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Linearized $\beta-$deformation of $\mathcal{N} = 4$ SYM

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LINEARIZED β –DEFORMATION OF N = 4 SYM

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To those who have passed away due to Covid-19

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"What we observe is not nature itself, but nature exposed to our method of questioning."

Werner Heisenberg

Resumo

Esta tese está dividida em duas partes. Na primeira, estudamos o supermultipleto das deformações β da teoria Supersimétrica $\mathcal{N} = 4$ Yang-Mills (SYM) e comparamos duas representações finitas da álgebra superconforme. Na segunda parte, analisamos as deformações $\beta-$ linearizadas sob uma perspectiva geométrica. Nosso principal resultado é fornecer uma ponte entre a perspectiva de campo livre, a teoria de representação superconforme e a interpretação geométrica do espaço de deformações. Além disso, revisamos os conceitos utilizados ao longo deste trabalho.

Palavras Chaves: Deformação-β; Super-Grassmanniano; Supertwistors, Super Yang-Mills;

Áreas do conhecimento: Física Teórica; Teoria de Yang-Mills; Supersimetria.

Abstract

This thesis is divided into two parts. In the first part, we study the supermultiplet of $\beta-$ deformations of $\mathcal{N} = 4$ Supersymmetric Yang-Mills (SYM) theory and we compare two finite-dimensional representations of the superconformal algebra. In the second part, we analyze linearized $\beta-$deformations from a geometrical perspective. Our main result is the explicit construction of a bridge between the free-field perspective, the superconformal representation theory, and the geometric interpretation of the deformation space. We also review the concepts used throughout this thesis.

Keywords: β-deformation; Super-Grassmannian; Supertwistors, Super Yang-Mills;

Areas of knowledge: Theoretical Physics; Yang-Mills Theory; Supersymmetry.

Contents

Chapter 1

Introduction

1.1 Deformations by spacetime-dependent interactions

One crucially important theme in QFT is the deformation by local operators. We can change the action of a theory by adding an integral of local operator:

$$S \mapsto S + \int d^4 x U(x) \tag{1.1}$$

It is usually assumed that all the coupling constants are, indeed, **constants**. In particular, $U(x)$ depends on x only implicitly, *i.e.* through the field operators. For example:

$$U(x) = \lambda \phi(x)^4 \tag{1.2}$$

$$\lambda = \text{const} \tag{1.3}$$

But sometimes it is interesting to consider interaction operators with explicit spacetime dependence, *i.e.* $\lambda = \lambda(x)$.

In particular, in a supersymmetric theory, we can add conformally invariant $\int \lambda U(x)$ with constant λ, and then act on it by supersymmetry generators, and then by conformal transformations. Then, at the linearized level (*i.e.* small λ) way we will get the whole "multiplet" of deformations. (Although the word "multiplet" is usually reserved for describing the field components in a supersymmetric theory, we use it here to describe a representation of the supersymmetry algebra.) We can ask various questions about such deformations, in particular:

- What is the structure of the "multiplet". What kind of a representation of the supersymmetry is it?
- How does the deformation affect the S-matrix? (Is it even possible to **define** the S-matrix when the coupling constant is not constant?)

- What happens when we pass from infinitesimal deformations to finite λ?

In supersymmetric theories, these questions are expected to open up a rich mathematical structure. In this thesis, we will consider the simplest, and in some sense very fundamental, example. This is the so-called $\beta-$deformation of the $\mathcal{N}=4$ supersymmetric Yang-Mills theory in four dimensions. We will address the first and partially second question in this particular example.

1.2 Finite-dimensional deformations of supersymmetric Yang-Mills

Consider single trace deformations of $\mathcal{N}=4$ supersymmetric Yang-Mills theory (SYM):

$$S_{\text{SYM}} \;\mapsto\; S_{\text{SYM}} + \varepsilon \int d^4x U \tag{1.4}$$

where U is a single-trace operator. According to AdS/CFT [1], they correspond to some deformations of the Type IIB superstring theory on $AdS_5 \times S^5$. Let us restrict ourselves to infinitesimal deformations, *i.e.* compute only to the first order in the deformation parameter ε. Besides ε, there are two other parameters: the Yang-Mills coupling constant g_{YM} and the number of colors N. In this thesis, we will consider N being very large. As for the g_{YM}, there are two opposite limits: the limit $g_{YM}=0$ of free $\mathcal{N}=4$ super Yang-Mills theory (SYM) and the strong coupling limit. In the weak coupling limit, we can do perturbative calculations in SYM, and in the strong coupling limit, we can use the superstring theory on the classical supergravity (SUGRA) background $AdS_5 \times S^5$.

