PHYSICAL REVIEW D **79,** 023516 (2009)

Physical approximations for the nonlinear evolution of perturbations in inhomogeneous dark energy scenarios

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(Received 20 September 2008; published 22 January 2009)

The abundance and distribution of collapsed objects such as galaxy clusters will become an important tool to investigate the nature of dark energy and dark matter. Number counts of very massive objects are sensitive not only to the equation of state of dark energy, which parametrizes the smooth component of its pressure, but also to the sound speed of dark energy, which determines the amount of pressure in inhomogeneous and collapsed structures. Since the evolution of these structures must be followed well into the nonlinear regime, and a fully relativistic framework for this regime does not exist yet, we compare two approximate schemes: the widely used spherical collapse model and the pseudo-Newtonian approach. We show that both approximation schemes convey identical equations for the density contrast, when the pressure perturbation of dark energy is parametrized in terms of an effective sound speed. We also make a comparison of these approximate approaches to general relativity in the linearized regime, which lends some support to the approximations.

DOI: 10.1103/PhysRevD.79.023516 PACS numbers: 98.80.Cq, 95.36.+x

I. INTRODUCTION

We now have overwhelming evidence that the Universe is accelerating, possibly under the influence of some type of negative-pressure substance—dark energy (DE) [1–3]. However, even though DE may be directly responsible for this enhanced expansion, it is widely believed that the direct impact of perturbations in DE density and pressure on structure formation is very weak. This is strictly correct only for a cosmological constant model of DE, which does not have perturbations.

For most scalar field models of DE, this component remains very homogeneous even on galaxy and cluster scales. Heuristically, this can be understood as follows. In these models the scalar field cannot have relaxed to its minimum energy state, and one must require that the time scale for the variation of the field is longer than the Hubble time, implying a very flat potential. Therefore, the scalar field must be extraordinarily light, $m < H_0$, where H_0 is the Hubble parameter today. The mass of the scalar field sets the scale for its spatial variation, and hence one usually expects small perturbations in the scalar field for scales $\lambda < 1/m$ (the Compton wavelength), which are of the order of the Hubble radius. However, this argument may not apply to more general models of dark energy.

If our only concern is the evolution of the background, then the role of dark energy in the evolution of dark matter perturbations is completely determined by its equation of state $w = p_e/\rho_e$, where p_e is the homogeneous pressure and ρ_e is the homogeneous energy density of dark energy [4–6]. At this level, dark energy affects structure formation indirectly because, as it starts to dominate the background, very large structures are ripped apart by the ensuing accelerated expansion [7,8].

However, dark energy can influence structure formation in an additional manner. If it is a dynamical field or fluid, then dark energy must possess inhomogeneities, and these perturbations will interact gravitationally both with themselves and with clumps of dark matter [9]. This means that, unless dark energy is just a cosmological constant, it will both feel and create local gravitational potentials.

Although the effect of these inhomogeneities in the dark energy component becomes small as $w \to -1$, in many models with $w \neq -1$ it can be non-negligible when evolved in the nonlinear regime [10–23]. Since the effects of dark energy perturbations on the cosmic microwave background are quite small (see, e.g., [24]), structure formation is the only remaining probe of the nature of dark energy on small and intermediate scales.

Nevertheless, a fully relativistic method to treat nonlinear perturbations is not available. When there is a pressure ingredient the nonlinear relativistic equations take a very complicated form. The Lemaître-Tolman-Bondi (LTB) model [25–27] is the closest one can get to a working formalism, but it only works if matter is pressureless. The problem is not with the gravity side of the equations, but with the nonlinear evolution of matter and the relativistic treatment of pressure.

In this respect, the only well-studied models with inhomogeneous dark energy are those involving canonical scalar fields [15–20], for which the equations of motion and the pressure follow directly from a given Lagrangian. For these models the free parameters are the scalar potential and some set of initial conditions. In this approach, the equation of state, the density perturbations, and the pressure perturbations are derived quantities. Hence, a more kinematical and model-independent approach to structure formation, closer in spirit to the homogeneous description of dark energy in terms of a parametrized equation of state $w(z)$, is sorely lacking.

There are two very different approximations to full-blown general relativity that have been frequently used. They are the spherical collapse (SC) model [28–30] and the pseudo-Newtonian (PN) approach [31–34]. In a previous work [21] we showed the equivalence between the SC/PN approaches for a dark energy model where the effective sound speed was equal to the equation of state, $c_{\rm eff}^2 = w$, and we computed the number counts for that model. In this particular case the equation of state of dark energy is homogeneous.

More recently, we have used these approximations in the nonlinear regime in order to show that the "effective equation of state" of dark energy inside a collapsed region could be very different from its background value, a phenomenon we dubbed "dark energy mutation" [23].

