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Plethysms and interacting boson models

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A short review of the plethysm technique aiming to its application in finding branching rules for the reduction of an irreducible representation of a group under the restriction to one of its subgroups is given. The algebraic structure of the interacting boson model and some of its extensions is given together with the branching rules needed to classify their basis states, obtained by the use of plethysms. © *2003 American Institute of Physics.* [DOI: 10.1063/1.1611265]

I. INTRODUCTION

In the study of irreducible representations (irreps) of the full linear group GL(n) in n dimensions an important role is played by the so called Schur functions.[1–3] In a given irrep $\{\lambda\}$ of GL(n) the character of each of its elements A is the Schur function $\{\lambda\}$ evaluated with the eigenvalues of A.

A Schur function is expressed in terms of *fundamental symmetrical quantities* a_i, h_i, and s_i,[4] polynomials in n unknowns that are left invariant under permutations of these unknowns. The plethysm of Schur functions turned out a powerful tool to determine branching rules for the reduction of irreps of GL(n) subgroups under restriction to some of their subgroups.[4,5] The plethysm operation of Schur functions was discovered by Littlewood[6] as a third way of combining two Schur function to obtain a linear combination of Schur functions of a same degree. With few exceptions,[4,7,8] it remainded almost unknown to physicists due to the great difficulties involved in its calculation. With the appearence of powerful computers the tedious labor of computing plethysms was no more a problem and new efforts were made in order to find algorithms for computing them.[5,9–13]

In most applications to physical problems such as in nuclear structure[5,7] and in the present work only a particular class of plethysm is needed, namely that in which the left factor is a symmetric Schur function and in the expansion only those Schur functions with no more than a given number of rows are considered. In that case, using an induction formula for computing plethysms with both factors being symmetric Schur functions given in Ref. 13, we developed an algorithm[5] to compute plethysms with a symmetric Schur function in the left and all Schur functions of a given degree at right.

A field in which the plethysm technique can show all its power is the interacting boson model (IBM) and its generalizations.

The IBM when originally introduced by Arima and Iachello[14] is 1975 takes the nucleons outside a core of an even–even nucleus couple then into pairs to form bosons with angular momentum 2(d-bosons) and 0(s-bosons), no other degree of freedom, besides their z-component being taken into account. To work in the second quantization formalism they introduce 5 (for d-bosons) $+1$(for s-bosons)$=6$ creation and annihilation boson operators. The space of states is taken as polynomials of degree N (number of boson pairs) in creation operators acting on a vacuum realizing in this way the basis states of irreducible symmetric representations of U(6).

These bosons interact among themselves by interactions that preserve angular momentum and number of boson pairs so that their Hamiltonian can be written in terms of Casimir invariants of U(6) subgroups.

This original verson is nowadays referred to as IBM-1. Some extensions of the model appeared[15] in order to account for other degrees of freedom and the inclusion of bosons with other angular momenta. The unitary group is enlarged and a very rich algebraic structure arises. The basis states of the irreps of these unitary groups are labeled by labels of irreps of their subgroups in chains ending with $O^+(3)$, the rotation group in three dimensions. To this end one needs to know how an irrep of a group branches into irreps of some of its subgroups. We will show in this paper how plethysms can be used to find these branching rules. Besides, the cases here studied can serve as examples for applications in other areas.

II. SUMMARY OF PLETHYSMS

A partition $(\lambda)\equiv(\lambda_1,\lambda_2,\ldots,\lambda_n)$ is a set of nonnegative integers (parts) λ_i such that $\lambda_1+\lambda_2+\cdots+\lambda_n=n$. If, in addition, they satisfy $\lambda_1\geqslant\lambda_2\geqslant\cdots\geqslant\lambda_n$, the partition is called *standard*. Since we will deal only with standard partitions we will omit the word standard. Usually in a partition the null parts are omitted and the repeated ones are exponentiated. We will use greek letters to denote a partition and italic letters to denote a single part of a partition. To each partition one associates a Young diagram, an array of n boxes with λ_1 boxes in the first row, λ_2 in the second and so on. Due to that the nonzero parts of a partition are referred to as *rows* and the conjugate partition of a given partition $(\lambda_1,\lambda_2,\ldots,\lambda_p,0,\ldots,0)$ is defined as the partition whose Young diagram is obtained from that of (λ) by interchange of rows and columns, i.e.,

$$(\tilde{\lambda})=(p^{\lambda_p},(p-1)^{\lambda_{p-1}-\lambda_p},\ldots,2^{\lambda_2-\lambda_3},1^{\lambda_1-\lambda_2}). \tag{1}$$

Given a set of n variables $x_1,x_2,\ldots,x_n$ and a partition (λ) of r, the Schur function $\{\lambda\}$ associated to (λ) is defined as[4,6]

$$\{\lambda\}=\frac{1}{r!}|Z_r|^{[\lambda]}, \tag{2}$$

where Z is the matrix

$$Z_r=\begin{pmatrix} s_1 & 1 & 0 & 0 & \ldots & \ldots & \ldots & 0 \\ s_2 & s_1 & 2 & 0 & \ldots & \ldots & \ldots & 0 \\ s_3 & s_2 & s_1 & 3 & 0 & \ldots & \ldots & 0 \\ \ldots & \ldots & \ldots & .. & \ldots & \ldots & \ldots & 0 \\ s_{r-1} & s_{r-2} & \ldots & \ldots & \ldots & \ldots & s_1 & r-1 \\ s_r & s_{r-1} & \ldots & \ldots & \ldots & \ldots & s_2 & s_1 \end{pmatrix} \tag{3}$$

and $s_i\equiv s_i(x_1,x_2,\ldots,x_n)$ is the sum of ith powers of each variable $x_1,x_2,\ldots,x_n$. In this notation of Schur function the variables and their number n are implied while r, called its *degree*, is obtained by $r=\lambda_1+\lambda_2+\cdots+\lambda_r$.

In (2), $|Z|^{(\lambda)}$ is the immanant of Z, an extension of the concept of determinant, given by

$$|Z|^{(\lambda)}=\sum_P \chi^{[\lambda]}(P)z_{1p_1}z_{2p_2}\cdots z_{rp_r}, \tag{4}$$

where the sum is over all permutations $P=(p_1,p_2,\ldots,p_r)$ of the integers $1,2,\ldots,r$ and $\chi^{[\lambda]}(P)$ is the character of permutation P in the irrep $[\lambda]$ of the symmetric group $S(r)$.

As a consequence of definition (2) a Schur function is an homogeneous polynomial of degree r in the variables $x_1, x_2, \ldots, x_n$, being identically null for partitions (λ) of r into more than n nonzero parts.

Expression (2) can be worked out to produce an alternative definition[6,9] of the Schur function

$$\{\lambda\} = \frac{|(x_i)^{\lambda_j+n-j}|}{|(x_i)^{n-j}|}, \tag{5}$$

where $|f_j(x_i)|$ denotes the determinant of a matrix M with elements $M_{i,j} = f_j(x_i)$.

A pair of Schur functions $\{\lambda'\}$, $\{\lambda''\}$ of degrees r' and r'' can be combined into three different ways to produce linear combination of Schur functions $\{\lambda'''\}$ of degree r''': inner (or direct) product, outer product and plethysm. These three operations will be denoted, respectively, as

$$\{\lambda'\}\times\{\lambda''\} = \sum_{\lambda'''} \alpha(\{\lambda'\}\times\{\lambda''\}\rightarrow\{\lambda'''\})\{\lambda'''\},$$

$$\{\lambda'\}\{\lambda''\} = \sum_{\lambda'''} \alpha(\{\lambda'\}\{\lambda''\}\rightarrow\{\lambda'''\})\{\lambda'''\}, \tag{6}$$

$$\{\lambda'\}\otimes\{\lambda''\} = \sum_{\lambda'''} \alpha(\{\lambda\}\otimes\{\lambda''\}\rightarrow\{\lambda'''\})\{\lambda'''\},$$

where $\alpha(\cdots)$ is a non-negative integer denoting the multiplicity of $\{\lambda'''\}$ in the expansion. For clarity we attach to it an argument denoting the kind of operation that produced it.

In the inner product the degrees of the Schur functions involved are all equal, i.e., $r''' = r' = r'' = n$, and the expansion coefficients α are the coefficients of reduction of the Kronecker product of $S(n)$ irreps $[\lambda']$ and $[\lambda'']$.

In the outer product one has $r''' = r' + r''$ and the coefficients α are obtained by making the product of a Schur function in variables $(x_1, x_2, \ldots, x_{n'})$ by another in variables $(y_1, y_2, \ldots, y_{n''})$ and expressing it as a linear combination of Schur functions in variables $(z_1, z_2, \ldots, z_{n'''})$ with $z_i = x_i$ for $1 \leqslant i \leqslant n'$ and $z_{n'+i} = y_i$ for $1 \leqslant i \leqslant n''$. Littlewood obtained a procedure to find the coefficients of the outer product known in the literature as "Littlewood's rules."

To define plethysm one needs first to introduce the concept of *invariant matrix*.

Let $T(A)$ be an $m\times m$ matrix whose elements t_{ij} are given homogeneous polynomials of degree r in the elements of A. Let $T(B)$ be a matrix built with the *same* polynomials t_{ij} now in the elements of B. If

$$T(A)T(B) = T(AB) \tag{7}$$

for any nonsingular $m\times m$ matrices A,B then the matrix $T(A)$ is called an invariant matrix(of degree r) of A.

It follows from (7) that, once the set of polynomial t_{ij} is fixed, the set of matrices $\mathcal{D}^T(A) \equiv T(A)$ is a representation of $\mathrm{GL}(n)$.

As the Kronecker product of two representations of a group is also a representation of this group, the Kronecker product of invariant matrices is also an invariant matrix, in general reducible. Schur[1] demonstrated that *if A is an $n\times n$ matrix, there are as many irreducible invariant matrices of A of degree r as are the partitions of r with no more than n nonzero parts and the trace of them are the Schur functions of degree r in the eigenvalues of A*. These irreducible invariant matrices are then labeled by those partitions and denoted by $A^{[\mu]}$. The details of construction of irreducible invariant matrices can be found in Refs. 16 and 17.

Since an invariant matrix of an invariant matrix is also an invariant matrix of the original matrix, it can be decomposed into irreducible components

$$[A^{[\mu]}]^{[\nu]}=\sum_{\lambda} \kappa_{\lambda\mu\nu}A^{[\lambda]}. \tag{8}$$

Let us denote by r_μ, r_ν, and r_λ, the degrees of $\{\mu\}$, $\{\nu\}$, and $\{\lambda\}$, respectively. Since the elements of $A^{[\mu]}$ are polynomials of degree r_μ in the elements of A and those of $[A^{[\mu]}]^{[\nu]}$ are polynomials of degree r_ν in the components of $A^{[\mu]}$, it follows that $r_\lambda = r_\mu r_\nu$.

Equation (8) led Littlewood[6] to define a third composition rule of Schur functions denoted by the symbol $\otimes$ and defined as

$$\{\mu\}\otimes\{\nu\}=\sum_{\lambda} \kappa_{\lambda\mu\nu}\{\lambda\}, \tag{9}$$

where the Schur functions $\{\lambda\}$ and the numerical coefficients $\kappa_{\lambda\mu\nu}$ are those given in (8). This operation was later on named *plethysm*.