Both SYM theory and superstring on $AdS_5 \times S^5$ are invariant under the superconformal group $PSU(2,2|4)$, and it is natural to ask how the deformations of the form (1.4) transform under this group. In particular, some deformations of the form (1.4) transform in finite-dimensional representations [2]. As an example of a finite-dimensional representation, consider U of Eq. (1.4) of the form [2]:

$$\int d^4x \; \mathrm{tr}(\Phi_1 + i\Phi_2)^{n+4} \; + \; \text{c.c.} \tag{1.5}$$

where $\Phi_1, \ldots, \Phi_6$ are the scalar fields of the $\mathcal{N}=4$-super-Yang-Mills theory. Consider the linear space of all deformations obtained from Eq. (1.5) by acting with all possible polynomials of generators of $psu(2,2|4)$. In the free theory, this gives a finite-dimensional representation. Let us call it E^{free}_{n+4}.

Most of finite-dimensional deformations of the free theory combine into infinite-dimensional representations in the quantum interacting theory. But E^{free}_{n+4} stays finite-dimensional. We will call it just E_{n+4}. This follows from the finiteness of the operator $\text{tr}(\Phi_1 + i\Phi_2)^{n+4}$ and its descendants. Finiteness implies that the conformal transformations of this operators are same in quantum theory as in classical theory (no countertems $\Rightarrow$ no anomalous dimension). Therefore, the subspace generated by acting on Eq. (1.5) with the conformal generators is the same in classical and quantum theory. In particular, it is finite-dimensional. To generate the full representation, it remains to act with supersymmetry and superconformal generators. However, those are nilpotent modulo elements of $su(2,2) \oplus su(4)$, and therefore cannot change the property of the representation being finite-dimensional.

We will now argue that E_{n+4} is isomorphic to the representation E^{free}_{n+4} which is generated by acting with $psu(2,2|4)$ on Eq. (1.5) in the **free** $\mathcal{N} = 4$ super-Yang-Mills theory. Indeed, let us consider the **chiral primary representation**, which is defined as the infinite-dimensional representation generated by the local operator $\text{tr}(\Phi_1 + i\Phi_2)^{n+4}(0)$ by acting on it with $psu(2,2|4)$. This is a unitary representation. Let us call it F_{n+4}.

Similarly, there is a free version F^{free}_{n+4} which is defined in the same way as F_{n+4}, but in the free theory. We start by pointing out that F_{n+4} is isomorphic to F^{free}_{n+4}, because both are highest weight unitary irreducible representations of $psu(2,2|4)$ with the same highest weight $\text{tr}(\Phi_1 + i\Phi_2)^{n+4}(0)$. On the other hand, our finite-dimensional representations E_{n+4} and E^{free}_{n+4} are fully determined by F_{n+4} and F^{free}_{n+4}, respectively, in the following way. Elements of E_{n+4} are of the form $\int d^4x\; \rho(x)\; \mathcal{O}(x)$ where $\mathcal{O}$ is an element of F_{n+4} (a local operator), but inserted at the point x instead of 0. When we act on $\int d^4x\; \rho(x)\; \mathcal{O}(x)$ with an element of $psu(2,2|4)$, the result is determined by how this element acts on $\mathcal{O}(x)$, *i.e.* by the structure of F_{n+4}. The isomorphism $F_{n+4} \xrightarrow{\simeq} F^{\text{free}}_{n+4}$ implies a map $\mathcal{O} \mapsto \mathcal{O}^{\text{free}}$ commuting with the action of $psu(2,2|4)$. This establishes an isomoprhism $E_{n+4} \xrightarrow{\simeq} E^{\text{free}}_{n+4}$.

1.3 Supermultiplet of $\beta-$deformation

On the string theory side, the $\beta-$deformation of the string worldsheet is just wedge product of two symmetry currents:

$$\int dz d\bar{z}\; B^{ab}\; j_a \wedge j_b \tag{1.6}$$

Therefore, the corresponding multiplet is "more or less" the exterior product of two adjoint representations [3]:

$$\frac{(\mathfrak{g}\wedge\mathfrak{g})_0}{\mathfrak{g}} \tag{1.7}$$

where we abbreviate:

$$\mathfrak{g} = psu(2,2|4) \tag{1.8}$$

(The constant tensor $B^{ab} = -B^{ba}$ has two adjoint indices ^{ab}, thus it is in $\mathfrak{g}\wedge\mathfrak{g}$.) But on the field theory side, only the "top" of the supermultiplet is known. It is a cubic interaction:

$$\int d^4x\ \phi_{IJ}(x)\ B^{IJ}_{KL}\ \epsilon^{\alpha\beta}\psi^K_\alpha(x)\psi^L_\beta(x) \tag{1.9}$$

where B^{IJ}_{KL} is a constant tensor in the top component of the wedge-product in Eq. (1.7). It is far from being obvious, that the action of the superconformal generators on this "top" component actually generates the full Eq. (1.7). It is not at all clear why these deformations would transform under superconformal transformations as a product of two currents. We have proven this, by explicit computation, in [4]. The computation uses the oscillator formalism (the "doubleton"). Then, in [5], we have proceeded with studying the properties of the supermultiplet, from the point of view of supergeometry of harmonic superspace.