Some of the statements, such as the equivalence between the SC and PN approaches in a general dark energy model, were made without proof in [23] due to lack of space. The purpose of this paper is to fill this gap. We show explicitly that, even though the underlying assumptions for either the SC or the PN approach are rather different, they yield exactly the same nonperturbative equations as long as the pressure perturbations are treated in the same way. They also have an important advantage: they allow for a completely parametrized approach to dark energy. Furthermore, we compare the growth of perturbations in the linear regime with a linearized relativistic analysis and show that they are similar, lending support to the approximations.

This paper is organized as follows. In Sec. II we review both the PN and SC approaches and show that they are equivalent. In Sec. III we study the linear evolution of perturbations in DE in this approximation. The linear evolution of perturbations in a universe with a two-component fluid is studied in general relativity in Sec. IV. We present a comparison between the relativistic analysis and the approximate analysis in the linear regime in Sec. V. Section VI concludes.

II. SPHERICAL COLLAPSE AND PSEUDO-NEWTONIAN COSMOLOGY

In linear perturbation theory there are essentially 3 degrees of freedom for scalar perturbations: the energy density perturbation $\delta\rho$, the pressure perturbation δp, and the scalar anisotropic stress π [35,36]. An alternative set is given by the density contrast $\delta\rho/\rho$, the velocity potential $\theta = \vec{\nabla} \cdot \vec{v}$, and the anisotropic stress [37]. Since large-scale anisotropic stresses decay rapidly, they can only become relevant again inside structures which have collapsed. This means that anisotropic stress should not influence the mass of these structures, and therefore it is unlikely that dark energy models can be differentiated on the basis of anisotropic stress. For this reason we do not consider it any further in this work (see, however, [38]).

We will parametrize the pressure perturbation using the so-called effective sound velocity [39], defined as $c_{\rm eff}^2 \equiv \delta p_e/\delta\rho_e$. We will assume that $c_{\rm eff}^2$ is a function of time only, even though this simplification lacks any formal basis in cosmological perturbation theory. This should be clear from the fact that δp_e is an independent degree of freedom whose time and spatial dependences can be, and often are, completely different from $\delta\rho_e$. Only in a particular gauge (the so-called "rest frame" of the fluid, where $T_0^i = 0$) does the effective sound speed coincide with the universal sound speed of linear relativistic perturbations, c_X^2 [39,40]. It may be difficult to realize this parametrization in a natural model, but the situation is not much different from what happens when we parametrize the equation of state.

The main reason that we use the effective sound speed, though, is that it allows us to study nonlinear structure formation within the spherical collapse model [28]. In this extremely simple model, a spherically symmetric region of homogeneous overdensity evolves inside the homogeneous expanding Universe (this is the so-called "top-hat" density profile). General relativistic arguments show that one can regard the overdense region as a mini-universe of positive curvature, and then we use the Friedmann and the Raychaudhury equations to evolve the density and radius of the spherical region [29,30].

It is therefore extremely interesting that this simplified relativistic approach coincides with a pseudo-Newtonian approach to cosmology. In fact, we will show below that, as long as the pressure perturbations are described in terms of an effective pressure, the two approximations are completely equivalent. This means that the main physical characteristics of gravitational collapse of structures such as galaxy clusters is probably well described within this framework.

The argument is as follows. First, the SC approach should be a good approximation for large scales (where relativistic effects should matter most), but not necessarily for small scales, where the "mini-universe" argument is less persuasive. On the other hand, the PN approach is well motivated by the physics of gravity in small scales, but is not assured to work for large scales. That the two approaches coincide shows that, at least in some limited sense, the equations of the SC/PN approach should give a good description of the gravitational interactions on scales smaller than the Hubble radius.

A. Pseudo-Newtonian cosmology

In PN cosmology, particles in a comoving grid attract each other gravitationally with a Newtonian potential. The positions of the particles in the grid are the perturbed variables. Although obviously limited, this approach can be used for any configuration, not just the spherically symmetric ones. But in order to bring the PN approach closer to the SC model, we will adopt the same basic assumptions of the SC model for the PN cosmological perturbations.

We consider an admixture of two fluids, cold dark matter and dark energy. The key assumptions of the SC model (see the next subsection) are that the density of each fluid is homogeneous at all times in the spherical region (this is the top-hat density profile) and that the velocity profile preserves this homogeneity.