The plethysm operation has the following properties:[4,6,8]

$$\{\lambda\}\otimes(\{\mu\}\otimes\{\nu\})=(\{\lambda\}\otimes\{\mu\})\otimes\{\nu\}, \tag{10}$$

$$\{\lambda\}\otimes(\{\mu\}\pm\{\nu\})=\{\lambda\}\otimes\{\mu\}\pm\{\lambda\}\otimes\{\nu\}, \tag{11}$$

$$(\{\lambda\}+\{\mu\})\otimes\{\nu\}=\sum_{\lambda'\lambda''} \alpha(\{\lambda'\}\{\lambda''\}\to\{\nu\})(\{\lambda\}\otimes\{\lambda'\})(\{\mu\}\otimes\{\lambda''\}), \tag{12}$$

$$(\{\lambda\}-\{\mu\})\otimes\{\nu\}=\sum_{\lambda'\,\lambda''} (-)^{r''}\alpha(\{\lambda'\}\{\lambda''\}\to\{\nu\})(\{\lambda\}\otimes\{\lambda'\})(\{\mu\}\otimes\{\widetilde{\lambda}''\}), \tag{13}$$

$$\{\lambda\}\otimes(\{\mu\}\{\nu\})=(\{\lambda\}\otimes\{\mu\})(\{\lambda\}\otimes\{\nu\}), \tag{14}$$

$$(\{\lambda\}\{\mu\})\otimes\{\nu\}=\sum_{\lambda'\,\lambda''} \alpha(\{\lambda'\}\times\{\lambda''\}\to\{\nu\})(\{\lambda\}\otimes\{\lambda'\})(\{\mu\}\otimes\{\lambda''\}), \tag{15}$$

$$[\{\lambda\}\otimes\{\mu\}]^T=\begin{cases}\{\widetilde{\lambda}\}\otimes\{\mu\} & \text{for } r_\lambda \text{ even,}\\ \{\widetilde{\lambda}\}\otimes\{\widetilde{\mu}\} & \text{for } r_\lambda \text{ odd.}\end{cases} \tag{16}$$

The sum in Eqs. (12), (13), and (15) includes the cases $\{\lambda'\}=\{0\}\equiv 1$, $\{\lambda''\}=\{\nu\}$ and $\{\lambda'\}=\{\nu\}$, $\{\lambda''\}=\{0\}\equiv 1$. Also, r'' and r_λ are the degrees of $\{\lambda''\}$ and $\{\lambda\}$.

In Eq. (16) we used the notation

$$\left[\sum_i a_i\{\lambda\}^{(i)}\right]^T=\sum_i a_i\widetilde{\{\lambda\}^{(i)}}, \tag{17}$$

where a_i are numerical factors, $\{\lambda\}^{(i)}$ Schur functions and $\widetilde{\{\lambda\}^{(i)}}$ their conjugate.

A. Special plethysms

The plethysm calculation is, in general, a hard and tedious task. Nevertheless there are special cases with closed and simple expressions

$$\{\lambda\}\otimes\{1\}=\{1\}\otimes\{\lambda\}=\{\lambda\}, \tag{18}$$

$$\{\lambda\}\otimes\{0\}=\{0\}, \quad \{0\}\otimes\{\lambda\}=\delta_{\{\lambda\},\{r_\lambda\}}\{0\}, \tag{19}$$

$$\{r\}\otimes\{2\}=\sum_{i=0}^{[r/2]}\{2r-2i,2i\},\tag{20}$$

$$\{r\}\otimes\{1^2\}=\sum_{i=1}^{[(r+1)/2]}\{2r-(2i-1),(2i-1)\},\tag{21}$$

$$\{2\}\otimes\{r\}=\sum_{\lambda}\{\lambda\}_{\text{even}},\tag{22}$$

$$\{1^2\}\otimes\{r\}=\sum_{\lambda}\widetilde{\{\lambda\}_{\text{even}}},\tag{23}$$

$$\{1^r\}\otimes\{2\}=\{1^{2r}\}+\sum_{i=1}^{[r/2]}\{2^{2i},1^{2r-4i}\},\quad r\ \text{even},\tag{24}$$

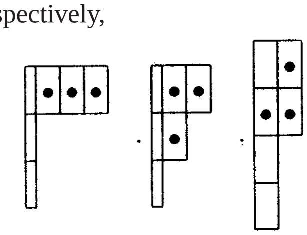
$$\{1^r\}\otimes\{1^2\}=\{1^{2r}\}+\sum_{i=1}^{[r/2]}\{2^{2i},1^{2r-4i}\},\quad r\ \text{odd},\tag{25}$$

$$\{1^r\}\otimes\{1^2\}=\sum_{i=1}^{[(r+1)/2]}\{2^{2i-1},1^{2r-2(2i-1)}\},\quad r\ \text{even},\tag{26}$$

$$\{1^r\}\otimes\{2\}=\sum_{i=1}^{[(r+1)/2]}\{2^{2i-1},1^{2r-2(2i-1)}\},\quad r\ \text{odd}.\tag{27}$$

Equation (18) follows from plethysms definition while Eq. (19) is set for consistency. In Eq. (22) $\{\lambda\}_{\text{even}}$ means partition of $2r$ with all parts even. Equations (23)–(27) follow from conjugation of Eqs. (22), (20), and (21). In Ref. 13 there are formulas for the calculation of plethysms $\{\lambda\}\otimes\{\mu\}$ when both Schur functions are symmetric or/and antisymmetric. To explain them we need the following definition: a *k-border strip* of a Young diagram associated to a given partition (λ) is a sequence of k squares in which the first of them is the last one of the first line of (λ) and the next square to a given one is the one below it, if it exists, or the one to its left, otherwise.

For example, the three-border strips of (41^2), (321), and (2^21^2) are the squares with the symbol $\cdot$ in the figures below, respectively,



When $\{\lambda\}$ and $\{\mu\}$ are both symmetric, one has

$$\{n\}\otimes\{m\}=\frac{1}{m}\sum_{k=1}^{m}\{n\}(x^k)(\{n\}\otimes\{m-k\}),\quad m\geqslant 1,\tag{28}$$

with

$$\{n\}(x^k)=\sum_{\nu}C_{n,k,\nu}\{\nu\}.\tag{29}$$

In (29) the $\{\nu\}$'s are all Schur functions of degree nk. The coefficients $C_{n,k,\nu}$ are obtained from the Young diagram associated to (ν) removing, in sequence, n k-border strips. If in all steps the resulting diagram represents a standard partition then

$$C_{n,k,\nu}=(-)^l \tag{30}$$

with $l=$(number of lines in the k-border strips)$-n$. If in some step the resulting diagram does not represent a standard partition, then $C_{n,k,\nu}=0$. As example, from the figures above one has

$$C_{2,3,\{41^2\}}=(-)^{4-2}=1, \quad C_{2,3,\{321\}}=(-)^{5-2}=-1, \quad C_{2,3,\{2^21^2\}}=0.$$

Equation (28) allows one to relate the plethysm of two symmetric Schur functions with the plethysms of symmetric Schur functions of smaller degrees. In this way, using $\{n\}\otimes\{1\}\equiv\{n\}$ as starting point one computes all the plethysms of type $\{n\}\otimes\{m\}$. This equation, together with

$$\{n\}\otimes\{1^m\}=(-)^{m+1}\{n\}\otimes\{m\}+\sum_{k=1}^{m-1}(-)^{k+1}(\{n\}\otimes\{k\})(\{n\}\otimes\{1^{m-k}\}), \tag{31}$$

$$\{1^n\}\otimes\{1^m\}=(-)^{m+1}\{1^n\}\otimes\{m\}+\sum_{k=1}^{m-1}(-)^{k+1}(\{1^n\}\otimes\{k\})(\{1^n\}\otimes\{1^{m-k}\}), \tag{32}$$

$$\{1^n\}\otimes\{m\}=\begin{cases}[\{n\}\otimes\{m\}]^T & \text{for } n \text{ even},\\ [\{n\}\otimes\{1^m\}]^T & \text{for } n \text{ odd}\end{cases} \tag{33}$$

allows us to compute plethysms with both Schur functions symmetric and/or antisymmetric.

A very common situation which arises in applications is when one needs to compute plethysms of a same Schur function by many (sometimes all) Schur functions of a given degree to the right. (This is the case of the applications that we will make in Secs. III–VII.) For such cases we proposed in Ref. 5 the following algorithm that allows to compute, in a build up way, all plethysms $\{\lambda\}\otimes\{\mu\}_r$ with $\{\lambda\}$ a fixed Schur function and $\{\mu\}_r$ all Schur functions of degree r, once the plethysms $\{\lambda\}\otimes\{r\}$ and $\{\lambda\}\otimes\{\mu\}_{r'}$, with $r'<r$ have already been computed.

(1) Find all partitions of r and order them in descending order of all their parts read from left to right.

(2) For each partition $\{\mu\}=\{\mu_1,\mu_2,\ldots,\mu_t,0,\ldots,0\}$ perform the outer product $\{\mu_1,\mu_2,\ldots,\mu_{t-1}\}\{\mu_t\}$, order the irreps in the reduction as in item (1), then use Eqs. (11) to obtain the equation

$$\{\lambda\}\otimes\{\mu\}=(\{\lambda\}\otimes\{\mu_1,\mu_2,\ldots,\mu_{t-1}\})(\{\lambda\}\otimes\{\mu_t\})-\sum_{\{\mu'\}<\{\mu\}}\alpha(\{\mu_1,\mu_2,\ldots,\mu_{t-1}\}\{\mu_t\}\\ \to\{\mu'\})\{\lambda\}\otimes\{\mu'\}, \tag{34}$$

where the symbol $<$ means preceding, following the ordering in item (1).

Since $\{\mu_1,\mu_2,\ldots,\mu_{t-1}\}$ and $\{\mu_t\}$ have smaller degree than $\{\mu\}$, the plethysms $\{\lambda\}\otimes\{\mu_1,\mu_2,\ldots,\mu_{t-1}\}$ and $\{\lambda\}\otimes\{\mu_t\}$ have already been computed in the induction process. On the other hand, the plethysms $\{\lambda\}\otimes\{\mu'\}$ also have been computed since $\{\mu'\}$ precedes $\{\mu\}$.

The formulas here given and the above algorithm suffice for calculating all plethysms needed in this work.

B. Special branching rules

The use of plethysms to compute branching rules is based in the theorem.[4]

If under the restriction $G\to H$ the character $[1]$ of group G decomposes as

$$[1] = \langle\alpha\rangle + \langle\beta\rangle + \cdots + \langle\omega\rangle, \tag{35}$$

then the character $[\lambda]$ *of G decomposes into the characters* $\langle\rho\rangle$ *of H according to the characters contained in the plethysm*

$$[\langle\alpha\rangle + \langle\beta\rangle + \cdots + \langle\omega\rangle] \otimes [\lambda]. \tag{36}$$

This plethysm can be obtained expressing the characters of G and H in terms of characters of $\mathrm{GL}(n)$, computing the resulting plethysms of $\mathrm{GL}(n)$ characters and re-expressing the result in terms of characters of H in order to obtain the final result.

Using the association irrep $\leftrightarrow$ character this theorem gives us the coefficients of the reduction of the irrep $[\lambda]$ of G in the direct sum of irreps $\langle\rho\rangle$ of H.

To illustrate the use of this theorem, let us consider some general cases that will be used later on. The first step toward the use of Eq. (36) is to find the decomposition (35). One way of finding it is by constructing a realization of basis states of irreps and generators of groups G and H.

One such realization is provided by the *boson calculus*[18] in which a set of boson operators $b_i^\dagger$ (creation) and b_i (annihilation) is introduced and the generators and basis states of irreps are written in terms of them.

The boson operators satisfy the usual commutation relations

$$[b_i, b_j^\dagger] = \delta_{ij}, \quad [b_i, b_j] = [b_i^\dagger, b_j^\dagger] = 0, \quad i,j = 1,2,\ldots,n, \tag{37}$$

and the b_i's annihilate the vacuum state $|0\rangle$.