1.4 Generalizations of S-matrix?

We have not worked out the appropriate definition of S-matrix in the theory with spacetime dependent coupling constants, although we think that it might be possible. Instead, we will simply **define** the **deformation** of S-matrix as the evaluation of the interaction term $\int \lambda U(x)$ on on-shell solutions. As we argued in Section 1.2 the structure of the representation does not depend on the Yang-Mills coupling constant. The deformation of the scattering matrix probably does. But, let us continue with deforming just the **free** $\mathcal{N} = 4$ SYM theory. This is already very interesting, and we can study the effect of Yang-Mills coupling constant later.

In 2+2 dimensions, we can consider free solutions in the form of delta-functions localized on null-planes. Then, the product of free solutions has compact support, and therefore the integral $\int d^4x\lambda U(x)$ localizes on one point. Therefore the integral is a **well-defined generalized function** of those planes. We will compute it for one particular set of quantum numbers, and conjecture the general formula.

We want to stress, that our computation is somewhat non-standard. Scattering amplitudes for the "usual" $\beta-$deformation [6] is certainly well-defined, and its physical meaning is transparent. (Although we are not sure if it has been explicitly studied in the literature.) But here we go further and study the scattering for the full supermultiplet of $\beta-$deformation, which involves spacetime dependent couplings in the Lagrangian. We believe that it is physically meaningful, although to the best of our knowledge has not been done. In any case, we obtain (or at least conjecture) Section 4.2 some interesting formulas. Moreover, these formulas seem to generalize rather naturally to other finite-dimensional representations. If we allow non-constant coupling "constants", then the space of deformations is very rich. The $\beta-$deformation is just the first in the infinite series of finite-dimensional representations. We conjecture a formula for the corresponding deformation of the scattering amplitude for all those deformations Section 4.3.

We will now briefly describe how we have proceeded. We will first remind the twistor description of the Minkowski space.

1.5 Twistor geometry of Minkowski space

We use the twistor approach, and the closely related oscillator formalism (the "doubleton"), mostly following [7]. In this approach the four-dimensional space is realized as the "configuration space" of two-dimensional planes in the four dimensional "twistor" space $\mathbb{T}$, passing through the origin. Each point of space corresponds to some 2-plane $L \subset \mathbb{T}$ in the twistor space. Remarkably, this construction defines a conformal structure; There is a conformal metric on the space of 2-planes. Therefore the group $PGL(4,\mathbb{R})$ of projective transformations of $\mathbb{T}$ is identified with the conformal group.

The space of planes is called Grassmannian manifold:

$$M = \mathrm{Gr}(2,4) \tag{1.10}$$

We will now explain how the conformal structure on M arizes. The conformal structure is metric modulo rescaling. Metric is a section of $S^2\mathcal{T}^*M$ — the vector bundle of symmetric rank two tensors in the cotangent space $\mathcal{T}^*M$. One key point is the identification of $\mathcal{T}^*M$ with the tensor product of two spinor bundles:

$$\mathcal{T}^*M = \mathcal{S}^\perp \otimes \mathcal{S} \tag{1.11}$$

Here $\mathcal{S}$ is the "tautological" vector bundle over M; its fiber over $L \in M$ is L itself (a two-dimensional vector space). Sections of $\mathcal{S}$ are spinor fields. The second bundle $\mathcal{S}^\perp$ is the space of linear functionals on $\mathbb{T}$ which are zero on L:

$$\mathcal{S}^\perp \subset \mathbb{T}^* \tag{1.12}$$

$$\psi(L) \in \mathcal{S}^\perp \Leftrightarrow \langle \psi_L, v \rangle = 0 \;\; \forall v \in L \tag{1.13}$$

Eq. (1.11) should be read in the following way. Given $\xi \in \mathcal{S}$ and $\psi \in \mathcal{S}^\perp$, we can construct a linear functional α on $\mathcal{T}M$ in the following way. An element of $\mathcal{T}M$ is an infinitesimal variation δL of L. We let our $\xi \in L$ vary with L. Let $\delta\xi$ be a variation of ξ, "following" the variation of L. We then define:

$$\langle \alpha, \delta L \rangle = \langle \psi, \delta\xi \rangle \tag{1.14}$$

Notice that $\delta\xi$ is not uniquely defined. In fact we can modify $\delta\xi$ by adding to it any vector in L, and the resulting $\delta\xi$ will still follow δL. However, this ambiguity in $\delta\xi$ drops out from Eq. (1.14), because of Eq. (1.13).