The comoving coordinates are $\vec{x}_0 = \vec{r}_0/a$, where $\vec{r}_0$ is the homogeneous (unperturbed) physical distance—here, the radius of a spherically symmetric region. Under the assumption of the SC model, the perturbed physical distance (physical radius) can be written as

$$\vec{r} = [a(t) + f(t, \vec{x}_0)]\vec{x}_0, \tag{1}$$

where a is the usual scale factor and f is the function that accounts for the deviations from the background evolution. The physical velocity is then given by

$$\vec{u} = \frac{d\vec{r}}{dt} = (\dot{a} + \dot{f})\vec{x}_0 = \left(H + \frac{\dot{f}}{a}\right)\vec{r}_0, \tag{2}$$

where $\dot{} = \partial/\partial t$ and $H = \dot{a}/a$ is the Hubble parameter. From the last equality we can define an effective rate of expansion for the spherical region:

$$h = H + \frac{\dot{f}}{a}. \tag{3}$$

Since the perturbed velocity is related to the peculiar velocity $\vec{v}$ by

$$\vec{u} = \dot{a}\vec{x}_0 + \vec{v}, \tag{4}$$

we obtain from Eq. (2) that

$$\vec{v} = \dot{f}\vec{x}_0. \tag{5}$$

In particular, the divergence of this velocity field is given by

$$\theta \equiv \vec{\nabla} \cdot \vec{v} = 3\dot{f} + \vec{x}_0 \cdot \vec{\nabla}\dot{f}. \tag{6}$$

But, for a top-hat profile the last term vanishes, and we obtain a simple relation between the local expansion rate h and the background expansion rate H:

$$h = H + \frac{\dot{f}}{a} = H + \frac{\theta}{3a}. \tag{7}$$

The PN cosmological model is described by the equations [31]

$$\frac{\partial \rho_j}{\partial t} + \vec{\nabla} \cdot (\vec{u}_j \rho_j) + p_j \vec{\nabla} \cdot \vec{u}_j = 0, \tag{8}$$

$$\frac{\partial \vec{u}_j}{\partial t} + (\vec{u}_j \cdot \vec{\nabla})\vec{u}_j = -\vec{\nabla}\Phi - \frac{\vec{\nabla} p_j}{\rho_j + p_j}, \tag{9}$$

$$\nabla^2 \Phi = 4\pi G \sum_k (\rho_k + 3p_k), \tag{10}$$

where ρ_j, p_j, and $\vec{u}_j$ denote, respectively, the density, pressure, and velocity of a given cosmic fluid and Φ is the Newtonian gravitational potential due to all the components; the equations are written in physical coordinates. The corresponding perturbations above the background are denoted by $\delta\rho_j$, δp_j, $\vec{v}_j$, and ϕ. These equations are, respectively, generalizations for fluids with pressure of the continuity equation, of the Euler equation (both valid for each fluid species j), and of the Poisson equation. Notice the absence of an equation that dictates the evolution of pressure: in this hydrodynamical approach, pressure is a thermodynamical function of the energy, temperature, etc.

For cold dark matter and baryons the pressure is zero, but for dark energy there is a homogeneous as well as an inhomogeneous pressure. The homogeneous pressure is usually described in terms of a parametrized equation of state $w_e(t)$, such that $p_e(t) = w_e(t)\rho_e(t)$. As for the pressure perturbations, we have chosen to specify another free function, the effective sound speed $c^2_{\rm eff}$, so $\delta p_e = c^2_{\rm eff}\delta\rho_e$. Within the SC description, this means that we consider an effective equation of state w_c inside the spherical region which is not necessarily equal to the background equation of state.

With the assumptions of the SC model, the equations of PN cosmology assume a simple form. Using the density contrast $\delta_j \equiv \delta\rho_j/\rho_j$ we obtain, after some algebra,

$$\dot{\delta}_j + 3H(c^2_{{\rm eff}j} - w_j)\delta_j + \frac{\theta_j}{a}[1 + w_j + (1 + c^2_{{\rm eff}j})\delta_j] = 0, \tag{11}$$

$$\dot{\theta}_j + H\theta_j + \frac{\theta_j^2}{3a} = -4\pi G a \sum_k \rho_{0k}\delta_k(1 + 3c^2_{{\rm eff}k}). \tag{12}$$

Equation (11) follows from the continuity equation, and Eq. (12) is the divergence of the Euler equation. The last equality in Eq. (12) is found by using the Poisson equation. Note that, in general, we have separate Euler equations for each fluid [21], but for a top-hat profile ($\vec{\nabla}\delta_j = 0$) they turn out to be identical, so there is only one θ. The reason for that is obvious: in order to preserve the top-hat profile, all fluids must flow in the same way. Hence, in this approximation we have something similar to an effective single fluid description [41].

B. The spherical collapse model

Let us now briefly review the spherical collapse model. This formalism describes a spherically symmetric region of uniform energy density $\rho_c = \rho_0 + \delta\rho$ immersed in a homogeneous universe of energy density ρ_0. This spherical region will detach from the expansion of the Universe and eventually collapse.