For $\mathrm{U}(n)$ the generators are realized by

$$\mathcal{C}_i^j = b_i^\dagger b_j, \quad i,j = 1,2,\ldots,n, \tag{38}$$

while the maximum weight basis states of symmetric irreps $\{N,0,\ldots,0\} \equiv \{N\}$ are realized by

$$|\{N\}\mathrm{m.w.}\rangle = \frac{1}{\sqrt{N!}} (b_1^\dagger)^N |0\rangle, \tag{39}$$

from which it follows that the basis states of irrep $\{1\}$ of $\mathrm{U}(n)$ are realized by

$$|\{1\}i\rangle = b_i^\dagger |0\rangle, \quad i = 1,2,\ldots,n. \tag{40}$$

The generators of $\mathrm{U}(n-1)$ are the $\mathcal{C}_i^j$ given in Eq. (38) for $i,j = 1,2,\ldots,n-1$. Acting then in (40) one sees that the $\mathrm{U}(n)$ irrep $\{1\}$ splits into two $\mathrm{U}(n-1)$ irreps $\{1\}$ and $\{0\}$ with basis states

$$|\{1\}i\rangle = b_i^\dagger |0\rangle, \quad i = 1,2,\ldots,n-1 \quad \text{and} \quad |\{0\}\rangle = b_n^\dagger |0\rangle. \tag{41}$$

Therefore one obtains

$$\{1\} = \{1\} + \{0\} \quad \text{for } \mathrm{U}(n) \supset \mathrm{U}(n-1). \tag{42}$$

For $\mathrm{O}(n)$ the generators are $\mathcal{L}_i^j = \mathcal{C}_i^j - \mathcal{C}_j^i$ and reduction (35) read as

$$\{1\} = (1). \tag{43}$$

[We denote the irreps of unitary (U) and orthogonal (O) groups as quantities inside braces and parentheses, respectively.]

Consider the case in which $\mathrm{U}(n)$ acts on a vector space $\mathcal{E} = \mathcal{E}' + \mathcal{E}''$ with dimensions n' and n'' such that $n = n' + n''$. We then split n into two terms n' and n'' and consider $\mathrm{U}(n')$ as the group with generators $\mathcal{C}_{i'}^{j'}$ for $i', j' = 1,2,\ldots,n'$ and $\mathrm{U}(n'')$ that with generators $\mathcal{C}_{i''}^{j''}$ with $i'', j'' = n' + 1, n'+2, \ldots, n' + n'' = n$. The basis (40) splits into two

$$|\{1\}i'\rangle = b^{\dagger}_{i'}|0\rangle, \quad i'=1,2,\ldots,n',$$

$$|\{1\}i''\rangle = b^{\dagger}_{i''}|0\rangle, \quad i''=n'+1,n'+2,\ldots,n'+n''=n, \tag{44}$$

realizing the basis states of irreps $\{1\}'\{0\}''$ and $\{0\}'\{1\}''$ of $\mathrm{U}(n')\oslash\mathrm{U}(n'')$, respectively. We then have

$$\{1\}=\{1\}'\{0\}''+\{0\}'\{1\}'' \quad \text{for } \mathrm{U}(n'+n'')\supset\mathrm{U}(n')\oslash\mathrm{U}(n''). \tag{45}$$

For the case in which $\mathrm{U}(n)$ acts on a vector space $\mathcal{E}=\mathcal{E}'\otimes\mathcal{E}''$ with dimensions n' and n'' one uses boson operators with two indices, each one associated to transformations in each subspace,

$$[b_{is},b^{\dagger}_{jt}]=\delta_{ij}\delta_{st}, \quad [b_{is},b_{jt}]=[b^{\dagger}_{is},b^{\dagger}_{jt}]=0, \quad i,j=1,2,\ldots,n', \quad s,t=1,2,\ldots,n''. \tag{46}$$

The basis states of irrep $\{1\}$ are realized by

$$|\{1\}is\rangle = b^{\dagger}_{is}|0\rangle, \quad i=1,2,\ldots,n', \quad s=1,2,\ldots,n''. \tag{47}$$

Since the $\mathrm{U}(n')$ generators $\mathcal{C}^{j}_{i}=\Sigma_s b^{\dagger}_{is}b_{js}$ act on the first index and those $C^{t}_{s}=\Sigma_i b^{\dagger}_{is}b_{it}$ of $\mathrm{U}(n'')$ on the second, one concludes that

$$\{1\}=\{1\}'\{1\}'' \quad \text{for } \mathrm{U}(n'n'')\supset\mathrm{U}(n')\times\mathrm{U}(n''). \tag{48}$$

Using Eq. (42) in Eq. (36), the branching rule for the reduction $\mathrm{U}(n)\supset\mathrm{U}(n-1)$ is given by computing the plethysm

$$(\{1\}+\{0\})\otimes\{\lambda\}=\sum_{\lambda'\lambda''}\alpha(\{\lambda'\}\{\lambda''\}\to\{\lambda\})(\{1\}\otimes\{\lambda'\})(\{0\}\otimes\{\lambda''\})$$

$$=\sum_{\lambda' n''}\alpha(\{\lambda'\}\{n''\}\to\{\lambda\})\{\lambda'\}, \tag{49}$$

where use was made of (12), (18), and (19). By Littlewood rules, one sees that the Schur functions that contain $\{\lambda\}$ in the expansion of its outer product by a symmetric Schur function are those $\{\lambda'\}$ satisfying

$$\lambda_i \geqslant \lambda'_i \geqslant \lambda_{i+1}, \quad i=1,2,\ldots,i-1. \tag{50}$$

Then one concludes that under restriction $\mathrm{U}(n)\supset\mathrm{U}(n-1)$ the $\mathrm{U}(n)$ irrep $\{\lambda\}$ reduces as

$$\{\lambda\}=\sum_{\lambda'}\{\lambda'\}, \tag{51}$$

where $\{\lambda'\}$ are the $\mathrm{U}(n-1)$ irreps satisfying Eq. (50). These are the well known *in-betweeness* conditions introduced by Gelfand[19] in the labeling of basis states of $\mathrm{U}(n)$ irreps.

To compute the branching of irrep $\{\lambda\}$ of $\mathrm{U}(n'+n'')$ into irreps of $\mathrm{U}(n')\oslash\mathrm{U}(n'')$, according to Eqs. (36) and (45) we need to compute the plethysm

$$(\{1\}'\{0\}''+\{0\}'\{1\}'')\otimes\{\lambda\}=\sum_{\mu\nu}\alpha(\{\mu\}\{\nu\}\to\{\lambda\})((\{1\}'\{0\}'')\otimes\{\mu\})(\{0\}'\{1\}''\otimes\{\nu\}), \tag{52}$$

where use was made of Eq. (12). Using Eq. (15) one has

$$((\{1\}'\{0\}'')\otimes\{\mu\})=\sum_{\gamma\rho}\alpha(\{\gamma\}\times\{\rho\}\rightarrow\{\mu\})(\{1\}'\otimes\{\gamma\})(\{0\}''\otimes\{\rho\})=\{\mu\}\{0\}=\{\mu\} \quad (53)$$

using Eqs. (18) and (19) and the result $\{\mu\}\times\{n\}=\{\mu\}$ on inner product of Schur functions. Analogously $(\{0\}'\{1\}'')\otimes\{\nu\}$ gives $\{\nu\}$ and one concludes that

$$\{\lambda\}=\sum_{\mu\nu}\alpha(\{\mu\}\{\nu\}\rightarrow\{\lambda\})\{\mu\}'\{\nu\}'' \quad \text{for } \mathrm{U}(n'+n'')\supset\mathrm{U}(n')\oslash\mathrm{U}(n''), \quad (54)$$

$\{\mu\}',\{\nu\}''$ being irreps of $\mathrm{U}(n')$ and $\mathrm{U}(n'')$, respectively.

To compute the branching of irrep $\{\lambda\}$ of $\mathrm{U}(n'n'')$ into irreps of $\mathrm{U}(n')\times\mathrm{U}(n'')$, according to Eqs. (36) and (48) we need to compute the plethysm,

$$(\{1\}'\{1\}'')\otimes\{\lambda\}=\sum_{\mu\nu}\alpha(\{\mu\}\times\{\nu\}\rightarrow\{\lambda\})(\{1\}'\otimes\{\mu\})(\{1\}''\otimes\{\nu\}), \quad (55)$$

where use was made of Eq. (15). Using Eq. (18) one concludes that

$$\{\lambda\}=\sum_{\mu,\nu}\alpha(\{\mu\}\times\{\nu\}\rightarrow\{\lambda\})\{\mu\}'\{\nu\}'' \quad \text{for } \mathrm{U}(n')\times\mathrm{U}(n''), \quad (56)$$

$\{\mu\}',\{\nu\}''$ being irreps of $\mathrm{U}(n')$ and $\mathrm{U}(n'')$, respectively.

When $\{\lambda\}$ is a symmetric representation the inner product in Eq. (56) requires that the irreps of $\mathrm{U}(n')$ and $\mathrm{U}(n'')$ be the same.

Equation (43) is of no use for producing branching rules since it gives a trivial result. For this case one uses the known result.[4,6]

The character $\{\lambda\}$ *of* $\mathrm{U}(n)$ *decomposes into* $\mathrm{O}(n)$ *characters* (λ'') *by the relation*

$$\{\lambda\}=\sum_{\lambda''}\left[\sum_{\lambda'}\alpha(\{\lambda'\}\{\lambda''\}\rightarrow\{\lambda\})\right](\lambda''), \quad (57)$$

where the sum is made in the irreps $\{\lambda'\}$ *with even parts.*

When $\{\lambda\}$ is a symmetric representation both Schur functions $\{\lambda'\}$ and $\{\lambda''\}$ are symmetric and Eq. (57) gives

$$\{N\}=(N)+(N-2)+\cdots+(0) \text{ or } (1) \quad \text{for } \mathrm{U}(n)\supset\mathrm{O}(n). \quad (58)$$

We give in Table I the branching rules in the reduction $\mathrm{U}(n)\supset\mathrm{O}(n)$ for the lowest degree $\mathrm{U}(n)$ irreps with no more than three rows.

For small values of n some $\mathrm{O}(n)$ characters in Eq. (57) may have more than the allowed number $[n/2]$ of rows. In this case, they are worked out using *modified rules*.[20] For $\mathrm{U}(3)\supset\mathrm{O}^+(3)$, Eq. (57) and the corresponding modification rules are equivalent to the *Elliott rules*[7] for the branching of U(3) irrep $\{f_1,f_2,f_3\}$ into $\mathrm{O}^+(3)$ irreps (L).

(1) Define

$$\lambda=f_1-f_2, \quad \mu=f_2-f_3, \quad \overline{\lambda}=\max(\lambda,\mu), \quad \overline{\mu}=\min(\lambda,\mu). \quad (59)$$

(2) Introduce an extra label K that can assume the values

$$K=\overline{\mu},\overline{\mu}-2,\ldots,0 \text{ or } 1. \quad (60)$$

(3) To each K corresponds a set of L values

$$L=K,K+1,\ldots,K+\overline{\lambda}, \quad \text{for } K\neq 0,$$

TABLE I. $U(n)\supset O(n)$ branching rules for $U(n)$ irreps with no more than three rows and the lowest degrees.

