space, $n \in \mathbb{N}$;
$[\underline{a}, \bar{a}]$	element of $\mathcal{K}_C$ and such that $\underline{a} \leq \bar{a}$;
$\underline{a}$	real left extremum of the interval $[\underline{a}, \bar{a}]$;
$\bar{a}$	real right extremum of the interval $[\underline{a}, \bar{a}]$;
$[a, a]$	degenerate interval;
$\ll_{LU}$	order relation LU, where $\ll_{LU} \in \{\leq_{LU}, \leq_{LU}, \leq_{LU}, \leq_{LU}, <_{LU}\}$;
$M(x^*)$	set of Fritz–John nonzero multipliers associated to x^* ;
f	real–valued function of one variable or several variables and sometimes, if there is no confusion, it also means a vector–valued function of one variable or several variables;
$\begin{bmatrix} f_1 & f_2 & \dots & f_q \end{bmatrix}^\top$	vector function $f : \mathbb{R}^p \rightarrow \mathbb{R}^q$;
F	interval–valued function of one variable or several variables;
f'	classic derivative of a real–valued function of one variable;
f'_-, f'_+	classical left and right lateral derivative, respectively;
$\frac{\partial f}{\partial x_i}$	classical i –th partial derivative;
$\frac{\partial f}{\partial x_i^-}, \frac{\partial f}{\partial x_i^+}$	i –th left–partial and i –th right–partial derivative, respectively;

Description

$\frac{\partial f}{\partial x_i^\sharp}$	i -th partial derivative sharp, that means the classical i -th partial derivative, or the i -th left-partial derivative, or the i -th right-partial derivative;
df	classical diferencial function;
∇f	classical gradient;
$(\nabla f)_-, (\nabla f)_+$	left gradient and right gradient, respectively;
$\nabla_\sharp f$	gradient sharp, that means the classical gradient, or the left gradient, or the right gradient;
Jf	classical Jacobian;
Df	generalized Jacobian;
$D_x f$	partial generalized Jacobian with respect to x ;
F'_E	extremal derivative, E -derivative, of an interval-valued function of one variable or several variables;
F'_{gH}	generalized Hukuhara derivative, gH -derivative, of an interval-valued function of one variable or several variables;
$\frac{\partial_{gH} F}{\partial x_i}$	i -th partial interval-valued gH -derivative of F ;
$\frac{\partial_{gH} F}{\partial x_i^-}, \frac{\partial_{gH} F}{\partial x_i^+}$	i -th left-partial and i -th right-partial interval-valued gH -derivative of F , respectively;
$D_{gH} F$	interval-valued gH -differential function (or interval-valued total gH -derivative function) of F ;
$\nabla_{gH} F$	gH -gradient of an interval-valued function of several variables;
$f'(x; d)$	directional derivative of f at x in the direction d ;
$F'_{gH}(x; d)$	directional gH -derivative of F at x in direction d ;
$\langle a, b \rangle$	classical inner product between $a, b \in \mathbb{R}^n$;
$A + B$	addition between $A, B \in \mathcal{K}_C$;
$\lambda \cdot A$	product between $\lambda \in \mathbb{R}$ and $A \in \mathcal{K}_C$;
$A - B$	difference between $A, B \in \mathcal{K}_C$;
$A \otimes B$	product between $A, B \in \mathcal{K}_C$;
$A \ominus_{gH} B$	gH -difference between $A, B \in \mathcal{K}_C$;
$A \oplus B$	addition between $A, B \in \mathcal{K}_C^n$;
$\lambda \odot A$	product between $\lambda \in \mathbb{R}$ and $A \in \mathcal{K}_C^n$;
S°	polar cone of S ;
S^a	set of accumulation points of S ;
$\top$	transpose of a matrix;

Description

$\mathcal{L}$	Lebesgue subsets of a given interval $[a, b]$;
$\mathcal{B}^m$	Borel sets of $\mathbb{R}^m$;
$\mathcal{L} \times \mathcal{B}^m$	product σ -algebra;
$\text{Gr } U$	graph of U ;
$W^{1,1}([0, T]; \mathbb{R}^n), W^{1,1}$	space of all absolutely continuous functions $f : [0, T] \rightarrow \mathbb{R}^n$;
$\mathcal{M}([0, T]; \mathbb{R}^m)$	space of all Lebesgue-measurable functions $f : [0, T] \rightarrow \mathbb{R}^m$;
$L_\infty([0, T]; \mathbb{R}^m), L_\infty$	space of essentially bounded functions $f : [0, T] \rightarrow \mathbb{R}^m$;
$L_p([a, b]; \mathbb{R}^n), L_p$	space of n -tuples of measurable functions $f(t) = (f_1(t), \dots, f_n(t))$ defined on $[a, b]$ with summable p -th power, where $1 \leq p < +\infty$;
$C([0, T]; \mathbb{R}^n), C$	space of continuous functions $f : [0, T] \rightarrow \mathbb{R}^n$;
$ \cdot $	norm in $\mathbb{R}$;
$\ \cdot\ _{\mathbb{R}^n}$	norm in $\mathbb{R}^n$;
$\ \cdot\ _{W^{1,1}}$	norm in $W^{1,1}$;
$\ \cdot\ _{L_\infty}$	norm in L_∞ ;
ess sup	essential supremum;
$\ \cdot\ _C$	norm in C ;
$\ \cdot\ _{C \times L_\infty}$	norm in $C \times L_\infty$;
$\ \cdot\ _{\mathcal{K}_C}$	quasi-norm in $\mathcal{K}_C$;
N_S^P	proximal normal cone to S ;
N_S	limiting normal cone to S ;
$\overline{T}_S$	Clarke tangent cone to S ;
∂f	limiting Mordukhovich's subdifferential of f ;
$\partial_x f$	partial limiting Mordukhovich's subdifferential of f with respect to x ;
$\text{dom } f$	domain of f ;
$\text{epi } f$	epigraph of f ;
$\text{co}(S)$	convex hull of S ;
a.e.	almost everywhere. A property holds almost everywhere if the set for which the property does not hold has measure zero (in the sense of Lebesgue);
w.r.t.	with respect to;
$\text{int}(S)$	interior of the set S ;
$\text{int}(A)$	interior of the n -tuple of intervals A ;
$\text{cl}(S)$	closure of the set S ;

Description

0	element 0 of $\mathbb{R}$;
$\mathbf{0}$	element $(0, \dots, 0)$ of $\mathbb{R}^n$;
$[0, 0]$	zero interval of $\mathcal{K}_C$;
$\mathbb{B}_{\mathbb{R}^n}$	open unit ball of center $\mathbf{0}$ and radius 1 in $\mathbb{R}^n$;
$\overline{\mathbb{B}_{\mathbb{R}^n}}$	closed unit ball of center $\mathbf{0}$ and radius 1 in $\mathbb{R}^n$;
$\mathbb{B}_{\mathbb{R}^n}(x, \epsilon)$	open ball of center x and radius ϵ in $\mathbb{R}^n$;
$\overline{\mathbb{B}_{\mathbb{R}^n}}(x, \epsilon)$	closed ball of center x and radius ϵ in $\mathbb{R}^n$;
$\mathbb{B}_{C \times L_\infty}((x, u), \epsilon)$	open ball of center (x, u) and radius ϵ in $C \times L_\infty$;
$\Delta_\beta f$	difference operator of f ;
$\mathcal{F}_\beta$	β –admissible directions set;
d_S	distance function associated with the set S ;
d_H	Pompeiu–Hausdorff metric;
$\text{len}(A)$	length of the interval $A = [\underline{a}, \bar{a}]$;
a^C	center of the interval $A = [\underline{a}, \bar{a}]$;
a^R	radius of the interval $A = [\underline{a}, \bar{a}]$;
$(a^C; a^R)$	center–radius representation of the interval $A = [\underline{a}, \bar{a}]$;
$(0; 0)$	center–radius representation of the interval $[0, 0]$;
$[\underline{a} \vee \bar{a}]$	means that $\underline{a}$ and $\bar{a}$ are the extremum of the interval, but $\underline{a} \leq \bar{a}$ is not necessarily satisfied;
E	topological linear space;
E'	topological dual space;
$\mathcal{K}$	cone;
$\mathcal{K}^*$	dual cone;
S^*	annihilator of $S \subset E'$.

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CHAPTER 1

INTRODUCTION

The theory of interval optimization problems and interval optimal control problems, in which the objective functions are interval-valued and possibly of multiple variables, plays an increasingly important role since such problems deal with numerically inexact data and uncertain environments, they allow developments of physical theories as well as descriptions of real-life situations, and they have a large number of applications.

For instance, consider a portfolio management problem and assume that there are 2 investment, see Osuna-Gómez et al., [65].

Denote the return of the j -th investment as R_j , $j \in \{1, 2\}$, and the proportion of the total investment funds to this investment as x_j , and let us say that

$$x_1 + x_2 = 1 \text{ million dollars.}$$

Since R_j 's vary due to uncertainties, these are assumed to be the random variables which can be represented by the average vector

$$\gamma = (\gamma_1, \gamma_2),$$

where γ_j is the average rate for R_j , and the covariance matrix is denoted by

$$\sigma_{jk},$$

where σ_{jk} is the covariance between R_j and R_k with $\sigma_{jk} = \sigma_{kj}$.

The average return associated with the portfolio $x = (x_1, x_2)$ is given by

$$\gamma_1 x_1 + \gamma_2 x_2, \quad \text{which it is desirable to be as much as possible!}$$

and the risk of investment can be formulated by

$$f(x) = \frac{1}{2} \sum_{j=1}^2 \sum_{k=1}^2 \sigma_{jk} x_j x_k, \quad \text{which it is desirable to be as little as possible!}$$

From a practical point, it is difficult to specify the coefficients, γ_j and σ_{jk} . But, there are cases where one can estimate the upper and lower bound of the coefficients.

Then, the *nonlinear multiobjective interval-valued optimization* is:

$$(IOP) \begin{cases} \text{minimize} & \left(-([\underline{\gamma}_1, \bar{\gamma}_1]x_1 + [\underline{\gamma}_2, \bar{\gamma}_2]x_2), \sum_{j=1}^2 \sum_{k=1}^2 [\underline{\sigma}_{jk}, \bar{\sigma}_{jk}]x_j x_k \right) \\ \text{subject to} & x_1 + x_2 = 1 \text{ million dollars,} \\ & x_1, x_2 \geq 0, \end{cases}$$

$$\Leftrightarrow \begin{cases} \text{minimize} & F(x) = (F_1(x), F_2(x)) \\ \text{subject to} & -x \leq 0 \\ & x - 1 \leq 0, \\ & x \in \mathbb{R}, \end{cases}$$

where F_1 and F_2 are interval-valued functions (wich will be explained later) defined by $F_1(x) = [\underline{f}_1(x), \bar{f}_1(x)]$, $F_2(x) = [\underline{f}_2(x), \bar{f}_2(x)]$, and

$$\begin{aligned} x &= x_1, & x_2 &= 1 - x, \\ \underline{f}_1(x) &= (\bar{\gamma}_2 - \bar{\gamma}_1)x - \bar{\gamma}_2, & \bar{f}_1(x) &= (\underline{\gamma}_2 - \underline{\gamma}_1)x - \underline{\gamma}_2, \\ \underline{f}_2(x) &= (\underline{\sigma}_{11} - 2\underline{\sigma}_{12} + \underline{\sigma}_{22})x^2 + 2(\underline{\sigma}_{12} - \underline{\sigma}_{22})x - \underline{\sigma}_{22}, \end{aligned}$$

and

$$\bar{f}_2(x) = (\bar{\sigma}_{11} - 2\bar{\sigma}_{12} + \bar{\sigma}_{22})x^2 + 2(\bar{\sigma}_{12} - \bar{\sigma}_{22})x - \bar{\sigma}_{22}.$$

Now, consider the example of the temperature control of an oven where the conditions for control existence and uniqueness of solutions are satisfied (namely, the compactness of the admissible control set, the convexity and continuity of the cost functional regarding the control), see Campos, [12], and Campos et al., [13].

A certain material goes through a sequence of five ovens. Denote $x_0 = 40^\circ C$ the

initial temperature of the material, x_k , $k \in \{1, \dots, 5\}$, the interval state that represents the material temperature at the exit of the oven k , and u_{k-1} , $k \in \{1, \dots, 5\}$, the interval control that represents the oven temperature at the stage k . The interval difference equation is given by

$$x_{k+1} = [0.2, 0.8] \otimes x_k + 0.5 \cdot u_k, \quad k \in \{0, \dots, 4\}.$$

For the analyzed interval model, the loss of the material temperature at the exit of the oven from one stage to another is uncertain and is represented by the interval $[0.2, 0.8]$ instead of a real and fixed number. This interval shows a great uncertainty regarding the loss of the material temperature and may come from the lack of knowledge of the modeler.

The objective of the problem is to get x_5 as close as possible to, let us say, the temperature $T = 150^\circ C$, spending the least amount of energy, and where the interval cost functional has the form

$$\text{minimize } C(x, u) = 500 \cdot (x_5 - [150, 150])^2 + \sum_{k=0}^4 u_k^2.$$

To get x_5 to the temperature $T = 150^\circ C$ means that the distance between them is “small” according to the definition of the distance between two intervals given in Chalco-Cano et al., [15], for example. The value 500 that appears in the objective functional represents a penalty for the model. In addition, a lower cost for the objective functional is expected if the loss of the material temperature is small.

Then, the *interval optimal control problem* is given by

$$(IOCP) \begin{cases} \text{minimize } C(x, u) = 500 \cdot (x_5 - [150, 150])^2 + \sum_{k=0}^4 u_k^2 \\ \text{subject to } x_{k+1} = [0.2, 0.8] \otimes x_k + 0.5 \cdot u_k, & k \in \{0, \dots, 4\}, \end{cases}$$

where x_k is the state variable with initial condition equals to $T = 40^\circ C$, u_k is the control variable and $C(x, u)$ is the interval cost functional.

In the last decades, the interval optimization and the interval optimal control theory have been investigated by many authors, see, e. g., Ahmad et al., [1]; Chalco-Cano et al., [16]; Chalco-Cano et al., [17]; Chanas and Kuchta, [21]; Costa et al., [23]; Inuiguchi and Kume, [38]; Ishibuchi and Tanaka, [39]; Jiang et al., [41]; Montemanni and Gambardella, [56]; Osuna-Gómez et al., [64]; Osuna-Gómez et al., [65]; Pal et al., [66], [67]; Singh et al., [77]; Stefanini and Arana-Jiménez, [81]; Wu, [85], [87]; Zhang et al., [88]; and the references therein.

Necessary optimality conditions for classical optimal control problems are present in the well known Pontryagin Maximum Principle, which is a generalization of the classic Euler–Lagrange and Weierstrass conditions. This principle gives first order conditions which brings the characterization of optimal (extremal) control processes candidates, see e. g., Athans and Falb, [3]; Bellaassali and Jourani, [7]; Bonnel and Kaya, [10]; Clarke, [22]; Dubovitskii and Milyutin [27], [28]; Gamkrelidze, [35]; Girsanov, [36]; Kien et al., [44]; Mordukhovich, [58], [59]; Pinch, [68]; Pontryagin et al., [69], [70]; Vinter, [83]; Zhu, [89]; among others. Now, regarding necessary optimality conditions for interval optimal control problems that are in the Pontryagin Maximum Principle form, are seldom at hand in the literature. Nonetheless, it can be found necessary optimality conditions for fuzzy optimal control problems, see e. g., Fard et al., [29]; Fard and Salehi, [30]; Fard and Zadeh, [31]; Farhadinia, [32], [33]; Najariyan and Farahi, [60], [61]; Leal, [46], [47]; Soolaki et al., [78], [79]; and its references.

On the one hand, Farhadinia, [32], studied necessary optimality conditions for fuzzy variational problems using the fuzzy differentiability concept of Buckley and Feuring, see [11], but this work was generalized by Fard et al., [29]; Fard and Salehi, [30]; Fard and Zadeh, [31]. In Fard et al., [29], it was presented the fuzzy Euler–Lagrange condition for fuzzy constrained and unconstrained variational problems under the generalized Hukuhara differentiability of several variables. Fard and Salehi, [30], investigated fuzzy fractional Euler–Lagrange equations for fuzzy fractional variational problems using their generalized fuzzy fractional Caputo–type derivatives. In Fard and Zadeh, [31], it was used the α –differentiability concept, obtained and extended the fuzzy Euler–Lagrange condition. Farhadinia, [33], derived the Pontryagin Minimum Principle for fuzzy optimality control problems based on the concepts of differentiability and integrability of a fuzzy mapping parameterized by the left and right–hand functions of its α –level sets. In Najariyan and Farahi, [60], it was used a procedure to solve linear time invariant dynamical systems with fuzzy parameters, and they applied it to solve a fuzzy optimal controlled system by using the α –cuts and the Pontryagin Maximum Principle of Pinch, [68]. Najariyan and Farahi, [61], studied and solved fuzzy linear time varying control systems and fuzzy optimal control problems, with fuzzy boundary conditions governed by fuzzy differential equations, using the Heaviside functions concept, the generalized Hukuhara derivative, and the Pontryagin Maximum Principle of Pinch, [68]. In Leal, [46], it was considered interval optimal control problems and given optimality conditions using the extremal differentiability, the gH –differentiability for several variables, the λ –single–level differentiability for several variables, and assumptions of convexity. Soolaki et al., [78], studied

fuzzy problems with boundary conditions and fuzzy problems with holonomic constraints, and obtained Euler–Lagrange equations and transversality conditions. In Soolaki et al., [79], using the Pontryagin Maximum Principle of Pinch, [68], a form of the Hamiltonian, and the generalized Hukuhara differentiability, it was obtained fuzzy solutions by solving a system of differential equations.

On the other hand, Ahmad et al., [1]; Chalco-Cano et al., [16]; Chalco-Cano et al., [17]; Osuna-Gómez et al., [64]; Singh et al., [77]; Stefanini and Arana-Jiménez, [81]; Wu, [85], [87]; Zhang et al., [88], studied optimality conditions for interval optimization problems in several variables. For instance, in Wu, [85], [87], it is presented sufficient optimality conditions using the Hukuhara differentiability and convexity assumptions. In Zhang et al., [88], it is also presented sufficient optimality conditions using the same concept of differentiability, but under invexity assumptions.

Since the Hukuhara differentiability is restrictive (see Chalco-Cano et al., [20], and Diamond, [26]), the authors Ahmad et al., [1]; Chalco-Cano et al., [16]; Chalco-Cano et al., [17]; Osuna-Gómez et al., [64]; Singh et al., [77], have been obtained optimality conditions using the concept of continuously generalized Hukuhara differentiability for several variables (see Chalco-Cano et al., [17]) or gH –differentiability for several variables (see Chalco-Cano et al., [16]; Osuna-Gómez et al., [64]). In Chalco-Cano et al., [17]; Singh et al., [77], sufficient conditions are obtained through convexity concepts. In Ahmad et al., [1], sufficient conditions are given, but with invexity and pre–invexity concepts. In Chalco-Cano et al., [16], sufficient conditions are obtained using concepts of pseudoinvexity. In Osuna-Gómez et al., [64], necessary and sufficient conditions are given, but for the unconstrained case, where the sufficient conditions are obtained using concepts of pseudoinvexity different from Chalco-Cano et al., [16].

Despite the use of the continuously generalized Hukuhara differentiability has the purpose to deal with interval optimization problems using a less restrictive differentiability concept, this concept is also restrictive.

In Stefanini and Arana-Jiménez, [81], it is presented a new concept of differentiability: the total generalized Hukuhara differentiability for interval–valued functions with several variables, which extends the generalized Hukuhara differentiability given in Stefanini and Bede, [82], and is more convenient than the concept of continuously gH –differentiability of Ahmad et al., [1]; Chalco-Cano et al., [16]; Chalco-Cano et al., [17]; Osuna-Gómez et al., [64]. Stefanini and Arana-Jiménez, [81], gave optimality conditions for interval optimization problems, where they used the linear independence constraint qualification

for the necessary conditions and the convexity concept for the sufficient ones.

This Thesis is focused mainly in the use of:

1. the total and directional generalized Hukuhara differentiability, a more suitable concept, given by Stefanini and Arana-Jiménez, [81];
2. the positive linear independence constraint qualification or Mangasarian–Fromovitz constraint qualification (introduced by Mangasarian and Fromovitz, [54], and used by many authors such as Davis, [24]; Qi and Wei, [71]; Robinson, [74]; Robinson and Meyer, [75]; among others) and the positive linear independence regularity condition (an adaptation of Maciel et al., [52]). The Mangasarian–Fromovitz constraint qualification is more general than the linear independence constraint qualification used in Stefanini and Arana-Jiménez, [81], but the positive linear independence regularity condition is not comparable with the linear independence constraint qualification, since the positive linear independence regularity condition is not a constraint qualification, the objective functions are present in the positive linear independence regularity condition;
3. generalized convexity;
4. Dubovitskii–Milyutin approach, see Girsanov, [36].

The main purposes of this thesis are present optimality conditions for interval optimization problems and interval optimal control problems, both with a single interval-valued objective function of several variables.

This Thesis is organized in the following sequence.

In order to reach our important results, it will be required some notations and mathematical background that will be used throughout the Thesis: the optimality for multi-objective programming problems: the Karush–Kuhn–Tucker conditions; the optimality for multi-objective control problems: the Maximum Principle and the *MP*–pseudoinvexity; the interval space $\mathcal{K}_C$; the limit, the continuity, the differentiability, and the integrability of interval-valued functions; classical integral equations; interval optimization problems; topological properties, cones, and dual cones. These are in Chapter 2.

In the third chapter, it will be considered the interval optimization problem with inequality constraints. It will be shown that to determining the *LU*–solutions of an

interval optimization problem is equivalent to determining the solutions of a classical bi-objective optimization problem. Moreover, using the concept of E -differentiability, it will be given necessary conditions of Fritz John and Karush–Kuhn–Tucker–types.

The fourth chapter is dedicated to present an interval optimal control problem. In this chapter, it will be used the Lipschitz concept of an interval-valued function of several variables. It will be presented the weak (strong) optimal LU -processes, the optimal LU -processes, and their local $W^{1,1}$ and L_∞ definitions, respectively. Next, the Hamilton function, and the (locally) Lipschitz continuous concept of interval-valued functions it will be given. It will be shown that to determining the optimal LU -processes of an interval optimal control problem is equivalent to determining the solutions of a classical bi-objective control problem. Therefore, the necessary and sufficient optimality conditions for the weak optimal LU -processes are easily presented.

In the fifth chapter, it will be used the total and directional generalized Hukuhara differentiability. The necessary optimality conditions are Fritz John-type, KKT -type and strict KKT -type, and the sufficient optimality conditions are of generalized convexity type. To obtain the necessary optimality conditions it will be used the positive linear independence constraint qualification and the positive linear independence regularity condition. Considering the sufficient ones, it will be given definitions of weak (strong) KKT -interval and KKT -interval pseudoinvexities, and with these it will be shown that KKT -solutions are weak (strong) LU -solutions and LU -solutions, respectively. Also, it will be given definitions of strict KKT -interval and partial- $\underline{LU}(\overline{LU})$ strict KKT -interval pseudoinvexities, and with these it will be shown that strict KKT -solutions are LU -solutions and $\underline{LU}(\overline{LU})$ -solutions, respectively. Furthermore, it will be given the converses of each one of these results. The $\underline{LU}$ and $\overline{LU}$ partial order relation given here, are adaptations of Arana-Jiménez and Neto, [2], but are new in interval optimization. Also, the constraint qualification that is used in this chapter is more general than the constraint qualification used in Stefanini and Arana-Jiménez, [81], so, the $KKKT$ -types optimality conditions are original in the interval Theory. As far as it is known, the pseudoinvexities defined here, are similar to the pseudoinvexities of Chalco-Cano et al., [16], but in Chalco-Cano et al., [16], it is used different assumptions such as the differentiability. Finally, the pseudoinvexities of this chapter, lead to get sufficient optimality conditions under more general assumptions than those used in Stefanini and Arana-Jiménez, [81], since the authors Stefanini and Arana-Jiménez of [81], used just convexity.

The sixth chapter, using the total and directional generalized Hukuhara differentia-

bility with several variables, is devoted to show an interesting original result for interval optimization problems with abstract constraints. It will be defined optimal LU –solution for the problem in question. Next, it will be considered some basic assumptions, definitions, and properties of them. It will be described important cones, and it will be presented some properties. Using well known results of the Theory of extremum problems, based on the Dubovitskii–Milyutin formalism (see Girsanov, [36]), it will be achieved a desired result.

In the seventh chapter, it will be obtained necessary optimality conditions for interval optimal control problems in the Lagrange form using again the total and directional generalized Hukuhara differentiability, for problems with several variables. First, it will be given the optimal LU –process concepts for the considered problems: with right endpoint fixed and then with free right endpoint. Next, it will be given some basic assumptions, definitions, and properties of them. It will be characterized important cones and it will be given some properties. Using again results of the Theory of extremum problems, it will be obtained an interval local Pontryagin Maximum Principle. All this study is based on the Dubovitskii–Milyutin formalism given in Girsanov, [36]. The originality resides in the fact that in the Dubovitskii–Milyutin formalism, the objective functional must take values in a topological linear space, which is not the case of the interval control problem. Therefore, it is necessary to define directions of decrease for interval–valued functions and characterize geometrically the optimal solutions, for then apply the Lemma of Dubovitskii–Milyutin.

Finally, the eighth chapter is committed to give general conclusions of this Thesis and some suggestions of future research.

CHAPTER 2

PRELIMINARIES

This chapter is dedicated to set notations, give key definitions, and mention important results, in order to reach our important results.

2.1 Optimality for multi-objective programming problems: The KKT conditions

The following conventions for equality and inequalities are assumed:

Let $\mathbb{R}^p$ be the p -dimensional Euclidean space. Let $x = (x_1, \dots, x_p)$ and $y = (y_1, \dots, y_p) \in \mathbb{R}^p$,

- (i) $x = y$ means $x_i = y_i$ for all $i \in \{1, \dots, p\}$;
- (ii) $x \leq y$ means $x_i \leq y_i$ for all $i \in \{1, \dots, p\}$;
- (iii) $x \leq y$ means $x \leq y$ and $x \neq y$, i.e., $x_i \leq y_i$ for all $i \in \{1, \dots, p\}$ and there exist $j \in \{1, \dots, p\}$ such that $x_j < y_j$ (that is, at least one strict inequality is satisfied);
- (iv) $x \leq_j y$ means $x_i \leq y_i$ for all $i \in \{1, \dots, p\}$ and $x_j < y_j$, with $j \in \{1, \dots, p\}$;
- (v) $x < y$ means $x_i < y_i$ for all $i \in \{1, \dots, p\}$.

The *nonnegative orthant* of $\mathbb{R}^p$ is denoted by

$$\mathbb{R}_+^p := \{y \in \mathbb{R}^p \mid 0 \leq y_i \text{ for } i \in \{1, \dots, p\}\}.$$

The *positive orthant* of $\mathbb{R}^p$ is denoted by

$$\mathring{\mathbb{R}}_+^p := \{y \in \mathbb{R}^p \mid 0 < y_i \quad \text{for } i \in \{1, \dots, p\}\}.$$

Consider the *abstract multi-objective optimization problem*:

$$(AP) \begin{cases} \text{minimize} & f(x) = (f_1(x), \dots, f_p(x)) \\ \text{subject to} & x \in X \subset \mathbb{R}^n, \end{cases}$$

where $f : \mathbb{R}^n \rightarrow \mathbb{R}^p$.

Definition 2.1.1. A point $x^* \in X$ is said to be a (globally) *Pareto optimal solution* (or *efficient* or *non-dominated* or *non-inferior solution*) of (AP) if there does not exist $x \in X$ such that $f(x) \leq f(x^*)$.

Definition 2.1.2. A point $x^* \in X$ is said to be a *partial- i Pareto optimal solution* of (AP), with $i \in \{1, \dots, p\}$, if there does not exist $x \in X$ such that $f(x) \leq_i f(x^*)$.

In some cases a weaker solution concept than *Pareto optimality* is used. It is called *weak Pareto optimality*, which corresponds to the case in which it is considered the positive orthant

$$\mathring{\mathbb{R}}_+^p = \{y \in \mathbb{R}^p \mid \mathbf{0} < y\}.$$

Definition 2.1.3. A point $x^* \in X$ is said to be a *weakly Pareto optimal solution* of (AP) if there does not exist $x \in X$ such that $f(x) < f(x^*)$.

It is easy to see that any Pareto optimal solution is a partial- i Pareto optimal solution for any $i \in \{1, \dots, p\}$, and a partial- i Pareto optimal solution is a weakly Pareto optimal solution. In general, the reverses are not true, see M. Arana-Jiménez and L. L. S. Neto, [2].

Definition 2.1.4 (See E. L. Lima, [49]). Let $f : S \rightarrow \mathbb{R}$ be a function on an open set $S \subset \mathbb{R}^n$ and assume that f is differentiable at $x \in S$. The differential of f at x is the linear functional $df(x) : \mathbb{R}^n \rightarrow \mathbb{R}$, whose value at $h = (h_1, \dots, h_n) \in \mathbb{R}^n$ is given by

$$df(x)(h) = \frac{\partial f}{\partial h}(x) = \lim_{t \rightarrow 0} \frac{f(x + th) - f(x)}{t} = \sum_{i=1}^n \frac{\partial f}{\partial x_i}(x) h_i.$$

Similar to all linear transformation $\mathbb{R}^n \rightarrow \mathbb{R}$, the linear functional $df(x)$ has a $1 \times n$ -matrix in relation to the canonic basis of $\mathbb{R}^n$. If we identify the functional with its

matrix, we will have that

$$df(x) = \left[\frac{\partial f}{\partial x_1}(x) \quad \dots \quad \frac{\partial f}{\partial x_n}(x) \right]_{1 \times n}.$$

Definition 2.1.5. Let $f : \mathbb{R}^n \rightarrow \mathbb{R}^p$ be a vector function and $x \in \mathbb{R}^n$ be a point. If the partial derivatives $\frac{\partial f_i}{\partial x_j}(x)$, $i \in \{1, \dots, p\}$, $j \in \{1, \dots, n\}$, exist, then the $p \times n$ -matrix $Jf(x)$, the Jacobian of f at x , is given by

$$Jf(x) = \begin{bmatrix} \frac{\partial f_1}{\partial x_1}(x) & \dots & \frac{\partial f_1}{\partial x_n}(x) \\ \vdots & \ddots & \vdots \\ \frac{\partial f_p}{\partial x_1}(x) & \dots & \frac{\partial f_p}{\partial x_n}(x) \end{bmatrix}_{p \times n}.$$

If $p = 1$, then the Jacobian of f at x will be a matrix formed by a row vector that it is identified with the gradient of f , which is denoted by $\nabla f(x)$ and given by

$$\nabla f(x) = \left[\frac{\partial f}{\partial x_1}(x) \quad \dots \quad \frac{\partial f}{\partial x_n}(x) \right]_{1 \times n}.$$

Consider the *multi-objective programming problem* with inequality functional constraints:

$$(P) \begin{cases} \text{minimize} & f(x) = (f_1(x), \dots, f_p(x)) \\ \text{subject to} & x \in \mathcal{X} = \{x \in \mathbb{R}^n \mid g(x) = (g_1(x), \dots, g_m(x)) \leq \mathbf{0}\}, \end{cases}$$

where $f_i : \mathbb{R}^n \rightarrow \mathbb{R}$, $i \in \{1, \dots, p\}$ and $g_j : \mathbb{R}^n \rightarrow \mathbb{R}$, $j \in \{1, \dots, m\}$, are continuously differentiable.

The next result states a necessary condition for weak Pareto optimality.

Theorem 2.1.1 (Fritz—John Conditions, see M. C. Maciel, S. A. Santos, G. N. Sotomanto, [52]). Consider problem (P) and let the functions involved to the problem be continuously differentiable at $x^* \in \mathcal{X}$. Then, a necessary condition for x^* to be a weak Pareto optimal solution of (P) is that there exist multipliers $\mathbf{0} \leq (\lambda, \mu) \in \mathbb{R}^{p+m}$ with $(\lambda, \mu) \neq \mathbf{0}$, such that

$$\langle \lambda, \nabla f(x^*) \rangle + \langle \mu, \nabla g(x^*) \rangle = 0 \quad (2.1)$$

and

$$\langle \mu, g(x^*) \rangle = 0, \quad (2.2)$$

where $\langle \lambda, \nabla f(x^*) \rangle$ and $\langle \mu, \nabla g(x^*) \rangle$ stand for $\sum_{i=1}^p \lambda_i \nabla f_i(x^*)$ and $\sum_{j=1}^m \mu_j \nabla g_j(x^*)$, respectively.

The nonzero multiplier, $(\lambda, \mu) \neq \mathbf{0}$, satisfying (2.1)–(2.2) is known as Fritz–John multiplier. Let $M(x^*)$ denote the set of Fritz–John nonzero multipliers associated to x^* .

Definition 2.1.6 (See Y. Sawaragi, H. Nakayama, T. Tanino, [76]). *Regarding Problem (P), $x^* \in \mathcal{X}$ is said to satisfy the Karush–Kuhn–Tucker constraint’s qualification, KKT constraint’s qualification in short if, for any $h \in \mathbb{R}^n$ such that $\langle \nabla g_j(x^*), h \rangle \leq 0$ and for any $j \in I(x^*) = \{j \mid g_j(x^*) = 0\}$, there exist $t^* > 0$, a vector-valued function θ on $[0, t^*]$ differentiable at $t = 0$, and a real number $\alpha > 0$*

$$[0, t^*] \subseteq \mathbb{R} \xrightarrow{\theta} \mathbb{R}^n \xrightarrow{g} \mathbb{R}^m,$$

such that

$$\theta(0) = x^*, \quad g(\theta(t)) \leq \mathbf{0} \quad \text{for any } t \in [0, t^*], \quad \theta'(0) = \alpha h.$$

Next, the necessary conditions for multi-objective programming problems for weak Pareto optimality is stated:

Theorem 2.1.2 (Karush–Kuhn–Tucker Conditions, see Y. Sawaragi, H. Nakayama, T. Tanino, [76]). *Suppose that at problem (P), $x^* \in \mathcal{X}$ satisfies the Karush–Kuhn–Tucker constraint’s qualification. Then, a necessary condition for x^* to be a weak Pareto optimal solution of (P) is that there exist $\lambda \in \mathbb{R}^p$ and $\mu \in \mathbb{R}^m$ such that*

$$\langle \lambda, \nabla f(x^*) \rangle + \langle \mu, \nabla g(x^*) \rangle = 0 \quad (2.3)$$

$$\langle \mu, g(x^*) \rangle = 0, \quad (2.4)$$

and

$$\lambda \geq \mathbf{0} \quad \text{and} \quad \mu \geq \mathbf{0}, \quad (2.5)$$

where $\langle \lambda, \nabla f(x^*) \rangle$ and $\langle \mu, \nabla g(x^*) \rangle$ stand for $\sum_{i=1}^p \lambda_i \nabla f_i(x^*)$ and $\sum_{j=1}^m \mu_j \nabla g_j(x^*)$, respectively.

Definition 2.1.7 (See M. Arana-Jiménez and L. L. S. Neto, [2]). *Let f_i and g_j be continuously differentiable, $i \in \{1, \dots, p\}$ and $j \in \{1, \dots, m\}$, on the set $\mathcal{X}$. A point $x^* \in \mathcal{X}$ is said to be a Karush–Kuhn–Tucker solution, KKT-solution in short, or stationary solution, for (P) if there exist $\lambda \in \mathbb{R}^p$ and $\mu \in \mathbb{R}^m$, satisfying conditions (2.3)–(2.5).*

Definition 2.1.8 (See M. Arana-Jiménez and L. L. S. Neto, [2]). *Let f_i and g_j be continuously differentiable, $i \in \{1, \dots, p\}$ and $j \in \{1, \dots, m\}$, on the set $\mathcal{X}$. A point $x^* \in \mathcal{X}$ is said to be a strict Karush–Kuhn–Tucker solution, strict KKT-solution in short, or strict stationary solution, for (P) if there exist $\lambda \in \mathbb{R}^p$, with $\lambda > \mathbf{0}$ and $\mu \in \mathbb{R}^m$, satisfying equations (2.3)–(2.5).*

From previous definitions, it is derived that for (P) , every strict Karush–Kuhn–Tucker solution is a Karush–Kuhn–Tucker solution.

In the following section, it will be given optimality conditions for control problems of several variables.

2.2 Optimality for multi-objective control problems

The results of this section can be found in V. A. de Oliveira and G. N. Silva, [25].

Notation

Given a set $S \subset \mathbb{R}^n$, the *polar cone of S* is

$$S^\circ := \{\zeta \in \mathbb{R}^n \mid \langle \zeta, v \rangle \leq 0 \quad \forall v \in S\}.$$

Given a multifunction $U : [a, b] \rightrightarrows \mathbb{R}^n$, the *graph of U* is

$$\text{Gr}(U) := \{(t, u) \in [a, b] \times \mathbb{R}^n \mid u \in U(t)\}.$$

The *space of the absolutely continuous functions* is denoted by $W^{1,1}([a, b]; \mathbb{R}^n)$ ¹.

Basic definitions (See R. Vinter, [83])

Given a closed set $S \subset \mathbb{R}^n$ and a point $x \in S$, the set of all directions $v \in \mathbb{R}^n$ such that there exists $M > 0$ satisfying

$$\langle v, y - x \rangle \leq M \|y - x\|^2 \quad \forall y \in S,$$

is said to be the *proximal normal cone to S at x* and it is denoted by $N_S^P(x)$.

The set of all directions $v \in \mathbb{R}^n$ such that there exist sequences $x_i \xrightarrow{S} x$ and $v_i \rightarrow v$ satisfying $v_i \in N_S^P(x_i)$ for all i , is said to be the *limiting normal cone to S at x* and

¹ $W^{1,1}([a, b]; \mathbb{R}^n) := \{\gamma : [a, b] \rightarrow \mathbb{R}^n \mid \gamma \text{ is an absolutely continuous function}\}$, where a function $f : [a, b] \rightarrow \mathbb{R}$ is called *absolutely continuous*, when it satisfies one of the following cases:

(I) $\forall \epsilon > 0 \exists \delta > 0$ such that for every collection $\{[a_i, b_i]\}_i$ of disjoint sub-intervals of $[a, b]$ we have

$$\sum_i (b_i - a_i) < \delta \Rightarrow \sum_i |f(b_i) - f(a_i)| < \epsilon.$$

(II) $\exists v \in L([a, b])$ (the space of integrable functions defined at the interval $[a, b]$) such that

$$f(t) = f(a) + \int_a^t v(\tau) d\tau, \quad t \in [a, b] \quad \Rightarrow \quad f'(t) = \frac{d}{dt} f(t) = v(t) \quad \text{a. e. } t \in [a, b].$$

denoted by $N_S(x)$.

The set of all directions $w \in \mathbb{R}^n$ such that for any sequences $x_i \xrightarrow{S} x$ and $t_i \downarrow 0$ there exists a sequence $\{s_i\} \subset S$ satisfying $t_i^{-1}(s_i - x_i) \rightarrow w$, is said to be the *Clarke tangent cone to S at x* and denoted by $\overline{T}_S(x)$.

The Clarke tangent cone and the limiting normal cone are related as follows

$$\overline{T}_S(x) = [N_S(x)]^\circ = \{\zeta \in \mathbb{R}^n \mid \langle \zeta, v \rangle \leq 0 \quad \forall v \in N_S(x)\}.$$

Let $\gamma : \mathbb{R}^n \rightarrow \mathbb{R} \cup \{+\infty\}$ be a lower semi-continuous function and $x \in \text{dom } \gamma$. The *limiting Mordukhovich's subdifferential of γ at x* is

$$\partial\gamma(x) = \{\zeta \mid (\zeta, -1) \in N_{\text{epi } \gamma}(x, \gamma(x))\}.$$

Let $L : \mathbb{R}^n \rightarrow \mathbb{R}^m$ be a Lipschitz continuous function defined in a neighborhood of $y \in \mathbb{R}^n$. The *generalized Jacobian of L at y* is the set of $m \times n$ matrices

$$DL(y) = \text{co} \{\vartheta \mid \exists y_i \rightarrow y \text{ such that } \nabla L(y_i) \text{ exists } \forall i \text{ and } \nabla L(y_i) \rightarrow \vartheta\}.$$

Let $v \in \mathbb{R}^n$. The generalized Jacobian has the following property:

$$v^\top DL(y) = \text{co} \{\partial(\langle v, L \rangle)(y)\}.$$

The multi-objective optimal control problem

A *multi-objective optimal control problem (MCP)* is posed as follows:

$$(MCP) \left\{ \begin{array}{l} \text{minimize } g(x(a), x(b)) \\ \text{subject to } x'(t) = \gamma(t, x(t), u(t)) \text{ a. e. } t \in [a, b] \\ (x(a), x(b)) \in S, \\ u(t) \in U(t) \text{ a. e. } t \in [a, b], \end{array} \right.$$

where $g = [g_1 \ g_2 \ \dots \ g_k]^\top : \mathbb{R}^n \times \mathbb{R}^n \rightarrow \mathbb{R}^k$, $\gamma : [a, b] \times \mathbb{R}^n \times \mathbb{R}^m \rightarrow \mathbb{R}^n$, S is a closed subset of $\mathbb{R}^n \times \mathbb{R}^n$, U is a multifunction mapping $[a, b]$ to nonempty subsets of $\mathbb{R}^m$, a and b are fixed.

A measurable function $u : [a, b] \rightarrow \mathbb{R}^m$ such that $u(t) \in U(t)$ for a. e. $t \in [a, b]$ is said to be a *control function*.

A pair (x, u) consisting of an arc $x \in W^{1,1}([a, b]; \mathbb{R}^n)$ and a control function u obeying

the differential equation above is said to be a *process*.

Given a process (x, u) , its first component is said to be a *state trajectory*.

(x, u) is an *admissible process* if x is a trajectory corresponding to the control $u \in U$ which satisfies $(x(a), x(b)) \in S$.

An admissible process (x^*, u^*) is a *Pareto optimal process* (or efficient or non-dominated or non-inferior process) if there exists no other admissible process (x, u) such that

$$g(x(a), x(b)) \leq g(x^*(a), x^*(b)).$$

An admissible process (x^*, u^*) is a *weak Pareto optimal process* (or weakly efficient or weakly non-dominated or weakly non-inferior process) if there exists no other admissible process (x, u) such that

$$g(x(a), x(b)) < g(x^*(a), x^*(b)).$$

Hypothesis

Let (x^*, u^*) be a reference admissible process of (MCP) . For some $\delta > 0$, the following hypothesis are satisfied:

(H1) The function $\gamma(\cdot, x, \cdot)$ is $\mathcal{L} \times \mathcal{B}^m$ -measurable, for each $x \in \mathbb{R}^n$.

(H2) There exists a $\mathcal{L} \times \mathcal{B}^m$ -measurable function $k : [a, b] \times \mathbb{R}^m \rightarrow \mathbb{R}$ such that $t \mapsto k(t, u^*(t))$ is integrable, and for a. e. $t \in [a, b]$,

$$\|\gamma(t, x, u) - \gamma(t, \check{x}, u)\|_{\mathbb{R}^n} \leq k(t, u) \|x - \check{x}\|_{\mathbb{R}^n} \quad \forall x, \check{x} \in x^*(t) + \delta \mathbb{B}_{\mathbb{R}^n} \quad \forall u \in U(t),$$

where $\mathbb{B}_{\mathbb{R}^n} := \mathbb{B}_{\mathbb{R}^n}(\mathbf{0}, 1) = \{x \in \mathbb{R}^n \mid \|x\|_{\mathbb{R}^n} < 1\}$.

(H3) $\text{Gr}(U)$ is $\mathcal{L} \times \mathcal{B}^m$ measurable.

(H4) g is locally Lipschitz continuous.

2.2.1 Necessary optimality for multi-objective control problems: The Maximum Principle

Let $\mathcal{H} : [a, b] \times \mathbb{R}^n \times \mathbb{R}^n \times \mathbb{R}^m \rightarrow \mathbb{R}$ denote the *Unmaximized Hamiltonian function*

$$\mathcal{H}(t, x, p, u) := \langle p, \gamma(t, x, u) \rangle.$$

Now will be stated:

Theorem 2.2.1 (Necessary optimality conditions of Pontryagin Maximum Principle—type for **(MCP)**). *If (x^*, u^*) is a weak Pareto optimal process of the (MCP), then there exist a scalar λ (equal to 0 or 1), a nonzero vector $\omega \in \mathbb{R}^k$, and an absolutely continuous function $p : [a, b] \rightarrow \mathbb{R}^n$ such that*

- (i) $-p'(t) \in \text{co} \{ \partial_x \mathcal{H}(t, x^*(t), p(t), u^*(t)) \}$ a. e. $t \in [a, b]$,
- (ii) $(p(a), -p(b)) \in \lambda \partial(\langle \omega, g \rangle)(x^*(a), x^*(b)) + N_S(x^*(a), x^*(b))$,
- (iii) $\max_{u \in U(t)} \mathcal{H}(t, x^*(t), p(t), u) = \mathcal{H}(t, x^*(t), p(t), u^*(t))$ a. e. $t \in [a, b]$,
- (iv) $\|p\|_{\mathbb{R}^n} + \lambda > 0$, $\|\omega\|_{\mathbb{R}^k} = 1$, $\omega \geq 0$.

Remark 2.2.1. 1. Observe that condition (i) can be equivalently stated as

$$-p'(t)^\top \in p(t)^\top D_x \gamma(t, x^*(t), u^*(t)) \quad \text{a. e. } t \in [a, b].$$

- 2. Every Pareto optimal process is a weak Pareto optimal one, so the maximum principle above is applicable to Pareto optimal solutions as well.

When there exists (λ, ω, p) satisfying (i) – (iv), (x^*, u^*) is an *extremal or stationary process of (MCP)*.

If $\lambda = 1$, (x^*, u^*) is a *normal extremal process of (MCP)* and (MCP) is said to be *normal at (x^*, u^*)* .

(MCP) is *normal* if it is normal at any extremal process (x^*, u^*) , and it is *abnormal*, otherwise.

It will be used the following types of *difference operators*: Given $(t, x, u) \in [a, b] \times \mathbb{R}^n \times \mathbb{R}^m$, $(t, x, p, u) \in [a, b] \times \mathbb{R}^n \times \mathbb{R}^n \times \mathbb{R}^m$ and $\beta \in (0, 1)$, $\Delta_\beta \gamma(t, x, u)$, $\Delta_\beta \mathcal{H}(t, x, p, u) : \mathbb{R}^m \rightarrow \mathbb{R}$, are defined, respectively, as

$$\begin{aligned} \Delta_\beta \gamma(t, x, u)(\xi) &:= \frac{\gamma(t, x, u + \beta \xi) - \gamma(t, x, u)}{\beta}, \\ \Delta_\beta \mathcal{H}(t, x, p, u)(\xi) &:= \frac{\mathcal{H}(t, x, p, u + \beta \xi) - \mathcal{H}(t, x, p, u)}{\beta}. \end{aligned}$$

2.2.2 Sufficient optimality for multi-objective control problems: The *MP*–pseudoinvexity

Definition 2.2.1. Let $\beta \in (0, 1)$ be a scalar, (x^*, u^*) be an admissible process of (MCP) and $\zeta(t) \in D_x \gamma(t, x^*(t), u^*(t))$ a. e. $t \in [a, b]$. The set of β –admissible directions, denoted

by $\mathcal{F}_\beta(x^*, u^*; \zeta)$, is the set of all pairs (y, v) where $y : [a, b] \rightarrow \mathbb{R}^n, v : [a, b] \rightarrow \mathbb{R}^m$ are such that

$$\begin{aligned} \dot{y}(t) &= \zeta(t)y(t) + \Delta_\beta \gamma(t, x^*(t), u^*(t))(v(t)) \quad \text{a. e. } t \in [a, b], \\ (y(a), y(b)) &\in \overline{T}_S(x^*(a), x^*(b)), \\ u^*(t) + \beta v(t) &\in U(t) \quad \text{a. e. } t \in [a, b]. \end{aligned}$$

Remark 2.2.2. Observe that $u^*(t) + \beta v(t) \in U(t)$ a. e. $t \in [a, b]$ may be seen as a variation on the control variable and a solution of $\dot{y}(t) = \zeta(t)y(t) + \Delta_\beta \gamma(t, x^*(t), u^*(t))(v(t))$ a. e. $t \in [a, b]$ with $(y(a), y(b)) \in \overline{T}_S(x^*(a), c^*(b))$, as the corresponding variation on the state variable.

Definition 2.2.2. (i) Let (x^*, u^*) be an admissible process. (MCP) is said to be MP-pseudoinvex at (x^*, u^*) if for every admissible process (x, u) with $g(x(a), x(b)) \leq g(x^*(a), x^*(b))$ and all $\beta \in (0, 1)$, given $\tau = [\tau_1 \tau_2 \dots \tau_k]^\top \in Dg(x^*(a), x^*(b))$ and $\zeta(t) \in D_x \gamma(t, x^*(t), u^*(t))$ a. e. $t \in [a, b]$, there exist maps $(\eta, \xi) \in \mathcal{F}_\beta(x^*, u^*; \zeta)$ such that, for each $j \in \{1, 2, \dots, k\}$,

$$\begin{cases} g_j(x(a), x(b)) = g_j(x^*(a), x^*(b)) \Rightarrow \langle \tau_j, (\eta(a), \eta(b)) \rangle \leq 0 \\ \text{or} \\ g_j(x(a), x(b)) < g_j(x^*(a), x^*(b)) \Rightarrow \langle \tau_j, (\eta(a), \eta(b)) \rangle < 0. \end{cases}$$

(ii) (MCP) is said to be MP-pseudoinvex if it is MP-pseudoinvex at every admissible process (x^*, u^*) .

Note that in the definition above

$$\eta := \eta(x, u, x^*, u^*) \quad \text{and} \quad \xi := \xi(x, u, x^*, u^*).$$

Hypothesis 1. Let $\mathbf{0} \neq \omega \in \mathbb{R}_+^k$. Hypothesis $(H(\omega))$ is satisfied if

$$\{j \mid g_j(x(a), x(b)) < g_j(y(a), y(b))\} \cap \{j \mid \omega_j > 0\} \neq \emptyset$$

whenever (x, u) and (y, v) are admissible processes of (MCP) with

$$g(x(a), x(b)) \leq g(y(a), y(b)).$$

Remark 2.2.3. Hypothesis $(H(\omega))$ is satisfied when $\omega > \mathbf{0}$. It is also satisfied if $k = 1$, that is, if the problem is mono-objective.

Theorem 2.2.2 (Sufficient optimality conditions for **(MCP)**). *Let (x^*, u^*) be a normal extremal process of (MCP) with multipliers $(1, \omega, p)$. Assume that (MCP) is MP-pseudoinvex at (x^*, u^*) and the Hypothesis $(H(\omega))$ holds. Then (x^*, u^*) is a Pareto optimal process.*

In the case of weak Pareto optimality, Hypothesis $(H(\omega))$ is not required to be valid:

Theorem 2.2.3 (Sufficient weak optimality conditions for **(MCP)**). *Let (x^*, u^*) be a normal extremal process of (MCP). Assume that (MCP) is MP-pseudoinvex at (x^*, u^*) . Then (x^*, u^*) is a weak Pareto optimal process.*

Next, it will be introduced the interval space and it will be recalled relevant results that will be used throughout the Thesis.

2.3 The interval space $\mathcal{K}_C$

In this section, it will be considered the interval arithmetic operations of the interval space, the relation between the elements of the interval space, and some important properties.

Definition 2.3.1. *Let $\underline{a}, \bar{a} \in \mathbb{R}$ with $\underline{a} \leq \bar{a}$. The closed line segment joining $\underline{a}$ and $\bar{a}$, that is, the closed interval is defined by*

$$[\underline{a}, \bar{a}] = \{a \mid a = (1 - t)\underline{a} + t\bar{a}, \quad 0 \leq t \leq 1\},$$

or equivalently,

$$[\underline{a}, \bar{a}] = \{a \mid a = \zeta_1 \underline{a} + \zeta_2 \bar{a}, \quad \zeta_1, \zeta_2 \geq 0, \quad \text{and} \quad \zeta_1 + \zeta_2 = 1\}.$$

When $\underline{a} = \bar{a}$, the closed interval is called degenerate interval, which now, is a set consisting of a single element $\underline{a} = \bar{a} = a$.

Definition 2.3.2. *Let $\underline{a}, \bar{a} \in \mathbb{R}$ with $\underline{a} < \bar{a}$. The open line segment joining $\underline{a}$ and $\bar{a}$, that is, the open interval is defined by*

$$(\underline{a}, \bar{a}) = \{a \mid a = (1 - t)\underline{a} + t\bar{a}, \quad 0 < t < 1\},$$

or equivalently,

$$(\underline{a}, \bar{a}) = \{a \mid a = \zeta_1 \underline{a} + \zeta_2 \bar{a}, \quad \zeta_1, \zeta_2 > 0, \quad \text{and} \quad \zeta_1 + \zeta_2 = 1\}.$$

Consider the space $\mathcal{K}_C = \{[\underline{a}, \bar{a}] \mid \underline{a}, \bar{a} \in \mathbb{R} \text{ and } \underline{a} \leq \bar{a}\}$. Given $A = [\underline{a}, \bar{a}]$, $B = [\underline{b}, \bar{b}]$, $C = [\underline{c}, \bar{c}] \in \mathcal{K}_C$ and $\lambda \in \mathbb{R}$, the *interval arithmetic operations* are defined by

$$A + B = [\underline{a} + \underline{b}, \bar{a} + \bar{b}] \quad \text{and} \quad \lambda \cdot A = \begin{cases} [\lambda \underline{a}, \lambda \bar{a}], & \text{if } \lambda \geq 0 \\ [\lambda \bar{a}, \lambda \underline{a}], & \text{if } \lambda < 0. \end{cases} \quad (2.6)$$

Then, $A - B = [\underline{a} - \bar{b}, \bar{a} - \underline{b}]$ and $A \otimes B = [\min P, \max P]$, where $P = \{\underline{a}\underline{b}, \underline{a}\bar{b}, \bar{a}\underline{b}, \bar{a}\bar{b}\}$.

The *gH-difference of two intervals* $A, B \in \mathcal{K}_C$, which it is recalled from L. Stefanini, [80], is defined by:

$$A \ominus_{gH} B = C \Leftrightarrow \begin{cases} A = B + C, & \text{if } \text{len}(B) \leq \text{len}(A) \\ B = A + (-1)C, & \text{if } \text{len}(B) > \text{len}(A), \end{cases} \quad (2.7)$$

where $\text{len}(D)$ denotes the *length of an interval* $D = [\underline{d}, \bar{d}] \in \mathcal{K}_C$, i.e., $\text{len}(D) = \bar{d} - \underline{d}$. Note that $A \ominus_{gH} A = [0, 0]$. Moreover, the *gH-difference* of two intervals always exists and, if $A = [\underline{a}, \bar{a}]$, $B = [\underline{b}, \bar{b}] \in \mathcal{K}_C$, it holds that

$$A \ominus_{gH} B = \left[\underline{a} - \underline{b} \bigvee \bar{a} - \bar{b} \right], \quad (2.8)$$

where $[\underline{a} \bigvee \bar{a}]$ means that $\underline{a}$ and $\bar{a}$ are the extremum of the interval, but $\underline{a} \leq \bar{a}$ is not necessarily satisfied.

Given $n \in \mathbb{N}$, an application $F : S \subseteq \mathbb{R}^n \rightarrow \mathcal{K}_C$ such that $F(x) = [\underline{f}(x), \bar{f}(x)]$ for all $x \in S$, is called an *interval-valued function of several variables*, and $\underline{f}, \bar{f} : S \rightarrow \mathbb{R}$, real-valued functions, are its *extremum functions*. If $n = 1$, then F is called an *interval-valued function of a real variable*.

Since the space $(\mathcal{K}_C, d_H)$ is a complete and separable metric space (see P. Diamond and P. Kloeden, [26]), where $d_H(A, B) = \max\{|\underline{a} - \underline{b}|, |\bar{a} - \bar{b}|\}$ is the *Pompeiu-Hausdorff metric between* $A = [\underline{a}, \bar{a}]$, $B = [\underline{b}, \bar{b}] \in \mathcal{K}_C$, it is natural to consider the concepts of limit and continuity of interval-valued functions defined by means of the Pompeiu-Hausdorff metric d_H .

The *quasi-norm* of $A = [\underline{a}, \bar{a}] \in \mathcal{K}_C$ is the real number $\|A\|_{\mathcal{K}_C}$ defined by

$$\|A\|_{\mathcal{K}_C} = d_H(A, [0, 0]) = \max\{|\underline{a}|, |\bar{a}|\}.$$

The next order relation was proposed by U. W. Kulish and W. L. Miranker, [45], and adapted using M. Arana-Jiménez and L. L. S. Neto, [2].

Definition 2.3.3. Let $A = [\underline{a}, \bar{a}]$ and $B = [\underline{b}, \bar{b}] \in \mathcal{K}_C$. The order relation lower and upper, LU in short, is defined by

- (i) $A \leq_{LU} B$ if and only if $\underline{a} \leq \underline{b}$ and $\bar{a} \leq \bar{b}$.
- (ii) $A \leq_{LU} B$ if and only if $A \leq_{LU} B$ and $A \neq B$, that is, either $\underline{a} < \underline{b}$ and $\bar{a} \leq \bar{b}$ or $\underline{a} \leq \underline{b}$ and $\bar{a} < \bar{b}$.
- (iii) $A \leq_{LU} B$ means $\underline{a} < \underline{b}$ and $\bar{a} \leq \bar{b}$.
- (iv) $A \leq_{LU} B$ means $\underline{a} \leq \underline{b}$ and $\bar{a} < \bar{b}$.
- (v) $A <_{LU} B$ if and only if $\underline{a} < \underline{b}$ and $\bar{a} < \bar{b}$.

This relation of order will be extremely important, since this Thesis will study interval optimization problems and interval optimal control problems using this order relation.

The order relation LU has been widely used, see H.-C. Wu, [84], [85], [86], [87], and has been compared with other order relations, for instance, see the order relation UC and LC of H. Ishibuchi and H. Tanaka, [39], and see the order relation LS of Y. Chalco-Cano, W. A. Lodwick, A. Rufián-Lizana, [17], among others.

Remark 2.3.1. (i) The concept of the order relation proposed by H. Ishibuchi and H. Tanaka, [39], differs from the concept of Y. Chalco-Cano, W. A. Lodwick, A. Rufián-Lizana, [17], since the order relation UC and LC seeks only the minimum or the maximum of the intervals, without the intention of minimize or maximize the uncertainty of the same. Thus, the order relation UC and LC must be applied in the comparison of intervals, where the minimization or maximization of uncertainty is not required, as the order relation LU .

(ii) The concept of the order relation proposed by Y. Chalco-Cano, W. A. Lodwick, A. Rufián-Lizana, [17], seeks intervals with the least uncertainty, since the two cases, minimization and maximization, seek minimize the interval spread. Thus, the order relation LS should be applied when comparing intervals, where the objective is to find intervals with the least uncertainty.

(iii) All the order relations presented here are partial order relations.

(iv) For the minimization case, we have the following implications:

$$\boxed{\text{Order } LS \xrightarrow{(a)} \text{Order } LU \xrightarrow{(b)} \text{Order } UC.}$$

The implication (a) can be found at Y. Chalco-Cano, W. A. Lodwick, A. Rufián-Lizana, [17], and the implication (b) can be found at H. Ishibuchi and H. Tanaka, [39]. With these in hands, the order relation LS implies the order relation UC .

Proposition 2.3.1. *Given $A = [\underline{a}, \bar{a}]$, $B = [\underline{b}, \bar{b}]$, $C = [\underline{c}, \bar{c}] \in \mathcal{K}_C$, and $\ll_{LU} \in \{\leq_{LU}, \leq_{LU}, \leq_{LU}, \leq_{LU}, <_{LU}\}$, it follows that*

$$(i) \ A \ll_{LU} B \Leftrightarrow A \ominus_{gH} B \ll_{LU} [0, 0] \Leftrightarrow [0, 0] \ll_{LU} B \ominus_{gH} A.$$

$$(ii) \ A \ll_{LU} B \Leftrightarrow A + C \ll_{LU} B + C.$$

Proof. (i) For $\ll_{LU} \in \{\leq_{LU}, \leq_{LU}, <_{LU}\}$, see L. Stefanini and M. Arana-Jiménez, [81], and for $\ll_{LU} \in \{\leq_{LU}, \leq_{LU}\}$ the proofs are easily obtained.

(ii) Immediately. □

Proposition 2.3.2. *Given $A = [\underline{a}, \bar{a}]$, $B = [\underline{b}, \bar{b}] \in \mathcal{K}_C$ it follows that*

$$A \ominus_{gH} B \leq_{LU} [d_H(A, B), d_H(A, B)].$$

Proof. We have that $A \ominus_{gH} B = [\underline{a} - \underline{b} \vee \bar{a} - \bar{b}] \leq_{LU} [|\underline{a} - \underline{b}| \vee |\bar{a} - \bar{b}|] \leq_{LU} [\max\{|\underline{a} - \underline{b}|, |\bar{a} - \bar{b}|\}, \max\{|\underline{a} - \underline{b}|, |\bar{a} - \bar{b}|\}] = [d_H(A, B), d_H(A, B)]$. □

On the other hand, all interval $A = [\underline{a}, \bar{a}] \in \mathcal{K}_C$, can be expressed through its *center–radius representation*

$$A = (a^C; a^R),$$

see M. Ishibuchi and H. Tanaka, [39], L. Stefanini and M. Arana-Jiménez, [81], where its center a^C and radius a^R are defined by

$$a^C = \frac{\underline{a} + \bar{a}}{2} \quad \text{and} \quad a^R = \frac{\bar{a} - \underline{a}}{2},$$

respectively. In this case, $\underline{a} = a^C - a^R$ and $\bar{a} = a^C + a^R$. So, the concepts of limit and continuity of interval–valued functions defined by means of the Pompeiu–Hausdorff metric d_H can be interpreted through its respective versions for real–valued functions, see L. Stefanini and B. Bede, [82]. Based on the center–radius representation of intervals the definitions of gH –derivative, gH –gradient, the gH –directional derivative and the gH –partial derivatives are defined in L. Stefanini and M. Arana-Jiménez, [81], as it will be seen.

Remark 2.3.2 (See R. E. Moore, R. B. Kearfott, M. J. Cloud, [57]). *The notion of degenerate interval permits us to regard the closed interval number system as an extension of the real number system. Indeed, there is an obvious one-to-one pairing*

$$[a, a] \longleftrightarrow a$$

between the elements of the two systems, where $[a, a] \in \mathcal{K}_C$ and $a \in \mathbb{R}$.

The interior of a non-degenerate interval $A = [\underline{a}, \bar{a}] \in \mathcal{K}_C$, is denoted by

$$\text{int}(A) = \text{int}([\underline{a}, \bar{a}]) = (\underline{a}, \bar{a}). \quad (2.9)$$

Proposition 2.3.3. *Given $A = [\underline{a}, \bar{a}] \in \mathcal{K}_C$ and $\lambda \in \mathbb{R}$, we have that*

$$(i) \quad -\lambda \in [\underline{a}, \bar{a}] \Leftrightarrow 0 \in [\underline{a}, \bar{a}] + [\lambda, \lambda].$$

$$(ii) \quad -\lambda \in \text{int}([\underline{a}, \bar{a}]) \Leftrightarrow 0 \in \text{int}([\underline{a}, \bar{a}] + [\lambda, \lambda]).$$

Proof. (i) $-\lambda \in [\underline{a}, \bar{a}] \Leftrightarrow \underline{a} \leq -\lambda \leq \bar{a} \Leftrightarrow \underline{a} + \lambda \leq 0 \leq \bar{a} + \lambda$
 $\Leftrightarrow 0 \in [\underline{a} + \lambda, \bar{a} + \lambda] \Leftrightarrow 0 \in [\underline{a}, \bar{a}] + [\lambda, \lambda].$

$$(ii) \quad -\lambda \in \text{int}([\underline{a}, \bar{a}]) \Leftrightarrow -\lambda \in (\underline{a}, \bar{a}) \Leftrightarrow \underline{a} < -\lambda < \bar{a}$$

$$\Leftrightarrow \underline{a} + \lambda < 0 < \bar{a} + \lambda \Leftrightarrow 0 \in (\underline{a} + \lambda, \bar{a} + \lambda) \Leftrightarrow 0 \in \text{int}([\underline{a} + \lambda, \bar{a} + \lambda])$$

$$\Leftrightarrow 0 \in \text{int}([\underline{a}, \bar{a}] + [\lambda, \lambda]).$$

□

The cartesian product $\mathcal{K}_C \times \dots \times \mathcal{K}_C$ of n -factors $\mathcal{K}_C$, is denoted by $\mathcal{K}_C^n$. Thus, the last remark can be generalized to:

Remark 2.3.3. *The notion of degenerate intervals permits us to regard the n -dimensional closed interval number system as an extension of the n -dimensional real number system. There is an one-to-one pairing*

$$A = ([a_1, a_1], \dots, [a_n, a_n]) \longleftrightarrow (a_1, \dots, a_n) = a$$

between the elements of the two systems, where $A = ([a_1, a_1], \dots, [a_n, a_n]) \in \mathcal{K}_C^n$ and $a = (a_1, \dots, a_n) \in \mathbb{R}^n$.

The interior of a non-degenerate n -dimensional interval $A = (A_1, \dots, A_n) \in \mathcal{K}_C^n$, where $A_i = [\underline{a}_i, \bar{a}_i]$ is a non-degenerate interval, $i \in \{1, \dots, n\}$, is denoted by

$$\begin{aligned}
\text{int}(A) &= \text{int}((A_1, \dots, A_n)) \\
&= (\text{int}(A_1), \dots, \text{int}(A_n)) && \text{(by the product topology in } \mathbb{R}^n) \\
&= (\text{int}([\underline{a}_1, \overline{a}_1]), \dots, \text{int}([\underline{a}_n, \overline{a}_n])) \\
&= ((\underline{a}_1, \overline{a}_1), \dots, (\underline{a}_n, \overline{a}_n)).
\end{aligned} \tag{2.10}$$

It is not difficult to see that:

Proposition 2.3.4. *Given $A = ([\underline{a}_1, \overline{a}_1], \dots, [\underline{a}_n, \overline{a}_n])$ and $\lambda = (\lambda_1, \dots, \lambda_n) = ([\lambda_1, \lambda_1], \dots, [\lambda_n, \lambda_n]) \in \mathcal{K}_C^n$, where $\lambda_i \in \mathbb{R}$, $i \in \{1, \dots, n\}$, we have that*

$$(i) \quad -\lambda \in A \Leftrightarrow \mathbf{0} \in A \oplus \lambda.$$

$$(ii) \quad -\lambda \in \text{int}(A) \Leftrightarrow \mathbf{0} \in \text{int}(A + \lambda).$$

Definition 2.3.4. *Let $A = (A_1, \dots, A_n) \in \mathcal{K}_C^n$, where $A_i = [\underline{a}_i, \overline{a}_i] \in \mathcal{K}_C$, $i \in \{1, \dots, n\}$, and $v = (v_1, \dots, v_n) \in \mathbb{R} \times \dots \times \mathbb{R} = \mathbb{R}^n$. The addition between a n -dimensional interval vector and a n -dimensional vector is defined by*

$$\begin{aligned}
A \oplus v &= (A_1, \dots, A_n) \oplus (v_1, \dots, v_n) \\
&= ([\underline{a}_1, \overline{a}_1], \dots, [\underline{a}_n, \overline{a}_n]) \oplus ([v_1, v_1], \dots, [v_n, v_n]) \\
&= ([\underline{a}_1, \overline{a}_1] + [v_1, v_1], \dots, [\underline{a}_n, \overline{a}_n] + [v_n, v_n]) \\
&= ([\underline{a}_1 + v_1, \overline{a}_1 + v_1], \dots, [\underline{a}_n + v_n, \overline{a}_n + v_n]).
\end{aligned}$$

Definition 2.3.5. *Let $A = (A_1, \dots, A_n), B = (B_1, \dots, B_n) \in \mathcal{K}_C^n$, where $A_i = [\underline{a}_i, \overline{a}_i]$, $B_i = [\underline{b}_i, \overline{b}_i] \in \mathcal{K}_C$, $i \in \{1, \dots, n\}$. The addition between n -dimensional interval vectors is defined by*

$$\begin{aligned}
A \oplus B &= (A_1, \dots, A_n) \oplus (B_1, \dots, B_n) \\
&= ([\underline{a}_1, \overline{a}_1], \dots, [\underline{a}_n, \overline{a}_n]) \oplus ([\underline{b}_1, \overline{b}_1], \dots, [\underline{b}_n, \overline{b}_n]) \\
&= ([\underline{a}_1, \overline{a}_1] + [\underline{b}_1, \overline{b}_1], \dots, [\underline{a}_n, \overline{a}_n] + [\underline{b}_n, \overline{b}_n]) \\
&= ([\underline{a}_1 + \underline{b}_1, \overline{a}_1 + \overline{b}_1], \dots, [\underline{a}_n + \underline{b}_n, \overline{a}_n + \overline{b}_n]).
\end{aligned}$$

Definition 2.3.6. *Let $A = (A_1, \dots, A_n) \in \mathcal{K}_C^n$, where $A_i = [\underline{a}_i, \overline{a}_i] \in \mathcal{K}_C$, $i \in \{1, \dots, n\}$, and $\lambda \in \mathbb{R}$. The product between a scalar and a n -dimensional interval vector is defined by*

$$\begin{aligned}
\lambda \odot A &= \lambda \odot (A_1, \dots, A_n) \\
&= \lambda \odot ([a_1, \bar{a}_1], \dots, [a_n, \bar{a}_n]) \\
&= (\lambda \cdot [a_1, \bar{a}_1], \dots, \lambda \cdot [a_n, \bar{a}_n]) \\
&= \left([\lambda a_1 \vee \lambda \bar{a}_1], \dots, [\lambda a_n \vee \lambda \bar{a}_n] \right).
\end{aligned}$$

The symbol $v \in A$, where $v = (v_1, \dots, v_n) \in \mathbb{R}^n$ and $A = (A_1, \dots, A_n) \in \mathcal{K}_C^n$, is used to mean that $v_i \in A_i$ for all $i \in \{1, \dots, n\}$.