Consider the one-dimensional subbundle in $\mathcal{T}^*M$:

$$\Lambda^2\mathcal{S}^\perp \otimes \Lambda^2\mathcal{S} \subset S^2\mathcal{T}^*M \tag{1.15}$$

We define the metric as any section of this subbundle. Since it is one-dimensional, any two sections are related by rescaling, and therefore define the same conformal structure.

Given that $\det\mathcal{S}\ \det(\mathbb{T}/\mathcal{S}) = 1$ and $\det(\mathbb{T}/\mathcal{S}) = (\det\mathcal{S}^\perp)^{-1}$, Eq. (1.11) also implies:

$$\det T^*M = (\det\mathcal{S})^2(\det\mathcal{S}^\perp)^2 = (\det\mathcal{S})^4 \tag{1.16}$$

The scalar field $\phi(x)$ is actually not a scalar, but a density of weight one:

$$\phi \in \Gamma(\det\mathcal{S}) \tag{1.17}$$

(Here Γ(bundle) denotes the space of sections of a bundle. We do allow generalized-functions-sections.) And the spinor field, too:

$$\psi \in \Gamma(\mathcal{S} \otimes \det\mathcal{S}) \tag{1.18}$$

This implies that we have integration densities:

$$\phi^4 \in (\det\mathcal{S})^4 \tag{1.19}$$

$$\phi\,\psi\wedge\psi \in (\det\mathcal{S})^4 \tag{1.20}$$

This proves that $\int\phi^4$ and $\int\phi\psi\wedge\psi$ are conformally-invariant deformations of the action.

1.6 Coherent wave functions

For $(\lambda,\mu)\in\mathbb{T}$, consider the special section of $\det\mathcal{S}$, which we call $\delta_{(\mu,\lambda)}$:

$$\delta_{(\mu,\lambda)} \in \Gamma(\det\mathcal{S}) \tag{1.21}$$

defined so that for all $\rho\in\Gamma(\det\mathcal{S}^*)$ and for all $x\in M$:

$$\int_{(\mu,\lambda)\in\mathbb{T}} f(\mu,\lambda)\left\langle\rho(x),\delta_{(\mu,\lambda)}(x)\right\rangle = \int_{(\mu,\lambda)\in L(x)}\rho(x)f(\mu,\lambda) \tag{1.22}$$

In coordinates:

$$\delta_{(\mu,\lambda)} = \delta^2(\mu - x\lambda) \tag{1.23}$$

It is not immediately obvious from the definition, but we can verify that this ϕ satisfies the free wave equation:

$$\epsilon^{\alpha\beta}\epsilon^{\hat\alpha\hat\beta}\frac{\partial}{\partial x^{\alpha\hat\alpha}}\frac{\partial}{\partial x^{\beta\hat\beta}}\delta^2(\mu - x\lambda) = 0 \tag{1.24}$$

(We will give an explanation of this fact in Sections 2.2 and 2.3.4 .) We now explain why the field ϕ defined in Eq. (1.23) is actually a section of $\det\mathcal{S}$. This can be seen as follows. Let us consider the space $\widehat{M}$, whose points are pairs $(x,\{E_1,E_2\})$ where x is a point of $Gr(2,4)$ and $\{E_1,E_2\}$ is a basis in $\mathcal{S}^\perp$ — the space of linear equations specifying the plane corresponding to x:

$$E_1 = (\mu_1^\vee,\lambda_1^\vee) \tag{1.25}$$

$$E_2 = (\mu_2^\vee,\lambda_2^\vee) \tag{1.26}$$

For example, $E_1 = (1,0,x^{11},x^{12})$ and $E_2 = (1,0,x^{21},x^{22})$. Then:

$$\delta_{(\mu,\lambda)}(x,E_1,E_2) = \delta((\mu_1^\vee,\mu) + (\lambda_1^\vee,\lambda))\,\delta((\mu_2^\vee,\mu) + (\lambda_2^\vee,\lambda)) \tag{1.27}$$

(This is just the "unfolding" of Eq. (1.23).) Notice that:

$$\phi_{\mu,\lambda}(x, AE) \;=\; |\det A|^{-1}\phi_{\mu,\lambda}(x, E) \tag{1.28}$$

Therefore ϕ can be seen as a linear function on $(\det\mathcal{S}^{\perp})^{-1}$. More precisely, there is a map

$$u\; : \widehat{M} \to (\det\mathcal{S}^{\perp})^{*} \tag{1.29}$$

$$u(\{E_1, E_2\}) \;= v \tag{1.30}$$

$$\text{where } \langle v, E_1 \wedge E_2\rangle = 1 \tag{1.31}$$

and ϕ factorizes through this map. This means that we can think of ϕ as a section of $(\det\mathcal{S}^{\perp})$; let us denote it $\hat{\phi}$:

$$\hat{\phi} \;\in \Gamma((\det\mathcal{S}^{\perp})) \tag{1.32}$$

$$\phi(x, \{E_1, E_2\}) \;=\; \Big\langle u(\{E_1, E_2\})\,,\; \hat{\phi}(x)\Big\rangle \tag{1.33}$$