$\{0\}=(0)$
$\{1\}=(1)$
$\{2\}=(2)+(0)$
$\{1^2\}=(1^2)$
$\{3\}=(3)+(1)$
$\{21\}=(21)+(1)$
$\{1^3\}=(1^3)$
$\{4\}=(4)+(2)+(0)$
$\{31\}=(31)+(2)+(1^2)$
$\{2^2\}=(2^2)+(2)+(0)$
$\{21^2\}=(21^2)+(1^2)$
$\{5\}=(5)+(3)+(1)$
$\{41\}=(41)+(3)+(21)+(1)$
$\{32\}=(32)+(3)+(21)+(1)$
$\{31^2\}=(31^2)+(21)+(1^3)$
$\{2^21\}=(2^21)+(21)+(1)$
$\{6\}=(6)+(4)+(2)+(0)$
$\{51\}=(51)+(4)+(31)+(2)+(1^2)$
$\{42\}=(42)+(4)+(31)+(2^2)+2(2)+(0)$
$\{3^2\}=(3^2)+(31)+(1^2)$
$\{41^2\}=(41^2)+(31)+(21^2)+(1^2)$
$\{321\}=(321)+(31)+(2^2)+(21^2)+(2)+(1^2)$
$\{2^3\}=(2^3)+(2^2)+(2)+(0)$
$\{7\}=(7)+(5)+(3)+(1)$
$\{61\}=(61)+(5)+(41)+(3)+(21)+(1)$
$\{52\}=(52)+(5)+(41)+(32)+2(3)+(21)+(1)$
$\{43\}=(43)+(41)+(32)+(3)+(21)+(1)$
$\{51^2\}=(51^2)+(41)+(31^2)+(21)+(1^3)$
$\{421\}=(421)+(41)+(32)+(31^2)+(2^21)+(3)+2(21)+(1)$
$\{3^21\}=(3^21)+(32)+(31^2)+(21)+(1^3)$
$\{32^2\}=(32^2)+(32)+(2^21)+(3)+(21)+(1)$
$\{8\}=(8)+(6)+(4)+(2)+(0)$
$\{71\}=(71)+(6)+(51)+(4)+(31)+(2)+(1^2)$
$\{62\}=(62)+(6)+(51)+(42)+2(4)+(31)+(2^2)+2(2)+(0)$

$$L=\overline{\lambda},\overline{\lambda}-2,\ldots,0 \quad \text{or} \quad 1 \quad \text{for } K=0. \tag{61}$$

The inverse result, that is, the expression of $O(n)$ characters in terms of those of $U(n)$ is also needed. It can be obtained by subtractions using tables of $U(n)\supset O(n)$ reductions or by use of the result[4,21]

$$(\lambda)=\{\lambda\}+\sum_{\eta}\left[\sum_{\gamma}(-)^{r/2}\alpha(\{\gamma\}\{\eta\}\rightarrow\{\lambda\}\right]\{\eta\}, \tag{62}$$

where r is the degree of $\{\gamma\}$ and these are taken among the set of Schur functions that in Frobenius' notation[4] assume the form

$$\begin{pmatrix} a+1 \\ a \end{pmatrix}, \quad \begin{pmatrix} a+1 & b+1 \\ a & b \end{pmatrix}, \quad \begin{pmatrix} a+1 & b+1 & c+1 \\ a & b & c \end{pmatrix}, \ldots . \tag{63}$$

When (λ) is symmetric one obtains from Eq. (58) and also from Eq. (62),

$$(N)=\{N\}-\{N-2\} \quad \text{for } N\geqslant 2. \tag{64}$$

Table B-4 in Ref. 4 gives a list of reductions (62) for irreps $\{\lambda\}$ of degree up to 16 and parts not greater than 4.

III. IBM-1

In the original IBM, now named IBM-1, the valence nucleons of even–even nuclei are joined in pairs to form a s- or d-boson, without distinguishing protons from neutrons. Then the building blocks are creation $(s^\dagger, d_\mu^\dagger)$ and annihilation (s, d_μ) boson operators satisfying the commutation relations

$$[s,s^\dagger]=1, \quad [d_\mu, d_{\mu'}^\dagger]=\delta_{\mu\mu'}, \quad \mu,\mu'=0,\pm 1,\pm 2, \tag{65}$$

all other commutators vanishing. In a compact notation one can define, say,

$$b_\rho^\dagger \text{ with } b_\rho^\dagger = d_{\rho-3}^\dagger \quad \text{for } \rho=1,2,3,4,5 \quad \text{and} \quad b_6^\dagger = s^\dagger \tag{66}$$

and analogously for b_ρ, recovering Eq. (37). Using linear combinations of creation and annihilation operators that preserve the number of bosons, it is possible to construct $O^+(3)$ Racah tensors of ranks $\ell=0,1,2,3,4$. Linear combinations of these tensors realize[15,22] the infinitesimal generators of U(6) subgroups in the three chains ending with $O^+(3)\supset O^+(2)$,

$$\begin{array}{llllllllll} & \nearrow & U(5) & \supset & O^+(5) & \supset & O^+(3) & \supset & O^+(2) & (I), \\ U(6) & \rightarrow & SU(3) & \supset & O^+(3) & \supset & O^+(2) & & & (II), \\ & \searrow & O^+(6) & \supset & O^+(5) & \supset & O^+(3) & \supset & O^+(2) & (III). \end{array} \tag{67}$$

With one-index boson operators only symmetrical irreps can be realized. Then the U(6) irrep is $\{N\}$ where N denotes the number of bosons.

Let us examine the branching rules in chain (I) of Eq. (67). The U(5) labels are given by the general result (51). Then the U(5) irrep is symmetrical $\{N_d\}$, where N_d is the number of d-bosons and can assume the values

$$N_d = N, N-1, \ldots, 0. \tag{68}$$

Each U(5) irrep $\{N_d\}$, being symmetric, branches as (58) into $O^+(5)$ irreps

$$\{N_d\} = (N_d) + (N_d-2) + \cdots + (0) \text{ or } (1). \tag{69}$$

To find the branching in $O^+(5)\supset O^+(3)$ one observes that the generators of U(5) were constructed only with operators $d_\mu^\dagger$ and d_μ so one has

$$(1)=(2) \quad \text{for } O^+(5)\supset O^+(3). \tag{70}$$

According to Eq. (36) the branching of a *general* $O^+(5)$ irrep $\{\lambda\}$ into $O^+(3)$ irreps is found computing the plethysm $(2)\otimes(\lambda)$. The character (2) of $O^+(3)$ is given by $(2)=\{2\}-\{0\}$. Since (λ) is an $O^+(5)$ irrep it has at most two lines, then we expand it using Eq. (62) in terms of Schur functions with up to two rows:

$$(\lambda) = \sum_k \alpha_k\{k\} + \sum_{\mu_1,\mu_2} \alpha_{\mu_1,\mu_2}\{\mu_1,\mu_2\}. \tag{71}$$

The plethysm $(2)\otimes(\lambda)$ is then

$$\begin{aligned}(2)\otimes(\lambda) &= (\{2\}-\{0\})\otimes\left[\sum_k \alpha_k\{k\} + \sum_{\mu_1,\mu_2}\{\mu_1,\mu_2\}\right] \\ &= \sum_k \alpha_k[\{2\}\otimes\{k\} - \{2\}\otimes\{k-1\}] + \sum_{\mu_1\mu_2} \alpha_{\mu_1\mu_2}[\{2\}\otimes\{\mu_1,\mu_2\} - \{2\}\otimes\{\mu_1-1,\mu_2\} \\ &\quad - \{2\}\otimes\{\mu_1,\mu_2-1\} + \{2\}\otimes\{\mu_1-1,\mu_2-1\}],\end{aligned} \tag{72}$$

TABLE II. $O^+(5)\supset O^+(3)$ branching rules for $O^+(5)$ irreps of degrees up to 8.

$(0)=(0)$
$(1)=(2)$
$(2)=(2)+(4)$
$(1^2)=(1)+(3)$
$(3)=(0)+(3)+(4)+(6)$
$(21)=(1)+(2)+(3)+(4)+(5)$
$(4)=(2)+(4)+(5)+(6)+(8)$
$(31)=(1)+(2)+2(3)+(4)+2(5)+(6)+(7)$
$(2^2)=(0)+(2)+(3)+(4)+(6)$
$(5)=(2)+(4)+(5)+(6)+(7)+(8)+(10)$
$(41)=(1)+(2)+2(3)+2(4)+2(5)+2(6)+2(7)+(8)+(9)$
$(32)=(1)+2(2)+(3)+2(4)+2(5)+(6)+(7)+(8)$
$(6)=(0)+(3)+(4)+2(6)+(7)+(8)+(9)+(10)+(12)$
$(51)=(1)+(2)+2(3)+2(4)+3(5)+2(6)+3(7)+2(8)+2(9)+(10)+(11)$
$(42)=(0)+(1)+2(2)+2(3)+3(4)+2(5)+3(6)+2(7)+2(8)+(9)+(10)$
$(3^2)=(1)+2(3)+(4)+(5)+(6)+(7)+(9)$
$(7)=(2)+(4)+(5)+(6)+(7)+2(8)+(9)+(10)+(11)+(12)+(14)$
$(61)=(1)+(2)+2(3)+2(4)+3(5)+3(6)+3(7)+3(8)+3(9)+2(10)+2(11)+(12)+(13)$
$(52)=(0)+(1)+2(2)+3(3)+3(4)+3(5)+4(6)+3(7)+3(8)+3(9)+2(10)++(11)+(12)$
$(43)=(1)+2(2)+2(3)+2(4)+3(5)+2(6)+2(7)+2(8)+(9)+(10)+(11)$
$(8)=(2)+(4)+(5)+(6)+(7)+2(8)+(9)+2(10)+(11)+(12)+(13)+(14)+(16)$
$(71)=(1)+(2)+2(3)+2(4)+3(5)+3(6)+4(7)+3(8)+4(9)+3(10)+3(11)+2(12)+2(13)+(14)$ $+(15)$
$(62)=(1)+3(2)+2(3)+4(4)+4(5)+4(6)+4(7)+5(8)+3(9)+4(10)+3(11)+2(12)+(13)+(14)$
$(53)=2(1)+2(2)+3(3)+3(4)+4(5)+3(6)+4(7)+3(8)+3(9)+2(10)+2(11)+(12)+(13)$
$(4^2)=(0)+(2)+(3)+2(4)+(5)+2(6)+(7)+(8)+(9)+(10)+(12)$

where plethysms with Schur functions associated to nonstandard partitions are disregarded. The final result is obtained by expressing the Schur functions resulting from plethysms in terms of $O^+(3)$ irreps (L) using Eqs. (59)–(61).

In IBM-1 the $O^+(5)$ irrep (λ) is symmetric, then one uses Eqs. (71) and (72) with $\alpha_{\mu_1,\mu_2}=0$. The terms with $\alpha_{\mu_1,\mu_2}\neq 0$ will be used in IBM-2 and 3. In Table II the $O^+(5)\supset O^+(3)$ branching rules for $O^+(5)$ irreps with the lowest degrees are given.

Now let us find the branching rules in chain (II) of Eq. (67). To find the decomposition (35) for $U(6)\supset SU(3)$ in chain (II) one observes that the U(3) irreps must have the (L) multiplets (2) and (0) contained in irrep {1} of U(6) [the reduction $U(3)\supset SU(3)$ has only one SU(3) irrep with labels given by Eq. (59)]. Using Elliott's rules (59)–(61) one sees that the U(3) irrep must be {2}. We then have

$$\{1\}=\{2\}\equiv(2,0) \quad \text{for } U(6)\supset U(3)\ [\text{or} \quad SU(3)]. \tag{73}$$

Using Eq. (73) and Eq. (36) one has that the U(3)[SU(3)] irreps contained in the irrep $\{\lambda\}$ of U(6) are

$$\{\lambda\}=\sum_{\mu}\alpha(\{2\}\otimes\{\lambda\}\rightarrow\{\mu\})\{\mu\}, \tag{74}$$

where in the plethysms only irreps with no more than three rows are considered and these produce SU(3) irreps $(\mu_1-\mu_2,\mu_2-\mu_3)$ in Elliott's notation. Table III presents the branching $U(6)\supset SU(3)$ for U(6) irreps with no more than three rows and the lowest degrees.

The branching rule in $SU(3)\supset O^+(3)\supset O^+(2)$ is given by Elliott's rules (59)–(61).

Since in IBM-1 the U(6) irrep $\{\lambda\}$ is a symmetric irrep $\{N\}$, the plethysm in Eq. (74) is given by Eq. (22) and one obtains

TABLE III. Branching rules for $U(6)\supset SU(3)$ for $U(6)$ irreps with no more than three rows and lowest degrees.