The following five sections will remind some concepts and results about the calculus of interval-valued functions.

2.4 Limit of interval-valued functions

Since the differentiability and the directional derivative concepts of interval-valued functions are in terms of the limit of interval-valued functions, see Definition 2.7.1 and Definition 2.7.5, respectively, it will be considered the following:

Definition 2.4.1 (See J. P. Aubin and A. Cellina, [4]). *Let $S \subseteq \mathbb{R}$, $F : S \rightarrow \mathcal{K}_C$ be an interval-valued function and $x_0 \in S^a$ an accumulation point of S , that is, $x_0 \in S^a \cap S$. Then $L \in \mathcal{K}_C$ is called a limit of F at $x_0 \in S$ if for every $\epsilon > 0$ there exists $\delta(\epsilon, x_0) = \delta > 0$ such that $d_H(F(x), L) < \epsilon$ for all $x \in S$ with $0 < |x - x_0| < \delta$. This is denoted by*

$$\lim_{x \rightarrow x_0} F(x) = L.$$

The following results establish the relationship between the limit of an interval-valued function and the limit of its extremum functions.

Theorem 2.4.1 (See J. P. Aubin and A. Cellina, [4]). *Let $F : S \subseteq \mathbb{R} \rightarrow \mathcal{K}_C$ be an interval-valued function such that $F(x) = [\underline{f}(x), \bar{f}(x)]$ for all $x \in S$. Given $x_0 \in S$, it follows that*

$$\lim_{x \rightarrow x_0} F(x) = \left[\lim_{x \rightarrow x_0} \underline{f}(x), \lim_{x \rightarrow x_0} \bar{f}(x) \right].$$

Corollary 2.4.1 (See R. Osuna-Goméz, Y. Chalco-Cano, B. Hernández-Jiménez, and T. M. Costa, [63]). *Let $F(x) = [\underline{f}(x) \vee \bar{f}(x)]$ be an interval-valued function, where $\underline{f}, \bar{f} : S \subseteq \mathbb{R} \rightarrow \mathbb{R}$. Given $x_0 \in S$, then $\lim_{x \rightarrow x_0} F(x)$ exists if $\lim_{x \rightarrow x_0} \underline{f}(x)$ and $\lim_{x \rightarrow x_0} \bar{f}(x)$*

exist. In this case

$$\lim_{x \rightarrow x_0} F(x) = \left[\lim_{x \rightarrow x_0} \underline{f}(x) \bigvee \lim_{x \rightarrow x_0} \overline{f}(x) \right].$$

The limit definition of interval-valued functions of a real variable (Definition 2.4.1) is easily extended to *limit of interval-valued functions of several variables*.

The next result is a natural extension of Theorem 2.4.1.

Theorem 2.4.2 (See R. Osuna-Gómez, Y. Chalco-Cano, B. Hernández-Jiménez, T. M. Costa, [63]). *Let $F : S \subseteq \mathbb{R}^n \rightarrow \mathcal{K}_C$ be an interval-valued function such that $F(x) = [\underline{f}(x), \overline{f}(x)]$ for all $x \in S$. Given $x_0 \in S$, it follows that*

$$\lim_{x \rightarrow x_0} F(x) = \left[\lim_{x \rightarrow x_0} \underline{f}(x), \lim_{x \rightarrow x_0} \overline{f}(x) \right].$$

Next, it will be recalled some concepts and results about the continuity of interval-valued function.

2.5 Continuity of interval-valued functions

Since the notion of continuity is essential, it will be looked at the coming results.

Definition 2.5.1 (See J. P. Aubin and A. Cellina, [4]). *An interval-valued function $F : S \subseteq \mathbb{R} \rightarrow \mathcal{K}_C$ is d_H -continuous at $x_0 \in S$ if and only if*

$$\lim_{x \rightarrow x_0} F(x) = F(x_0).$$

If F is d_H -continuous at all $x \in S$, then F is said to be d_H -continuous.

Theorem 2.5.1 (See J. P. Aubin and A. Cellina, [4]). *Let $F : S \subseteq \mathbb{R} \rightarrow \mathcal{K}_C$ be an interval-valued function such that $F(x) = [\underline{f}(x), \overline{f}(x)]$, where $\underline{f}, \overline{f} : S \rightarrow \mathbb{R}$. F is d_H -continuous at $x_0 \in S$ if and only if $\underline{f}, \overline{f}$ are continuous at x_0 . Moreover, if F is d_H -continuous at x_0 , then*

$$\lim_{x \rightarrow x_0} F(x) = \left[\lim_{x \rightarrow x_0} \underline{f}(x), \lim_{x \rightarrow x_0} \overline{f}(x) \right] = [\underline{f}(x_0), \overline{f}(x_0)].$$

The continuity definition of interval-valued functions of a real variable (Definition 2.5.1) is easily extended to continuity of interval-valued functions of several variables.

Definition 2.5.2 (See M. K. Luhandjula and M. J. Rangoaga, [51]). *An interval-valued function $F : S \subseteq \mathbb{R}^n \rightarrow \mathcal{K}_C$ such that $F(x) = [\underline{f}(x), \overline{f}(x)]$, where $\underline{f}, \overline{f} : S \rightarrow \mathbb{R}$, is said to*

be d_H -continuous at $x_0 \in S$ if for every $\epsilon > 0$, there exists a $\delta > 0$ such that $\|x - x_0\| < \delta$ implies

$$d_H(F(x), F(x_0)) < \epsilon.$$

The next result is a natural extension of Theorem 2.5.1 and follows directly from the definitions.

Theorem 2.5.2 (See J. P. Aubin and H. Franskowska, [5]; M. K. Luhandjula and M. J. Rangoaga, [51]). *Let $F : S \subseteq \mathbb{R}^n \rightarrow \mathcal{K}_C$ be an interval-valued function such that $F(x) = [\underline{f}(x), \overline{f}(x)]$, where $\underline{f}, \overline{f} : S \rightarrow \mathbb{R}$. F is d_H -continuous at $x_0 \in S$ if and only if $\underline{f}, \overline{f}$ are continuous at x_0 . Moreover, if F is d_H -continuous at x_0 , then*

$$\lim_{x \rightarrow x_0} F(x) = \left[\lim_{x \rightarrow x_0} \underline{f}(x), \lim_{x \rightarrow x_0} \overline{f}(x) \right] = [\underline{f}(x_0), \overline{f}(x_0)].$$

In this Thesis, it will be considered two differentiability of interval-valued functions: the E -differentiability and the gH -differentiability.

2.6 E -differentiability of interval-valued functions

First, it will be considered the following differentiability:

Definition 2.6.1 (See U. A. S. Leal, [46]; H. C. Wu, [85]). *Let $S \subseteq \mathbb{R}$ be an open and nonempty set. An interval-valued function $F : S \rightarrow \mathcal{K}_C$ such that $F(x) = [\underline{f}(x), \overline{f}(x)]$ is called extremal differentiable, E -differentiable in short, or weakly differentiable at $x_0 \in S$ if the extreme functions $\underline{f}, \overline{f} : S \rightarrow \mathbb{R}$ are differentiable at x_0 (in the classical sense).*

Now we are going to consider the interval-valued function F defined on $\mathbb{R}^n$, i.e., $F(x) = F(x_1, \dots, x_n)$ is an interval in $\mathbb{R}$ for each $x = (x_1, \dots, x_n) \in \mathbb{R}^n$. Therefore, we also have the corresponding real-valued functions $\underline{f}(x) = \underline{f}(x_1, \dots, x_n)$ and $\overline{f}(x) = \overline{f}(x_1, \dots, x_n)$ defined on $\mathbb{R}^n$.

Definition 2.6.2 (See H. C. Wu, [85]). *Let $S \subseteq \mathbb{R}^n$ be an open and nonempty set. An interval-valued function $F : S \rightarrow \mathcal{K}_C$ such that $F(x) = [\underline{f}(x), \overline{f}(x)]$ is called extremal differentiable, E -differentiable in short, or weakly differentiable at $x_0 \in S$ if the extreme functions $\underline{f}, \overline{f} : S \rightarrow \mathbb{R}$ are differentiable at x_0 .*

Then, naturally, it can be given the next definition.

Definition 2.6.3. Let $S \subseteq \mathbb{R}^n$ be an open and nonempty set. An interval-valued function $F : S \rightarrow \mathcal{K}_C$ is continuously E -differentiable at x_0 if all the partial derivatives

$$\frac{\partial \underline{f}}{\partial x_i} \quad \text{and} \quad \frac{\partial \bar{f}}{\partial x_i}, \quad i \in \{1, \dots, n\},$$

exist on some neighborhood of x_0 and are continuous at x_0 .

The E -differentiability can be generalized by the differentiability below.

2.7 gH -differentiability of interval-valued functions

Based on the gH -difference and the limit of interval-valued functions, it will be given the following differentiability concept:

Definition 2.7.1 (See L. Stefanini and B. Bede, [82]). Let $S \subseteq \mathbb{R}$ be an open and nonempty set, and let $F : S \rightarrow \mathcal{K}_C$ be an interval-valued function. Then, the generalized Hukuhara derivative, gH -derivative, for short, of F at $x_0 \in S$ is defined by

$$F'_{gH}(x_0) = \lim_{h \rightarrow 0} \frac{1}{h} \cdot (F(x_0 + h) \ominus_{gH} F(x_0)). \quad (2.11)$$

If $F'_{gH}(x_0) \in \mathcal{K}_C$ satisfying (2.11) exist, F is said to be generalized Hukuhara differentiable, gH -differentiable, for short, at x_0 .

The relationships between the gH -differentiability of an interval-valued function $F(x) = [\underline{f}(x), \bar{f}(x)]$ and the differentiability of its extremum functions, $\underline{f}(x)$ and $\bar{f}(x)$, have been studied. Let us see:

Theorem 2.7.1 (See Y. Chalco-Cano, H. Román-Flores, M. D. Jiménez-Gamero, [20]). Let $S \subseteq \mathbb{R}$ be an open and nonempty set, and let $F : S \rightarrow \mathcal{K}_C$ be an interval-valued function. Then F is gH -differentiable at $x_0 \in S$ if and only if one of the following cases holds:

(i) $\underline{f}$ and $\bar{f}$ are differentiable at x_0 , and

$$F'_{gH}(x_0) = [\underline{f}'(x_0) \vee \bar{f}'(x_0)].$$

(ii) $\underline{f}'_-(x_0)$, $\underline{f}'_+(x_0)$, $\bar{f}'_-(x_0)$, $\bar{f}'_+(x_0)$ exist, satisfy $\underline{f}'_-(x_0) = \bar{f}'_+(x_0)$, $\underline{f}'_+(x_0) = \bar{f}'_-(x_0)$ and

$$F'_{gH}(x_0) = [\underline{f}'_-(x_0) \vee \bar{f}'_-(x_0)] = [\underline{f}'_+(x_0) \vee \bar{f}'_+(x_0)].$$

Definition 2.7.2 (See Definition 3 in Y. Chalco-Cano and H. Román-Flores, [19]). *Let $S \subseteq \mathbb{R}$ be an open and nonempty set, and let $F : S \rightarrow \mathcal{K}_C$ be an interval-valued function. We have the following definitions:*

(i) F is gH -differentiable at $x_0 \in S$ in the first form if $\underline{f}$ and $\overline{f}$ are differentiable at x_0 , and

$$F'_{gH}(x_0) = [\underline{f}'(x_0), \overline{f}'(x_0)];$$

(ii) F is gH -differentiable at $x_0 \in S$ in the second form if $\underline{f}$ and $\overline{f}$ are differentiable at x_0 , and

$$F'_{gH}(x_0) = [\overline{f}'(x_0), \underline{f}'(x_0)];$$

(iii) F is gH -differentiable at $x_0 \in S$ in the third form if $\underline{f}'_-(x_0)$, $\underline{f}'_+(x_0)$, $\overline{f}'_-(x_0)$, $\overline{f}'_+(x_0)$ exist and satisfy $\underline{f}'_-(x_0) = \overline{f}'_+(x_0)$, $\underline{f}'_+(x_0) = \overline{f}'_-(x_0)$, and

$$F'_{gH}(x_0) = [\underline{f}'_-(x_0), \overline{f}'_-(x_0)] = [\overline{f}'_+(x_0), \underline{f}'_+(x_0)];$$

(iv) F is gH -differentiable at $x_0 \in S$ in the fourth form if $\underline{f}'_-(x_0)$, $\underline{f}'_+(x_0)$, $\overline{f}'_-(x_0)$, $\overline{f}'_+(x_0)$ exist and satisfy $\underline{f}'_-(x_0) = \overline{f}'_+(x_0)$, $\underline{f}'_+(x_0) = \overline{f}'_-(x_0)$, and

$$F'_{gH}(x_0) = [\overline{f}'_-(x_0), \underline{f}'_-(x_0)] = [\underline{f}'_+(x_0), \overline{f}'_+(x_0)].$$

Definition 2.7.3 (See Definition 3 in Y. Chalco-Cano and H. Román-Flores, [19]; L. Stefanini and B. Bede, [82]). *A point $x_0 \in S$ is said to be a switching point for the gH -differentiability of F , if in any neighborhood N of x_0 there exist points $x_1 < x_0 < x_2$, such that*

(type 1) F is gH -differentiable at x_1 in the first form while it is not gH -differentiable in the second form, and F is gH -differentiable at x_2 in the second form while it is not gH -differentiable in the first form, i.e., at x_1 happens $F'_{gH}(x_1) = [\underline{f}'(x_1), \overline{f}'(x_1)]$ and at x_2 happens $F'_{gH}(x_2) = [\overline{f}'(x_2), \underline{f}'(x_2)]$ or

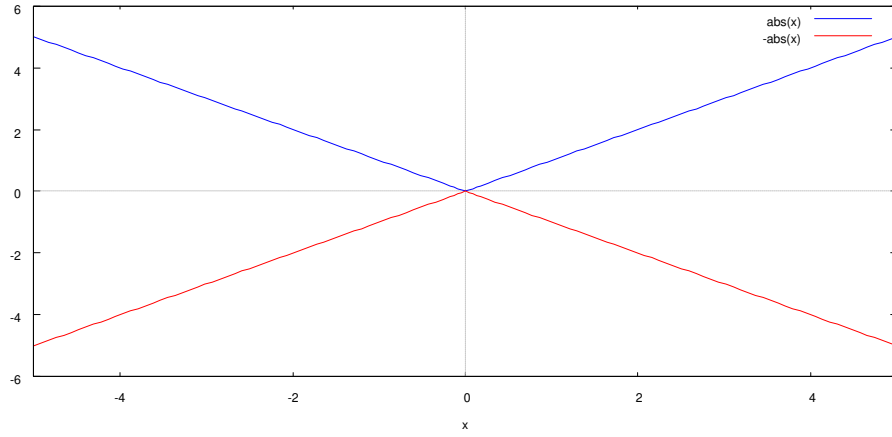
(type 2) F is gH -differentiable at x_1 in the second form while it is not gH -differentiable in the first form, and F is gH -differentiable at x_2 in the first form while it is not gH -differentiable in the second form, i.e., at x_1 happens $F'_{gH}(x_1) = [\overline{f}'(x_1), \underline{f}'(x_1)]$ and at x_2 happens $F'_{gH}(x_2) = [\underline{f}'(x_2), \overline{f}'(x_2)]$.

Remark 2.7.1 (See Y. Chalco-Cano and A. Flores-Franulić, [14]). *Note that if $\text{len}(F(x)) = \overline{f}(x) - \underline{f}(x)$ is decreasing on the interval $(0, +\infty)$ whereas*

$\text{len}(F(x)) = \bar{f}(x) - \underline{f}(x)$ is increasing on the interval $(-\infty, 0)$. Then, $x = 0$ is a switching point for the monotonicity of $\text{len}(F(x))$, that is, $\text{len}(F(x))$ changes its monotonicity at $x = 0$.

Example 2.7.1. Let the interval-valued function $F : \mathbb{R} \rightarrow \mathcal{K}_C$ be defined by $F(x) = [\underline{f}(x), \bar{f}(x)] = [-|x|, |x|]$ for all $x \in \mathbb{R}$. Then F is gH -differentiable in $\mathbb{R}$ but the extremum functions $\underline{f}$ and $\bar{f}$ are not differentiable at 0, see Figure 1. Moreover, from (i) of Theorem 2.7.1, it follows that $F'_{gH}(x) = [\bar{f}'(x), \underline{f}'(x)] = [-1, 1]$ for all $x \in (-\infty, 0)$ and $F'_{gH}(x) = [\underline{f}'(x), \bar{f}'(x)] = [-1, 1]$ for all $x \in (0, +\infty)$. From (ii) of Theorem 2.7.1, it follows that $F'_{gH}(0) = [-1, 1]$. Then, $x = 0$ is a switching point.

Figure 1: $F : \mathbb{R} \rightarrow \mathcal{K}_C$ defined by $F(x) = [\underline{f}(x), \bar{f}(x)] = [-|x|, |x|]$ of Example 2.7.1.



Source: Elaborated by the author herself.

In the following, the gH -differentiability for interval-valued functions of several variables will be an extension of the gH -differentiability for interval-valued functions of a real variable given by L. Stefanini and B. Bede, [82].

Definition 2.7.4 (See Definition 6 in L. Stefanini and M. Arana Jiménez, [81]). Let $F : S \subseteq \mathbb{R}^n \rightarrow \mathcal{K}_C$, $F(x) = (f^C(x); f^R(x))$, and let $x_0 \in S$ such that $x_0 + h \in S$ for all $h \in \mathbb{R}^n$ with $\|h\|_{\mathbb{R}^n} < \delta$ for a given $\delta > 0$. F is said to be gH -differentiable at x_0 if and only if there exist two vectors $w^C, w^R \in \mathbb{R}^n$, $w^C = (w_1^C, \dots, w_n^C)$, $w^R = (w_1^R, \dots, w_n^R)$ and two functions $\epsilon^C, \epsilon^R : \mathbb{R}^n \rightarrow \mathbb{R}$ with $\lim_{h \rightarrow 0} \epsilon^C(h) = \lim_{h \rightarrow 0} \epsilon^R(h) = 0$ such that for all $h \neq 0$,

$$f^C(x_0 + h) - f^C(x_0) = \sum_{j=1}^n h_j w_j^C + \|h\|_{\mathbb{R}^n} \epsilon^C(h) \quad (2.12)$$

and

$$|f^R(x_0 + h) - f^R(x_0)| = \left| \sum_{j=1}^n h_j w_j^R + \|h\|_{\mathbb{R}^n} \epsilon^R(h) \right|. \quad (2.13)$$

The interval-valued function $D_{gH}F(x_0) : \mathbb{R}^n \rightarrow \mathcal{K}_C$ defined, for $h = (h_1, \dots, h_n) \in \mathbb{R}^n$, by

$$D_{gH}F(x_0)(h) = \left(\sum_{j=1}^n h_j w_j^C; \left| \sum_{j=1}^n h_j w_j^R \right| \right)$$

is called the gH -differential (or the total gH -derivative) of F at x_0 and $D_{gH}F(x_0)(h)$ is the interval-valued differential of F at x_0 with respect to h .

Definition 2.7.5 (See Definition 10 in L. Stefanini and M. Arana Jiménez, [81]). Let S be a nonempty open subset of $\mathbb{R}^n$ and $F : S \rightarrow \mathcal{K}_C$ be an interval-valued function, with $x_0 \in S$. If $d \in \mathbb{R}^n$ is any admissible direction at $x_0 \in S$ (that is, $x_0 \in S$ is such that $x_0 + hd \in S$ for all $h \in \mathbb{R}$ with $\|h\|_{\mathbb{R}^n} < \delta$ for a given $\delta > 0$), then the directional generalized Hukuhara derivative, directional gH -derivative, for short, of F at x_0 in the direction d is defined by

$$F'_{gH}(x_0; d) = \lim_{h \rightarrow 0^+} \frac{1}{h} \cdot (F(x_0 + hd) \ominus_{gH} F(x_0))$$

if it exists.

The partial (interval-valued) gH -derivative of F at $x_0 \in S$ with respect to x_j , $j \in \{1, \dots, n\}$, is defined by

$$\frac{\partial_{gH} F}{\partial x_j}(x_0) = \lim_{h \rightarrow 0} \frac{1}{h} \cdot (F(x_0 + h e_j) \ominus_{gH} F(x_0))$$

if it exists.

Proposition 2.7.1 (See Proposition 9 in L. Stefanini and M. Arana Jiménez, [81]). Let $F : S \rightarrow \mathcal{K}_C$, S an open subset of $\mathbb{R}^n$. If F is gH -differentiable at $x_0 \in S$, then all the interval-valued partial gH -derivatives of F exist and, with reference to (2.12)-(2.13), it holds

$$\frac{\partial_{gH} F}{\partial x_j}(x_0) = (w_j^C; |w_j^R|), \quad j \in \{1, \dots, n\}.$$

Proposition 2.7.2 (See Proposition 11 in L. Stefanini and M. Arana Jiménez, [81]). Let

$F : S \rightarrow \mathcal{K}_C$ and S an open subset of $\mathbb{R}^n$. Assume that $F = [f, \bar{f}]$ is gH -differentiable at $x_0 \in S$ and suppose that the lateral partial derivatives of $\underline{f}$ and $\bar{f}$ exist at x_0 for a given index $j \in \{1, \dots, n\}$. Denote $\underline{u}_j^- = \frac{\partial \underline{f}}{\partial x_j^-}(x_0)$, $\underline{u}_j^+ = \frac{\partial \underline{f}}{\partial x_j^+}(x_0)$, $\bar{u}_j^- = \frac{\partial \bar{f}}{\partial x_j^-}(x_0)$ and $\bar{u}_j^+ = \frac{\partial \bar{f}}{\partial x_j^+}(x_0)$.

(i) If $\underline{u}_j^- = \underline{u}_j^+ =: m_j$ and $\bar{u}_j^- = \bar{u}_j^+ =: n_j$, then

$$\frac{\partial_{gH} F}{\partial x_j}(x_0) = \left[m_j \bigvee n_j \right] = \left(\frac{m_j + n_j}{2}; \left| \frac{m_j - n_j}{2} \right| \right).$$

(ii) If $\underline{u}_j^- = \bar{u}_j^+ =: p_j$ and $\bar{u}_j^- = \underline{u}_j^+ =: q_j$, then

$$\frac{\partial_{gH} F}{\partial x_j}(x_0) = \left[p_j \bigvee q_j \right] = \left(\frac{p_j + q_j}{2}; \left| \frac{p_j - q_j}{2} \right| \right).$$

Theorem 2.7.2 (See Theorem 4 in L. Stefanini and M. Arana Jiménez, [81]). Let $F = (f^C; f^R)$ be an interval-valued function on an open set $S \subseteq \mathbb{R}^n$ and assume that F is gH -differentiable at x_0 ; if f^R has all the (standard) partial derivatives at x_0 and $d = (d_1, \dots, d_n) \in \mathbb{R}^n$ is any fixed direction, then the directional gH -derivative of F at x_0 is given by

$$F'_{gH}(x_0; d) = \left(\sum_{j=1}^n \frac{\partial f^C}{\partial x_j}(x_0) d_j; \left| \sum_{j=1}^n \frac{\partial f^R}{\partial x_j}(x_0) d_j \right| \right) = D_{gH} F(x_0)(d).$$

Theorem 2.7.3 (See Theorem 5 in L. Stefanini and M. Arana Jiménez, [81]). Let $F = (f^C; f^R)$ be an interval-valued function on an open set $S \subseteq \mathbb{R}^n$. If F is gH -differentiable at x_0 , then all the interval-valued partial gH -derivatives of F exist and, following the notation used in Definition 2.7.4,

$$\frac{\partial_{gH} F}{\partial x_j}(x_0) = F'_{gH}(x_0; e_j) = (w_j^C; |w_j^R|) \quad (2.14)$$

for $j \in \{1, \dots, n\}$. Furthermore, all directional gH -derivatives of F exist and

$$F'_{gH}(x_0; d) = \left(\sum_{j=1}^n w_j^C d_j; \left| \sum_{j=1}^n w_j^R d_j \right| \right). \quad (2.15)$$

Let $f : \mathbb{R}^n \rightarrow \mathbb{R}$ and $x_0 \in \mathbb{R}^n$. Next, it will be used the following notation:

$$(\nabla f)_-(x_0) = \left(\frac{\partial f}{\partial x_1^-}(x_0), \dots, \frac{\partial f}{\partial x_n^-}(x_0) \right)$$

and

$$(\nabla f)_+(x_0) = \left(\frac{\partial f}{\partial x_1^+}(x_0), \dots, \frac{\partial f}{\partial x_n^+}(x_0) \right).$$

The theorem below follows directly from Proposition 2.7.2 and Theorem 2.7.3.

Theorem 2.7.4. *Let $S \subseteq \mathbb{R}^n$ be an open and nonempty set, and let $F : S \rightarrow \mathcal{K}_C$ be an interval-valued function with $F(x) = [f(x), \bar{f}(x)]$ for all $x \in S$. If F is gH -differentiable at $x_0 \in S$, then for all $d \in \mathbb{R}^n$, one of the following cases holds:*

(i) $\nabla \underline{f}(x_0)$ and $\nabla \bar{f}(x_0)$ exist, and

$$F'_{gH}(x_0; d) = \left[\langle \nabla \underline{f}(x_0), d \rangle \vee \langle \nabla \bar{f}(x_0), d \rangle \right].$$

Particularly,

$$\frac{\partial_{gH} F}{\partial x_i}(x_0) = \left[\frac{\partial f}{\partial x_i}(x_0) \vee \frac{\partial \bar{f}}{\partial x_i}(x_0) \right] \text{ for all } i \in \{1, \dots, n\}.$$

(ii) $(\nabla \underline{f})_-(x_0)$, $(\nabla \bar{f})_-(x_0)$, $(\nabla \underline{f})_+(x_0)$, $(\nabla \bar{f})_+(x_0)$ exist, satisfy $\langle (\nabla \underline{f})_-(x_0), d \rangle = \langle (\nabla \bar{f})_+(x_0), d \rangle$ and $\langle (\nabla \bar{f})_-(x_0), d \rangle = \langle (\nabla \underline{f})_+(x_0), d \rangle$, and

$$\begin{aligned} F'_{gH}(x_0; d) &= \left[\langle (\nabla \underline{f})_-(x_0), d \rangle \vee \langle (\nabla \bar{f})_-(x_0), d \rangle \right] \\ &= \left[\langle (\nabla \underline{f})_+(x_0), d \rangle \vee \langle (\nabla \bar{f})_+(x_0), d \rangle \right]. \end{aligned}$$

Particularly,

$$\frac{\partial_{gH} F}{\partial x_i}(x_0) = \left[\frac{\partial f}{\partial x_i^-}(x_0) \vee \frac{\partial \bar{f}}{\partial x_i^-}(x_0) \right] = \left[\frac{\partial f}{\partial x_i^+}(x_0) \vee \frac{\partial \bar{f}}{\partial x_i^+}(x_0) \right] \text{ for all } i \in \{1, \dots, n\}.$$

Let $x^* \in \mathbb{R}^n$. It will be used the following notation:

$$\begin{aligned} \nabla_{\#} \underline{f}(x^*) &= \begin{cases} \nabla \underline{f}(x^*), & \text{if it exists,} \\ \nabla \underline{f}_-(x^*), & \text{otherwise,} \end{cases} \\ \nabla_{\#} \bar{f}(x^*) &= \begin{cases} \nabla \bar{f}(x^*), & \text{if it exists,} \\ \nabla \bar{f}_-(x^*), & \text{otherwise,} \end{cases} \\ \frac{\partial f}{\partial x_i^{\#}}(x^*) &= \begin{cases} \frac{\partial f}{\partial x_i}(x^*), & \text{if it exists,} \\ \frac{\partial f}{\partial x_i^-}(x^*), & \text{otherwise,} \end{cases} \\ \frac{\partial \bar{f}}{\partial x_i^{\#}}(x^*) &= \begin{cases} \frac{\partial \bar{f}}{\partial x_i}(x^*), & \text{if it exists,} \\ \frac{\partial \bar{f}}{\partial x_i^-}(x^*), & \text{otherwise,} \end{cases} \end{aligned}$$

$$\begin{aligned} \frac{\partial_{gH} F}{\partial x_i}(x^*) &= \begin{cases} \left[\frac{\partial f}{\partial x_i}(x^*) \vee \frac{\partial \bar{f}}{\partial x_i}(x^*) \right], & \text{if it exists,} \\ \left[\frac{\partial f}{\partial x_i^-}(x^*) \vee \frac{\partial \bar{f}}{\partial x_i^-}(x^*) \right] = \left[\frac{\partial f}{\partial x_i^+}(x_0) \vee \frac{\partial \bar{f}}{\partial x_i^+}(x_0) \right], & \text{otherwise} \end{cases} \\ &= \left[\frac{\partial f}{\partial x_i^\#}(x^*) \vee \frac{\partial \bar{f}}{\partial x_i^\#}(x^*) \right] \end{aligned}$$

and

$$\nabla_{gH} F(x^*) = \left(\frac{\partial_{gH} F}{\partial x_1}(x^*), \dots, \frac{\partial_{gH} F}{\partial x_n}(x^*) \right).$$

Definition 2.7.6 (See M. K. Luhandjula and M. J. Rangoaga, [51]). *Let $S \subseteq \mathbb{R}^n$ be an open and nonempty set. An interval-valued function $F : S \rightarrow \mathcal{K}_C$ is continuously gH -differentiable at x_0 if all the partial gH -derivatives*

$$\frac{\partial_{gH} F}{\partial x_i}(x_0) \quad \text{for all } i \in \{1, \dots, n\},$$

exist on some neighborhood of x_0 and are d_H -continuous at x_0 , in the sense of Definition 2.5.2.

Throughout the Thesis, it will be given more importance to the gH -differentiability, the continuously gH -differentiability and the directional gH -derivative.

In the following section, it will be brought to mind some notions and results of integrability of interval-valued functions.

2.8 Integrability of interval-valued functions

An interval-valued function $F : [a, b] \subseteq \mathbb{R} \rightarrow \mathcal{K}_C$ is said to be *measurable* if and only if

$$\{(t, x) \mid x \in F(t)\} \in \mathcal{L} \times \mathcal{B},$$

where $\mathcal{L}$ denotes the σ -algebra composed of all Lebesgue-measurable subsets of $\mathbb{R}$ and $\mathcal{B}$ denotes the σ -algebra composed of all Borel-measurable subsets of $\mathbb{R}$. F is said to be *integrably bounded on $[a, b]$* if and only if there exists a Lebesgue-integrable function $h : [a, b] \rightarrow [0, +\infty)$ such that

$$|x| \leq h(t)$$

for all x and t such that $x \in F(t)$.

Definition 2.8.1 (See R. J. Aumann, [6]). *Given an interval-valued function*

$F : [a, b] \rightarrow \mathcal{K}_C$, the Aumann integral ((IA)–integral, for short) of F over $[a, b]$ is defined by

$$(IA) \quad \int_a^b F(t)dt = \left\{ \int_a^b f(t)dt \mid f \in S(F) \right\},$$

where $S(F) := \{f \in L_1([a, b]; \mathbb{R}) \mid f(t) \in F(t) \text{ for almost every } t \in [a, b]\}$ and $L_1([a, b]; \mathbb{R})$ is the space of all functions $f : [a, b] \rightarrow \mathbb{R}$ that are Lebesgue–integrable over $[a, b]$. The (IA)–integral of F over $[a, b]$ exists (or F is (IA)–integrable over $[a, b]$) if $S(F) \neq \emptyset$.

For further details, the following two results, can be seen in P. Diamond and P. E. Kloeden, [26]; R. Goetschel and W. Voxman, [37]; J. L. W. V. Jensen, [40]; O. Kaleva, [42].

Theorem 2.8.1. *If an interval–valued function $F : [a, b] \rightarrow \mathcal{K}_C$ is measurable and integrably bounded on $[a, b]$, then F is (IA)–integral over $[a, b]$, and*

$$\int_a^b F(t)dt \in \mathcal{K}_C.$$

Theorem 2.8.2. *Let $F : [a, b] \rightarrow \mathcal{K}_C$ be an interval–valued function such that $F(t) = [\underline{f}(t), \bar{f}(t)]$, where $\underline{f}, \bar{f} : [a, b] \rightarrow \mathbb{R}$. Then F is (IA)–integrable over $[a, b]$ if and only if $\underline{f}$ and $\bar{f}$ belongs to $L_1([a, b]; \mathbb{R})$. Moreover, if F is (IA)–integrable over $[a, b]$, then*

$$\int_a^b F(t)dt = \left[\int_a^b \underline{f}(t)dt, \int_a^b \bar{f}(t)dt, \right].$$

Remark 2.8.1. *By Theorem 2.5.1 and Theorem 2.8.2, if F is d_H –continuous, then F is (IA)–integrable over $[a, b]$.*

Next, it will be remembered the main classical types of integral equations.

2.9 Classical integral equations

An *integral equation* is an equation in which an unknown function appears under one or more integration signs, such as

$$y = \int_a^b \phi(x)dx \quad \text{or} \quad y(x) = \int_a^x \psi(x)dx.$$

A general type of integral equation,

$$g(x)y(x) = f(x) + \lambda \int_a^\square K(x, t)y(t)dt \quad (2.16)$$

is called *linear integral equation* as only linear operations are performed in the equation. The one, which is not linear, is obviously called *non-linear integral equation*. In this Thesis, when we write “integral equation”, we want to say “linear integral equation.”

In the general type of linear equation 2.16, it was used a box, $\square$, to indicate the higher limit of the integration. Integral equations can be of two types according to whether the box, $\square$,

- (i) is a constant “ b ”
- (ii) or a variable “ x .”

First type of integral equations which involve constants as both the limits are called *Fredholm type integral equations*:

$$g(x)y(x) = f(x) + \lambda \int_a^b K(x,t)y(t)dt.$$

On the other hand, when one of the limits is a variable (x , the independent variable of which y , f , and K are functions), the integral equation is called *Volterra type integral equations*:

$$g(x)y(x) = f(x) + \lambda \int_a^x K(x,t)y(t)dt.$$

In an integral equation, y is to be determined with g , f , K being known, and λ being a non-zero complex parameter. The function $K(x,t)$ is called *Kernel of the integral equation*.

2.9.1 Types of Fredholm integral equations

As the general form of Fredholm integral equation is

$$g(x)y(x) = f(x) + \lambda \int_a^b K(x,t)y(t)dt,$$

there may be other types of it according to the valued of g and f :

- (i) *Fredholm Integral Equation of First Kind*, when $g(x) = 0$:

$$f(x) + \lambda \int_a^b K(x,t)y(t)dt = 0.$$

- (ii) *Fredholm Integral Equation of Second Kind*, when $g(x) = 1$:

$$y(x) = f(x) + \lambda \int_a^b K(x,t)y(t)dt.$$

- (iii) *Fredholm Integral Equation of Homogeneous Second Kind*, when $g(x) = 1$ and $f(x) = 0$:

$$y(x) = \lambda \int_a^b K(x, t)y(t)dt.$$

- (iv) *Fredholm Integral Equation of Third (Final) Kind*, when $1 \neq g(x) \neq 0$ and $f(x) \neq 0$.

2.9.2 Types of Volterra integral equations

As the general form of Volterra integral equation is

$$g(x)y(x) = f(x) + \lambda \int_a^x K(x, t)y(t)dt,$$

there may be other types of it according to the valued of g and f :

- (i) *Volterra Integral Equation of First Kind*, when $g(x) = 0$:

$$f(x) + \lambda \int_a^x K(x, t)y(t)dt = 0.$$

- (ii) *Volterra Integral Equation of Second Kind*, when $g(x) = 1$:

$$y(x) = f(x) + \lambda \int_a^x K(x, t)y(t)dt. \quad (2.17)$$

- (iii) *Volterra Integral Equation of Homogeneous Second Kind*, when $g(x) = 1$ and $f(x) = 0$:

$$y(x) = \lambda \int_a^x K(x, t)y(t)dt.$$

- (iv) *Volterra Integral Equation of Third (Final) Kind*, when $1 \neq g(x) \neq 0$ and $f(x) \neq 0$.

In the following, it will be studied interval optimization problems, their kinds of solutions, some Alternative Theorems, a constraint qualification, and a regularity condition.

2.10 The interval optimization problem

Now, consider the following *interval optimization problem* with inequality constraints:

$$(IOP) \begin{cases} \text{minimize} & F(x) = [\underline{f}(x), \bar{f}(x)] \\ \text{subject to} & g_j(x) \leq 0, \quad j \in \{1, \dots, m\} \\ & x \in S \subseteq \mathbb{R}^n, \end{cases}$$

where $F : S \subseteq \mathbb{R}^n \rightarrow \mathcal{K}_C$ is an interval-valued function given by $F(x) = [\underline{f}(x), \overline{f}(x)]$ with $\underline{f}, \overline{f} : S \subseteq \mathbb{R}^n \rightarrow \mathbb{R}$ and $\underline{f}(x) \leq \overline{f}(x)$ for all $x \in S \subseteq \mathbb{R}^n$, and $g_j : S \subseteq \mathbb{R}^n \rightarrow \mathbb{R}$ are the constraints of (IOP), $j \in \{1, \dots, m\}$.

By simplicity, the *indices set* is defined by

$$I = \{1, \dots, m\}$$

and the *admissible set* is defined by

$$\mathcal{X} = \{x \in S \subseteq \mathbb{R}^n \mid g_j(x) \leq 0, j \in I\}.$$

For each *admissible point* $x \in \mathcal{X}$, the set of *indices of active constraints at x* is defined by

$$I(x) = \{j \in I \mid g_j(x) = 0\}.$$

Definition 2.10.1 (See H. C. Wu, [85]). *Let x^* be an admissible solution of (IOP), i.e., $x^* \in \mathcal{X}$. Then,*

- (i) x^* is said to be a **local strong LU-solution** of (IOP) if there exists $\epsilon > 0$ such that there does not exist $x \in N_\epsilon(x^*) \cap \mathcal{X}$, $x \neq x^*$, with $F(x) \leq_{LU} F(x^*)$.
- (ii) x^* is said to be a **local LU-solution** of (IOP) if there exists $\epsilon > 0$ such that there does not exist $x \in N_\epsilon(x^*) \cap \mathcal{X}$ with $F(x) \leq_{LU} F(x^*)$.
- (iii) x^* is said to be a **local $\underline{LU}$ -solution** of (IOP) if there exists $\epsilon > 0$ such that there does not exist $x \in N_\epsilon(x^*) \cap \mathcal{X}$ with $F(x) \leq_{\underline{LU}} F(x^*)$.
- (iv) x^* is said to be a **local $\overline{LU}$ -solution** of (IOP) if there exists $\epsilon > 0$ such that there does not exist $x \in N_\epsilon(x^*) \cap \mathcal{X}$ with $F(x) \leq_{\overline{LU}} F(x^*)$.
- (v) x^* is said to be a **local weak LU-solution** of (IOP) if there exists $\epsilon > 0$ such that there does not exist $x \in N_\epsilon(x^*) \cap \mathcal{X}$ with $F(x) <_{LU} F(x^*)$.

If $N_\epsilon(x^*)$ above is replaced by the whole space $\mathbb{R}^n$, we have the corresponding definitions of global solutions for (IOP).

Remark 2.10.1. Note that

- (i) The following relations are immediate:

$\text{strong LU-solution} \Rightarrow \text{LU-solution} \Rightarrow \underline{LU}/\overline{LU}\text{-solution} \Rightarrow \text{weak LU-solution}.$
--

(ii) The theory that will be developed for weak LU -solutions is applicable to $\underline{LU}/\overline{LU}$ -solutions, to LU -solutions and to strong LU -solutions as well.

Now, let us remember some important results:

Theorem 2.10.1 (Motzkin's Alternative Theorem, see O. L. Mangasarian, [53]). *Let A , C , and D be given matrices, with A being nonvacuous. Then either*

$$Ax > 0, \quad Cx \geq 0, \quad \text{and} \quad Dx = 0 \quad \text{has a solution } x$$

or

$$\begin{cases} A^\top y_1 + C^\top y_3 + D^\top y_4 = 0 \\ y_1 \geq 0, \quad y_3 \geq 0 \end{cases} \quad \text{has a solution } y_1, y_3, y_4$$

but never both.

Theorem 2.10.2 (Tucker's Alternative Theorem, see O. L. Mangasarian, [53]). *Let B , C , and D be given matrices, with B being nonvacuous. Then either*

$$Bx \geq 0, \quad Cx \geq 0, \quad \text{and} \quad Dx = 0 \quad \text{has a solution } x$$

or

$$\begin{cases} B^\top y_2 + C^\top y_3 + D^\top y_4 = 0 \\ y_2 > 0, \quad y_3 \geq 0 \end{cases} \quad \text{has a solution } y_2, y_3, y_4$$

but never both.

Theorem 2.10.3 (Gordan Transposition Theorem, see O. L. Mangasarian, [53]). *For each given matrix A , either*

$$Ax > 0 \quad \text{has a solution } x$$

or

$$A^\top y = 0, \quad y \geq 0 \quad \text{has a solution } y$$

but never both.

Definition 2.10.2. *The constraints of (IOP) are said to satisfy the positive linear independence constraint qualification (PLICQ), or the Mangasarian–Fromovitz constraint qualification, at an admissible point $x^* \in \mathcal{X}$ if there do not exist $\beta_j \geq 0$, $j \in I(x^*)$, not all zero, such that*

$$\sum_{j \in I(x^*)} \beta_j \nabla g_j(x^*) = \mathbf{0}.$$

The qualification conditions involving only the problem constraints are not enough to ensure, in general, the existence of the nonzero multipliers associated to the objective functions. Therefore, one needs conditions involving the problem functions. So, it will be considered another qualification condition called: the positive linear independence regularity condition. This condition was given, for example, in M. C. Maciel, S. A. Santos, G. N. Sottosanto, [52], among others, to deal with vector optimization problems. This condition results to be necessary and sufficient for the total regularity ², and next it is adapted to deal with the classe of interval optimization problems herein considered in order to derive a more accurate *KKT*–type theorem.

Definition 2.10.3. *The constraints of (IOP) are said to satisfy the positive linear independence regularity condition (PLIRC) at an admissible point $x^* \in \mathcal{X}$ if*

(i) *The constraints of (IOP) satisfy the positive linear independence constraint qualification (PLICQ) at $x^* \in \mathcal{X}$.*

(ii) *There does not exist $\lambda_1 > 0$ and $\mu_j \geq 0$, $j \in I(x^*)$, such that*

$$\lambda_1 \nabla_{\#} \underline{f}(x^*) + \sum_{j \in I(x^*)} \mu_j \nabla g_j(x^*) = \mathbf{0},$$

and there does not exist $\lambda_2 > 0$ and $\mu_j \geq 0$, $j \in I(x^*)$, such that

$$\lambda_2 \nabla_{\#} \bar{f}(x^*) + \sum_{j \in I(x^*)} \mu_j \nabla g_j(x^*) = \mathbf{0}.$$

²The following can be found in G. Bigi and M. Pappalardo, [9]; M. C. Maciel, S. A. Santos, G. N. Sottosanto, [52]:

(i) Given $x \in \mathcal{X}$ such that $M(x) \neq \emptyset$,

(I) x is said to be *weak-regular* if there exists $(\lambda, \mu) \in M(x)$ with $\lambda \neq 0$.

(II) x is said to be *totally weak-regular* if, for all $(\lambda, \mu) \in M(x)$, there exists $i \in \{1, 2\}$ such that $\lambda_i \neq 0$.

(III) x is said to be *regular* if there exists $(\lambda, \mu) \in M(x)$ with $\lambda_i > 0$ for all $i \in \{1, 2\}$.

(IV) x is said to be *totally regular* if, for all $(\lambda, \mu) \in M(x)$, one has $\lambda_i > 0$ for all $i \in \{1, 2\}$.

(ii) The following relations are natural:

$$\boxed{(IV) \Rightarrow (II) \Rightarrow (I)}$$

and

$$\boxed{(IV) \Rightarrow (III) \Rightarrow (I)}.$$

(iii) Note that (II) and (III) are not related.

Remark 2.10.2. (i) Due to Motzkin's Alternative Theorem, see Theorem 2.10.1, the second condition of (PLIRC) is equivalent to: "There exists $\underline{d}, \bar{d} \in \mathbb{R}^n$ such that

$$\begin{aligned}\langle \nabla_{\#} f(x^*), \underline{d} \rangle &< 0, \\ \langle \nabla g_j(x^*), \underline{d} \rangle &\leq 0, \quad j \in I(x^*),\end{aligned}$$

and

$$\begin{aligned}\langle \nabla_{\#} \bar{f}(x^*), \bar{d} \rangle &< 0, \\ \langle \nabla g_j(x^*), \bar{d} \rangle &\leq 0, \quad j \in I(x^*).\end{aligned}$$

(ii) In item (i), it may happen that $\underline{d} = \bar{d} \in \mathbb{R}^n$.

It is important to mention that a similar result to G. Bigi, [8] (see Theorem 2.3.2), on the boundedness of a normalized set of multipliers, in the case that (PLIRC) holds, is also valid.

Next, it will be reminded some definitions and important results given in I. V. Girsanov, [36], that will be used in Chapter 6 and Chapter 7.

2.11 Some topological properties, cones, and dual cones

We start with a lemma related to closed sets:

Lemma 2.11.1 (See Lemma 2.6 in I. V. Girsanov, [36]). *Let A and B be closed sets in a topological linear space, and let B be bicomact³. Then $A + B$ is closed. If L_1, L_2 are closed subspaces and L_2 is finite-dimensional, then $L_1 + L_2$ is closed.*

Definition 2.11.1. *Let E a topological linear space. A nonzero continuous linear functional f is said to be a supporting functional for a set $A \subset E$ at $x_0 \in A$ if $f(x) \geq f(x_0)$ for all $x \in A$.*

The set of all linear functionals on E is a linear space which is denoted by E' and known as the *topological dual space*.

Definition 2.11.2. *Let E a topological linear space.*

³A set B is said to be *bicomact* if, from every covering of B by open sets, one can extract a finite covering.

- (i) A set $\mathcal{K}$ in E is called a cone if $\lambda\mathcal{K} = \mathcal{K}$ for any $\lambda > 0$, i.e., if $x \in \mathcal{K}$ implies that $\lambda x \in \mathcal{K}$ for all $\lambda > 0$.
- (ii) A cone which is a convex set is called a convex cone.
- (iii) $\mathcal{K}$ a cone in E . The dual cone $\mathcal{K}^* \subset E'$ is defined as the set of all continuous linear functionals nonnegative on $\mathcal{K}$, i.e.,

$$\mathcal{K}^* = \{f \in E' \mid f(x) \geq 0 \quad \text{for all } x \in \mathcal{K}\}.$$

Lemma 2.11.2 (See Lemma 5.8 in I. V. Girsanov, [36]). Let $\mathcal{K}_\alpha$, $\alpha \in I$, where I is an index set, be weakly closed and convex cones. Then

$$\left(\bigcap_{\alpha \in I} \mathcal{K}_\alpha\right)^* = \text{cl} \left(\text{co} \left(\sum_{\alpha \in I} \mathcal{K}_\alpha^* \right) \right),$$

i.e., the dual cone of the intersection is the weakly* closed convex hull of the sum of dual cones.

Corollary 2.11.1 (See Corollary of Lemma 5.8 in I. V. Girsanov, [36]). If the cones $\mathcal{K}_\alpha$, $\alpha \in I$, where I is an index set, are weakly closed and convex, while $\sum_{\alpha \in I} \mathcal{K}_\alpha^*$ is weakly* closed, then

$$\left(\bigcap_{\alpha \in I} \mathcal{K}_\alpha\right)^* = \sum_{\alpha \in I} \mathcal{K}_\alpha^*.$$

Lemma 2.11.3 (Dubovitskii–Milyutin, see Lemma 5.11 in I. V. Girsanov, [36]). Let $\mathcal{K}_1, \dots, \mathcal{K}_n, \mathcal{K}_{n+1}$ be convex cones with apex at 0, where $\mathcal{K}_1, \dots, \mathcal{K}_n$ are open. Then $\bigcap_{i=1}^{n+1} \mathcal{K}_i = \emptyset$ if and only if there exist linear functionals $f_i \in \mathcal{K}_i^*$, $i \in \{1, \dots, n+1\}$, not all zero, such that $f_1 + \dots + f_n + f_{n+1} = 0$.

Definition 2.11.3. Let E a topological linear space.

- (i) A vector h is a direction of decrease of the classical functional $f(x)$ at the point x_0 if there exist a neighborhood U of the vector h and a number $\alpha = \alpha(f, x_0, h)$, $\alpha < 0$, such that for all $0 < \epsilon < \epsilon_0$ and any $\bar{h} \in U$ such that $f(x_0 + \epsilon \bar{h}) \leq f(x_0) + \epsilon \alpha$.
- (ii) Given inequality constraints, a vector h is a feasible direction for Q at a point x_0 if there exists a neighborhood U of the vector h such that, for all $0 < \epsilon < \epsilon_0$ and all $\bar{h} \in U$, the vectors $x_0 + \epsilon \bar{h}$ are in Q .
- (iii) A vector h is a tangent direction to Q at a point x_0 if, for any $0 < \epsilon < \epsilon_0$, there exists a point $x(\epsilon) \in Q$ such that, if we set $x(\epsilon) = x_0 + \epsilon h + r(\epsilon)$, then the vector $r(\epsilon) \in E$

has the following property: for any neighborhood U of zero, $\frac{1}{\epsilon}r(\epsilon) \in U$ for all small $\epsilon > 0$ (in a Banach space this is replaced by the simpler condition $\|r(\epsilon)\| = o(\epsilon)$).

Theorem 2.11.1 (See Theorem 8.2 in I. V. Girsanov, [36]). *Let Q be a convex set. Then the cone of feasible directions $\mathcal{K}_b$ at the point x_0 is given by*

$$\mathcal{K}_b = \{h \mid h = \lambda(x - x_0), \quad x \in \text{int}(Q) \quad \text{and} \quad \lambda > 0\} = \{\lambda[\text{int}(Q) - x_0] \mid \lambda > 0\}.$$

Theorem 2.11.2 (Lyusternik, see Theorem 9.1 in I. V. Girsanov, [36]). *Let $P(x)$ be an operator mapping E_1 into E_2 , differentiable in a neighborhood of a point x_0 , $P(x_0) = 0$. Let $P'(x)$ be continuous in a neighborhood of x_0 and suppose that $P'(x_0)$ maps E_1 onto E_2 (i.e., the linear equation $P'(x_0)h = b$ has a solution h for any $b \in E_2$). Then the set of tangent directions $\mathcal{K}$ to the set $Q = \{x \mid P(x) = 0\}$ at the point x_0 is the subspace*

$$\mathcal{K} = \{h \mid P'(x_0)h = 0\}.$$

Theorem 2.11.3 (See Theorem 10.1 in I. V. Girsanov, [36]). *Let S be a subspace. Then the annihilator of S is*

$$S^* = \{f \in E' \mid f(x) = 0 \quad \text{for all } x \in S\}.$$

Theorem 2.11.4 (See Theorem 10.2 in I. V. Girsanov, [36]). *Let $f \in E'$,*

$$\mathcal{K}_I = \{x \mid f(x) = 0\}, \quad \mathcal{K}_{II} = \{x \mid f(x) \geq 0\}, \quad \text{and} \quad \mathcal{K}_{III} = \{x \mid f(x) > 0\},$$

then

$$\mathcal{K}_I^* = \{\lambda f, \quad \lambda \in \mathbb{R}\}, \quad \mathcal{K}_{II}^* = \{\lambda f, \quad 0 \leq \lambda < \infty\}, \quad \text{and} \quad \mathcal{K}_{III}^* = \begin{cases} E', & \text{if } f = 0 \\ \mathcal{K}_{II}^*, & \text{if } f \neq 0. \end{cases}$$

Let $Q \in E$, $x_0 \in Q$, $\mathcal{K}_b$ the cone of feasible directions for Q at x_0 and $\mathcal{K}_k$ the cone of tangent directions for Q at x_0 , in the sense of Lecture 6 of I. V. Girsanov, [36]. It turns out that in some cases the duals of these cones coincide with the set of linear functionals which are *supports* for Q at the point x_0 , for further details, see Lecture 4 of I. V. Girsanov, [36]. Denote the latter set by

$$Q^* = \{f \in E' \mid f(x) \geq f(x_0) \quad \text{for all } x \in Q\}.$$

Theorem 2.11.5 (See Theorem 10.5 in I. V. Girsanov, [36]). *Let Q be a closed convex set. Then $\mathcal{K}_k^* = Q^*$. If moreover $\text{int } Q \neq \emptyset$, then $\mathcal{K}_b^* = Q^*$.*

Example 2.11.1 (See Example 10.5 in I. V. Girsanov, [36]). *Let*

$Q = \{u \in L_\infty([0, T]; \mathbb{R}^m) \mid u(t) \in \mathcal{M} \text{ for almost all } t \in [0, T]\}$, where $\mathcal{M} \subset \mathbb{R}^m$ and $u^* \in Q$. Then, if the linear functional defined by $f(x) = \int_0^T \langle a(t), u(t) \rangle dt$, $a(t) \in L_1([0, T]; \mathbb{R}^m)$, is a supporting functional to Q at the point u^* , then

$$\langle a(t), u - u^*(t) \rangle \geq 0$$

for all $u \in \mathcal{M}$ and almost all $t \in [0, T]$, i.e., for almost all $t \in [0, T]$, the vector $a(t) \in \mathbb{R}^m$ is a supporting functional to $\mathcal{M}$ at the point $u^*(t)$, that is, $\langle a(t), u \rangle \geq \langle a(t), u^*(t) \rangle$ for all $u \in \mathcal{M}$.

CHAPTER 3

INTERVAL OPTIMIZATION USING *E*–DIFFERENTIABILITY

This chapter addresses constrained optimization problems in which the objective function is interval-valued, and since there are no difficulties in working with inequality constraints that are interval-valued functions, it will consider inequality constraints that are classical functions. In this chapter, using the fact that a solution of an interval optimization problem is a solution of a bi-objective optimization problem and conversely, and using the *E*-derivative introduced in H. C. Wu, [85], and in other works, it will be given necessary conditions of Fritz John type and then with a regularity condition, it will be obtained the Karush–Kuhn–Tucker type conditions for the interval-valued optimization problem. Finally, it will be presented an example.

3.1 Necessary optimality conditions for (*IOP*)

Consider the interval optimization problem:

$$(IOP) \begin{cases} \text{minimize } F(x) = [\underline{f}(x), \bar{f}(x)] \\ \text{subject to } g_j(x) \leq 0, \quad j \in \{1, \dots, m\} \\ x \in S \subseteq \mathbb{R}^n, \end{cases}$$

where $F : S \subseteq \mathbb{R}^n \rightarrow \mathcal{K}_C$ is an interval-valued function given by $F(x) = [\underline{f}(x), \bar{f}(x)]$ with $\underline{f}, \bar{f} : S \subseteq \mathbb{R}^n \rightarrow \mathbb{R}$ and $\underline{f}(x) \leq \bar{f}(x)$ for all $x \in S \subseteq \mathbb{R}^n$, and $g_j : S \subseteq \mathbb{R}^n \rightarrow \mathbb{R}$ are the constraints of (*IOP*), $j \in \{1, \dots, m\}$.

The method to obtain the weak LU –solutions is equivalent to the resolution of a classical bi–objective optimization problem, derived from the interval problem in question. This is seen in the next Theorem:

Theorem 3.1.1 (See U. A. S. Leal, [48]). *If x^* is a weak LU –solution of (IOP), then x^* is a weakly Pareto optimal solution of the following classical bi–objective problem:*

$$(BP)_{LU} \begin{cases} \text{minimize} & f(x) = (\underline{f}(x), \bar{f}(x)) \\ \text{subject to} & g_j(x) \leq 0, \quad j \in \{1, \dots, m\}. \end{cases}$$

Conversely, if x^* is a weakly Pareto optimal solution of $(BP)_{LU}$, then x^* is a weak LU –solution of (IOP).

The next two theorems present the necessary optimality conditions of Fritz John and Karush–Kuhn–Tucker type for weak LU –solutions.

Theorem 3.1.2 (Necessary optimality conditions of Fritz John–type for (IOP) using E –differentiability). *Suppose that the interval–valued function $F : \mathbb{R}^n \rightarrow \mathcal{K}_C$ is continuously E –differentiable and $g : \mathbb{R}^n \rightarrow \mathbb{R}^m$ is continuously differentiable. If $x^* \in \mathcal{X}$ is a local weak LU –solution of (IOP), then there exist multipliers $\mathbf{0} \leq (\lambda, \mu) \in \mathbb{R}^{2+m}$ with $(\lambda, \mu) \neq \mathbf{0}$, such that*

$$\lambda_1 \nabla \underline{f}(x^*) + \lambda_2 \nabla \bar{f}(x^*) + \sum_{j=1}^m \mu_j \nabla g_j(x^*) = \mathbf{0}$$

and

$$\mu_j g_j(x^*) = 0, \quad j \in \{1, \dots, m\}.$$

Proof. If x^* is a local weak LU –solution of (IOP), then, by Theorem 3.1.1, x^* is a local weakly Pareto optimal solution of $(BP)_{LU}$. Applying the necessary conditions of Fritz John at x^* (Theorem 2.1) of M. C. Maciel, S. A. Santos, G. N. Sottosanto, [52], there exist multipliers $\mathbf{0} \leq (\lambda, \mu) \in \mathbb{R}^{2+m}$ with $(\lambda, \mu) \neq \mathbf{0}$, such that

$$\lambda_1 \nabla \underline{f}(x^*) + \lambda_2 \nabla \bar{f}(x^*) + \sum_{j=1}^m \mu_j \nabla g_j(x^*) = \mathbf{0}$$

and

$$\mu_j g_j(x^*) = 0, \quad j \in \{1, \dots, m\}.$$

□

To ensure that at least one entry of λ is positive, some regularity condition must be presupposed (see D. G. Luenberger, [50]; M. C. Maciel, S. A. Santos, G. N. Sottosanto, [52]; K. M. Miettinen, [55], among others), and these regularity conditions are called *constraints qualification*. It will be considered the Karush–Kuhn–Tucker’s constraint qualification of Y. Sawaragi, H. Nakayama, T. Tanino, [76], presented at Definition 2.1.6 of Chapter 2. Others constraint qualifications can be used, it depends on the constraints, but we will use this Karush–Kuhn–Tucker’s constraint qualification. The following theorem will present the necessary optimality conditions of Karush–Kuhn–Tucker type for a local weak LU –solution of the interval–valued optimization problem.

Theorem 3.1.3 (Necessary optimality conditions of Karush–Kuhn–Tucker–type for (IOP) using $\mathbf{E}$ –differentiability). *Suppose that the interval–valued function $F : \mathbb{R}^n \rightarrow \mathcal{K}_C$ is continuously $\mathbf{E}$ –differentiable and $g : \mathbb{R}^n \rightarrow \mathbb{R}^m$ is continuously differentiable. If $x^* \in \mathcal{X}$ satisfies the Karush–Kuhn–Tucker’s constraint qualification (see Definition 2.1.6) and is a local weak LU –solution of (IOP), then there exist (Lagrange) multipliers $\mathbf{0} \leq (\lambda, \mu) \in \mathbb{R}^{2+m}$ with $\lambda_1 + \lambda_2 = 1$, such that*

$$\lambda_1 \nabla \underline{f}(x^*) + \lambda_2 \nabla \bar{f}(x^*) + \sum_{j=1}^m \mu_j \nabla g_j(x^*) = \mathbf{0} \quad (3.1)$$

and

$$\mu_j g_j(x^*) = 0, \quad j \in \{1, \dots, m\}. \quad (3.2)$$

Proof. If x^* is a local weak LU –solution of (IOP), then, by Theorem 3.1.1, x^* is a local weakly Pareto optimal solution of $(BP)_{LU}$. Applying Theorem 2.1.2 at x^* , there exist (Lagrange) multipliers $\mathbf{0} \leq (\lambda, \mu) \in \mathbb{R}^{2+m}$ such that (3.1) and (3.2) hold, and also satisfying the property that $\lambda_1 + \lambda_2 = 1$. Also, we have that

$$\tilde{\lambda}_1 + \tilde{\lambda}_2 = c\lambda_1 + c\lambda_2 = c(\lambda_1 + \lambda_2) = \frac{1}{\lambda_1 + \lambda_2}(\lambda_1 + \lambda_2) = 1$$

and

$$\begin{aligned} \lambda_1 \nabla \underline{f}(x^*) + \lambda_2 \nabla \bar{f}(x^*) + \sum_{j=1}^m \mu_j \nabla g_j(x^*) &= \mathbf{0} \\ \Leftrightarrow c\lambda_1 \nabla \underline{f}(x^*) + c\lambda_2 \nabla \bar{f}(x^*) + \sum_{j=1}^m c\mu_j \nabla g_j(x^*) &= \mathbf{0} \\ \Leftrightarrow \tilde{\lambda}_1 \nabla \underline{f}(x^*) + \tilde{\lambda}_2 \nabla \bar{f}(x^*) + \sum_{j=1}^m \tilde{\mu}_j \nabla g_j(x^*) &= \mathbf{0}, \end{aligned}$$

where $\tilde{\mu}_j = c\mu_j$, $j \in \{1, \dots, m\}$. It is clear that $\tilde{\mu}_j \geq 0$, $j \in \{1, \dots, m\}$, and

$$\tilde{\mu}_j g_j(x^*) = 0 \quad \text{for all } j \in \{1, \dots, m\}.$$

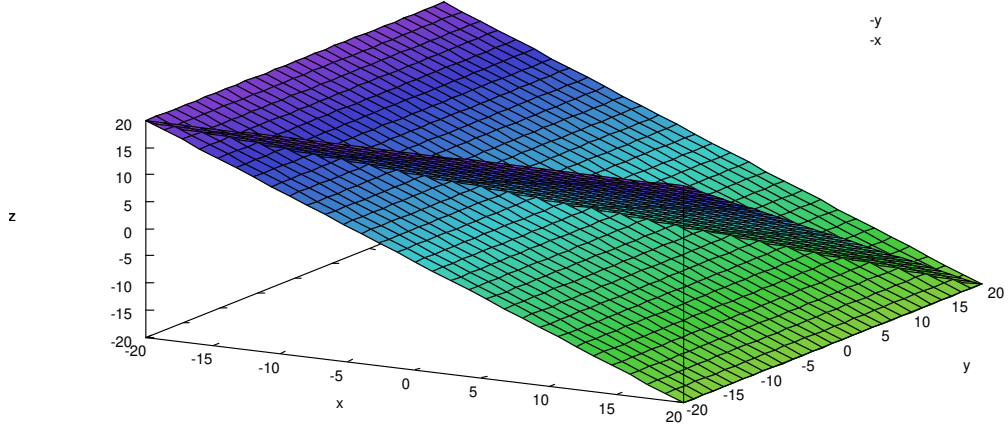
□

Example 3.1.1. Consider the interval-valued problem:

$$\begin{cases} \text{minimize} & F(x) = [\underline{f}(x), \bar{f}(x)] \\ \text{subject to} & g_j(x) \leq 0, \quad j \in \{1, 2\}, \end{cases}$$

where $F : \mathbb{R}^2 \rightarrow \mathcal{K}_C$ is given by $F(x) = [\underline{f}(x), \bar{f}(x)]$, with $\underline{f}(x) = -x_1^2$, $\bar{f}(x) = x_1^2$, $g_1(x) = -x_1$, and $g_2(x) = -x_2$, see Figure 2.

Figure 2: $g_1(x) = -x_1$ and $g_2(x) = -x_2$ of Example 3.1.1.



Source: Elaborated by the author herself.

Then, $\nabla \underline{f}(x) = (-2x_1, 0)$, $\nabla \bar{f}(x) = (2x_1, 0)$, $\nabla g_1(x) = (-1, 0)$, and $\nabla g_2(x) = (0, -1)$.

We have that $x^* = \mathbf{0} \in \mathcal{X} = \{x \in \mathbb{R}^2 \mid (g_1(x), g_2(x)) \leq \mathbf{0}\}$, see Figure 3, satisfy the Karush–Kuhn–Tucker’s Constraint Qualification, see Definition 2.1.6. Indeed, for $h = (h_1, h_2) \in \mathbb{R}^2$ with $h_1 \geq 0$, $h_2 \geq 0$,

$$\langle \nabla g_1(x^*), (h_1, h_2) \rangle = \langle (-1, 0), (h_1, h_2) \rangle = -h_1 \leq 0$$

and

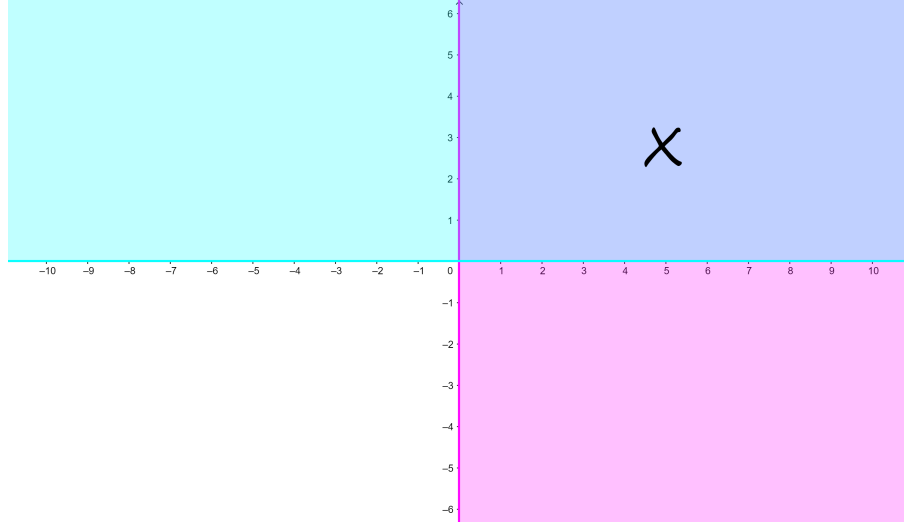
$$\langle \nabla g_2(x^*), (h_1, h_2) \rangle = \langle (0, -1), (h_1, h_2) \rangle = -h_2 \leq 0,$$

and for any $j \in I(x^*) = \{j \mid g_j(x^*) = 0\} = \{1, 2\}$, there exists $t^* := 1 > 0$, there exists $\theta : [0, t^*] \rightarrow \mathbb{R}^2$ given by $\theta(t) := th$, $t \geq 0$, which is differentiable at $t = 0$, and there exists $\alpha := 1 > 0$ such that $\theta(0) = \mathbf{0} = x^*$,

$$g(\theta(t)) = g(th) = (-th_1, -th_2) = -t(h_1, h_2) \leq \mathbf{0} \quad \text{for any } t \in [0, t^* = 1],$$

and also $\theta'(t) = h = \alpha h$.