Consider the continuity equation for each fluid denoted by an index j in the spherical region:

$$\dot{\rho}_{c_j} + 3h(1 + w_{c_j})\rho_{c_j} = 0, \tag{13}$$

where $h = \dot{r}/r$ is the local expansion rate of that region and w_{c_j} denotes the equation of state in the perturbed region. We can regard this spherical region as a Friedmann universe with spatial curvature [28]. The dynamics of the coordinate r is then given by the second Friedmann equation applied to this collapsing region:

$$\frac{\ddot{r}}{r} = -\frac{4\pi G}{3}\sum_j(\rho_{c_j} + 3p_{c_j}). \tag{14}$$

Equations (13) and (14), which were obtained using general relativistic arguments, are the basic equations of the SC model. Note that there is only one dynamical equation for the collapsing region, which is in agreement with the single Euler equation that we found for the velocity field in the PN description, Eq. (12).

The pressure and the energy density outside the spherical region are related by the background equation of state, $p_{0_j} = w_{0_j}\rho_{0_j}$. Inside the spherical region these quantities can be different from their background values, so we have now $p_{c_j} = w_{c_j}\rho_{c_j}$ for the collapsing region. In order to compare the SC formalism with the PN equations derived in the last section, we will employ here the same effective sound speed we used before in order to describe the pressure perturbations. Hence, we need to express the equation of state w_{c_j} in terms of $c^2_{\mathrm{eff}j}$. Using the density contrast $\delta_j = \delta\rho_j/\rho_{0_j}$, we have that

$$\rho_{c_j} = (1 + \delta_j)\rho_{0_j}, \tag{15}$$

from which it follows that

$$w_{c_j} = \frac{p_{c_j}}{\rho_{c_j}} = \frac{p_{0_j} + \delta p_j}{\rho_{0_j} + \delta\rho_j} = w_j + (c^2_{\mathrm{eff}j} - w_j)\frac{\delta_j}{1 + \delta_j}. \tag{16}$$

This equation relates the equation of state in the perturbed region to the background equation of state, the effective sound speed, and the size of perturbations. It is possible that the nature of dark energy can be significantly changed in the perturbed region [23]. This effect is general, occurring even at the level of linear evolution, and its magnitude depends on the dynamical evolution of DM and DE fluctuations—see also Refs. [16,20].

Using now Eqs. (15) and (16) we can recast Eq. (13) as

$$\dot{\delta}_j + (3h - 3H)(1 + w_j)(1 + \delta_j) + 3h(c^2_{\mathrm{eff}j} - w_j)\delta_j = 0.$$

We can eliminate h using Eq. (7), with the result

$$\dot{\delta}_j + 3H(c^2_{\mathrm{eff}j} - w_j)\delta_j + [1 + w_j + (1 + c^2_{\mathrm{eff}j})\delta_j]\frac{\theta}{a} = 0. \tag{17}$$

Now consider the dynamical equation (14). From Eq. (7) we can write

$$\dot{h} = \frac{\ddot{r}}{r} - h^2 = \dot{H} + \frac{\dot{\theta}}{3a} - H\frac{\theta}{3a}, \tag{18}$$

and substituting this expression into Eq. (14) we obtain, with the help of Eqs. (15) and (16), that

$$\dot{\theta} + H\theta + \frac{\theta^2}{3a} = -4\pi G a\sum_k \rho_{0_k}\delta_k(1 + 3c^2_{\mathrm{eff}k}). \tag{19}$$

Equations (17) and (19) are identical to Eqs. (11) and (12). This means that both approaches are identical. The relations (7) and (16) enable us to translate the PN variables into the SC variables, and now it becomes clear that the two different descriptions give the same dynamics for a top-hat perturbation where pressure gradients are absent.

III. LINEAR EVOLUTION IN THE SC/PN APPROACH

Even though we showed that the PN and SC approaches are equivalent, that still does not mean that they are correct. Unfortunately, presently there is no fully nonlinear general treatment of the evolution of perturbations in general relativity (GR). For this reason, we will compare our linearized results with those obtained from linearized GR. We will compute the linear evolution of an overdense region well inside the matter-dominated era, and will compare the growing mode obtained in the PN/SC formalism with the relativistic growing mode.

The first-order equations can be linearized and recast as a single, second-order differential equation for the density contrast of each fluid species. We will assume that there is always a dominant (d) and a subdominant (s) fluid. Using the scale factor a for the time evolution ($' = d/da$), we obtain for the dominant species

$$\delta''_d + \frac{\delta'_d}{a}\left[3\Delta_d + \frac{3}{2}(1 - w_d)\right] + \frac{3\delta_d}{2a^2}[\Delta_d(1 - 3w_d) - (1 + w_d)(1 + 3c^2_{\mathrm{eff}_d})] = 0, \tag{20}$$

where

$$\Delta_d = (c^2_{\mathrm{eff}d} - w_d). \tag{21}$$

For cold dark matter ($c_{\mathrm{eff}} = w = 0$) this equation reduces to

$$\delta''_d + \frac{3}{2}\frac{\delta'_d}{a} + \frac{3\delta_d}{2a^2} = 0, \tag{22}$$

with the well-known growing solution $\delta(a) \propto a$. Hence, when cold dark matter is dominant, which should be the case in the linear regime, the linear evolution of its density perturbations is the standard one.