$\{0\}=(0,0)$
$\{1\}=(2,0)$
$\{2\}=(4,0)+(0,2)$
$\{1^2\}=(2,1)$
$\{3\}=(6,0)+(2,2)+(0,0)$
$\{21\}=(4,1)+(2,2)+(1,1)$
$\{1^3\}=(3,0)+(0,3)$
$\{4\}=(8,0)+(4,2)+(0,4)+(2,0)$
$\{31\}=(6,1)+(4,2)+(2,3)+(1,2)+(2,0)+(3,1)$
$\{2^2\}=(4,2)+(0,4)+(2,0)+(3,1)$
$\{21^2\}=(5,0)+(2,3)+(1,2)+(0,1)+(3,1)$
$\{5\}=(10,0)+(6,2)+(2,4)+(4,0)+(0,2)$
$\{41\}=(8,1)+(6,2)+(4,3)+(5,1)+(2,4)+(3,2)+(4,0)+(1,3)+(2,1)+(0,2)$
$\{32\}=(6,2)+(4,3)+(5,1)+(2,4)+(3,2)+2(4,0)+(1,3)+(2,1)+(0,2)$
$\{31^2\}=(7,0)+(4,3)+(5,1)+2(3,2)+(0,5)+(1,3)+2(2,1)+(1,0)$
$\{2^2 1\}=(5,1)+(2,4)+(3,2)+(4,0)+(1,3)+(2,1)+(0,2)$
$\{6\}=(12,0)+(8,2)+(4,4)+(6,0)+(0,6)+(2,2)+(0,0)$
$\{51\}=(10,1)+(8,2)+(6,3)+(7,1)+(4,4)+(5,2)+(6,0)+(2,5)+(3,3)+(4,1)+(1,4)+2(2,2)+(1,1)$
$\{42\}=(8,2)+(6,3)+(7,1)+2(4,4)+(5,2)+2(6,0)+2(3,3)+2(4,1)+(0,6)+(1,4)+3(2,2)+(1,1)+(0,0)$
$\{41^2\}=(9,0)+(6,3)+(7,1)+2(5,2)+(2,5)+2(3,3)+2(4,1)+(1,4)+(2,2)+2(3,0)+2(0,3)+(1,1)$
$\{3^2\}=(6,3)+(5,2)+(6,0)+(2,5)+(3,3)+(4,1)+(2,2)+(3,0)+(0,3)$
$\{321\}=(7,1)+(4,4)+2(5,2)+(6,0)+(2,5)+2(3,3)+3(4,1)+2(1,4)+3(2,2)+(3,0)+(0,3)+2(1,1)$
$\{2^3\}=(6,0)+(3,3)+(0,6)+2(2,2)+(0,0)$
$\{7\}=(14,0)+(10,2)+(6,4)+(8,0)+(2,6)+(4,2)+(0,4)+(2,0)$
$\{61\}=(12,1)+(10,2)+(8,3)+(9,1)+(6,4)+(7,2)+(8,0)+(4,5)+(5,3)+(6,1)+(2,6)+(3,4)+2(4,2)$
$\quad+(1,5)+(2,3)+(3,1)+(0,4)+(1,2)+(2,0)$
$\{52\}=(10,2)+(8,3)+(9,1)+2(6,4)+(7,2)+2(8,0)+(4,5)+2(5,3)+2(6,1)+(2,6)+2(3,4)+4(4,2)+(1,5)$
$\quad+2(2,3)+2(3,1)+2(0,4)+(1,2)+2(2,0)$
$\{51^2\}=(11,0)+(8,3)+(9,1)+2(7,2)+(4,5)+2(5,3)+2(6,1)+2(3,4)+(4,2)+2(5,0)+(0,7)+(1,5)$
$\quad+3(2,3)+2(3,1)+2(1,2)+(0,1)$

$$\{N\}=\sum_{\mu_1,\mu_2,\mu_3}\{2\mu_1,2\mu_2,2\mu_3\}\equiv\sum_{\mu_1\mu_2\mu_3}(2(\mu_1-\mu_2),2(\mu_2-\mu_3)), \tag{75}$$

where (μ_1,μ_2,μ_3) are (standard)partitions of N into three parts.

The branching for the first link $U(6)\supset O^+(6)$ in chain (III) is found using Eq. (57). Note that in (57) the branching is for $U(n)\supset O(n)$ and we need a further reduction $O(n)\supset O^+(n)$. For the cases treated here the O(6) and $O^+(6)$ irreps are the same.

In IBM-1 the U(6) irrep $\{\lambda\}$ being symmetric implies that Eq. (57) has a simple expression:

$$\{N\}=(N)+(N-2)+\cdots+(0) \quad \text{or} \quad (1) \quad \text{for } U(6)\supset O^+(6). \tag{76}$$

To find the branching rule in the link $O^+(6)\supset O^+(5)$ one first observes that Eq. (42) gives

$$(1)=(1)+(0) \quad \text{for } O^+(6)\supset O^+(5). \tag{77}$$

Next one writes the $O^+(6)$ irrep (λ) in terms of U(6) irreps $\{\mu\}$:

$$(\lambda)=\sum_{\mu} \alpha_{\mu}\{\mu\} \tag{78}$$

and computes the plethysm

$$((1)+(0))\otimes(\lambda)=\sum_{\mu} \alpha_{\mu}((1)+(0))\otimes\{\mu\}=\sum_{\mu} \alpha_{\mu}\left[\sum_{\nu',k} \alpha(\{\nu'\}\{k\}\rightarrow\{\mu\})\{\nu'\}\right]. \tag{79}$$

The resulting U(5) irreps are then converted into $O^+(5)$ irreps by use of Eq. (57).

In IBM-1 the $O^+(6)$ irrep is symmetric and, in this case, Eq. (78) becomes

$$(0)=\{0\}, \quad (1)=\{1\} \quad \text{and } (k)=\{k\}-\{k-2\} \quad \text{for } k\geqslant 2 \tag{80}$$

and Eq. (79) gives

$$((1)+(0))\otimes(k)=\sum_{p,q} [\alpha(\{p\}\{q\}\rightarrow\{k\})-\alpha(\{p\}\{q\}\rightarrow\{k-2\})]\{p\}. \tag{81}$$

Expressing $\{p\}$ in terms of $O^+(5)$ irreps by means of Eq. (58) one has the final result

$$(k)=(k)+(k-1)+\cdots+(0) \quad \text{for } O^+(6)\supset O^+(5). \tag{82}$$

For general $O^+(6)$ irreps $(\sigma)=(\sigma_1,\sigma_2,\sigma_3)$, computer calculations using Eq. (79) shows the branching rule

$$(\sigma)=\sum_{\rho_1=\sigma_2}^{\sigma_1}\sum_{\rho_2=\sigma_3}^{\sigma_2}(\rho_1,\rho_2) \quad \text{for } O^+(6)\supset O^+(5), \tag{83}$$

the usual inbetweeness conditions for Gelfand labels.

IV. IBM-2

In IBM-2 two kinds of bosons are considered, one formed by proton pairs and other by neutron pairs, denoted by

$$s_{\pi}^{\dagger},\ d_{\pi,\mu}^{\dagger}, \quad s_{\nu}^{\dagger},\ d_{\nu,\mu}^{\dagger}, \quad \mu=0,\pm 1,\pm 2, \quad \pi \quad \text{for protons}, \quad \nu \quad \text{for neutrons} \tag{84}$$

and similarly for annihilation operators. The commutation relations are the same as Eq. (65) concerning angular momentum labels and neutron operators commute with proton operators.

Using the compact notation

$$b_{\rho,\alpha}^{\dagger}, \quad b_{\rho,\alpha}, \quad \rho=\pi,\nu, \quad \alpha=1,2,\ldots,6, \tag{85}$$

the commutation relations become

$$[b_{\rho\alpha},b_{\rho'\alpha'}^{\dagger}]=\delta_{\rho\rho'}\delta_{\alpha\alpha'}, \quad [b_{\rho\alpha},b_{\rho'\alpha'}]=[b_{\rho\alpha}^{\dagger},b_{\rho'\alpha'}^{\dagger}]=0. \tag{86}$$

With these operators one constructs operators

$$\mathcal{C}_{\rho\alpha}^{\rho'\alpha'}=b_{\rho\alpha}^{\dagger}b_{\rho'\alpha'} \tag{87}$$

that under commutation close the Lie algebra of U(12). The operators $C_{\alpha}^{\alpha'}=\mathcal{C}_{\pi\alpha}^{\pi\alpha'}$ and $\mathbb{C}_{\alpha}^{\alpha'}=\mathcal{C}_{\nu\alpha}^{\nu\alpha'}$ generate the Lie algebras of $\mathrm{U}_{\pi}(6)$ and $\mathrm{U}_{\nu}(6)$, respectively. We have then a particular case of the reduction $\mathrm{U}(n_1+n_2)\supset \mathrm{U}(n_1)\oslash \mathrm{U}(n_2)$ studied in Sec. II B. Using the results there obtained one has the branching rule

$$\{\lambda\}=\sum_{\lambda',\lambda''}\alpha(\{\lambda'\}\{\lambda''\}\rightarrow\{\lambda\})\{\lambda'\}_{\pi}\{\lambda''\}_{\nu}\quad \text{for } \mathrm{U}(12)\supset \mathrm{U}_{\pi}(6)\oslash \mathrm{U}_{\nu}(6). \tag{88}$$

With operators (84) [or (85)] one can construct only symmetrical irreps $\{N\}$ of U(12) and Eq. (88) reduces to

$$\{N\}=\sum_{k=0}^{N}\{N-k\}_{\pi}\{k\}_{\nu}\quad \text{for } \mathrm{U}(12)\supset \mathrm{U}_{\pi}(6)\oslash \mathrm{U}_{\nu}(6). \tag{89}$$

The basis states of irrep $\{N\}$ must be also basis states for an irrep of $\mathrm{O}^{+}_{\pi+\nu}(3)$, the group of simultaneous rotations of protons and neutrons. This can be achieved by use of *lattice of algebras*, in contrast with *chains of algebras* in IBM-1.

The simplest lattice is obtained when we use chains (I), (II), and (III) separately for protons and for neutrons and only in the last step one couples $\mathrm{O}^{+}_{\pi}(3)$ with $\mathrm{O}^{+}_{\nu}(3)$ to obtain $\mathrm{O}^{+}_{\pi+\nu}(3)$:

$$\begin{array}{ccccccc}
 & & \nearrow \mathrm{U}_{\pi}(5)\supset & \mathrm{O}^{+}_{\pi}(5)\searrow & & & \\
 & \mathrm{U}_{\pi}(6) & \rightarrow \mathrm{SU}_{\pi}(3) & \rightarrow & \mathrm{O}^{+}_{\pi}(3) & & \\
\nearrow & & \searrow \mathrm{O}^{+}_{\pi}(6)\supset & \mathrm{O}^{+}_{\pi}(5)\nearrow & & \searrow & \\
\mathrm{U}(12) & & & & & & \mathrm{O}^{+}_{\pi+\nu}(3)\supset \quad \mathrm{O}^{+}_{\pi+\nu}(2). \\
\searrow & & \nearrow \mathrm{U}_{\nu}(5)\supset & \mathrm{O}^{+}_{\nu}(5)\searrow & & \nearrow & \\
 & \mathrm{U}_{\nu}(6) & \rightarrow \mathrm{SU}_{\nu}(3) & \rightarrow & \mathrm{O}^{+}_{\nu}(3) & & \\
 & & \searrow \mathrm{O}^{+}_{\nu}(6)\supset & \mathrm{O}^{+}_{\nu}(5)\nearrow & & &
\end{array} \tag{90}$$

This is a trivial extension of IBM-1 and L_{π} and L_{ν} are coupled to give

$$L_{\pi+\nu}=L_{\pi}+L_{\nu},\quad L_{\pi}+L_{\nu}-1,\ldots,|L_{\pi}-L_{\nu}|. \tag{91}$$

Another lattice is

$$\begin{array}{cccccccc}
\mathrm{U}_{\pi}(6) & & & & \mathrm{U}_{\pi+\nu}(5)\supset \mathrm{O}^{+}_{\pi+\nu}(5) & & & (\mathrm{I}_1) \\
\searrow & & & \nearrow & & \searrow & & \\
 & \mathrm{U}_{\pi+\nu}(6) & \rightarrow & \mathrm{SU}_{\pi+\nu}(3) & & \rightarrow & \mathrm{O}^{+}_{\pi+\nu}(3)\supset \mathrm{O}^{+}_{\pi+\nu}(2) & (\mathrm{II}_1) \\
\nearrow & & & \searrow & & \nearrow & & \\
\mathrm{U}_{\nu}(6) & & & & \mathrm{O}^{+}_{\pi+\nu}(6)\supset \mathrm{O}^{+}_{\pi+\nu}(5) & & & (\mathrm{III}_1)
\end{array} \tag{92}$$

in which the algebras of $\mathrm{U}_{\pi}(6)$ and $\mathrm{U}_{\nu}(6)$ are joined in the first step. In the first link one has $\mathrm{U}_{\pi}(6)\times \mathrm{U}_{\nu}(6)\rightarrow \mathrm{U}_{\pi+\nu}(6)$ and the branching rules are given by the Kronecker product of U(6) irreps. In this case, the irreps of $\mathrm{U}_{\pi}(6)$ and $\mathrm{U}_{\nu}(6)$ are both symmetric by Eq. (89) and the irreps of $\mathrm{U}_{\pi+\nu}(6)$ can have one or two rows. Chains (I_1), (II_1), and (III_1) are the same as (I), (II), and (III) but now the U(6), U(5), $\mathrm{O}^{+}(6)$, and $\mathrm{O}^{+}(6)$ irreps can be two-rowed.