Figure 3: Admissible set $\mathcal{X} = \{x \mid g_1(x) \leq 0 \text{ and } g_2(x) \leq 0\}$ in Example 3.1.1.



Source: Elaborated by the author herself.

If there exists $x \in N_\epsilon(x^*) \cap \mathcal{X}$ such that $F(x) <_{LU} F(x^*)$, we would have that

$$[\underline{f}(x), \overline{f}(x)] <_{LU} [\underline{f}(x^*), \overline{f}(x^*)] \Leftrightarrow \begin{cases} -x_1^2 = \underline{f}(x) < \underline{f}(x^*) = \underline{f}(0) = 0 \\ \text{and} \\ x_1^2 = \overline{f}(x) < \overline{f}(x^*) = \overline{f}(0) = 0, \end{cases}$$

and $x_1^2 < 0$ would occur, which is an absurd. Therefore there does not exist $x \in N_\epsilon(x^*) \cap \mathcal{X}$ such that $F(x) <_{LU} F(x^*)$, that is, $x^* = \mathbf{0}$ is a local weak LU -solution of the interval-valued problem.

Consequently, since $x^* = \mathbf{0} \in \mathcal{X}$ satisfies the Karush–Kuhn–Tucker's Constraint Qualification and is a local weak LU -solution of the interval-valued problem, by Theorem 3.1.3, there exist (Lagrange) multipliers $\mathbf{0} \leq (\lambda, \mu) \in \mathbb{R}^{2+2}$ with $\lambda_1 + \lambda_2 = 1$, such that

$$\lambda_1 \nabla \underline{f}(x^*) + \lambda_2 \nabla \overline{f}(x^*) + \mu_1 \nabla g_1(x^*) + \mu_2 \nabla g_2(x^*) = \mathbf{0},$$

$$\mu_1 g_1(x^*) = 0$$

and

$$\mu_2 g_2(x^*) = 0.$$

Clearly, the KKT conditions hold, for example, with $\lambda_1 = \lambda_2 = \frac{1}{2}$ and $\mu_1 = \mu_2 = 0$.

3.2 Conclusion

We considered an interval optimization problem and we used the partial order relation LU . Then, using a method to determine the solutions and the continuous differentiability of all the involved functions and classical necessary conditions, we gave the necessary conditions of Fritz John and Karush–Kuhn–Tucker types for the interval optimization problem.

CHAPTER 4

LIPSCHITZ INTERVAL OPTIMAL CONTROL PROBLEMS

It will be considered an optimal control problem with interval-valued objective functional. For this purpose, it will be defined optimal LU -process concepts. It will be presented a method to determine optimal LU -processes. Using the Lipschitz concept for interval-valued functions and some hypotheses, it will be obtained necessary and sufficient optimality conditions for LU -process of the problem in question.

4.1 The interval optimal control problem (*IOCP*)

Next, a brief presentation of an interval-valued optimal control problem will be done, where the objective is considered uncertain, with uncertainty of interval type.

In an optimal control problem, a state variable $x = x(t) \in \mathbb{R}^n$, dependent of time, evolves according to a given dynamics

$$x'(t) = \gamma(t, x(t), u(t)) \quad \text{a. e. } t \in [a, b], \quad (4.1)$$

where $\gamma : [a, b] \times \mathbb{R}^n \times \mathbb{R}^m \rightarrow \mathbb{R}^n$ corresponds to the studied model, $x(a) = x_a$ is the *initial state of the system*, $x(b) \in \mathbb{R}^n$ is the *final state of the system*, and $u : [a, b] \rightarrow \mathbb{R}^m$ is a free parameter influencing the dynamics. Equation (4.1) is called the *state equation*.

In view of the interval-valued optimal control problem considered in this chapter, the objective is to minimize the interval-valued function, of type

$$G(x(b)) := [\underline{g}(x(b)), \bar{g}(x(b))],$$

where $G : \mathbb{R}^n \rightarrow \mathcal{K}_C$, with $\underline{g}, \bar{g} : \mathbb{R}^n \rightarrow \mathbb{R}$, and $\underline{g}(x) \leq \bar{g}(x)$ for all $x \in \mathbb{R}^n$, $x(t)$ and $u(t)$ are related by the dynamics $x'(t) = \gamma(t, x(t), u(t))$ for a. e. $t \in [a, b]$, $(x(a), x(b)) \in \{x_a\} \times \mathbb{R}^n =: S$, where x_a is a fixed point in $\mathbb{R}^n$, a and b are fixed, and u is a Lebesgue–measurable function ¹ for a. e. $t \in [a, b]$.

The *interval optimal control problem* is posed as follows:

$$(IOCP) \begin{cases} \text{minimize } G(x(b)) = [\underline{g}(x(b)), \bar{g}(x(b))] \\ \text{subject to } x'(t) = \gamma(t, x(t), u(t)) \text{ a. e. } t \in [a, b] \\ (x(a), x(b)) \in \{x_a\} \times \mathbb{R}^n = S, \\ u(t) \in U(t) \text{ a. e. } t \in [a, b]. \end{cases}$$

The set of *admissible controls* $\mathcal{U}_{ad}$ is given by

$$\mathcal{U}_{ad} = \{u \in \mathcal{M}([a, b]; \mathbb{R}^m) \mid u(t) \in U(t) \text{ a. e. } t \in [a, b]\},$$

where $\mathcal{M}([a, b]; \mathbb{R}^m)$ ² is the *space of the Lebesgue–measurable functions* and since the problem in question has no restriction on the control variables.

The *set of admissible trajectories* $\mathcal{X}_{ad}$ is given by:

$$\mathcal{X}_{ad} = \{x \in W^{1,1}([a, b]; \mathbb{R}^n) \mid x'(t) = \gamma(t, x(t), u(t)) \text{ a. e. } t, (x(a), x(b)) \in S, \text{ and } u \in \mathcal{U}_{ad}\}.$$

The candidates to be the solution of the problem in question will be the admissible processes, $(x, u) \in \mathcal{X}_{ad} \times \mathcal{U}_{ad}$.

Therefore, the optimal control problem with interval–valued objective function that will be considered, is the next:

$$\begin{cases} \text{minimize } G(x(b)) = [\underline{g}(x(b)), \bar{g}(x(b))] \\ \text{subject to } (x, u) \in \mathcal{X}_{ad} \times \mathcal{U}_{ad}. \end{cases} \quad (4.2)$$

Thus, it is aimed to determine the set of optimal *LU*–processes for the interval optimal control problem (IOCP):

$$\begin{cases} \mathcal{X}^* \times \mathcal{U}^* = \{(x^*, u^*) \in W^{1,1}([a, b]; \mathbb{R}^n) \times \mathcal{M}([a, b]; \mathbb{R}^m)\} \\ (x^*, u^*) = \arg \min_{(x, u) \in \mathcal{X}_{ad} \times \mathcal{U}_{ad}} G(x(b)). \end{cases}$$

¹Let $f : X \rightarrow Y$ be a function, where $(X, \mathcal{M}_X)$ and $(Y, \mathcal{M}_Y)$ are Lebesgue–measurable spaces. The function f is called *Lebesgue–measurable* if $f^{-1}(E) \in \mathcal{M}_X \forall E \in \mathcal{M}_Y$.

² $\mathcal{M}([a, b]; \mathbb{R}^m) = \{u : [a, b] \rightarrow \mathbb{R}^m \mid u \text{ is a Lebesgue–measurable function}\}.$

The purpose is to determine the set $\mathcal{X}^* \times \mathcal{U}^*$, belonging to the set of admissible LU –processes, $\mathcal{X}_{ad} \times \mathcal{U}_{ad}$, such that it minimizes, somehow, the interval–valued function.

In the following section, it will be seen some kinds of processes for the problem in question.

4.2 Optimal LU –processes for (IOCP)

From the order relation LU , it will be defined the optimal LU –process concepts for optimal control problems with interval uncertainty.

Definition 4.2.1. Let (x^*, u^*) be an admissible process of (4.2), i.e., $(x^*, u^*) \in \mathcal{X}_{ad} \times \mathcal{U}_{ad}$.

- (i) (a) (x^*, u^*) is a **$W^{1,1}$ –local strong optimal LU –process** of (4.2) if there exists $\delta > 0$ such that there does not exist $(x, u) \in \mathcal{X}_{ad} \times \mathcal{U}_{ad}$, with $(x, u) \neq (x^*, u^*)$ and ³ $\|x - x^*\|_{W^{1,1}} \leq \delta$, such that

$$G(x(b)) \leq_{LU} G(x^*(b)).$$

- (b) Furthermore, (x^*, u^*) is a **L_∞ –local strong optimal LU –process** of (4.2) if there exists $\delta > 0$ such that there does not exist $(x, u) \in \mathcal{X}_{ad} \times \mathcal{U}_{ad}$, with $(x, u) \neq (x^*, u^*)$ and ⁴ $\|x - x^*\|_{L_\infty} \leq \delta$, such that

$$G(x(b)) \leq_{LU} G(x^*(b)).$$

- (ii) (c) (x^*, u^*) is a **$W^{1,1}$ –local optimal LU –process** of (4.2) if there exists $\delta > 0$ such that there does not exist $(x, u) \in \mathcal{X}_{ad} \times \mathcal{U}_{ad}$, and $\|x - x^*\|_{W^{1,1}} \leq \delta$, such that

$$G(x(b)) \leq_{LU} G(x^*(b)).$$

- (d) Furthermore, (x^*, u^*) is a **L_∞ –local optimal LU –process** of (4.2) if there exists $\delta > 0$ such that there does not exist $(x, u) \in \mathcal{X}_{ad} \times \mathcal{U}_{ad}$, and $\|x - x^*\|_{L_\infty} \leq \delta$, such that

$$G(x(b)) \leq_{LU} G(x^*(b)).$$

³ $\|x - x^*\|_{W^{1,1}} = \|(x - x^*)(a)\|_{\mathbb{R}^n} + \int_a^b \|(x - x^*)'(t)\|_{\mathbb{R}^n} dt.$

⁴ $\|x - x^*\|_{L_\infty} = \text{ess sup}_{t \in [a, b]} \|(x - x^*)(t)\|_{\mathbb{R}^n}$ and also ess sup denotes the smaller number λ such that $\|x(t)\|_{\mathbb{R}^n} \leq \lambda$ for a. e. $t \in [a, b]$.

- (iii) (e) (x^*, u^*) is a **$W^{1,1}$ –local weak optimal LU–process** of (4.2) if there exists $\delta > 0$ such that there does not exist $(x, u) \in \mathcal{X}_{ad} \times \mathcal{U}_{ad}$, and $\|x - x^*\|_{W^{1,1}} \leq \delta$, such that

$$G(x(b)) <_{LU} G(x^*(b)).$$

- (f) Furthermore, (x^*, u^*) is a **L_∞ –local weak optimal LU–process** of (4.2) if there exists $\delta > 0$ such that there does not exist $(x, u) \in \mathcal{X}_{ad} \times \mathcal{U}_{ad}$, and $\|x - x^*\|_{L_\infty} \leq \delta$, such that

$$G(x(b)) <_{LU} G(x^*(b)).$$

If the neighborhood of x^* above is replaced by the whole space $\mathbb{R}^n$, we have the corresponding definitions of global solutions for (4.2).

A question to be highlighted is that the optimality conditions usually found in the literature are of local character and therefore do not guarantee a global optimality, which is our main interest.

The $W^{1,1}$ norm is stronger than the L_∞ norm and therefore the class of $W^{1,1}$ –local (strong, weak) optimal LU–processes is greater than the class of L_∞ –local (strong, weak) optimal LU–processes. It follows that, by choosing to work with $W^{1,1}$ –local (strong, weak) optimal LU–processes, it is carried out a deeper analysis of the local nature of the Maximum Principle than would be the case if it is chosen to derive conditions satisfied by L_∞ –local (strong, weak) optimal LU–processes.

Remark 4.2.1. (i) The following relations are immediate:

$$\boxed{\text{strong optimal LU–process} \Rightarrow \text{optimal LU–process} \Rightarrow \text{weak optimal LU–process.}}$$

- (ii) The theory that will be developed for the weak optimal LU–process is applicable to the optimal LU–process and to the strong optimal LU–process as well.
- (iii) Note that the Hamiltonian function, $\mathcal{H}$, for (4.2) is the same as the Hamiltonian function for (MCP), since the dynamics of both problems are the same, then

$$\mathcal{H}(t, x, p, u) := \langle p, \gamma(t, x, u) \rangle.$$

Later, the optimality conditions will be presented for the weak optimal LU–process concept of interval–valued optimal control problems. However, before presenting the optimality conditions, it will be presented the following section.

4.3 Hypotheses

Here, it will be considered some hypothesis, but before doing it, it will be defined the Lipschitz continuity of interval-valued functions and this concept will be characterized.

Definition 4.3.1. *Given $A \subseteq \mathbb{R}^n$ an open and nonempty set, let $F : A \rightarrow \mathcal{K}_C$ be an interval-valued function.*

(i) *We say that F is Lipschitz continuous of rank $L > 0$ if we have that*

$$d_H(F(x_1), F(x_2)) \leq L \|x_1 - x_2\|_{\mathbb{R}^n}$$

for all $x_1, x_2 \in A$.

(ii) *We say that F is Lipschitz continuous locally near a given point $x \in A$ of rank $L > 0$ if for some $\epsilon > 0$, we have*

$$d_H(F(x_1), F(x_2)) \leq L \|x_1 - x_2\|_{\mathbb{R}^n}$$

for all $x_1, x_2 \in \mathbb{B}_{\mathbb{R}^n}(x, \epsilon)$, where $\mathbb{B}_{\mathbb{R}^n}(x, \epsilon) = \{x' \in \mathbb{R}^n \mid \|x' - x\|_{\mathbb{R}^n} < \epsilon\}$.

Proposition 4.3.1. *Let $S \subseteq \mathbb{R}^n$ be an open and nonempty set, and $F : S \rightarrow \mathcal{K}_C$ be an interval-valued function such that $F(x) = [\underline{f}(x), \overline{f}(x)]$, where $\underline{f}, \overline{f} : S \rightarrow \mathbb{R}$. Then,*

(i) *F is Lipschitz continuous if and only if the extreme functions $\underline{f}$ and $\overline{f}$ are both Lipschitz continuous.*

(ii) *F is Lipschitz continuous locally near a given point $x \in \mathbb{R}^n$ if and only if the extreme functions $\underline{f}$ and $\overline{f}$ are both Lipschitz continuous locally near a given point $x \in \mathbb{R}^n$.*

Proof. We have that

$$\begin{aligned} F \text{ is Lipschitz continuous} &\Leftrightarrow d_H(F(x), F(y)) \leq L \|x - y\|_{\mathbb{R}^n} \quad \forall x, y \in S \\ &\Leftrightarrow \max \{ |\underline{f}(x) - \underline{f}(y)|, |\overline{f}(x) - \overline{f}(y)| \} \leq L \|x - y\|_{\mathbb{R}^n} \quad \forall x, y \in S \\ &\Leftrightarrow |\underline{f}(x) - \underline{f}(y)| \leq L \|x - y\|_{\mathbb{R}^n} \quad \forall x, y \in S \\ &\quad \text{and } |\overline{f}(x) - \overline{f}(y)| \leq L \|x - y\|_{\mathbb{R}^n} \quad \forall x, y \in S \\ &\Leftrightarrow \underline{f} \text{ and } \overline{f} \text{ are Lipschitz continuous.} \end{aligned}$$

The second part is proved in an analogous reasoning. □

Let (x^*, u^*) be a reference admissible process of (4.2). As before, for some $\delta > 0$, we are going to suppose that γ and U satisfy (H1) – (H3), see Chapter 2, but instead of (H4), we are going to suppose

(H4)' G is locally Lipschitz continuous.

In the next two sections, optimality conditions for optimal weak LU –processes of the optimal control problem will be obtained. First, it will be presented necessary optimality conditions for the interval optimal control problem.

4.4 Necessary optimality conditions for (IOCP)

The following result of this section, says how to determine the optimal LU –processes.

Theorem 4.4.1. *If (x^*, u^*) is a weak optimal LU –process of (4.2), then (x^*, u^*) is a weak Pareto optimal process of the following classical bi–objective control problem:*

$$(BCP)_{LU} \begin{cases} \text{minimize } g(x(b)) = (\underline{g}(x(b)), \bar{g}(x(b))) \\ \text{subject to } (x, u) \in \mathcal{X}_{ad} \times \mathcal{U}_{ad}. \end{cases}$$

Conversely, if (x^*, u^*) is a weak Pareto optimal process of $(BCP)_{LU}$, then (x^*, u^*) is a weak optimal LU –process of (4.2).

Proof. Let (x^*, u^*) be a weak optimal LU –process of (4.2), that is, there exists no $(x, u) \in \mathcal{X}_{ad} \times \mathcal{U}_{ad}$ such that

$$\begin{aligned} & G(x(b)) <_{LU} G(x^*(b)) \\ & \Leftrightarrow [\underline{g}(x(b)), \bar{g}(x(b))] <_{LU} [\underline{g}(x^*(b)), \bar{g}(x^*(b))] \\ & \Leftrightarrow \underline{g}(x(b)) < \underline{g}(x^*(b)) \text{ and } \bar{g}(x(b)) < \bar{g}(x^*(b)). \end{aligned} \quad (4.3)$$

Suppose that (x^*, u^*) does not solve the classical biobjective control problem $(BCP)_{LU}$. Then there exists a solution $(x, u) \in \mathcal{X}_{ad} \times \mathcal{U}_{ad}$ such that

$$\begin{aligned} & g(x(b)) < g(x^*(b)) \\ & \Leftrightarrow (\underline{g}(x(b)), \bar{g}(x(b))) < (\underline{g}(x^*(b)), \bar{g}(x^*(b))) \\ & \Leftrightarrow \underline{g}(x(b)) < \underline{g}(x^*(b)) \text{ and } \bar{g}(x(b)) < \bar{g}(x^*(b)), \end{aligned}$$

which contradicts the weak LU –optimality of (x^*, u^*) (see (4.3)). The converse, it follows similarly. $\square$

Remark 4.4.1. (i) At Theorem 4.4.1, it can be replaced the weak optimal LU –process by the strong optimal LU –process (optimal LU –process) and the weak Pareto optimal process by the strong Pareto optimal process (Pareto optimal process).

(ii) Note that the Hamiltonian function, $\mathcal{H}$, for $(BCP)_{LU}$ is the same as the Hamiltonian function for (MCP) , since the dynamics of both problems are the same.

The next result deals with the necessary optimality conditions for problem (4.2), which is a variant of the optimality Maximum Principle, but for the case in which G is an interval–valued function and considering the Lipschitz concept for interval–valued functions.

Theorem 4.4.2 (Necessary optimality conditions of Pontryagin Maximum Principle–type for (4.2)). *If $(x^*, u^*) \in \mathcal{X}_{ad} \times \mathcal{U}_{ad}$ is a weak optimal LU –process of (4.2), then there exist a multiplier $p^* \in W^{1,1}([a, b], \mathbb{R}^n)$, a scalar $\lambda^* \in \{0, 1\}$, and a nonzero vector $\omega^* = (\omega_1^*, \omega_2^*) \in \mathbb{R}^2$, such that:*

$$(x^*)'(t) = \frac{\partial \mathcal{H}}{\partial p}(t, x^*(t), p^*(t), u^*(t)), \quad (4.4)$$

$$-(p^*)'(t) \in \text{co} \{ \partial_x \mathcal{H}(t, x^*(t), p^*(t), u^*(t)) \}, \quad \text{a. e. } t \in [a, b], \quad (4.5)$$

$$x^*(a) = x_a \quad \text{and} \quad -p^*(b) \in \lambda^* \omega_1^* \frac{\partial \underline{g}}{\partial x}(b)(x^*(b)) + \lambda^* \omega_2^* \frac{\partial \bar{g}}{\partial x}(b)(x^*(b)). \quad (4.6)$$

Furthermore, the control u^* satisfies the following optimality condition

$$\mathcal{H}(t, x^*(t), p^*(t), u^*(t)) = \max_{u \in U(t)} \mathcal{H}(t, x^*(t), p^*(t), u) \quad \text{a. e. } t \in [a, b]. \quad (4.7)$$

Proof. Since (x^*, u^*) is a weak optimal LU –process, then by Theorem 4.4.1, (x^*, u^*) is also a weak Pareto optimal process of $(BCP)_{LU}$, by Theorem 2.2.1, there exist a scalar λ^* (equal to 0 or 1), a nonzero vector $\omega^* = (\omega_1^*, \omega_2^*) \in \mathbb{R}^2$, and an absolutely continuous function $p^* : [a, b] \rightarrow \mathbb{R}^n$, that is, $p^* \in W^{1,1}([a, b], \mathbb{R}^n)$, such that

$$(i) \quad -(p^*)'(t) \in \text{co} \{ \partial_x \mathcal{H}(t, x^*(t), p^*(t), u^*(t)) \} \quad \text{a. e. } t \in [a, b].$$

$$(ii) \quad (p^*(a), -p^*(b)) \in \lambda^* \partial(\langle \omega^*, \widehat{G} \rangle)(x^*(a), x^*(b)) + N_S(x^*(a), x^*(b)), \text{ where } S = \{x_a\} \times \mathbb{R} \text{ and } \widehat{G} : \mathbb{R}^n \times \mathbb{R}^n \rightarrow \mathbb{R}^2 \text{ is defined by}$$

$$\widehat{G}(x(a), x(b)) := (\underline{g}(x(b)), \bar{g}(x(b))).$$

Since

$$\langle \omega^*, \widehat{G}(x(a), x(b)) \rangle = \langle (\omega_1^*, \omega_2^*), (\underline{g}(x(b)), \bar{g}(x(b))) \rangle = \omega_1^* \underline{g}(x(b)) + \omega_2^* \bar{g}(x(b)),$$

we have that by the Sum Rule, see Theorem 5.4.1. of R. Vinter, [83],

$$\partial(\langle \omega^*, \widehat{G} \rangle)(x^*(a), x^*(b)) \subset \{\mathbf{0}\} \times \left[\omega_1^* \frac{\partial \underline{g}}{\partial x(b)}(x^*(b)) + \omega_2^* \frac{\partial \overline{g}}{\partial x(b)}(x^*(b)) \right] \subseteq \mathbb{R}^{2n},$$

and since $(x^*, u^*) \in \mathcal{X}_{ad} \times \mathcal{U}_{ad}$ is a weak optimal LU -process of (4.2), then $x^*(a) = x_a$ and

$$N_S(x^*(a), x^*(b)) = N_{\{x_a\} \times \mathbb{R}^n}(x_a, x^*(b)) = N_{\{x_a\}}(x_a) \times N_{\mathbb{R}^n}(x^*(b)) = \mathbb{R}^n \times \{\mathbf{0}\}.$$

Therefore,

$$\begin{aligned} (p^*(a), -p^*(b)) &\in \lambda^* \partial(\langle \omega^*, \widehat{G} \rangle)(x^*(a), x^*(b)) + N_S(x^*(a), x^*(b)) \\ &\subset \lambda^* \left(\{\mathbf{0}\} \times \left[\omega_1^* \frac{\partial \underline{g}}{\partial x(b)}(x^*(b)) + \omega_2^* \frac{\partial \overline{g}}{\partial x(b)}(x^*(b)) \right] \right) + \mathbb{R}^n \times \{\mathbf{0}\} \\ &= \{\mathbf{0}\} \times \left[\lambda^* \omega_1^* \frac{\partial \underline{g}}{\partial x(b)}(x^*(b)) + \lambda^* \omega_2^* \frac{\partial \overline{g}}{\partial x(b)}(x^*(b)) \right] + \mathbb{R}^n \times \{\mathbf{0}\} \\ &= \mathbb{R}^n \times \left[\lambda^* \omega_1^* \frac{\partial \underline{g}}{\partial x(b)}(x^*(b)) + \lambda^* \omega_2^* \frac{\partial \overline{g}}{\partial x(b)}(x^*(b)) \right]. \end{aligned}$$

Then

$$p^*(a) \in \mathbb{R}^n \quad \text{and} \quad -p^*(b) \in \lambda^* \omega_1^* \frac{\partial \underline{g}}{\partial x(b)}(x^*(b)) + \lambda^* \omega_2^* \frac{\partial \overline{g}}{\partial x(b)}(x^*(b)).$$

$$(iii) \quad \max_{u \in U(t)} \mathcal{H}(t, x^*(t), p^*(t), u) = \mathcal{H}(t, x^*(t), p^*(t), u^*(t)) \quad \text{a. e. } t \in [a, b].$$

$$(iv) \quad \|p^*\| + \lambda^* > 0, \quad \|\omega^*\| = 1, \quad \omega^* \geq \mathbf{0}.$$

Moreover, since $\mathcal{H}(t, x(t), p(t), u(t)) = \langle p(t), \gamma(t, x(t), u(t)) \rangle$,

$$\frac{\partial \mathcal{H}}{\partial p}(t, x^*(t), p^*(t), u^*(t)) = \gamma(t, x^*(t), u^*(t)) = (x^*)'(t).$$

□

In the following, it will be given sufficient optimality conditions for the interval optimal control problem.

4.5 Sufficient optimality conditions for (IOCP)

The next result, using Theorem 2.2.3 of Chapter 2, will present the sufficient optimality conditions for the problem (4.2).

Theorem 4.5.1 (Sufficient optimality conditions for (4.2)). *If $(x^*, p^*, u^*) \in W^{1,1}([a, b]; \mathbb{R}^n) \times W^{1,1}([a, b]; \mathbb{R}^n) \times \mathcal{M}([a, b]; \mathbb{R}^m)$ is solution of System (4.8)-(4.11):*

$$-(p^*)'(t) \in \text{co} \{ \partial_x \mathcal{H}(t, x^*(t), p^*(t), u^*(t)) \} \quad a. e. t \in [a, b], \quad (4.8)$$

$$(p^*(a), -p^*(b)) \in \lambda^* \partial(\langle \omega^*, \widehat{G} \rangle)(x^*(a), x^*(b)) + N_C(x^*(a), x^*(b)), \quad (4.9)$$

$$\max_{u \in U(t)} \mathcal{H}(t, x^*(t), p^*(t), u) = \mathcal{H}(t, x^*(t), p^*(t), u^*(t)) \quad a. e. t \in [a, b], \quad (4.10)$$

$$\|p^*\| + \lambda^* > 0, \quad \|\omega^*\| = 1, \quad \omega^* \geq 0, \quad (4.11)$$

with $\lambda^* = 1$, and if for every admissible process (x, u) with $\widehat{G}(x(a), x(b)) \leq \widehat{G}(x^*(a), x^*(b))$ and all $\beta \in (0, 1)$, given $\tau = [\tau_1 \tau_2]^\top \in D \widehat{G}(x^*(a), x^*(b))$ and $\zeta(t) \in D_x \gamma(t, x^*(t), u^*(t))$ a. e. $t \in [a, b]$, there exist maps $(\eta, \xi) \in \mathcal{F}_\beta(x^*, u^*; \zeta)$ such that

$$\begin{cases} \underline{g}(x(a), x(b)) = \underline{g}(x^*(a), x^*(b)) \Rightarrow \langle \tau_1, (\eta(a), \eta(b)) \rangle \leq 0 \\ \text{or} \\ \underline{g}(x(a), x(b)) < \underline{g}(x^*(a), x^*(b)) \Rightarrow \langle \tau_1, (\eta(a), \eta(b)) \rangle < 0 \end{cases} \quad (4.12)$$

and

$$\begin{cases} \overline{g}(x(a), x(b)) = \overline{g}(x^*(a), x^*(b)) \Rightarrow \langle \tau_2, (\eta(a), \eta(b)) \rangle \leq 0 \\ \text{or} \\ \overline{g}(x(a), x(b)) < \overline{g}(x^*(a), x^*(b)) \Rightarrow \langle \tau_2, (\eta(a), \eta(b)) \rangle < 0, \end{cases}$$

then (x^*, u^*) is a weak optimal LU -process of (4.2).

Proof. If (x^*, p^*, u^*) satisfy conditions (4.8)-(4.11) with $\lambda^* = 1$ and also condition (4.12), then by Theorem 2.2.3, $(x^*, u^*) \in \mathcal{X}_{ad} \times \mathcal{U}_{ad}$ is a weak Pareto optimal process of $(BCP)_{LU}$, and by Theorem 4.4.1, (x^*, u^*) is also a weak optimal LU -process of (4.2). $\square$

4.6 Conclusion

We considered an interval optimal control problem and we used the partial order relation LU for defining the optimal process concepts. Next, we presented a technique to determine the optimal LU -processes. Then, applying the Lipschitz concept for interval-valued functions, the *Maximum Principle* and the called *PM-pseudoinvexity* for classical multi-objective optimal control problems, we obtained necessary and sufficient optimality conditions for interval optimal control problems.

CHAPTER 5

INTERVAL OPTIMIZATION USING *GH*–DIFFERENTIABILITY

In this chapter, it will be used the gH –gradient and the gH –directional derivative of an interval–valued function of several variables. For the necessary optimality conditions, it will be given a geometric characterization of local weak optimality for the interval problem. It will be obtained a Fritz John–type condition and assuming a constraint qualification or a regularity condition, it will be obtained interval Karush–Kuhn–Tucker conditions. Moreover, it will be defined KKT and strict KKT –solutions. For the sufficient optimality conditions, it will be defined several types of pseudoinvexities, which differ one from another by the type of partial order relation used (that is, LU , weak LU , $\underline{LU}$, and $\overline{LU}$). It will be proved that points that satisfy the KKT conditions are optimal LU –solutions for each of the various classes of pseudo–invexity problems. Moreover, it will be proved that this classes of pseudoinvexities are the most general classes, in the sense that if the problem holds the condition that every (strict) KKT –solution is an optimal solution, then necessarily, it satisfies one of the definitions of KKT –pseudoinvexities. Finally, it will be presented some examples.

Consider the interval optimization problem:

$$(IOP) \begin{cases} \text{minimize} & F(x) = [\underline{f}(x), \overline{f}(x)] \\ \text{subject to} & g_j(x) \leq 0, \quad j \in \{1, \dots, m\} \\ & x \in S \subseteq \mathbb{R}^n, \end{cases}$$

where $F : S \subseteq \mathbb{R}^n \rightarrow \mathcal{K}_C$ is an interval–valued function given by $F(x) = [\underline{f}(x), \overline{f}(x)]$

with $\underline{f}, \bar{f} : S \subseteq \mathbb{R}^n \rightarrow \mathbb{R}$, and $\underline{f}(x) \leq \bar{f}(x)$ for all $x \in S \subseteq \mathbb{R}^n$, S an open and nonempty set, and $g_j : S \subseteq \mathbb{R}^n \rightarrow \mathbb{R}$ are the constraints of (IOP), $j \in \{1, \dots, m\}$. Herein it will be assumed that the interval-valued function $F : S \rightarrow \mathcal{K}_C$ is gH -differentiable at $x^* \in S$ and that the functions g_j , $j \in I$, are differentiable.

5.1 Necessary optimality conditions for (IOP)

The following is a geometric characterization of weak optimality for (IOP):

Proposition 5.1.1. *If $x^* \in \mathcal{X}$ is a local weak LU-solution of (IOP), then the following system*

$$(S_n) \begin{cases} F'_{gH}(x^*; d) <_{LU} [0, 0] \\ \langle \nabla g_j(x^*), d \rangle < 0, \end{cases} \quad j \in I(x^*),$$

does not have solution $d \in \mathbb{R}^n$.

Proof. Suppose that there exists a direction $\widehat{d} \in \mathbb{R}^n$ that solves System (S_n) . From the second inequality of System (S_n) ,

$$0 > \lim_{\alpha \rightarrow 0^+} \frac{g_j(x^* + \alpha \widehat{d}) - g_j(x^*)}{\alpha} = \langle \nabla g_j(x^*), \widehat{d} \rangle, \quad j \in I(x^*),$$

it follows that

$$0 > g_j(x^* + \alpha \widehat{d}) - g_j(x^*) = g_j(x^* + \alpha \widehat{d}), \quad j \in I(x^*), \quad (5.1)$$

for all $\alpha > 0$ small enough. For $j \in I \setminus I(x^*)$, we have $g_j(x^*) < 0$, so, by continuity of g_j ,

$$g_j(x^* + \alpha \widehat{d}) < 0 \quad (5.2)$$

for all $\alpha > 0$ small enough.

Then, from (5.1) and (5.2), there exists $\widehat{\alpha} > 0$ such that

$$g_j(x^* + \alpha \widehat{d}) < 0, \quad j \in I, \text{ and } \alpha \in (0, \widehat{\alpha}). \quad (5.3)$$

From the first inequality of System (S_n) , we have, by Definition 2.7.5,

$$F'_{gH}(x^*; \widehat{d}) = \lim_{\alpha \rightarrow 0^+} \frac{1}{\alpha} \cdot (F(x^* + \alpha \widehat{d}) \ominus_{gH} F(x^*)) <_{LU} [0, 0].$$

Let $F'_{gH}(x^*; \widehat{d}) = L = [\underline{l}, \bar{l}] \in \mathcal{K}_C$. Then, given $\epsilon \in \mathbb{R}$ with $0 < \epsilon < -\bar{l}$, there exists $\widetilde{\alpha} > 0$ such that

$$d_H \left(\frac{1}{\alpha} \cdot (F(x^* + \alpha \widehat{d}) \ominus_{gH} F(x^*)), L \right) < \epsilon$$

for all $\alpha \in (0, \widetilde{\alpha})$. By setting

$$\frac{1}{\alpha} \cdot (F(x^* + \alpha \widehat{d}) \ominus_{gH} F(x^*)) =: K_\alpha = [\underline{k}_\alpha, \overline{k}_\alpha],$$

where $\alpha \in (0, \widetilde{\alpha})$, we have that $\max\{|\underline{k}_\alpha - \underline{l}|, |\overline{k}_\alpha - \bar{l}|\} = d_H(K_\alpha, L) < \epsilon$

$$\begin{aligned} &\Rightarrow \underline{k}_\alpha - \underline{l} < |\underline{k}_\alpha - \underline{l}| < \epsilon \quad \text{and} \quad \overline{k}_\alpha - \bar{l} < |\overline{k}_\alpha - \bar{l}| < \epsilon \\ &\Rightarrow \underline{k}_\alpha < \underline{l} + \epsilon < \underline{l} - \bar{l} < 0 \quad \text{and} \quad \overline{k}_\alpha < \bar{l} + \epsilon < \bar{l} - \bar{l} = 0 \\ &\Rightarrow \underline{k}_\alpha < 0 \quad \text{and} \quad \overline{k}_\alpha < 0 \\ &\Leftrightarrow K_\alpha = [\underline{k}_\alpha, \overline{k}_\alpha] <_{LU} [0, 0] \\ &\Leftrightarrow \frac{1}{\alpha} \cdot (F(x^* + \alpha \widehat{d}) \ominus_{gH} F(x^*)) <_{LU} [0, 0] \\ &\Leftrightarrow F(x^* + \alpha \widehat{d}) \ominus_{gH} F(x^*) <_{LU} [0, 0] \\ &\Leftrightarrow F(x^* + \alpha \widehat{d}) <_{LU} F(x^*), \quad \alpha \in (0, \widetilde{\alpha}), \text{ by Proposition 2.3.1 - (i).} \end{aligned} \quad (5.4)$$

Given $\delta := \min\{\widehat{\alpha}, \widetilde{\alpha}\}$, from (5.3) and (5.4), it follows that $x^* + \alpha \widehat{d} \in \mathcal{X}$ with $F(x^* + \alpha \widehat{d}) <_{LU} F(x^*)$ for all $\alpha \in (0, \delta)$ and, consequently, $x^* \in \mathcal{X}$ is not a local weak LU -solution of (IOP). Therefore, if $x^* \in \mathcal{X}$ is a local weak LU -solution of (IOP), then System (S_n) does not have any solution $d \in \mathbb{R}^n$. $\square$

Since the “quasi interval system” (S_n) does not have a solution, the next result shows that the classical system $(S_n)'$ below does not have a solution too.

Proposition 5.1.2. *Let $x^* \in \mathcal{X}$ be a local weak LU -solution of (IOP). If the following system*

$$(S_n) \begin{cases} F'_{gH}(x^*; d) <_{LU} [0, 0] \\ \langle \nabla g_j(x^*), d \rangle < 0, \end{cases} \quad j \in I(x^*),$$

does not have solution $d \in \mathbb{R}^n$, then also the next system

$$(S_n)' \begin{cases} \langle \nabla_{\#} \underline{f}(x^*), d \rangle < 0, \\ \langle \nabla_{\#} \overline{f}(x^*), d \rangle < 0, \\ \langle \nabla g_j(x^*), d \rangle < 0, \end{cases} \quad j \in I(x^*),$$

does not have solution $d \in \mathbb{R}^n$.

Proof. The result follows directly from the fact that, by Theorem 2.7.4,

$$F'_{gH}(x^*; d) <_{LU} [0, 0] \Leftrightarrow \langle \nabla_{\#} \underline{f}(x^*), d \rangle < 0 \quad \text{and} \quad \langle \nabla_{\#} \bar{f}(x^*), d \rangle < 0.$$

□

If it is applied the Gordan Transposition Theorem to the System $(S_n)'$, it can be obtained:

Theorem 5.1.1 (Necessary optimality conditions of Fritz John–type for **(IOP)** using **gH**–differentiability). *Let $x^* \in \mathcal{X}$ be a local weak LU–solution of (IOP). Then, there exist $(\lambda_1, \lambda_2) \in \mathbb{R}^2$ and $\mu = (\mu_1, \dots, \mu_m) \in \mathbb{R}^m$ not all zero such that*

$$\lambda_1 \nabla_{\#} \underline{f}(x^*) + \lambda_2 \nabla_{\#} \bar{f}(x^*) + \sum_{j=1}^m \mu_j \nabla g_j(x^*) = \mathbf{0}, \quad (5.5)$$

$$\mu_j g_j(x^*) = 0, \quad j \in I, \quad (5.6)$$

and

$$\lambda_1, \lambda_2, \mu_j \geq 0, \quad j \in I. \quad (5.7)$$

Proof. From Proposition 5.1.2, System $(S_n)'$ does not have solution $d \in \mathbb{R}^n$. Then applying Gordan Transposition Theorem, see Theorem 2.10.3, to System $(S_n)'$:

$$(S_n)' \begin{cases} \langle \nabla_{\#} \underline{f}(x^*), d \rangle < 0, \\ \langle \nabla_{\#} \bar{f}(x^*), d \rangle < 0, \\ \langle \nabla g_j(x^*), d \rangle < 0, \quad j \in I(x^*), \end{cases} \Leftrightarrow \begin{cases} \nabla_{\#} \underline{f}(x^*) d^\top < 0 \\ \nabla_{\#} \bar{f}(x^*) d^\top < 0 \\ Jg(x^*) d^\top < \mathbf{0} \end{cases} \Leftrightarrow \begin{bmatrix} \nabla_{\#} \underline{f}(x^*) \\ \nabla_{\#} \bar{f}(x^*) \\ Jg(x^*) \end{bmatrix} d^\top < \begin{bmatrix} 0 \\ 0 \\ \mathbf{0} \end{bmatrix},$$

there exist $\lambda_l \geq 0$, $l \in \{1, 2\}$, $\mu_j \geq 0$, $j \in \{1, \dots, k\}$, and $(\lambda, \mu) \neq \mathbf{0}$ such that

$$\begin{aligned} & \begin{bmatrix} \nabla_{\#} \underline{f}(x^*) \\ \nabla_{\#} \bar{f}(x^*) \end{bmatrix}^\top \lambda^\top + Jg(x^*)^\top \mu^\top = \mathbf{0}^\top \\ \Leftrightarrow & \nabla_{\#} \underline{f}(x^*) \lambda_1 + \nabla_{\#} \bar{f}(x^*) \lambda_2 + \sum_{j \in I(x^*)} \nabla g_j(x^*) \mu_j = \mathbf{0}. \end{aligned}$$

For $j \in I \setminus I(x^*)$, we define $\mu_j := 0$, so that

$$\nabla_{\#} \underline{f}(x^*) \lambda_1 + \nabla_{\#} \bar{f}(x^*) \lambda_2 + \sum_{j=1}^m \nabla g_j(x^*) \mu_j = \mathbf{0}.$$

Additionally,

$$\mu_j g_j(x^*) = \begin{cases} 0, & \text{for } j \in I(x^*) \text{ (since } g_j(x^*) = 0) \\ 0, & \text{for } j \in I \setminus I(x^*) \text{ (since } \mu_j = 0) \end{cases} \Rightarrow \mu_j g_j(x^*) = 0, \quad \text{for } j \in I.$$

It is clear that $\mu_j \geq 0$, $j \in I$. Therefore, there exist $(\lambda_1, \lambda_2) \in \mathbb{R}^2$ and $\mu = (\mu_1, \dots, \mu_m) \in \mathbb{R}^m$ not all zero such that

$$\lambda_1 \nabla_{\#} \underline{f}(x^*) + \lambda_2 \nabla_{\#} \bar{f}(x^*) + \sum_{j=1}^m \mu_j \nabla g_j(x^*) = \mathbf{0},$$

$$\mu_j g_j(x^*) = 0, \quad j \in I,$$

and

$$\lambda_1, \lambda_2, \mu_j \geq 0, \quad j \in I.$$

□

The elements λ_1 , λ_2 , and μ_j , $j \in I$, considered at Theorem 5.1.1 are called *Fritz–John, Lagrange multipliers*.

It can be observed that at Theorem 5.1.1, it does not exist a guarantee that at least one of the multipliers λ_1/λ_2 associated with the objective interval-valued function will be different from zero, i.e., nothing guarantees that $(\lambda_1, \lambda_2) \neq (0, 0)$. In that way, if that multipliers were zero, it would be lost information contained at the gH -derivative of the objective interval-valued function. To solve this inconvenience, it will be added to the hypothesis of the last theorem, a *constraint qualification* or a *regularity condition*. For simplicity, it will be admitted the positive linear independence constraint qualification (*PLICQ*) and the positive linear independence regularity condition (*PLIRC*), see Definition 2.10.2 and Definition 2.10.3 of Chapter 2, respectively.

If at the hypotheses of Theorem 5.1.1 is added the positive linear independence constraint qualification (*PLICQ*), it can be obtained the following interval *KKT* condition:

Theorem 5.1.2 (Necessary optimality conditions of Karush–Kuhn–Tucker–type 1 for (IOP) using gH -differentiability). *Let $x^* \in \mathcal{X}$ be a weak LU-solution of (IOP). Assume that the constraints of (IOP) satisfy (PLICQ) at x^* . Then there exists $\mu = (\mu_1, \dots, \mu_m) \in \mathbb{R}^m$ such that*

$$\mathbf{0} \in \nabla_{gH} F(x^*) \oplus \sum_{j=1}^m \mu_j \nabla g_j(x^*), \quad (5.8)$$

$$\mu_j g_j(x^*) = 0, \quad j \in I, \quad (5.9)$$

and

$$\mu_j \geq 0, \quad j \in I. \quad (5.10)$$

Proof. From the Fritz John–type condition, there exist $\lambda_l \in \mathbb{R}$, $l \in \{1, 2\}$, and $\mu = (\mu_1, \dots, \mu_m) \in \mathbb{R}^m$ not all zero such that

$$\lambda_1 \nabla_{\#} \underline{f}(x^*) + \lambda_2 \nabla_{\#} \bar{f}(x^*) + \sum_{j=1}^m \mu_j \nabla g_j(x^*) = \mathbf{0}, \quad (5.11)$$

$$\mu_j g_j(x^*) = 0, \quad j \in I, \quad (5.12)$$

and

$$\lambda_1, \lambda_2, \mu_j \geq 0, \quad j \in I. \quad (5.13)$$

If $\lambda_1 = 0$ and $\lambda_2 = 0$, then from Equation (5.11), we would have that

$$\sum_{j=1}^m \mu_j \nabla g_j(x^*) = \mathbf{0},$$

but Equation (5.12) is equivalent to saying that $\mu_j = 0$ when $g_j(x^*) < 0$, $j \in I \setminus I(x^*)$, i.e., when the constraint $g_j(x) \leq 0$ is inactive at x^* . Then, we would have

$$\sum_{j \in I(x^*)} \mu_j \nabla g_j(x^*) = \mathbf{0},$$

with $\mu_j \geq 0$ and not all zero, which contradicts (PLICQ). Therefore, $(\lambda_1, \lambda_2) \neq (0, 0)$, so that $\lambda_1 + \lambda_2 > 0$. By Equation (5.11), we have that

$$\begin{aligned} & \lambda_1 \nabla_{\#} \underline{f}(x^*)^\top + \lambda_2 \nabla_{\#} \bar{f}(x^*)^\top + \sum_{j=1}^m \mu_j \nabla g_j(x^*)^\top = \mathbf{0}^\top \\ \Leftrightarrow & \lambda_1 \begin{bmatrix} \frac{\partial f}{\partial x_1^\#}(x^*) \\ \vdots \\ \frac{\partial f}{\partial x_n^\#}(x^*) \end{bmatrix}_{n \times 1} + \lambda_2 \begin{bmatrix} \frac{\partial \bar{f}}{\partial x_1^\#}(x^*) \\ \vdots \\ \frac{\partial \bar{f}}{\partial x_n^\#}(x^*) \end{bmatrix}_{n \times 1} + \mu_1 \begin{bmatrix} \frac{\partial g_1}{\partial x_1}(x^*) \\ \vdots \\ \frac{\partial g_1}{\partial x_n}(x^*) \end{bmatrix}_{n \times 1} + \dots + \mu_m \begin{bmatrix} \frac{\partial g_m}{\partial x_1}(x^*) \\ \vdots \\ \frac{\partial g_m}{\partial x_n}(x^*) \end{bmatrix}_{n \times 1} \\ = & \begin{bmatrix} 0 \\ \vdots \\ 0 \end{bmatrix}_{n \times 1} \end{aligned}$$

$$\begin{aligned}
& \Leftrightarrow \begin{cases} \lambda_1 \frac{\partial f}{\partial x_1^\#}(x^*) + \lambda_2 \frac{\partial \bar{f}}{\partial x_1^\#}(x^*) + \mu_1 \frac{\partial g_1}{\partial x_1}(x^*) + \dots + \mu_m \frac{\partial g_m}{\partial x_1}(x^*) = 0 \\ \vdots \\ \lambda_1 \frac{\partial f}{\partial x_n^\#}(x^*) + \lambda_2 \frac{\partial \bar{f}}{\partial x_n^\#}(x^*) + \mu_1 \frac{\partial g_1}{\partial x_n}(x^*) + \dots + \mu_m \frac{\partial g_m}{\partial x_n}(x^*) = 0 \end{cases} \\
& \Leftrightarrow \lambda_1 \frac{\partial f}{\partial x_i^\#}(x^*) + \lambda_2 \frac{\partial \bar{f}}{\partial x_i^\#}(x^*) + \sum_{j=1}^m \mu_j \frac{\partial g_j}{\partial x_i}(x^*) = 0, \quad i \in \{1, \dots, n\}. \quad (5.14)
\end{aligned}$$

Since $\lambda_1 + \lambda_2 > 0$, we can divide all the multipliers of Equation (5.14) by $\lambda_1 + \lambda_2$:

$$\begin{aligned}
& \frac{\lambda_1}{\lambda_1 + \lambda_2} \frac{\partial f}{\partial x_i^\#}(x^*) + \frac{\lambda_2}{\lambda_1 + \lambda_2} \frac{\partial \bar{f}}{\partial x_i^\#}(x^*) + \sum_{j=1}^m \frac{\mu_j}{\lambda_1 + \lambda_2} \frac{\partial g_j}{\partial x_i}(x^*) = 0 \\
& \Leftrightarrow \zeta_1 \frac{\partial f}{\partial x_i^\#}(x^*) + \zeta_2 \frac{\partial \bar{f}}{\partial x_i^\#}(x^*) = - \sum_{j=1}^m \hat{\mu}_j \frac{\partial g_j}{\partial x_i}(x^*), \quad i \in \{1, \dots, n\},
\end{aligned}$$

where $\zeta_l := \frac{\lambda_l}{\lambda_1 + \lambda_2} \geq 0$ and $\hat{\mu}_j := \frac{\mu_j}{\lambda_1 + \lambda_2} \geq 0$, by Equation (5.13), and since we already have that $\lambda_1 + \lambda_2 > 0$. Moreover, $\zeta_1 + \zeta_2 = \frac{\lambda_1}{\lambda_1 + \lambda_2} + \frac{\lambda_2}{\lambda_1 + \lambda_2} = \frac{\lambda_1 + \lambda_2}{\lambda_1 + \lambda_2} = 1$.

Then, by Definition 2.3.1 and by Proposition 2.3.3–(i), we have that

$$\begin{aligned}
& - \sum_{j=1}^m \hat{\mu}_j \frac{\partial g_j}{\partial x_i}(x^*) \in \left[\frac{\partial f}{\partial x_i^\#}(x^*) \vee \frac{\partial \bar{f}}{\partial x_i^\#}(x^*) \right] \\
& \Leftrightarrow 0 \in \left[\frac{\partial f}{\partial x_i^\#}(x^*) \vee \frac{\partial \bar{f}}{\partial x_i^\#}(x^*) \right] + \left[\sum_{j=1}^m \hat{\mu}_j \frac{\partial g_j}{\partial x_i}(x^*), \sum_{j=1}^m \hat{\mu}_j \frac{\partial g_j}{\partial x_i}(x^*) \right] \\
& \Leftrightarrow 0 \in \frac{\partial_{gH} F}{\partial x_i}(x^*) + \sum_{j=1}^m \hat{\mu}_j \frac{\partial g_j}{\partial x_i}(x^*), \quad i \in \{1, \dots, n\}. \quad (5.15)
\end{aligned}$$

Then, by Definition 2.3.4,

$$\begin{aligned}
\mathbf{0} = (0, \dots, 0) & \in \left(\frac{\partial_{gH} F}{\partial x_1}(x^*) + \sum_{j=1}^m \hat{\mu}_j \frac{\partial g_j}{\partial x_1}(x^*), \dots, \frac{\partial_{gH} F}{\partial x_n}(x^*) + \sum_{j=1}^m \hat{\mu}_j \frac{\partial g_j}{\partial x_n}(x^*) \right) \\
& = \left(\frac{\partial_{gH} F}{\partial x_1}(x^*), \dots, \frac{\partial_{gH} F}{\partial x_n}(x^*) \right) \oplus \left(\sum_{j=1}^m \hat{\mu}_j \frac{\partial g_j}{\partial x_1}(x^*), \dots, \sum_{j=1}^m \hat{\mu}_j \frac{\partial g_j}{\partial x_n}(x^*) \right) \\
& = \nabla_{gH} F(x^*) \oplus \sum_{j=1}^m \hat{\mu}_j \left(\frac{\partial g_j}{\partial x_1}(x^*), \dots, \frac{\partial g_j}{\partial x_n}(x^*) \right) \\
& = \nabla_{gH} F(x^*) \oplus \sum_{j=1}^m \hat{\mu}_j \nabla g_j(x^*).
\end{aligned}$$

Additionally, by Equation (5.12), we have that $\mu_j g_j(x^*) = 0$, $j \in I$, then

$$\widehat{\mu}_j g_j(x^*) = \frac{\mu_j}{\lambda_1 + \lambda_2} g_j(x^*) = \frac{\mu_j g_j(x^*)}{\lambda_1 + \lambda_2} = 0, \quad j \in I.$$

Therefore, there exists $\widehat{\mu} = (\widehat{\mu}_1, \dots, \widehat{\mu}_m) = \left(\frac{\mu_1}{\lambda_1 + \lambda_2}, \dots, \frac{\mu_m}{\lambda_1 + \lambda_2} \right) \in \mathbb{R}^m$ such that

$$\mathbf{0} \in \nabla_{gH} F(x^*) \oplus \sum_{j=1}^m \widehat{\mu}_j \nabla g_j(x^*),$$

$$\widehat{\mu}_j g_j(x^*) = 0, \quad j \in I,$$

and

$$\widehat{\mu}_j \geq 0, \quad j \in I.$$

□

Definition 5.1.1. An admissible point $x^* \in \mathcal{X}$ is said to be a Karush–Kuhn–Tucker solution, *KKT–solution in short*, or stationary solution, for (IOP) if there exists $\mu_j \in \mathbb{R}$, $j \in I$, satisfying equations (5.8)–(5.10).

Then, Theorem 5.1.2 becomes:

Theorem 5.1.3. Let $x^* \in \mathcal{X}$ be a weak LU–solution of (IOP). Assume that the constraints of (IOP) satisfy (PLICQ) at x^* . Then x^* is a KKT–solution.

If at the hypotheses of Theorem 5.1.1, is added the positive linear independence regularity condition (PLIRC), it can be obtained the following interval KKT condition:

Theorem 5.1.4 (Necessary optimality conditions of Karush–Kuhn–Tucker–type **2** for (IOP) using gH –differentiability). Let $x^* \in \mathcal{X}$ be a weak LU–solution of (IOP). Assume that (PLIRC) holds at x^* and that $\text{int} \left(\frac{\partial_{gH} F}{\partial x_i}(x^*) \right) \neq \emptyset$, $i \in \{1, \dots, n\}$. Then there exists $\mu = (\mu_1, \dots, \mu_m) \in \mathbb{R}^m$ such that

$$\mathbf{0} \in \text{int} \left(\nabla_{gH} F(x^*) \oplus \sum_{j=1}^m \mu_j \nabla g_j(x^*) \right), \quad (5.16)$$

$$\mu_j g_j(x^*) = 0, \quad j \in I, \quad (5.17)$$

and

$$\mu_j \geq 0, \quad j \in I. \quad (5.18)$$

Proof. From the Fritz John–type condition, there exist $\lambda_l \in \mathbb{R}$, $l \in \{1, 2\}$, and $\mu = (\mu_1, \dots, \mu_m) \in \mathbb{R}^m$ not all zero such that

$$\lambda_1 \nabla_{\#} \underline{f}(x^*) + \lambda_2 \nabla_{\#} \bar{f}(x^*) + \sum_{j=1}^m \mu_j \nabla g_j(x^*) = \mathbf{0}, \quad (5.19)$$

$$\mu_j g_j(x^*) = 0, \quad j \in I, \quad (5.20)$$

and

$$\lambda_1, \lambda_2, \mu_j \geq 0, \quad j \in I. \quad (5.21)$$

(i) If $\lambda_1 = 0$ and $\lambda_2 = 0$, then from Equation (5.19)

$$\sum_{j=1}^m \mu_j \nabla g_j(x^*) = \mathbf{0},$$

but Equation (5.20) is equivalent to saying that $\mu_j = 0$ when $g_j(x^*) < 0$, $j \in I \setminus I(x^*)$, i.e., when the constraint $g_j(x) \leq 0$ is inactive at x^* . Then, we would have

$$\sum_{j \in I(x^*)} \mu_j \nabla g_j(x^*) = \mathbf{0},$$

with $\mu_j \geq 0$ and not all zero, which contradicts (PLIRC) – (i). Therefore, $(\lambda_1, \lambda_2) \neq (0, 0)$, so that $\lambda_1 + \lambda_2 > 0$.

(ii) If $\lambda_1 = 0$ and $\lambda_2 > 0$, then from Equation (5.19)

$$\lambda_2 \nabla_{\#} \bar{f}(x^*) + \sum_{j \in I(x^*)} \mu_j \nabla g_j(x^*) = \mathbf{0},$$

with $\lambda_2 > 0$ and $\mu_j \geq 0$, $j \in I(x^*)$, which contradicts (PLIRC) – (ii).

(iii) If $\lambda_1 > 0$ and $\lambda_2 = 0$, then from Equation (5.19)

$$\lambda_1 \nabla_{\#} \underline{f}(x^*) + \sum_{j \in I(x^*)} \mu_j \nabla g_j(x^*) = \mathbf{0},$$

with $\lambda_1 > 0$ and $\mu_j \geq 0$, $j \in I(x^*)$, which contradicts (PLIRC) – (iii).

(iv) Therefore,

$$\lambda_1 > 0 \quad \text{and} \quad \lambda_2 > 0. \quad (5.22)$$

By Equation (5.19), we have that

$$\begin{aligned}
& \lambda_1 \nabla_{\#} \underline{f}(x^*)^\top + \lambda_2 \nabla_{\#} \bar{f}(x^*)^\top + \sum_{j=1}^m \mu_j \nabla g_j(x^*)^\top = \mathbf{0}^\top \\
& \Leftrightarrow \lambda_1 \begin{bmatrix} \frac{\partial \underline{f}}{\partial x_1^\#}(x^*) \\ \vdots \\ \frac{\partial \underline{f}}{\partial x_n^\#}(x^*) \end{bmatrix}_{n \times 1} + \lambda_2 \begin{bmatrix} \frac{\partial \bar{f}}{\partial x_1^\#}(x^*) \\ \vdots \\ \frac{\partial \bar{f}}{\partial x_n^\#}(x^*) \end{bmatrix}_{n \times 1} + \mu_1 \begin{bmatrix} \frac{\partial g_1}{\partial x_1}(x^*) \\ \vdots \\ \frac{\partial g_1}{\partial x_n}(x^*) \end{bmatrix}_{n \times 1} + \dots + \mu_m \begin{bmatrix} \frac{\partial g_m}{\partial x_1}(x^*) \\ \vdots \\ \frac{\partial g_m}{\partial x_n}(x^*) \end{bmatrix}_{n \times 1} \\
& = \begin{bmatrix} 0 \\ \vdots \\ 0 \end{bmatrix}_{n \times 1} \\
& \Leftrightarrow \begin{cases} \lambda_1 \frac{\partial \underline{f}}{\partial x_1^\#}(x^*) + \lambda_2 \frac{\partial \bar{f}}{\partial x_1^\#}(x^*) + \mu_1 \frac{\partial g_1}{\partial x_1}(x^*) + \dots + \mu_m \frac{\partial g_m}{\partial x_1}(x^*) = 0 \\ \vdots \\ \lambda_1 \frac{\partial \underline{f}}{\partial x_n^\#}(x^*) + \lambda_2 \frac{\partial \bar{f}}{\partial x_n^\#}(x^*) + \mu_1 \frac{\partial g_1}{\partial x_n}(x^*) + \dots + \mu_m \frac{\partial g_m}{\partial x_n}(x^*) = 0 \end{cases} \\
& \Leftrightarrow \lambda_1 \frac{\partial \underline{f}}{\partial x_i^\#}(x^*) + \lambda_2 \frac{\partial \bar{f}}{\partial x_i^\#}(x^*) + \sum_{j=1}^m \mu_j \frac{\partial g_j}{\partial x_i}(x^*) = 0, \quad i \in \{1, \dots, n\}. \quad (5.23)
\end{aligned}$$

Since $\lambda_1 > 0$ and $\lambda_2 > 0$, we have that $\lambda_1 + \lambda_2 > 0$, then we can divide all the multipliers of Equation (5.23) by $\lambda_1 + \lambda_2$:

$$\begin{aligned}
& \frac{\lambda_1}{\lambda_1 + \lambda_2} \frac{\partial \underline{f}}{\partial x_i^\#}(x^*) + \frac{\lambda_2}{\lambda_1 + \lambda_2} \frac{\partial \bar{f}}{\partial x_i^\#}(x^*) + \sum_{j=1}^m \frac{\mu_j}{\lambda_1 + \lambda_2} \frac{\partial g_j}{\partial x_i}(x^*) = 0 \\
& \Leftrightarrow \zeta_1 \frac{\partial \underline{f}}{\partial x_i^\#}(x^*) + \zeta_2 \frac{\partial \bar{f}}{\partial x_i^\#}(x^*) = - \sum_{j=1}^m \hat{\mu}_j \frac{\partial g_j}{\partial x_i}(x^*), \quad i \in \{1, \dots, n\},
\end{aligned}$$

where $\zeta_l := \frac{\lambda_l}{\lambda_1 + \lambda_2} > 0$ and $\hat{\mu}_j := \frac{\mu_j}{\lambda_1 + \lambda_2} \geq 0$, by Equation (5.22) and by Equation (5.21), respectively, and since we already have that $\lambda_1 + \lambda_2 > 0$. Moreover, $\zeta_1 + \zeta_2 = \frac{\lambda_1}{\lambda_1 + \lambda_2} + \frac{\lambda_2}{\lambda_1 + \lambda_2} = \frac{\lambda_1 + \lambda_2}{\lambda_1 + \lambda_2} = 1$.

Then, by Definition 2.3.2 and by Proposition 2.3.3-(ii), we have that

$$\begin{aligned}
& - \sum_{j=1}^m \hat{\mu}_j \frac{\partial g_j}{\partial x_i}(x^*) \in \left(\frac{\partial \underline{f}}{\partial x_i^\#}(x^*) \vee \frac{\partial \bar{f}}{\partial x_i^\#}(x^*) \right) \\
& \Leftrightarrow 0 \in \text{int} \left(\left[\frac{\partial \underline{f}}{\partial x_i^\#}(x^*) \vee \frac{\partial \bar{f}}{\partial x_i^\#}(x^*) \right] + \left[\sum_{j=1}^m \hat{\mu}_j \frac{\partial g_j}{\partial x_i}(x^*), \sum_{j=1}^m \hat{\mu}_j \frac{\partial g_j}{\partial x_i}(x^*) \right] \right) \\
& \Leftrightarrow 0 \in \text{int} \left(\frac{\partial g_H F}{\partial x_i}(x^*) + \sum_{j=1}^m \hat{\mu}_j \frac{\partial g_j}{\partial x_i}(x^*) \right), \quad i \in \{1, \dots, n\}. \quad (5.24)
\end{aligned}$$

Then, by Definition 2.3.4 and Equality (2.10),

$$\begin{aligned}
\mathbf{0} &= (0, \dots, 0) \\
&\in \left(\text{int} \left(\frac{\partial_{gH}F}{\partial x_1}(x^*) + \sum_{j=1}^m \widehat{\mu}_j \frac{\partial g_j}{\partial x_1}(x^*) \right), \dots, \text{int} \left(\frac{\partial_{gH}F}{\partial x_n}(x^*) + \sum_{j=1}^m \widehat{\mu}_j \frac{\partial g_j}{\partial x_n}(x^*) \right) \right) \\
&= \text{int} \left(\left(\frac{\partial_{gH}F}{\partial x_1}(x^*) + \sum_{j=1}^m \widehat{\mu}_j \frac{\partial g_j}{\partial x_1}(x^*), \dots, \frac{\partial_{gH}F}{\partial x_n}(x^*) + \sum_{j=1}^m \widehat{\mu}_j \frac{\partial g_j}{\partial x_n}(x^*) \right) \right) \\
&= \text{int} \left(\left(\frac{\partial_{gH}F}{\partial x_1}(x^*), \dots, \frac{\partial_{gH}F}{\partial x_n}(x^*) \right) \oplus \left(\sum_{j=1}^m \widehat{\mu}_j \frac{\partial g_j}{\partial x_1}(x^*), \dots, \sum_{j=1}^m \widehat{\mu}_j \frac{\partial g_j}{\partial x_n}(x^*) \right) \right) \\
&= \text{int} \left(\nabla_{gH}F(x^*) \oplus \sum_{j=1}^m \widehat{\mu}_j \left(\frac{\partial g_j}{\partial x_1}(x^*), \dots, \frac{\partial g_j}{\partial x_n}(x^*) \right) \right) \\
&= \text{int} \left(\nabla_{gH}F(x^*) \oplus \sum_{j=1}^m \widehat{\mu}_j \nabla g_j(x^*) \right).
\end{aligned}$$

Additionally, by Equation (5.20), we have that $\mu_j g_j(x^*) = 0$, $j \in I$, then

$$\widehat{\mu}_j g_j(x^*) = \frac{\mu_j}{\lambda_1 + \lambda_2} g_j(x^*) = \frac{\mu_j g_j(x^*)}{\lambda_1 + \lambda_2} = 0, \quad j \in I.$$

Therefore, there exists $\widehat{\mu} = (\widehat{\mu}_1, \dots, \widehat{\mu}_m) = \left(\frac{\mu_1}{\lambda_1 + \lambda_2}, \dots, \frac{\mu_m}{\lambda_1 + \lambda_2} \right) \in \mathbb{R}^m$ such that

$$\mathbf{0} \in \text{int} \left(\nabla_{gH}F(x^*) \oplus \sum_{j=1}^m \widehat{\mu}_j \nabla g_j(x^*) \right),$$

$$\widehat{\mu}_j g_j(x^*) = 0, \quad j \in I,$$

and

$$\widehat{\mu}_j \geq 0, \quad j \in I.$$

□

Definition 5.1.2. An admissible point $x^* \in \mathcal{X}$ is said to be a strict Karush–Kuhn–Tucker solution, strict KKT–solution in short, or strict stationary solution, for (IOP) if there exists $\mu_j \in \mathbb{R}$, $j \in I$, satisfying equations (5.16)–(5.18).

Then, Theorem 5.1.4 becomes:

Theorem 5.1.5. Let $x^* \in \mathcal{X}$ be a weak LU–solution of (IOP). Assume that (PLIRC) holds at x^* and that $\text{int} \left(\frac{\partial_{gH}F}{\partial x_i}(x^*) \right) \neq \emptyset$, $i \in \{1, \dots, n\}$. Then x^* is a strict KKT–solution.

The last Theorem is a strong result, so naturally it has the strong hypothesis:

$$\text{int} \left(\frac{\partial_{gH} F}{\partial x_i}(x^*) \right) \neq \emptyset. \quad (5.25)$$

In simple examples, we can have that Condition (5.25) is violated.

Example 5.1.1. Given $F : \mathbb{R} \rightarrow \mathcal{K}_C$ by $F(x) = [\underline{f}(x), \bar{f}(x)] = [0, x^2]$, then $F'_{gH}(x) = [0 \vee 2x]$. If $x^* = 0$, we would have $F'_{gH}(x^*) = [0, 0]$, so $\text{int} (F'_{gH}(x^*)) = \emptyset$.

We can see that Condition (5.25) for Example 5.1.1 is not satisfied. As a future result, it will be interesting to remove this hypothesis or in a some way weaken it.

Remark 5.1.1. By Remark 2.10.1–(i), every strong LU–solution is a LU–solution, and every LU–solution is a $\underline{LU}$ –solution (respectively $\overline{LU}$ –solution), and every $\underline{LU}$ –solution (respectively $\overline{LU}$ –solution) is a weak LU–solution, so that Theorem 5.1.3 and 5.1.5 that are applicable to weak LU–solutions are also applicable to the other Types of LU–solutions.

In the following section, it will be given sufficient optimality conditions for the interval optimization problem.

5.2 Sufficient optimality conditions for (IOP)

The sufficient optimality conditions herein presented are obtained through generalized convexity. For this reason, it will be extended to the intervalar environment some concepts of pseudo–invexity.

In the case of necessary conditions, it was enough to obtain results for weak LU–solutions (since they are also valid for the other types of solutions). Concerning sufficient conditions, each type of solution requires a specific definition of pseudo–invexity and, consequently, a specific result is obtained.