In other words:

$$\phi \in \Gamma(\det\mathcal{S}^{\perp}) = \Gamma(\det\mathcal{S}) \tag{1.34}$$

Actually, this is not quite correct, because our ϕ, as we have so far defined it, transforms in $|\det\mathcal{S}|$. And we need it to transform in $\det\mathcal{S}$. We really need to keep track of the sign:

$$\phi_{\mu,\lambda}(x, E_1, E_2) \;=\; s(E_1, E_2)\,\delta((\mu_1^{\vee}, \mu) + (\lambda_1^{\vee}, \lambda))\,\delta((\mu_2^{\vee}, \mu) + (\lambda_2^{\vee}, \lambda)) \tag{1.35}$$

where $s(E_1, E_2)$ is the sign corresponding to the orientation of E_1, E_2.

1.7 Harmonic superspace

So far we talked about scalar field and spinor field, but we have not yet discussed the supersymmetry. But in this thesis we study the $N = 4$ **supersymmetric** Yang-Mills theory. Fortunately, a supersymmetric extension of the twistor theory has been developed, known as "harmonic superspace". In this formalism, the twistor is replaced with super-twistor:

$$(\mu, \lambda) \longrightarrow (\mu, \lambda, \zeta) = Z \tag{1.36}$$

where $\zeta = (\zeta^1, \zeta^2, \zeta^3, \zeta^4)$ are Grassmann odd variables. Together μ, λ, ζ parameterize the super-twistor space:

$$Z \in \mathbb{T} = \mathbb{R}^{4|4} \tag{1.37}$$

Correspondingly, there is a super-analogue of $\delta_{(\mu,\lambda)}$. This is the product of two "usual" delta-functions (as in Eq. (1.27)) and two Grassmann-odd delta-functions. A Grassmann-odd delta-function $\delta(\theta)$ is just θ itself. Therefore we have, schematically:

$$\begin{aligned}
\delta_Z(X, E_1, E_2, \Psi_1, \Psi_2) &= ((\Theta_1, \mu) + (\Xi_1, \lambda) + (u_1, \zeta))\times & (1.38)\\
&\quad ((\Theta_2, \mu) + (\Xi_2, \lambda) + (u_2, \zeta))\times & (1.39)\\
&\quad \delta((\mu_1^\vee, \mu) + (\lambda_1^\vee, \lambda) + (\zeta_1^\vee, \zeta))\times & (1.40)\\
&\quad \delta((\mu_2^\vee, \mu) + (\lambda_2^\vee, \lambda) + (\zeta_2^\vee, \zeta))\times & (1.41)
\end{aligned}$$

where

$$\begin{aligned}
E_i &= (\mu_i^\vee, \lambda_i^\vee, \zeta_i^\vee) & (1.42)\\
\Psi_i &= (\Theta_i, \Xi_i, u_i) & (1.43)
\end{aligned}$$

are four elements of $\mathbb{T}^*$ (the dual space to twistors), which vanish on the $2|2$-dimensional space corresponding to X. This is somewhat schematic. The thorough explanation follows in the body of the thesis Section 4.1. But, importantly:

$$\delta_Z = \delta_{\kappa Z} \tag{1.44}$$

for all $\kappa \in \mathbb{R}$.

1.8 Evaluation of the interaction on coherent wave functions

The idea is to "probe" the $\beta-$deformation interaction by evaluating it on solutions of free wave equations. Importantly, the color factor enters through the symmetric coefficient:

$$d_{abc} = \mathrm{tr}(t_a t_b t_c) \tag{1.45}$$

Let us consider the solution of the free wave equation, given by the sum of three delta functions:

$$\phi = (\varepsilon_1 \delta_{Z_1} + \varepsilon_2 \delta_{Z_2} + \varepsilon_3 \delta_{Z_3}) M \tag{1.46}$$

where M is the color matrix. Since color factor is completely symmetric Eq. (1.45), it is enough to put the same color matrix M for all three scattering waves. The solution given by Eq. (1.46) is parametrized by the manifold:

$$\mathbb{PT} \times \mathbb{PT} \times \mathbb{PT} \tag{1.47}$$

These three copies of projective twistor space $\mathbb{PT}$ parametrize Z_1, Z_2, Z_3.

The space of linearized $\beta-$deformations, as a representation of $psl(4|4)$, is parametrized by a matrix B in the antisymmetric product of two adjoint representations.

Thus, the evaluation of the interaction gives us a function (actually generalized function) on the space:

$$\mathbb{PT} \times \mathbb{PT} \times \mathbb{PT} \times \frac{\mathfrak{g} \otimes \mathfrak{g}}{\mathfrak{g}} \tag{1.48}$$

which depends linearly on $\frac{\mathfrak{g} \otimes \mathfrak{g}}{\mathfrak{g}}$.