Another type of lattice of algebras is obtained by joining the neutron and proton algebras at the second step:

$$\begin{array}{lll} \mathrm{U}_\pi(6)\supset \mathrm{U}_\pi(5) & & \\ & \searrow & \\ & & \mathrm{U}_{\pi+\nu}(5)\supset \mathrm{O}^+_{\pi+\nu}(5)\supset \mathrm{O}^+_{\pi+\nu}(3)\supset \mathrm{O}^+_{\pi+\nu}(2) \quad (\mathrm{I}_2), \\ & \nearrow & \\ \mathrm{U}_\nu(6)\supset \mathrm{U}_\nu(5) & & \\ \mathrm{U}_\pi(6)\supset \mathrm{SU}_\pi(3) & & \\ & \searrow & \\ & & \mathrm{SU}_{\pi+\nu}(3)\supset \mathrm{O}^+_{\pi+\nu}(3\supset \mathrm{O}^+_{\pi+\nu}(2), \qquad (\mathrm{II}_2), \\ & \nearrow & \\ \mathrm{U}_\nu(6)\supset \mathrm{SU}_\nu(3) & & \\ \mathrm{U}_\pi(6)\supset \mathrm{O}^+_\pi(6) & & \\ & \searrow & \\ & & \mathrm{O}^+_{\pi+\nu}(6)\supset \mathrm{O}^+_{\pi+\nu}(5)\supset \mathrm{O}^+_{\pi+\nu}(3)\supset \mathrm{O}^+_{\pi+\nu}(2) \quad (\mathrm{III}_2). \\ & \nearrow & \\ \mathrm{U}_\nu(6)\supset \mathrm{O}^+_\nu(6) & & \end{array} \tag{93}$$

The branching rules for the irreps of the joined algebras are obtained by Kronecker products and the resulting irreps can be one- and two-rowed. The Kronecker product expansion of irreps of unitary groups are given by the outer product of Schur functions:

$$\{\lambda\}_\pi\{\mu\}_\nu=\sum_\rho \alpha(\{\lambda\}\{\mu\}\rightarrow\{\rho\})\{\rho\}_{\pi+\nu}. \tag{94}$$

The Kronecker product of O(6) irreps is done by expressing the O(6) characters in terms of Schur functions, making the outer products and re-expressing the result in terms of O(6) irreps. In Table IV we give the Kronecker product of O(6) irreps with the lowest product degrees.

V. IBM-3

This model was proposed by Elliott and White[23] in order to take into account the isospin degree of freedom. It differs from IBM-2 by the inclusion of a third kind of boson, the δ-boson, formed by a proton–neutron pair. There are 18 creation operators

$$s^\dagger_\pi,\ d^\dagger_{\pi,\mu},\ s^\dagger_\nu,\ d^\dagger_{\nu,\mu},\ s^\dagger_\delta,\ d^\dagger_{\delta,\mu} \quad (\mu=0,\pm1,\pm2) \tag{95}$$

and the corresponding annihilation operators. Operators of different pairs of bosons commute among themselves while each set π, ν and δ satisfies bose commutation relations.

One has again lattices of algebras now starting with

$$\mathrm{U}(18)\supset \mathrm{U}_\pi(6)\oslash \mathrm{U}_\nu(6)\oslash \mathrm{U}_\delta(6) \tag{96}$$

and ending with $\mathrm{O}^+_{\pi+\nu+\delta}(3)\supset \mathrm{O}^+_{\pi+\nu+\delta}(2)$.

By an extension of the calculation done to obtain Eq. (54) one obtains

$$\{\lambda\}=\sum_{\mu,\sigma,\rho}\alpha(\{\mu\}\{\sigma\}\{\rho\}\rightarrow\{\lambda\})\{\mu\}_\pi\{\sigma\}_\nu\{\rho\}_\delta \quad \text{for } \mathrm{U}_\pi(6)\oslash \mathrm{U}_\nu(6)\oslash \mathrm{U}_\delta(6). \tag{97}$$

Since in IBM-3 the U(18) irrep is symmetric, Eq. (97) reduces to

TABLE IV. Kronecker product of O(6) irreps with lowest total degrees.

$(1)(1)=(0)+(1^2)+(2)$
$(2)(1)=(1)+(21)+(3)$
$(1^2)(1)=(1)+(1^3)+(21)$
$(3)(1)=(2)+(31)+(4)$
$(21)(1)=(1^2)+(2)+(21^2)+(2^2)+(31)$
$(1^3)(1)=(1^2)+(21^2)$
$(2)(2)=(0)+(1^2)+(2)+(2^2)+(31)+(4)$
$(1^2)(2)=(1^2)+(2)+(21^2)+(31)$
$(1^2)(1^2)=(0)+(1^2)+(2)+(21^2)+(2^2)$
$(4)(1)=(3)+(41)+(5)$
$(31)(1)=(21)+(3)+(31^2)+(32)+(41)$
$(2^2)(1)=(21)+(2^21)+(32)$
$(21^2)(1)=(1^3)+(21)+(2^21)+(31^2)$
$(3)(2)=(1)+(21)+(3)+(32)+(41)+(5)$
$(21)(2)=(1)+(1^3)+2(21)+(2^21)+(3)+(31^2)+(32)+(41)$
$(1^3)(2)=(1^3)+(21)+(31^2)$
$(3)(1^2)=(21)+(3)+(31^2)+(41)$
$(21)(1^2)=(1)+(1^3)+2(21)+(2^21)+(3)+(31^2)+(32)$
$(1^3)(1^2)=(1)+(1^3)+(21)+(2^21)$
$(5)(1)=(4)+(51)+(6)$
$(41)(1)=(31)+(4)+(41^2)+(42)+(51)$
$(32)(1)=(2^2)+(31)+(321)+(3^2)+(42)$
$(31^2)(1)=(21^2)+(31)+(321)+(41^2)$
$(2^21)(1)=(21^2)+(2^2)+(2^3)+(321)$
$(4)(2)=(2)+(31)+(4)+(42)+(51)+(6)$
$(31)(2)=(1^2)+(2)+(21^2)+(2^2)+2(31)+(321)+(3^2)+(4)+(41^2)+(42)+(51)$
$(2^2)(2)=(2)+(21^2)+(2^2)+(2^3)+(31)+(321)+(42)$
$(21^2)(2)=(1^2)+2(21^2)+(2^2)+(31)+(321)+(41^2)$
$(4)(1^2)=(31)+(4)+(41^2)+(51)$
$(31)(1^2)=(2)+(21^2)+(2^2)+2(31)+(321)+(4)+(41^2)+(42)$
$(2^2)(1^2)=(1^2)+(21^2)+(2^2)+(31)+(321)+(3^2)$
$(21^2)(1^2)=(1^2)+(2)+2(21^2)+(2^2)+(2^3)+(31)+(321)$
$(3)(3)=(0)+(1^2)+(2)+(2^2)+(31)+(3^2)+(4)+(42)+(51)+(6)$

$$\{N\}=\sum_{p=0}^{N}\sum_{q=0}^{N-p}\{p\}_{\pi}\{q\}_{\nu}\{N-p-q\}_{\delta}\quad\text{for }\ \mathrm{U}_{\pi}(6)\oslash\mathrm{U}_{\nu}(6)\oslash\mathrm{U}_{\delta}(6). \tag{98}$$

As in IBM-2, a trivial lattice is obtained joining the three algebras in the first step by the link

$$\mathrm{U}_{\pi}(6)\oslash\mathrm{U}_{\nu}(6)\oslash\mathrm{U}_{\delta}(6)\supset\mathrm{U}_{\pi+\nu+\delta}(6). \tag{99}$$

In this case we will have a triple Kronecker product of U(6) irreps and the resulting $\mathrm{U}_{\pi+\nu+\delta}(6)$ irreps can be three-rowed. From this point on one follows chains (I), (II), and (III) in which the irreps of U(5) and O(6) can be three-rowed and those of O(5) two-rowed.

A more interesting lattice, from the physical point of view, is the one that works separately with space and isospin degrees of freedom and joins then at the end. To this end let us denote creation and annihilation operators by

$$b^{\dagger}_{\rho\ell m},\quad b_{\rho\ell m}\qquad(\rho=\pi,\nu,\delta,\quad m=-\ell,-\ell+1,\dots,\ell),$$

or

$$b^{\dagger}_{\rho\alpha},\quad b_{\rho\alpha}\quad (\rho=\pi,\nu,\delta,\quad \alpha=1,2,\ldots,6) \tag{100}$$

so that the commutation relations read as

$$[b_{\rho\alpha},b^{\dagger}_{\rho'\alpha'}]=\delta_{\rho\rho'}\delta_{\alpha\alpha'},\quad [b_{\rho\alpha},b_{\rho'\alpha'}]=[b^{\dagger}_{\rho\alpha},b^{\dagger}_{\rho'\alpha'}]=0. \tag{101}$$

The U(18) infinitesimal generators will then be realized by

$$\mathcal{C}^{\rho'\alpha'}_{\rho\alpha}=b^{\dagger}_{\rho\alpha}b_{\rho'\alpha'}, \tag{102}$$

while

$$C^{\rho'}_{\rho}=\sum_{\alpha=1}^{6}\mathcal{C}^{\rho'\alpha}_{\rho\alpha}\quad\text{and}\quad \mathbb{C}^{\alpha'}_{\alpha}=\sum_{\rho=1}^{3}\mathcal{C}^{\rho\alpha'}_{\rho\alpha} \tag{103}$$

are generator of the Lie algebras of $U_S(6)$ (space) and $U_T(3)$ (isospin), respectively.

We then have as first link in this lattice,

$$U(18)\supset U_S(6)\times U_T(3). \tag{104}$$

The U(18) irrep will be symmetric and, according to Eq. (56), the branching law in Eq. (104) will be

$$\{N\}=\sum_{n_1,n_2,n_3}\{n_1,n_2,n_3\}_S\{n_1,n_2,n_3\}_T, \tag{105}$$

where (n_1,n_2,n_3) is a (standard) partition of N into three parts.

For $U_T(3)$ one uses the chain $U_T(3)\supset O^{+}_T(3)\supset O^{+}_T(2)$ and the branching rule is given by Elliott's rules, Eqs. (59)–(61).

From $U_S(6)$ one can follow each of chains (I), (II), and (III) and use the results of Sec. III for three-rowed U(6) irreps.