We start with the following technical lemma:

Lemma 5.2.1. Let $x^* \in \mathcal{X}$ and $\mu = (\mu_1, \dots, \mu_m) \geq 0$. We have the following

(i) $\mathbf{0} \in \nabla_{gH} F(x^*) \oplus \sum_{j \in I(x^*)} \mu_j \nabla g_j(x^*)$ if and only if there exist $\lambda_1 \geq 0$ and $\lambda_2 \geq 0$ with $\lambda_1 + \lambda_2 = 1$ such that

$$\lambda_1 \nabla_{\#} \underline{f}(x^*) + \lambda_2 \nabla_{\#} \bar{f}(x^*) + \sum_{j \in I(x^*)} \mu_j \nabla g_j(x^*) = \mathbf{0}.$$

(ii) $\mathbf{0} \in \text{int} \left(\nabla_{gH} F(x^*) \oplus \sum_{j \in I(x^*)} \mu_j \nabla g_j(x^*) \right)$ if and only if there exist $\lambda_1 > 0$ and $\lambda_2 > 0$ with $\lambda_1 + \lambda_2 = 1$ such that

$$\lambda_1 \nabla_{\#} \underline{f}(x^*) + \lambda_2 \nabla_{\#} \bar{f}(x^*) + \sum_{j \in I(x^*)} \mu_j \nabla g_j(x^*) = \mathbf{0}.$$

Proof. (i) In fact, if $\mathbf{0} \in \nabla_{gH} F(x^*) \oplus \sum_{j \in I(x^*)} \mu_j \nabla g_j(x^*)$ we have that,

$$\begin{aligned} & - \sum_{j \in I(x^*)} \mu_j \nabla g_j(x^*) \in \nabla_{gH} F(x^*), \text{ by Proposition 2.3.4 - (i)} \\ \Leftrightarrow & \left(- \sum_{j \in I(x^*)} \mu_j \frac{\partial g_j}{\partial x_1}(x^*), \dots, - \sum_{j \in I(x^*)} \mu_j \frac{\partial g_j}{\partial x_n}(x^*) \right) \in \left(\frac{\partial_{gH} F}{\partial x_1}(x^*), \dots, \frac{\partial_{gH} F}{\partial x_n}(x^*) \right) \\ \Leftrightarrow & - \sum_{j \in I(x^*)} \mu_j \frac{\partial g_j}{\partial x_i}(x^*) \in \frac{\partial_{gH} F}{\partial x_i}(x^*) = \left[\frac{\partial \underline{f}}{\partial x_i^{\#}}(x^*) \vee \frac{\partial \bar{f}}{\partial x_i^{\#}}(x^*) \right], \quad i \in \{1, \dots, n\}, \end{aligned}$$

then, by Definition 2.3.1, there exist $\lambda_1 \geq 0$ and $\lambda_2 \geq 0$ with $\lambda_1 + \lambda_2 = 1$ such that

$$\begin{aligned} & \lambda_1 \frac{\partial \underline{f}}{\partial x_i^{\#}}(x^*) + \lambda_2 \frac{\partial \bar{f}}{\partial x_i^{\#}}(x^*) = - \sum_{j \in I(x^*)} \mu_j \frac{\partial g_j}{\partial x_i}(x^*), \quad i \in \{1, \dots, n\} \\ \Leftrightarrow & \left(\lambda_1 \frac{\partial \underline{f}}{\partial x_1^{\#}}(x^*) + \lambda_2 \frac{\partial \bar{f}}{\partial x_1^{\#}}(x^*) + \sum_{j \in I(x^*)} \mu_j \frac{\partial g_j}{\partial x_1}(x^*), \dots, \right. \\ & \left. \lambda_1 \frac{\partial \underline{f}}{\partial x_n^{\#}}(x^*) + \lambda_2 \frac{\partial \bar{f}}{\partial x_n^{\#}}(x^*) + \sum_{j \in I(x^*)} \mu_j \frac{\partial g_j}{\partial x_n}(x^*) \right) = (0, \dots, 0) \\ \Leftrightarrow & \lambda_1 \nabla_{\#} \underline{f}(x^*) + \lambda_2 \nabla_{\#} \bar{f}(x^*) + \sum_{j \in I(x^*)} \mu_j \nabla g_j(x^*) = \mathbf{0}. \end{aligned}$$

(ii) In fact, if $\mathbf{0} \in \text{int} \left(\nabla_{gH} F(x^*) \oplus \sum_{j \in I(x^*)} \mu_j \nabla g_j(x^*) \right)$ we have that,

$$\begin{aligned} & - \sum_{j \in I(x^*)} \mu_j \nabla g_j(x^*) \in \text{int}(\nabla_{gH} F(x^*)), \text{ by Proposition 2.3.4 - (ii)} \\ \Leftrightarrow & \left(- \sum_{j \in I(x^*)} \mu_j \frac{\partial g_j}{\partial x_1}(x^*), \dots, - \sum_{j \in I(x^*)} \mu_j \frac{\partial g_j}{\partial x_n}(x^*) \right) \\ & \in \left(\text{int} \left(\frac{\partial_{gH} F}{\partial x_1}(x^*) \right), \dots, \text{int} \left(\frac{\partial_{gH} F}{\partial x_n}(x^*) \right) \right) \end{aligned}$$

$$\begin{aligned}
&\Leftrightarrow - \sum_{j \in I(x^*)} \mu_j \frac{\partial g_j}{\partial x_i}(x^*) \in \text{int} \left(\frac{\partial_{gH} F}{\partial x_i}(x^*) \right) \\
&= \text{int} \left(\left[\frac{\partial f}{\partial x_i^\#}(x^*) \vee \frac{\partial \bar{f}}{\partial x_i^\#}(x^*) \right] \right) = \left(\frac{\partial f}{\partial x_i^\#}(x^*) \vee \frac{\partial \bar{f}}{\partial x_i^\#}(x^*) \right), \quad i \in \{1, \dots, n\},
\end{aligned}$$

then, by Definition 2.3.2, there exist $\lambda_1 > 0$ and $\lambda_2 > 0$ with $\lambda_1 + \lambda_2 = 1$ such that

$$\begin{aligned}
&\lambda_1 \frac{\partial f}{\partial x_i^\#}(x^*) + \lambda_2 \frac{\partial \bar{f}}{\partial x_i^\#}(x^*) = - \sum_{j \in I(x^*)} \mu_j \frac{\partial g_j}{\partial x_i}(x^*), \quad i \in \{1, \dots, n\} \\
&\Leftrightarrow (\lambda_1 \frac{\partial f}{\partial x_1^\#}(x^*) + \lambda_2 \frac{\partial \bar{f}}{\partial x_1^\#}(x^*) + \sum_{j \in I(x^*)} \mu_j \frac{\partial g_j}{\partial x_1}(x^*), \dots, \\
&\quad \lambda_1 \frac{\partial f}{\partial x_n^\#}(x^*) + \lambda_2 \frac{\partial \bar{f}}{\partial x_n^\#}(x^*) + \sum_{j \in I(x^*)} \mu_j \frac{\partial g_j}{\partial x_n}(x^*)) = (0, \dots, 0) \\
&\Leftrightarrow \lambda_1 \nabla_{\#} f(x^*) + \lambda_2 \nabla_{\#} \bar{f}(x^*) + \sum_{j \in I(x^*)} \mu_j \nabla g_j(x^*) = \mathbf{0}.
\end{aligned}$$

□

Given $y \in \mathcal{X}$, we denote the cone of linearized feasible directions at y by

$$\mathcal{L}(y) = \{d \in \mathbb{R}^n \mid \langle \nabla g_j(y), d \rangle \leq 0, \quad j \in I(y)\},$$

where $I(y) = \{j \in I \mid g_j(y) = 0\}$. This set coincides with the tangent cone to $\mathcal{X}$ at y when a constraint qualification is satisfied ((PLICQ), for example, for further details see O. L. Mangasarian and S. Fromovitz, [54]).

Definition 5.2.1. (IOP) is said to be a weak Karush–Kuhn–Tucker–interval pseudoinvex problem, a weak KKT–interval pseudoinvex problem in short, if for each pair $(x, y) \in \mathcal{X} \times \mathcal{X}$ there exists $\eta = \eta(x, y) \in \mathcal{L}(y)$ such that

$$F(x) <_{LU} F(y) \Rightarrow F'_{gH}(y; \eta(x, y)) <_{LU} [0, 0].$$

Theorem 5.2.1 (Sufficient Interval Optimality Condition for weak **LU**–solutions). If (IOP) is a weak KKT–interval pseudoinvex, then every KKT–solution is a weak **LU**–solution of (IOP).

Proof. Let $x^* \in \mathcal{X}$ be a KKT–solution of (IOP), then by Definition 5.1.1, it follows that

$$\mathbf{0} \in \nabla_{gH} F(x^*) \oplus \sum_{j=1}^m \mu_j \nabla g_j(x^*) \quad (5.26)$$

$$\mu_j g_j(x^*) = 0, \quad j \in I, \quad (5.27)$$

and

$$\mu_j \geq 0, \quad j \in I. \quad (5.28)$$

Assume that x^* is not a weak LU -solution, that is, assume that there exists $x \in \mathcal{X}$ such that $F(x) <_{LU} F(x^*)$.

Provided (IOP) is weak KKT -interval pseudoinvex, by Definition 5.2.1, there exists $\eta(x, x^*) \in \mathcal{L}(x^*)$ such that

$$F'_{gH}(x^*; \eta(x, x^*)) <_{LU} [0, 0]$$

and

$$\langle \nabla g_j(x^*), \eta(x, x^*) \rangle \leq 0, \quad j \in I(x^*). \quad (5.29)$$

By (5.28) and (5.29), we have that

$$\sum_{j \in I(x^*)} \langle \mu_j \nabla g_j(x^*), \eta(x, x^*) \rangle \leq 0, \quad j \in I(x^*). \quad (5.30)$$

By Theorem 2.7.4, we have that

$$\left[\langle \nabla_{\#} \underline{f}(x^*), \eta(x, x^*) \rangle \vee \langle \nabla_{\#} \bar{f}(x^*), \eta(x, x^*) \rangle \right] <_{LU} [0, 0],$$

so that

$$\langle \nabla_{\#} \underline{f}(x^*), \eta(x, x^*) \rangle < 0 \quad \text{and} \quad \langle \nabla_{\#} \bar{f}(x^*), \eta(x, x^*) \rangle < 0. \quad (5.31)$$

From Equations (5.26) and (5.27),

$$\mathbf{0} \in \nabla_{gH} F(x^*) \oplus \sum_{j \in I(x^*)} \mu_j \nabla g_j(x^*),$$

which is equivalent to

$$\begin{aligned} & \lambda_1 \nabla_{\#} \underline{f}(x^*) + \lambda_2 \nabla_{\#} \bar{f}(x^*) + \sum_{j \in I(x^*)} \mu_j \nabla g_j(x^*) = \mathbf{0} \\ \Rightarrow & \lambda_1 \langle \nabla_{\#} \underline{f}(x^*), \eta(x, x^*) \rangle + \lambda_2 \langle \nabla_{\#} \bar{f}(x^*), \eta(x, x^*) \rangle + \sum_{j \in I(x^*)} \langle \mu_j \nabla g_j(x^*), \eta(x, x^*) \rangle = 0 \\ \Leftrightarrow & \sum_{j \in I(x^*)} \langle \mu_j \nabla g_j(x^*), \eta(x, x^*) \rangle = -\lambda_1 \langle \nabla_{\#} \underline{f}(x^*), \eta(x, x^*) \rangle - \lambda_2 \langle \nabla_{\#} \bar{f}(x^*), \eta(x, x^*) \rangle, \end{aligned}$$

for some $\lambda_1 \geq 0$ and $\lambda_2 \geq 0$ with $\lambda_1 + \lambda_2 = 1$, so that $(\lambda_1, \lambda_2) \neq (0, 0)$, by Lemma 5.2.1-(i). Then, by inequalities (5.31),

$$\sum_{j \in I(x^*)} \langle \mu_j \nabla g_j(x^*), \eta(x, x^*) \rangle > 0,$$

which contradicts (5.30). $\square$

Now, we establish the converse of Theorem 5.2.1:

Theorem 5.2.2. *If (IOP) is such that every KKT-solution is a weak LU-solution, then (IOP) is a weak KKT-interval pseudoinvex problem.*

Proof. Let $(x, x^*) \in \mathcal{X} \times \mathcal{X}$ be a pair of admissible points such that $F(x) <_{LU} F(x^*)$. Then x^* is not a weak LU-solution, so by hypothesis, x^* is not a KKT-solution. By Lemma 5.2.1-(i), there do not exist $\lambda_1 \geq 0$ and $\lambda_2 \geq 0$ with $\lambda_1 + \lambda_2 = 1$, and $\mu_j \in \mathbb{R}$, $j \in I$, such that

$$\lambda_1 \nabla_{\#} \underline{f}(x^*) + \lambda_2 \nabla_{\#} \bar{f}(x^*) + \sum_{j \in I(x^*)} \mu_j \nabla g_j(x^*) = \mathbf{0}.$$

Then, by Motzkin's Alternative Theorem, see Theorem 2.10.1, there exist a vector $u \in \mathbb{R}^n$ such that the following system has a solution

$$\begin{cases} \langle \nabla_{\#} \underline{f}(x^*), u \rangle < 0, \\ \langle \nabla_{\#} \bar{f}(x^*), u \rangle < 0, \\ \langle \nabla g_j(x^*), u \rangle \leq 0, \quad j \in I(x^*). \end{cases} \quad (5.32)$$

Defining $u := \eta(x, x^*) \in \mathbb{R}^n$, we see that $\eta(x, x^*) \in \mathcal{L}(x^*)$ and, by Theorem 2.7.4, $F'_{gH}(x^*; \eta(x, x^*)) <_{LU} [0, 0]$.

Thus, supposing that every KKT-solution is a weak LU-solution, for each pair $(x, x^*) \in \mathcal{X} \times \mathcal{X}$ there exists $\eta(x, x^*) \in \mathcal{L}(x^*)$ such that $F(x) <_{LU} F(x^*)$ implies that $F'_{gH}(x^*; \eta(x, x^*)) <_{LU} [0, 0]$, that is, (IOP) is a weak KKT-interval pseudoinvex problem. $\square$

Consequently, as a direct consequence of the last two theorems, we have the following result.

Theorem 5.2.3. *(IOP) is such that every KKT-solution is a weak LU-solution if and only if (IOP) is a weak KKT-interval pseudoinvex problem.*

Definition 5.2.2. (IOP) is said to be a Karush–Kuhn–Tucker–interval pseudoinvex problem, a KKT–interval pseudoinvex problem in short, if for each pair $(x, y) \in \mathcal{X} \times \mathcal{X}$ there exists $\eta = \eta(x, y) \in \mathcal{L}(y)$ such that

$$F(x) \leq_{LU} F(y) \Rightarrow F'_{gH}(y; \eta(x, y)) <_{LU} [0, 0].$$

Theorem 5.2.4. (IOP) is such that every KKT–solution is a LU–solution if and only if (IOP) is a KKT–interval pseudoinvex problem.

Proof. Similar to the proofs of Theorems 5.2.1 and 5.2.2 (just change $F(x) <_{LU} F(x^*)$ by $F(x) \leq_{LU} F(x^*)$). $\square$

Definition 5.2.3. (IOP) is said to be a strong Karush–Kuhn–Tucker–interval pseudoinvex problem, a strong KKT–interval pseudoinvex problem in short, if for each pair $(x, y) \in \mathcal{X} \times \mathcal{X}$ there exists $\eta = \eta(x, y) \in \mathcal{L}(y)$ such that

$$F(x) \leq_{LU} F(y) \Rightarrow F'_{gH}(y; \eta(x, y)) <_{LU} [0, 0].$$

Theorem 5.2.5. (IOP) is such that every KKT–solution is a strong LU–solution if and only if (IOP) is a strong KKT–interval pseudoinvex problem.

Proof. Similar to the proofs of Theorems 5.2.1 and 5.2.2 (just change $F(x) <_{LU} F(x^*)$ by $F(x) \leq_{LU} F(x^*)$). $\square$

Definition 5.2.4. (IOP) is said to be a strict Karush–Kuhn–Tucker–interval pseudoinvex problem, a strict KKT–interval pseudoinvex problem in short, if for each pair $(x, y) \in \mathcal{X} \times \mathcal{X}$ there exists $\eta = \eta(x, y) \in \mathcal{L}(y)$ such that

$$F(x) \leq_{LU} F(y) \Rightarrow F'_{gH}(y; \eta(x, y)) \leq_{LU} [0, 0].$$

Theorem 5.2.6 (Sufficient Interval Optimality Condition for **LU**–solutions). If (IOP) is a strict KKT–interval pseudoinvex problem, then every strict KKT–solution is a LU–solution of (IOP).

Proof. Let $x^* \in \mathcal{X}$ be a strict KKT–solution of (IOP). By Definition 5.1.2, it follows that

$$\mathbf{0} \in \text{int} \left(\nabla_{gH} F(x^*) \oplus \sum_{j=1}^m \mu_j \nabla g_j(x^*) \right) \quad (5.33)$$

$$\mu_j g_j(x^*) = 0, \quad j \in I, \quad (5.34)$$

and

$$\mu_j \geq 0, \quad j \in I. \quad (5.35)$$

Assume that x^* is not a LU -solution, that is, assume that there exists $x \in \mathcal{X}$ such that $F(x) \leq_{LU} F(x^*)$.

Provided (IOP) is strict KKT -interval pseudoinvex, by Definition 5.2.4, there exists $\eta(x, x^*) \in \mathcal{L}(x^*)$ such that

$$F'_{gH}(x^*; \eta(x, x^*)) \leq_{LU} [0, 0] \quad (5.36)$$

and

$$\langle \nabla g_j(x^*), \eta(x, x^*) \rangle \leq 0, \quad j \in I(x^*). \quad (5.37)$$

By (5.35) and (5.37), we have that

$$\sum_{j \in I(x^*)} \langle \mu_j \nabla g_j(x^*), \eta(x, x^*) \rangle \leq 0, \quad j \in I(x^*). \quad (5.38)$$

By Theorem 2.7.4, we have that

$$\left[\langle \nabla_{\#} \underline{f}(x^*), \eta(x, x^*) \rangle \vee \langle \nabla_{\#} \bar{f}(x^*), \eta(x, x^*) \rangle \right] \leq_{LU} [0, 0],$$

so that

$$\begin{cases} \langle \nabla_{\#} \underline{f}(x^*), \eta(x, x^*) \rangle < 0 & \text{and} & \langle \nabla_{\#} \bar{f}(x^*), \eta(x, x^*) \rangle \leq 0 \\ \text{or} \\ \langle \nabla_{\#} \underline{f}(x^*), \eta(x, x^*) \rangle \leq 0 & \text{and} & \langle \nabla_{\#} \bar{f}(x^*), \eta(x, x^*) \rangle < 0. \end{cases} \quad (5.39)$$

From Equations (5.33) and (5.34), then

$$\mathbf{0} \in \text{int} \left(\nabla_{gH} F(x^*) \oplus \sum_{j \in I(x^*)} \mu_j \nabla g_j(x^*) \right),$$

which is equivalent to

$$\begin{aligned}
& \lambda_1 \nabla_{\#} \underline{f}(x^*) + \lambda_2 \nabla_{\#} \bar{f}(x^*) + \sum_{j \in I(x^*)} \mu_j \nabla g_j(x^*) = 0 \\
& \Rightarrow \lambda_1 \langle \nabla_{\#} \underline{f}(x^*), \eta(x, x^*) \rangle + \lambda_2 \langle \nabla_{\#} \bar{f}(x^*), \eta(x, x^*) \rangle + \sum_{j \in I(x^*)} \langle \mu_j \nabla g_j(x^*), \eta(x, x^*) \rangle = 0 \\
& \Leftrightarrow \sum_{j \in I(x^*)} \langle \mu_j \nabla g_j(x^*), \eta(x, x^*) \rangle = -\lambda_1 \langle \nabla_{\#} \underline{f}(x^*), \eta(x, x^*) \rangle - \lambda_2 \langle \nabla_{\#} \bar{f}(x^*), \eta(x, x^*) \rangle,
\end{aligned} \tag{5.40}$$

for some $\lambda_1 > 0$ and $\lambda_2 > 0$ with $\lambda_1 + \lambda_2 = 1$, by Lemma 5.2.1-(ii). Then, by inequalities (5.39),

$$\sum_{j \in I(x^*)} \langle \mu_j \nabla g_j(x^*), \eta(x, x^*) \rangle > 0,$$

which contradicts (5.38). $\square$

Now, we establish the converse of Theorem 5.2.6:

Theorem 5.2.7. *If (IOP) is such that every strict KKT-solution is a LU-solution, then (IOP) is a strict KKT-interval pseudoinvex problem.*

Proof. Let $(x, x^*) \in \mathcal{X} \times \mathcal{X}$ be a pair of admissible points such that $F(x) \leq_{LU} F(x^*)$, so that x^* is not a LU-solution, and, by hypothesis, x^* is not a strict KKT-solution. By Lemma 5.2.1-(ii), there do not exist $\lambda_1 > 0$ and $\lambda_2 > 0$ with $\lambda_1 + \lambda_2 = 1$, and $\mu_j \in \mathbb{R}$, $j \in I$, such that

$$\lambda_1 \nabla_{\#} \underline{f}(x^*) + \lambda_2 \nabla_{\#} \bar{f}(x^*) + \sum_{j \in I(x^*)} \mu_j \nabla g_j(x^*) = \mathbf{0}.$$

Then, by Tucker's Alternative Theorem, see Theorem 2.10.2, there exist a vector $u \in \mathbb{R}^n$ such that one of the following system has a solution

$$\begin{cases} \langle \nabla_{\#} \underline{f}(x^*), u \rangle < 0, \\ \langle \nabla_{\#} \bar{f}(x^*), u \rangle \leq 0, \\ \langle \nabla g_j(x^*), u \rangle \leq 0, \quad j \in I(x^*) \end{cases} \quad \text{or} \quad \begin{cases} \langle \nabla_{\#} \underline{f}(x^*), u \rangle \leq 0, \\ \langle \nabla_{\#} \bar{f}(x^*), u \rangle < 0, \\ \langle \nabla g_j(x^*), u \rangle \leq 0, \quad j \in I(x^*). \end{cases} \tag{5.41}$$

Defining $u := \eta(x, x^*) \in \mathbb{R}^n$, we see that $\eta(x, x^*) \in \mathcal{L}(x^*)$ and, by Theorem 2.7.4, $F'_{gH}(x^*; \eta(x, x^*)) <_{LU} [0, 0]$.

Thus, supposing that every strict KKT-solution is a LU-solution, for each pair $(x, x^*) \in \mathcal{X} \times \mathcal{X}$ there exists $\eta(x, x^*) \in \mathcal{L}(x^*)$ such that $F(x) \leq_{LU} F(x^*)$ implies that

$F'_{gH}(x^*; \eta(x, x^*)) \leq_{LU} [0, 0]$, that is, (IOP) is a strict KKT -interval pseudoinvex problem. $\square$

Consequently, as a direct consequence of the last two theorems, we have the following result.

Theorem 5.2.8. (IOP) is such that every strict KKT -solution is a LU -solution if and only if (IOP) is strict KKT -interval pseudoinvex.

Definition 5.2.5. (IOP) is said to be a partial- $\underline{LU}$ strict Karush–Kuhn–Tucker–interval pseudoinvex problem, a partial- $\underline{LU}$ strict KKT -interval pseudoinvex problem in short, if for each pair $(x, y) \in \mathcal{X} \times \mathcal{X}$ there exists $\eta = \eta(x, y) \in \mathcal{L}(y)$ such that

$$F(x) \leq_{\underline{LU}} F(y) \Rightarrow F'_{gH}(y; \eta(x, y)) \leq_{LU} [0, 0].$$

Theorem 5.2.9. (IOP) is such that every strict KKT -solution is a $\underline{LU}$ -solution if and only if (IOP) is partial- $\underline{LU}$ strict KKT -interval pseudoinvex.

Proof. Similar to the proofs of Theorems 5.2.6 and 5.2.7 (just change $F(x) \leq_{LU} F(x^*)$ by $F(x) \leq_{\underline{LU}} F(x^*)$). $\square$

Definition 5.2.6. (IOP) is said to be a partial- $\overline{LU}$ strict Karush–Kuhn–Tucker–interval pseudoinvex problem, a partial- $\overline{LU}$ strict KKT -interval pseudoinvex problem in short, if for each pair $(x, y) \in \mathcal{X} \times \mathcal{X}$ there exists $\eta = \eta(x, y) \in \mathcal{L}(y)$ such that

$$F(x) \leq_{\overline{LU}} F(y) \Rightarrow F'_{gH}(y; \eta(x, y)) \leq_{LU} [0, 0].$$

Theorem 5.2.10. (IOP) is such that every strict KKT -solution is a $\overline{LU}$ -solution if and only if (IOP) is partial- $\overline{LU}$ strict KKT -interval pseudoinvex.

Proof. Similar to the proofs of Theorems 5.2.6 and 5.2.7 (just change $F(x) \leq_{LU} F(x^*)$ by $F(x) \leq_{\overline{LU}} F(x^*)$). $\square$

Remark 5.2.1. (i) The Definitions 5.2.1, 5.2.2, 5.2.3, 5.2.4, 5.2.5 and 5.2.6 do not involve neither interval sums nor interval differences as the definitions of some other generalized convexity found in the literature.

(ii) The relationship between the definitions of KKT -interval pseudoinvexities are:

$$\boxed{\text{Definition 5.2.3} \Rightarrow \text{Definition 5.2.2} \Rightarrow \text{Definition 5.2.1.}}$$

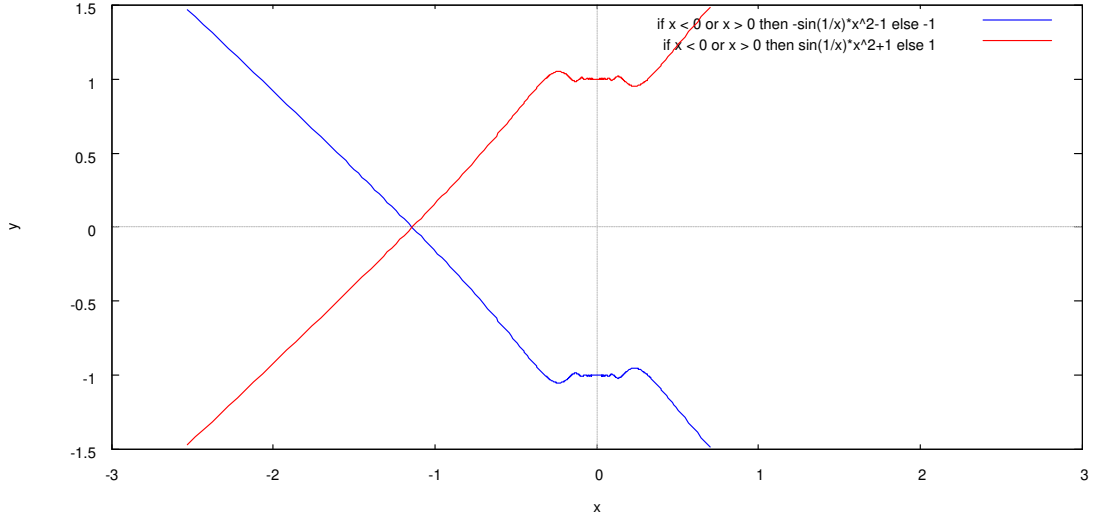
$$\boxed{\text{Definition 5.2.3} \Rightarrow \text{Definition 5.2.2} \Rightarrow \text{Definition 5.2.1.}}$$

Example 5.2.1. Given $F : \mathbb{R} \rightarrow \mathcal{K}_C$ by $F(x) = [\underline{f}(x), \overline{f}(x)]$, where

$$\underline{f}(x) = \begin{cases} -1 - x^2 \sin(\frac{1}{x}), & \text{if } x \neq 0, \\ -1, & \text{if } x = 0, \end{cases}$$

and $\overline{f}(x) = -\underline{f}(x)$ for all x , see Figure 4.

Figure 4: $\underline{f}(x)$ and $\overline{f}(x)$ of Example 5.2.1.



Source: Elaborated by the author herself.

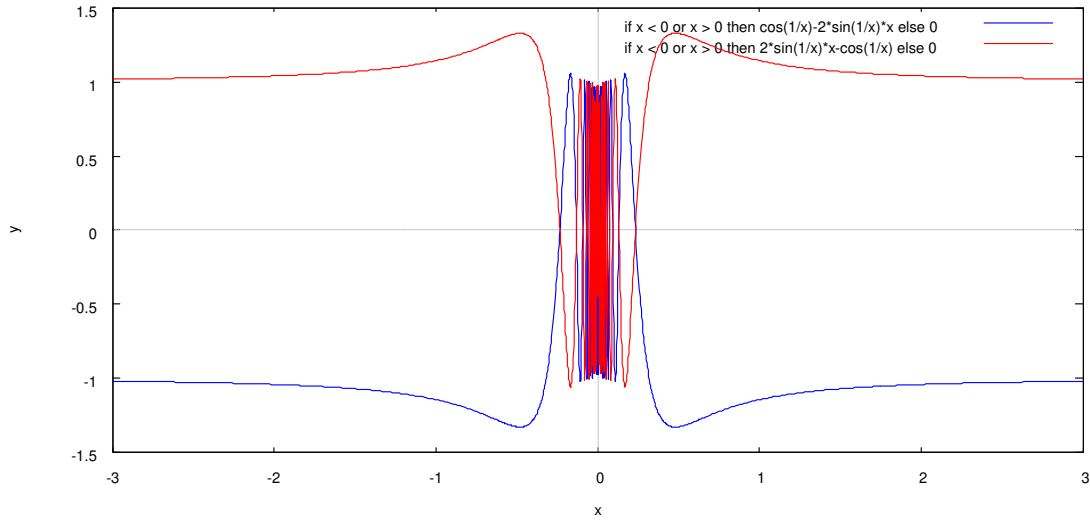
Note that $\underline{f}$ is differentiable, then $\overline{f}$ is also differentiable and, consequently, F is gH -differentiable.

We have that

$$\underline{f}'(x) = \begin{cases} -2x \sin(\frac{1}{x}) + \cos(\frac{1}{x}), & \text{if } x \neq 0, \\ 0, & \text{if } x = 0, \end{cases}$$

and $\overline{f}'(x) = -\underline{f}'(x)$, see Figure 5.

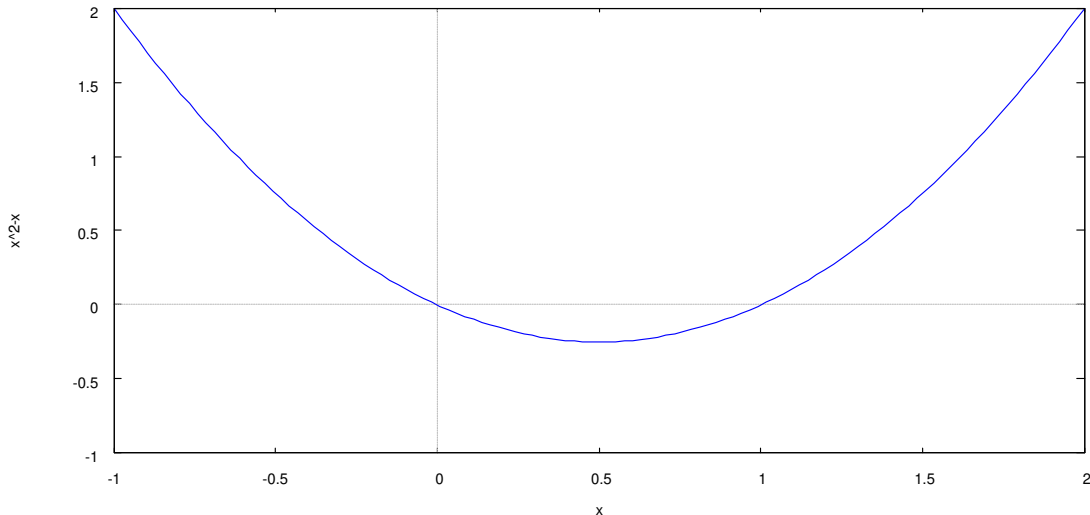
On the other hand, by Theorem 2.5.1, an interval function is continuous at $x \in \mathbb{R}$ in the d_H -metric if and only if $\underline{f}$ and $\overline{f}$ are continuous at $x \in \mathbb{R}$. Thus, F'_{gH} is not d_H -continuous at $x = 0$. Therefore, F is gH -differentiable at $x = 0$, but it is not continuously gH -differentiable at $x = 0$.

Figure 5: $\underline{f}'(x)$ and $\bar{f}'(x)$ of Example 5.2.1.

Source: Elaborated by the author herself.

Based on Example 5.2.1, the next example presents an (IOP) that cannot be tractable under the approaches given in I. Ahmad, D. Singh, B. A. Dar, [1]; Y. Chalco-Cano, W. A. Lodwick, R. Osuna-Gómez, A. Rufián-Lizana, [16]; Y. Chalco-Cano, W. A. Lodwick, A. Rufián-Lizana, [17]; and R. Osuna-Gómez, Y. Chalco-Cano, B. Hernández-Jiménez, G. Ruiz-Garzón, [64], since the objective is not a continuously gH -differentiable function.

Example 5.2.2. Consider the interval optimization problem where $S = \mathbb{R}$, F is the interval-valued function given in Example 5.2.1, and $g : \mathbb{R} \rightarrow \mathbb{R}$ is given by $g(x) = x^2 - x$, see Figure 6.

Figure 6: $g(x) = x^2 - x$ of Example 5.2.2.

Source: Elaborated by the author herself.

In this case, the feasible set is given by $\mathcal{X} = \{x \in \mathbb{R} \mid 0 \leq x \leq 1\}$. Thus, all $x \in \mathcal{X}$ is a weak LU -solution of the problem. Indeed, consider $x^* \in \mathcal{X}$ arbitrarily fixed. Suppose that there exists $y \in \mathcal{X}$ such that $F(y) <_{LU} F(x^*)$, thus

(i) if $x^* \neq 0 \neq y$, then

$$\begin{cases} -1 - y^2 \sin(\frac{1}{y}) < -1 - (x^*)^2 \sin(\frac{1}{x^*}), \\ 1 + y^2 \sin(\frac{1}{y}) < 1 + (x^*)^2 \sin(\frac{1}{x^*}), \end{cases} \quad (5.42)$$

which is not possible since the first inequality in (5.42) contradicts the second inequality in (5.42).

(ii) if $x^* = 0 \neq y$, then

$$\begin{cases} -1 - y^2 \sin(\frac{1}{y}) < -1, \\ 1 + y^2 \sin(\frac{1}{y}) < 1, \end{cases} \quad (5.43)$$

which is not possible since the first inequality in (5.43) contradicts the second inequality in (5.43).

(iii) if $x^* \neq 0 = y$, then similarly to case (i) and (ii), we obtain an impossible system of inequalities. Therefore, there does not exist $y \in \mathcal{X}$ such that $F(y) <_{LU} F(x^*)$. Consequently, every $x^* \in \mathcal{X}$ is a weak LU -solution of the interval optimization problem considered. In particular, $x^* = 0$ is a weak LU -solution of the problem.

The constraint of the problem is (PLICQ) at $x^* = 0$, since by supposing the existence of a $\beta > 0$ such that $\beta g'(x^*) = 0$, it implies that $\beta = 0$. Thus, there does not exist $\beta > 0$ such that $\beta g'(x^*) = 0$. Consequently, from Theorem 5.1.2, it follows that there exists $\mu \in \mathbb{R}$ such that

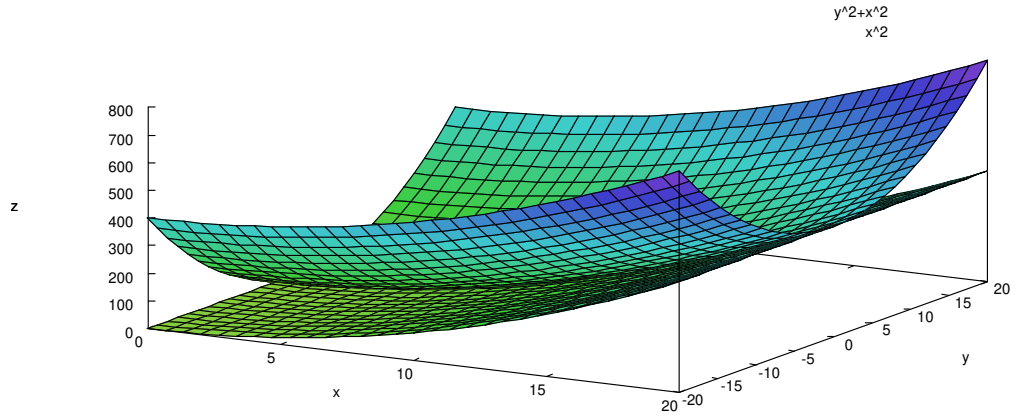
$$\begin{aligned} 0 &\in \nabla_{gH} F(x^*) + \mu g'(x^*), \\ \mu g(x^*) &= 0, \\ \mu &\geq 0, \end{aligned}$$

and from Definition 5.1.1, we conclude that $x^* = 0$ is a KKT -solution of the problem. Lastly, since all $x \in \mathcal{X}$ is a weak LU -solution of the problem, then all KKT -solution is also a weak LU -solution and, from Theorem 5.2.3, it follows that the interval optimization problem considered is a weak KKT -interval pseudoinvex problem.

Example 5.2.3. Consider the interval optimization problem (IOP):

$$(IOP) \begin{cases} \text{minimize} & F(x) = [x_1^2, x_1^2 + x_2^2] \\ \text{subject to} & g_1(x_1, x_2) = -x_1 - x_2 + 1 \leq 0, \\ & g_2(x_1, x_2) = -x_1 \leq 0, \\ & (x_1, x_2) \in S = \mathbb{R}^2. \end{cases}$$

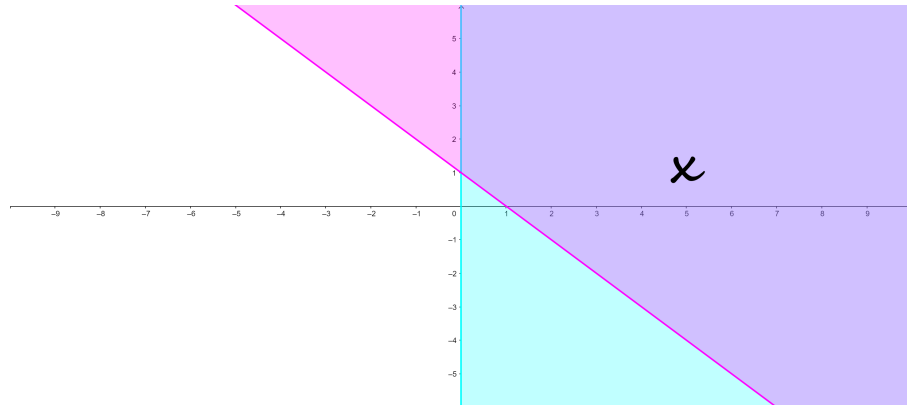
Figure 7: $\underline{f}(x_1, x_2)$ and $\bar{f}(x_1, x_2)$ of Example 5.2.3.



Source: Elaborated by the author herself.

We claim that $(x_1^*, x_2^*) = (\frac{1}{2}, \frac{1}{2})$ is a strong LU-solution, i.e., there does not exist $(x_1, x_2) \in \mathcal{X}$, see Figure 8, $(x_1, x_2) \neq (\frac{1}{2}, \frac{1}{2})$, such that $[x_1^2, x_1^2 + x_2^2] = F(x_1, x_2) \leq_{LU} F(\frac{1}{2}, \frac{1}{2}) = [\frac{1}{4}, \frac{1}{2}]$, which is equivalent to for all $(x_1, x_2) \in \mathcal{X}$, $x_1^2 \not\leq \frac{1}{4}$ or $x_1^2 + x_2^2 \not\leq \frac{1}{2}$. Indeed, if $(x_1, x_2) \in \mathcal{X}$ and $(x_1, x_2) \neq (\frac{1}{2}, \frac{1}{2})$, clearly $x_1^2 + x_2^2 > \frac{1}{2}$.

Figure 8: Admissible set $\mathcal{X} = \{(x_1, x_2) \mid g_1(x_1, x_2) \leq 0 \text{ and } g_2(x_1, x_2) \leq 0\}$ in Example 5.2.3.



Source: Elaborated by the author herself.

We have that $\nabla_{gH} F\left(\frac{1}{2}, \frac{1}{2}\right) = ([1, 1], [0, 1])$, $\nabla g_1\left(\frac{1}{2}, \frac{1}{2}\right) = (-1, -1)$, and $\nabla g_2\left(\frac{1}{2}, \frac{1}{2}\right) = (-1, 0)$.

Since $g_1\left(\frac{1}{2}, \frac{1}{2}\right) = 0$ and $g_2\left(\frac{1}{2}, \frac{1}{2}\right) \neq 0$, $I\left(\frac{1}{2}, \frac{1}{2}\right) = \{j \in I \mid g_j\left(\frac{1}{2}, \frac{1}{2}\right) = 0\} = \{1\}$, so (PLICQ) holds at $\left(\frac{1}{2}, \frac{1}{2}\right)$.

If $(x_1^*, x_2^*) = \left(\frac{1}{2}, \frac{1}{2}\right)$ is a KKT-solution, by Theorem 5.1.2, there is $(\mu_1, \mu_2) \in \mathbb{R}^2$ such that $((x_1^*, x_2^*), (\mu_1, \mu_2))$ satisfy conditions (1) – (3):

$$(1) \quad (0, 0) \in ([1, 1], [0, 1]) \oplus [\mu_1(-1, -1) + \mu_2(-1, 0)] = ([1, 1], [0, 1]) \oplus ([-\mu_1 - \mu_2, -\mu_1 - \mu_2], [-\mu_1, -\mu_1]) = ([1 - \mu_1 - \mu_2, 1 - \mu_1 - \mu_2], [-\mu_1, 1 - \mu_1]).$$

Then, $1 - \mu_1 - \mu_2 \leq 0 \leq 1 - \mu_1 - \mu_2 \Leftrightarrow 1 - \mu_1 - \mu_2 = 0 \Leftrightarrow \mu_2 = 1 - \mu_1$ and $-\mu_1 \leq 0 \leq 1 - \mu_1 \Leftrightarrow 0 \leq \mu_1 \leq 1 \Leftrightarrow 0 \geq -\mu_1 \geq -1 \Leftrightarrow 1 \geq \mu_2 \geq 0$.

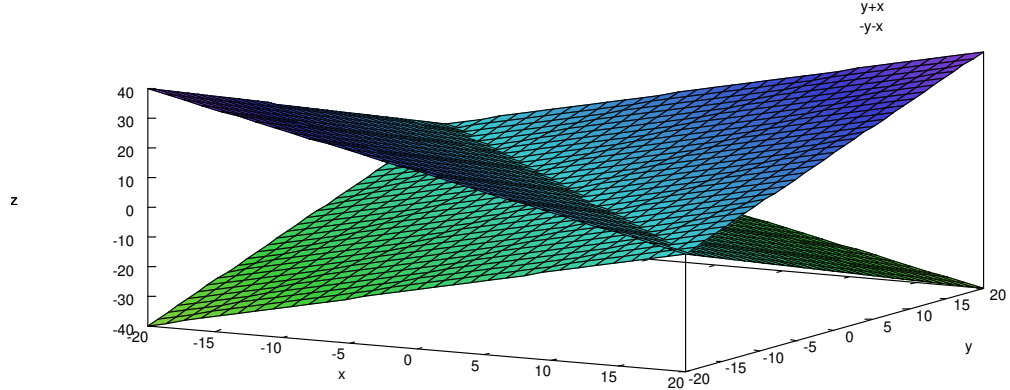
$$(2) \quad \mu_1 g_1\left(\frac{1}{2}, \frac{1}{2}\right) = 0 \Leftrightarrow \mu_1(0) = 0 \text{ and } \mu_2 g_2\left(\frac{1}{2}, \frac{1}{2}\right) = 0 \Leftrightarrow \mu_2(-1) = 0 \Leftrightarrow \mu_2 = 0.$$

$$(3) \quad \mu_1 \geq 0 \text{ and } \mu_2 \geq 0.$$

Therefore, $x^* = (x_1^*, x_2^*) = \left(\frac{1}{2}, \frac{1}{2}\right)$ is a strong LU-solution (particular is a LU-solution and then is a weak LU-solution) and is a KKT-solution with $\mu_1 = 1$ and $\mu_2 = 0$.

Example 5.2.4. Consider the interval optimization problem (IOP):

$$(IOP) \left\{ \begin{array}{l} \text{minimize } F(x_1, x_2) = [-x_1 - x_2, x_1 + x_2] \\ \text{subject to } g_1(x_1, x_2) = x_1 - 1 \leq 0, \\ g_2(x_1, x_2) = -x_1 \leq 0, \\ g_3(x_1, x_2) = x_2 - 1 \leq 0, \\ g_4(x_1, x_2) = -x_2 \leq 0, \\ (x_1, x_2) \in S = \mathbb{R}^2. \end{array} \right.$$

Figure 9: $\underline{f}(x_1, x_2)$ and $\bar{f}(x_1, x_2)$ of Example 5.2.4.

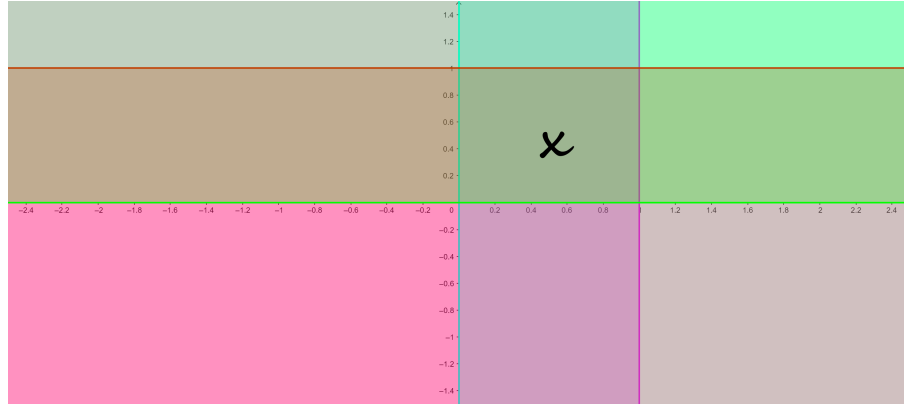
Source: Elaborated by the author herself.

We claim that $(x_1^*, x_2^*) = (1, 0)$ is a weak LU-solution. Indeed, assume that there exists $(x_1, x_2) \in \mathcal{X}$, see Figure 10, such that $F(x_1, x_2) <_{LU} F(1, 0)$. Then,

$$\left\{ \begin{array}{l} -x_1 - x_2 < -1 \\ x_1 + x_2 < 1 \end{array} \right\},$$

which is an absurd.

Figure 10: Admissible set $\mathcal{X} = \{(x_1, x_2) \mid g_1(x_1, x_2) \leq 0, g_2(x_1, x_2) \leq 0, g_3(x_1, x_2) \leq 0 \text{ and } g_4(x_1, x_2) \leq 0\}$ in Example 5.2.4 and in Example 5.2.5.



Source: Elaborated by the author herself.

We have that $\nabla_{gH} F(1, 0) = ([-1, 1], [-1, 1])$, $\nabla \underline{f}(1, 0) = (-1, -1)$, $\nabla \bar{f}(1, 0) = (1, 1)$, $\nabla g_1(1, 0) = (1, 0)$, $\nabla g_2(1, 0) = (-1, 0)$, $\nabla g_3(1, 0) = (0, 1)$, and $\nabla g_4(1, 0) = (0, -1)$.

Since $g_1(1, 0) = g_4(1, 0) = 0$ and $g_2(1, 0), g_3(1, 0) \neq 0$, $I(1, 0) = \{j \in I \mid g_j(1, 0) = 0\} = \{1, 4\}$. (PLIRC) holds at $(1, 0)$. In fact,

(i) Let $\alpha \geq 0$ and $\beta \geq 0$ be such that $\alpha(1, 0) + \beta(0, -1) = (0, 0)$. Then, $(\alpha, -\beta) = (0, 0)$

$\Leftrightarrow \alpha = 0$ and $\beta = 0$. Thus $\{\nabla g_j(1,0) \mid j \in I(1,0)\} = \{(1,0), (0,-1)\}$ is (PLICQ).

(ii) There exists $\underline{d} = (\underline{d}_1, \underline{d}_2) = (0,1) \in \mathbb{R}^2$ such that $\nabla f(1,0)(\underline{d}_1, \underline{d}_2) < 0$, $\nabla g_1(1,0)(\underline{d}_1, \underline{d}_2) \leq 0$, $\nabla g_4(1,0)(\underline{d}_1, \underline{d}_2) \leq 0$ and there exists $\bar{d} = (\bar{d}_1, \bar{d}_2) = (-1,0) \in \mathbb{R}^2$ such that $\nabla f(1,0)(\bar{d}_1, \bar{d}_2) < 0$, $\nabla g_1(1,0)(\bar{d}_1, \bar{d}_2) \leq 0$, $\nabla g_4(1,0)(\bar{d}_1, \bar{d}_2) \leq 0$.

If $(x_1^*, x_2^*) = (1,0)$ is a strict KKT-solution, by Theorem 5.1.4, then there is $(\mu_1, \mu_2, \mu_3, \mu_4) \in \mathbb{R}^4$ such that $((x_1^*, x_2^*), (\mu_1, \mu_2, \mu_3, \mu_4))$ have to satisfy conditions (1)–(3):

$$\begin{aligned} (1) \quad & (0,0) \in \text{int}([[-1,1], [-1,1]] \oplus [\mu_1(1,0) + \mu_2(-1,0) + \mu_3(0,1) + \mu_4(0,-1)]) \\ & = \text{int}([[-1,1], [-1,1]] \oplus ([\mu_1 - \mu_2, \mu_1 - \mu_2], [\mu_3 - \mu_4, \mu_3 - \mu_4])) \\ & = ((-1 + \mu_1 - \mu_2, 1 + \mu_1 - \mu_2), (-1 + \mu_3 - \mu_4, 1 + \mu_3 - \mu_4)). \end{aligned}$$

Then, $-1 + \mu_1 - \mu_2 < 0 < 1 + \mu_1 - \mu_2 \Leftrightarrow -1 < \mu_2 - \mu_1 < 1$ and $-1 + \mu_3 - \mu_4 < 0 < 1 + \mu_3 - \mu_4 \Leftrightarrow -1 < \mu_4 - \mu_3 < 1$.

$$(2) \quad \mu_1 g_1(1,0) = 0 \Leftrightarrow \mu_1(0) = 0, \quad \mu_2 g_2(1,0) = 0 \Leftrightarrow \mu_2(-1) = 0 \Leftrightarrow \mu_2 = 0, \\ \mu_3 g_3(1,0) = 0 \Leftrightarrow \mu_3(-1) = 0 \Leftrightarrow \mu_3 = 0 \text{ and } \mu_4 g_4(1,0) = 0 \Leftrightarrow \mu_4(0) = 0.$$

$$(3) \quad \mu_1 \geq 0, \mu_2 \geq 0, \mu_3 \geq 0 \text{ and } \mu_4 \geq 0.$$

Therefore, $x^* = (x_1^*, x_2^*) = (1,0)$ is a weak LU-solution and is a strict KKT-solution with $\mu_2 = \mu_3 = 0$ and $\mu_1, \mu_4 \in [0,1)$.

Example 5.2.5. Consider the interval problem (IOP):

$$(IOP) \left\{ \begin{array}{l} \text{minimize } F(x_1, x_2) = [x_1 - 2x_2, 2x_1 + x_2] \\ \text{subject to } g_1(x_1, x_2) = x_1 - 1 \leq 0, \\ \phantom{\text{subject to }} g_2(x_1, x_2) = -x_1 \leq 0, \\ \phantom{\text{subject to }} g_3(x_1, x_2) = x_2 - 1 \leq 0, \\ \phantom{\text{subject to }} g_4(x_1, x_2) = -x_2 \leq 0, \\ \phantom{\text{subject to }} (x_1, x_2) \in S = \mathbb{R}^2. \end{array} \right.$$

We claim that $(x_1^*, x_2^*) = (0, \frac{1}{2})$ is a strong LU-solution. In fact, if we assume that there exist $(x_1, x_2) \in \mathcal{X}$, see Figure 10, $(x_1, x_2) \neq (x_1^*, x_2^*) = (0, \frac{1}{2})$, such that $[x_1 - 2x_2, 2x_1 + x_2] = F(x_1, x_2) \leq_{LU} F(0, \frac{1}{2}) = [-1, \frac{1}{2}]$, then we would have $(x_1, x_2) = (0, \frac{1}{2}) = (x_1^*, x_2^*)$, which is a contradiction.

We have that $\nabla_{gH}F(0, \frac{1}{2}) = ([1, 2], [-2, 1])$, $\nabla \underline{f}(0, \frac{1}{2}) = (1, -2)$, $\nabla \bar{f}(0, \frac{1}{2}) = (2, 1)$, $\nabla g_1(0, \frac{1}{2}) = (1, 0)$, $\nabla g_2(0, \frac{1}{2}) = (-1, 0)$, $\nabla g_3(0, \frac{1}{2}) = (0, 1)$, and $\nabla g_4(0, \frac{1}{2}) = (0, -1)$.

Since $g_2(0, \frac{1}{2}) = 0$ and $g_1(0, \frac{1}{2}), g_3(0, \frac{1}{2}), g_4(0, \frac{1}{2}) \neq 0$, $I(0, \frac{1}{2}) = \{j \in I \mid g_j(0, \frac{1}{2}) = 0\} = \{2\}$. (PLIRC) holds at $(0, \frac{1}{2})$. In fact,

(i) Let $\alpha \geq 0$ be such that $\alpha(-1, 0) = (0, 0)$. Then, $(-\alpha, 0) = (0, 0) \Leftrightarrow \alpha = 0$. Thus, $\{\nabla g_j(0, \frac{1}{2}) \mid j \in I(0, \frac{1}{2})\} = \{\nabla g_2(0, \frac{1}{2})\} = \{(-1, 0)\}$ is (PLICQ).

(ii) There exists $\underline{d} = (\underline{d}_1, \underline{d}_2) = (0, 1) \in \mathbb{R}^2$ such that $\nabla \underline{f}(0, \frac{1}{2})(\underline{d}_1, \underline{d}_2) < 0$, $\nabla g_2(0, \frac{1}{2})(\underline{d}_1, \underline{d}_2) \leq 0$ and there exists $\bar{d} = (\bar{d}_1, \bar{d}_2) = (0, -1) \in \mathbb{R}^2$ such that $\nabla \bar{f}(0, \frac{1}{2})(\bar{d}_1, \bar{d}_2) < 0$, $\nabla g_2(0, \frac{1}{2})(\bar{d}_1, \bar{d}_2) \leq 0$.

If $(x_1^*, x_2^*) = (0, \frac{1}{2})$ is a strict KKT-solution, by Theorem 5.1.4, there is $(\mu_1, \mu_2, \mu_3, \mu_4) \in \mathbb{R}^4$ such that $((x_1^*, x_2^*), (\mu_1, \mu_2, \mu_3, \mu_4))$ have to satisfy conditions (1)–(3):

$$\begin{aligned} (1) \quad & (0, 0) \in \text{int}((([1, 2], [-2, 1]) \oplus [\mu_1(1, 0) + \mu_2(-1, 0) + \mu_3(0, 1) + \mu_4(0, -1)])) \\ & = \text{int}((([1, 2], [-2, 1]) \oplus ([\mu_1 - \mu_2, \mu_1 - \mu_2], [\mu_3 - \mu_4, \mu_3 - \mu_4]))) \\ & = ((1 + \mu_1 - \mu_2, 2 + \mu_1 - \mu_2), (-2 + \mu_3 - \mu_4, 1 + \mu_3 - \mu_4)). \end{aligned}$$

$$\begin{aligned} \text{Then, } & 1 + \mu_1 - \mu_2 < 0 < 2 + \mu_1 - \mu_2 \Leftrightarrow 1 < \mu_2 - \mu_1 < 2 \text{ and} \\ & -2 + \mu_3 - \mu_4 < 0 < 1 + \mu_3 - \mu_4 \Leftrightarrow -2 < \mu_4 - \mu_3 < 1. \end{aligned}$$

$$\begin{aligned} (2) \quad & \mu_1 g_1(0, \frac{1}{2}) = 0 \Leftrightarrow \mu_1(-1) = 0 \Leftrightarrow \mu_1 = 0, \mu_2 g_2(0, \frac{1}{2}) = 0 \Leftrightarrow \mu_2(0) = 0, \\ & \mu_3 g_3(0, \frac{1}{2}) = 0 \Leftrightarrow \mu_3(-\frac{1}{2}) = 0 \Leftrightarrow \mu_3 = 0 \text{ and } \mu_4 g_4(0, \frac{1}{2}) = 0 \\ & \Leftrightarrow \mu_4(-\frac{1}{2}) = 0 \Leftrightarrow \mu_4 = 0. \end{aligned}$$

$$(3) \quad \mu_1 \geq 0, \mu_2 \geq 0, \mu_3 \geq 0 \text{ and } \mu_4 \geq 0.$$

Therefore, $x^* = (x_1^*, x_2^*) = (0, \frac{1}{2})$ is a strong LU-solution (particular is a LU-solution and then is a weak LU-solution) and is a strict KKT-solution with $\mu_1 = \mu_3 = \mu_4 = 0$ and $\mu_2 \in (1, 2)$.

Remark 5.2.2. The point $(0, x_2)$, $0 < x_2 < 1$, is a strict KKT-solution and with it, it holds (PLIRC). Also, the point $(0, 1)$, is a strict KKT-solution, but with it, it does not hold (PLIRC). Indeed, since $g_2(0, 1) = g_3(0, 1) = 0$ and $g_1(0, 1) = g_4(0, 1) = -1 \neq 0$, $I(0, 1) = \{j \in I \mid g_j(0, 1) = 0\} = \{2, 3\}$, and since $\nabla g_2(0, 1) = (-1, 0)$ and $\nabla g_3(0, 1) = (0, 1)$, we have that

(i) Given $\alpha \geq 0$ and $\beta \geq 0$ be such that $\alpha(-1, 0) + \beta(0, 1) = (0, 0)$. Then, $(-\alpha, \beta) = (0, 0) \Leftrightarrow \alpha = 0$ and $\beta = 0$. Thus, $\{\nabla g_j(0, 1) \mid j \in I(0, 1)\} = \{\nabla g_2(0, 1), \nabla g_3(0, 1)\} = \{(-1, 0), (0, 1)\}$ is (PLICQ).

(ii) Since $\nabla \underline{f}(0, 1) = (1, -2)$, $\nabla \bar{f}(0, 1) = (2, 1)$, $\nabla g_2(0, 1) = (-1, 0)$, $\nabla g_3(0, 1) = (0, 1)$ there does not exist $\underline{d} = (\underline{d}_1, \underline{d}_2) \in \mathbb{R}^2$ such that $\nabla \underline{f}(0, 1)(\underline{d}_1, \underline{d}_2) < 0$, $\nabla g_2(0, 1)(\underline{d}_1, \underline{d}_2) \leq 0$, and $\nabla g_3(0, 1)(\underline{d}_1, \underline{d}_2) \leq 0$, simultaneously. Also, there does not exist $\bar{d} = (\bar{d}_1, \bar{d}_2) \in \mathbb{R}^2$ such that $\nabla \bar{f}(0, 1)(\bar{d}_1, \bar{d}_2) < 0$, $\nabla g_2(0, 1)(\bar{d}_1, \bar{d}_2) \leq 0$, and $\nabla g_3(0, 1)(\bar{d}_1, \bar{d}_2) \leq 0$, simultaneously.

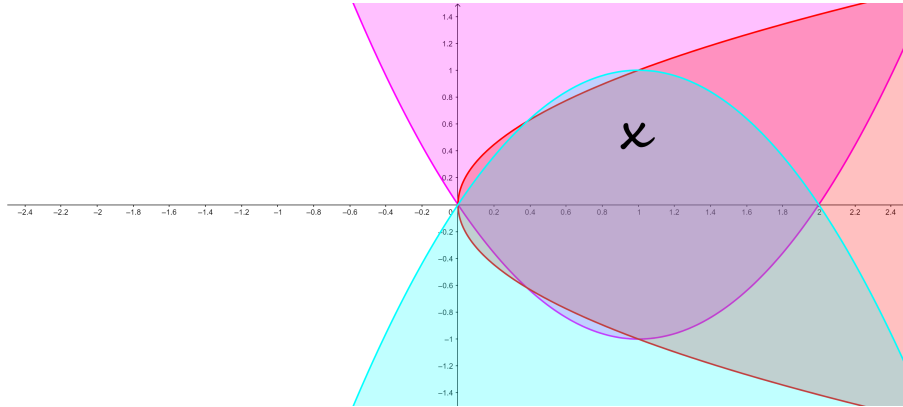
Therefore, (PLIRC) does not holds at $(0, 1)$.

Example 5.2.6. The interval optimization problem is given as

$$\begin{aligned} & \text{minimize} && F(x_1, x_2) = [x_1 \vee 2x_1] \\ & \text{subject to} && g_1(x_1, x_2) = x_1^2 - 2x_1 - x_2 \leq 0, \\ & && g_2(x_1, x_2) = x_1^2 - 2x_1 + x_2 \leq 0, \\ & && g_3(x_1, x_2) = -x_1 + x_2^2 \leq 0, \\ & && (x_1, x_2) \in S = \mathbb{R}^2. \end{aligned}$$

Note that $x_1 \geq 0$ for all feasible point (x_1, x_2) , see Figure 11.

Figure 11: Admissible set $\mathcal{X} = \{(x_1, x_2) \mid g_1(x_1, x_2) \leq 0, g_2(x_1, x_2) \leq 0, \text{ and } g_3(x_1, x_2) \leq 0\}$ in Example 5.2.6.



Source: Elaborated by the author herself.

Therefore, $x^* = (0, 0)$ is clearly a LU-solution of the problem. Note that $x^* = (0, 0)$ is a switching point of F in $S = \mathbb{R}^2$. Moreover, F is gH -differentiable at x^* while it is not H -differentiable at x^* . We have that $I(0, 0) = \{j \in I \mid g_j(0, 0) = 0\} = \{1, 2, 3\}$. Then, the gradients of the active constraints are not linearly independent at x^* . But it is easily verifiable they are positively linearly independent, that is, (PLICQ) is satisfied at x^* . It follows from Theorem 5.1.2 that x^* is a KKT-solution of the problem. In fact, the conditions (5.8)-(5.10) hold, for instance, with $\mu_1 = \mu_2 = \frac{1}{4}$ and $\mu_3 = \frac{1}{2}$.

In the example below, we have a weak KKT -interval pseudoinvex problem which is not convex (in the sense defined of L. Stefanini and M. Arana-Jiménez, [81]). Indeed, it is not even invex.

Example 5.2.7. Consider the following interval optimization problem:

$$\begin{aligned} & \text{Minimize } F(x) = [1 - \exp(-x) \vee 2 - 2\exp(-x)] \\ & \text{subject to } g(x) = -x \leq 0, \\ & x \in S = \mathbb{R}. \end{aligned}$$

In this case, $\mathcal{X} = \mathbb{R}_+$. Also, $\mathcal{L}(y) = \mathbb{R}_+$ for $y \in \mathcal{X}$ such that $I(y) \neq \emptyset$.

Define $\eta(x, y) = 1 - \exp(-x + y)$, $x, y \in \mathbb{R}$. Given $(x, y) \in \mathcal{X} \times \mathcal{X}$, if $1 \in I(y)$, we have $\eta(x, y) = 1 - \exp(-x) \geq 0$, so that $\eta(x, y) \in \mathcal{L}(y)$. Assume that $F(x) <_{LU} F(y)$. Then, $x < y$, which implies in $\eta(x, y) < 0$. Therefore, $F'_{gH}(y; \eta(x, y)) <_{LU} [0, 0]$. Thus, the problem is weak KKT -interval pseudoinvex. Let us show that it is not invex in $\mathcal{X}$.

Let $\eta \in \mathbb{R}$ and suppose that

$$F'_{gH}(y; \eta) \leq_{LU} F(x) \ominus_{gH} F(y), \quad (5.44)$$

$$\langle g'_1(y), \eta \rangle \leq g_1(x) - g_2(y) \quad (5.45)$$

for all $x, y \in \mathcal{X}$. By (5.44), either

$$F'_{gH}(y; \eta) \leq_{LU} [(1 - \exp(-x)) - (1 - \exp(-y)), (2 - 2\exp(-x)) - (2 - 2\exp(-y))] \quad (5.46)$$

or

$$F'_{gH}(y; \eta) \leq_{LU} [(2 - 2\exp(-x)) - (2 - 2\exp(-y)), (1 - \exp(-x)) - (1 - \exp(-y))]. \quad (5.47)$$

By (5.45), $x - y \leq \eta$. If $\eta \geq 0$, by (5.46), for all $x, y \in \mathcal{X}$,

$$\exp(-y)(x - y) \leq \exp(-y)\eta \leq (1 - \exp(-x)) - (1 - \exp(-y))$$

and by (5.47),

$$\exp(-y)(x - y) \leq \exp(-y)\eta \leq 2\exp(-y)\eta \leq (1 - \exp(-x)) - (1 - \exp(-y))$$

which are both a contradiction, since $1 - \exp(-x)$ is not a convex function. If $\eta < 0$, from (5.46), for all $x, y \in \mathcal{X}$,

$$2\exp(-y)(x - y) \leq \exp(-y)(x - y) \leq \exp(-y)\eta \leq (2 - 2\exp(-x)) - (2 - 2\exp(-y))$$

and by (5.47),

$$2 \exp(-y)(x - y) \leq 2 \exp(-y)\eta \leq (2 - 2 \exp(-x)) - (2 - 2 \exp(-y)),$$

which contradicts the fact that $2 - 2 \exp(-x)$ is not a convex function. Thus the problem is weak KKT-interval pseudoinvex and is not invex (neither convex).

It is easy to see that $x^* = 0$ is a KKT-solution (with $\mu = \frac{3}{2}$, for example). Then, by Theorem 5.2.1, it is a weak LU-solution.

Example 5.2.8. Consider Example 5.2.3. We claim that the interval optimization problem (IOP) is weak KKT-interval-pseudoinvex.

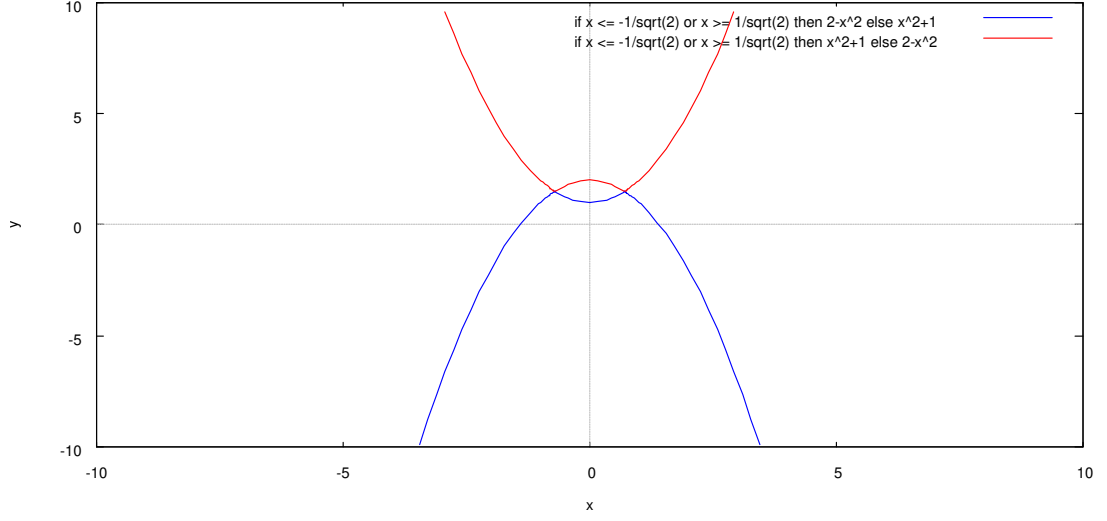
Let $(x_1, x_2), (y_1, y_2) \in \mathcal{X}$ be such that $F(x_1, x_2) <_{LU} F(y_1, y_2)$. If $y_1 = 0$, it does not hold that $F(x_1, x_2) <_{LU} F(y_1, y_2)$, since $F(x_1, x_2) <_{LU} F(0, y_2) \Leftrightarrow [x_1^2, x_1^2 + x_2^2] <_{LU} [0, y_2^2] \Leftrightarrow x_1^2 < 0$ and $x_1^2 + x_2^2 < y_2^2$, which never holds.

If $y_1 \neq 0$, with $y_1 < 0$, take $\eta(x, y) := (-y_1, 0)$, so $\eta(x, y) \in \mathcal{L}(y)$. Then $x \in \mathbb{R}^2$,
 $F'_{gH}(y; \eta(x, y)) = F'_{gH}((y_1, y_2); (-y_1, 0)) = \lim_{h \rightarrow 0^+} \frac{1}{h} \cdot (F((y_1, y_2) + h(-y_1, 0)) \ominus_{gH} F(y_1, y_2))$
 $= \lim_{h \rightarrow 0^+} \frac{1}{h} \cdot ([(y_1 - hy_1)^2, (y_1 - hy_1)^2 + y_2^2] \ominus_{gH} [y_1^2, y_1^2 + y_2^2])$
 $= \lim_{h \rightarrow 0^+} \frac{1}{h} \cdot \left([-2hy_1^2 + h^2y_1^2 \vee -2hy_1^2 + h^2y_1^2] \right) = [-2y_1^2, -2y_1^2] <_{LU} [0, 0].$

Example 5.2.9. Consider the interval problem (IOP):

$$(IOP) \begin{cases} \text{minimize} & F(x) = [\underline{f}(x) \vee \bar{f}(x)] \\ \text{subject to} & g(x) = 3 - x \leq 0, \end{cases}$$

where $\underline{f}(x) = \begin{cases} -x^2 + 2, & x \leq \bar{x} \text{ or } x \geq \tilde{x} \\ x^2 + 1, & \bar{x} \leq x \leq \tilde{x} \end{cases}$, $\bar{f}(x) = \begin{cases} x^2 + 1, & x \leq \bar{x} \text{ or } x \geq \tilde{x} \\ -x^2 + 2, & \bar{x} \leq x \leq \tilde{x} \end{cases}$,
 $\bar{x} = -\frac{\sqrt{2}}{2}$, and $\tilde{x} = \frac{\sqrt{2}}{2}$, see Figure 12.