For the more general case, of a dominant fluid with constant equation of state and constant speed of sound, the solution is given by

$$\delta_d(a) = c_1 a^{1+3w_d} + c_2 a^{-3(1+2c^2_{\mathrm{eff}d}-w_d)/2}. \tag{23}$$

Turning now to the subdominant fluid, its perturbations obey the equation

$$\delta_s'' + \frac{\delta_s'}{a}\left[3\Delta_s + \frac{3}{2}(1-w_d)\right] + \frac{3\delta_s}{2a^2}[\Delta_s(1-3w_d)] = \frac{3\delta_d}{2a^2}(1+w_s)(1+3c^2_{\mathrm{eff}_d}), \tag{24}$$

where

$$\Delta_s = (c^2_{\mathrm{eff}s} - w_s). \tag{25}$$

Assuming again that w_s and $c^2_{\mathrm{eff}s}$ are constants, one has the solution

$$\delta_s(a) = c_3 a^{-3\Delta_s} + c_4 a^{(3w_s-1)/2} + c_5 a^{1+3w_d}, \tag{26}$$

where

$$c_5 = \tfrac{1}{2}c_1(1+w_s)(1+3c^2_{\mathrm{eff}d})[(1+3w_d)(\Delta_s + \tfrac{1}{2}(1+w_d)) + \tfrac{1}{2}(1-3w_d)\Delta_s]^{-1} \tag{27}$$

arises from a particular solution of the inhomogeneous equation.

In particular, if matter is the dominant fluid it follows that

$$c_5 = \frac{c_1(1+w_s)}{3\Delta_s + 1}, \tag{28}$$

and the dark energy density contrast grows in the same way as the dark matter density contrast. In addition, for the case in which $c^2_{\mathrm{eff}} = w$ one has an adiabatic condition satisfied, namely, $\delta_e = (1+w_e)\delta_m$. In general, however, the perturbations have a nonadiabatic component and the dark energy density contrast evolves as

$$\delta_s(a) = \frac{(1+w_s)}{3\Delta_s+1}\delta_d(a) + c_3 a^{-3\Delta_s}, \tag{29}$$

where the last term in the right-hand side is a decreasing mode in most cases.

IV. LINEAR EVOLUTION IN GR

In a previous paper [21] we showed that, for a single perfect fluid with no pressure gradients, the growing modes in the linearized SC/PN approach coincide with those found in GR. Now we want to compare the PN and the GR solutions for the dark energy perturbations in the linear regime, including pressure gradients. We will consider these perturbations during the matter-dominated period, i.e., while DE is subdominant. This is motivated by the fact that most observed structures were formed well into the matter-dominated period.

We consider scalar perturbations to the metric in the Newtonian gauge without anisotropic stress:

$$ds^2 = (1+2\phi)dt^2 - a^2(1-2\phi)d\vec{x}^2. \tag{30}$$

The (00) and (ii) components of the Einstein equations in Fourier space are

$$\frac{k^2}{a^2}\phi + 3H(\dot{\phi} + H\phi) = -4\pi G\sum_j \delta\rho_j, \tag{31}$$

$$\ddot{\phi} + 4H\dot{\phi} + \left(2\frac{\ddot{a}}{a} + H^2\right)\phi = 4\pi G\sum_j \delta p_j, \tag{32}$$

and the conservation equations $T^{\mu}_{0;\mu} = 0$ and $T^{\mu}_{i;\mu} = 0$ yield

$$\dot{\delta}_j + 3H(c^2_{\mathrm{eff}j} - w_j)\delta_j + (1+w_j)\left(\frac{\theta_j}{a} - 3\dot{\phi}\right) = 0, \tag{33}$$

$$\dot{\theta}_j + H(1-3c^2_{sj})\theta_j - \frac{k^2\delta p_j}{(1+w_j)\rho_j a} - \frac{k^2}{a}\phi = 0, \tag{34}$$

where $c^2_{sj} = \dot{p}_j/\dot{\rho}_j$ is the adiabatic speed of sound.