For $U_S(6)\supset U_S(5)$, Eqs. (50) and (51) give

$$\{f_1,f_2,f_3\}=\sum_{f_1'=f_2}^{f_1}\sum_{f_2'=f_3}^{f_2}\sum_{f_3'=0}^{f_3}\{f_1',f_2',f_3'\}. \tag{106}$$

For $U(5)\supset O(5)$ one uses Eq. (57) and Table I. The three-rowed $O^{+}(5)$ irreps $(\omega_1,\omega_2,\omega_3)$ in Eq. (57) must be interpreted using the modification rules

$$(\omega_1,\omega_2,1)\equiv(\omega_1,\omega_2),\quad (\omega_1,\omega_2,\omega_3>1)\ \text{disregarded}. \tag{107}$$

For $O^{+}(6)\supset O^{+}(5)$ one uses Eq. (83).

For $U_S(6)\supset SU_S(3)$ one uses Eq. (74) where now Schur functions $\{\mu\}$ with up to three rows must be considered.

VI. IBM-4

In IBM-4, proposed by Elliott and Evans,[24] the bosonic pairs, besides the spatial degree of freedom, have also spin–isospin degrees of freedom in the combination $S=0$, $T=1$ and $S=1$, $T=0$. The model has thus $6\times 6=36$ bosonic creation operators

$$b^{\dagger}_{(\ell m_\ell)(S m_S)(T m_T)} \quad \text{with } \ell=0,2,\ -\ell\leqslant m_\ell \leqslant \ell,$$

$$S=m_S=0,\ T=1,\ m_T=0,\pm 1,$$

$$S=1,\ m_S=0,\pm 1,\ \ T=m_T=0 \tag{108}$$

and corresponding annihilation operators. The operators

$$\mathcal{C}^{(\ell' m'_\ell)(S' m'_S)(T' m'_T)}_{(\ell m_\ell)(S m_S)(T m_T)} = b^{\dagger}_{(\ell m_\ell)(S m_S)(T m_T)} b_{(\ell' m'_\ell)(S' m'_S)(T' m'_T)} \tag{109}$$

generate the Lie algebra of U(36) while

$$C^{(S' m'_S)(T' m'_T)}_{(S m_S)(T m_T)} = \sum_{\ell m_\ell} \mathcal{C}^{(\ell m_\ell)(S' m'_S)(T' m_T)'}_{(\ell m_\ell)(S m_S)(T m_T)} \quad \text{and } \mathbb{C}\ {}^{\ell' m'_\ell}_{\ell m_\ell} = \sum_{S m_S T m_T} \mathcal{C}^{(\ell' m'_\ell)(S m_S)(T m_T)}_{(\ell m_\ell)(S m_S)(T m_T)} \tag{110}$$

generate the Lie algebras of $\mathrm{U}_{ST}(6)$ and $\mathrm{U}_L(6)$ in the chain

$$\mathrm{U}(36) \supset \mathrm{U}_L(6) \times \mathrm{U}_{ST}(6). \tag{111}$$

An arbitrary irrep of U(36) branches into irreps of $\mathrm{U}_L(6)\times \mathrm{U}_{ST}(6)$ according to Eq. (56). Since the U(36) irreps that one can realize with (108) and (109) are only symmetric ones, Eq. (56) gives

$$\{N\} = \sum_{\mu} \{\mu\}_L \{\mu\}_{ST}, \tag{112}$$

where (μ) are (standard) partitions of N into six parts. For $\mathrm{U}_L(6)$ one follows chains (I), (II), and (III), now with all irreps in their greatest generality.

To treat $\mathrm{U}_{ST}(6)$ one observes that

$$C^{(00)(1m')}_{(00)(1m)} \quad \text{and } C^{(1m)(00)}_{(1m)(00)}$$

generate the Lie algebras of $\mathrm{U}_S(6)$ and $\mathrm{U}_T(3)$ in the link

$$\mathrm{U}_{ST}(6) \supset \mathrm{U}_S(3) \oslash \mathrm{U}_T(3), \tag{113}$$

which allows us to treat spin and isospin separately. The branching rules in this link are given by Eq. (54):

$$\{\lambda\} = \sum_{\mu\nu} \alpha(\{\mu\}\{\nu\} \to \{\lambda\}) \{\mu\}_S \{\nu\}_T \quad \text{for } \mathrm{U}_{ST}(6) \supset \mathrm{U}_S(3) \oslash \mathrm{U}_T(3). \tag{114}$$

The simplest case is when $\{\lambda\}$ is symmetric,

$$\{N\}_{ST} = \sum_{k=0}^{N} \{N-k\}_S \{k\}_T. \tag{115}$$

The next one is

$$\{N-1,1\} = \sum_{k=1}^{N-1} \{k\}_S \{N-k\}_T + \sum_{k=2}^{N} [\{N-k\}_S \{k-1,1\}_T + \{k-1,1\}_S \{N-k\}_T]. \tag{116}$$

Another chain of interest is

$$\mathrm{U}_{ST}(6)\supset \mathrm{SU}_{ST}(4)\supset \mathrm{SU}_{S}(2)\times \mathrm{SU}_{T}(2). \tag{117}$$

To find the reduction (35) for this chain one observes that the basis states for irrep $\{1\}$ of $\mathrm{U}_{ST}(6)$ are

$$\sum_{\ell m_\ell} b^{\dagger}_{(\ell m_\ell)(S m_S)(T m_T)}|0\rangle \quad \text{with } S=M_S=0, \quad T=1, \quad M_T=0,\pm 1,$$
$$S=1, \quad m_S=0,\pm 1, \quad T=m_T=0. \tag{118}$$

The states with the first and second sets of labels are basis states for irreps $\{0\}_S\{1\}_T$ and $\{1\}_S\{0\}_T$ of $\mathrm{SU}_S(2)\times \mathrm{SU}_T(2)$, respectively. The U(4) irrep $\{11\}$ has exactly this $\mathrm{SU}_S(2)\times \mathrm{SU}_T(2)$ reduction, so one has

$$\{1\}=\{11\} \text{ for } \mathrm{U}_{ST}(6)\supset \mathrm{U}_{ST}(4)[\mathrm{SU}_{ST}(4)] \tag{119}$$

and according to Eq. (36) one has the branching rule

$$\{\lambda\}=\sum_{\mu} \alpha(\{11\}\otimes\{\mu\}\rightarrow\{\lambda\})\{\mu\}, \quad \text{for} \quad \mathrm{U}_{ST}(6)\supset \mathrm{U}_{ST}(4)[\mathrm{SU}_{ST}(4)], \tag{120}$$

where only the Schur functions with up to four rows are considered in the plethysm. This plethysm can be computed by use of Eq. (23) as input in the algorithm given in Sec. II A. In Table V one gives the branching rules for the reduction $\mathrm{U}_{ST}(6)\supset \mathrm{U}_{ST}(4)$ for $\mathrm{U}_{ST}(6)$ irreps with the lowest degrees.

The branching rules for the reduction $\mathrm{U}(4)\supset \mathrm{SU}(2)\times \mathrm{SU}(2)$ are given by Eq. (56). Table 11-18 in Ref. 25 gives the branching rules for this reduction for U(3) irreps of degrees up to 10.

VII. IBM-1 G AND F

IBM-1 can be extended by introducing bosons with angular momenta 3,4,... . In order to deal with states of positive parity only bosons with even angular momenta are introduced. Bosons with odd angular momenta are used to deal with spectra with even and odd parity levels.

The inclusion of boson pairs of angular momenta $\ell=4$ in IBM-1 gave birth to IBM-1G. In this model one has the boson creation operators

$$b_{\ell m} \quad \text{with} \quad \ell=0,2,4, \quad -\ell\leqslant m\leqslant \ell \tag{121}$$

and the corresponding annihilation operators and the group involved will be U(15). One then has to search for chains ending with $\mathrm{O}^+(3)\supset \mathrm{O}^+(2)$. One of such chains,

$$\mathrm{U}(15)\supset \mathrm{SU}(3)\supset \mathrm{O}^+(3)\supset \mathrm{O}^+(2) \tag{122}$$

was studied in Ref. 26. There, the generators of the Lie algebra of SU(3) in chain (122) are realized as

$$X^{(1)}_{\mu}=\sqrt{1/7}\ [d^{\dagger}\times\tilde{d}]^{(1)}_{\mu}+\sqrt{6/7}\ [g^{\dagger}\times\tilde{g}]^{(1)}_{\mu},$$
$$X^{(2)}_{\mu}=\sqrt{1/70}\ \{4\sqrt{7/15}\ [d^{\dagger}\times\tilde{s}+s^{\dagger}\times\tilde{d}\ ]^{(2)}_{\mu}-11\sqrt{2/21}\ [d^{\dagger}\times\tilde{d}]^{(2)}_{\mu}$$
$$+36\sqrt{1/105}\ [d^{\dagger}\times\tilde{g}+g^{\dagger}\times\tilde{d}]^{(2)}_{\mu}-2\sqrt{33/7}\ [g^{\dagger}\times\tilde{g}]^{(2)}_{\mu}]\}, \tag{123}$$

where $[T^{(k_1)}\times T^{(k_2)}]^{(k)}_m$ denotes coupling of $\mathrm{O}^+(3)$ Racah tensors via Clebsch–Gordan coefficients to produce m components of $\mathrm{O}^+(3)$ tensors of rank k and $\tilde{b}$, as in Ref. 15, is defined by $\tilde{b}_{\ell m}=(-)^{\ell-m}b_{\ell-m}$.

TABLE V. $U_{ST}(6) \supset U_{ST}(4)$ branching rules for $U_{ST}(6)$ irreps with no more than 4 rows and lowest degrees.

$\{1\}=\{1^2\}$
$\{2\}=\{1^4\}+\{2^2\}$
$\{1^2\}=\{21^2\}$
$\{3\}=\{3^2\}+\{2^2 1^2\}$
$\{21\}=\{321\}+\{2^2 1^2\}$
$\{1^3\}=\{31^3\}+\{2^3\}$
$\{4\}=\{4^2\}+\{3^2 1^2\}+\{2^4\}$
$\{31\}=\{431\}+\{3^2 1^2\}+\{32^2 1\}$
$\{2^2\}=\{42^2\}+\{3^2 1^2\}+\{2^4\}$
$\{21^2\}=\{421^2\}+\{3^2 2\}+\{32^2 1\}$
$\{1^4\}=\{32^2 1\}$
$\{5\}=\{3^2 2^2\}+\{4^2 1^2\}+\{5^2\}$
$\{41\}=\{541\}+\{4^2 1^2\}+\{4321\}+\{3^2 2^2\}$
$\{32\}=\{532\}+\{4^2 1^2\}+\{4321\}+\{3^2 2^2\}$
$\{31^2\}=\{531^2\}+\{4^2 2\}+\{4321\}+\{42^3\}+\{3^3 1\}$
$\{2^2 1\}=\{52^2 1\}+\{43^2\}+\{4321\}+\{3^2 2^2\}$
$\{21^3\}=\{4321\}+\{42^3\}+\{3^3 1\}$
$\{6\}=\{3^4\}+\{4^2 2^2\}+\{5^2 1^2\}+\{6^2\}$
$\{51\}=\{651\}+\{5^2 1^2\}+\{5421\}+\{4^2 2^2\}+\{43^2 2\}$
$\{42\}=\{642\}+\{5^2 1^2\}+\{5421\}+\{53^2 1\}+2\{4^2 2^2\}+\{3^4\}$
$\{41^2\}=\{641^2\}+\{5^2 2\}+\{5421\}+\{532^2\}+\{4^2 31\}+\{43^2 2\}$
$\{3^2\}=\{63^2\}+\{5421\}+\{43^2 2\}$
$\{321\}=\{6321\}+\{543\}+\{5421\}+\{53^2 1\}+\{532^2\}$ $+\{4^2 31\}+\{4^2 2^2\}+\{43^2 2\}$
$\{31^3\}=\{5421\}+\{532^2\}+\{4^2 31\}+\{43^2 2\}$
$\{2^3\}=\{62^3\}+\{53^2 1\}+\{4^3\}+\{4^2 2^2\}+\{3^4\}$
$\{2^2 1^2\}=\{53^2 1\}+\{532^2\}+\{4^2 31\}+\{43^2 2\}$
$\{7\}=\{4^2 3^2\}+\{5^2 2^2\}+\{6^2 1^2\}+\{7^2\}$
$\{61\}=\{761\}+\{6^2 1^2\}+\{6521\}+\{5^2 2^2\}+\{5432\}+\{4^2 3^2\}$
$\{52\}=\{752\}+\{6^2 1^2\}+\{6521\}+\{6431\}+2\{5^2 2^2\}$ $+\{5432\}+\{4^2 3^2\}$
$\{51^2\}=\{751^2\}+\{6^2 2\}+\{6521\}+\{642^2\}+\{5^2 31\}+\{5432\}$ $+\{53^3\}+\{4^3 2\}$
$\{43\}=\{743\}+\{6521\}+\{6431\}+\{5^2 2^2\}+\{5432\}+\{4^2 3^2\}$