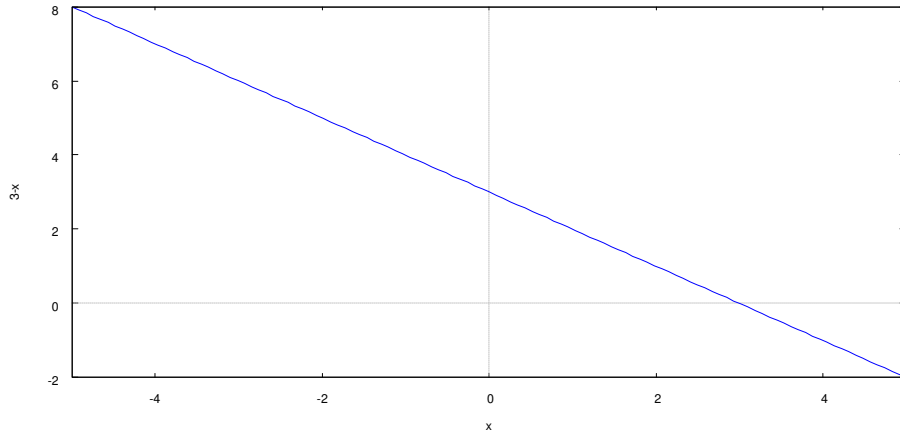
Figure 12: $\underline{f}(x)$ and $\bar{f}(x)$ of Example 5.2.9.

Source: Elaborated by the author herself.

Let $x, x^* \in \mathcal{X}$, then $\underline{f}(x) = -x^2 + 2$ and $\bar{f}(x) = x^2 + 1$. We claim that (IOP) is KKT–interval–pseudoconvex problem. We have that $F(x) \leq_{LU} F(x^*) \Leftrightarrow [-x^2 + 2, x^2 + 1] \leq_{LU} [-(x^*)^2 + 2, (x^*)^2 + 1] \Leftrightarrow (-x^2 < -(x^*)^2 \text{ and } x^2 \leq (x^*)^2) \text{ or } (-x^2 \leq -(x^*)^2 \text{ and } x^2 < (x^*)^2)$, which, clearly, never holds.

Therefore, by vacuity, (IOP) is KKT–interval pseudoconvex problem.

Consider $x \geq 3$, see Figure 13,

Figure 13: $g(x) = 3 - x$ of Example 5.2.9.

Source: Elaborated by the author herself.

then we have that $F'_{gH}(x) = [\underline{f}'(x) \vee \bar{f}'(x)] = [-2x, 2x]$ and $g'(x) = -1$.

If x is a KKT–solution, then there is $\mu \in \mathbb{R}$ such that (x, μ) has to satisfy the conditions (1) – (3):

- (1) $0 \in [-2x, 2x] + \mu(-1) = [-2x - \mu, 2x - \mu]$, then $-2x - \mu \leq 0 \leq 2x - \mu$, i.e., $-2x \leq \mu \leq 2x$, that is $\mu \in [-2x, 2x]$.
- (2) $\mu g(x) = 0 \Leftrightarrow \mu(3 - x) = 0 \Leftrightarrow \mu = 0$ or $3 - x = 0 \Leftrightarrow \mu = 0$ or $x = 3$.
- (3) $\mu \geq 0$.

If $\mu = 0$, then $0 \in [-2x, 2x]$. Therefore, given x and $\mu = 0$, we obtain (x, μ) which verifies Equations (1) – (3). So, all admissible points x are *KKT*–solutions.

Given x, x^* admissible points, it is not verified that $F(x) \leq_{LU} F(x^*)$, i.e., there does not exist $x \in \mathcal{X}$ such that $F(x) \leq_{LU} F(x^*)$, i.e., x^* is a *LU*–solution of (IOP).

Therefore, (IOP) is a *KKT*–interval pseudoinvex problem, where all *KKT*–solutions are *LU*–solutions.

5.3 Conclusion

With the *gH*–differentiability of an interval–valued function of several variables, we obtained a local optimality characterization for the interval problem, then we got the Fritz John–type and the Karush–Kuhn–Tucker conditions. Moreover, we proposed six new pseudoinvexities that helped us to obtain *LU*–solutions, under two new concepts: *KKT*–solutions and strict *KKT*–solutions. Finally, we show that the concepts of generalized invexity obtained are the most general ones and we gave some examples.

CHAPTER 6

ABSTRACT INTERVAL OPTIMIZATION USING *GH*–DIFFERENTIABILITY

This chapter addresses interval optimization problems in which the objective function is interval-valued while the constraints are abstract, we do not know how are this constraints. Necessary optimality conditions are derived and they are obtained through the *gH*–gradient of the interval objective function. Firstly, it will be given the *LU*–solutions for the problem. Then, it will be considered some basic assumptions, it will be presented two new definitions for sets of directions, and some properties of them. Next, it will be provided two cones and it will be shown some features of them. Moreover, it will be given two classical cones from one of the cones defined before and it will be presented some properties of them. Finally, it will be provided an important result: the necessary optimality conditions for the problem, in which is analyzed two functionals and the abstract constraint of the problem, just to use the Dubovitskii–Milyutin formalism and obtain the Euler equation.

In the following, it will be established the abstract interval optimization problem.

6.1 The abstract interval optimization problem

Consider the next *abstract interval optimization problem*:

$$(AIOP) \begin{cases} \text{minimize} & F(x) = [\underline{f}(x), \bar{f}(x)] \\ \text{subject to} & x \in S \subseteq \mathbb{R}^n, \end{cases}$$

where $F : \mathbb{R}^n \rightarrow \mathcal{K}_C$ is an interval-valued function given by $F(x) = [\underline{f}(x), \overline{f}(x)]$ with $\underline{f}, \overline{f} : \mathbb{R}^n \rightarrow \mathbb{R}$, $\underline{f}(x) \leq \overline{f}(x)$ for all $x \in \mathbb{R}^n$, and S is a nonempty set in $\mathbb{R}^n$.

Next, it will be given some solutions for the problem $(AIOP)$.

6.2 Optimal LU -solutions for $(AIOP)$

Given $\epsilon > 0$, $N_\epsilon(x^*)$ denotes the ϵ -neighborhood of $x^* \in \mathbb{R}^n$, that is,

$$N_\epsilon(x^*) = \{x \in \mathbb{R}^n \mid |x - x^*| < \epsilon\}.$$

Aiming to obtain LU -solutions, it will be used the order relation LU .

Definition 6.2.1 (See H. C. Wu, [85]). *Let x^* be an admissible solution of $(AIOP)$, i.e., $x^* \in S$. Then,*

- (i) x^* is said to be a **local strong LU -solution** of $(AIOP)$ if there exists $\epsilon > 0$ such that there does not exist $x \in N_\epsilon(x^*) \cap S$, $x \neq x^*$ with $F(x) \leq_{LU} F(x^*)$.
- (ii) x^* is said to be a **local LU -solution** of $(AIOP)$ if there exists $\epsilon > 0$ such that there does not exist $x \in N_\epsilon(x^*) \cap S$ with $F(x) \leq_{LU} F(x^*)$.
- (iii) x^* is said to be a **local $\underline{LU}$ -solution** of $(AIOP)$ if there exists $\epsilon > 0$ such that there does not exist $x \in N_\epsilon(x^*) \cap S$ with $F(x) \leq_{\underline{LU}} F(x^*)$.
- (iv) x^* is said to be a **local $\overline{LU}$ -solution** of $(AIOP)$ if there exists $\epsilon > 0$ such that there does not exist $x \in N_\epsilon(x^*) \cap S$ with $F(x) \leq_{\overline{LU}} F(x^*)$.
- (v) x^* is said to be a **local weak LU -solution** of $(AIOP)$ if there exists $\epsilon > 0$ such that there does not exist $x \in N_\epsilon(x^*) \cap S$ with $F(x) <_{LU} F(x^*)$.

If $N_\epsilon(x^*)$ above is replaced by the whole space $\mathbb{R}^n$, we have the corresponding definitions of global solutions for $(AIOP)$.

In the following, to reach relevant results, it will be required the next assumptions and definitions:

6.3 Some assumptions and definitions

Let $x^* \in \mathbb{R}^n$ be a reference optimal LU -solution of $(AIOP)$, then we have the following assumptions:

- (A1) The interval-valued function $F(x)$ satisfy the Lipschitz continuous condition in a neighborhood of x^* .
- (A2) The interval-valued function F is continuous and continuously gH -differentiable.
- (A3) For all bounded x , the partial derivatives $\nabla_{\#} \underline{f}(x)$ and $\nabla_{\#} \bar{f}(x)$ exist, and are bounded, where

$$\nabla_{\#} \underline{f}(x) = \begin{bmatrix} \frac{\partial \underline{f}}{\partial x_1}(x) \\ \vdots \\ \frac{\partial \underline{f}}{\partial x_n}(x) \end{bmatrix}_{n \times 1} \quad \text{and} \quad \nabla_{\#} \bar{f}(x) = \begin{bmatrix} \frac{\partial \bar{f}}{\partial x_1}(x) \\ \vdots \\ \frac{\partial \bar{f}}{\partial x_n}(x) \end{bmatrix}_{n \times 1}.$$

- (A4) S is a closed convex set in $\mathbb{R}^n$ such that $\text{int } S \neq \emptyset$.

Now, two directions will be defined and some properties of them will be considered.

Definition 6.3.1. We give the definitions of directions of decrease and feasible directions

- (i) We say that a vector $\bar{x}$ is a direction of decrease of $F(x)$ at x^* if there exist a neighborhood $N \subset \mathbb{R}^n$ of the vector $\bar{x}$, a positive scalar ϵ^* , and an interval $[\underline{\alpha}, \bar{\alpha}] <_{LU} [0, 0]$ such that for all ϵ with $0 < \epsilon < \epsilon^*$ and all $\tilde{x} \in N$, we have that

$$F(x^* + \epsilon \tilde{x}) <_{LU} F(x^*) + \epsilon [\underline{\alpha}, \bar{\alpha}].$$

We denote the set of all directions of decrease at x^* for $F(x)$ by $\mathcal{K}_y$ ¹.

- (ii) Given abstract constraints, we say that a vector $\bar{x}$ is a feasible direction to S at x^* if there exists a neighborhood $N \subset \mathbb{R}^n$ of the vector $\bar{x}$ and a scalar $\epsilon_1 > 0$ such that for all ϵ with $0 < \epsilon < \epsilon_1$ and all $\tilde{x} \in N$, we have that the vectors

$$x^* + \epsilon \tilde{x} \in S.$$

We denote the set of all feasible directions at x^* for S by $\mathcal{K}_b$ ².

Remark 6.3.1. In Definition 6.3.1-(i), both $\underline{\alpha}$ and $\bar{\alpha}$ depends on F , x^* , and $\bar{x}$.

Proposition 6.3.1. Let $\mathcal{K}_y$ be the set of all directions of decrease. Then,

¹In Russian the initials of “decrease cone” are $\mathcal{K}_y$.

²In Russian the initials of “feasible cone” are $\mathcal{K}_b$.

- (i) $\mathcal{K}_y$ is open. (iii) $\mathcal{K}_y$ is a cone.
(ii) $\mathcal{K}_y$ is a convex.

Remark 6.3.2. The proof of Proposition 6.3.1, it will be obviously seen later, since later it will be proved that $\mathcal{K}_y = \mathcal{K}_0 = \underline{\mathcal{K}}_0 \cap \overline{\mathcal{K}}_0$ and using the fact that the intersection of open convex cones is also an open convex cone.

Next, it will be included the following proposition, that is in I. V. Girsanov, [36], for the completeness of the chapter.

Proposition 6.3.2. Let $\mathcal{K}_b$ be the set of all feasible directions. Then,

- (i) $\mathcal{K}_b$ is open. (ii) $\mathcal{K}_b$ is a cone.

Proof. (i) $\mathcal{K}_b \subset \text{int } \mathcal{K}_b$. Indeed, let $\bar{x} \in \mathcal{K}_b$, then there exists a neighborhood $N \subset \mathbb{R}^n$ of $\bar{x}$ and a scalar $\epsilon_1 > 0$ such that for all ϵ with $0 < \epsilon < \epsilon_1$ and all $\tilde{x} \in N$, we have that $x^* + \epsilon\tilde{x} \in S$. Then, $N \subset \mathcal{K}_b$. Indeed, let $\tilde{x} \in N$. It is clear that N is a neighborhood of $\tilde{x}$. Let $\hat{x} \in N$ and $0 < \epsilon < \epsilon_1$. By the fact that $\bar{x} \in \mathcal{K}_b$, we have that

$$x^* + \epsilon\hat{x} \in S.$$

Therefore $\tilde{x} \in \mathcal{K}_b$. It follows that $\mathcal{K}_b$ is open.

- (ii) $\mathcal{K}_b$ is a cone, that is, if $\bar{x} \in \mathcal{K}_b$, then $\alpha\bar{x} \in \mathcal{K}_b$, for $\alpha > 0$. Indeed, if $\bar{x} \in \mathcal{K}_b$, there exists a neighborhood $N \subset \mathbb{R}^n$ of $\bar{x}$ and a scalar $\epsilon_1 > 0$ such that for all ϵ with $0 < \epsilon < \epsilon_1$ and all $\tilde{x} \in N$, we have that

$$x^* + \epsilon\tilde{x} \in S. \tag{6.1}$$

Let $V := \alpha N \subset \mathbb{R}^n$ be a neighborhood of $\alpha\bar{x}$ and $\epsilon'_1 := \frac{\epsilon_1}{\alpha}$, $\alpha > 0$. Given $\hat{x} \in V$, we have $\hat{x} = \alpha\tilde{x}$ for some $\tilde{x} \in N$. Then, for all $0 < \epsilon < \epsilon'_1$, we have $\alpha\epsilon < \epsilon_1$, and

$$x^* + \epsilon\hat{x} = x^* + \epsilon\alpha\tilde{x} \in S, \quad \text{by (6.1),}$$

i.e., $\alpha\bar{x} \in \mathcal{K}_b$, for $\alpha > 0$.

□

Next, it will be characterized some cones and it will be presented some properties of them.

6.4 Characterizations of the cones $\mathcal{K}_0$ and $\mathcal{K}_1$

In this section, it will be defined two cones, it will be described them, it will be shown that they are exactly the cone of directions of decrease and the cone of feasible directions, see Theorem 6.4.3 and Lemma 6.4.1. To the end, it will be proved that the intersection of the cones initially considered, is empty.

Let $x^* \in S$. Set

$$\mathcal{K}_0 := \{\bar{x} \in \mathbb{R}^n \mid F'_{gH}(x^*; \bar{x}) <_{LU} [0, 0]\}$$

and

$$\begin{aligned} \mathcal{K}_1 &:= \{\bar{x} \in \mathbb{R}^n \mid \bar{x} = \lambda[x - x^*], \quad x \in \text{int } S, \quad \text{and } \lambda > 0\} \\ &= \{\lambda[\text{int}(S) - x^*] \mid \lambda > 0\}, \quad \text{see Theorem 2.11.1.} \end{aligned}$$

Theorem 6.4.1. *If $\bar{x} \in \mathcal{K}_y$ and $F'_{gH}(x^*; \bar{x})$ exists, then $\bar{x} \in \mathcal{K}_0$.*

Proof. Let $\bar{x} \in \mathcal{K}_y$. Then, there exists a neighborhood $N \subset \mathbb{R}^n$ of $\bar{x}$, a positive scalar ϵ^* , and an interval

$$[\underline{\alpha}, \bar{\alpha}] <_{LU} [0, 0] \tag{6.2}$$

such that for all ϵ with $0 < \epsilon < \epsilon^*$ and all $\tilde{x} := \bar{x} \in N$, we have that

$$F(x^* + \epsilon\bar{x}) <_{LU} F(x^*) + \epsilon[\underline{\alpha}, \bar{\alpha}] \tag{6.3}$$

Affirmation. $\lim_{\epsilon \rightarrow 0^+} \frac{1}{\epsilon} \cdot (F(x^* + \epsilon\bar{x}) \ominus_{gH} F(x^*)) <_{LU} [0, 0]$.

Indeed, let us set, for $\epsilon \in (0, \epsilon^*)$,

$$A_\epsilon := [\underline{a}_\epsilon, \bar{a}_\epsilon] = F(x^* + \epsilon\bar{x}),$$

$$B_\epsilon := [\underline{b}_\epsilon, \bar{b}_\epsilon] = F(x^*),$$

and

$$C_\epsilon := [\underline{c}_\epsilon, \bar{c}_\epsilon] = A_\epsilon \ominus_{gH} B_\epsilon = F(x^* + \epsilon\bar{x}) \ominus_{gH} F(x^*).$$

Since $A_\epsilon \ominus_{gH} B_\epsilon = C_\epsilon$, we have that

$$\begin{cases} A_\epsilon = B_\epsilon + C_\epsilon \\ \text{or} \\ B_\epsilon = A_\epsilon - C_\epsilon \end{cases} \Leftrightarrow \begin{cases} \underline{c}_\epsilon = \underline{a}_\epsilon - \underline{b}_\epsilon \quad \text{and} \quad \bar{c}_\epsilon = \bar{a}_\epsilon - \bar{b}_\epsilon \\ \text{or} \\ \bar{c}_\epsilon = \underline{a}_\epsilon - \underline{b}_\epsilon \quad \text{and} \quad \underline{c}_\epsilon = \bar{a}_\epsilon - \bar{b}_\epsilon. \end{cases} \tag{6.4}$$

By (6.3), we have that

$$A_\epsilon <_{LU} B_\epsilon + \epsilon[\underline{\alpha}, \bar{\alpha}] \Leftrightarrow [\underline{a}_\epsilon, \bar{a}_\epsilon] <_{LU} [\underline{b}_\epsilon + \epsilon\underline{\alpha}, \bar{b}_\epsilon + \epsilon\bar{\alpha}],$$

that is,

$$\begin{aligned} \underline{a}_\epsilon - \underline{b}_\epsilon &< \epsilon\underline{\alpha} \quad \text{and} \quad \bar{a}_\epsilon - \bar{b}_\epsilon < \epsilon\bar{\alpha} \\ \Leftrightarrow \frac{1}{\epsilon}(\underline{a}_\epsilon - \underline{b}_\epsilon) &< \underline{\alpha} \quad \text{and} \quad \frac{1}{\epsilon}(\bar{a}_\epsilon - \bar{b}_\epsilon) < \bar{\alpha}. \end{aligned}$$

Let us assume that the first case in (6.4) holds, if the second case occurs, we proceed similarly. Then,

$$\begin{aligned} &\frac{1}{\epsilon}\underline{c}_\epsilon < \underline{\alpha} \quad \text{and} \quad \frac{1}{\epsilon}\bar{c}_\epsilon < \bar{\alpha} \\ \Rightarrow \lim_{\epsilon \rightarrow 0^+} \frac{1}{\epsilon}\underline{c}_\epsilon &< \underline{\alpha} \quad \text{and} \quad \lim_{\epsilon \rightarrow 0^+} \frac{1}{\epsilon}\bar{c}_\epsilon < \bar{\alpha} \\ \Leftrightarrow \left[\lim_{\epsilon \rightarrow 0^+} \frac{1}{\epsilon}\underline{c}_\epsilon, \lim_{\epsilon \rightarrow 0^+} \frac{1}{\epsilon}\bar{c}_\epsilon \right] &<_{LU} [\underline{\alpha}, \bar{\alpha}] <_{LU} [0, 0], & \text{by (6.2)} \\ \Rightarrow \left[\lim_{\epsilon \rightarrow 0^+} \frac{1}{\epsilon}\underline{c}_\epsilon, \lim_{\epsilon \rightarrow 0^+} \frac{1}{\epsilon}\bar{c}_\epsilon \right] &<_{LU} [0, 0] \\ \Leftrightarrow \lim_{\epsilon \rightarrow 0^+} \frac{1}{\epsilon} \cdot C_\epsilon &<_{LU} [0, 0], & \text{by Theorem 2.4.1} \\ \Leftrightarrow \lim_{\epsilon \rightarrow 0^+} \frac{1}{\epsilon} \cdot (F(x^* + \epsilon\bar{x}) \ominus_{gH} F(x^*)) &<_{LU} [0, 0] \\ \Leftrightarrow F'_{gH}(x^*; \bar{x}) &<_{LU} [0, 0], & \text{by Definition 2.7.5,} \end{aligned}$$

that is, $\bar{x} \in \mathcal{K}_0$. □

Now, we establish the converse of Theorem 6.4.1:

Theorem 6.4.2. *Suppose that (A1) holds and that $\bar{x} \in \mathcal{K}_0$. Then, $\bar{x} \in \mathcal{K}_y$.*

Proof. By Definition 2.7.5:

$$\begin{aligned} F'_{gH}(x^*; \bar{x}) &= \lim_{\epsilon \rightarrow 0^+} \frac{1}{\epsilon} \cdot (F(x^* + \epsilon\bar{x}) \ominus_{gH} F(x^*)) =: [-\underline{\delta}, -\bar{\delta}] \in \mathcal{K}_C, \quad \bar{\delta} < \underline{\delta} \\ \Rightarrow [0, 0] &>_{LU} F'_{gH}(x^*; \bar{x}), \quad \text{since } \bar{x} \in \mathcal{K}_0 \\ &= [-\underline{\delta}, -\bar{\delta}], \end{aligned}$$

which implies that $[-\underline{\delta}, -\bar{\delta}] <_{LU} [0, 0]$, that is, $\underline{\delta} > 0$ and $\bar{\delta} > 0$.

Since $F(x)$ satisfy the Lipschitz continuous condition in a neighborhood of x^* , there

exists $L > 0$ and there exists $\epsilon^* > 0$ such that

$$d_H(F(x_1), F(x_2)) \leq L \|x_1 - x_2\|_{\mathbb{R}^n} \quad (6.5)$$

for all x_1 and x_2 such that $\|x_1 - x^*\|_{\mathbb{R}^n} < \epsilon^*$ and $\|x_2 - x^*\|_{\mathbb{R}^n} < \epsilon^*$.

Let $\tilde{x} \in \mathbb{R}^n$ be arbitrary such that

$$\|\tilde{x} - \bar{x}\|_{\mathbb{R}^n} < \frac{1}{4L} \bar{\delta}, \quad (6.6)$$

that is, $\tilde{x} \in \mathbb{B}_{\mathbb{R}^n}(\bar{x}, \frac{1}{4L} \bar{\delta})$.

By definition of limit, given $\epsilon' \in \mathbb{R}$ with $0 < \epsilon' < \frac{1}{2} \bar{\delta}$, there exists $\delta' > 0$ such that

$$d_H\left(\frac{1}{\epsilon} \cdot (F(x^* + \epsilon \bar{x}) \ominus_{gH} F(x^*)), [-\underline{\delta}, -\bar{\delta}]\right) < \epsilon'$$

for all $\epsilon \in (0, \delta')$. By setting

$$\frac{1}{\epsilon} \cdot (F(x^* + \epsilon \bar{x}) \ominus_{gH} F(x^*)) =: K_\epsilon = [\underline{k}_\epsilon, \overline{k}_\epsilon],$$

where $\epsilon \in (0, \delta')$, we have that

$$\begin{aligned} \max\{|\underline{k}_\epsilon + \underline{\delta}|, |\overline{k}_\epsilon + \bar{\delta}|\} &= d_H(K_\epsilon, [-\underline{\delta}, -\bar{\delta}]) < \epsilon' \\ \Rightarrow \underline{k}_\epsilon + \underline{\delta} < |\underline{k}_\epsilon + \underline{\delta}| < \epsilon' &\quad \text{and} \quad \overline{k}_\epsilon + \bar{\delta} < |\overline{k}_\epsilon + \bar{\delta}| < \epsilon' \\ \Rightarrow \underline{k}_\epsilon < \epsilon' - \underline{\delta} < \frac{1}{2} \bar{\delta} - \underline{\delta} < \frac{1}{2} \underline{\delta} - \underline{\delta} = -\frac{1}{2} \underline{\delta} &\quad \text{and} \quad \overline{k}_\epsilon < \epsilon' - \bar{\delta} < \frac{1}{2} \bar{\delta} - \bar{\delta} = -\frac{1}{2} \bar{\delta} \\ \Rightarrow K_\epsilon = [\underline{k}_\epsilon, \overline{k}_\epsilon] <_{LU} \left[-\frac{1}{2} \underline{\delta}, -\frac{1}{2} \bar{\delta}\right] \\ \Leftrightarrow \frac{1}{\epsilon} \cdot (F(x^* + \epsilon \bar{x}) \ominus_{gH} F(x^*)) <_{LU} \left[-\frac{1}{2} \underline{\delta}, -\frac{1}{2} \bar{\delta}\right] \\ \Leftrightarrow F(x^* + \epsilon \bar{x}) \ominus_{gH} F(x^*) <_{LU} \epsilon \left[-\frac{1}{2} \underline{\delta}, -\frac{1}{2} \bar{\delta}\right], &\quad \epsilon \in (0, \delta'). \end{aligned} \quad (6.7)$$

Whenever $0 < \epsilon < \hat{\epsilon}$, where $\hat{\epsilon} < \min\left\{\frac{4\epsilon^* L}{\bar{\delta} + 4\|\bar{x}\|_{\mathbb{R}^n} L}, \frac{\epsilon^*}{\|\bar{x}\|_{\mathbb{R}^n}}\right\}$, we obtain that

$$\begin{aligned} \|(x^* + \epsilon \tilde{x}) - x^*\|_{\mathbb{R}^n} &= \epsilon \|(\tilde{x} - \bar{x}) + \bar{x}\|_{\mathbb{R}^n} \\ &\leq \epsilon (\|\tilde{x} - \bar{x}\|_{\mathbb{R}^n} + \|\bar{x}\|_{\mathbb{R}^n}) \\ &< \frac{4\epsilon^* L}{\bar{\delta} + 4\|\bar{x}\|_{\mathbb{R}^n} L} \left(\frac{1}{4L} \bar{\delta} + \|\bar{x}\|_{\mathbb{R}^n}\right), &\quad \text{by (6.6)} \\ &= \epsilon^* &\quad (6.8) \end{aligned}$$

and

$$\|(x^* + \epsilon \bar{x}) - x^*\|_{\mathbb{R}^n} = \epsilon \|\bar{x}\|_{\mathbb{R}^n} < \frac{\epsilon^*}{\|\bar{x}\|_{\mathbb{R}^n}} \|\bar{x}\|_{\mathbb{R}^n} = \epsilon^*. \quad (6.9)$$

Now, since it holds (6.8) and (6.9), by (6.5)

$$d_H(F(x^* + \epsilon \tilde{x}), F(x^* + \epsilon \bar{x})) \leq L \|(x^* + \epsilon \tilde{x}) - (x^* + \epsilon \bar{x})\|_{\mathbb{R}^n} \quad (6.10)$$

for all $0 < \epsilon < \hat{\epsilon}$. Then, we have, by Proposition 2.3.2, for all ϵ that

$$\begin{aligned} & F(x^* + \epsilon \tilde{x}) \ominus_{gH} F(x^* + \epsilon \bar{x}) \\ & \leq_{LU} [d_H(F(x^* + \epsilon \tilde{x}), F(x^* + \epsilon \bar{x})), d_H(F(x^* + \epsilon \tilde{x}), F(x^* + \epsilon \bar{x}))] \\ & \leq_{LU} [L \|(x^* + \epsilon \tilde{x}) - (x^* + \epsilon \bar{x})\|_{\mathbb{R}^n}, L \|(x^* + \epsilon \tilde{x}) - (x^* + \epsilon \bar{x})\|_{\mathbb{R}^n}], \quad \text{by (6.10)} \\ & = [L\epsilon \|\tilde{x} - \bar{x}\|_{\mathbb{R}^n}, L\epsilon \|\tilde{x} - \bar{x}\|_{\mathbb{R}^n}] \\ & <_{LU} \left[L\epsilon \frac{1}{4L} \bar{\delta}, L\epsilon \frac{1}{4L} \bar{\delta} \right], \quad \text{by (6.6)} \\ & = \epsilon \left[\frac{1}{4} \bar{\delta}, \frac{1}{4} \bar{\delta} \right], \end{aligned}$$

i.e.,

$$F(x^* + \epsilon \tilde{x}) \ominus_{gH} F(x^* + \epsilon \bar{x}) <_{LU} \epsilon \left[\frac{1}{4} \bar{\delta}, \frac{1}{4} \bar{\delta} \right]. \quad (6.11)$$

for all $0 < \epsilon < \hat{\epsilon}$.

Affirmation. Given $0 < \epsilon < (\epsilon^*)'$, where $(\epsilon^*)' := \min \{\delta', \hat{\epsilon}\}$,

$$F(x^* + \epsilon \tilde{x}) <_{LU} F(x^*) + \epsilon [\underline{\alpha}', \bar{\alpha}'],$$

with $\underline{\alpha}' = \bar{\alpha}' = -\frac{1}{4} \bar{\delta}$.

Indeed, let us set, for $\epsilon \in (0, \epsilon^*)$,

$$\begin{aligned} A_\epsilon &:= [\underline{a}_\epsilon, \bar{a}_\epsilon] = F(x^* + \epsilon \bar{x}), \\ B_\epsilon &:= [\underline{b}_\epsilon, \bar{b}_\epsilon] = F(x^*), \\ C_\epsilon &:= [\underline{c}_\epsilon, \bar{c}_\epsilon] = A_\epsilon \ominus_{gH} B_\epsilon = F(x^* + \epsilon \bar{x}) \ominus_{gH} F(x^*), \\ D_\epsilon &:= [\underline{d}_\epsilon, \bar{d}_\epsilon] = F(x^* + \epsilon \tilde{x}) \end{aligned}$$

and

$$E_\epsilon := [\underline{e}_\epsilon, \bar{e}_\epsilon] = D_\epsilon \ominus_{gH} A_\epsilon = F(x^* + \epsilon \tilde{x}) \ominus_{gH} F(x^* + \epsilon \bar{x}).$$

Since $A_\epsilon \ominus_{gH} B_\epsilon = C_\epsilon$, by (6.7), we have that

$$\left\{ \begin{array}{l} A_\epsilon = B_\epsilon + C_\epsilon \\ \text{or} \\ B_\epsilon = A_\epsilon - C_\epsilon, \\ \text{with} \\ C_\epsilon <_{LU} \left[-\frac{1}{2}\epsilon\bar{\delta}, -\frac{1}{2}\epsilon\bar{\delta} \right] \end{array} \right. \Leftrightarrow \left\{ \begin{array}{l} [\underline{a}_\epsilon, \bar{a}_\epsilon] = [\underline{b}_\epsilon, \bar{b}_\epsilon] + [\underline{c}_\epsilon, \bar{c}_\epsilon] \\ \text{or} \\ [\underline{b}_\epsilon, \bar{b}_\epsilon] = [\underline{a}_\epsilon, \bar{a}_\epsilon] + [-\bar{c}_\epsilon, -\underline{c}_\epsilon], \\ \text{with} \\ [\underline{c}_\epsilon, \bar{c}_\epsilon] <_{LU} \left[-\frac{1}{2}\epsilon\bar{\delta}, -\frac{1}{2}\epsilon\bar{\delta} \right] \end{array} \right.$$

and since $D_\epsilon \ominus_{gH} A_\epsilon = E_\epsilon$, by (6.11), we have that

$$\left\{ \begin{array}{l} D_\epsilon = A_\epsilon + E_\epsilon \\ \text{or} \\ A_\epsilon = D_\epsilon - E_\epsilon, \\ \text{with} \\ E_\epsilon <_{LU} \left[\frac{1}{4}\epsilon\bar{\delta}, \frac{1}{4}\epsilon\bar{\delta} \right] \end{array} \right. \Leftrightarrow \left\{ \begin{array}{l} [\underline{d}_\epsilon, \bar{d}_\epsilon] = [\underline{a}_\epsilon, \bar{a}_\epsilon] + [\underline{e}_\epsilon, \bar{e}_\epsilon] \\ \text{or} \\ [\underline{a}_\epsilon, \bar{a}_\epsilon] = [\underline{d}_\epsilon, \bar{d}_\epsilon] + [-\bar{e}_\epsilon, -\underline{e}_\epsilon], \\ \text{with} \\ [\underline{e}_\epsilon, \bar{e}_\epsilon] <_{LU} \left[\frac{1}{4}\epsilon\bar{\delta}, \frac{1}{4}\epsilon\bar{\delta} \right]. \end{array} \right.$$

We have one of the following cases:

- (I) $[\underline{a}_\epsilon, \bar{a}_\epsilon] = [\underline{b}_\epsilon, \bar{b}_\epsilon] + [\underline{c}_\epsilon, \bar{c}_\epsilon]$ and $[\underline{d}_\epsilon, \bar{d}_\epsilon] = [\underline{a}_\epsilon, \bar{a}_\epsilon] + [\underline{e}_\epsilon, \bar{e}_\epsilon]$, with $[\underline{c}_\epsilon, \bar{c}_\epsilon] <_{LU} \left[-\frac{1}{2}\epsilon\bar{\delta}, -\frac{1}{2}\epsilon\bar{\delta} \right]$ and $[\underline{e}_\epsilon, \bar{e}_\epsilon] <_{LU} \left[\frac{1}{4}\epsilon\bar{\delta}, \frac{1}{4}\epsilon\bar{\delta} \right]$.
- (II) $[\underline{a}_\epsilon, \bar{a}_\epsilon] = [\underline{b}_\epsilon, \bar{b}_\epsilon] + [\underline{c}_\epsilon, \bar{c}_\epsilon]$ and $[\underline{a}_\epsilon, \bar{a}_\epsilon] = [\underline{d}_\epsilon, \bar{d}_\epsilon] + [-\bar{e}_\epsilon, -\underline{e}_\epsilon]$, with $[\underline{c}_\epsilon, \bar{c}_\epsilon] <_{LU} \left[-\frac{1}{2}\epsilon\bar{\delta}, -\frac{1}{2}\epsilon\bar{\delta} \right]$ and $[\underline{e}_\epsilon, \bar{e}_\epsilon] <_{LU} \left[\frac{1}{4}\epsilon\bar{\delta}, \frac{1}{4}\epsilon\bar{\delta} \right]$.
- (III) $[\underline{b}_\epsilon, \bar{b}_\epsilon] = [\underline{a}_\epsilon, \bar{a}_\epsilon] + [-\bar{c}_\epsilon, -\underline{c}_\epsilon]$ and $[\underline{d}_\epsilon, \bar{d}_\epsilon] = [\underline{a}_\epsilon, \bar{a}_\epsilon] + [\underline{e}_\epsilon, \bar{e}_\epsilon]$, with $[\underline{c}_\epsilon, \bar{c}_\epsilon] <_{LU} \left[-\frac{1}{2}\epsilon\bar{\delta}, -\frac{1}{2}\epsilon\bar{\delta} \right]$ and $[\underline{e}_\epsilon, \bar{e}_\epsilon] <_{LU} \left[\frac{1}{4}\epsilon\bar{\delta}, \frac{1}{4}\epsilon\bar{\delta} \right]$.
- (IV) $[\underline{b}_\epsilon, \bar{b}_\epsilon] = [\underline{a}_\epsilon, \bar{a}_\epsilon] + [-\bar{c}_\epsilon, -\underline{c}_\epsilon]$ and $[\underline{a}_\epsilon, \bar{a}_\epsilon] = [\underline{d}_\epsilon, \bar{d}_\epsilon] + [-\bar{e}_\epsilon, -\underline{e}_\epsilon]$, with $[\underline{c}_\epsilon, \bar{c}_\epsilon] <_{LU} \left[-\frac{1}{2}\epsilon\bar{\delta}, -\frac{1}{2}\epsilon\bar{\delta} \right]$ and $[\underline{e}_\epsilon, \bar{e}_\epsilon] <_{LU} \left[\frac{1}{4}\epsilon\bar{\delta}, \frac{1}{4}\epsilon\bar{\delta} \right]$.

In all of the cases, (I), (II), (III), and (IV), we always have $[\underline{d}_\epsilon, \bar{d}_\epsilon] <_{LU} [\underline{b}_\epsilon, \bar{b}_\epsilon] + \epsilon[\underline{\alpha}', \bar{\alpha}']$, with $\underline{\alpha}' = \bar{\alpha}' = -\frac{1}{4}\bar{\delta}$, that is, $D_\epsilon <_{LU} B_\epsilon + \epsilon[\underline{\alpha}', \bar{\alpha}']$, which is equivalent to

$$F(x^* + \epsilon\tilde{x}) <_{LU} F(x^*) + \epsilon[\underline{\alpha}', \bar{\alpha}'],$$

with $\underline{\alpha}' = \bar{\alpha}' = -\frac{1}{4}\bar{\delta} < 0$, since $\bar{\delta} > 0$.

Therefore, there exists a neighborhood $N := \mathbb{B}_{\mathbb{R}^n}(\bar{x}, \frac{1}{4L}\bar{\delta})$ of $\bar{x}$, a positive scalar $(\epsilon^*)' = \min\{\delta', \hat{\epsilon}\}$, and an interval

$$[\underline{\alpha}', \bar{\alpha}'] = \left[-\frac{1}{4}\bar{\delta}, -\frac{1}{4}\bar{\delta} \right] <_{LU} [0, 0]$$

such that for all ϵ with $0 < \epsilon < (\epsilon^*)'$, and all $\tilde{x} \in N = \mathbb{B}_{\mathbb{R}^n}(\bar{x}, \frac{1}{4L}\bar{\delta})$, we have that

$$F(x^* + \epsilon\tilde{x}) <_{LU} F(x^*) + \epsilon[\underline{\alpha}', \bar{\alpha}'], \quad (6.12)$$

that is, by Definition 6.3.1–(i), $\bar{x} \in \mathcal{K}_y$. $\square$

Therefore, by Theorems 6.4.1 and 6.4.2, we have:

Theorem 6.4.3. *Suppose that (A1) holds. Then, the cone of directions of decrease $\mathcal{K}_y$ at x^* is given by $\mathcal{K}_y = \mathcal{K}_0$.*

The next result is in I. V. Girsanov, [36], but he did not do the proof, he just said that the proof follows directly from the definitions.

Lemma 6.4.1. *Suppose that (A4) holds. Then, the cone of feasible directions $\mathcal{K}_b$ at x^* is given by $\mathcal{K}_b = \mathcal{K}_1$.*

Proof. First, we are going to prove that $\mathcal{K}_b \subseteq \mathcal{K}_1$. Let $\bar{x} \in \mathcal{K}_b$, by Definition 6.3.1–(ii), there exists a neighborhood $N \subset \mathbb{R}^n$ of $\bar{x}$ and a scalar $\epsilon_1 > 0$ such that for all ϵ with $0 < \epsilon < \epsilon_1$ and all $\tilde{x} \in N$, we have that $x^* + \epsilon\tilde{x} \in S$. It is clear that $x^* + \epsilon N$ is a neighborhood of $x^* + \epsilon\bar{x}$, $0 < \epsilon < \epsilon_1$, and that $x^* + \epsilon N \subseteq S$, $0 < \epsilon < \epsilon_1$, that is,

$$x^* + \epsilon\bar{x} \in \text{int } S. \quad (6.13)$$

So $\bar{x} = \frac{1}{\epsilon}[x - x^*]$, where $x := x^* + \epsilon\bar{x} \in \text{int } S$, by (6.13), and set $\lambda := \frac{1}{\epsilon} > 0$. Therefore,

$$\mathcal{K}_b \subseteq \{\bar{x} \in \mathbb{R}^n \mid \bar{x} = \lambda[x - x^*], \ x \in \text{int } S, \text{ and } \lambda > 0\} = \mathcal{K}_1.$$

Now, we are going to prove that $\mathcal{K}_1 \subseteq \mathcal{K}_b$. Let $\bar{x} \in \mathcal{K}_1$, then

$$\bar{x} = \lambda[x - x^*], \quad x \in \text{int } S, \text{ and } \lambda > 0.$$

Since $x \in \text{int } S$, there exists V a neighborhood of x such that $x \in V \subseteq S$. Set $N := \lambda[V - x^*]$ and $\epsilon_1 := \frac{1}{\lambda} > 0$, then N is a neighborhood of $\bar{x}$, since V is a neighborhood of x , such that for all ϵ with $0 < \epsilon < \frac{1}{\lambda} = \epsilon_1$ and all $\tilde{x} \in N$, we have that

$$\begin{aligned} x^* + \epsilon\tilde{x} &= x^* + \epsilon\lambda[v - x^*], & v \in V \subseteq S \\ &= (\epsilon\lambda)v + (1 - \epsilon\lambda)x^*, & \text{with } 0 < \epsilon < \epsilon_1 \text{ and } \epsilon\lambda + (1 - \epsilon\lambda) = 1 \\ &\in S, & \text{by the convexity of } S. \end{aligned} \quad (6.14)$$

Then, there exists a neighborhood $N = \lambda[V - x^*]$ of $\bar{x}$ and a scalar $\epsilon_1 > 0$ such that for all ϵ with $0 < \epsilon < \epsilon_1 = \frac{1}{\lambda}$ and all $\tilde{x} \in N$, we have that

$$x^* + \epsilon\tilde{x} \in S,$$

that is, by Definition 6.3.1–(ii), $\bar{x} \in \mathcal{K}_b$. $\square$

Theorem 6.4.4. *Let $x^* \in S$ be a weak LU –solution of (AIOP). Assume that (A1) and (A4) hold, then*

$$\mathcal{K}_0 \cap \mathcal{K}_1 = \emptyset.$$

Proof. Suppose that there exists $\bar{x} \in \mathcal{K}_0 \cap \mathcal{K}_1$. Then, $\bar{x} \in \mathcal{K}_0 = \mathcal{K}_y$, by Theorem 6.4.3, that is, $\bar{x}$ is a direction of decrease of $F(x)$ at x^* , so by Definition 6.3.1–(i), there exists a neighborhood N_0 of $\bar{x}$, a positive scalar ϵ_0 , and an interval $[\underline{\alpha}, \bar{\alpha}] <_{LU} [0, 0]$ such that for all ϵ with $0 < \epsilon < \epsilon_0$ and all $\tilde{x} \in N_0$, we have that

$$F(x^* + \epsilon\tilde{x}) <_{LU} F(x^*) + \epsilon[\underline{\alpha}, \bar{\alpha}]. \quad (6.15)$$

We have also that $\bar{x} \in \mathcal{K}_1 = \mathcal{K}_b$, by Lemma 6.4.1, that is, $\bar{x}$ is a feasible direction to Q_1 at x^* , so by Definition 6.3.1–(ii), there exists a neighborhood N_1 of $\bar{x}$ and a scalar $\epsilon_1 > 0$ such that for all ϵ with $0 < \epsilon < \epsilon_1$ and all $\tilde{x} \in N_1$, we have that

$$x_\epsilon := x^* + \epsilon\tilde{x} \in S. \quad (6.16)$$

Setting $N := N_0 \cap N_1$ and $\epsilon_2 := \min\{\epsilon_0, \epsilon_1\}$, since there exists a neighborhood N_0 of $\bar{x}$, a positive scalar ϵ_0 , and an interval $[\underline{\alpha}, \bar{\alpha}] <_{LU} [0, 0]$ such that for all ϵ with $0 < \epsilon < \epsilon_0$ and $\tilde{x} \in N_0 \subset N$, we have that, for all $0 < \epsilon < \epsilon_2$ and $\tilde{x} \in N$:

$$\begin{aligned} F(x_\epsilon) &= F(x^* + \epsilon\tilde{x}), && \text{by (6.16)} \\ &<_{LU} F(x^*) + \epsilon[\underline{\alpha}, \bar{\alpha}], && \text{by using (6.15)} \\ &<_{LU} F(x^*), && \text{since } \epsilon > 0 \text{ and } [\underline{\alpha}, \bar{\alpha}] <_{LU} [0, 0]. \end{aligned}$$

Therefore, $x_\epsilon \in S$ satisfies $F(x_\epsilon) <_{LU} F(x^*)$, which is a contradiction to the assumption that $x^* \in S$ is a weak LU –solution. $\square$

In the following section, it will be established other cones and it will be given some characteristics of them.

6.5 Characterizations of classical cones derived from $\mathcal{K}_0$

It will be considered two classical cones and it will be characterized these cones together with other classical one too. Finally, it will be proved an important result that says that the intersection of all these classical cones is empty.

Now, let $T \subseteq \mathbb{R}^n$ be an open and nonempty set, and let $F : T \rightarrow \mathcal{K}_C$ be an interval-valued function with $F(x) = [\underline{f}(x), \overline{f}(x)]$ for all $x \in T$, if F is gH -differentiable at $x^* \in T$, then for all $\bar{x} \in \mathbb{R}^n$, by Theorem 2.7.4, we have

$$F'_{gH}(x^*; \bar{x}) = \left[\langle \nabla_{\#} \underline{f}(x^*), \bar{x} \rangle \vee \langle \nabla_{\#} \overline{f}(x^*), \bar{x} \rangle \right]$$

$$\begin{aligned} \Rightarrow \mathcal{K}_0 &= \{ \bar{x} \in \mathbb{R}^n \mid F'_{gH}(x^*; \bar{x}) <_{LU} [0, 0] \} \\ &= \left\{ \bar{x} \in \mathbb{R}^n \mid \left[\langle \nabla_{\#} \underline{f}(x^*), \bar{x} \rangle \vee \langle \nabla_{\#} \overline{f}(x^*), \bar{x} \rangle \right] <_{LU} [0, 0] \right\}. \end{aligned}$$

Now, let us set

$$\underline{\mathcal{K}}_0 := \{ \bar{x} \in \mathbb{R}^n \mid \langle \nabla_{\#} \underline{f}(x^*), \bar{x} \rangle < 0 \}$$

and

$$\overline{\mathcal{K}}_0 := \{ \bar{x} \in \mathbb{R}^n \mid \langle \nabla_{\#} \overline{f}(x^*), \bar{x} \rangle < 0 \}.$$

(6.17)

Proposition 6.5.1. *Assume that (A3) and (A4) hold. Then,*

- (i) $\underline{\mathcal{K}}_0$ is an open convex cone.
- (iii) $\mathcal{K}_1$ is an open convex cone.
- (ii) $\overline{\mathcal{K}}_0$ is an open convex cone.

Proof. (i) **$\underline{\mathcal{K}}_0$ is open.** Let $\bar{x} \in \underline{\mathcal{K}}_0$, then $\bar{x} \in \mathbb{R}^n$ is such that

$$\langle \nabla_{\#} \underline{f}(x^*), \bar{x} \rangle < 0. \quad (6.18)$$

We want to prove that $\bar{x} \in \text{int } \underline{\mathcal{K}}_0$, that is, we want to prove that there exists $r > 0$ such that $\mathbb{B}_{\mathbb{R}^n}(\bar{x}, r) \subset \underline{\mathcal{K}}_0$. Define

$$\underline{\Psi}(x) := \langle \nabla_{\#} \underline{f}(x^*), x \rangle.$$

By (A3), $\nabla_{\#} \underline{f}(x^*)$ exists and is bounded, let us say by K , that is, $\|\nabla_{\#} \underline{f}(x^*)\|_{\mathbb{R}^n} \leq K$, so we have that

$$|\underline{\Psi}(x)| = |\langle \nabla_{\#} \underline{f}(x^*), x \rangle| \leq \|\nabla_{\#} \underline{f}(x^*)\|_{\mathbb{R}^n} \|x\|_{\mathbb{R}^n} \leq K \|x\|_{\mathbb{R}^n} \quad \forall x \in \mathbb{R}^n,$$

i.e., $|\underline{\Psi}| \leq K$, which implies that $\underline{\Psi}$ is bounded, that is, $\underline{\Psi}$ is continuous. By (6.18), $\underline{\Psi}(\bar{x}) = \langle \nabla_{\#} \underline{f}(x^*), \bar{x} \rangle < 0$, then there exists $r > 0$ such that

$$\underline{\Psi}(x) = \langle \nabla_{\#} \underline{f}(x^*), x \rangle < 0 \quad \forall x \in \mathbb{B}_{\mathbb{R}^n}(\bar{x}, r).$$

So, $\mathbb{B}_{\mathbb{R}^n}(\bar{x}, r) \subset \underline{\mathcal{K}}_0$. Therefore, $x \in \underline{\mathcal{K}}_0$.

$\mathcal{K}_0$ is convex. Let $x_1, x_2 \in \underline{\mathcal{K}}_0$, i.e., by (6.17), $x_1, x_2 \in \mathbb{R}^n$ satisfy

$$\langle \nabla_{\#} \underline{f}(x^*), x_1 \rangle < 0 \tag{6.19}$$

and

$$\langle \nabla_{\#} \underline{f}(x^*), x_2 \rangle < 0. \tag{6.20}$$

Then, given $\lambda \in [0, 1]$, $(1 - \lambda)x_1 + \lambda x_2 \in \mathbb{R}^n$ is such that

$$\begin{aligned} \langle \nabla_{\#} \underline{f}(x^*), ((1 - \lambda)x_1 + \lambda x_2) \rangle &= (1 - \lambda) \langle \nabla_{\#} \underline{f}(x^*), x_1 \rangle + \lambda \langle \nabla_{\#} \underline{f}(x^*), x_2 \rangle \\ &< 0 + 0 = 0, \quad \text{by (6.19), (6.20), and since } \lambda \in [0, 1]. \end{aligned}$$

$\mathcal{K}_0$ is a cone. If $\bar{x} \in \underline{\mathcal{K}}_0$, then $\alpha \bar{x} \in \underline{\mathcal{K}}_0$, for $\alpha > 0$. Indeed, if $\bar{x} \in \underline{\mathcal{K}}_0$,

$$\bar{x} \in \mathbb{R}^n \text{ is such that } \langle \nabla_{\#} \underline{f}(x^*), \bar{x} \rangle < 0 \Leftrightarrow \alpha \bar{x} \in \mathbb{R}^n \text{ is such that } \langle \nabla_{\#} \underline{f}(x^*), \alpha \bar{x} \rangle < 0,$$

that is, $\alpha \bar{x} \in \underline{\mathcal{K}}_0$, for $\alpha > 0$.

(ii) The proof is similar to (i).

(iii) **$\mathcal{K}_1$ is open.** By Lemma 6.4.1 and by Proposition 6.3.2-(i), we have that $\mathcal{K}_1 = \mathcal{K}_b$ is open.

$\mathcal{K}_1$ is convex. Since S is convex, by a known property, $\text{int } S$ is also convex.

Now, we are going to prove that $\mathcal{K}_1$ is convex. Indeed, let $\bar{x}_1, \bar{x}_2 \in \mathcal{K}_1$, that is,

$$\bar{x}_i = \lambda_i [x_i - x^*], \quad x_i \in \text{int } S, \quad \lambda_i > 0, \quad \text{and } i \in \{1, 2\}. \tag{6.21}$$

Let $\alpha \in (0, 1)$,

$$\begin{aligned} \alpha \bar{x}_1 + (1 - \alpha) \bar{x}_2 &= \alpha \lambda_1 [x_1 - x^*] + (1 - \alpha) \lambda_2 [x_2 - x^*], & \text{by (6.21),} \\ &= \alpha \lambda_1 x_1 + (1 - \alpha) \lambda_2 x_2 - (\alpha \lambda_1 + (1 - \alpha) \lambda_2) x^* \\ &= \alpha \lambda_1 x_1 + (1 - \alpha) \lambda_2 x_2 - \lambda x^*, \end{aligned}$$

where $\lambda := \alpha\lambda_1 + (1 - \alpha)\lambda_2 > 0$, since $\lambda_1 > 0$, $\lambda_2 > 0$, and $\alpha \in (0, 1)$, so

$$\alpha\bar{x}_1 + (1 - \alpha)\bar{x}_2 = \lambda \left(\frac{\alpha\lambda_1}{\lambda}x_1 + \frac{(1 - \alpha)\lambda_2}{\lambda}x_2 \right) - \lambda x^*.$$

By setting, $\alpha_1 := \frac{\alpha\lambda_1}{\lambda} = \frac{\alpha\lambda_1}{\alpha\lambda_1 + (1 - \alpha)\lambda_2} \in (0, 1)$, we have that $1 - \alpha_1 = 1 - \frac{\alpha\lambda_1}{\alpha\lambda_1 + (1 - \alpha)\lambda_2} = \frac{\alpha\lambda_1 + (1 - \alpha)\lambda_2 - \alpha\lambda_1}{\alpha\lambda_1 + (1 - \alpha)\lambda_2} = \frac{(1 - \alpha)\lambda_2}{\lambda}$.

Then,

$$\alpha\bar{x}_1 + (1 - \alpha)\bar{x}_2 = \lambda[(\alpha_1 x_1 + (1 - \alpha_1)x_2) - x^*],$$

where, $\alpha_1 x_1 + (1 - \alpha_1)x_2 \in \text{int } S$ (since $x_1, x_2 \in \text{int } S$, and $\text{int } S$ is a convex set) and $\lambda = \alpha\lambda_1 + (1 - \alpha)\lambda_2 > 0$. Therefore,

$$\alpha\bar{x}_1 + (1 - \alpha)\bar{x}_2 \in \mathcal{K}_1,$$

that is, $\mathcal{K}_1$ is convex.

$\mathcal{K}_1$ is a cone. By Lemma 6.4.1 and by Proposition 6.3.2-(ii), we have that $\mathcal{K}_1 = \mathcal{K}_b$ is a cone.

□

Theorem 6.5.1. *Let $x^* \in S$ be a weak LU-solution of (AIOP). Assume that (A1)–(A4) hold, then*

$$\underline{\mathcal{K}}_0 \cap \overline{\mathcal{K}}_0 \cap \mathcal{K}_1 = \emptyset.$$

Proof. Let $x^* \in S$ be a weak LU-solution of (AIOP), then by Theorem 6.4.4, we have that

$$\mathcal{K}_0 \cap \mathcal{K}_1 = \emptyset. \quad (6.22)$$

Suppose that there exists $\bar{x} \in \underline{\mathcal{K}}_0 \cap \overline{\mathcal{K}}_0 \cap \mathcal{K}_1$, then, it is not difficult to see that $\bar{x} \in \mathcal{K}_0 \cap \mathcal{K}_1$, which contradicts (6.22). □

Next, necessary optimality conditions for (AIOP) and the maximizing–minimizing equivalence property will be given.

6.6 Necessary optimality conditions for (AIOP)

Theorem 6.6.1 (Necessary optimality conditions for (AIOP)). *Let x^* be a weak LU-solution of (AIOP). Assume that (A1)–(A4) hold. Then, there exists $y \in \mathbb{R}^n$ such that $y \in \nabla_{gH} F(x^*)$ and*

$$\langle y, x - x^* \rangle \geq 0 \quad (6.23)$$

for all $x \in S$.

Proof. We have the following three cases:

(i) Suppose that $\underline{\mathcal{K}}_0 \neq \emptyset$ and $\overline{\mathcal{K}}_0 \neq \emptyset$ holds.

(A) Analysis of the functional. By (6.17),

$$\underline{\mathcal{K}}_0 = \{ \bar{x} \in \mathbb{R}^n \mid \underline{\Psi}(\bar{x}) := \langle \nabla_{\#} \underline{f}(x^*), \bar{x} \rangle < 0 \}$$

and

$$\overline{\mathcal{K}}_0 = \{ \bar{x} \in \mathbb{R}^n \mid \overline{\Psi}(\bar{x}) := \langle \nabla_{\#} \overline{f}(x^*), \bar{x} \rangle < 0 \},$$

then by Theorem 2.11.4, we have that

$$\underline{\mathcal{K}}_0^* = \begin{cases} \mathbb{R}^n, & \text{if } \underline{\Psi} = 0 \\ \{ -\underline{\lambda}^* \underline{\Psi}, 0 \leq \underline{\lambda}^* < \infty \}, & \text{if } \underline{\Psi} \neq 0 \end{cases}$$

and

$$\overline{\mathcal{K}}_0^* = \begin{cases} \mathbb{R}^n, & \text{if } \overline{\Psi} = 0 \\ \{ -\overline{\lambda}^* \overline{\Psi}, 0 \leq \overline{\lambda}^* < \infty \}, & \text{if } \overline{\Psi} \neq 0. \end{cases}$$

Since $\underline{\mathcal{K}}_0 \neq \emptyset$ and $\overline{\mathcal{K}}_0 \neq \emptyset$, the cases $\underline{\Psi} = 0$ and $\overline{\Psi} = 0$ are excluded, so that for any $\underline{f}_0 \in \underline{\mathcal{K}}_0^*$ and $\overline{f}_0 \in \overline{\mathcal{K}}_0^*$, we have

$$\underline{f}_0(\bar{x}) = -\underline{\lambda}^* \langle \nabla_{\#} \underline{f}(x^*), \bar{x} \rangle, \quad \underline{\lambda}^* \geq 0 \quad (6.24)$$

and

$$\overline{f}_0(\bar{x}) = -\overline{\lambda}^* \langle \nabla_{\#} \overline{f}(x^*), \bar{x} \rangle, \quad \overline{\lambda}^* \geq 0. \quad (6.25)$$

(B) Analysis of the constraint S . S is a closed convex set in $\mathbb{R}^n$ such that $\text{int } S \neq \emptyset$. As $\mathcal{K}_1$ is the cone of feasible directions for S at x^* , by Theorem 2.11.5,

$$\mathcal{K}_1^* = S^* = \{ f : \mathbb{R}^n \rightarrow \mathbb{R} \mid f \text{ is linear with } f(x) \geq f(x^*) \text{ for all } x \in S \},$$

and if $f_1 \in S^*$,

$$f_1(x) \geq f_1(x^*) \text{ for all } x \in S, \quad (6.26)$$

i.e., f_1 is a supporting functional to $S \subset \mathbb{R}^n$ at $x^* \in S$.

Since $S^* \subset (\mathbb{R}^n)^* \approx \mathbb{R}^n$, by the Riesz Representation Theorem (see M. Fréchet, [34]; F. Riesz, [72], [73]), there exists $y \in \mathbb{R}^n$ such that

$$f_1(x) = \langle y, x \rangle \quad \text{for all } x \in \mathbb{R}^n. \quad (6.27)$$

Then,

$$\langle y, x \rangle \geq \langle y, x^* \rangle \Leftrightarrow \langle y, x - x^* \rangle \geq 0 \quad \text{for all } x \in S.$$

(C) Euler equation. Applying Proposition 6.5.1, Theorem 6.5.1, and Lemma 2.11.3 (Dubovitskii–Milyutin), we have the following non-triviality condition: there exist linear functionals $\underline{f}_0 \in \underline{\mathcal{K}}_0^*$, $\bar{f}_0 \in \bar{\mathcal{K}}_0^*$, and $f_1 \in \mathcal{K}_1^*$, not all zero, such that, for all $\bar{x} \in \mathbb{R}^n$,

$$\underline{f}_0(\bar{x}) + \bar{f}_0(\bar{x}) + f_1(\bar{x}) = 0, \quad (6.28)$$

where $\underline{f}_0(\bar{x})$, $\bar{f}_0(\bar{x})$ are given by (6.24) and (6.25), respectively, and $f_1(\bar{x})$ is a supporting functional to $S \subset \mathbb{R}^n$ at $x^* \in S$. Then, we have that

$$\begin{aligned} & f_1(\bar{x}) - \underline{\lambda}^* \langle \nabla_{\#} \underline{f}(x^*), \bar{x} \rangle - \bar{\lambda}^* \langle \nabla_{\#} \bar{f}(x^*), \bar{x} \rangle = 0, \quad \text{by (6.28), (6.24), and (6.25)} \\ \Leftrightarrow & \langle y, \bar{x} \rangle - \underline{\lambda}^* \langle \nabla_{\#} \underline{f}(x^*), \bar{x} \rangle - \bar{\lambda}^* \langle \nabla_{\#} \bar{f}(x^*), \bar{x} \rangle = 0, \quad \text{by (6.27)} \\ \Leftrightarrow & \left\langle y - \underline{\lambda}^* \nabla_{\#} \underline{f}(x^*) - \bar{\lambda}^* \nabla_{\#} \bar{f}(x^*), \bar{x} \right\rangle = 0 \quad \text{for all } \bar{x} \in \mathbb{R}^n \\ \Leftrightarrow & y - \underline{\lambda}^* \nabla_{\#} \underline{f}(x^*) - \bar{\lambda}^* \nabla_{\#} \bar{f}(x^*) = 0 \\ \Leftrightarrow & y = \underline{\lambda}^* \nabla_{\#} \underline{f}(x^*) + \bar{\lambda}^* \nabla_{\#} \bar{f}(x^*), \quad \text{with } \underline{\lambda}^* \geq 0 \text{ and } \bar{\lambda}^* \geq 0. \end{aligned} \quad (6.29)$$

Now, under these assumptions, the case $\underline{\lambda}^* = 0$ and $\bar{\lambda}^* = 0$ cannot occur simultaneously, since by (6.24) and (6.25), we would have that for all $\bar{x} \in \mathbb{R}^n$, $\underline{f}_0(\bar{x}) = 0$ and $\bar{f}_0(\bar{x}) = 0$, respectively, but then by (6.28), $f_1(\bar{x}) = 0$ for all $\bar{x} \in \mathbb{R}^n$. Then, we would have that $\underline{f}_0 = \bar{f}_0 = f_1 = 0$, which contradicts the non-triviality condition that says that there exist $\underline{f}_0$, $\bar{f}_0$, f_1 , not all zero, see item (C). Therefore, we can assume that $\underline{\lambda}^* + \bar{\lambda}^* = 1$, so, from (6.29), where $\underline{\lambda}^* \geq 0$ and $\bar{\lambda}^* \geq 0$, we have that

$$y = \underline{\lambda}^* \nabla_{\#} \underline{f}(x^*) + \bar{\lambda}^* \nabla_{\#} \bar{f}(x^*) \in \nabla_{gH} F(x^*), \quad \text{by Lemma 5.2.1 - (i).}$$

(ii) Suppose that $\underline{\mathcal{K}}_0 = \emptyset$, that is,

$$\langle \nabla_{\#} \underline{f}(x^*), \bar{x} \rangle = 0 \quad \text{for all } \bar{x} \in \mathbb{R}^n. \quad (6.30)$$

Taking $\underline{\lambda}^* = 1$ and $\bar{\lambda}^* = 0$, there exists $y \in \mathbb{R}^n$ such that

$$\begin{aligned} y &= \underline{\lambda}^* \nabla_{\#} \underline{f}(x^*) + \bar{\lambda}^* \nabla_{\#} \bar{f}(x^*) = \nabla_{\#} \underline{f}(x^*) \\ &\in \nabla_{gH} F(x^*), \quad \text{by Lemma 5.2.1 - (i).} \end{aligned}$$

Then, (6.30) becomes

$$\langle y, \bar{x} \rangle = 0 \quad \text{for all } \bar{x} \in \mathbb{R}^n,$$

which implies that $y = 0$ so,

$$\langle y, x - x^* \rangle = \langle 0, x - x^* \rangle = 0 \geq 0$$

for all $x \in S$.

(iii) Suppose that $\bar{\mathcal{K}}_0 = \emptyset$, that is,

$$\langle \nabla_{\#} \bar{f}(x^*), \bar{x} \rangle = 0 \quad \text{for all } \bar{x} \in \mathbb{R}^n. \quad (6.31)$$

Taking $\underline{\lambda}^* = 0$ and $\bar{\lambda}^* = 1$, there exists $y \in \mathbb{R}^n$ such that

$$\begin{aligned} y &= \underline{\lambda}^* \nabla_{\#} \underline{f}(x^*) + \bar{\lambda}^* \nabla_{\#} \bar{f}(x^*) = \nabla_{\#} \bar{f}(x^*) \\ &\in \nabla_{gH} F(x^*), \quad \text{by Lemma 5.2.1 - (i).} \end{aligned}$$

Then, (6.31) becomes

$$\langle y, \bar{x} \rangle = 0 \quad \text{for all } \bar{x} \in \mathbb{R}^n,$$

which implies that $y = 0$ so,

$$\langle y, x - x^* \rangle = \langle 0, x - x^* \rangle = 0 \geq 0$$

for all $x \in S$.

Therefore, the proof is complete for all the Cases (i), (ii), and (iii). $\square$

Proposition 6.6.1. *Let $x^* \in S$ be a weak LU -solution. Then, maximizing the interval-valued function F is equivalent to minimizing $-F$.*

Proof. We have that

$$\begin{aligned} &x^* \text{ is a weak maximum } LU\text{-solution of } F \\ &\Leftrightarrow \nexists x \in S \text{ such that } F(x^*) <_{LU} F(x) \\ &\Leftrightarrow \nexists x \in S \text{ such that } (-F)(x) <_{LU} (-F)(x^*) \\ &\Leftrightarrow x^* \text{ is a weak minimum } LU\text{-solution of } -F. \end{aligned}$$

$\square$

6.7 Conclusion

We considered an abstract interval optimization problem and we defined LU –solution for it. Next, we gave two definitions of cones and some properties of them. Also, we settled two more cones: $\mathcal{K}_0$, $\mathcal{K}_1$, and we described them. Moreover, we defined classical cones: $\underline{\mathcal{K}}_0$, $\overline{\mathcal{K}}_0$, and we gave some features of them. We analyzed some functionals and the abstract constraint, and we obtained the Euler equation thanks to the Dubovitskii–Milyutin formalism. Therefore, we presented authentic necessary optimality conditions for interval optimization problems with abstract constraints.

CHAPTER 7

INTERVAL OPTIMAL CONTROL USING
GH—DIFFERENTIABILITY

The simplest version of the interval optimal control problem is to determine a phase trajectory and a control satisfying a differential equation with boundary conditions in such a way to minimize an interval integral functional, where the control satisfies some constraint. There are several versions of this problem, they differ in the form of the boundary conditions and/or the constraints imposed on the control; in some problems, the time variable is not fixed, in others there may be constraints imposed on the phase coordinates, among others. Here, the problems considered are such that the left endpoint and the right endpoint conditions are fixed, and such that the left endpoint condition is fixed but the right endpoint condition is free. First, it will be defined the optimal *LU*—processes for the problems. Then, it will be considered some basic assumptions, it will be given some new definitions for sets of directions, and some properties of them. Next, it will be considered some cones, it will be obtained some characterizations of them, and also it will be used nondegeneracy conditions in order to obtain some useful results. Moreover, it will be obtained classical cones from one of the cones defined before, it will be given some properties of them, and it will be obtained important results, which will help to obtain necessary optimality conditions in the form of a local Pontryagin Maximum Principle for both problems (with fixed right endpoint and free right endpoint). The study of the necessary optimality conditions, is based in the Dubovitskii—Milyutin formalism. In order to apply this formalism, it will be analyzed some functionals, the constraints of the problems, and the Euler equation, which in turn use previous results.

Herein, consider the following Banach space

$$E = C([0, T]; \mathbb{R}^n) \times L_\infty([0, T]; \mathbb{R}^m).$$

By simplicity of notation, it will be denoted $E = C \times L_\infty$. It will also be considered the state variable as a continuous function:

$$\begin{aligned} x : [0, T] &\rightarrow \mathbb{R}^n \\ t &\mapsto x(t) \in \mathbb{R}^n \end{aligned}$$

and the control variable as an essentially bounded and measurable function:

$$\begin{aligned} u : [0, T] &\rightarrow \mathbb{R}^m \\ t &\mapsto u(t) \in \mathbb{R}^m. \end{aligned}$$

In the following section, it will be considered the Lagrange interval optimal control problem with both left endpoint and right endpoint fixed.

7.1 The Lagrange interval optimal control problem (LIOCP)

Now, the *interval optimal control problem* is posed as follows:

$$\text{minimize } F(x, u) = \int_0^T \Phi(t, x(t), u(t)) dt \quad (7.1)$$

$$\text{subject to } x'(t) = \gamma(t, x(t), u(t)), \quad (7.2)$$

$$\begin{aligned} (x(0), x(T)) &\in \{c\} \times \{d\} \subset \mathbb{R}^n \times \mathbb{R}^n, \\ u(t) &\in \mathcal{M} \text{ a. e. } t \in [0, T], \end{aligned} \quad (7.3)$$

where $\gamma : [0, T] \times \mathbb{R}^n \times \mathbb{R}^m \rightarrow \mathbb{R}^n$, $\Phi : [0, T] \times \mathbb{R}^n \times \mathbb{R}^m \rightarrow \mathcal{K}_C$ is given by $\Phi(t, x, u) = [\underline{\phi}(t, x, u), \bar{\phi}(t, x, u)]$ with $\underline{\phi}, \bar{\phi} : [0, T] \times \mathbb{R}^n \times \mathbb{R}^m \rightarrow \mathbb{R}$, $\underline{\phi}(t, x, u) \leq \bar{\phi}(t, x, u)$ for all (t, x, u) , $\mathcal{M} \subset \mathbb{R}^m$, $[0, T] \subset \mathbb{R}$, and T fixed.