In summary, the evolution of perturbations in a system consisting of dark energy and dark matter in linearized GR is described by the following set of five coupled differential equations:

$$\ddot{\phi} + 4H\dot{\phi} + \left(2\frac{\ddot{a}}{a} + H^2\right)\phi = \frac{3}{2}H^2\Omega_e c^2_{\mathrm{eff}}\delta_e, \tag{35}$$

$$\dot{\delta}_m + \frac{\theta_m}{a} - 3\dot{\phi} = 0, \tag{36}$$

$$\dot{\delta}_e + (1+w_e)\left(\frac{\theta_e}{a} - 3\dot{\phi}\right) + 3H(c^2_{\mathrm{eff}} - w_e)\delta_e = 0, \tag{37}$$

$$\dot{\theta}_m + H\theta_m - \frac{k^2}{a}\phi = 0, \tag{38}$$

$$\dot{\theta}_e + H(1-3c^2_{se})\theta_e - \frac{k^2 c^2_{\mathrm{eff}}\delta_e}{(1+w_e)a} - \frac{k^2}{a}\phi = 0. \tag{39}$$

V. COMPARISON BETWEEN GR AND PN

In PN cosmology the linear evolution of DM and DE is determined by the system of equations that arise from Eqs. (8) and (9) for each fluid, namely,

$$\dot{\delta}_m + \frac{\theta_m}{a} = 0, \tag{40}$$

$$\dot{\delta}_e + (1+w_e)\frac{\theta_e}{a} + 3H(c_{\rm eff}^2 - w_e)\delta_e = 0, \quad (41)$$

$$\dot{\theta}_m + H\theta_m - \frac{k^2}{a}\phi = 0, \quad (42)$$

$$\dot{\theta}_e + H\theta_e - \frac{k^2 c_{\rm eff}^2 \delta_e}{(1+w_e)a} - \frac{k^2}{a}\phi = 0. \quad (43)$$

Notice the absence of a dynamical equation for ϕ. To eliminate the $k^2\phi$ terms we can use the constraint implied by the Poisson equation in PN cosmology, Eq. (10). Then the time variation of the potential is determined by the evolution of the density contrasts. Also notice that these equations lack some terms when compared with their relativistic counterparts, as already pointed out in Ref. [42]. However, as we will show, during the matter-dominated era and on small scales, this discrepancy changes only the velocity that DE perturbations decay but does not modify their late-time behavior.

In the matter-dominated regime, $\phi = \text{const}$ is a solution of Eq. (32), which also arises from the system of Eqs. (40)–(43). In this case, it is interesting to notice that Eq. (37) becomes identical to Eq. (41). However, Eq. (39) coincides with Eq. (43) only in the case $c_{se} = 0$.

As we see, the equations for the growth of perturbations are different in GR and PN already in the linear regime. Now we perform a quantitative study of this difference. We will work out the case of a matter-dominated universe with a small DE component, as expected in the linear regime, in which case ϕ is a constant. Furthermore, to avoid further complications, we assume constant values for w and $c_{\rm eff}^2$.

Under these conditions we can write a second-order differential equation for the linear growth of the dark energy density perturbation as a function of the scale factor, $\delta_e(a)$:

$$\delta_e'' + \alpha\frac{\delta_e'}{a} + \left[\beta + \frac{k^2 c_{\rm eff}^2}{a^2H^2}\right]\frac{\delta_e}{a^2} = -(1+w)\frac{k^2}{a^2H^2}\frac{\phi}{a^2}. \quad (44)$$

This equation arises both in GR and PN: in the latter case, we keep the pressure gradient in the Euler equation (9), which was dropped in the case of a top-hat perturbation. Only the parameters α and β are different in the two cases:

$$\alpha_{\rm GR} = \tfrac{3}{2} + 3\Delta - 3w; \qquad \beta_{\rm GR} = 3\Delta(\tfrac{1}{2} - 3w), \quad (45)$$

$$\alpha_{\rm PN} = \tfrac{3}{2} + 3\Delta; \qquad \beta_{\rm PN} = \tfrac{3}{2}\Delta, \quad (46)$$

where Δ was defined in Eq. (25).

We will make a comparison focusing on small scales, where the PN approximation is supposed to be more accurate. In this case, we can neglect the β term in the square brackets of Eq. (44), and we immediately write a constant particular solution:

$$\delta_e = -\frac{(1+w)}{c_{\rm eff}^2}\phi. \quad (47)$$

In order to solve the homogeneous equation we perform the following change of variables:

$$\delta_e(a) = x^{1-\alpha}y(x), \quad (48)$$

where x is defined in terms of the conformal time η as $x = kc_{\rm eff}\eta$. Then Eq. (44) becomes

$$\frac{d^2y}{dx^2} + \frac{1}{x}\frac{dy}{dx} + \left[1 - \frac{\mu^2}{x^2}\right]y = 0, \quad (49)$$

where, according to the different coefficients in Eqs. (45) and (46), μ assumes different values:

$$\mu_{\rm GR} = \pm\tfrac{1}{2}(1 - 6c_{\rm eff}^2), \quad (50)$$

$$\mu_{\rm PN} = \pm\tfrac{1}{2}(1 + 6w - 6c_{\rm eff}^2). \quad (51)$$

The solutions are Bessel functions of the first kind, $J_{\pm\mu}(x)$. The dark energy density contrast behaves as

$$\delta_e(x) = x^{1-\alpha}J_{\pm\mu}(x) - \frac{(1+w)}{c_{\rm eff}^2}\phi. \quad (52)$$

These solutions both have an oscillatory behavior with a decreasing amplitude proportional to $x^{1-\alpha-1/2}$, and they eventually reach the constant value $\delta_e = -(1+w)\phi/c_{\rm eff}^2$.