We conjecture the exact expression for this function in Section 4.1.

It should be possible to derive these results using twistor string theory [8], from a generalization of the results of [9], [10], [11]. Notice that twistor string theory requires complex twistor suprspace $\mathbf{CP}^{3|4}$, while for our considerations here real twistors $\mathbf{RP}^{3|4}$ are enough (*cp.* Section 3.2 of [8]).

1.9 Notations

- (a), (b), (c) — Gauge indices,
- μ, ν four dimensional spacetime indices,
- $A, B, C, \ldots = 1,\ 2,\ 3,\ 4$ — internal SU(4) symmetry indices
- $D_\mu = \partial_\mu \pm i g_{YM} A_\mu$ denotes the covariant derivative,
- $f_{abc} = Tr(T_a[T_b, T_c])$
- $d_{abc} = Tr(T_a\{T_b, T_c\})$
- $\sigma^\mu = (-\mathbb{1}, \sigma^i)$ with σ_i being the Pauli matrices, and $\bar{\sigma}^\mu = (-\mathbb{1}, -\sigma^i)$
- $\mathcal{V}$: vector bundle and $\mathcal{S}_M$ vector bundle over M
- Ber: Berezinian

- SDet: SuperDeterminant
- $(\mathrm{Ber}\,\mathcal{V})^{-1}$, $(\mathrm{Ber}\,\mathcal{V})'$: the line bundle dual to $\mathrm{Ber}\,\mathcal{V}$

Chapter 2

The Supertwistor space and the superconformal algebra

This chapter introduces the geometric preliminaries underlying the twistor formulation of $\mathcal{N}=4$ supersymmetric Yang Mills theory. In the section 2.1, we review the superconformal algebra, with emphasis on $gl(4|4,\mathbf{R})$ and their realizations in oscillator and twistor representations. In section 2.2, we review the supergrassmannian $Gr(2|2,4|4)$ and its relation with the harmonic superspace. Finally, in section 2.3 we review the $\mathcal{N}=4$ SYM field theory, the field content and composite operators are written in oscillator representation. To end this chapter, we review how solutions of free classical super-Maxwell equations from twistor space can be written.

2.1 Four dimensional Supertwistors and the superconformal algebra

At four dimensions and $\mathcal{N}=4$, we will denote the supertwistor space $\mathbf{T}=\mathbf{R}^{4|4}$. Elements of $\mathbf{T}$ are vectors Z parametrized by even $\lambda_\alpha, \mu^{\dot\alpha}$ and odd ζ^I

$$Z = \left(\mu^{\dot\alpha}, \lambda_\alpha, \zeta^I\right)^T \tag{2.1}$$

where $\alpha = 1,2$, $\dot\alpha = \dot 1, \dot 2$ and $I = 1,\ldots,4$. T in Eq. (2.1) means transposition, since we think of Z as a "column vector".

The above supertwistor variables are related to the superspace coordinates $x^\mu, \theta^{\alpha I}, \bar\theta_I^{\dot\alpha}$ by

$$\mu^{\dot\alpha} = x^{\alpha\dot\alpha}\lambda_\alpha + \bar\theta_J^{\dot\alpha}\zeta^J \;\text{ and }\; \zeta^J = \theta^{J\alpha}\lambda_\alpha \tag{2.2}$$

2.1.1 About Signature and Superconformal Group

For a real-valued action, the signature of spacetime and the supersymmetry algebra are correlated. This fact was discussed by Beisert [12]

1. In the $1+3$ signature, the spacetime symmetry algebra is $sl(2,\mathbf{C})$ and the internal space symmetry algebra is $su(4)$. The superconformal group is the real form $PSU(2,2|4)$ of $PSL(4|4)$ [8].

2. In the signature $2+2$. The spacetime symmetry algebra is $sl(2,\mathbf{R})\times sl(2,\mathbf{R})$, and the internal space symmetry algebra is $sl(4,\mathbf{R})$. Therefore, the superconformal group is the real form $PSL(4|4,\mathbf{R})$ of $PSL(4|4)$. This signature is often used in twistor theory [8].

3. In the Euclidean signature, the spacetime symmetry algebra is $su(2)\times su(2)$ and the conformal group is the real form $SO(1,5)$ of $SL(4)$

We work with finite-dimensional representations and the corresponding deformations of the action. The formal expression for the deformations does not depend on the signature of spacetime. We choose the $2+2$ signature, where we can use some special wavefunctions, which allow us to probe the interaction term and relate it to a geometrical construction.

A further simplification arises from the fact, in this signature, the reality conditions for symmetry generators in matrix representation become particularly simple: all matrix entries can be taken to be real.Therefore, it is convenient to work with the $2+2$ signature, where conjugations does not play a role [12], leading to a completely real formulation of the theory. From now, we refer to $PSL(4|4,\mathbf{R})$ as the superconformal group and $psl(4|4,\mathbf{R})$ as the corresponding superconformal algebra.