Then, the boundary conditions are

$$x(0) = c \in \mathbb{R}^n \quad (7.4)$$

and

$$x(T) = d \in \mathbb{R}^n. \quad (7.5)$$

Instead of Equation (7.2), we shall consider the equivalent integral equation, see Section 2.9 of Chapter 2, i.e., a solution will be a pair $(x(t), u(t)) \in C \times L_\infty$ satisfying the integral equation

$$x(t) = c + \int_0^t \gamma(\tau, x(\tau), u(\tau)) d\tau, \quad t \in [0, T].$$

Let Q'_1 be the set of all functions u satisfying (7.3), let Q_1 be the set of all $(x, u) \in C \times L_\infty$ satisfying (7.3), and let Q_2 be the set of all $(x, u) \in C \times L_\infty$ satisfying Equations (7.2), (7.4), and (7.5), that is,

$$Q'_1 := \{u \in L_\infty \mid (7.3) \text{ is satisfied}\},$$

$$Q_1 := \{(x, u) \in C \times L_\infty \mid (7.3) \text{ is satisfied}\} = C \times Q'_1,$$

$$Q_2 := \{(x, u) \in C \times L_\infty \mid (7.2), (7.4), \text{ and } (7.5) \text{ are satisfied}\}$$

$$= \{(x, u) \in C \times L_\infty \mid x(t) = c + \int_0^t \gamma(\tau, x(\tau), u(\tau)) d\tau, \quad t \in [0, T], \quad x(T) = d\},$$

where Q_1 is a set with nonempty interior and Q_2 has no interior points. In view of the formalism of Dubovitskii–Milyutin, Q_1 can be regarded, let us say, as an “inequality constraint,” that to be honest is an abstract constraint, and Q_2 can be regarded as an equality constraint, so that, as a rule Q_2 is like a manifold of lower dimension than the space.

Therefore, the problem is minimize the integral $F(x, u) = \int_0^T \Phi(t, x(t), u(t)) dt$ on the set $Q := Q_1 \cap Q_2$, that is, the *Lagrange interval optimal control problem* is:

$$(LIOCP) \begin{cases} \text{minimize } F(x, u) = \int_0^T \Phi(t, x(t), u(t)) dt \\ \text{subject to } (x, u) \in Q = Q_1 \cap Q_2. \end{cases}$$

Next, it will be defined some processes for the problem $(LIOCP)$.

7.2 Optimal LU –processes for $(LIOCP)$

From the order relation LU already studied for the interval space, it will be defined the optimal LU –process concepts for Lagrange interval optimal control problems.

Definition 7.2.1. Let (x^*, u^*) be an admissible process of $(LIOCP)$, i.e., $(x^*, u^*) \in Q$.

- (i) (a) (x^*, u^*) is a **$W^{1,1}$ –local strong optimal LU –process** of $(LIOCP)$ if there exists $\delta > 0$ such that there does not exist $(x, u) \in Q$, with $(x, u) \neq (x^*, u^*)$ and $\|x - x^*\|_{W^{1,1}} \leq \delta$, such that

$$F(x, u) \leq_{LU} F(x^*, u^*).$$

(b) Furthermore, (x^*, u^*) is a **L_∞ –local strong optimal LU–process** of (LIOCP) if there exists $\delta > 0$ such that there does not exist $(x, u) \in Q$, with $(x, u) \neq (x^*, u^*)$ and $\|x - x^*\|_{L_\infty} \leq \delta$, such that

$$F(x, u) \leq_{LU} F(x^*, u^*).$$

(ii) (c) (x^*, u^*) is a **$W^{1,1}$ –local optimal LU–process** of (LIOCP) if there exists $\delta > 0$ such that there does not exist $(x, u) \in Q$, and $\|x - x^*\|_{W^{1,1}} \leq \delta$, such that

$$F(x, u) \leq_{LU} F(x^*, u^*).$$

(d) Furthermore, (x^*, u^*) is a **L_∞ –local optimal LU–process** of (LIOCP) if there exists $\delta > 0$ such that there does not exist $(x, u) \in Q$, and $\|x - x^*\|_{L_\infty} \leq \delta$, such that

$$F(x, u) \leq_{LU} F(x^*, u^*).$$

(iii) (e) (x^*, u^*) is a **$W^{1,1}$ –local weak optimal LU–process** of (LIOCP) if there exists $\delta > 0$ such that there does not exist $(x, u) \in Q$, and $\|x - x^*\|_{W^{1,1}} \leq \delta$, such that

$$F(x, u) <_{LU} F(x^*, u^*).$$

(f) Furthermore, (x^*, u^*) is a **L_∞ –local weak optimal LU–process** of (LIOCP) if there exists $\delta > 0$ such that there does not exist $(x, u) \in Q$, and $\|x - x^*\|_{L_\infty} \leq \delta$, such that

$$F(x, u) <_{LU} F(x^*, u^*).$$

If the neighborhood of (x^*, u^*) above is replaced by the whole space $E = C \times L_\infty$, we have the corresponding definitions of global solutions for (LIOCP).

In the following, it will be established the Lagrange interval optimal control problem with left endpoint fixed and right endpoint free.

7.3 The Lagrange interval optimal control problem ($\widetilde{LIOCP}$)

It will be considered a variation of ($LIOCP$) where the left endpoint condition (7.4) is fixed as before, $x(0) = c \in \mathbb{R}^n$, but the right endpoint condition (7.5) is free,

$$x(T) \in \mathbb{R}^n. \quad (7.6)$$

Let $\widetilde{Q}_2$ be the set of all $(x, u) \in E$ satisfying Equations (7.2), (7.4), and (7.6)

$$\begin{aligned} \widetilde{Q}_2 &:= \{(x, u) \in C \times L_\infty \mid (7.2), (7.4), \text{ and } (7.6) \text{ are satisfied}\} \\ &= \{(x, u) \in C \times L_\infty \mid x(t) = c + \int_0^t \gamma(\tau, x(\tau), u(\tau)) d\tau, \quad t \in [0, T], \quad x(T) \in \mathbb{R}^n\}. \end{aligned}$$

Therefore, the Lagrange interval optimal control problem now considered is:

$$(\widetilde{LIOCP}) \begin{cases} \text{minimize} & F(x, u) = \int_0^T \Phi(t, x(t), u(t)) dt \\ \text{subject to} & (x, u) \in \widetilde{Q} = Q_1 \cap \widetilde{Q}_2. \end{cases}$$

Remark 7.3.1. *The definitions for optimal LU-processes for ($\widetilde{LIOCP}$) are similar to the definitions for optimal LU-processes for ($LIOCP$) (just change Q by $\widetilde{Q}$).*

Next, in order to establish our purposes, it will be needed the following definitions and assumptions:

7.4 Some definitions and assumptions

The Lipschitz continuity will be defined, some assumptions will be considered, and some directions and properties of them will be given.

Definition 7.4.1. *Given $A = [0, T] \times \mathbb{R}^{n+m}$, let $\Phi : A \rightarrow \mathcal{K}_C$ be an interval-valued function.*

- (i) *We say that Φ is Lipschitz continuous with respect to (x, u) of rank L , where $L : [0, T] \rightarrow \mathbb{R}$ is a $\mathcal{L}$ -measurable integrable function with $L(t) > 0$ for a. e. $t \in [0, T]$, if we have*

$$d_H(\Phi(t, x_1, u_1), \Phi(t, x_2, u_2)) \leq L(t) \|(x_1, u_1) - (x_2, u_2)\|_{\mathbb{R}^{n+m}}$$

for all $(x_1, u_1), (x_2, u_2) \in \mathbb{R}^{n+m}$ and for a. e. $t \in [0, T]$.

- (ii) We say that Φ is Lipschitz continuous with respect to (x, u) locally near a given point $(x, u) \in \mathbb{R}^{n+m}$ of rank L , where $L : [0, T] \rightarrow \mathbb{R}$ is a $\mathcal{L}$ -measurable integrable function with $L(t) > 0$ for a. e. $t \in [0, T]$, if for some $\epsilon > 0$, we have

$$d_H(\Phi(t, x_1, u_1), \Phi(t, x_2, u_2)) \leq L(t) \|(x_1, u_1) - (x_2, u_2)\|_{\mathbb{R}^{n+m}}$$

for all $(x_1, u_1), (x_2, u_2) \in \mathbb{B}_{\mathbb{R}^{n+m}}((x, u), \epsilon)$ and for a. e. $t \in [0, T]$, where $\mathbb{B}_{\mathbb{R}^{n+m}}((x, u), \epsilon) = \{(x', u') \in \mathbb{R}^n \times \mathbb{R}^m \mid \|(x', u') - (x, u)\|_{\mathbb{R}^{n+m}} < \epsilon\}$.

Let $(x^*, u^*) \in C \times L_\infty$ be a reference optimal LU -process of $(LIOCP)$, then we have the following assumptions:

- (A1) The interval-valued function $\Phi(t, x, u)$ satisfy the Lipschitz continuous condition in a neighborhood of (x^*, u^*) .
- (A2) The interval-valued function Φ is continuously gH -differentiable (with respect to x and u), $\gamma(t, x(t), u(t))$ is continuous (with respect to x and u), continuously differentiable (with respect to x and u), and measurable (with respect to t).
- (A3) For all bounded x and u , and all $t \in [0, T]$, the partial derivatives $\nabla_{\#x} \underline{\phi}(t, x(t), u(t))$, $\nabla_{\#x} \bar{\phi}(t, x(t), u(t))$, $\nabla_{\#u} \underline{\phi}(t, x(t), u(t))$, and $\nabla_{\#u} \bar{\phi}(t, x(t), u(t))$ exist, and are bounded, where

$$\begin{aligned} \nabla_{\#x} \underline{\phi}(t, x(t), u(t)) &= \begin{bmatrix} \frac{\partial \underline{\phi}}{\partial x_1}(t, x(t), u(t)) \\ \vdots \\ \frac{\partial \underline{\phi}}{\partial x_n}(t, x(t), u(t)) \end{bmatrix}_{n \times 1}, \\ \nabla_{\#x} \bar{\phi}(t, x(t), u(t)) &= \begin{bmatrix} \frac{\partial \bar{\phi}}{\partial x_1}(t, x(t), u(t)) \\ \vdots \\ \frac{\partial \bar{\phi}}{\partial x_n}(t, x(t), u(t)) \end{bmatrix}_{n \times 1}, \\ \nabla_{\#u} \underline{\phi}(t, x(t), u(t)) &= \begin{bmatrix} \frac{\partial \underline{\phi}}{\partial u_1}(t, x(t), u(t)) \\ \vdots \\ \frac{\partial \underline{\phi}}{\partial u_m}(t, x(t), u(t)) \end{bmatrix}_{m \times 1}, \end{aligned}$$

and

$$\nabla_{\#u} \bar{\phi}(t, x(t), u(t)) = \begin{bmatrix} \frac{\partial \bar{\phi}}{\partial u_1}(t, x(t), u(t)) \\ \vdots \\ \frac{\partial \bar{\phi}}{\partial u_m}(t, x(t), u(t)) \end{bmatrix}_{m \times 1}$$

(herein we are going to see the gradients such as column vectors and not any more such as row vectors, see the Chapter 5).

- (A4) For all bounded x and u , and all $t \in [0, T]$, the partial derivatives $\gamma_x(t, x(t), u(t))$ and $\gamma_u(t, x(t), u(t))$ are measurable (with respect to t), and are bounded, where

$$\gamma_x(t, x(t), u(t)) = \begin{bmatrix} \frac{\partial \gamma_1}{\partial x_1}(t, x(t), u(t)) & \dots & \frac{\partial \gamma_1}{\partial x_n}(t, x(t), u(t)) \\ \vdots & \ddots & \vdots \\ \frac{\partial \gamma_n}{\partial x_1}(t, x(t), u(t)) & \dots & \frac{\partial \gamma_n}{\partial x_n}(t, x(t), u(t)) \end{bmatrix}_{n \times n}$$

and

$$\gamma_u(t, x(t), u(t)) = \begin{bmatrix} \frac{\partial \gamma_1}{\partial u_1}(t, x(t), u(t)) & \dots & \frac{\partial \gamma_1}{\partial u_m}(t, x(t), u(t)) \\ \vdots & \ddots & \vdots \\ \frac{\partial \gamma_n}{\partial u_1}(t, x(t), u(t)) & \dots & \frac{\partial \gamma_n}{\partial u_m}(t, x(t), u(t)) \end{bmatrix}_{n \times m}.$$

- (A5) $\mathcal{M}$ is a closed convex set in $\mathbb{R}^m$ such that $\text{int } \mathcal{M} \neq \emptyset$.

Definition 7.4.2. We give the definitions of directions of decrease, feasible directions, and tangent direction

- (i) We say that a vector $(\bar{x}, \bar{u})$ is a direction of decrease of $F(x, u)$ at (x^*, u^*) if there exists a neighborhood $N \subset E$ of the vector $(\bar{x}, \bar{u})$, a positive scalar ϵ^* , and an interval $[\underline{\alpha}, \bar{\alpha}] <_{LU} [0, 0]$ such that for all ϵ with $0 < \epsilon < \epsilon^*$ and all $(\tilde{x}, \tilde{u}) \in N$, we have that

$$F((x^*, u^*) + \epsilon(\tilde{x}, \tilde{u})) <_{LU} F(x^*, u^*) + \epsilon[\underline{\alpha}, \bar{\alpha}].$$

We denote the set of all directions of decrease at (x^*, u^*) for $F(x, u)$ by $\mathcal{K}_y$ ¹.

- (ii) Given inequality constraints, we say that a vector $(\bar{x}, \bar{u})$ is a feasible direction to Q_1 at (x^*, u^*) if there exists a neighborhood $N \subset E$ of the vector $(\bar{x}, \bar{u})$ and a positive scalar ϵ_1 such that for all ϵ with $0 < \epsilon < \epsilon_1$ and all $(\tilde{x}, \tilde{u}) \in N$, we have that the vectors

$$(x^*, u^*) + \epsilon(\tilde{x}, \tilde{u}) \in Q_1.$$

We denote the set of all feasible directions at (x^*, u^*) for Q_1 by $\mathcal{K}_b$ ².

- (iii) We say that a vector $(\bar{x}, \bar{u})$ is a tangent direction to Q_2 at (x^*, u^*) if there exists $\epsilon_2 > 0$ such that, for all ϵ with $0 < \epsilon < \epsilon_2$, there exists $(x_\epsilon, u_\epsilon) \in Q_2$ such that, if we set $(x_\epsilon, u_\epsilon) := (x^*, u^*) + \epsilon(\bar{x}, \bar{u}) + (r_\epsilon^1, r_\epsilon^2)$, then the vector $(r_\epsilon^1, r_\epsilon^2) \in E$ has the

¹In Russian the initials of “decrease cone” are $\mathcal{K}_y$.

²In Russian the initials of “feasible cone” are $\mathcal{K}_b$.

following property: for any neighborhood $N \subset E$ of zero, we have that

$$\frac{1}{\epsilon} (r_\epsilon^1, r_\epsilon^2) \in N \quad (7.7)$$

for all small $\epsilon > 0$. We denote the set of all tangent directions at (x^*, u^*) for Q_2 by $\mathcal{K}_k$ ³

Remark 7.4.1. 1. In Definition 7.4.2–(i), both $\underline{\alpha}$ and $\bar{\alpha}$ depends on F , (x^*, u^*) , and $(\bar{x}, \bar{u})$.

2. In Definition 7.4.2–(ii), since $C \times L_\infty$ is a Banach space, (7.7) can be replaced by $\|(r_\epsilon^1, r_\epsilon^2)\| = o(\epsilon)$, i.e., by the condition: $\lim_{\epsilon \rightarrow 0} \frac{\|(r_\epsilon^1, r_\epsilon^2)\|_E}{\epsilon} = 0$.

3. Every feasible direction is also a tangent direction, but the converse is false.

Proposition 7.4.1. Let $\mathcal{K}_y$ be the set of all directions of decrease. Then,

(i) $\mathcal{K}_y$ is open.

(iii) $\mathcal{K}_y$ is a cone.

(ii) $\mathcal{K}_y$ is a convex.

Remark 7.4.2. The proof of Proposition 7.4.1, it will be obviously seen later, since later it will be proved that $\mathcal{K}_y = \mathcal{K}_0 = \underline{\mathcal{K}}_0 \cap \bar{\mathcal{K}}_0$ and using the fact that the intersection of open convex cones is also an open convex cone.

Next, it will be included the following two propositions, that are in I. V. Girsanov, [36], for the completeness of the chapter.

Proposition 7.4.2. Let $\mathcal{K}_b$ be the set of all feasible directions. Then,

(i) $\mathcal{K}_b$ is open.

(ii) $\mathcal{K}_b$ is a cone.

Proof. (i) $\mathcal{K}_b \subset \text{int } \mathcal{K}_b$. Indeed, let $(\bar{x}, \bar{u}) \in \mathcal{K}_b$, then there exists a neighborhood $N \subset E$ of $(\bar{x}, \bar{u})$ and a positive scalar ϵ_1 such that for all ϵ with $0 < \epsilon < \epsilon_1$ and all $(\tilde{x}, \tilde{u}) \in N$, we have that $(x^*, u^*) + \epsilon(\tilde{x}, \tilde{u}) \in Q_1$. Then, $N \subset \mathcal{K}_b$. Indeed, let $(\tilde{x}, \tilde{u}) \in N$. It is clear that N is a neighborhood of $(\tilde{x}, \tilde{u})$. Let $(\hat{x}, \hat{u}) \in N$ and $0 < \epsilon < \epsilon_1$. By the fact that $(\bar{x}, \bar{u}) \in \mathcal{K}_b$, we have that

$$(x^*, u^*) + \epsilon(\hat{x}, \hat{u}) \in Q_1.$$

Therefore $(\tilde{x}, \tilde{u}) \in \mathcal{K}_b$. It follows that $\mathcal{K}_b$ is open.

³In Russian the initials of “tangent cone” are $\mathcal{K}_k$.

- (ii) $\mathcal{K}_b$ is a cone, that is, if $(\bar{x}, \bar{u}) \in \mathcal{K}_b$, then $\alpha(\bar{x}, \bar{u}) \in \mathcal{K}_b$, for $\alpha > 0$. Indeed, if $(\bar{x}, \bar{u}) \in \mathcal{K}_b$, there exists a neighborhood $N \subset E$ of $(\bar{x}, \bar{u})$ and a positive scalar ϵ_1 such that for all ϵ with $0 < \epsilon < \epsilon_1$ and all $(\tilde{x}, \tilde{u}) \in N$, we have that

$$(x^*, u^*) + \epsilon(\tilde{x}, \tilde{u}) \in Q_1. \quad (7.8)$$

Let $V := \alpha N \subset E$ be a neighborhood of $\alpha(\bar{x}, \bar{u})$ and $\epsilon'_1 := \frac{\epsilon_1}{\alpha}$, $\alpha > 0$. Given $(\hat{x}, \hat{u}) \in V$, we have $(\hat{x}, \hat{u}) = \alpha(\tilde{x}, \tilde{u})$ for some $(\tilde{x}, \tilde{u}) \in N$. Then, for all $0 < \epsilon < \epsilon'_1$, we have $\alpha\epsilon < \epsilon_1$, and

$$(x^*, u^*) + \epsilon(\hat{x}, \hat{u}) = (x^*, u^*) + \epsilon\alpha(\tilde{x}, \tilde{u}) \in Q_1, \quad \text{by (7.8),}$$

i.e., $\alpha(\bar{x}, \bar{u}) \in \mathcal{K}_b$, for $\alpha > 0$.

□

Proposition 7.4.3. *The set of all tangent directions $\mathcal{K}_k$ is a cone.*

Proof. If $(\bar{x}, \bar{u}) \in \mathcal{K}_k$, then $\alpha(\bar{x}, \bar{u}) \in \mathcal{K}_k$, for $\alpha > 0$. Indeed, if $(\bar{x}, \bar{u}) \in \mathcal{K}_k$, there exists $\epsilon_2 > 0$ such that, for all ϵ with $0 < \epsilon < \epsilon_2$, there exists $(x_\epsilon, u_\epsilon) \in Q_2$ such that, if we set $(x_\epsilon, u_\epsilon) := (x^*, u^*) + \epsilon(\bar{x}, \bar{u}) + (r_\epsilon^1, r_\epsilon^2)$, then $(r_\epsilon^1, r_\epsilon^2) \in E$ has the following property: for any neighborhood $N \subset E$ of zero, we have that $\frac{1}{\epsilon}(r_\epsilon^1, r_\epsilon^2) \in N$ for all small $\epsilon > 0$. Let $\epsilon'_2 := \frac{\epsilon_2}{\alpha}$. Then, for $0 < \sigma < \epsilon'_2$,

$$(x_\sigma, u_\sigma) = (x^*, u^*) + \sigma\alpha(\bar{x}, \bar{u}) + (r_\sigma^1, r_\sigma^2) \in Q_2$$

and

$$\frac{1}{\sigma}(r_\sigma^1, r_\sigma^2) \in N$$

for all neighborhood N of zero and all small σ . It follows that $\alpha(\bar{x}, \bar{u}) \in \mathcal{K}_y$. □

In the following section, it will be described some cones, it will be given some properties of them, and it will be recalled a nondegeneracy condition.

7.5 Characterizations of the cones $\mathcal{K}_0$, $\mathcal{K}_1$ and $\mathcal{K}_2$

Next, it will be considered three cones, it will be characterized them, it will be seen that they are exactly the cone of directions of decrease, the cone of feasible directions, and the cone of tangent directions, see Theorem 7.5.3, Lemma 7.5.1, and Lemma 7.5.2

(or Lemma 7.5.3). Finally, it will be shown that the intersection of the cones initially considered, is empty.

We begin with:

Proposition 7.5.1. *Suppose that (A5) holds. Then, Q_1 is a convex set.*

Proof. Let $(x_1, u_1), (x_2, u_2) \in Q_1$, and $\alpha \in (0, 1)$. Then, $u_1(t), u_2(t) \in \mathcal{M}$, and since $\mathcal{M}$ is a convex set,

$$\alpha u_1(t) + (1 - \alpha)u_2(t) \in \mathcal{M}. \quad (7.9)$$

Therefore,

$$\alpha(x_1, u_1) + (1 - \alpha)(x_2, u_2) = (\alpha x_1 + (1 - \alpha)x_2, \alpha u_1 + (1 - \alpha)u_2) \in Q_1, \quad \text{by (7.9),}$$

that is, Q_1 is convex. $\square$

Now, let $(x^*, u^*) \in Q_1$ and set

$$\begin{aligned} \mathcal{K}_0 &:= \left\{ (\bar{x}, \bar{u}) \in C \times L_\infty \mid \int_0^T \Phi'_{gH}((t, x^*(t), u^*(t)); (\bar{x}(t), \bar{u}(t))) dt <_{LU} [0, 0] \right\}, \\ \mathcal{K}_1 &:= \{ (\bar{x}, \bar{u}) \in C \times L_\infty \mid (\bar{x}, \bar{u}) = \lambda[(x, u) - (x^*, u^*)], \quad (x, u) \in \text{int } Q_1, \text{ and } \lambda > 0 \} \\ &= \{ \lambda[\text{int}(Q_1) - (x^*, u^*)] \mid \lambda > 0 \}, \quad \text{see Theorem 2.11.1,} \end{aligned}$$

consider the space

$$Q_2 = \left\{ (x, u) \in C \times L_\infty \mid x(t) = c + \int_0^t \gamma(\tau, x(\tau), u(\tau)) d\tau, \quad 0 \leq t \leq T, \quad x(T) = d \right\},$$

i.e., x and u satisfy the differential equation

$$x'(t) = \gamma(t, x(t), u(t)), \quad x(0) = c \in \mathbb{R}^n, \quad \text{and} \quad x(T) = d \in \mathbb{R}^n,$$

and set

$$\begin{aligned} \mathcal{K}_2 &:= \left\{ (\bar{x}, \bar{u}) \in C \times L_\infty \mid \bar{x}(t) = \int_0^t [\gamma_x(\tau, x^*(\tau), u^*(\tau))\bar{x}(\tau) + \gamma_u(\tau, x^*(\tau), u^*(\tau))\bar{u}(\tau)] d\tau, \quad \bar{x}(T) = \mathbf{0} \right\} \\ &= \{ (\bar{x}, \bar{u}) \in C \times L_\infty \mid \bar{x}'(t) = \gamma_x(t, x^*(t), u^*(t))\bar{x}(t) + \gamma_u(t, x^*(t), u^*(t))\bar{u}(t), \quad \bar{x}(0) = \bar{x}(T) = \mathbf{0} \}. \end{aligned}$$

Theorem 7.5.1. *If $(\bar{x}, \bar{u}) \in \mathcal{K}_y$ and $\int_0^T \Phi'_{gH}((t, x^*, u^*); (\bar{x}, \bar{u})) dt$ exists, then $(\bar{x}, \bar{u}) \in \mathcal{K}_0$.*

Proof. Let $(\bar{x}, \bar{u}) \in \mathcal{K}_y$. Then, there exist a neighborhood $N \subset E$ of $(\bar{x}, \bar{u})$, a positive

scalar ϵ^* , and an interval

$$[\underline{\alpha}, \bar{\alpha}] <_{LU} [0, 0] \quad (7.10)$$

such that for all ϵ with $0 < \epsilon < \epsilon^*$ and all $(\tilde{x}, \tilde{u}) := (\bar{x}, \bar{u}) \in N$, we have that

$$\begin{aligned} & F((x^*, u^*) + \epsilon(\bar{x}, \bar{u})) <_{LU} F(x^*, u^*) + \epsilon[\underline{\alpha}, \bar{\alpha}] \\ \Leftrightarrow & \int_0^T \Phi(t, x^*(t) + \epsilon\bar{x}(t), u^*(t) + \epsilon\bar{u}(t)) dt <_{LU} \int_0^T \Phi(t, x^*(t), u^*(t)) dt + \epsilon[\underline{\alpha}, \bar{\alpha}]. \end{aligned} \quad (7.11)$$

Affirmation.

$$\int_0^T \lim_{\epsilon \rightarrow 0^+} \frac{1}{\epsilon} \cdot (\Phi(t, x^*(t) + \epsilon\bar{x}(t), u^*(t) + \epsilon\bar{u}(t)) \ominus_{gH} \Phi(t, x^*(t), u^*(t))) dt <_{LU} [0, 0].$$

Indeed, let us set, for $t \in [0, T]$ and $\epsilon \in (0, \epsilon^*)$,

$$A_\epsilon(t) := [\underline{a}_\epsilon(t), \bar{a}_\epsilon(t)] = \Phi(t, x^*(t) + \epsilon\bar{x}(t), u^*(t) + \epsilon\bar{u}(t)),$$

$$B_\epsilon(t) := [\underline{b}_\epsilon(t), \bar{b}_\epsilon(t)] = \Phi(t, x^*(t), u^*(t)),$$

and

$$\begin{aligned} C_\epsilon(t) &:= [\underline{c}_\epsilon(t), \bar{c}_\epsilon(t)] = A_\epsilon(t) \ominus_{gH} B_\epsilon(t) \\ &= \Phi(t, x^*(t) + \epsilon\bar{x}(t), u^*(t) + \epsilon\bar{u}(t)) \ominus_{gH} \Phi(t, x^*(t), u^*(t)). \end{aligned}$$

Since $A_\epsilon(t) \ominus_{gH} B_\epsilon(t) = C_\epsilon(t)$, we have that

$$\begin{aligned} \begin{cases} A_\epsilon(t) = B_\epsilon(t) + C_\epsilon(t) \\ \text{or} \\ B_\epsilon(t) = A_\epsilon(t) - C_\epsilon(t) \end{cases} & \Leftrightarrow \begin{cases} \underline{a}_\epsilon(t) = \underline{b}_\epsilon(t) + \underline{c}_\epsilon(t) \quad \text{and} \quad \bar{a}_\epsilon(t) = \bar{b}_\epsilon(t) + \bar{c}_\epsilon(t) \\ \text{or} \\ \underline{b}_\epsilon(t) = \underline{a}_\epsilon(t) - \bar{c}_\epsilon(t) \quad \text{and} \quad \bar{b}_\epsilon(t) = \bar{a}_\epsilon(t) - \underline{c}_\epsilon(t) \end{cases} \\ & \Leftrightarrow \begin{cases} \underline{c}_\epsilon(t) = \underline{a}_\epsilon(t) - \underline{b}_\epsilon(t) \quad \text{and} \quad \bar{c}_\epsilon(t) = \bar{a}_\epsilon(t) - \bar{b}_\epsilon(t) \\ \text{or} \\ \bar{c}_\epsilon(t) = \underline{a}_\epsilon(t) - \underline{b}_\epsilon(t) \quad \text{and} \quad \underline{c}_\epsilon(t) = \bar{a}_\epsilon(t) - \bar{b}_\epsilon(t). \end{cases} \end{aligned} \quad (7.12)$$

By (7.11), we have that $\int_0^T A_\epsilon(t) dt <_{LU} \int_0^T B_\epsilon(t) dt + \epsilon[\underline{\alpha}, \bar{\alpha}]$, and by Theorem 2.8.2:

$$\begin{aligned} & \left[\int_0^T \underline{a}_\epsilon(t) dt, \int_0^T \bar{a}_\epsilon(t) dt \right] <_{LU} \left[\int_0^T \underline{b}_\epsilon(t) dt, \int_0^T \bar{b}_\epsilon(t) dt \right] + [\epsilon\underline{\alpha}, \epsilon\bar{\alpha}] \\ \Leftrightarrow & \left[\int_0^T \underline{a}_\epsilon(t) dt, \int_0^T \bar{a}_\epsilon(t) dt \right] <_{LU} \left[\int_0^T \underline{b}_\epsilon(t) dt + \epsilon\underline{\alpha}, \int_0^T \bar{b}_\epsilon(t) dt + \epsilon\bar{\alpha} \right], \end{aligned}$$

that is,

$$\begin{aligned} & \int_0^T (\underline{a}_\epsilon(t) - \underline{b}_\epsilon(t))dt < \epsilon \underline{\alpha} \quad \text{and} \quad \int_0^T (\bar{a}_\epsilon(t) - \bar{b}_\epsilon(t))dt < \epsilon \bar{\alpha} \\ \Leftrightarrow & \int_0^T \frac{1}{\epsilon} (\underline{a}_\epsilon(t) - \underline{b}_\epsilon(t))dt < \underline{\alpha} \quad \text{and} \quad \int_0^T \frac{1}{\epsilon} (\bar{a}_\epsilon(t) - \bar{b}_\epsilon(t))dt < \bar{\alpha}. \end{aligned}$$

Let us assume that the first case in (7.12) holds, if the second case occurs, we proceed similarly. Then,

$$\begin{aligned} & \int_0^T \frac{1}{\epsilon} \underline{c}_\epsilon(t)dt < \underline{\alpha} \quad \text{and} \quad \int_0^T \frac{1}{\epsilon} \bar{c}_\epsilon(t)dt < \bar{\alpha} \\ \Rightarrow & \lim_{\epsilon \rightarrow 0^+} \int_0^T \frac{1}{\epsilon} \underline{c}_\epsilon(t)dt < \underline{\alpha} \quad \text{and} \quad \lim_{\epsilon \rightarrow 0^+} \int_0^T \frac{1}{\epsilon} \bar{c}_\epsilon(t)dt < \bar{\alpha}. \end{aligned}$$

Since for all ϵ with $0 < \epsilon < \epsilon^*$, $\{\frac{1}{\epsilon} \underline{c}_\epsilon(t)\}$ and $\{\frac{1}{\epsilon} \bar{c}_\epsilon(t)\}$ are monotone sequences of measurable, continuous, and integrable functions, then by the Generalized Monotone Convergence Theorem ⁴,

$$\begin{aligned} & \int_0^T \lim_{\epsilon \rightarrow 0^+} \frac{1}{\epsilon} \underline{c}_\alpha(t)dt < \underline{\alpha} \quad \text{and} \quad \int_0^T \lim_{\epsilon \rightarrow 0^+} \frac{1}{\epsilon} \bar{c}_\alpha(t)dt < \bar{\alpha} \\ \Leftrightarrow & \left[\int_0^T \lim_{\epsilon \rightarrow 0^+} \frac{1}{\epsilon} \underline{c}_\alpha(t)dt, \int_0^T \lim_{\epsilon \rightarrow 0^+} \frac{1}{\epsilon} \bar{c}_\alpha(t)dt \right] <_{LU} [\underline{\alpha}, \bar{\alpha}] <_{LU} [0, 0], \quad \text{by (7.10)}. \end{aligned}$$

Then,

$$\begin{aligned} & \left[\int_0^T \lim_{\epsilon \rightarrow 0^+} \frac{1}{\epsilon} \underline{c}_\alpha(t)dt, \int_0^T \lim_{\epsilon \rightarrow 0^+} \frac{1}{\epsilon} \bar{c}_\alpha(t)dt \right] <_{LU} [0, 0] \\ \Leftrightarrow & \int_0^T \lim_{\epsilon \rightarrow 0^+} \frac{1}{\epsilon} \cdot C_\alpha(t)dt <_{LU} [0, 0], \quad \text{by Theorem 2.4.1 and by Theorem 2.8.2} \\ \Leftrightarrow & \int_0^T \lim_{\epsilon \rightarrow 0^+} \frac{1}{\epsilon} \cdot (\Phi(t, x^*(t) + \epsilon \bar{x}(t), u^*(t) + \epsilon \bar{u}(t)) \ominus_{gH} \Phi(t, x^*(t), u^*(t)))dt <_{LU} [0, 0] \\ \Leftrightarrow & \int_0^T \Phi'_{gH}((t, x^*(t), u^*(t)); (\bar{x}(t), \bar{u}(t)))dt <_{LU} [0, 0], \quad \text{by Definition 2.7.5,} \end{aligned}$$

that is, $(\bar{x}, \bar{u}) \in \mathcal{K}_0$. □

Now, we establish the converse of Theorem 7.5.1:

Theorem 7.5.2. *Suppose that (A1) holds and that $(\bar{x}, \bar{u}) \in \mathcal{K}_0$. Then, $(\bar{x}, \bar{u}) \in \mathcal{K}_y$.*

⁴**Generalized Monotone Convergence Theorem.** Let $\{f_k\}$ a monotone sequence (increasing or decreasing sequence) of measurable functions. If some of the functions of this sequence is integrable, then the two expressions, $\lim_{k \rightarrow \infty} \int f_k$ and $\int \lim_{k \rightarrow \infty} f_k$, exist and

$$\lim_{k \rightarrow \infty} \int f_k = \int \lim_{k \rightarrow \infty} f_k.$$

Proof. By Definition 2.7.5:

$$\begin{aligned}
& \Phi'_{gH}((t, x^*(t), u^*(t)); (\bar{x}(t), \bar{u}(t))) \\
&= \lim_{\epsilon \rightarrow 0^+} \frac{1}{\epsilon} \cdot (\Phi(t, x^*(t) + \epsilon \bar{x}(t), u^*(t) + \epsilon \bar{u}(t)) \ominus_{gH} \Phi(t, x^*(t), u^*(t))) \\
&=: [-\underline{\sigma}(t), -\bar{\sigma}(t)] \in \mathcal{K}_C, \quad \bar{\sigma}(t) < \underline{\sigma}(t) \text{ for a. e. } t \in [0, T] \\
\\
&\Rightarrow \int_0^T \Phi'_{gH}((t, x^*(t), u^*(t)); (\bar{x}(t), \bar{u}(t))) dt \\
&= \int_0^T \lim_{\epsilon \rightarrow 0^+} \frac{1}{\epsilon} \cdot (\Phi(t, x^*(t) + \epsilon \bar{x}(t), u^*(t) + \epsilon \bar{u}(t)) \ominus_{gH} \Phi(t, x^*(t), u^*(t))) dt \\
&= \int_0^T [-\underline{\sigma}(t), -\bar{\sigma}(t)] dt \\
&= \left[-\int_0^T \underline{\sigma}(t) dt, -\int_0^T \bar{\sigma}(t) dt \right], \quad \text{by Theorem 2.8.2} \\
&= [-\underline{\delta}, -\bar{\delta}] \in \mathcal{K}_C, \quad \bar{\delta} < \underline{\delta}, \quad -\int_0^T \underline{\sigma}(t) dt =: -\underline{\delta} \text{ and } -\int_0^T \bar{\sigma}(t) dt =: -\bar{\delta} \\
\\
&\Rightarrow [0, 0] >_{LU} \int_0^T \Phi'_{gH}((t, x^*(t), u^*(t)); (\bar{x}(t), \bar{u}(t))) dt, \quad \text{since } (\bar{x}, \bar{u}) \in \mathcal{K}_0 \\
&= [-\underline{\delta}, -\bar{\delta}],
\end{aligned}$$

which implies that $[-\underline{\delta}, -\bar{\delta}] <_{LU} [0, 0]$, that is, $\underline{\delta} > 0$ and $\bar{\delta} > 0$.

Since $\Phi(t, x, u)$ satisfy the Lipschitz continuous condition with respect to (x, u) in a neighborhood of (x^*, u^*) , there exists $L : [0, T] \rightarrow \mathbb{R}$ a $\mathcal{L}$ -measurable integrable function with $L(t) > 0$ for a. e. $t \in [0, T]$, and there exists $\epsilon^* > 0$ such that

$$d_H(\Phi(t, x_1, u_1), \Phi(t, x_2, u_2)) \leq L(t) \|(x_1, u_1) - (x_2, u_2)\|_{\mathbb{R}^{n+m}} \quad (7.13)$$

for all (x_1, u_1) and (x_2, u_2) such that $\|(x_1, u_1) - (x^*, u^*)\|_{\mathbb{R}^{n+m}} < \epsilon^*$ and $\|(x_2, u_2) - (x^*, u^*)\|_{\mathbb{R}^{n+m}} < \epsilon^*$ for a. e. $t \in [0, T]$.

Let $(\tilde{x}, \tilde{u}) \in C \times L_\infty$ be arbitrary such that

$$\|(\tilde{x}, \tilde{u}) - (\bar{x}, \bar{u})\|_{C \times L_\infty} < \frac{1}{4 \int_0^T L(t) dt} \bar{\delta}, \quad (7.14)$$

that is, $(\tilde{x}, \tilde{u}) \in \mathbb{B}_{C \times L_\infty} \left((\bar{x}, \bar{u}), \frac{1}{4 \int_0^T L(t) dt} \bar{\delta} \right)$.

By definition of limit, given $\epsilon' \in \mathbb{R}$ with $0 < \epsilon' < \frac{1}{2T} \bar{\delta}$, there exists $\delta' > 0$ such that

$$d_H \left(\frac{1}{\epsilon} \cdot (\Phi(t, x^*(t) + \epsilon \bar{x}(t), u^*(t) + \epsilon \bar{u}(t)) \ominus_{gH} \Phi(t, x^*(t), u^*(t))), [-\underline{\sigma}(t), -\bar{\sigma}(t)] \right) < \epsilon'$$

for all $\epsilon \in (0, \delta')$ and for all $t \in [0, T]$. By setting

$$\frac{1}{\epsilon} \cdot (\Phi(t, x^*(t) + \epsilon \bar{x}(t), u^*(t) + \epsilon \bar{u}(t)) \ominus_{gH} \Phi(t, x^*(t), u^*(t))) =: K_\epsilon(t) = [\underline{k}_\epsilon(t), \bar{k}_\epsilon(t)],$$

where $\epsilon \in (0, \delta')$ and $t \in [0, T]$, we have that

$$\begin{aligned} \max\{|\underline{k}_\epsilon(t) + \underline{\sigma}(t)|, |\bar{k}_\epsilon(t) + \bar{\sigma}(t)|\} &= d_H(K_\epsilon(t), [-\underline{\sigma}(t), -\bar{\sigma}(t)]) < \epsilon' \\ \Rightarrow \underline{k}_\epsilon(t) + \underline{\sigma}(t) &< |\underline{k}_\epsilon(t) + \underline{\sigma}(t)| < \epsilon' \quad \text{and} \quad \bar{k}_\epsilon(t) + \bar{\sigma}(t) < |\bar{k}_\epsilon(t) + \bar{\sigma}(t)| < \epsilon' \\ \Rightarrow \underline{k}_\epsilon(t) &< \epsilon' - \underline{\sigma}(t) \quad \text{and} \quad \bar{k}_\epsilon(t) < \epsilon' - \bar{\sigma}(t) \\ \Rightarrow \begin{cases} \int_0^T \underline{k}_\epsilon(t) dt < \int_0^T \epsilon' dt - \int_0^T \underline{\sigma}(t) dt = \epsilon' T - \underline{\delta} < \frac{1}{2T} \bar{\delta} T - \underline{\delta} < \frac{1}{2} \underline{\delta} - \underline{\delta} = -\frac{1}{2} \underline{\delta} \\ \text{and} \\ \int_0^T \bar{k}_\epsilon(t) dt < \int_0^T \epsilon' dt - \int_0^T \bar{\sigma}(t) dt = \epsilon' T - \bar{\delta} < \frac{1}{2T} \bar{\delta} T - \bar{\delta} = -\frac{1}{2} \bar{\delta} \end{cases} \\ \Leftrightarrow \int_0^T K_\epsilon(t) dt &= \int_0^T [\underline{k}_\epsilon(t), \bar{k}_\epsilon(t)] dt = \left[\int_0^T \underline{k}_\epsilon(t) dt, \int_0^T \bar{k}_\epsilon(t) dt \right], \quad \text{by Theorem 2.8.2} \\ &<_{LU} \left[-\frac{1}{2} \underline{\delta}, -\frac{1}{2} \bar{\delta} \right] \\ \Leftrightarrow \int_0^T \frac{1}{\epsilon} \cdot (\Phi(t, x^*(t) + \epsilon \bar{x}(t), u^*(t) + \epsilon \bar{u}(t)) \ominus_{gH} \Phi(t, x^*(t), u^*(t))) dt &<_{LU} \left[-\frac{1}{2} \underline{\delta}, -\frac{1}{2} \bar{\delta} \right] \\ \Leftrightarrow \int_0^T \Phi(t, x^*(t) + \epsilon \bar{x}(t), u^*(t) + \epsilon \bar{u}(t)) \ominus_{gH} \Phi(t, x^*(t), u^*(t)) dt & \\ &<_{LU} \epsilon \left[-\frac{1}{2} \underline{\delta}, -\frac{1}{2} \bar{\delta} \right], \end{aligned} \tag{7.15}$$

$\epsilon \in (0, \delta')$. Whenever $0 < \epsilon < \hat{\epsilon}$, where $\hat{\epsilon} < \min \left\{ \frac{4\epsilon^* \int_0^T L(t) dt}{\bar{\delta} + 4\|(\bar{x}, \bar{u})\|_{C \times L_\infty} \int_0^T L(t) dt}, \frac{\epsilon^*}{\|(\bar{x}, \bar{u})\|_{C \times L_\infty}} \right\}$, we obtain that

$$\begin{aligned} \|((x^*, u^*) + \epsilon(\tilde{x}, \tilde{u})) - (x^*, u^*)\|_{C \times L_\infty} &= \epsilon \|(\tilde{x}, \tilde{u}) - (\bar{x}, \bar{u}) + (\bar{x}, \bar{u})\|_{C \times L_\infty} \\ &\leq \epsilon (\|(\tilde{x}, \tilde{u}) - (\bar{x}, \bar{u})\|_{C \times L_\infty} + \|(\bar{x}, \bar{u})\|_{C \times L_\infty}) \\ &< \frac{4\epsilon^* \int_0^T L(t) dt}{\bar{\delta} + 4\|(\bar{x}, \bar{u})\|_{C \times L_\infty} \int_0^T L(t) dt} \left(\frac{1}{4 \int_0^T L(t) dt} \bar{\delta} + \|(\bar{x}, \bar{u})\|_{C \times L_\infty} \right), \quad \text{by (7.14)} \\ &= \epsilon^* \end{aligned} \tag{7.16}$$

and

$$\begin{aligned}
\|((x^*, u^*) + \epsilon(\bar{x}, \bar{u})) - (x^*, u^*)\|_{C \times L} &= \epsilon \|(\bar{x}, \bar{u})\|_{C \times L_\infty} \\
&< \frac{\epsilon^*}{\|(\bar{x}, \bar{u})\|_{C \times L_\infty}} \|(\bar{x}, \bar{u})\|_{C \times L_\infty} \\
&= \epsilon^*.
\end{aligned} \tag{7.17}$$

Now, since it holds (7.16) and (7.17), by (7.13)

$$\begin{aligned}
&d_H(\Phi(t, x^*(t) + \epsilon\tilde{x}(t), u^*(t) + \epsilon\tilde{u}(t)), \Phi(t, x^*(t) + \epsilon\bar{x}(t), u^*(t) + \epsilon\bar{u}(t))) \\
&\leq L(t) \|(x^*(t) + \epsilon\tilde{x}(t), u^*(t) + \epsilon\tilde{u}(t)) - (x^*(t) + \epsilon\bar{x}(t), u^*(t) + \epsilon\bar{u}(t))\|_{\mathbb{R}^{n+m}}
\end{aligned} \tag{7.18}$$

for all $0 < \epsilon < \hat{\epsilon}$ and for a. e. $t \in [0, T]$. Then, we have, by Proposition 2.3.2, for all ϵ and for all t that

$$\begin{aligned}
&\Phi(t, x^*(t) + \epsilon\tilde{x}(t), u^*(t) + \epsilon\tilde{u}(t)) \ominus_{gH} \Phi(t, x^*(t) + \epsilon\bar{x}(t), u^*(t) + \epsilon\bar{u}(t)) \\
&\leq_{LU} [d_H(\Phi(t, x^*(t) + \epsilon\tilde{x}(t), u^*(t) + \epsilon\tilde{u}(t)), \Phi(t, x^*(t) + \epsilon\bar{x}(t), u^*(t) + \epsilon\bar{u}(t))), \\
&\quad d_H(\Phi(t, x^*(t) + \epsilon\tilde{x}(t), u^*(t) + \epsilon\tilde{u}(t)), \Phi(t, x^*(t) + \epsilon\bar{x}(t), u^*(t) + \epsilon\bar{u}(t)))] \\
&\leq_{LU} [L(t) \|(x^*(t) + \epsilon\tilde{x}(t), u^*(t) + \epsilon\tilde{u}(t)) - (x^*(t) + \epsilon\bar{x}(t), u^*(t) + \epsilon\bar{u}(t))\|_{\mathbb{R}^{n+m}}, \\
&\quad L(t) \|(x^*(t) + \epsilon\tilde{x}(t), u^*(t) + \epsilon\tilde{u}(t)) - (x^*(t) + \epsilon\bar{x}(t), u^*(t) + \epsilon\bar{u}(t))\|_{\mathbb{R}^{n+m}}], \quad \text{by (7.18)} \\
&= [L(t)\epsilon \|(\tilde{x}(t), \tilde{u}(t)) - (\bar{x}(t), \bar{u}(t))\|_{\mathbb{R}^{n+m}}, L(t)\epsilon \|(\tilde{x}(t), \tilde{u}(t)) - (\bar{x}(t), \bar{u}(t))\|_{\mathbb{R}^{n+m}}] \\
&<_{LU} \left[L(t)\epsilon \frac{1}{4 \int_0^T L(t)dt} \bar{\delta}, L(t)\epsilon \frac{1}{4 \int_0^T L(t)dt} \bar{\delta} \right], \quad \text{by (7.14),} \\
&= \epsilon \left[\frac{L(t)}{4 \int_0^T L(t)dt} \bar{\delta}, \frac{L(t)}{4 \int_0^T L(t)dt} \bar{\delta} \right]
\end{aligned}$$

i.e.,

$$\begin{aligned}
&\Phi(t, x^*(t) + \epsilon\tilde{x}(t), u^*(t) + \epsilon\tilde{u}(t)) \ominus_{gH} \Phi(t, x^*(t) + \epsilon\bar{x}(t), u^*(t) + \epsilon\bar{u}(t)) \\
&<_{LU} \epsilon \left[\frac{L(t)}{4 \int_0^T L(t)dt} \bar{\delta}, \frac{L(t)}{4 \int_0^T L(t)dt} \bar{\delta} \right],
\end{aligned}$$

then

$$\begin{aligned}
& \int_0^T \Phi(t, x^*(t) + \epsilon \tilde{x}(t), u^*(t) + \epsilon \tilde{u}(t)) \ominus_{gH} \Phi(t, x^*(t) + \epsilon \bar{x}(t), u^*(t) + \epsilon \bar{u}(t)) dt \\
& \leq_{LU} \epsilon \int_0^T \left[\frac{L(t)}{4 \int_0^T L(t) dt} \bar{\delta}, \frac{L(t)}{4 \int_0^T L(t) dt} \bar{\delta} \right] dt \\
& = \epsilon \left[\frac{\int_0^T L(t) dt}{4 \int_0^T L(t) dt} \bar{\delta}, \frac{\int_0^T L(t) dt}{4 \int_0^T L(t) dt} \bar{\delta} \right], \quad \text{by Theorem 2.8.2} \\
& = \epsilon \left[\frac{1}{4} \bar{\delta}, \frac{1}{4} \bar{\delta} \right],
\end{aligned}$$

i.e.,

$$\begin{aligned}
& \int_0^T \Phi(t, x^*(t) + \epsilon \tilde{x}(t), u^*(t) + \epsilon \tilde{u}(t)) \ominus_{gH} \Phi(t, x^*(t) + \epsilon \bar{x}(t), u^*(t) + \epsilon \bar{u}(t)) dt \\
& \leq_{LU} \epsilon \left[\frac{1}{4} \bar{\delta}, \frac{1}{4} \bar{\delta} \right]
\end{aligned} \tag{7.19}$$

for all $0 < \epsilon < \hat{\epsilon}$.

Affirmation. Given $0 < \epsilon < (\epsilon^*)'$, where $(\epsilon^*)' := \min \{\delta', \hat{\epsilon}\}$,

$$\int_0^T \Phi(t, x^*(t) + \epsilon \tilde{x}(t), u^*(t) + \epsilon \tilde{u}(t)) dt \leq_{LU} \int_0^T \Phi(t, x^*(t), u^*(t)) dt + \epsilon [\underline{\alpha}', \bar{\alpha}'],$$

with $\underline{\alpha}' = \bar{\alpha}' = -\frac{1}{4} \bar{\delta}$.

Indeed, let us set, for $t \in [0, T]$ and $\epsilon \in (0, \epsilon^*)$,

$$\begin{aligned}
A_\epsilon(t) &:= [\underline{a}_\epsilon(t), \bar{a}_\epsilon(t)] = \Phi(t, x^*(t) + \epsilon \bar{x}(t), u^*(t) + \epsilon \bar{u}(t)), \\
B_\epsilon(t) &:= [\underline{b}_\epsilon(t), \bar{b}_\epsilon(t)] = \Phi(t, x^*(t), u^*(t)), \\
C_\epsilon(t) &:= [\underline{c}_\epsilon(t), \bar{c}_\epsilon(t)] = A_\epsilon(t) \ominus_{gH} B_\epsilon(t) \\
&= \Phi(t, x^*(t) + \epsilon \bar{x}(t), u^*(t) + \epsilon \bar{u}(t)) \ominus_{gH} \Phi(t, x^*(t), u^*(t)), \\
D_\epsilon(t) &:= [\underline{d}_\epsilon(t), \bar{d}_\epsilon(t)] = \Phi(t, x^*(t) + \epsilon \tilde{x}(t), u^*(t) + \epsilon \tilde{u}(t)),
\end{aligned}$$

and

$$\begin{aligned}
E_\epsilon(t) &:= [\underline{e}_\epsilon(t), \bar{e}_\epsilon(t)] = D_\epsilon(t) \ominus_{gH} A_\epsilon(t) \\
&= \Phi(t, x^*(t) + \epsilon \tilde{x}(t), u^*(t) + \epsilon \tilde{u}(t)) \ominus_{gH} \Phi(t, x^*(t) + \epsilon \bar{x}(t), u^*(t) + \epsilon \bar{u}(t)).
\end{aligned}$$

Since $A_\epsilon(t) \ominus_{gH} B_\epsilon(t) = C_\epsilon(t)$, by (7.15), we have that

$$\left\{ \begin{array}{l} A_\epsilon(t) = B_\epsilon(t) + C_\epsilon(t) \\ \text{or} \\ B_\epsilon(t) = A_\epsilon(t) - C_\epsilon(t), \\ \text{with} \\ \int_0^T C_\epsilon(t) dt <_{LU} \left[-\frac{1}{2}\epsilon\bar{\delta}, -\frac{1}{2}\epsilon\bar{\delta} \right] \end{array} \right\} \Leftrightarrow \left\{ \begin{array}{l} [\underline{a}_\epsilon(t), \bar{a}_\epsilon(t)] = [\underline{b}_\epsilon(t), \bar{b}_\epsilon(t)] + [\underline{c}_\epsilon(t), \bar{c}_\epsilon(t)] \\ \text{or} \\ [\underline{b}_\epsilon(t), \bar{b}_\epsilon(t)] = [\underline{a}_\epsilon(t), \bar{a}_\epsilon(t)] + [-\bar{c}_\epsilon(t), -\underline{c}_\epsilon(t)], \\ \text{with} \\ \int_0^T [\underline{c}_\epsilon(t), \bar{c}_\epsilon(t)] dt <_{LU} \left[-\frac{1}{2}\epsilon\bar{\delta}, -\frac{1}{2}\epsilon\bar{\delta} \right] \end{array} \right.$$

and since $D_\epsilon(t) \ominus_{gH} A_\epsilon(t) = E_\epsilon(t)$, by (7.19), we have that

$$\left\{ \begin{array}{l} D_\epsilon(t) = A_\epsilon(t) + E_\epsilon(t) \\ \text{or} \\ A_\epsilon(t) = D_\epsilon(t) - E_\epsilon(t), \\ \text{with} \\ \int_0^T E_\epsilon(t) dt <_{LU} \left[\frac{1}{4}\epsilon\bar{\delta}, \frac{1}{4}\epsilon\bar{\delta} \right] \end{array} \right\} \Leftrightarrow \left\{ \begin{array}{l} [\underline{d}_\epsilon(t), \bar{d}_\epsilon(t)] = [\underline{a}_\epsilon(t), \bar{a}_\epsilon(t)] + [\underline{e}_\epsilon(t), \bar{e}_\epsilon(t)] \\ \text{or} \\ [\underline{a}_\epsilon(t), \bar{a}_\epsilon(t)] = [\underline{d}_\epsilon(t), \bar{d}_\epsilon(t)] + [-\bar{e}_\epsilon(t), -\underline{e}_\epsilon(t)], \\ \text{with} \\ \int_0^T [\underline{e}_\epsilon(t), \bar{e}_\epsilon(t)] dt <_{LU} \left[\frac{1}{4}\epsilon\bar{\delta}, \frac{1}{4}\epsilon\bar{\delta} \right]. \end{array} \right.$$

We have one of the following cases:

$$(I) \quad [\underline{a}_\epsilon(t), \bar{a}_\epsilon(t)] = [\underline{b}_\epsilon(t), \bar{b}_\epsilon(t)] + [\underline{c}_\epsilon(t), \bar{c}_\epsilon(t)] \text{ and } [\underline{d}_\epsilon(t), \bar{d}_\epsilon(t)] = [\underline{a}_\epsilon(t), \bar{a}_\epsilon(t)] + [\underline{e}_\epsilon(t), \bar{e}_\epsilon(t)],$$

with $\int_0^T [\underline{c}_\epsilon(t), \bar{c}_\epsilon(t)] dt <_{LU} \left[-\frac{1}{2}\epsilon\bar{\delta}, -\frac{1}{2}\epsilon\bar{\delta} \right]$ and $\int_0^T [\underline{e}_\epsilon(t), \bar{e}_\epsilon(t)] dt <_{LU} \left[\frac{1}{4}\epsilon\bar{\delta}, \frac{1}{4}\epsilon\bar{\delta} \right]$,

$$\begin{aligned} \Rightarrow \int_0^T [\underline{d}_\epsilon(t), \bar{d}_\epsilon(t)] dt &= \int_0^T [\underline{b}_\epsilon(t), \bar{b}_\epsilon(t)] dt + \int_0^T [\underline{c}_\epsilon(t), \bar{c}_\epsilon(t)] dt + \int_0^T [\underline{e}_\epsilon(t), \bar{e}_\epsilon(t)] dt \\ &<_{LU} \int_0^T [\underline{b}_\epsilon(t), \bar{b}_\epsilon(t)] dt + \left[-\frac{1}{2}\epsilon\bar{\delta}, -\frac{1}{2}\epsilon\bar{\delta} \right] + \left[\frac{1}{4}\epsilon\bar{\delta}, \frac{1}{4}\epsilon\bar{\delta} \right] \\ &= \int_0^T [\underline{b}_\epsilon(t), \bar{b}_\epsilon(t)] dt + \left[-\frac{1}{2}\epsilon\bar{\delta} + \frac{1}{4}\epsilon\bar{\delta}, -\frac{1}{2}\epsilon\bar{\delta} + \frac{1}{4}\epsilon\bar{\delta} \right] \\ &\leq_{LU} \int_0^T [\underline{b}_\epsilon(t), \bar{b}_\epsilon(t)] dt + \left[-\frac{1}{2}\epsilon\bar{\delta} + \frac{1}{4}\epsilon\bar{\delta}, -\frac{1}{2}\epsilon\bar{\delta} + \frac{1}{4}\epsilon\bar{\delta} \right] \\ &= \int_0^T [\underline{b}_\epsilon(t), \bar{b}_\epsilon(t)] dt + \epsilon \left[-\frac{1}{4}\bar{\delta}, -\frac{1}{4}\bar{\delta} \right]. \end{aligned}$$

$$(II) \quad [\underline{a}_\epsilon(t), \bar{a}_\epsilon(t)] = [\underline{b}_\epsilon(t), \bar{b}_\epsilon(t)] + [\underline{c}_\epsilon(t), \bar{c}_\epsilon(t)] \text{ and } [\underline{a}_\epsilon(t), \bar{a}_\epsilon(t)] = [\underline{d}_\epsilon(t), \bar{d}_\epsilon(t)] + [-\bar{e}_\epsilon(t), -\underline{e}_\epsilon(t)],$$

with $\int_0^T [\underline{c}_\epsilon(t), \bar{c}_\epsilon(t)] dt <_{LU} \left[-\frac{1}{2}\epsilon\bar{\delta}, -\frac{1}{2}\epsilon\bar{\delta} \right]$ and $\int_0^T [\underline{e}_\epsilon(t), \bar{e}_\epsilon(t)] dt <_{LU} \left[\frac{1}{4}\epsilon\bar{\delta}, \frac{1}{4}\epsilon\bar{\delta} \right]$, so

$$\begin{aligned} [\underline{d}_\epsilon(t), \bar{d}_\epsilon(t)] + [-\bar{e}_\epsilon(t), -\underline{e}_\epsilon(t)] &= [\underline{b}_\epsilon(t), \bar{b}_\epsilon(t)] + [\underline{c}_\epsilon(t), \bar{c}_\epsilon(t)] \\ \Leftrightarrow [\underline{d}_\epsilon(t) - \bar{e}_\epsilon(t), \bar{d}_\epsilon(t) - \underline{e}_\epsilon(t)] &= [\underline{b}_\epsilon(t) + \underline{c}_\epsilon(t), \bar{b}_\epsilon(t) + \bar{c}_\epsilon(t)]. \end{aligned}$$

$$\begin{aligned}
& \Rightarrow \left\{ \begin{aligned} & \int_0^T \underline{d}_\epsilon(t) dt - \int_0^T \bar{e}_\epsilon(t) dt = \int_0^T \underline{b}_\epsilon(t) dt + \int_0^T \underline{c}_\epsilon(t) dt \\ & \Rightarrow \int_0^T \underline{d}_\epsilon(t) dt = \int_0^T \underline{b}_\epsilon(t) dt + \int_0^T \underline{c}_\epsilon(t) dt + \int_0^T \bar{e}_\epsilon(t) dt \\ & < \int_0^T \underline{b}_\epsilon(t) dt - \frac{1}{2}\epsilon\bar{\delta} + \frac{1}{4}\epsilon\bar{\delta} < \int_0^T \underline{b}_\epsilon(t) dt - \frac{1}{2}\epsilon\bar{\delta} + \frac{1}{4}\epsilon\bar{\delta} = \int_0^T \underline{b}_\epsilon(t) dt - \frac{1}{4}\epsilon\bar{\delta} \end{aligned} \right. \\
& \Rightarrow \left\{ \begin{aligned} & \text{and} \\ & \int_0^T \bar{d}_\epsilon(t) dt - \int_0^T \underline{e}_\epsilon(t) dt = \int_0^T \bar{b}_\epsilon(t) dt + \int_0^T \bar{c}_\epsilon(t) dt \\ & \Rightarrow \int_0^T \bar{d}_\epsilon(t) dt = \int_0^T \bar{b}_\epsilon(t) dt + \int_0^T \bar{c}_\epsilon(t) dt + \int_0^T \underline{e}_\epsilon(t) dt \\ & < \int_0^T \bar{b}_\epsilon(t) dt - \frac{1}{2}\epsilon\bar{\delta} + \frac{1}{4}\epsilon\bar{\delta} < \int_0^T \bar{b}_\epsilon(t) dt - \frac{1}{4}\epsilon\bar{\delta} \end{aligned} \right. \\
& \Rightarrow \left\{ \begin{aligned} & \int_0^T \underline{d}_\epsilon(t) dt < \int_0^T \underline{b}_\epsilon(t) dt - \frac{1}{4}\epsilon\bar{\delta} \\ & \text{and} \\ & \int_0^T \bar{d}_\epsilon(t) dt < \int_0^T \bar{b}_\epsilon(t) dt - \frac{1}{4}\epsilon\bar{\delta} \end{aligned} \right. \\
& \Leftrightarrow \int_0^T [\underline{d}_\epsilon(t), \bar{d}_\epsilon(t)] dt <_{LU} \left[\int_0^T \underline{b}_\epsilon(t) dt - \frac{1}{4}\epsilon\bar{\delta}, \int_0^T \bar{b}_\epsilon(t) dt - \frac{1}{4}\epsilon\bar{\delta} \right] \\
& = \int_0^T [\underline{b}_\epsilon(t), \bar{b}_\epsilon(t)] dt + \epsilon \left[-\frac{1}{4}\bar{\delta}, -\frac{1}{4}\bar{\delta} \right], \quad \text{by Theorem 2.8.2.}
\end{aligned}$$

$$\begin{aligned}
\text{(III)} \quad & [\underline{b}_\epsilon(t), \bar{b}_\epsilon(t)] = [\underline{a}_\epsilon(t), \bar{a}_\epsilon(t)] + [-\bar{c}_\epsilon(t), -\underline{c}_\epsilon(t)] \text{ and } [\underline{d}_\epsilon(t), \bar{d}_\epsilon(t)] = [\underline{a}_\epsilon(t), \bar{a}_\epsilon(t)] + [\underline{e}_\epsilon(t), \bar{e}_\epsilon(t)], \\
& \text{with } \int_0^T [\underline{c}_\epsilon(t), \bar{c}_\epsilon(t)] dt <_{LU} \left[-\frac{1}{2}\epsilon\bar{\delta}, -\frac{1}{2}\epsilon\bar{\delta} \right] \text{ and } \int_0^T [\underline{e}_\epsilon(t), \bar{e}_\epsilon(t)] dt <_{LU} \left[\frac{1}{4}\epsilon\bar{\delta}, \frac{1}{4}\epsilon\bar{\delta} \right], \\
& \Rightarrow [\underline{b}_\epsilon(t), \bar{b}_\epsilon(t)] = [\underline{a}_\epsilon(t) - \bar{c}_\epsilon(t), \bar{a}_\epsilon(t) - \underline{c}_\epsilon(t)] \text{ and } [\underline{d}_\epsilon(t), \bar{d}_\epsilon(t)] = [\underline{a}_\epsilon(t) + \underline{e}_\epsilon(t), \bar{a}_\epsilon(t) + \bar{e}_\epsilon(t)],
\end{aligned}$$

that is,

$$\underline{b}_\epsilon(t) = \underline{a}_\epsilon(t) - \bar{c}_\epsilon(t) \Rightarrow \underline{a}_\epsilon(t) = \underline{b}_\epsilon(t) + \bar{c}_\epsilon(t)$$

$$\bar{b}_\epsilon(t) = \bar{a}_\epsilon(t) - \underline{c}_\epsilon(t) \Rightarrow \bar{a}_\epsilon(t) = \bar{b}_\epsilon(t) + \underline{c}_\epsilon(t)$$

$$\underline{d}_\epsilon(t) = \underline{a}_\epsilon(t) + \underline{e}_\epsilon(t),$$

and

$$\bar{d}_\epsilon(t) = \bar{a}_\epsilon(t) + \bar{e}_\epsilon(t)$$

$$\begin{aligned}
& \Rightarrow \begin{cases} \int_0^T \underline{d}_\epsilon(t) dt = \int_0^T \underline{b}_\epsilon(t) dt + \int_0^T \bar{c}_\epsilon(t) dt + \int_0^T \underline{e}_\epsilon(t) dt < \int_0^T \underline{b}_\epsilon(t) dt - \frac{1}{2}\epsilon\bar{\delta} + \frac{1}{4}\epsilon\bar{\delta} \\ = \int_0^T \underline{b}_\epsilon(t) dt - \frac{1}{4}\epsilon\bar{\delta} \end{cases} \\
& \Rightarrow \begin{cases} \text{and} \\ \int_0^T \bar{d}_\epsilon(t) dt = \int_0^T \bar{b}_\epsilon(t) dt + \int_0^T \underline{c}_\epsilon(t) dt + \int_0^T \bar{e}_\epsilon(t) dt < \int_0^T \bar{b}_\epsilon(t) dt - \frac{1}{2}\epsilon\bar{\delta} + \frac{1}{4}\epsilon\bar{\delta} \\ < \int_0^T \bar{b}_\epsilon(t) dt - \frac{1}{2}\epsilon\bar{\delta} + \frac{1}{4}\epsilon\bar{\delta} = \int_0^T \bar{b}_\epsilon(t) dt - \frac{1}{4}\epsilon\bar{\delta} \end{cases} \\
& \Rightarrow \begin{cases} \int_0^T \underline{d}_\epsilon(t) dt < \int_0^T \underline{b}_\epsilon(t) dt - \frac{1}{4}\epsilon\bar{\delta} \\ \text{and} \\ \int_0^T \bar{d}_\epsilon(t) dt < \int_0^T \bar{b}_\epsilon(t) dt - \frac{1}{4}\epsilon\bar{\delta} \end{cases} \\
& \Leftrightarrow \int_0^T [\underline{d}_\epsilon(t), \bar{d}_\epsilon(t)] dt <_{LU} \left[\int_0^T \underline{b}_\epsilon(t) dt - \frac{1}{4}\epsilon\bar{\delta}, \int_0^T \bar{b}_\epsilon(t) dt - \frac{1}{4}\epsilon\bar{\delta} \right] \\
& = \int_0^T [\underline{b}_\epsilon(t), \bar{b}_\epsilon(t)] dt + \epsilon \left[-\frac{1}{4}\bar{\delta}, -\frac{1}{4}\bar{\delta} \right], \quad \text{by Theorem 2.8.2.}
\end{aligned}$$

(IV) $[\underline{b}_\epsilon(t), \bar{b}_\epsilon(t)] = [\underline{a}_\epsilon(t), \bar{a}_\epsilon(t)] + [-\bar{c}_\epsilon(t), -\underline{c}_\epsilon(t)]$ and $[\underline{a}_\epsilon(t), \bar{a}_\epsilon(t)] = [\underline{d}_\epsilon(t), \bar{d}_\epsilon(t)] + [-\bar{e}_\epsilon(t), -\underline{e}_\epsilon(t)]$,
with $\int_0^T [\underline{c}_\epsilon(t), \bar{c}_\epsilon(t)] dt <_{LU} \left[-\frac{1}{2}\epsilon\bar{\delta}, -\frac{1}{2}\epsilon\bar{\delta} \right]$ and $\int_0^T [\underline{e}_\epsilon(t), \bar{e}_\epsilon(t)] dt <_{LU} \left[\frac{1}{4}\epsilon\bar{\delta}, \frac{1}{4}\epsilon\bar{\delta} \right]$, so

$$\begin{aligned}
[\underline{b}_\epsilon(t), \bar{b}_\epsilon(t)] &= [\underline{d}_\epsilon(t), \bar{d}_\epsilon(t)] + [-\bar{e}_\epsilon(t), -\underline{e}_\epsilon(t)] + [-\bar{c}_\epsilon(t), -\underline{c}_\epsilon(t)] \\
&= [\underline{d}_\epsilon(t) - \bar{e}_\epsilon(t) - \bar{c}_\epsilon(t), \bar{d}_\epsilon(t) - \underline{e}_\epsilon(t) - \underline{c}_\epsilon(t)]
\end{aligned}$$

$$\begin{aligned}
& \Rightarrow \begin{cases} \int_0^T \underline{b}_\epsilon(t) dt = \int_0^T \underline{d}_\epsilon(t) dt - \int_0^T \bar{e}_\epsilon(t) dt - \int_0^T \bar{c}_\epsilon(t) dt \\ \Rightarrow \int_0^T \underline{d}_\epsilon(t) dt = \int_0^T \underline{b}_\epsilon(t) dt + \int_0^T \bar{c}_\epsilon(t) dt + \int_0^T \bar{e}_\epsilon(t) dt < \int_0^T \underline{b}_\epsilon(t) dt - \frac{1}{2}\epsilon\bar{\delta} + \frac{1}{4}\epsilon\bar{\delta} \\ = \int_0^T \underline{b}_\epsilon(t) dt - \frac{1}{4}\epsilon\bar{\delta} \end{cases} \\
& \Rightarrow \begin{cases} \text{and} \\ \int_0^T \bar{b}_\epsilon(t) dt = \int_0^T \bar{d}_\epsilon(t) dt - \int_0^T \underline{e}_\epsilon(t) dt - \int_0^T \underline{c}_\epsilon(t) dt \\ \Rightarrow \int_0^T \bar{d}_\epsilon(t) dt = \int_0^T \bar{b}_\epsilon(t) dt + \int_0^T \underline{c}_\epsilon(t) dt + \int_0^T \underline{e}_\epsilon(t) dt \\ < \int_0^T \bar{b}_\epsilon(t) dt - \frac{1}{2}\epsilon\bar{\delta} + \frac{1}{4}\epsilon\bar{\delta} < \int_0^T \bar{b}_\epsilon(t) dt - \frac{1}{2}\epsilon\bar{\delta} + \frac{1}{4}\epsilon\bar{\delta} = \int_0^T \bar{b}_\epsilon(t) dt - \frac{1}{4}\epsilon\bar{\delta} \end{cases}
\end{aligned}$$

$$\begin{aligned}
& \Rightarrow \begin{cases} \int_0^T \underline{d}_\epsilon(t) dt < \int_0^T \underline{b}_\epsilon(t) dt - \frac{1}{4}\epsilon\bar{\delta} \\ \text{and} \\ \int_0^T \bar{d}_\epsilon(t) dt < \int_0^T \bar{b}_\epsilon(t) dt - \frac{1}{4}\epsilon\bar{\delta} \end{cases} \\
& \Leftrightarrow \int_0^T [\underline{d}_\epsilon(t), \bar{d}_\epsilon(t)] dt <_{LU} \left[\int_0^T \underline{b}_\epsilon(t) dt - \frac{1}{4}\epsilon\bar{\delta}, \int_0^T \bar{b}_\epsilon(t) dt - \frac{1}{4}\epsilon\bar{\delta} \right] \\
& = \int_0^T [\underline{b}_\epsilon(t), \bar{b}_\epsilon(t)] dt + \epsilon \left[-\frac{1}{4}\bar{\delta}, -\frac{1}{4}\bar{\delta} \right], \quad \text{by Theorem 2.8.2.}
\end{aligned}$$

In all of the cases, (I), (II), (III), and (IV), we always have $\int_0^T [\underline{d}_\epsilon(t), \bar{d}_\epsilon(t)] dt <_{LU} \int_0^T [\underline{b}_\epsilon(t), \bar{b}_\epsilon(t)] dt + \epsilon[\underline{\alpha}', \bar{\alpha}']$, with $\underline{\alpha}' = \bar{\alpha}' = -\frac{1}{4}\bar{\delta}$, that is,

$$\begin{aligned}
& \int_0^T D_\epsilon(t) dt <_{LU} \int_0^T B_\epsilon(t) dt + \epsilon[\underline{\alpha}', \bar{\alpha}'] \\
& \Leftrightarrow \int_0^T \Phi(t, x^*(t) + \epsilon\tilde{x}(t), u^*(t) + \epsilon\tilde{u}(t)) dt <_{LU} \int_0^T \Phi(t, x^*(t), u^*(t)) dt + \epsilon[\underline{\alpha}', \bar{\alpha}'] \\
& \Leftrightarrow F((x^*, u^*) + \epsilon(\tilde{x}, \tilde{u})) <_{LU} F(x^*, u^*) + \epsilon[\underline{\alpha}', \bar{\alpha}'],
\end{aligned}$$

with $\underline{\alpha}' = \bar{\alpha}' = -\frac{1}{4}\bar{\delta} < 0$, since $\bar{\delta} > 0$.