In order to check this analytical behavior, we numerically solve the complete system of coupled differential equations (35)–(39). We used as an illustration $c_{\rm eff}^2 = -w_e = 0.8$, $\Omega_{de}^{(0)} = 1 - \Omega_m^{(0)} = 0.75$, and we evolved the equations from an initial redshift of $z_i = 100$. We examined the mode $k = 100H_0 = 0.0333h\ \text{Mpc}^{-1}$, corresponding to a physical scale of $\lambda = 188.5h^{-1}$ Mpc, well inside the Hubble radius at z_i and large enough to be in the linear part of the matter power spectrum. As initial conditions we chose $\phi_i = -10^{-4}$, $\dot{\phi}_i = 0$, and

$$\begin{aligned}\delta_m(z_i) &= -2\phi_i\left[1 + \frac{k^2(1+z_i)^2}{3H(z_i)^2}\right];\\ \delta_e(z_i) &= (1+w)\delta_m(z_i);\end{aligned} \quad (53)$$

$$\theta_m(z_i) = \frac{2(1+z_i)k^2}{3H(z_i)}\phi_i; \qquad \theta_e(z_i) = 0, \quad (54)$$

which are consistent with Einstein's equation and adiabaticity. The result is presented in Fig. 1 and compared to the decay factor and the final value given by Eq. (47).

We also perform the same exercise for the PN approximation. The numerical solution of the system of Eqs. (40)–(43) with the same parameters and initial conditions is presented in Fig. 2 and compared to the decay factor and the final value given by Eq. (47). Again we see that the qualitative analytical behavior is reproduced by the numerical solution.

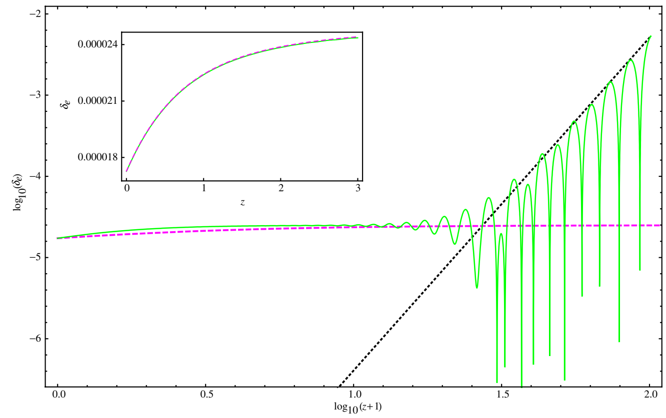

FIG. 1 (color online). Linear evolution of dark energy perturbations in GR at small scales ($k = 0.0333h$ Mpc^{-1}) for $c_{\rm eff}^2 = -w_e = 0.8$. The solid line is the solution of the complete set of five coupled differential equations. The dotted line is the decay factor according to Eq. (52). The dashed line is the particular solution Eq. (47). The inset shows in more detail the decay of the perturbations due to the recent acceleration of the Universe.

Therefore, even though the equations from GR and PN are not the same already at the linear level, the results are not qualitatively very different. In particular, both approaches predict the same asymptotic behavior for the DE perturbation. Because the decay rate of the transient is slightly different in each case, the time when the asymptotic regime is reached differs—in the PN approach this happens at a later time. We note that, even though we have shown comparisons between solutions of the PN and GR equations for constant equation of state and sound speed only, we have also checked that for dynamical dark energy models there is no stark difference between the two solutions. However, it is clear that one could engineer models with rapidly varying parameters for which the two solutions may be very different.

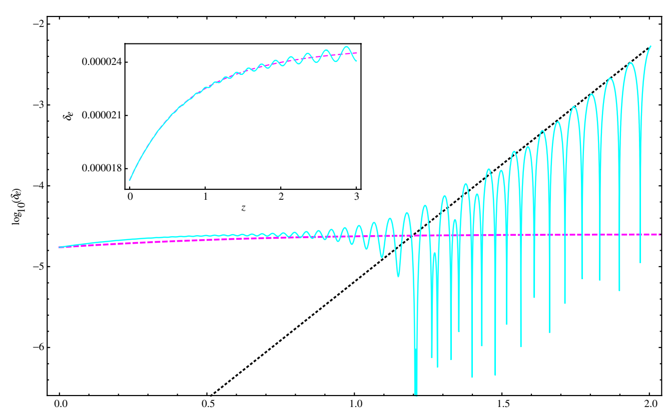

FIG. 2 (color online). Linear evolution of dark energy perturbations in the PN approximation at small scales ($k = 0.0333h$ Mpc^{-1}) for $c_{\rm eff}^2 = -w_e = 0.8$. The solid line is the solution of the complete set of four coupled differential equations. The dotted line is the decay factor according to Eq. (52). The dashed line is the particular solution Eq. (47). The inset shows in more detail the decay of the perturbations due to the recent acceleration of the Universe.