2.1.2 The superalgebra $psl(4|4,\mathbf{R})$

Let us start with the superalgebra $gl(4|4)$, its generators include

$$\mathcal{I} \;=\; \left\{ l_{\alpha}^{\ \beta},\ \dot{l}_{\dot{\alpha}}^{\ \dot{\beta}},\ r^{A}_{\ B},\ p_{\alpha\dot{\alpha}},\ k^{\dot{\alpha}\alpha},\ D,\ B,\ C|q_{A\alpha},\ \dot{q}^{A}_{\dot{\alpha}},\ s^{A\alpha},\ \dot{s}^{\dot{\alpha}}_{A} \right\}, \tag{2.3}$$

where $l_{\alpha}^{\ \beta}$, $\dot{l}_{\dot{\alpha}}^{\ \dot{\beta}}$, $r^{A}_{\ B}$ are the $sl(2)\times sl(2)$ and $sl(4)$ rotations, respectively. $p_{\alpha\dot{\alpha}}, q_{A\alpha}$, $\dot{q}^{A}_{\dot{\alpha}}$ are the (super)translation generators and $k^{\dot{\alpha}\alpha}, s^{A\alpha}$, $\dot{s}^{\dot{\alpha}}_{A}$ are the (super)conformal generators. As well as the dilatation generator D, the outer derivation B and the central charge C. The commutation relations of these generators are left to Appendix A.

The superalgebra $psl(4|4)$ is obtained from $gl(4|4)$ as follows. The generator B does not appear in the right hand side of the commutation relations of the generators of $gl(4|4)$. So, it can be dropped. This leads to $sl(4|4)$ superalgebra. Moreover, when restricting to representations with vanishing central charge C the resulting algebra is $pgl(4|4)$. Combining both facts we get the $psl(4|4)$ algebra.

2.1.3 Action of $GL(4|4,\mathbf{R})$ transformations on twistors

The twistor vector space $\mathbf{T}$ is a fundamental representation of $GL(4|4,\mathbf{R})$. We have:

$$GL(4|4)/\text{center} \;=\; PGL(4|4) \;=\; PSL(4|4,\mathbf{R}) \rtimes U(1)_B \tag{2.4}$$

The $u(1)_B$ acts as follows:

$$BZ = \begin{pmatrix} -\mathbf{1}_{4\times4} & \mathbf{0}_{4\times4} \\ \\ \mathbf{0}_{4\times4} & \mathbf{1}_{4\times4} \end{pmatrix} Z \tag{2.5}$$

In terms of the parametrization (2.1):

$$B \;=\; -\mu_{\dot\alpha}\frac{\partial}{\partial\mu_{\dot\alpha}} - \lambda^\alpha\frac{\partial}{\partial\lambda^\alpha} + \zeta^I\frac{\partial}{\partial\zeta^I} \tag{2.6}$$

B is the hypercharge generator of the algebra. The generators of $psl(4|4,\mathbf{R})$ in terms of supertwistor variables (2.1) are first-order differential operators [8], and are left to the appendix section A.2.

2.1.4 The superalgebra $psl(4|4)$ in oscillator representation

The oscillator representation was introduced for the first time by Bars et. al. [13] in order to construct unitary irreducible representations of non-compact supergroups G.

The key idea of the oscillator representation consists to identify a compact subgroup H of G and construct the unitary irreducible representations(UIR) of G in terms of finite dimensional unitary representations of H [13, 14, 15]. Gunaydin and Marcus[15] used this method to construct UIR of $U(2,2|4)$.

For an oscillator representation of the generators of $gl(4|4, \mathbf{R})$, two sets of bosonic oscillators and one set of fermionic oscillators are introduced

$$(a^{\alpha}, a^{\dagger}_{\alpha}), \ (b^{\dot{\alpha}}, b^{\dagger}_{\dot{\alpha}}), \ \text{ and } \ (c_A, c^{\dagger A}) \tag{2.7}$$

with $\alpha, \dot{\alpha} = 1, 2$ and $A = 1, 2, 3, 4$. These set of oscillators obey the commutation relations

$$[a^{\alpha}, a^{\dagger}_{\beta}] = \delta^{\alpha}{}_{\beta}, \qquad [b^{\dot{\alpha}}, b^{\dagger}_{\dot{\beta}}] = \delta^{\dot{\alpha}}{}_{\dot{\beta}} \qquad \{c_A, c^{\dagger B}\} = \delta_A{}^B. \tag{2.8}$$