Therefore, there exists a neighborhood $N =: \mathbb{B}_{C \times L^\infty} \left((\bar{x}, \bar{u}), \frac{1}{4 \int_0^T L(t) dt} \bar{\delta} \right)$ of $(\bar{x}, \bar{u})$, a positive scalar $(\epsilon^*)' = \min\{\delta', \hat{\epsilon}\}$, and an interval

$$[\underline{\alpha}', \bar{\alpha}'] = \left[-\frac{1}{4}\bar{\delta}, -\frac{1}{4}\bar{\delta} \right] <_{LU} [0, 0]$$

such that for all ϵ with $0 < \epsilon < (\epsilon^*)'$, and all $(\tilde{x}, \tilde{u}) \in N = \mathbb{B}_{C \times L^\infty} \left((\bar{x}, \bar{u}), \frac{1}{4 \int_0^T L(t) dt} \bar{\delta} \right)$, we have that

$$F((x^*, u^*) + \epsilon(\tilde{x}, \tilde{u})) <_{LU} F(x^*, u^*) + \epsilon[\underline{\alpha}', \bar{\alpha}'], \quad (7.20)$$

that is, by Definition 7.4.2–(i), $(\bar{x}, \bar{u}) \in \mathcal{K}_y$. $\square$

Therefore, by Theorems 7.5.1 and 7.5.2, we have:

Theorem 7.5.3. *Suppose that (A1) holds. Then, the cone of directions of decrease $\mathcal{K}_y$ at (x^*, u^*) is given by $\mathcal{K}_y = \mathcal{K}_0$.*

The next result is in I. V. Girsanov, [36], but he did not do the proof, he just said that the proof follows directly from the definitions.

Lemma 7.5.1. *Suppose that (A5) holds. Then, the cone of feasible directions $\mathcal{K}_b$ at (x^*, u^*) is given by $\mathcal{K}_b = \mathcal{K}_1$.*

Proof. First, we are going to prove that $\mathcal{K}_b \subseteq \mathcal{K}_1$. Let $(\bar{x}, \bar{u}) \in \mathcal{K}_b$, by Definition 7.4.2–(ii), there exists a neighborhood $N \subset E$ of $(\bar{x}, \bar{u})$ and a positive scalar ϵ_1 such that for all ϵ with $0 < \epsilon < \epsilon_1$ and all $(\tilde{x}, \tilde{u}) \in N$, we have that $(x^*, u^*) + \epsilon(\tilde{x}, \tilde{u}) \in Q_1$. It is clear that $(x^*, u^*) + \epsilon N$ is a neighborhood of $(x^*, u^*) + \epsilon(\bar{x}, \bar{u})$, $0 < \epsilon < \epsilon_1$, and that $(x^*, u^*) + \epsilon N \subseteq Q_1$, $0 < \epsilon < \epsilon_1$, that is,

$$(x^*, u^*) + \epsilon(\bar{x}, \bar{u}) \in \text{int } Q_1. \quad (7.21)$$

So $(\bar{x}, \bar{u}) = \frac{1}{\epsilon}[(x, u) - (x^*, u^*)]$, where $(x, u) := (x^*, u^*) + \epsilon(\bar{x}, \bar{u}) \in \text{int } Q_1$, by (7.21), and set $\lambda := \frac{1}{\epsilon} > 0$. Therefore,

$$\mathcal{K}_b \subseteq \{(\bar{x}, \bar{u}) \in E \mid (\bar{x}, \bar{u}) = \lambda[(x, u) - (x^*, u^*)], \quad (x, u) \in \text{int } Q_1, \quad \text{and} \quad \lambda > 0\} = \mathcal{K}_1.$$

Now, we are going to prove that $\mathcal{K}_1 \subseteq \mathcal{K}_b$. Let $(\bar{x}, \bar{u}) \in \mathcal{K}_1$, then

$$(\bar{x}, \bar{u}) = \lambda[(x, u) - (x^*, u^*)], \quad (x, u) \in \text{int } Q_1, \quad \text{and} \quad \lambda > 0.$$

Since $(x, u) \in \text{int } Q_1$, there exists V a neighborhood of (x, u) such that $(x, u) \in V \subseteq Q_1$. Set $N := \lambda[V - (x^*, u^*)]$ and $\epsilon_1 := \frac{1}{\lambda} > 0$, then N is a neighborhood of $(\bar{x}, \bar{u})$, since V is a neighborhood of (x, u) , such that for all ϵ with $0 < \epsilon < \frac{1}{\lambda} = \epsilon_1$ and all $(\tilde{x}, \tilde{u}) \in N$, we have that

$$\begin{aligned} (x^*, u^*) + \epsilon(\tilde{x}, \tilde{u}) &= (x^*, u^*) + \epsilon\lambda[v - (x^*, u^*)], \quad v \in V \subseteq Q_1 \\ &= (\epsilon\lambda)v + (1 - \epsilon\lambda)(x^*, u^*), \quad \text{with } 0 < \epsilon < \epsilon_1 \quad \text{and} \quad \epsilon\lambda + (1 - \epsilon\lambda) = 1 \\ &\in Q_1, \quad \text{by Proposition 7.5.1.} \end{aligned}$$

Then, there exists a neighborhood $N = \lambda[V - (x^*, u^*)]$ of $(\bar{x}, \bar{u})$ and a positive scalar ϵ_1 such that for all ϵ with $0 < \epsilon < \epsilon_1 = \frac{1}{\lambda}$ and all $(\tilde{x}, \tilde{u}) \in N$, we have that

$$(x^*, u^*) + \epsilon(\tilde{x}, \tilde{u}) \in Q_1,$$

that is, by Definition 7.4.2–(ii), $(\bar{x}, \bar{u}) \in \mathcal{K}_b$. □

Definition 7.5.1 (Nondegeneracy Condition 1, see I. V. Girsanov, [36]). *Let $D \subset \mathbb{R}^n$ be the set of all vectors $\bar{x}(T)$, where $\bar{x}(T)$ satisfies the differential equation*

$$\bar{x}'(t) = A(t)\bar{x}(t) + B(t)\bar{u}(t), \quad \bar{x}(0) = \mathbf{0}, \quad (7.22)$$

with $\bar{u}(t)$ ranging over L_∞ , $A(t)$ and $B(t)$ are $n \times n$ and $n \times m$ -matrices, respectively,

$\bar{u}(t) \in \mathbb{R}^m$ and $\bar{x}(t) \in \mathbb{R}^n$. This differential equation is said to be nondegenerate of type 1 or completely controllable if $D = \mathbb{R}^n$.

Next, it will be included the following two lemmas, that are in I. V. Girsanov, [36], for the completeness of the chapter.

Lemma 7.5.2. *Suppose that (A2) and (A4) hold and that the system satisfies the Non-degeneracy Condition 1. Then, the tangent subspace $\mathcal{K}_k$ at (x^*, u^*) is given by*

$$\begin{aligned} \mathcal{K}_k &= \{(\bar{x}, \bar{u}) \in C \times L_\infty \mid \bar{x}(t) = \int_0^t [\gamma_x(\tau, x^*(\tau), u^*(\tau))\bar{x}(\tau) + \gamma_u(\tau, x^*(\tau), u^*(\tau))\bar{u}(\tau)] d\tau, \\ &\quad t \in [0, T], \quad \bar{x}(T) = \mathbf{0}\} \\ &= \{(\bar{x}, \bar{u}) \in C \times L_\infty \mid \bar{x}'(t) = \gamma_x(t, x^*(t), u^*(t))\bar{x}(t) + \gamma_u(t, x^*(t), u^*(t))\bar{u}(t), \\ &\quad t \in [0, T], \quad \bar{x}(0) = \bar{x}(T) = \mathbf{0}\} \\ &= \mathcal{K}_2. \end{aligned} \tag{7.23}$$

Proof. Consider the spaces $E = C \times L_\infty$, $E_2 = C \times \mathbb{R}^n$ and $Q_2 = \left\{ (x, u) \in E \mid x(t) = c + \int_0^t \gamma(\tau, x(\tau), u(\tau)) d\tau, \quad t \in [0, T], \quad x(T) = d \right\}$.

Introduce the operator

$$\begin{aligned} P : E &\rightarrow E_2 \\ (x, u) &\mapsto P(x, u) := (\tilde{P}(x, u), x(T) - d) \in E_2, \end{aligned}$$

$\tilde{P}(x, u)(t) = x(t) - c - \int_0^t \gamma(\tau, x(\tau), u(\tau)) d\tau$. Then, $Q_2 = \{(x, u) \in E \mid P(x, u) = (0, \mathbf{0})\}$. Since $\gamma(t, x(t), u(t))$ is continuously differentiable in x, u , by the Mean Value inequality for $\gamma(t, \cdot, \cdot)$ with $t \in [0, T]$ fixed, there exist $\theta_1(t), \theta_2(t) \in (0, 1)$ such that

$$\begin{aligned} &\gamma(\tau, x(\tau) + \bar{x}(\tau), u(\tau) + \bar{u}(\tau)) - \gamma(\tau, x(\tau), u(\tau)) \\ &\quad = \gamma_x(\tau, x(\tau) + \theta_1(\tau)\bar{x}(\tau), u(\tau))\bar{x}(\tau) + \gamma_u(\tau, x(\tau) + \theta_2(\tau)\bar{u}(\tau))\bar{u}(\tau) \\ &\Rightarrow \int_0^t [\gamma(\tau, x(\tau) + \bar{x}(\tau), u(\tau) + \bar{u}(\tau)) - \gamma(\tau, x(\tau), u(\tau))] d\tau \\ &= \int_0^t [\gamma_x(\tau, x(\tau) + \theta_1(\tau)\bar{x}(\tau), u(\tau))\bar{x}(\tau) + \gamma_u(\tau, x(\tau) + \theta_2(\tau)\bar{u}(\tau))\bar{u}(\tau)] d\tau \\ &\quad + \int_0^t [\gamma_x(\tau, x(\tau), u(\tau))\bar{x}(\tau) + \gamma_u(\tau, x(\tau), u(\tau))\bar{u}(\tau)] d\tau \\ &\quad - \int_0^t [\gamma_x(\tau, x(\tau), u(\tau))\bar{x}(\tau) + \gamma_u(\tau, x(\tau), u(\tau))\bar{u}(\tau)] d\tau \end{aligned}$$

$$\begin{aligned}
&= \int_0^t [\gamma_x(\tau, x(\tau), u(\tau))\bar{x}(\tau) + \gamma_u(\tau, x(\tau), u(\tau))\bar{u}(\tau)] d\tau \\
&\quad + \int_0^t [\gamma_x(\tau, x(\tau) + \theta_1(\tau)\bar{x}(\tau), u(\tau)) - \gamma_x(\tau, x(\tau), u(\tau))] \bar{x}(\tau) d\tau \\
&\quad + \int_0^t [\gamma_u(\tau, x(\tau), u(\tau) + \theta_2(\tau)\bar{u}(\tau)) - \gamma_u(\tau, x(\tau), u(\tau))] \bar{u}(\tau) d\tau \\
&= \int_0^t [\gamma_x(\tau, x(\tau), u(\tau))\bar{x}(\tau) + \gamma_u(\tau, x(\tau), u(\tau))\bar{u}(\tau)] d\tau + \delta'(t), \tag{7.24}
\end{aligned}$$

where

$$\begin{aligned}
\delta'(t) &:= \int_0^t [\gamma_x(\tau, x(\tau) + \theta_1(\tau)\bar{x}(\tau), u(\tau)) - \gamma_x(\tau, x(\tau), u(\tau))] \bar{x}(\tau) d\tau \\
&\quad + \int_0^t [\gamma_u(\tau, x(\tau), u(\tau) + \theta_2(\tau)\bar{u}(\tau)) - \gamma_u(\tau, x(\tau), u(\tau))] \bar{u}(\tau) d\tau.
\end{aligned}$$

Therefore,

$$\begin{aligned}
&P(x + \bar{x}, u + \bar{u}) - P(x, u) \\
&= \left(x(t) + \bar{x}(t) - c - \int_0^t \gamma(\tau, x(\tau) + \bar{x}(\tau), u(\tau) + \bar{u}(\tau)) d\tau, x(T) + \bar{x}(T) - d \right) \\
&\quad + \left(-x(t) + c + \int_0^t \gamma(\tau, x(\tau), u(\tau)) d\tau, -x(T) + d \right) \\
&= (x(t) + \bar{x}(t) - c - \int_0^t \gamma(\tau, x(\tau) + \bar{x}(\tau), u(\tau) + \bar{u}(\tau)) d\tau - x(t) + c \\
&\quad + \int_0^t \gamma(\tau, x(\tau), u(\tau)) d\tau, x(T) + \bar{x}(T) - d - x(T) + d) \\
&= \left(\bar{x}(t) - \int_0^t [\gamma(\tau, x(\tau) + \bar{x}(\tau), u(\tau) + \bar{u}(\tau)) - \gamma(\tau, x(\tau), u(\tau))] d\tau, \bar{x}(T) \right) \\
&= \left(\bar{x}(t) - \int_0^t [\gamma_x(\tau, x(\tau), u(\tau))\bar{x}(\tau) + \gamma_u(\tau, x(\tau), u(\tau))\bar{u}(\tau)] d\tau - \delta'(t), \bar{x}(T) \right), \quad \text{by (7.24)} \\
&= \left(\bar{x}(t) - \int_0^t [\gamma_x(\tau, x(\tau), u(\tau))\bar{x}(\tau) + \gamma_u(\tau, x(\tau), u(\tau))\bar{u}(\tau)] d\tau, \bar{x}(T) \right) + \delta(t), \tag{7.25}
\end{aligned}$$

where

$$\delta(t) := (-\delta'(t), 0) = \left(\int_0^t o(\|(\bar{x}(\tau), \bar{u}(\tau))\|_{\mathbb{R}^n \times \mathbb{R}^m}) d\tau, 0 \right)$$

is a remainder term for which an estimative is: $\delta(t) = \left(o\left(\sqrt{\|\bar{x}\|_C^2 + \|\bar{u}\|_{L_\infty}^2}\right), 0 \right)$.

Indeed,

$$\begin{aligned}
\|(\bar{x}(\tau), \bar{u}(\tau))\|_{\mathbb{R}^n \times \mathbb{R}^m} &= \sqrt{\|\bar{x}(\tau)\|_{\mathbb{R}^n}^2 + \|\bar{u}(\tau)\|_{\mathbb{R}^m}^2} \\
&\leq \sqrt{\|\bar{x}\|_C^2 + \|\bar{u}\|_{L_\infty}^2} \\
&= \|(\bar{x}, \bar{u})\|_{C \times L_\infty} \\
&\Rightarrow \frac{1}{\|(\bar{x}, \bar{u})\|_{C \times L_\infty}} \leq \frac{1}{\|(\bar{x}(\tau), \bar{u}(\tau))\|_{\mathbb{R}^n \times \mathbb{R}^m}} \\
&\Rightarrow \frac{o(\|(\bar{x}(\tau), \bar{u}(\tau))\|_{\mathbb{R}^n \times \mathbb{R}^m})}{\|(\bar{x}, \bar{u})\|_{C \times L_\infty}} \leq \frac{o(\|(\bar{x}(\tau), \bar{u}(\tau))\|_{\mathbb{R}^n \times \mathbb{R}^m})}{\|(\bar{x}(\tau), \bar{u}(\tau))\|_{\mathbb{R}^n \times \mathbb{R}^m}}.
\end{aligned} \tag{7.26}$$

Moreover, in (7.26), $\|(\bar{x}, \bar{u})\|_{C \times L_\infty} \rightarrow 0$, implies that $\|(\bar{x}(\tau), \bar{u}(\tau))\|_{\mathbb{R}^n \times \mathbb{R}^m} \rightarrow 0$. Therefore, since $\gamma(t, x, u)$ is differentiable at (x, u) , we have that

$$\frac{o(\|(\bar{x}(\tau), \bar{u}(\tau))\|_{\mathbb{R}^n \times \mathbb{R}^m})}{\|(\bar{x}, \bar{u})\|_{C \times L_\infty}} \rightarrow 0. \tag{7.27}$$

Then, by (7.27) we have that

$$\frac{\int_0^t o(\|(\bar{x}(\tau), \bar{u}(\tau))\|_{\mathbb{R}^n \times \mathbb{R}^m}) d\tau}{\|(\bar{x}, \bar{u})\|_{C \times L_\infty}} = \int_0^t \frac{o(\|(\bar{x}(\tau), \bar{u}(\tau))\|_{\mathbb{R}^n \times \mathbb{R}^m})}{\|(\bar{x}, \bar{u})\|_{C \times L_\infty}} d\tau \rightarrow 0.$$

Therefore, $\delta(t) = (o(\|(\bar{x}, \bar{u})\|_{C \times L_\infty}), 0) = \left(o\left(\sqrt{\|\bar{x}\|_C^2 + \|\bar{u}\|_{L_\infty}^2}\right), 0\right)$, and by (7.25) note that:

$$(\bar{x}, \bar{u}) \mapsto \left(\bar{x}(t) - \int_0^t [\gamma_x(\tau, x(\tau), u(\tau))\bar{x}(\tau) + \gamma_u(\tau, x(\tau), u(\tau))\bar{u}(\tau)] d\tau, \bar{x}(T)\right)$$

is a linear operator from E to E_2 . Therefore,

$$P'(x, u)(\bar{x}, \bar{u}) = \left(\bar{x}(t) - \int_0^t [\gamma_x(\tau, x(\tau), u(\tau))\bar{x}(\tau) + \gamma_u(\tau, x(\tau), u(\tau))\bar{u}(\tau)] d\tau, \bar{x}(T)\right). \tag{7.28}$$

Then, $P(x, u)$ is an operator mapping E to E_2 , differentiable in a neighborhood of a point (x, u) such that $P(x, u) = (0, \mathbf{0})$.

Since we are assuming that the derivatives $\gamma_x(t, x, u)$ and $\gamma_u(t, x, u)$ are bounded, $P'(x, u)$ is bounded, so, $P'(x, u)$ is continuous in a neighborhood of (x, u) , that is, $P(x, u)$ is continuously differentiable.

Now, let any $(a(t), b) \in E_2$. Take $z(t) \in C$ such that

$$\begin{aligned}
z(t) &= a(t) + \int_0^t \gamma_x(\tau, x(\tau), u(\tau))z(\tau) d\tau \\
\Leftrightarrow a(t) &= z(t) - \int_0^t \gamma_x(\tau, x(\tau), u(\tau))z(\tau) d\tau, \quad 0 \leq t \leq T.
\end{aligned}$$

Since the system is nondegenerate, by Definition 7.5.1,

$$D = \{\bar{x}(T) \mid \bar{x}'(t) = \gamma_x(t, x(t), u(t))\bar{x}(t) + \gamma_u(t, x(t), u(t))\bar{u}(t), \quad \bar{x}(0) = \mathbf{0}\} = \mathbb{R}^n,$$

then we can find $\bar{u}(t) \in L_\infty$ and $y(t) \in C$ such that

$$y'(t) = \gamma_x(t, x(t), u(t))y(t) + \gamma_u(t, x(t), u(t))\bar{u}(t), \quad y(0) = \mathbf{0}, \quad y(T) = b - z(T).$$

By setting $\bar{x}(t) := y(t) + z(t)$, we have that $\bar{x}(T) = y(T) + z(T) = b - z(T) + z(T) = b$, then at (7.28), we have that

$$\begin{aligned} P'(x, u)(\bar{x}, \bar{u}) &= \left(y(t) + z(t) - \int_0^t [\gamma_x(\tau, x(\tau), u(\tau))(y(\tau) + z(\tau)) + \gamma_u(\tau, x(\tau), u(\tau))\bar{u}(\tau)] d\tau, b \right) \\ &= \left(y(t) + z(t) - \int_0^t \gamma_x(\tau, x(\tau), u(\tau))z(\tau) d\tau - \int_0^t [\gamma_x(\tau, x(\tau), u(\tau))y(\tau) + \gamma_u(\tau, x(\tau), u(\tau))\bar{u}(\tau)] d\tau, b \right) \\ &= \left(y(t) + \left[z(t) - \int_0^t \gamma_x(\tau, x(\tau), u(\tau))z(\tau) d\tau \right] - \int_0^t [y'(\tau)] d\tau, b \right) \\ &= (y(t) + a(t) - [y(t) - y(0)], b) \\ &= (a(t), b), \quad \text{since } y(0) = \mathbf{0}. \end{aligned}$$

Then, the linear equation $P'(x, u)(\bar{x}, \bar{u}) = (a(t), b)$ has a solution $(\bar{x}, \bar{u})$ for any $(a(t), b) \in E_2$, i.e., the linear operator $P'(x, u)$ maps E onto E_2 .

Therefore, $P(x, u)$ is an operator mapping E into E_2 , differentiable in a neighborhood of a point (x^*, u^*) such that $P(x^*, u^*) = (0, 0)$, $P'(x, u)$ is continuous in a neighborhood of (x^*, u^*) and $P'(x^*, u^*)$ maps E onto E_2 (i.e., the linear equation $P'(x^*, u^*)(\bar{x}, \bar{u}) = (w_1, w_2)$ has a solution $(\bar{x}, \bar{u})$ for any $(w_1, w_2) \in E_2$). Then, by Lyusternik Theorem, see Theorem 2.11.2, the set of tangent direction $\mathcal{K}_k$ to the set $Q_2 = \{(x, u) \in E \mid P(x, u) = (0, \mathbf{0})\}$ at (x^*, u^*) is the subspace

$$\begin{aligned} \mathcal{K}_k &= \{(\bar{x}, \bar{u}) \in C \times L_\infty \mid P'(x^*, u^*)(\bar{x}, \bar{u}) = (0, \mathbf{0})\} \\ &= \{(\bar{x}, \bar{u}) \in C \times L_\infty \mid \bar{x}(t) = \int_0^t [\gamma_x(\tau, x^*(\tau), u^*(\tau))\bar{x}(\tau) + \gamma_u(\tau, x^*(\tau), u^*(\tau))\bar{u}(\tau)] d\tau, \\ &\quad t \in [0, T], \quad \bar{x}(T) = \mathbf{0}\} \\ &= \{(\bar{x}, \bar{u}) \in C \times L_\infty \mid \bar{x}'(t) = \gamma_x(t, x^*(t), u^*(t))\bar{x}(t) + \gamma_u(t, x^*(t), u^*(t))\bar{u}(t), \\ &\quad t \in [0, T], \quad \bar{x}(0) = \bar{x}(T) = \mathbf{0}\} \\ &= \mathcal{K}_2. \end{aligned}$$

□

Lemma 7.5.3. Suppose that (A2) and (A4) hold. If the left endpoint condition (7.4) is

fixed as before, $x(0) = c \in \mathbb{R}^n$, but if the right endpoint condition (7.5) is free, $x(T) \in \mathbb{R}^n$. Then, the tangent subspace $\mathcal{K}_k$ at (x^*, u^*) is given by

$$\begin{aligned} \mathcal{K}_k &= \{(\bar{x}, \bar{u}) \in C \times L_\infty \mid \bar{x}(t) = \int_0^t [\gamma_x(\tau, x^*(\tau), u^*(\tau))\bar{x}(\tau) + \gamma_u(\tau, x^*(\tau), u^*(\tau))\bar{u}(\tau)] d\tau, \\ &\quad t \in [0, T]\} \\ &= \{(\bar{x}, \bar{u}) \in C \times L_\infty \mid \bar{x}'(t) = \gamma_x(t, x^*(t), u^*(t))\bar{x}(t) + \gamma_u(t, x^*(t), u^*(t))\bar{u}(t), \\ &\quad t \in [0, T], \quad \bar{x}(0) = \mathbf{0}\}. \end{aligned}$$

Proof. Consider the spaces $E = C \times L_\infty$, C and $\widetilde{Q}_2 = \left\{ (x, u) \in E \mid x(t) = c + \int_0^t \gamma(\tau, x(\tau), u(\tau)) d\tau, \quad t \in [0, T] \right\}$.

Introduce the operator

$$\begin{aligned} \widetilde{P} : E &\rightarrow C \\ (x, u) &\mapsto \widetilde{P}(x, u)(t) := x(t) - c - \int_0^t \gamma(\tau, x(\tau), u(\tau)) d\tau \in C. \end{aligned}$$

Then $\widetilde{Q}_2 = \left\{ (x, u) \in E \mid \widetilde{P}(x, u) = 0 \right\}$.

In an analogous way to the last proof, we can obtain

$$\widetilde{P}'(x, u)(\bar{x}, \bar{u}) = \bar{x}(t) - \int_0^t [\gamma_x(\tau, x(\tau), u(\tau))\bar{x}(\tau) + \gamma_u(\tau, x(\tau), u(\tau))\bar{u}(\tau)] d\tau.$$

Then, $\widetilde{P}(x, u)$ is an operator mapping E to C , differentiable in a neighborhood of a point (x, u) such that $\widetilde{P}(x, u) = 0$.

Since we are assuming that the derivatives $\gamma_x(t, x, u)$ and $\gamma_u(t, x, u)$ are bounded, $\widetilde{P}'(x, u)$ is continuous in a neighborhood of (x, u) , that is, $\widetilde{P}(x, u)$ is continuously differentiable.

The linear operator $\widetilde{P}'(x, u)$ maps E onto C , that is, for any $a(t) \in C$, there exists $(\bar{x}, \bar{u}) \in E$ such that

$$\begin{aligned} \widetilde{P}'(x, u)(\bar{x}, \bar{u}) &= a(t) \\ \Leftrightarrow \bar{x}(t) - \int_0^t [\gamma_x(\tau, x(\tau), u(\tau))\bar{x}(\tau) + \gamma_u(\tau, x(\tau), u(\tau))\bar{u}(\tau)] d\tau &= a(t). \end{aligned}$$

Setting $\bar{u}(t) \equiv 0$, we have the integral equation:

$$\bar{x}(t) = a(t) + \int_0^t [\gamma_x(\tau, x(\tau), u(\tau))\bar{x}(\tau)] d\tau, \quad (7.29)$$

which is a *linear Volterra Integral Equation of Second Kind*, see Equation (2.17) of Sub-

section 2.9.2. The theory of integral equations says that for any $a(t) \in C$, Equation (7.29) has a unique solution $\bar{x}(t) \in C$, see L. V. Kantorovich and G. P. Akilov, [43].

Then, the linear equation $\tilde{P}'(x, u)(\bar{x}, \bar{u}) = a(t)$ has a solution $(\bar{x}, \bar{u})$ for any $a(t) \in C$, i.e., the linear operator $\tilde{P}'(x, u)$ maps E onto $\tilde{E}_2$.

Therefore, $\tilde{P}(x, u)$ is an operator mapping E into C , differentiable in a neighborhood of a point (x^*, u^*) such that $\tilde{P}(x^*, u^*) = 0$, $\tilde{P}'(x, u)$ is continuous in a neighborhood of (x^*, u^*) and $\tilde{P}'(x^*, u^*)$ maps E onto C (i.e., the linear equation $\tilde{P}'(x^*, u^*)(\bar{x}, \bar{u}) = w$ has a solution $(\bar{x}, \bar{u})$ for any $w \in C$), then by Lyusternik Theorem, see Theorem 2.11.2, the set of tangent direction $\mathcal{K}_k$ to the set $\widetilde{Q}_2 = \{(x, u) \in E \mid \tilde{P}(x, u) = 0\}$ at (x^*, u^*) is the subspace

$$\begin{aligned} \mathcal{K}_k &= \{(\bar{x}, \bar{u}) \in C \times L_\infty \mid \tilde{P}'(x^*, u^*)(\bar{x}, \bar{u}) = 0\} \\ &= \{(\bar{x}, \bar{u}) \in C \times L_\infty \mid \bar{x}(t) = \int_0^t [\gamma_x(\tau, x^*(\tau), u^*(\tau))\bar{x}(\tau) + \gamma_u(\tau, x^*(\tau), u^*(\tau))\bar{u}(\tau)] d\tau, \\ &\quad t \in [0, T]\} \\ &= \{(\bar{x}, \bar{u}) \in C \times L_\infty \mid \bar{x}'(t) = \gamma_x(t, x^*(t), u^*(t))\bar{x}(t) + \gamma_u(t, x^*(t), u^*(t))\bar{u}(t), \\ &\quad t \in [0, T], \quad \bar{x}(0) = \mathbf{0}\} \\ &=: \widetilde{\mathcal{K}}_2. \end{aligned} \tag{7.30}$$

□

Theorem 7.5.4. *Let $(x^*, u^*) \in Q$ be a weak optimal LU-process of (LIOCP). Assume that (A1), (A2), (A4), (A5) hold, and that the Nondegeneracy Condition 1 holds for System (7.22) with $A(t) = \gamma_x(t, x(t), u(t))$ and $B(t) = \gamma_u(t, x(t), u(t))$. Then,*

$$\mathcal{K}_0 \cap \mathcal{K}_1 \cap \mathcal{K}_2 = \emptyset.$$

Proof. Suppose that there exists $(\bar{x}, \bar{u}) \in \mathcal{K}_0 \cap \mathcal{K}_1 \cap \mathcal{K}_2$. Then, $(\bar{x}, \bar{u}) \in \mathcal{K}_0 = \mathcal{K}_y$, by Theorem 7.5.3, that is, $(\bar{x}, \bar{u})$ is a direction of decrease of $F(x, u)$ at (x^*, u^*) , so by Definition 7.4.2–(i), there exists a neighborhood N_0 of $(\bar{x}, \bar{u})$, a positive scalar ϵ_0 , and an interval $[\underline{\alpha}, \bar{\alpha}] <_{LU} [0, 0]$ such that for all ϵ with $0 < \epsilon < \epsilon_0$ and all $(\tilde{x}, \tilde{u}) \in N_0$, we have that

$$F((x^*, u^*) + \epsilon(\tilde{x}, \tilde{u})) <_{LU} F(x^*, u^*) + \epsilon[\underline{\alpha}, \bar{\alpha}]. \tag{7.31}$$

We have also that $(\bar{x}, \bar{u}) \in \mathcal{K}_1 = \mathcal{K}_b$, by Lemma 7.5.1, that is, $(\bar{x}, \bar{u})$ is a feasible direction to Q_1 at (x^*, u^*) , so by Definition 7.4.2–(ii), there exists a neighborhood N_1 of $(\bar{x}, \bar{u})$ and a positive scalar ϵ_1 such that for all ϵ with $0 < \epsilon < \epsilon_1$ and all $(\tilde{x}, \tilde{u}) \in N_1$, we have that

$$(x^*, u^*) + \epsilon(\tilde{x}, \tilde{u}) \in Q_1. \quad (7.32)$$

We have that $(\bar{x}, \bar{u}) \in \mathcal{K}_2 = \mathcal{K}_k$, by Lemma 7.5.2, that is, $(\bar{x}, \bar{u})$ is a tangent direction to Q_2 at (x^*, u^*) , so by Definition 7.4.2–(iii), there exists $\epsilon_2 > 0$ such that, for all ϵ with $0 < \epsilon < \epsilon_2$, there exists $(x_\epsilon, u_\epsilon) \in Q_2$ such that, if we set $(x_\epsilon, u_\epsilon) := (x^*, u^*) + \epsilon(\bar{x}, \bar{u}) + (r_\epsilon^1, r_\epsilon^2) \in Q_2$, then $(r_\epsilon^1, r_\epsilon^2) \in E$ has the following property: for any neighborhood, let say N_1 , of zero, we have that

$$\frac{1}{\epsilon} (r_\epsilon^1, r_\epsilon^2) \in N_1$$

for all small $\epsilon > 0$.

Let ϵ_3 be such that

$$\begin{aligned} & \frac{1}{\epsilon} (r_\epsilon^1, r_\epsilon^2) \in N_1 - (\bar{x}, \bar{u}), \quad \text{for } 0 < \epsilon < \epsilon_3 \\ \Leftrightarrow (\tilde{x}_\epsilon, \tilde{u}_\epsilon) &:= (\bar{x}, \bar{u}) + \frac{1}{\epsilon} (r_\epsilon^1, r_\epsilon^2) \in N_1, \quad \text{for } 0 < \epsilon < \epsilon_3. \end{aligned} \quad (7.33)$$

Then, whenever $0 < \epsilon < \min\{\epsilon_1, \epsilon_2, \epsilon_3\}$, since there exists a neighborhood N_1 of $(\bar{x}, \bar{u})$, a positive scalar ϵ_1 such that for all ϵ with $0 < \epsilon < \epsilon_1$ and $(\tilde{x}_\epsilon, \tilde{u}_\epsilon) \in N_1$:

$$\begin{aligned} & (x^*, u^*) + \epsilon(\tilde{x}_\epsilon, \tilde{u}_\epsilon) \in Q_1, \quad \text{by using (7.32)} \\ &= (x^*, u^*) + \epsilon \left[(\bar{x}, \bar{u}) + \frac{1}{\epsilon} (r_\epsilon^1, r_\epsilon^2) \right], \quad \text{by (7.33)} \\ &= (x^*, u^*) + \epsilon(\bar{x}, \bar{u}) + (r_\epsilon^1, r_\epsilon^2) \\ &= (x_\epsilon, u_\epsilon) \in Q_2, \end{aligned}$$

that is, $(x^*, u^*) + \epsilon(\tilde{x}_\epsilon, \tilde{u}_\epsilon) = (x_\epsilon, u_\epsilon) \in Q = Q_1 \cap Q_2$.

Setting $N := N_0 \cap N_1$ and $\epsilon_4 := \min\{\epsilon_0, \epsilon_1, \epsilon_2, \epsilon_3\}$, since there exists a neighborhood N_0 of $(\bar{x}, \bar{u})$, a positive scalar ϵ_0 , and an interval $[\underline{\alpha}, \bar{\alpha}] <_{LU} [0, 0]$ such that for all ϵ with $0 < \epsilon < \epsilon_0$ and $(\tilde{x}_\epsilon, \tilde{u}_\epsilon) \in N_0 \subset N$, we have that, for all $0 < \epsilon < \epsilon_4$ and $(\tilde{x}_\epsilon, \tilde{u}_\epsilon) \in N$:

$$\begin{aligned} F(x_\epsilon, u_\epsilon) &= F((x^*, u^*) + \epsilon(\tilde{x}_\epsilon, \tilde{u}_\epsilon)) \\ &<_{LU} F(x^*, u^*) + \epsilon[\underline{\alpha}, \bar{\alpha}], \quad \text{by using (7.31)} \\ &<_{LU} F(x^*, u^*), \quad \text{since } \epsilon > 0 \text{ and } [\underline{\alpha}, \bar{\alpha}] <_{LU} [0, 0]. \end{aligned}$$

Therefore, $(x_\epsilon, u_\epsilon) \in Q$ satisfy $F(x_\epsilon, u_\epsilon) <_{LU} F(x^*, u^*)$, which is a contradiction to the assumption that $(x^*, u^*) \in Q$ is a weak optimal LU -process. $\square$

Next, it will be defined other cones, it will be presented some features of them, and it will be remembered another nondegeneracy condition.

7.6 Characterizations of classical cones derived from $\mathcal{K}_0$

It will be considered two classical cones and it will be characterized these cones together with others classical ones too. To the end, it will be shown an important result that says that the intersection of all these classical cones is empty.

Now, let $S \subseteq \mathbb{R}^{1+n+m}$ be an open and nonempty set, and let $\Phi : S \rightarrow \mathcal{K}_C$ be an interval-valued function with $\Phi(t, x, u) = [\underline{\phi}(t, x, u), \bar{\phi}(t, x, u)]$ for all $(t, x, u) \in S$. If Φ is gH -differentiable at $(t, x^*, u^*) \in S$, then for all $(\bar{x}, \bar{u}) \in \mathbb{R}^n \times \mathbb{R}^m$, by Theorem 2.7.4, we have that

$$\Phi'_{gH}((t, x^*, u^*); (\bar{x}, \bar{u})) = \left[\langle \nabla_{\#} \underline{\phi}(t, x^*, u^*), (\bar{x}, \bar{u}) \rangle \vee \langle \nabla_{\#} \bar{\phi}(t, x^*, u^*), (\bar{x}, \bar{u}) \rangle \right]$$

Then, we see that

$$\begin{aligned} \mathcal{K}_0 &= \left\{ (\bar{x}, \bar{u}) \in E \mid \int_0^T \left[\langle \nabla_{\#} \underline{\phi}(t, x^*(t), u^*(t)), (\bar{x}(t), \bar{u}(t)) \rangle \vee \langle \nabla_{\#} \bar{\phi}(t, x^*(t), u^*(t)), (\bar{x}(t), \bar{u}(t)) \rangle \right] dt <_{LU} [0, 0] \right\} \\ &= \left\{ (\bar{x}, \bar{u}) \in E \mid \left[\int_0^T \langle \nabla_{\#} \underline{\phi}(t, x^*(t), u^*(t)), (\bar{x}(t), \bar{u}(t)) \rangle dt \vee \int_0^T \langle \nabla_{\#} \bar{\phi}(t, x^*(t), u^*(t)), (\bar{x}(t), \bar{u}(t)) \rangle dt \right] <_{LU} [0, 0] \right\}, \end{aligned}$$

by Theorem 2.8.2. Now, let us set

$$\underline{\mathcal{K}}_0 := \left\{ (\bar{x}, \bar{u}) \in E \mid \int_0^T \langle \nabla_{\#} \underline{\phi}(t, x^*(t), u^*(t)), (\bar{x}(t), \bar{u}(t)) \rangle dt < 0 \right\}$$

and

$$\bar{\mathcal{K}}_0 := \left\{ (\bar{x}, \bar{u}) \in E \mid \int_0^T \langle \nabla_{\#} \bar{\phi}(t, x^*(t), u^*(t)), (\bar{x}(t), \bar{u}(t)) \rangle dt < 0 \right\}.$$

(7.34)

Let $(t, x^*(t), u^*(t)) \in \mathbb{R} \times \mathbb{R}^n \times \mathbb{R}^m$. In what follows, we denote

$$\begin{aligned} \nabla_{\#x} \underline{\phi}(t, x^*(t), u^*(t)) &= \begin{cases} \nabla_x \underline{\phi}(t, x^*(t), u^*(t)), & \text{if } \underline{\phi} \text{ is differentiable w. r. t. } x \text{ at } q, \\ \nabla_x \underline{\phi}_-(t, x^*(t), u^*(t)), & \text{otherwise,} \end{cases} \\ \nabla_{\#x} \bar{\phi}(t, x^*(t), u^*(t)) &= \begin{cases} \nabla_x \bar{\phi}(x^*)(t, x^*(t), u^*(t)), & \text{if } \bar{\phi} \text{ is differentiable w. r. t. } x \text{ at } q, \\ \nabla_x \bar{\phi}_-(t, x^*(t), u^*(t)), & \text{otherwise,} \end{cases} \end{aligned}$$

$$\nabla_{\#u}\underline{\phi}(t, x^*(t), u^*(t)) = \begin{cases} \nabla_u\underline{\phi}(t, x^*(t), u^*(t)), & \text{if } \underline{\phi} \text{ is differentiable w. r. t. } u \text{ at } q, \\ \nabla_u\underline{\phi}_-(t, x^*(t), u^*(t)), & \text{otherwise,} \end{cases}$$

and

$$\nabla_{\#u}\overline{\phi}(t, x^*(t), u^*(t)) = \begin{cases} \nabla_u\overline{\phi}(x^*)(t, x^*(t), u^*(t)), & \text{if } \overline{\phi} \text{ is differentiable w. r. t. } u \text{ at } q, \\ \nabla_u\overline{\phi}_-(t, x^*(t), u^*(t)), & \text{otherwise,} \end{cases}$$

where $q = (t, x^*(t), u^*(t))$.

Proposition 7.6.1. *Assume that (A3) and (A5) hold. Then,*

- (i) $\underline{\mathcal{K}}_0$ is an open convex cone.
- (iii) $\mathcal{K}_1$ is an open convex cone.
- (ii) $\overline{\mathcal{K}}_0$ is an open convex cone.
- (iv) $\mathcal{K}_2$ is a convex cone.

Proof. (i) **$\underline{\mathcal{K}}_0$ is open.** Let $(\bar{x}, \bar{u}) \in \underline{\mathcal{K}}_0$, then $(\bar{x}, \bar{u}) \in E$ is such that

$$\int_0^T \langle \nabla_{\#}\underline{\phi}(t, x^*(t), u^*(t)), (\bar{x}(t), \bar{u}(t)) \rangle dt < 0. \quad (7.35)$$

We want to prove that $(\bar{x}, \bar{u}) \in \text{int } \underline{\mathcal{K}}_0$, that is, we want to prove that there exists $r > 0$ such that $\mathbb{B}_{C \times L_\infty}((\bar{x}, \bar{u}), r) \subset \underline{\mathcal{K}}_0$. Define

$$\underline{\Psi}(x, u) := \int_0^T \langle \nabla_{\#}\underline{\phi}(t, x^*(t), u^*(t)), (x(t), u(t)) \rangle dt.$$

By (A3), $\nabla_{\#x}\underline{\phi}(t, x(t), u(t))$ and $\nabla_{\#u}\underline{\phi}(t, x(t), u(t))$ exist and are bounded, then $\nabla_{\#}\underline{\phi}(t, x(t), u(t))$ exists and is bounded, let us say by K , that is, $\|\nabla_{\#}\underline{\phi}(t, x^*(t), u^*(t))\|_{\mathbb{R}^n \times \mathbb{R}^m} \leq K$ a. e. $t \in [0, T]$, so we have that

$$\begin{aligned} |\underline{\Psi}(x, u)| &= \left| \int_0^T \langle \nabla_{\#}\underline{\phi}(t, x^*(t), u^*(t)), (x(t), u(t)) \rangle dt \right| \\ &\leq \int_0^T \|\nabla_{\#}\underline{\phi}(t, x^*(t), u^*(t))\|_{\mathbb{R}^n \times \mathbb{R}^m} \|(x(t), u(t))\|_{\mathbb{R}^n \times \mathbb{R}^m} dt \\ &\leq \int_0^T K \|(x, u)\|_{C \times L_\infty} dt \\ &= M \|(x, u)\|_{C \times L_\infty} \quad \forall (x, u) \in E \text{ and where } M := KT, \end{aligned}$$

i.e., $|\underline{\Psi}| \leq M$, which implies that $\underline{\Psi}$ is bounded, that is, $\underline{\Psi}$ is continuous. By (7.35),

$$\underline{\Psi}(\bar{x}, \bar{u}) = \int_0^T \langle \nabla_{\#}\underline{\phi}(t, x^*(t), u^*(t)), (\bar{x}(t), \bar{u}(t)) \rangle dt < 0, \text{ then there exists } r > 0:$$

$$\underline{\Psi}(x, u) = \int_0^T \langle \nabla_{\#} \underline{\phi}(t, x^*(t), u^*(t)), (x(t), u(t)) \rangle dt < 0 \quad \forall (x, u) \in \mathbb{B}_{C \times L_\infty}((\bar{x}, \bar{u}), r).$$

So, $\mathbb{B}_{C \times L_\infty}((\bar{x}, \bar{u}), r) \subset \underline{\mathcal{K}}_0$. Therefore, $(x, u) \in \underline{\mathcal{K}}_0$.

$\mathcal{K}_0$ is convex. Let $(x_1, u_1), (x_2, u_2) \in \underline{\mathcal{K}}_0$, i.e., by (7.34), $(x_1, u_1), (x_2, u_2) \in E$ satisfy

$$\int_0^T \langle (\nabla_{\#x} \underline{\phi}(t, x^*(t), u^*(t)), \nabla_{\#u} \underline{\phi}(t, x^*(t), u^*(t))) , (x_1(t), u_1(t)) \rangle dt < 0 \quad (7.36)$$

and

$$\int_0^T \langle (\nabla_{\#x} \underline{\phi}(t, x^*(t), u^*(t)), \nabla_{\#u} \underline{\phi}(t, x^*(t), u^*(t))) , (x_2(t), u_2(t)) \rangle dt < 0. \quad (7.37)$$

Then, given $\lambda \in [0, 1]$,

$$(1 - \lambda)(x_1, u_1) + \lambda(x_2, u_2) = ((1 - \lambda)x_1 + \lambda x_2, (1 - \lambda)u_1 + \lambda u_2) \in E$$

is such that

$$\begin{aligned} & \int_0^T \langle (\nabla_{\#x} \underline{\phi}(t, x^*(t), u^*(t)), \nabla_{\#u} \underline{\phi}(t, x^*(t), u^*(t))) , \\ & \quad ((1 - \lambda)x_1(t) + \lambda x_2(t), (1 - \lambda)u_1(t) + \lambda u_2(t)) \rangle dt \\ &= \int_0^T [\langle \nabla_{\#x} \underline{\phi}(t, x^*(t), u^*(t)), ((1 - \lambda)x_1(t) + \lambda x_2(t)) \rangle dt \\ & \quad + \int_0^T \langle \nabla_{\#u} \underline{\phi}(t, x^*(t), u^*(t)), ((1 - \lambda)u_1(t) + \lambda u_2(t)) \rangle dt] \\ &= \int_0^T [\langle \nabla_{\#x} \underline{\phi}(t, x^*(t), u^*(t)), x_1(t) \rangle - \lambda(\langle \nabla_{\#x} \underline{\phi}(t, x^*(t), u^*(t)), x_1(t) \rangle - \langle \nabla_{\#x} \underline{\phi}(t, x^*(t), u^*(t)), x_2(t) \rangle)] dt \\ &+ \int_0^T [\langle \nabla_{\#u} \underline{\phi}(t, x^*(t), u^*(t)), u_1(t) \rangle - \lambda(\langle \nabla_{\#u} \underline{\phi}(t, x^*(t), u^*(t)), u_1(t) \rangle - \langle \nabla_{\#u} \underline{\phi}(t, x^*(t), u^*(t)), u_2(t) \rangle)] dt \\ &= \int_0^T [\langle \nabla_{\#x} \underline{\phi}(t, x^*(t), u^*(t)), x_1(t) \rangle + \langle \nabla_{\#u} \underline{\phi}(t, x^*(t), u^*(t)), u_1(t) \rangle] dt \\ & \quad - \int_0^T \lambda [\langle \nabla_{\#x} \underline{\phi}(t, x^*(t), u^*(t)), x_1(t) \rangle + \langle \nabla_{\#u} \underline{\phi}(t, x^*(t), u^*(t)), u_1(t) \rangle] dt \\ & \quad + \int_0^T \lambda [\langle \nabla_{\#x} \underline{\phi}(t, x^*(t), u^*(t)), x_2(t) \rangle + \langle \nabla_{\#u} \underline{\phi}(t, x^*(t), u^*(t)), u_2(t) \rangle] dt \\ &< 0 + 0 + 0 = 0, \quad \text{by (7.36), (7.37), and since } \lambda \in [0, 1]. \end{aligned}$$

$\mathcal{K}_0$ is a cone. If $(\bar{x}, \bar{u}) \in \underline{\mathcal{K}}_0$, then $\alpha(\bar{x}, \bar{u}) \in \underline{\mathcal{K}}_0$, for $\alpha > 0$. Indeed, if $(\bar{x}, \bar{u}) \in \underline{\mathcal{K}}_0$,

$$\begin{aligned} & (\bar{x}, \bar{u}) \in E \text{ is such that } \int_0^T \langle \nabla_{\#} \underline{\phi}(t, x^*(t), u^*(t)), (\bar{x}(t), \bar{u}(t)) \rangle dt < 0 \\ & \Leftrightarrow \alpha(\bar{x}, \bar{u}) \in E \text{ is such that } \int_0^T \langle \nabla_{\#} \underline{\phi}(t, x^*(t), u^*(t)), \alpha(\bar{x}(t), \bar{u}(t)) \rangle dt < 0, \quad \alpha > 0, \end{aligned}$$

that is, $\alpha(\bar{x}, \bar{u}) \in \underline{\mathcal{K}}_0$, for $\alpha > 0$.

(ii) The proof is similar to (i).

(iii) **$\mathcal{K}_1$ is open.** By Lemma 7.5.1 and by Proposition 7.4.2-(i), we have that $\mathcal{K}_1 = \mathcal{K}_b$ is open.

$\mathcal{K}_1$ is convex. By Proposition 7.5.1, Q_1 is convex, and by a known property, $\text{int } Q_1$ is also convex.

Now, we are going to prove that $\mathcal{K}_1$ is convex. Indeed, let $(\bar{x}_1, \bar{u}_1), (\bar{x}_2, \bar{u}_2) \in \mathcal{K}_1$, that is,

$$(\bar{x}_i, \bar{u}_i) = \lambda_i [(x_i, u_i) - (x^*, u^*)], \quad (x_i, u_i) \in \text{int } Q_1, \lambda_i > 0, \text{ and } i \in \{1, 2\}. \quad (7.38)$$

Let $\alpha \in (0, 1)$,

$$\begin{aligned} & \alpha(\bar{x}_1, \bar{u}_1) + (1 - \alpha)(\bar{x}_2, \bar{u}_2) \\ &= \alpha\lambda_1[(x_1, u_1) - (x^*, u^*)] + (1 - \alpha)\lambda_2[(x_2, u_2) - (x^*, u^*)], \quad \text{by (7.38),} \\ &= (\alpha\lambda_1x_1 + (1 - \alpha)\lambda_2x_2, \alpha\lambda_1u_1 + (1 - \alpha)\lambda_2u_2) - (\alpha\lambda_1 + (1 - \alpha)\lambda_2)(x^*, u^*) \\ &= (\alpha\lambda_1x_1 + (1 - \alpha)\lambda_2x_2, \alpha\lambda_1u_1 + (1 - \alpha)\lambda_2u_2) - \lambda(x^*, u^*), \end{aligned}$$

where $\lambda := \alpha\lambda_1 + (1 - \alpha)\lambda_2 > 0$, since $\lambda_1 > 0$, $\lambda_2 > 0$, and $\alpha \in (0, 1)$, so

$$\begin{aligned} & \alpha(\bar{x}_1, \bar{u}_1) + (1 - \alpha)(\bar{x}_2, \bar{u}_2) \\ &= \lambda \left(\frac{\alpha\lambda_1}{\lambda}x_1 + \frac{(1 - \alpha)\lambda_2}{\lambda}x_2, \frac{\alpha\lambda_1}{\lambda}u_1 + \frac{(1 - \alpha)\lambda_2}{\lambda}u_2 \right) - \lambda(x^*, u^*). \end{aligned}$$

By setting, $\alpha_1 := \frac{\alpha\lambda_1}{\lambda} = \frac{\alpha\lambda_1}{\alpha\lambda_1 + (1 - \alpha)\lambda_2} \in (0, 1)$, we have that $1 - \alpha_1 = 1 - \frac{\alpha\lambda_1}{\alpha\lambda_1 + (1 - \alpha)\lambda_2} = \frac{\alpha\lambda_1 + (1 - \alpha)\lambda_2 - \alpha\lambda_1}{\alpha\lambda_1 + (1 - \alpha)\lambda_2} = \frac{(1 - \alpha)\lambda_2}{\lambda}$.

Then,

$$\begin{aligned} \alpha(\bar{x}_1, \bar{u}_1) + (1 - \alpha)(\bar{x}_2, \bar{u}_2) &= \lambda[(\alpha_1x_1 + (1 - \alpha_1)x_2, \alpha_1u_1 + (1 - \alpha_1)u_2) - (x^*, u^*)] \\ &= \lambda[(\alpha_1(x_1, u_1) + (1 - \alpha_1)(x_2, u_2)) - (x^*, u^*)], \end{aligned}$$

where $\alpha_1(x_1, u_1) + (1 - \alpha_1)(x_2, u_2) \in \text{int } Q_1$ (since $(x_1, u_1), (x_2, u_2) \in \text{int } Q_1$ and $\text{int } Q_1$ is a convex set) and $\lambda = \alpha\lambda_1 + (1 - \alpha)\lambda_2 > 0$. Therefore,

$$\alpha(\bar{x}_1, \bar{u}_1) + (1 - \alpha)(\bar{x}_2, \bar{u}_2) \in \mathcal{K}_1,$$

that is, $\mathcal{K}_1$ is convex.

$\mathcal{K}_1$ is a cone. By Lemma 7.5.1 and by Proposition 7.4.2-(ii), we have that $\mathcal{K}_1 = \mathcal{K}_b$ is a cone.

(iv) Since $\mathcal{K}_2$ is a subspace, $\mathcal{K}_2$ is a convex cone. □

Theorem 7.6.1. *Let $(x^*, u^*) \in Q$ be a weak optimal LU-process of (LIOCP). Assume that (A1) – (A5) hold, and that the Nondegeneracy Condition 1 holds for System (7.22) with $A(t) = \gamma_x(t, x(t), u(t))$ and $B(t) = \gamma_u(t, x(t), u(t))$. Then,*

$$\underline{\mathcal{K}}_0 \cap \overline{\mathcal{K}}_0 \cap \mathcal{K}_1 \cap \mathcal{K}_2 = \emptyset.$$

Proof. The result follows by noting that $\mathcal{K}_0 = \underline{\mathcal{K}}_0 \cap \overline{\mathcal{K}}_0$. □

Definition 7.6.1 (Nondegeneracy Condition 2, see I. V. Girsanov, [36]). *The system (7.22) is said to be nondegenerate of type 2 if any nonzero solution $p(t)$ of the equation*

$$p'(t) = -A^\top(t)p(t), \quad (7.39)$$

satisfies the condition $B^\top(t)p(t) \not\equiv 0$ (more precisely, $B^\top(t)p(t)$ is nonzero on a set of positive measure), where $A(t)$, $B(t)$ are $n \times n$, $n \times m$ -matrices, respectively, and $p : [0, T] \rightarrow \mathbb{R}^n$.

Lemma 7.6.1 (See I. V. Girsanov, [36]). *The Nondegeneracy Condition 2 implies the Nondegeneracy Condition 1.*

Proof. See Lemma 9.1 of I. V. Girsanov, [36]. □

Remark 7.6.1. *The equation of the tangent subspace $\mathcal{K}_k = \mathcal{K}_2$ derived at Lemma 7.5.2 is also valid with the Nondegeneracy Condition 2.*

In the following, necessary optimality conditions for (LIOCP) will be presented and the Pontryagin function will be defined.

7.7 Necessary optimality conditions for (LIOCP)

Theorem 7.7.1 (Necessary optimality conditions for (LIOCP)). *Let $(x^*(t), u^*(t))$ be a weak optimal LU-process of (LIOCP). Assume that (A1) – (A5) hold. Then, there exist reals $\underline{\lambda}^* \geq 0$, $\overline{\lambda}^* \geq 0$, and a function $p : [0, T] \rightarrow \mathbb{R}^n$, satisfying*

$$p'(t) = \underline{\lambda}^* \nabla_{\#x} \phi(t, x^*(t), u^*(t)) + \overline{\lambda}^* \nabla_{\#x} \bar{\phi}(t, x^*(t), u^*(t)) - \gamma_x^\top(t, x^*(t), u^*(t))p(t), \quad (7.40)$$

such that $\underline{\lambda}^*$, $\bar{\lambda}^*$, and $p(t)$ cannot be simultaneously (identically) zero. Moreover,

$$\left\langle \underline{\lambda}^* \nabla_{\#u} \phi(t, x^*(t), u^*(t)) + \bar{\lambda}^* \nabla_{\#u} \bar{\phi}(t, x^*(t), u^*(t)) - \gamma_u^\top(t, x^*(t), u^*(t)) p(t), u - u^*(t) \right\rangle \geq 0 \quad (7.41)$$

for almost all $t \in [0, T]$ and all $u \in \mathcal{M}$.

Proof. We have the following four cases:

(i) Suppose that $\underline{\mathcal{K}}_0 \neq \emptyset$, $\bar{\mathcal{K}}_0 \neq \emptyset$, and that the Nondegeneracy Condition 2 holds.

(A) Analysis of the functional. By (7.34),

$$\underline{\mathcal{K}}_0 = \left\{ (\bar{x}, \bar{u}) \in E \mid \underline{\Psi}(\bar{x}, \bar{u}) := \int_0^T [\langle \nabla_{\#x} \phi(t, x^*(t), u^*(t)), \bar{x}(t) \rangle + \langle \nabla_{\#u} \phi(t, x^*(t), u^*(t)), \bar{u}(t) \rangle] dt < 0 \right\}$$

and

$$\bar{\mathcal{K}}_0 = \left\{ (\bar{x}, \bar{u}) \in E \mid \bar{\Psi}(\bar{x}, \bar{u}) := \int_0^T [\langle \nabla_{\#x} \bar{\phi}(t, x^*(t), u^*(t)), \bar{x}(t) \rangle + \langle \nabla_{\#u} \bar{\phi}(t, x^*(t), u^*(t)), \bar{u}(t) \rangle] dt < 0 \right\},$$

then by Theorem 2.11.4, we have that

$$\underline{\mathcal{K}}_0^* = \begin{cases} E', & \text{if } \underline{\Psi} = 0 \\ \{-\underline{\lambda}^* \underline{\Psi}, 0 \leq \underline{\lambda}^* < \infty\}, & \text{if } \underline{\Psi} \neq 0 \end{cases}$$

and

$$\bar{\mathcal{K}}_0^* = \begin{cases} E', & \text{if } \bar{\Psi} = 0 \\ \{-\bar{\lambda}^* \bar{\Psi}, 0 \leq \bar{\lambda}^* < \infty\}, & \text{if } \bar{\Psi} \neq 0. \end{cases}$$

Since $\underline{\mathcal{K}}_0 \neq \emptyset$ and $\bar{\mathcal{K}}_0 \neq \emptyset$, the cases $\underline{\Psi} = 0$ and $\bar{\Psi} = 0$ are excluded, so that for any $\underline{f}_0 \in \underline{\mathcal{K}}_0^*$ and $\bar{f}_0 \in \bar{\mathcal{K}}_0^*$, we have

$$\underline{f}_0(\bar{x}, \bar{u}) = -\underline{\lambda}^* \int_0^T [\langle \nabla_{\#x} \phi(t, x^*(t), u^*(t)), \bar{x}(t) \rangle + \langle \nabla_{\#u} \phi(t, x^*(t), u^*(t)), \bar{u}(t) \rangle] dt, \quad \underline{\lambda}^* \geq 0 \quad (7.42)$$

and

$$\bar{f}_0(\bar{x}, \bar{u}) = -\bar{\lambda}^* \int_0^T [\langle \nabla_{\#x} \bar{\phi}(t, x^*(t), u^*(t)), \bar{x}(t) \rangle + \langle \nabla_{\#u} \bar{\phi}(t, x^*(t), u^*(t)), \bar{u}(t) \rangle] dt, \quad \bar{\lambda}^* \geq 0. \quad (7.43)$$

(B) Analysis of the constraint Q_1 . Since $\mathcal{M}$ is a closed convex set in $\mathbb{R}^m$ such that $\text{int } \mathcal{M} \neq \emptyset$, Q'_1 , the set of all functions $u(t) \in \mathcal{M}$ for a. e. $t \in [0, T]$, is also a closed convex set in L_∞ such that $\text{int } Q'_1 \neq \emptyset$. Therefore, $Q_1 = C \times Q'_1$ is a closed convex set in $E = C \times L_\infty$, and

$$\text{int } Q_1 = \text{int } (C \times Q'_1) = C \times \text{int } Q'_1 \neq \emptyset.$$

Let $\mathcal{K}_1$ be the cone of feasible directions for Q_1 at (x^*, u^*) . Since Q_1 is a closed convex set with $\text{int } Q_1 \neq \emptyset$, by Theorem 2.11.5,

$$\mathcal{K}_1^* = Q_1^* = (C \times Q'_1)^* = C^* \times (Q'_1)^*,$$

then if $f_1 \in \mathcal{K}_1^*$, $f_1 = (e_1, f'_1)$ ⁵, where $e_1 \in C^*$ and $f'_1 \in (Q'_1)^*$. So, since C is a subspace, by Theorem 2.11.3, we have that $e_1(x^*) = 0$ for all $x^* \in C$, that is $f_1 = (0, f'_1)$. Thence,

$$f_1(x^*, u^*) = (0, f'_1)(x^*, u^*) = 0(x^*) + f'_1(u^*) = f'_1(u^*), \quad (7.44)$$

where $f'_1 \in (L_\infty)^*$ is a supporting functional to $Q'_1 \subset L_\infty$ at $u^* \in Q'_1$, that is, $f'_1(u) \geq f'_1(u^*)$ for all $u \in Q'_1$.

(C) Analysis of the constraint Q_2 . Since the Nondegeneracy Condition 2 holds, then $\gamma_u^\top(t, x^*(t), u^*(t))p(t) \not\equiv 0$ for any nonzero solution $p(t)$ of the equation

$$p'(t) = -\gamma_x^\top(t, x^*(t), u^*(t))p(t).$$

By Lemma 7.6.1, the Nondegeneracy Condition 1 also holds, so by (7.23) of Lemma 7.5.2, the tangent subspace $\mathcal{K}_2$ consist of all $(\bar{x}, \bar{u}) \in E$ such that

$$\bar{x}'(t) = \gamma_x(t, x^*(t), u^*(t))\bar{x}(t) + \gamma_u(t, x^*(t), u^*(t))\bar{u}(t), \quad \bar{x}(0) = \mathbf{0} \quad (7.45)$$

and

$$\bar{x}(T) = \mathbf{0}. \quad (7.46)$$

Let $L_2 \subset E$ denote the set of all pairs $(\bar{x}, \bar{u}) \in E$ satisfying (7.45) and $L_3 \subset E$ denote the set of all pairs $(\bar{x}, \bar{u}) \in E$ satisfying (7.46), that is,

$$L_2 := \{(\bar{x}, \bar{u}) \in E \mid (7.45) \text{ is satisfied}\}$$

and

$$L_3 := \{(\bar{x}, \bar{u}) \in E \mid (7.46) \text{ is satisfied}\}.$$

Then, L_2 and L_3 are subspaces, and $\mathcal{K}_2 = L_2 \cap L_3$. Since L_2 is a subspace, by Theorem 2.11.3, for any $f_2 \in L_2^*$, we have that

$$f_2(\bar{x}, \bar{u}) = 0 \quad \text{for all } (\bar{x}, \bar{u}) \in L_2. \quad (7.47)$$

We have that

⁵ f'_1 in this case does not denote the derivative of f_1 .

$$\begin{aligned}
L_3 &= \{(\bar{x}, \bar{u}) \in E \mid \bar{x}_i(T) = 0, \quad i \in \{1, \dots, n\}\} \\
&= \bigcap_{i=1}^n \{(\bar{x}, \bar{u}) \in E \mid \bar{x}_i(T) = 0\} \\
&= \bigcap_{i=1}^n \{(\bar{x}, \bar{u}) \in E \mid g_i(\bar{x}, \bar{u}) = 0\}, \quad g_i(\bar{x}, \bar{u}) := \bar{x}_i(T) \\
&= \bigcap_{i=1}^n L_{3_i}, \quad L_{3_i} := \{(\bar{x}, \bar{u}) \in E \mid g_i(\bar{x}, \bar{u}) = 0\},
\end{aligned}$$

where L_{3_i} , $i \in \{1, \dots, n\}$, is closed and convex, so that it is weakly closed and convex. Moreover, by Theorem 2.11.4, $L_{3_i}^* = \{a_i g_i \mid a_i \in \mathbb{R}\}$, then

$$\begin{aligned}
\sum_{i=1}^n L_{3_i}^* &= \{a_1 g_1 \mid a_1 \in \mathbb{R}\} + \dots + \{a_n g_n \mid a_n \in \mathbb{R}\} \\
&= \{a_1 g_1 + \dots + a_n g_n \mid a_i \in \mathbb{R}, \quad i \in \{1, \dots, n\}\} \\
&= \{\langle a, g \rangle \mid a \in \mathbb{R}^n\}, \quad a = (a_1, \dots, a_n) \text{ and } g = (g_1, \dots, g_n),
\end{aligned}$$

is weakly* closed.