At this point we should call attention to the origin of an apparent discrepancy between the results obtained in this section, namely, a constant behavior of the DE perturbations, Eq. (47), and the result obtained in Sec. III, where we showed that in the SC/PN approach the DE perturbations grow as DM perturbations, Eq. (29). The reason is that in Sec. III we assumed a top-hat profile of the perturbation, which amounts to setting $kc_{\rm eff} = 0$ in the square brackets of Eq. (44). In this case, the particular solution is

$$\delta_e(a) = -\frac{1+w}{\beta}\frac{k^2\phi}{H^2a^2} = \frac{3(1+w)}{2\beta}\delta_m(a), \qquad (55)$$

where in the last equality we used Poisson's equation for the case of a dominant dark matter component, $k^2\phi = -(3/2)H^2a^2\delta_m$. Hence, we see that indeed in this case, or in fact for perturbations with a small mode number k, the perturbations in DE grow at the same pace as the DM perturbations in the linear regime.

On super-Hubble scales, PN cosmology is not expected to be valid, due to its inherently instantaneous interactions: indeed, in that framework perturbations with scales larger than the Hubble radius would behave in the same way as those well inside it. However, since we are only interested in the evolution of perturbations which are initially in the linear regime *and* well inside the Hubble radius, this mismatch is irrelevant. Therefore, we do not compare the PN perturbations with the GR perturbations at large scales.

As a final remark, we recall that the analytic solution, Eq. (52), is valid only for linear perturbations during the matter-dominated period. In this regime the matter density contrast grows as $\delta_m \propto a$ and ϕ is constant in time. When the structures enter the nonlinear regime, matter fluctuations must grow faster, i.e., $\delta_m \propto a^n$, with $n > 1$; then the gravitational potential should grow in time. Hence DE behavior in nonlinear structures is expected to be different from the linear analysis results. However, the asymptotic constant solution in Eq. (52) is valid during the initial nonlinear process of matter collapse, and DE fluctuations can grow with ϕ.

Of course, we now live in a dark-energy-dominated period, so it is also important to understand whether the PN approach is a good approximation to GR at the present time. Notice that our numerical solutions include dark energy, which starts to dominate roughly around $z \sim 0.5$, and the PN solution is in excellent agreement with the GR solution on small scales. In fact, it is also trivial to show that, assuming as before that the equation of state and the effective sound speed are constant, for large scales the dark energy density contrast δ_e and the gravitational potential ϕ in the PN as well as in the GR approaches both scale as $\delta_e \propto a^{-(1-3w_e)/2}$. However, notice also that, because structures which are observable today were formed when the Universe was still, to a very good approximation, dominated by matter, the direct impact of late-time dark energy perturbations on large scale structure is indeed very limited.

VI. CONCLUSIONS

The study of perturbations in dark energy has received a great deal of attention recently. DE perturbations have the potential to alter the process of large-scale structure formation in the Universe. The existence of DE perturbations can, in principle, be tested in future surveys such as the Dark Energy Survey [43] and EUCLID [44]. These future observations may help to distinguish among different models of DE.

Structure formation occurs during the nonlinear stages of the evolution of perturbations. Unfortunately, there is no rigorous analytical description of this nonlinear stage in full GR. Approximation methods must be used. Possibly the most reliable method is through N-body simulations, but due to its very intensive computing requirements, it is not practical when one wants to study different models. Furthermore, N-body simulations employ Newtonian physics and do not allow for the possibility of DE fluctuations.

In this paper we study two different approximation schemes, namely, the spherical collapse and pseudo-Newtonian approaches. The advantage of these schemes is that DE can be fully characterized by two functions: the equation-of-state parameter $w(z)$ and the effective speed of sound $c_{\rm eff}(z)$. We show that, under a minimal set of assumptions, it is possible to translate one approach into the other, rendering them completely equivalent. In order to compare these approximations with GR, we study perturbations in the linearized regime with all approaches. When the assumptions about the pressure perturbations are the same in both GR and PN/SC, we find that the fluctuations present the same qualitative behavior, lending support to the approximations. However, in order to establish more firmly the validity of the approximations in the nonlinear regime, a comparison should be made with some nonperturbative model in GR, such as an extended LTB class of models, including fluids with pressure. Work along this direction is in progress.

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