To find how the U(15) irrep $\{1\}$ branches into irreps of chain (122) one observes that this irrep has three sub-bases with basis vectors $s^+|0\rangle, d_\mu^+|0\rangle$, and $g_\mu^+|0\rangle$ that are left invariant by the action of generators $X_\mu^{(1)}$, showing that $\{1\}$ contains $L=0,2,4$ multiplets. On the other hand, the generators $X_\mu^{(2)}$ mix these sub-bases showing that the SU(3) irrep contained in $\{1\}$ is irreducible. Examining the L-content of SU(3) irreps of dimension 15 one finds that the irrep $\{4\}$ has these L multiplets. Therefore one has

$$\{1\}=\{4\} \quad \text{for} \quad U(15)\supset U(3)[SU(3)] \tag{124}$$

in chain (122).

From Eq. (36) one then obtains

$$\{\lambda\}=\sum_\mu \alpha(\{4\}\otimes\{\lambda\}\rightarrow\{\mu\})\{\mu\} \quad \text{for} \quad U(15)\supset U(3), \tag{125}$$

where in the plethysm only the Schur functions $\{\mu\}$ with no more than three rows are considered. As before, only U(15) symmetric irreps are realized, so the only *reduced* plethysms needed are $\{4\}\otimes\{m\}$. Table VI lists the $U(15)\supset SU(3)$ branching rules for U(15) symmetric irreps with lowest degrees

TABLE VI. $U(15)\supset SU(3)$ branching rules for symmetric $U(15)$ irreps of lowest degrees.

$\{0\}=\{0\}$
$\{1\}=(4,0)$
$\{2\}=(0,4)+(8,0)+(4,2)$
$\{3\}=(0,0)+(0,6)+(2,2)+(3,3)+(4,4)+(6,3)+(8,2)+(6,0)+(12,0)$
$\{4\}=2(4,0)+(0,2)+(1,3)+2(2,4)+(3,5)+2(4,6)+(12,2)+2(8,4)+(5,1)+(4,3)+2(6,2)+(8,1)$
$+(10,0)+(5,4)+(7,3)+(0,8)+(16,0)+(10,3)$
$\{5\}=2(0,4)+(2,3)+4(4,2)+(6,1)+3(8,0)+2(2,0)+(3,1)+2(8,3)+(16,2)+2(3,4)+3(5,3)+(1,5)$
$+3(2,6)+2(4,5)+4(6,4)+(14,3)+(1,8)+(3,7)+(5,6)+2(7,5)+(0,10)+(20,0)+2(4,8)+(6,7)$
$+2(8,6)+3(10,2)+(9,4)+(12,1)+(11,3)+(10,5)+(14,0)+2(9,1)+2(12,4)+(7,2)$
$\{6\}=2(0,0)+2(9,6)+4(2,2)+4(3,3)+7(4,4)+(6,9)+2(5,8)+2(1,4)+2(4,1)+2(2,5)+2(5,2)+5(6,3)$
$+4(0,6)+5(6,0)+3(7,1)+7(8,2)+2(4,10)+2(10,1)+4(12,0)+5(5,5)+3(4,7)+4(7,4)+6(6,6)$
$+5(9,3)+4(8,5)+2(11,2)+5(10,4)+3(12,3)+3(14,2)+(13,4)+3(12,6)+(16,1)+(15,3)+(14,5)$
$+(18,0)+(24,0)+2(16,4)+3(8,8)+(18,3)+(20,2)+2(1,7)+4(3,6)+4(2,8)+2(7,7)+2(13,1)$
$+2(11,5)+(10,7)+2(0,12)+2(3,9)$
$\{7\}=5(4,0)+5(9,2)+13(8,4)+8(7,6)+8(6,8)+2(5,10)+8(10,3)+8(9,5)+6(8,7)+3(7,9)+6(11,4)$
$+8(10,6)+3(9,8)+5(12,5)+3(11,7)+2(13,6)+3(0,2)+(2,1)+3(1,3)+3(3,2)+9(2,4)+4(5,1)$
$+7(4,3)+11(6,2)+(28,0)+3(1,6)+8(3,5)+8(5,4)+9(7,3)+5(0,8)+4(2,7)+12(4,6)+9(6,5)$
$+8(5,7)+4(4,9)+2(3,11)+(19,3)+3(4,12)+(6,11)+3(8,10)+2(10,9)+5(3,8)+5(2,10)+(1,12)$
$+(24,2)+(22,3)+6(8,1)+7(10,0)+5(11,1)+9(12,2)+(0,14)+2(20,4)+3(14,1)+4(16,0)+4(1,9)$
$+(22,0)+6(13,3)+2(15,2)+6(14,4)+3(16,3)+2(17,1)+3(18,2)+2(15,5)+(17,4)+3(12,8)$
$+2(14,7)+3(16,6)+(18,5)+(20,1)$

In IBM-1F one considers boson pairs with angular momenta 0, 2, and 3 and the resulting group is $U(13)$. For physical reasons it is interesting to single out the bosons with odd angular momentum and then one will have lattices of groups like

$$\begin{array}{ccccccc} & & & \nearrow U(5)\supset O^+(5) & \searrow & & \\ & U_{sd}(6) & & \to SU(3) & \to & O^+_{sd}(3) & \\ \nearrow & & & \searrow O^+(6)\supset O^+(5) & \nearrow & & \searrow \\ U(13) & & & & & & O^+(3)\supset O^+(2) \\ \searrow & & & & & & \nearrow \\ & U_f(7)\supset O^+(7) & & & \supset & O^+_f(3) & \end{array} \qquad (126)$$

The branching rules for the reduction $U(13)\supset U_{sd}(6)\oslash U_f(7)$ are given by Eq. (54) that, in IBM-1F, reduces to

$$\{N\}=\sum_{N_f}\{N-N_f\}\{N_f\} \quad \text{for} \quad U(13)\supset U_{sd}(6)\oslash U_f(7). \qquad (127)$$

Since, by Eq. (128), the $U_{sd}(6)$ irrep is symmetric, the sd branch in (127) is exactly equal to IBM-1 up to $O^+_{sd}(3)$.

The $U_f(7)$ irrep, being symmetric, produces the branch

$$\{N_f\}=(N_f)+(N_f-2)+\cdots+(0) \quad \text{or} \quad (1) \quad \text{for} \quad U_f(7)\supset O^+(7). \qquad (128)$$

For the link $O^+(7)\supset O^+(3)$ one has obviously $(1)=(3)$ and the branching rule is obtained by calculating the plethysm

$$\begin{aligned}(3)\otimes(N_f)&=(\{3\}-\{1\})\otimes(\{N_f\}-\{N_f-2\})\\ &=\{3\}\otimes\{N_f\}-(\{3\}\otimes\{N_f-1\})\{1\}+(\{3\}\otimes\{N_f-2\})(\{1^2\}-\{0\})\\ &\quad+(\{3\}\otimes\{N_f-3\})(\{1\}-\{1^3\})\\ &\quad-(\{3\}\otimes\{N_f-4\})\{1^2\}+(\{3\}\otimes\{N_f-5\})\{1^3\},\end{aligned} \qquad (129)$$

TABLE VII. $O^+(7)\supset O^+(3)$ branching rules for $O^+(7)$ symmetric irreps of degrees up to 10.

$(0)=(0)$
$(1)=(3)$
$(2)=(2)+(4)+(6)$
$(3)=(1)+(3)+(4)+(5)+(6)+(7)+(9)$
$(4)=(0)+(2)+(3)+2(4)+(5)+2(6)+(7)+2(8)+(9)+(10)+(12)$
$(5)=(1)+(2)+2(3)+(4)+3(5)+2(6)+3(7)+2(8)+2(9)+2(10)+2(11)+(12)+(13)+(15)$
$(6)=(0)+2(2)+2(3)+3(4)+2(5)+4(6)+3(7)+4(8)+3(9)+4(10)+2(11)+3(12)+2(13)+2(14)+(15)+(16)+(18)$
$(7)=2(1)+(2)+3(3)+3(4)+4(5)+4(6)+5(7)+4(8)+6(9)+4(10)+5(11)+4(12)+4(13)+3(14)+3(15)+2(16)+2(17)+(18)+(19)+(21)$
$(8)=(0)+(1)+3(2)+2(3)+5(4)+4(5)+6(6)+5(7)+7(8)+6(9)+7(10)+6(11)+7(12)+5(13)+6(14)+4(15)+5(16)+3(17)+3(18)+2(19)+2(20)+(21)+(22)+(24)$
$(9)=2(1)+2(2)+5(3)+4(4)+6(5)+6(6)+8(7)+7(8)+9(9)+8(10)+9(11)+8(12)+9(13)+7(14)+8(15)+6(16)+6(17)+5(18)+5(19)+3(20)+3(21)+2(22)+2(23)+(24)+(25)+(27)$
$(10)=2(0)+(1)+4(2)+4(3)+6(4)+6(5)+9(6)+8(7)+10(8)+9(9)+12(10)+10(11)+12(12)+10(13)+11(14)+10(15)+10(16)+8(17)+9(18)+6(19)+7(20)+5(21)+5(22)+3(23)+3(24)+2(25)+2(26)+(27)+(28)+(30)$

where the plethysms $\{3\}\otimes\{m\}$ with negative m's are taken as null. After computing the plethysms and the outer products one uses Elliott's rules (59)–(61) to obtain the final L_f values. Branching rules for $O^+(7)\supset O^+(3)$ resulting from Eq. (129) are given in Table VII for $N=1,2,\ldots,10$.

One could obtain the L_f values using the reduction $U(7)\supset O^+(3)$ without the intermediate group $O^+(7)$. In this case, Eq. (35) will be $\{1\}=(3)$ and the result for U(7) symmetric irreps is

$$\{N_f\}=\sum_{k=0}^{3}\sum_{\mu}\alpha(\{3\}\otimes\{N_f-k\}\rightarrow\{\mu\})\{\mu\}\{1^k\}, \tag{130}$$

where in the plethysms only Schur functions with up to three rows are considered. The U(3) irreps resulting from the Kronecker products must be converted to $O^+(3)$ using Elliott's rules. Obviously the L_f values obtained using Eqs. (128) and (129) and Eq. (130) are the same.

VIII. FINAL COMMENTS

The branching rules for IBM-1 are known in the literature, each one being obtained by a different method. By the plethysm approach here presented all of them are obtained in a single unified way and the results obtained are used to other extensions of IBM. For some of these extensions the branching rules found in the literature are given only for simple cases without an explanation of how they were obtained, preventing the reader from extending tables when needed. The material presented in this paper provides to the reader all the material to check our tables and extend them as long as he needs. Besides, the approach here used can be applied, *mutatis mutandis* on other situations that need the knowledge of branching rules of subgroups of GL(n).

The tables here presented were obtained by computer programs which, by control, perform dimension tests. To avoid misprints, the output of these programs are read by another program that produces the latex source files of the tables.

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