Note that,

$$g(\bar{x}, \bar{u}) = (g_1(\bar{x}, \bar{u}), \dots, g_n(\bar{x}, \bar{u})) = (\bar{x}_1(T), \dots, \bar{x}_n(T)) = \bar{x}(T) \quad \forall (\bar{x}, \bar{u}) \in E.$$

Since the cone L_{3_i} , $i \in \{1, \dots, n\}$, is weakly closed and convex, and $\sum_{i=1}^n L_{3_i}^*$ is weakly* closed. By Corollary 2.11.1 of Lemma 2.11.2, it follows that

$$L_3^* = \left(\bigcap_{i=1}^n L_{3_i} \right)^* = \sum_{i=1}^n L_{3_i}^* = \{\langle a, g \rangle \mid a \in \mathbb{R}^n\}, \quad (7.48)$$

which is n -dimensional.

Since L_2^* and L_3^* are closed subspaces, and L_3^* is finite dimensional, by Lemma 2.11.1, we have that $L_2^* + L_3^*$ is closed, and then $L_2^* + L_3^*$ is weakly* closed.

Since L_2 and L_3 are weakly closed convex, and $L_2^* + L_3^*$ is weakly* closed, by Corollary 2.11.1 of Lemma 2.11.2, it follows that

$$\mathcal{K}_2^* = (L_2 \cap L_3)^* = L_2^* + L_3^*.$$

Then, if $f \in \mathcal{K}_2^*$, we have that

$$f = f_2 + f_3, \quad f_2 \in L_2^* \text{ and } f_3 \in L_3^*,$$

then by (7.47), $f_2(\bar{x}, \bar{u}) = 0$ for all $(\bar{x}, \bar{u}) \in L_2$, and by (7.48),

$$f_3(\bar{x}, \bar{u}) = \langle a, g(\bar{x}, \bar{u}) \rangle = \langle a, \bar{x}(T) \rangle, \text{ for some } a \in \mathbb{R}^n \text{ and for all } (\bar{x}, \bar{u}) \in E. \quad (7.49)$$

(D) Euler equation. Applying Proposition 7.6.1, Theorem 7.6.1, and Lemma 2.11.3 (Dubovitskii–Milyutin), we have the following optimality condition: there exist linear functionals $\underline{f}_0 \in \underline{\mathcal{K}}_0^*$, $\bar{f}_0 \in \bar{\mathcal{K}}_0^*$, $f_1 \in \mathcal{K}_1^*$, and $f \in \mathcal{K}_2^*$, where $f = f_2 + f_3$, $f_2 \in L_2^*$, and $f_3 \in L_3^*$, not all zero, such that, for all $(\bar{x}, \bar{u}) \in E$,

$$\underline{f}_0(\bar{x}, \bar{u}) + \bar{f}_0(\bar{x}, \bar{u}) + f_1(\bar{x}, \bar{u}) + f_2(\bar{x}, \bar{u}) + f_3(\bar{x}, \bar{u}) = 0, \quad (7.50)$$

where $\underline{f}_0(\bar{x}, \bar{u})$, $\bar{f}_0(\bar{x}, \bar{u})$, $f_1(\bar{x}, \bar{u})$, $f_2(\bar{x}, \bar{u})$, and $f_3(\bar{x}, \bar{u})$ are given by (7.42), (7.43), (7.44), (7.47), and (7.49), respectively.

(E) Analysis of the Euler equation. Equation (7.50) must hold for any $(\bar{x}, \bar{u}) \in E$. Let $\bar{u}$ be arbitrary, and determine a solution $\bar{x} = \bar{x}(\bar{u})$ of Equation (7.45). With this choice of $(\bar{x}, \bar{u})$, we have that $f_2(\bar{x}, \bar{u}) = 0$, and so condition (7.50) becomes

$$\begin{aligned} & -\underline{\lambda}^* \int_0^T [\langle \nabla_{\#x} \underline{\phi}(t, x^*(t), u^*(t)), \bar{x}(t) \rangle + \langle \nabla_{\#u} \underline{\phi}(t, x^*(t), u^*(t)), \bar{u}(t) \rangle] dt \\ & - \bar{\lambda}^* \int_0^T [\langle \nabla_{\#x} \bar{\phi}(t, x^*(t), u^*(t)), \bar{x}(t) \rangle + \langle \nabla_{\#u} \bar{\phi}(t, x^*(t), u^*(t)), \bar{u}(t) \rangle] dt \\ & + f_1'(\bar{u}) + 0 + \langle a, \bar{x}(T) \rangle = 0 \end{aligned}$$

$$\begin{aligned} & \Rightarrow f_1'(\bar{u}) \\ & = \underline{\lambda}^* \int_0^T [\langle \nabla_{\#x} \underline{\phi}(t, x^*(t), u^*(t)), \bar{x}(t) \rangle + \langle \nabla_{\#u} \underline{\phi}(t, x^*(t), u^*(t)), \bar{u}(t) \rangle] dt \\ & + \bar{\lambda}^* \int_0^T [\langle \nabla_{\#x} \bar{\phi}(t, x^*(t), u^*(t)), \bar{x}(t) \rangle + \langle \nabla_{\#u} \bar{\phi}(t, x^*(t), u^*(t)), \bar{u}(t) \rangle] dt - \langle a, \bar{x}(T) \rangle \\ & = \underline{\lambda}^* \int_0^T \langle \nabla_{\#u} \underline{\phi}(t, x^*(t), u^*(t)), \bar{u}(t) \rangle dt + \bar{\lambda}^* \int_0^T \langle \nabla_{\#u} \bar{\phi}(t, x^*(t), u^*(t)), \bar{u}(t) \rangle dt \\ & + \underline{\lambda}^* \int_0^T \langle \nabla_{\#x} \underline{\phi}(t, x^*(t), u^*(t)), \bar{x}(t) \rangle dt + \bar{\lambda}^* \int_0^T \langle \nabla_{\#x} \bar{\phi}(t, x^*(t), u^*(t)), \bar{x}(t) \rangle dt - \langle a, \bar{x}(T) \rangle. \quad (7.51) \end{aligned}$$

Now, we want to transform the right-hand side of (7.51) in such a way as to replace $\bar{x}$ by $\bar{u}$. Let $p(t)$ be a solution of the System (7.40), with the boundary condition

$$p(T) = a \in \mathbb{R}^n, \quad (7.52)$$

then

$$\begin{aligned}
& \underline{\lambda}^* \nabla_{\#x} \underline{\phi}(t, x^*(t), u^*(t)) + \bar{\lambda}^* \nabla_{\#x} \bar{\phi}(t, x^*(t), u^*(t)) = p'(t) + \gamma_x^\top(t, x^*(t), u^*(t))p(t) \\
& \Leftrightarrow \underline{\lambda}^* \langle \nabla_{\#x} \underline{\phi}(t, x^*(t), u^*(t)), \bar{x}(t) \rangle + \bar{\lambda}^* \langle \nabla_{\#x} \bar{\phi}(t, x^*(t), u^*(t)), \bar{x}(t) \rangle \\
& \quad = \langle p'(t) + \gamma_x^\top(t, x^*(t), u^*(t))p(t), \bar{x}(t) \rangle.
\end{aligned}$$

Integrating, we obtain

$$\begin{aligned}
& \underline{\lambda}^* \int_0^T \langle \nabla_{\#x} \underline{\phi}(t, x^*(t), u^*(t)), \bar{x}(t) \rangle dt + \bar{\lambda}^* \int_0^T \langle \nabla_{\#x} \bar{\phi}(t, x^*(t), u^*(t)), \bar{x}(t) \rangle dt \\
& \quad = \int_0^T \langle p'(t) + \gamma_x^\top(t, x^*(t), u^*(t))p(t), \bar{x}(t) \rangle dt. \tag{7.53}
\end{aligned}$$

By (7.45),

$$\begin{aligned}
& -\bar{x}'(t) + \gamma_x(t, x^*(t), u^*(t))\bar{x}(t) = -\gamma_u(t, x^*(t), u^*(t))\bar{u}(t) \\
& \Leftrightarrow -\langle p(t), \bar{x}'(t) \rangle + \langle p(t), \gamma_x(t, x^*(t), u^*(t))\bar{x}(t) \rangle = -\langle p(t), \gamma_u(t, x^*(t), u^*(t))\bar{u}(t) \rangle.
\end{aligned}$$

Integrating, we obtain

$$\begin{aligned}
& -\int_0^T \langle p(t), \bar{x}'(t) \rangle dt + \int_0^T \langle p(t), \gamma_x(t, x^*(t), u^*(t))\bar{x}(t) \rangle dt \\
& \quad = -\int_0^T \langle p(t), \gamma_u(t, x^*(t), u^*(t))\bar{u}(t) \rangle dt \\
& \quad = -\int_0^T \langle \gamma_u^\top(t, x^*(t), u^*(t))p(t), \bar{u}(t) \rangle dt. \tag{7.54}
\end{aligned}$$

From another part, we have that

$$\begin{aligned}
& \underline{\lambda}^* \int_0^T \langle \nabla_{\#x} \underline{\phi}(t, x^*(t), u^*(t)), \bar{x}(t) \rangle dt + \bar{\lambda}^* \int_0^T \langle \nabla_{\#x} \bar{\phi}(t, x^*(t), u^*(t)), \bar{x}(t) \rangle dt - \langle a, \bar{x}(T) \rangle \\
& \quad = \int_0^T \langle p'(t) + \gamma_x^\top(t, x^*(t), u^*(t))p(t), \bar{x}(t) \rangle dt - \langle a, \bar{x}(T) \rangle, \quad \text{by (7.53)} \\
& \quad = \langle \bar{x}(t), p(t) \rangle \Big|_0^T - \int_0^T \langle p(t), \bar{x}'(t) \rangle dt + \int_0^T \langle p(t), \gamma_x(t, x^*(t), u^*(t))\bar{x}(t) \rangle dt - \langle a, \bar{x}(T) \rangle \\
& \quad = \langle \bar{x}(T), p(T) \rangle - \langle \bar{x}(0), p(0) \rangle - \int_0^T \langle p(t), \bar{x}'(t) \rangle dt + \int_0^T \langle p(t), \gamma_x(t, x^*(t), u^*(t))\bar{x}(t) \rangle dt - \langle a, \bar{x}(T) \rangle \\
& \quad = \langle \bar{x}(T), a \rangle - 0 - \int_0^T \langle \gamma_u^\top(t, x^*(t), u^*(t))p(t), \bar{u}(t) \rangle dt - \langle \bar{x}(T), a \rangle, \quad \text{by (7.45), (7.52), and (7.54)} \\
& \quad = -\int_0^T \langle \gamma_u^\top(t, x^*(t), u^*(t))p(t), \bar{u}(t) \rangle dt. \tag{7.55}
\end{aligned}$$

Replacing (7.55) in (7.51), we obtain

$$\begin{aligned}
& f'_1(\bar{u}) \\
&= \underline{\lambda}^* \int_0^T \langle \nabla_{\#u} \underline{\phi}(t, x^*(t), u^*(t)), \bar{u}(t) \rangle dt + \bar{\lambda}^* \int_0^T \langle \nabla_{\#u} \bar{\phi}(t, x^*(t), u^*(t)), \bar{u}(t) \rangle dt \\
&\quad - \int_0^T \langle \gamma_u^\top(t, x^*(t), u^*(t)) p(t), \bar{u}(t) \rangle dt \\
&= \int_0^T \left\langle \underline{\lambda}^* \nabla_{\#u} \underline{\phi}(t, x^*(t), u^*(t)) + \bar{\lambda}^* \nabla_{\#u} \bar{\phi}(t, x^*(t), u^*(t)) - \gamma_u^\top(t, x^*(t), u^*(t)) p(t), \bar{u}(t) \right\rangle dt,
\end{aligned}$$

where $\bar{u}$ is arbitrary and $f'_1(\bar{u})$ is a supporting functional to $Q'_1 \subset L_\infty$ at $u^* \in Q'_1$, and since f'_1 is also a linear integral functional, by Example 2.11.1, we obtain

$$\left\langle \underline{\lambda}^* \nabla_{\#u} \underline{\phi}(t, x^*(t), u^*(t)) + \bar{\lambda}^* \nabla_{\#u} \bar{\phi}(t, x^*(t), u^*(t)) - \gamma_u^\top(t, x^*(t), u^*(t)) p(t), u - u^*(t) \right\rangle \geq 0$$

for all $u \in \mathcal{M}$ and almost all $t \in [0, T]$, i.e., inequality (7.41) is satisfied.

Now, under these assumptions, the case $\underline{\lambda}^* = 0$, $\bar{\lambda}^* = 0$, and $p(t) \equiv \mathbf{0}$ cannot occur simultaneously, since by (7.42), (7.43), and (7.52), we would have that for all $(\bar{x}, \bar{u}) \in E$, $\underline{f}_0(\bar{x}, \bar{u}) = 0$, $\bar{f}_0(\bar{x}, \bar{u}) = 0$, and $a = p(T) = \mathbf{0}$, respectively, but then by (7.51), $f'_1(\bar{u}) = 0$, so by (7.44), $f_1(\bar{x}, \bar{u}) = f'_1(\bar{u}) = 0$, and by (7.49), $f_3(\bar{x}, \bar{u}) = \langle a, \bar{x}(T) \rangle = 0$. Then, we would have that $\underline{f}_0 = \bar{f}_0 = f_1 = f_3 = 0$, and (7.50) would then imply also that $f_2 = 0$, which contradicts the non-triviality condition that says that there exist $\underline{f}_0, \bar{f}_0, f_1, f_2, f_3$, not all zero, see item (D).

(ii) Suppose that $\underline{K}_0 = \emptyset$, that is,

$$\underline{\Psi}(\bar{x}, \bar{u}) = \int_0^T [\nabla_{\#u} \underline{\phi}(t, x^*(t), u^*(t)), \bar{u}(t)] + \langle \nabla_{\#x} \underline{\phi}(t, x^*(t), u^*(t)), \bar{x}(t) \rangle dt \geq 0, \quad (7.56)$$

for all $(\bar{x}, \bar{u}) \in C \times L_\infty$, so since $(\tilde{x}, \tilde{u}) := -(\bar{x}, \bar{u}) \in C \times L_\infty$,

$$\underline{\Psi}(\tilde{x}, \tilde{u}) \geq 0 \Leftrightarrow \underline{\Psi}(-\bar{x}, -\bar{u}) \geq 0 \Leftrightarrow -\underline{\Psi}(\bar{x}, \bar{u}) \geq 0 \Leftrightarrow \underline{\Psi}(\bar{x}, \bar{u}) \leq 0. \quad (7.57)$$

By (7.56) and (7.57), $\underline{\Psi}(\bar{x}, \bar{u}) = 0$, then

$$\begin{aligned}
& \int_0^T [\langle \nabla_{\#u} \underline{\phi}(t, x^*(t), u^*(t)), \bar{u}(t) \rangle + \langle 0 \nabla_{\#u} \bar{\phi}(t, x^*(t), u^*(t)), \bar{u}(t) \rangle \\
& \quad \langle \nabla_{\#x} \underline{\phi}(t, x^*(t), u^*(t)), \bar{x}(t) \rangle + \langle 0 \nabla_{\#x} \bar{\phi}(t, x^*(t), u^*(t)), \bar{x}(t) \rangle] dt = 0.
\end{aligned} \quad (7.58)$$

In (7.53), taking $\underline{\lambda}^* = 1$, $\bar{\lambda}^* = 0$, and $p(T) = \mathbf{0}$,

$$\begin{aligned}
& \int_0^T [\langle \nabla_{\#x} \underline{\phi}(t, x^*(t), u^*(t)), \bar{x}(t) \rangle + \langle 0 \nabla_{\#x} \bar{\phi}(t, x^*(t), u^*(t)), \bar{x}(t) \rangle] dt \\
&= \int_0^T \langle p'(t) + \gamma_x^\top(t, x^*(t), u^*(t)) p(t), \bar{x}(t) \rangle dt
\end{aligned}$$

$$\begin{aligned}
&= \langle \bar{x}(t), p(t) \rangle|_0^T - \int_0^T \langle p(t), \bar{x}'(t) \rangle dt + \int_0^T \langle p(t), \gamma_x(t, x^*(t), u^*(t)) \bar{x}(t) \rangle dt \\
&= \langle \bar{x}(T), p(T) \rangle - \langle \bar{x}(0), p(0) \rangle - \int_0^T \langle p(t), \bar{x}'(t) \rangle dt + \int_0^T \langle p(t), \gamma_x(t, x^*(t), u^*(t)) \bar{x}(t) \rangle dt \\
&= \langle \bar{x}(T), \mathbf{0} \rangle - 0 - \int_0^T \langle \gamma_u^\top(t, x^*(t), u^*(t)) p(t), \bar{u}(t) \rangle dt, \quad \text{by (7.54)} \\
&= - \int_0^T \langle \gamma_u^\top(t, x^*(t), u^*(t)) p(t), \bar{u}(t) \rangle dt. \quad (7.59)
\end{aligned}$$

Replacing (7.59) in (7.58), we obtain

$$\int_0^T \langle \nabla_{\#u} \underline{\phi}(t, x^*(t), u^*(t)) + 0 \nabla_{\#u} \bar{\phi}(t, x^*(t), u^*(t)) - \gamma_u^\top(t, x^*(t), u^*(t)) p(t), \bar{u}(t) \rangle dt = 0$$

for all $\bar{u}$. Hence

$$\nabla_{\#u} \underline{\phi}(t, x^*(t), u^*(t)) + 0 \nabla_{\#u} \bar{\phi}(t, x^*(t), u^*(t)) - \gamma_u^\top(t, x^*(t), u^*(t)) p(t) = 0 \quad (7.60)$$

for almost all $t \in [0, T]$ and all $u \in \mathcal{M}$, i.e., Inequality (7.41), for $\underline{\lambda}^* = 1$, $\bar{\lambda}^* = 0$, and $p(T) = \mathbf{0}$ such that (7.60) holds, is satisfied.

(iii) Suppose that $\bar{\mathcal{K}}_0 = \emptyset$, and taking $\underline{\lambda}^* = 0$, $\bar{\lambda}^* = 1$, and $p(T) = \mathbf{0}$, the calculus are the same of Case (ii).

(iv) Suppose that Nondegeneracy Condition 2 does not hold, that is, there exists a nonzero $p(t)$ which is a solution of Equation (7.40) for $\underline{\lambda}^* = \bar{\lambda}^* = 0$ such that

$$- \gamma_u^\top(t, x^*(t), u^*(t)) p(t) \equiv 0, \quad (7.61)$$

i.e., Inequality (7.41), for $\underline{\lambda}^* = \bar{\lambda}^* = 0$ and $p(T) \neq \mathbf{0}$ such that (7.61) holds, is satisfied.

Therefore, the proof is complete for all the Cases (i), (ii), (iii), and (iv). $\square$

Definition 7.7.1. The Pontryagin function $\mathcal{H} : [0, T] \times \mathbb{R}^n \times \mathbb{R}^m \times \mathbb{R}^n \rightarrow \mathbb{R}$ is defined by

$$\mathcal{H}(t, x, p, u) := -\underline{\lambda}^* \underline{\phi}(t, x, u) - \bar{\lambda}^* \bar{\phi}(t, x, u) + \langle \gamma(t, x, u), p(t) \rangle. \quad (7.62)$$

Let us assume that $\mathcal{H}$ is differentiable with respect to u . Then,

$$\mathcal{H}_u(t, x^*(t), p(t), u^*(t)) = -\underline{\lambda}^* \nabla_u \underline{\phi}(t, x^*(t), u^*(t)) - \bar{\lambda}^* \nabla_u \bar{\phi}(t, x^*(t), u^*(t)) + \gamma_u^\top(t, x^*(t), u^*(t)) p(t),$$

and since a necessary condition for $\mathcal{H}(t, x^*(t), p(t), u(t))$ to have a maximum on $\mathcal{M}$, as a function of u , is that $-\mathcal{H}_u(t, x^*(t), p(t), u^*(t))$ be a support to $\mathcal{M}$ at $u^*(t)$, (for the proof,

see Theorem 11.4 of I. V. Girsanov, [36]), it follows that (7.41) may be rephrased as follows. If (x^*, u^*) is a weak optimal LU –process of $(LIOCP)$ and the assumptions of Theorem 7.7.1 are satisfied and, in addition, $\mathcal{H}$ is differentiable with respect to u , then $\mathcal{H}(t, x^*(t), p(t), u(t))$, as a function of u on $\mathcal{M}$, satisfies the necessary conditions for a maximum for almost all $t \in [0, T]$ at $u = u^*(t)$.

It can be observed that at Theorem 7.7.1, it does not exist a guarantee that at least one of the multipliers $\underline{\lambda}^*/\bar{\lambda}^*$, associated with the objective interval–valued function will be different from zero, i.e., nothing guarantees that $(\underline{\lambda}^*, \bar{\lambda}^*) \neq (0, 0)$. In that way, if that multipliers were zero, it would be lost information contained at the gH –derivative of the objective interval–valued function. To solve this inconvenience, it could be added to the hypothesis of the last theorem, a nondegeneracy condition, like the Definition 7.6.1, but this time it will not be the purpose of the Thesis. Next, it will be considered the $(\widetilde{LIOCP})$, which is a simpler problem where the inconvenience disappears. The boundary conditions should be naturally on $p(T)$ in the form: $p(T) = \mathbf{0} \in \mathbb{R}^n$, this comes from the nature of the problem, then the nondegeneracy condition is superfluous as is shown in Theorem 7.8.1 below.

Next, necessary optimality conditions and the Interval Local Pontryagin Maximum Principle for $(\widetilde{LIOCP})$ will be given.

7.8 Necessary optimality conditions for $(\widetilde{LIOCP})$

Theorem 7.8.1 (Necessary optimality conditions for $(\widetilde{LIOCP})$). *Let $(x^*(t), u^*(t))$ be a weak optimal LU –process of $(\widetilde{LIOCP})$. Assume that (A1) – (A5) hold. Then, there exist reals $\underline{\lambda}^* \geq 0$ and $\bar{\lambda}^* \geq 0$ with $\underline{\lambda}^* + \bar{\lambda}^* = 1$ and a function $p : [0, T] \rightarrow \mathbb{R}^n$, satisfying*

$$p'(t) = \underline{\lambda}^* \nabla_{\#x} \underline{\phi}(t, x^*(t), u^*(t)) + \bar{\lambda}^* \nabla_{\#x} \bar{\phi}(t, x^*(t), u^*(t)) - \gamma_x^\top(t, x^*(t), u^*(t))p(t) \quad (7.63)$$

and

$$p(T) = \mathbf{0}. \quad (7.64)$$

Moreover,

$$\left\langle \underline{\lambda}^* \nabla_{\#u} \underline{\phi}(t, x^*(t), u^*(t)) + \bar{\lambda}^* \nabla_{\#u} \bar{\phi}(t, x^*(t), u^*(t)) - \gamma_u^\top(t, x^*(t), u^*(t))p(t), u - u^*(t) \right\rangle \geq 0 \quad (7.65)$$

for almost all $t \in [0, T]$ and all $u \in \mathcal{M}$.

Proof. We have the following three cases:

(i) Suppose that $\underline{\mathcal{K}}_0 \neq \emptyset$ and $\overline{\mathcal{K}}_0 \neq \emptyset$.

(A) Analysis of the functional. As before,

$$\underline{f}_0(\bar{x}, \bar{u}) = -\underline{\lambda}^* \int_0^T [\langle \nabla_{\#x} \phi(t, x^*(t), u^*(t)), \bar{x}(t) \rangle + \langle \nabla_{\#u} \phi(t, x^*(t), u^*(t)), \bar{u}(t) \rangle] dt, \quad \underline{\lambda}^* \geq 0 \quad (7.66)$$

and

$$\overline{f}_0(\bar{x}, \bar{u}) = -\overline{\lambda}^* \int_0^T [\langle \nabla_{\#x} \bar{\phi}(t, x^*(t), u^*(t)), \bar{x}(t) \rangle + \langle \nabla_{\#u} \bar{\phi}(t, x^*(t), u^*(t)), \bar{u}(t) \rangle] dt, \quad \overline{\lambda}^* \geq 0. \quad (7.67)$$

(B) Analysis of the constraint Q_1 . As before,

$$f_1(x^*, u^*) = f'_1(u^*), \quad (7.68)$$

where $f'_1 \in (L_\infty)^*$ is a supporting functional to $Q'_1 \subset L_\infty$ at $u^* \in Q'_1$, that is, $f'_1(u) \geq f'_1(u^*)$ for all $u \in Q'_1$.

(C) Analysis of the constraint $\widetilde{Q}_2$. By (7.30) of Lemma 7.5.3, the tangent subspace $\widetilde{\mathcal{K}}_2$ consist of all $(\bar{x}, \bar{u}) \in E$ such that

$$\bar{x}'(t) = \gamma_x(t, x^*(t), u^*(t))\bar{x}(t) + \gamma_u(t, x^*(t), u^*(t))\bar{u}(t), \quad \bar{x}(0) = \mathbf{0}. \quad (7.69)$$

Since $\widetilde{\mathcal{K}}_2$ is a subspace, by Theorem 2.11.3, for any $f_2 \in \widetilde{\mathcal{K}}_2^*$, we have that

$$f_2(\bar{x}, \bar{u}) = 0 \quad \text{for all } (\bar{x}, \bar{u}) \in \widetilde{\mathcal{K}}_2. \quad (7.70)$$

(D) Euler equation. Applying Proposition 7.6.1, Theorem 7.6.1, and Lemma 2.11.3 (Dubovitskii–Milyutin), we have the following optimality condition: there exist linear functionals $\underline{f}_0 \in \underline{\mathcal{K}}_0^*$, $\overline{f}_0 \in \overline{\mathcal{K}}_0^*$, $f_1 \in \mathcal{K}_1^*$, $f_2 \in \widetilde{\mathcal{K}}_2^*$, not all zero, such that, for all $(\bar{x}, \bar{u}) \in E$,

$$\underline{f}_0(\bar{x}, \bar{u}) + \overline{f}_0(\bar{x}, \bar{u}) + f_1(\bar{x}, \bar{u}) + f_2(\bar{x}, \bar{u}) = 0, \quad (7.71)$$

where $\underline{f}_0(\bar{x}, \bar{u})$, $\overline{f}_0(\bar{x}, \bar{u})$, $f_1(\bar{x}, \bar{u})$, and $f_2(\bar{x}, \bar{u})$ are given by (7.66), (7.67), (7.68), and (7.70), respectively.

(E) Analysis of the Euler equation. Equation (7.71) must hold for any $(\bar{x}, \bar{u}) \in E$. Let $\bar{u}$ be arbitrary, and determine a solution $\bar{x} = \bar{x}(\bar{u})$ of Equation (7.69). With this choice of $(\bar{x}, \bar{u})$, we have that condition (7.71) becomes

$$\begin{aligned}
& -\underline{\lambda}^* \int_0^T [\langle \nabla_{\#x} \underline{\phi}(t, x^*(t), u^*(t)), \bar{x}(t) \rangle + \langle \nabla_{\#u} \underline{\phi}(t, x^*(t), u^*(t)), \bar{u}(t) \rangle] dt \\
& - \bar{\lambda}^* \int_0^T [\langle \nabla_{\#x} \bar{\phi}(t, x^*(t), u^*(t)), \bar{x}(t) \rangle + \langle \nabla_{\#u} \bar{\phi}(t, x^*(t), u^*(t)), \bar{u}(t) \rangle] dt \\
& + f'_1(\bar{u}) + 0 = 0
\end{aligned}$$

$$\begin{aligned}
& \Rightarrow f'_1(\bar{u}) \\
& = \underline{\lambda}^* \int_0^T [\langle \nabla_{\#x} \underline{\phi}(t, x^*(t), u^*(t)), \bar{x}(t) \rangle + \langle \nabla_{\#u} \underline{\phi}(t, x^*(t), u^*(t)), \bar{u}(t) \rangle] dt \\
& + \bar{\lambda}^* \int_0^T [\langle \nabla_{\#x} \bar{\phi}(t, x^*(t), u^*(t)), \bar{x}(t) \rangle + \langle \nabla_{\#u} \bar{\phi}(t, x^*(t), u^*(t)), \bar{u}(t) \rangle] dt \\
& = \underline{\lambda}^* \int_0^T \langle \nabla_{\#u} \underline{\phi}(t, x^*(t), u^*(t)), \bar{u}(t) \rangle dt + \bar{\lambda}^* \int_0^T \langle \nabla_{\#u} \bar{\phi}(t, x^*(t), u^*(t)), \bar{u}(t) \rangle dt \\
& + \underline{\lambda}^* \int_0^T \langle \nabla_{\#x} \underline{\phi}(t, x^*(t), u^*(t)), \bar{x}(t) \rangle dt + \bar{\lambda}^* \int_0^T \langle \nabla_{\#x} \bar{\phi}(t, x^*(t), u^*(t)), \bar{x}(t) \rangle dt. \quad (7.72)
\end{aligned}$$

Now, we want to transform the right-hand side of (7.72) in such a way as to replace $\bar{x}$ by $\bar{u}$. Let $p(t)$ be a solution of the System (7.63) with the boundary condition

$$p(T) = \mathbf{0} \in \mathbb{R}^n, \quad (7.73)$$

then

$$\begin{aligned}
& \underline{\lambda}^* \nabla_{\#x} \underline{\phi}(t, x^*(t), u^*(t)) + \bar{\lambda}^* \nabla_{\#x} \bar{\phi}(t, x^*(t), u^*(t)) = p'(t) + \gamma_x^\top(t, x^*(t), u^*(t))p(t) \\
& \Leftrightarrow \underline{\lambda}^* \langle \nabla_{\#x} \underline{\phi}(t, x^*(t), u^*(t)), \bar{x}(t) \rangle + \bar{\lambda}^* \langle \nabla_{\#x} \bar{\phi}(t, x^*(t), u^*(t)), \bar{x}(t) \rangle \\
& = \langle p'(t) + \gamma_x^\top(t, x^*(t), u^*(t))p(t), \bar{x}(t) \rangle.
\end{aligned}$$

Integrating, we obtain

$$\begin{aligned}
& \underline{\lambda}^* \int_0^T \langle \nabla_{\#x} \underline{\phi}(t, x^*(t), u^*(t)), \bar{x}(t) \rangle dt + \bar{\lambda}^* \int_0^T \langle \nabla_{\#x} \bar{\phi}(t, x^*(t), u^*(t)), \bar{x}(t) \rangle dt \\
& = \int_0^T \langle p'(t) + \gamma_x^\top(t, x^*(t), u^*(t))p(t), \bar{x}(t) \rangle dt. \quad (7.74)
\end{aligned}$$

By (7.69),

$$\begin{aligned}
& -\bar{x}'(t) + \gamma_x(t, x^*(t), u^*(t))\bar{x}(t) = -\gamma_u(t, x^*(t), u^*(t))\bar{u}(t) \\
& \Leftrightarrow -\langle p(t), \bar{x}'(t) \rangle + \langle p(t), \gamma_x(t, x^*(t), u^*(t))\bar{x}(t) \rangle = -\langle p(t), \gamma_u(t, x^*(t), u^*(t))\bar{u}(t) \rangle.
\end{aligned}$$

Integrating, we obtain

$$\begin{aligned}
& - \int_0^T \langle p(t), \bar{x}'(t) \rangle dt + \int_0^T \langle p(t), \gamma_x(t, x^*(t), u^*(t)) \bar{x}(t) \rangle dt \\
& = - \int_0^T \langle p(t), \gamma_u(t, x^*(t), u^*(t)) \bar{u}(t) \rangle dt \\
& = - \int_0^T \langle \gamma_u^\top(t, x^*(t), u^*(t)) p(t), \bar{u}(t) \rangle dt. \quad (7.75)
\end{aligned}$$

From another part, we have that

$$\begin{aligned}
& \underline{\lambda}^* \int_0^T \langle \nabla_{\#x} \phi(t, x^*(t), u^*(t)), \bar{x}(t) \rangle dt + \bar{\lambda}^* \int_0^T \langle \nabla_{\#x} \bar{\phi}(t, x^*(t), u^*(t)), \bar{x}(t) \rangle dt \\
& = \int_0^T \langle p'(t) + \gamma_x^\top(t, x^*(t), u^*(t)) p(t), \bar{x}(t) \rangle dt, \quad \text{by (7.74)} \\
& = \langle \bar{x}(t), p(t) \rangle \Big|_0^T - \int_0^T \langle p(t), \bar{x}'(t) \rangle dt + \int_0^T \langle p(t), \gamma_x(t, x^*(t), u^*(t)) \bar{x}(t) \rangle dt \\
& = \langle \bar{x}(T), p(T) \rangle - \langle \bar{x}(0), p(0) \rangle - \int_0^T \langle p(t), \bar{x}'(t) \rangle dt + \int_0^T \langle p(t), \gamma_x(t, x^*(t), u^*(t)) \bar{x}(t) \rangle dt \\
& = 0 - 0 - \int_0^T \langle \gamma_u^\top(t, x^*(t), u^*(t)) p(t), \bar{u}(t) \rangle dt, \quad \text{by (7.69) (7.73), and (7.75)} \\
& = - \int_0^T \langle \gamma_u^\top(t, x^*(t), u^*(t)) p(t), \bar{u}(t) \rangle dt. \quad (7.76)
\end{aligned}$$

Replacing (7.76) in (7.72), we obtain

$$\begin{aligned}
& f'_1(\bar{u}) \\
& = \underline{\lambda}^* \int_0^T \langle \nabla_{\#u} \phi(t, x^*(t), u^*(t)), \bar{u}(t) \rangle dt + \bar{\lambda}^* \int_0^T \langle \nabla_{\#u} \bar{\phi}(t, x^*(t), u^*(t)), \bar{u}(t) \rangle dt \\
& \quad - \int_0^T \langle \gamma_u^\top(t, x^*(t), u^*(t)) p(t), \bar{u}(t) \rangle dt \\
& = \int_0^T \left\langle \underline{\lambda}^* \nabla_{\#u} \phi(t, x^*(t), u^*(t)) + \bar{\lambda}^* \nabla_{\#u} \bar{\phi}(t, x^*(t), u^*(t)) - \gamma_u^\top(t, x^*(t), u^*(t)) p(t), \bar{u}(t) \right\rangle dt,
\end{aligned}$$

where $\bar{u}$ is arbitrary and $f'_1(\bar{u})$ is a supporting functional to $Q'_1 \subset L_\infty$ at $u^* \in Q'_1$, and since f'_1 is also a linear integral functional, by Example 2.11.1, we obtain

$$\left\langle \underline{\lambda}^* \nabla_{\#u} \phi(t, x^*(t), u^*(t)) + \bar{\lambda}^* \nabla_{\#u} \bar{\phi}(t, x^*(t), u^*(t)) - \gamma_u^\top(t, x^*(t), u^*(t)) p(t), u - u^*(t) \right\rangle \geq 0$$

for all $u \in \mathcal{M}$ and almost all $t \in [0, T]$, i.e., inequality (7.65) is satisfied.

Now, under these assumptions, the case $\underline{\lambda}^* = 0$ and $\bar{\lambda}^* = 0$ cannot occur simultaneously, since by (7.66), (7.67), and (7.72), we would have that for all $(\bar{x}, \bar{u}) \in E$, $\underline{f}_0(\bar{x}, \bar{u}) = 0$, $\bar{f}_0(\bar{x}, \bar{u}) = 0$, and $f'_1(\bar{u}) = 0$, respectively, so by (7.68), $f_1(\bar{x}, \bar{u}) = f'_1(\bar{u}) = 0$. Then, we would have that $\underline{f}_0 = \bar{f}_0 = f_1 = 0$, and (7.71) would then imply also that $f_2 = 0$, which contradicts the non-triviality condition that says that there exist $\underline{f}_0, \bar{f}_0, f_1, f_2$, not all zero, see item (D). Therefore, we can assume that $\underline{\lambda}^* + \bar{\lambda}^* = 1$.

(ii) Suppose that $\underline{\mathcal{K}}_0 = \emptyset$, that is,

$$\begin{aligned} & \int_0^T [\langle \nabla_{\#u} \underline{\phi}(t, x^*(t), u^*(t)), \bar{u}(t) \rangle + \langle 0 \nabla_{\#u} \bar{\phi}(t, x^*(t), u^*(t)), \bar{u}(t) \rangle \\ & \quad \langle \nabla_{\#x} \underline{\phi}(t, x^*(t), u^*(t)), \bar{x}(t) \rangle + \langle 0 \nabla_{\#x} \bar{\phi}(t, x^*(t), u^*(t)), \bar{x}(t) \rangle] dt = 0. \end{aligned} \quad (7.77)$$

In (7.74), by taking $\underline{\lambda}^* = 1$, $\bar{\lambda}^* = 0$, and $p(T) = \mathbf{0}$,

$$\begin{aligned} & \int_0^T [\langle \nabla_{\#x} \underline{\phi}(t, x^*(t), u^*(t)), \bar{x}(t) \rangle + \langle 0 \nabla_{\#x} \bar{\phi}(t, x^*(t), u^*(t)), \bar{x}(t) \rangle] dt \\ &= \int_0^T \langle p'(t) + \gamma_x^\top(t, x^*(t), u^*(t)) p(t), \bar{x}(t) \rangle dt \\ &= \langle \bar{x}(t), p(t) \rangle \Big|_0^T - \int_0^T \langle p(t), \bar{x}'(t) \rangle dt + \int_0^T \langle p(t), \gamma_x(t, x^*(t), u^*(t)) \bar{x}(t) \rangle dt \\ &= \langle \bar{x}(T), p(T) \rangle - \langle \bar{x}(0), p(0) \rangle - \int_0^T \langle p(t), \bar{x}'(t) \rangle dt + \int_0^T \langle p(t), \gamma_x(t, x^*(t), u^*(t)) \bar{x}(t) \rangle dt \\ &= 0 - 0 - \int_0^T \langle \gamma_u^\top(t, x^*(t), u^*(t)) p(t), \bar{u}(t) \rangle dt, \quad \text{by (7.75)} \\ &= - \int_0^T \langle \gamma_u^\top(t, x^*(t), u^*(t)) p(t), \bar{u}(t) \rangle dt. \end{aligned} \quad (7.78)$$

Replacing (7.78) in (7.77), we obtain

$$\int_0^T \langle \nabla_{\#u} \underline{\phi}(t, x^*(t), u^*(t)) + 0 \nabla_{\#u} \bar{\phi}(t, x^*(t), u^*(t)) - \gamma_u^\top(t, x^*(t), u^*(t)) p(t), \bar{u}(t) \rangle dt = 0$$

for all $\bar{u}$. Hence

$$\nabla_{\#u} \underline{\phi}(t, x^*(t), u^*(t)) + 0 \nabla_{\#u} \bar{\phi}(t, x^*(t), u^*(t)) - \gamma_u^\top(t, x^*(t), u^*(t)) p(t) = 0 \quad (7.79)$$

for almost all $t \in [0, T]$ and all $u \in \mathcal{M}$, i.e., inequality (7.65), for $\underline{\lambda}^* = 1$, $\bar{\lambda}^* = 0$, and $p(T) = \mathbf{0}$ such that (7.79) holds, is satisfied. Notice that, $\underline{\lambda}^* + \bar{\lambda}^* = 1 + 0 = 1$.

(iii) Suppose that $\bar{\mathcal{K}}_0 = \emptyset$, and taking $\underline{\lambda}^* = 0$, $\bar{\lambda}^* = 1$, and $p(T) = \mathbf{0}$, the calculus are the same of Case (ii).

Therefore, the proof is complete for all the Cases (i), (ii), and (iii) with $\underline{\lambda}^* + \bar{\lambda}^* = 1$. $\square$

Theorem 7.8.2 (Necessary optimality conditions of Local Pontryagin Maximum Principle—type for $(\widetilde{LIOCP})$). *Let $(x^*(t), u^*(t))$ be a weak optimal LU—process of $(\widetilde{LIOCP})$. Assume that (A1) – (A5) hold. Then, there exists a function $p : [0, T] \rightarrow \mathbb{R}^n$, satisfying*

$$p'(t) \in \nabla_{xgH} \Phi(t, x^*(t), u^*(t)) \oplus (-\gamma_x^\top(t, x^*(t), u^*(t)) p(t)). \quad (7.80)$$

Moreover, there exists $y = (y_1, \dots, y_m) \in \mathbb{R}^m$ such that

$y \in \nabla_{uqH}\Phi(t, x^*(t), u^*(t)) \oplus (-\gamma_u^\top(t, x^*(t), u^*(t))p(t))$ and

$$\langle y, u - u^*(t) \rangle \geq 0 \quad (7.81)$$

for almost all $t \in [0, T]$ and all $u \in \mathcal{M}$.

Proof. From Theorem 7.8.1, there exist reals

$$\underline{\lambda}^* \geq 0 \quad \text{and} \quad \bar{\lambda}^* \geq 0 \quad \text{with} \quad \underline{\lambda}^* + \bar{\lambda}^* = 1 \quad (7.82)$$

and a function $p : [0, T] \rightarrow \mathbb{R}^n$, satisfying

$$p'(t) = \underline{\lambda}^* \nabla_{\#x} \underline{\phi}(t, x^*(t), u^*(t)) + \bar{\lambda}^* \nabla_{\#x} \bar{\phi}(t, x^*(t), u^*(t)) - \gamma_x^\top(t, x^*(t), u^*(t))p(t) \quad (7.83)$$

and

$$p(T) = 0. \quad (7.84)$$

Moreover

$$0 \leq \left\langle \underline{\lambda}^* \nabla_{\#u} \underline{\phi}(t, x^*(t), u^*(t)) + \bar{\lambda}^* \nabla_{\#u} \bar{\phi}(t, x^*(t), u^*(t)) - \gamma_u^\top(t, x^*(t), u^*(t))p(t), u - u^*(t) \right\rangle \quad (7.85)$$

for almost all $t \in [0, T]$ and all $u \in \mathcal{M}$.

By Equation (7.83), we have that

$$\begin{aligned} \begin{bmatrix} p'_1(t) \\ \vdots \\ p'_n(t) \end{bmatrix}_{n \times 1} &= \underline{\lambda}^* \begin{bmatrix} \frac{\partial \underline{\phi}}{\partial x_1^\#}(t, x^*(t), u^*(t)) \\ \vdots \\ \frac{\partial \underline{\phi}}{\partial x_n^\#}(t, x^*(t), u^*(t)) \end{bmatrix}_{n \times 1} + \bar{\lambda}^* \begin{bmatrix} \frac{\partial \bar{\phi}}{\partial x_1^\#}(t, x^*(t), u^*(t)) \\ \vdots \\ \frac{\partial \bar{\phi}}{\partial x_n^\#}(t, x^*(t), u^*(t)) \end{bmatrix}_{n \times 1} \\ &\quad - \begin{bmatrix} \frac{\partial \gamma_1}{\partial x_1}(t, x^*(t), u^*(t)) & \dots & \frac{\partial \gamma_n}{\partial x_1}(t, x^*(t), u^*(t)) \\ \vdots & \ddots & \vdots \\ \frac{\partial \gamma_1}{\partial x_n}(t, x^*(t), u^*(t)) & \dots & \frac{\partial \gamma_n}{\partial x_n}(t, x^*(t), u^*(t)) \end{bmatrix}_{n \times n} \begin{bmatrix} p_1(t) \\ \vdots \\ p_n(t) \end{bmatrix}_{n \times 1} \\ &= \begin{bmatrix} \underline{\lambda}^* \frac{\partial \underline{\phi}}{\partial x_1^\#}(t, x^*(t), u^*(t)) \\ \vdots \\ \underline{\lambda}^* \frac{\partial \underline{\phi}}{\partial x_n^\#}(t, x^*(t), u^*(t)) \end{bmatrix}_{n \times 1} + \begin{bmatrix} \bar{\lambda}^* \frac{\partial \bar{\phi}}{\partial x_1^\#}(t, x^*(t), u^*(t)) \\ \vdots \\ \bar{\lambda}^* \frac{\partial \bar{\phi}}{\partial x_n^\#}(t, x^*(t), u^*(t)) \end{bmatrix}_{n \times 1} - \begin{bmatrix} \sum_{j=1}^n \frac{\partial \gamma_j}{\partial x_1}(t, x^*(t), u^*(t))p_j(t) \\ \vdots \\ \sum_{j=1}^n \frac{\partial \gamma_j}{\partial x_n}(t, x^*(t), u^*(t))p_j(t) \end{bmatrix}_{n \times 1} \\ &= \begin{bmatrix} \underline{\lambda}^* \frac{\partial \underline{\phi}}{\partial x_1^\#}(t, x^*(t), u^*(t)) + \bar{\lambda}^* \frac{\partial \bar{\phi}}{\partial x_1^\#}(t, x^*(t), u^*(t)) - \sum_{j=1}^n \frac{\partial \gamma_j}{\partial x_1}(t, x^*(t), u^*(t))p_j(t) \\ \vdots \\ \underline{\lambda}^* \frac{\partial \underline{\phi}}{\partial x_n^\#}(t, x^*(t), u^*(t)) + \bar{\lambda}^* \frac{\partial \bar{\phi}}{\partial x_n^\#}(t, x^*(t), u^*(t)) - \sum_{j=1}^n \frac{\partial \gamma_j}{\partial x_n}(t, x^*(t), u^*(t))p_j(t) \end{bmatrix}_{n \times 1}. \end{aligned}$$

Therefore,

$$\begin{aligned}
p'_i(t) &= \underline{\lambda}^* \frac{\partial \phi}{\partial x_i^\#}(t, x^*(t), u^*(t)) + \bar{\lambda}^* \frac{\partial \bar{\phi}}{\partial x_i^\#}(t, x^*(t), u^*(t)) - \sum_{j=1}^n \frac{\partial \gamma_j}{\partial x_i}(t, x^*(t), u^*(t)) p_j(t), \quad i \in \{1, \dots, n\} \\
&\Leftrightarrow \underline{\lambda}^* \frac{\partial \phi}{\partial x_i^\#}(t, x^*(t), u^*(t)) + \bar{\lambda}^* \frac{\partial \bar{\phi}}{\partial x_i^\#}(t, x^*(t), u^*(t)) = \sum_{j=1}^n \frac{\partial \gamma_j}{\partial x_i}(t, x^*(t), u^*(t)) p_j(t) + p'_i(t),
\end{aligned}$$

$i \in \{1, \dots, n\}$, with $\underline{\lambda}^* \geq 0$, $\bar{\lambda}^* \geq 0$, and $\underline{\lambda}^* + \bar{\lambda}^* = 1$, so by Definition 2.3.1 and by Proposition 2.3.3–(i), we have that

$$\begin{aligned}
\sum_{j=1}^n \frac{\partial \gamma_j}{\partial x_i}(t, x^*(t), u^*(t)) p_j(t) + p'_i(t) &\in \left[\frac{\partial \phi}{\partial x_i^\#}(t, x^*(t), u^*(t)) \vee \frac{\partial \bar{\phi}}{\partial x_i^\#}(t, x^*(t), u^*(t)) \right] \\
&\Leftrightarrow p'_i(t) \in \left[\frac{\partial \phi}{\partial x_i^\#}(t, x^*(t), u^*(t)) \vee \frac{\partial \bar{\phi}}{\partial x_i^\#}(t, x^*(t), u^*(t)) \right] - \sum_{j=1}^n \frac{\partial \gamma_j}{\partial x_i}(t, x^*(t), u^*(t)) p_j(t) \\
&\Leftrightarrow p'_i(t) \in \frac{\partial_{gH}\Phi}{\partial x_i}(t, x^*(t), u^*(t)) - \sum_{j=1}^n \frac{\partial \gamma_j}{\partial x_i}(t, x^*(t), u^*(t)) p_j(t), \quad i \in \{1, \dots, n\}. \quad (7.86)
\end{aligned}$$

Then, by Definition 2.3.4,

$$\begin{aligned}
p'(t) &= (p'_1(t), \dots, p'_n(t)) \\
&\in \left(\frac{\partial_{gH}\Phi}{\partial x_1}(t, x^*(t), u^*(t)) - \sum_{j=1}^n \frac{\partial \gamma_j}{\partial x_1}(t, x^*(t), u^*(t)) p_j(t), \dots, \frac{\partial_{gH}\Phi}{\partial x_n}(t, x^*(t), u^*(t)) - \sum_{j=1}^n \frac{\partial \gamma_j}{\partial x_n}(t, x^*(t), u^*(t)) p_j(t) \right) \\
&= \left(\frac{\partial_{gH}\Phi}{\partial x_1}(t, x^*(t), u^*(t)), \dots, \frac{\partial_{gH}\Phi}{\partial x_n}(t, x^*(t), u^*(t)) \right) \\
&\quad \oplus \left(- \left(\sum_{j=1}^n \frac{\partial \gamma_j}{\partial x_1}(t, x^*(t), u^*(t)) p_j(t), \dots, \sum_{j=1}^n \frac{\partial \gamma_j}{\partial x_n}(t, x^*(t), u^*(t)) p_j(t) \right) \right) \\
&= \nabla_{xgH}\Phi(t, x^*(t), u^*(t)) \oplus (-\gamma_x^\top(t, x^*(t), u^*(t)) p(t)).
\end{aligned}$$

Therefore, there exists a function $p : [0, T] \rightarrow \mathbb{R}^n$, satisfying

$$p'(t) \in \nabla_{xgH}\Phi(t, x^*(t), u^*(t)) \oplus (-\gamma_x^\top(t, x^*(t), u^*(t)) p(t)).$$

By Equation (7.85), we have that

$$\begin{aligned}
0 \leq \langle \underline{\lambda}^* &\begin{bmatrix} \frac{\partial \phi}{\partial u_1^\#}(t, x^*(t), u^*(t)) \\ \vdots \\ \frac{\partial \phi}{\partial u_m^\#}(t, x^*(t), u^*(t)) \end{bmatrix}_{m \times 1} + \bar{\lambda}^* \begin{bmatrix} \frac{\partial \bar{\phi}}{\partial u_1^\#}(t, x^*(t), u^*(t)) \\ \vdots \\ \frac{\partial \bar{\phi}}{\partial u_m^\#}(t, x^*(t), u^*(t)) \end{bmatrix}_{m \times 1} \\
&- \begin{bmatrix} \frac{\partial \gamma_1}{\partial u_1}(t, x^*(t), u^*(t)) & \dots & \frac{\partial \gamma_n}{\partial u_1}(t, x^*(t), u^*(t)) \\ \vdots & \ddots & \vdots \\ \frac{\partial \gamma_1}{\partial u_m}(t, x^*(t), u^*(t)) & \dots & \frac{\partial \gamma_n}{\partial u_m}(t, x^*(t), u^*(t)) \end{bmatrix}_{m \times n} \begin{bmatrix} p_1(t) \\ \vdots \\ p_n(t) \end{bmatrix}_{n \times 1}, \begin{bmatrix} u_1 - u_1^*(t) \\ \vdots \\ u_m - u_m^*(t) \end{bmatrix}_{n \times 1} \rangle
\end{aligned}$$

$$\begin{aligned}
& \Leftrightarrow 0 \leq \left\langle \begin{bmatrix} \underline{\lambda}^* \frac{\partial \phi}{\partial u_1^\#}(t, x^*(t), u^*(t)) \\ \vdots \\ \underline{\lambda}^* \frac{\partial \phi}{\partial u_m^\#}(t, x^*(t), u^*(t)) \end{bmatrix}_{m \times 1} + \begin{bmatrix} \bar{\lambda}^* \frac{\partial \bar{\phi}}{\partial u_1^\#}(t, x^*(t), u^*(t)) \\ \vdots \\ \bar{\lambda}^* \frac{\partial \bar{\phi}}{\partial u_m^\#}(t, x^*(t), u^*(t)) \end{bmatrix}_{m \times 1} - \begin{bmatrix} \sum_{j=1}^n \frac{\partial \gamma_j}{\partial u_1}(t, x^*(t), u^*(t)) p_j(t) \\ \vdots \\ \sum_{j=1}^n \frac{\partial \gamma_j}{\partial u_m}(t, x^*(t), u^*(t)) p_j(t) \end{bmatrix}_{m \times 1}, \right. \\
& \quad \left. \begin{bmatrix} u_1 - u_1^*(t) \\ \vdots \\ u_m - u_m^*(t) \end{bmatrix}_{n \times 1} \right\rangle \\
& \Leftrightarrow 0 \leq \left\langle \begin{bmatrix} \underline{\lambda}^* \frac{\partial \phi}{\partial u_1^\#}(t, x^*(t), u^*(t)) + \bar{\lambda}^* \frac{\partial \bar{\phi}}{\partial u_1^\#}(t, x^*(t), u^*(t)) - \sum_{j=1}^n \frac{\partial \gamma_j}{\partial u_1}(t, x^*(t), u^*(t)) p_j(t) \\ \vdots \\ \underline{\lambda}^* \frac{\partial \phi}{\partial u_m^\#}(t, x^*(t), u^*(t)) + \bar{\lambda}^* \frac{\partial \bar{\phi}}{\partial u_m^\#}(t, x^*(t), u^*(t)) - \sum_{j=1}^n \frac{\partial \gamma_j}{\partial u_m}(t, x^*(t), u^*(t)) p_j(t) \end{bmatrix}_{m \times 1}, \right. \\
& \quad \left. \begin{bmatrix} u_1 - u_1^*(t) \\ \vdots \\ u_m - u_m^*(t) \end{bmatrix}_{n \times 1} \right\rangle \\
& \Leftrightarrow 0 \leq \langle y, u - u^*(t) \rangle, \tag{7.87}
\end{aligned}$$

where $y = (y_1, \dots, y_m)$ and

$$y_k := \underline{\lambda}^* \frac{\partial \phi}{\partial u_k^\#}(t, x^*(t), u^*(t)) + \bar{\lambda}^* \frac{\partial \bar{\phi}}{\partial u_k^\#}(t, x^*(t), u^*(t)) - \sum_{j=1}^n \frac{\partial \gamma_j}{\partial u_k}(t, x^*(t), u^*(t)) p_j(t), \tag{7.88}$$

$k \in \{1, \dots, m\}$, for almost all $t \in [0, T]$ and all $u \in \mathcal{M}$.

We have that

$$\underline{\lambda}^* \frac{\partial \phi}{\partial u_k^\#}(t, x^*(t), u^*(t)) + \bar{\lambda}^* \frac{\partial \bar{\phi}}{\partial u_k^\#}(t, x^*(t), u^*(t)) = \sum_{j=1}^n \frac{\partial \gamma_j}{\partial u_k}(t, x^*(t), u^*(t)) p_j(t) + y_k,$$

$k \in \{1, \dots, m\}$, with $\underline{\lambda}^*, \bar{\lambda}^* \geq 0$, and $\underline{\lambda}^* + \bar{\lambda}^* = 1$, so by Definition 2.3.1 and by Proposition 2.3.3–(i), we have that

$$\begin{aligned}
& \sum_{j=1}^n \frac{\partial \gamma_j}{\partial u_k}(t, x^*(t), u^*(t)) p_j(t) + y_k \in \left[\frac{\partial \phi}{\partial u_k^\#}(t, x^*(t), u^*(t)) \vee \frac{\partial \bar{\phi}}{\partial u_k^\#}(t, x^*(t), u^*(t)) \right] \\
& \Leftrightarrow y_k \in \left[\frac{\partial \phi}{\partial u_k^\#}(t, x^*(t), u^*(t)) \vee \frac{\partial \bar{\phi}}{\partial u_k^\#}(t, x^*(t), u^*(t)) \right] - \sum_{j=1}^n \frac{\partial \gamma_j}{\partial u_k}(t, x^*(t), u^*(t)) p_j(t) \\
& \Leftrightarrow y_k \in \frac{\partial g_H \Phi}{\partial u_k}(t, x^*(t), u^*(t)) - \sum_{j=1}^n \frac{\partial \gamma_j}{\partial u_k}(t, x^*(t), u^*(t)) p_j(t), \quad k \in \{1, \dots, m\}. \tag{7.89}
\end{aligned}$$

Then, by Definition 2.3.4

$$\begin{aligned}
y &= (y_1, \dots, y_m) \\
&\in \left(\frac{\partial_{gH}\Phi}{\partial u_1}(t, x^*(t), u^*(t)) - \sum_{j=1}^n \frac{\partial \gamma_j}{\partial u_1}(t, x^*(t), u^*(t))p_j(t), \dots, \frac{\partial_{gH}\Phi}{\partial u_m}(t, x^*(t), u^*(t)) - \sum_{j=1}^n \frac{\partial \gamma_j}{\partial u_m}(t, x^*(t), u^*(t))p_j(t) \right) \\
&= \left(\frac{\partial_{gH}\Phi}{\partial u_1}(t, x^*(t), u^*(t)), \dots, \frac{\partial_{gH}\Phi}{\partial u_m}(t, x^*(t), u^*(t)) \right) \\
&\quad \oplus \left(- \left(\sum_{j=1}^n \frac{\partial \gamma_j}{\partial u_1}(t, x^*(t), u^*(t))p_j(t), \dots, \sum_{j=1}^n \frac{\partial \gamma_j}{\partial u_m}(t, x^*(t), u^*(t))p_j(t) \right) \right) \\
&= \nabla_{ugH}\Phi(t, x^*(t), u^*(t)) \oplus (-\gamma_u^\top(t, x^*(t), u^*(t))p(t)). \tag{7.90}
\end{aligned}$$

Therefore, by (7.87) and (7.90), there exists $y = (y_1, \dots, y_m) \in \mathbb{R}^m$ such that $y \in \nabla_{ugH}\Phi(t, x^*(t), u^*(t)) \oplus (-\gamma_u^\top(t, x^*(t), u^*(t))p(t))$ and $\langle y, u - u^*(t) \rangle \geq 0$ for almost all $t \in [0, T]$ and all $u \in \mathcal{M}$. $\square$

Remark 7.8.1. (i) It is natural to write the “interval-valued Pontryagin function”

$$\mathcal{H} : [0, T] \times \mathbb{R}^n \times \mathbb{R}^m \times \mathbb{R}^n \rightarrow \mathcal{K}_C \text{ as}$$

$$\mathcal{H}(t, x, p, u) = -\Phi(t, x, u) \oplus \langle \gamma(t, x, u), p(t) \rangle. \tag{7.91}$$

(ii) Let us assume that $\mathcal{H}$ and Φ are gH -differentiable with respect to u . In addition, let us assume that

$$\nabla_{ugH}\mathcal{H}(t, x^*(t), p(t), u^*(t)) = -\nabla_{ugH}\Phi(t, x^*(t), u^*(t)) \oplus \gamma_u^\top(t, x^*(t), u^*(t))p(t). \tag{7.92}$$

If (x^*, u^*) is a weak optimal LU -process of $(\widetilde{LIOCP})$ and the assumptions of Theorem 7.8.2 are satisfied, then $\mathcal{H}(t, x^*(t), p(t), u(t))$, as a function of u on $\mathcal{M}$, satisfies the necessary conditions for a maximum for almost all $t \in [0, T]$ at $u = u^*(t)$.

By Chapter 6, see Theorem 6.6.1, a necessary condition for the maximization of “the interval-valued Pontryagin function” is condition (7.81), provided that $\nabla_{ugH}\mathcal{H}(t, x^*(t), p(t), u^*(t))$ coincides with

$$-\nabla_{ugH}\Phi(t, x^*(t), u^*(t)) \oplus \gamma_u^\top(t, x^*(t), u^*(t))p(t).$$

(iii) The equality (7.92) may not be satisfied, since if F and G are two gH -differentiable interval-valued functions at x_0 , it is not always true that $F+G$ is gH -differentiable at x_0 , see Theorems 3.2 and 3.3 of Y. Chalco-Cano, G. G. Maqui-Huamán, G. N. Silva, M. D. Jiménez-Gamero, [18].

Example 7.8.1. Consider the following interval optimal control problem

$$\begin{cases} \text{minimize } F(x, u) = \frac{1}{2} \int_0^1 x[0.9, 1.1]x dt \\ \text{subject to } x'(t) = u^2, \\ (x(0), x(1)) \in \{1\} \times \mathbb{R}, \\ u \in \mathcal{M} = [-1, 0] \text{ a. e. } t \in [0, 1], \end{cases}$$

where $\gamma : [0, 1] \times \mathbb{R} \times \mathbb{R} \rightarrow \mathbb{R}$ is given by $\gamma(t, x, u) = u^2$, and $\Phi : [0, 1] \times \mathbb{R} \times \mathbb{R} \rightarrow \mathcal{K}_C$ is given by $\Phi(t, x, u) = \frac{1}{2} (x[0.9, 1.1]x) = [\frac{0.9}{2}x^2, \frac{1.1}{2}x^2] = [\underline{\phi}(t, x, u), \overline{\phi}(t, x, u)]$ with $\underline{\phi}, \overline{\phi} : [0, 1] \times \mathbb{R} \times \mathbb{R} \rightarrow \mathbb{R}$ defined by $\underline{\phi}(t, x, u) = \frac{0.9}{2}x^2$, $\overline{\phi}(t, x, u) = \frac{1.1}{2}x^2$ and $\text{int } \mathcal{M} = (-1, 0)$.

By Equality (7.91), we have that

$$\mathcal{H}(t, x, p, u) = -\Phi(t, x, u) \oplus \langle \gamma(t, x, u), p \rangle = -\left[\frac{0.9}{2}x^2, \frac{1.1}{2}x^2\right] \oplus \langle u^2, p \rangle.$$

Then

$$\nabla_{ugH} \mathcal{H}(t, x, p, u) = \nabla_{ugH} \left(-\left[\frac{0.9}{2}x^2, \frac{1.1}{2}x^2\right] \oplus \langle u^2, p \rangle \right) = 2up. \quad (7.93)$$

It is easily seen that

- (1) $(x^*(t), u^*(t)) = (1, 0)$ is a local weak optimal LU-process of the problem.
- (2) $\Phi(t, x, u)$ is Lipschitz continuous in a neighborhood of (x^*, u^*) .
- (3) Φ is continuously gH -differentiable, $\gamma(t, x, u)$ is continuous, continuously differentiable, and measurable (with respect to t).
- (4) For all bounded x and u , and all $t \in [0, 1]$, $\nabla_{\#x} \underline{\phi}(t, x, u) = 0.9x$, $\nabla_{\#x} \overline{\phi}(t, x, u) = 1.1x$, $\nabla_{\#u} \underline{\phi}(t, x, u) = 0$, and $\nabla_{\#u} \overline{\phi}(t, x, u) = 0$ exist, and are bounded.
- (5) For all bounded x and u , and all $t \in [0, 1]$, $\gamma_x(t, x, u) = 0$ and $\gamma_u(t, x, u) = 2u$ are measurable (with respect to t), and are bounded.
- (6) $\mathcal{M}$ is a closed convex set in $\mathbb{R}$ such that $\text{int } \mathcal{M} \neq \emptyset$.

Then, by Theorem 7.8.2, there exist a function $p : [0, 1] \rightarrow \mathbb{R}$ defined by $p(t) = t$, satisfying (7.80):

$$\begin{aligned} p'(t) &\in \nabla_{xgH} \Phi(t, x^*(t), u^*(t)) \oplus (-\gamma_x^\top(t, x^*(t), u^*(t))p(t)) \\ &\Leftrightarrow 1 \in [0.9x^*(t), 1.1x^*(t)] \oplus [-0(t), -0(t)] = [0.9, 1.1], \quad \text{since } x^*(t) = 1. \end{aligned}$$

Moreover there exists $y = 0 \in \mathbb{R}$, such that

$$\begin{aligned} y &\in \nabla_{u_g H} \Phi(t, x^*(t), u^*(t)) \oplus (-\gamma_u^\top(t, x^*(t), u^*(t))p(t)) \\ \Leftrightarrow 0 &\in [0, 0] \oplus (-[2u^*(t)t, 2u^*(t)t]) = [0, 0], \quad \text{since } u^*(t) = 0, \end{aligned}$$

and (7.81) holds:

$$\langle y, u - u^*(t) \rangle \geq 0 \Leftrightarrow \langle 0, u \rangle \geq 0, \quad \text{since } y = 0 \text{ and } u^*(t) = 0,$$

for almost all $t \in [0, 1]$ and all $u \in [-1, 0]$.

Finally, Equality (7.92) holds

$$\begin{aligned} \nabla_{u_g H} \mathcal{H}(t, x^*(t), p(t), u^*(t)) &= -\nabla_{u_g H} \Phi(t, x^*(t), u^*(t)) \oplus \gamma_u^\top(t, x^*(t), u^*(t))p(t) \\ \Leftrightarrow 0 = 2u^*(t)p(t) &= -[0, 0] \oplus [2u^*(t)t, 2u^*(t)t] = [0, 0], \quad \text{by (7.93) and since } u^*(t) = 0. \end{aligned}$$

Then, a necessary condition for the maximization of the interval-valued Pontryagin function is Condition (7.81), that is,

$$\langle 0, u \rangle \geq 0$$

for almost all $t \in [0, 1]$ and all $u \in \mathcal{M}$.

7.9 Conclusion

We considered two interval optimal control problems in the Lagrange form, with right endpoint condition first fixed and then free, and we defined optimal LU -processes for them. Next, we gave some definitions for cones and properties of them. We established the cones: $\mathcal{K}_0$, $\mathcal{K}_1$, $\mathcal{K}_2$, and we characterized them. Also, we defined classical cones: $\underline{\mathcal{K}}_0$, $\overline{\mathcal{K}}_0$, and we gave properties of them. We obtained some important results. We analyzed some functionals and the constraints considered, we used the Dubovitskii–Milyutin formalism to give the Euler equation, and then we analyzed it. Therefore, we gave optimality conditions for the problems. Finally, we obtained an interval local Pontryagin maximum principle.

CHAPTER 8

CONCLUSIONS AND FUTURE PERSPECTIVES

In this Thesis, applying classical results, necessary conditions of Fritz John and Karush–Kuhn–Tucker–types for interval optimization problems, using the E –differentiability, were obtained.

Also, applying classical results, we gave not only necessary but also sufficient optimality conditions for interval optimal control problems. To this, we showed that a solution of an interval optimal control problem is a solution of a classical bi–objective control problem and conversely.

We presented a new approach to interval optimization using the total and directional generalized Hukuhara differentiability, for problems with several variables. With the Gordan Transposition Theorem, we obtained a Fritz John–type condition and with a constraint qualification or a regularity condition, we obtained interval Karush–Kuhn–Tucker–type conditions. Also, we defined, for interval optimization problems, the following concepts: KKT –solutions, strict KKT –solutions, weak (strong) KKT –interval, KKT –interval, strict KKT –interval, and partial– $\underline{LU}(\overline{LU})$ strict KKT –interval pseudoinvexities. Moreover, we proved that a KKT –solution for a weak (strong) KKT –interval pseudoinvex problem and for a KKT –interval pseudoinvex problem is a weak (strong) LU –solution and LU –solution, respectively. Also, we proved that a strict KKT –solution for a strict KKT –interval pseudoinvex problem and for a partial– $\underline{LU}(\overline{LU})$ strict KKT –interval pseudoinvex problem is a LU –solution and $\underline{LU}(\overline{LU})$ –solution, respectively. As far as it knows, strict KKT –type optimality conditions are news in the interval optimization area. Furthermore, it is important to mention that these concepts of generalized invexity were proved

to be the most general as possible.

Using the total and directional generalized Hukuhara differentiability and known results from the general extremum problem theory, we obtained necessary optimality condition for abstract interval optimization problems, which is a novelty in the interval optimization.

We gave necessary optimality conditions for interval optimal control problems in the Lagrange forms using again the total and directional generalized Hukuhara differentiability. With general extremum problem theory, we obtained an interval local Pontryagin Maximum Principle. In the literature, this work is outstanding since there is nothing like this in the interval optimal control area.

In future researches, we would like:

1. to provide applications of the proposed results;
2. to obtain an interval Pontryagin Maximum Principle;
3. to consider a more general problem of interval optimal control, in which there are also constraints on the phase coordinates;
4. to extend the optimality conditions obtained for interval optimization problems and for interval optimal control problems to the case of Fuzzy optimization problems and to the case of Fuzzy optimal control problems, respectively;
5. to give sufficient conditions for interval control problems, in which beyond of the objective functional be interval-valued, the variables also be intervals, not only for interval control problems but also for interval optimization.

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