The $gl(4|4, \mathbf{R})$ generators can be written as products of two oscillators. The supercharges are expressed as

$$q_{A\alpha} \ = \ c_A a^{\dagger}_{\alpha}, \ \ s^{A\alpha} \ = \ c^{A\dagger} a^{\alpha}, \ \ \bar{q}^A_{\dot{\alpha}} \ = \ c^{A\dagger} b^{\dagger}_{\dot{\alpha}}, \ \text{ and } \ \bar{s}^{\dot{\alpha}}_A \ = \ b^{\dot{\alpha}} c_A \tag{2.9}$$

The other generators are listed in the appendix section A.1. The oscillator representation is related to supertwistors Z, by using the parametrization in Eq.2.1 this relation can be written

$$a^{\dagger}_{\rho} \ \rightarrow \ \lambda_{\rho}, \ \ b^{\dot{\alpha}} \rightarrow \mu^{\dot{\alpha}}, \ \text{ and } \ c^{\dagger A} \ \rightarrow \ \zeta^A, \tag{2.10}$$

and so on.

2.2 The Supergrassmannian $Gr(2|2, 4|4)$ and the Harmonic Superspace

The supergrassmannian $\mathrm{Gr}(2|2, 4|4)$ parametrizes $(2|2)$-dimensional subspaces inside the $(4|4)$-dimensional superspace. The embedding of such subspaces is described by a supermatrix,

$$M \ = \begin{pmatrix} A & B \\ & \\ C & D \end{pmatrix} \tag{2.11}$$

where A, D are 2×4 even matrices. And, B, C are 2×4 odd matrices. It seems, that in order to parametrize $\mathrm{Gr}(2|2, 4|4)$, we need $16|16$ coordinates, but these system coordinate is redundant. A $GL(2|2)$ transformation gauge away and we are left with a 8|8 dimensional space. Points of $\mathrm{Gr}(2|2, 4|4)$ are $2|2-$ planes L. They are parametrized by coordinates $x, \theta, \tilde{\theta}, u$

$$M = \begin{pmatrix} 1 & 0 & x^{1\dot{1}} & x^{2\dot{1}} & \tilde{\theta}_1^{\dot{1}} & \tilde{\theta}_2^{\dot{1}} & \tilde{\theta}_3^{\dot{1}} & \tilde{\theta}_4^{\dot{1}} \\ 0 & 1 & x^{1\dot{2}} & x^{2\dot{2}} & \tilde{\theta}_1^{\dot{2}} & \tilde{\theta}_2^{\dot{2}} & \tilde{\theta}_3^{\dot{2}} & \tilde{\theta}_4^{\dot{2}} \\ 0 & 0 & \theta^{11} & \theta^{21} & u_1^1 & u_2^1 & u_3^1 & u_4^1 \\ 0 & 0 & \theta^{12} & \theta^{22} & u_1^2 & u_2^2 & u_3^2 & u_4^2 \end{pmatrix} = \begin{pmatrix} \mathbb{1}_2 & x^{\alpha\dot{\alpha}} & \tilde{\theta}_I^{\dot{\alpha}} \\ \mathbb{0}_2 & \theta^{\alpha m} & u_I^m \end{pmatrix} \tag{2.12}$$

The eight u_I^m in Eq.(2.12) are redundant, only four of them are useful, i.e.

$$\begin{pmatrix} 1 & 0 & u_3^1 & u_4^1 \\ 0 & 1 & u_3^2 & u_4^2 \end{pmatrix}, \begin{pmatrix} 1 & u_2^1 & 0 & u_4^1 \\ 0 & u_2^2 & 1 & u_4^2 \end{pmatrix}, \begin{pmatrix} u_1^1 & u_2^1 & 1 & 0 \\ u_1^2 & u_2^2 & 0 & 1 \end{pmatrix}, \text{etc} \tag{2.13}$$

The coordinates $\tilde{\theta}_I^{\dot{\alpha}}$ in Eq(2.12) can be one of the following chooses

$$\begin{pmatrix} 0 & 0 & \tilde{\theta}_3^{\dot{1}} & \tilde{\theta}_4^{\dot{1}} \\ 0 & 0 & \tilde{\theta}_3^{\dot{2}} & \tilde{\theta}_4^{\dot{2}} \end{pmatrix}, \begin{pmatrix} 0 & \tilde{\theta}_2^{\dot{1}} & 0 & \tilde{\theta}_4^{\dot{1}} \\ 0 & \tilde{\theta}_2^{\dot{2}} & 0 & \tilde{\theta}_4^{\dot{2}} \end{pmatrix}, \begin{pmatrix} 0 & \tilde{\theta}_2^{\dot{1}} & \tilde{\theta}_3^{\dot{1}} & 0 \\ 0 & \tilde{\theta}_2^{\dot{2}} & \tilde{\theta}_3^{\dot{2}} & 0 \end{pmatrix}, \begin{pmatrix} \tilde{\theta}_1^{\dot{1}} & \tilde{\theta}_2^{\dot{1}} & 0 & 0 \\ \tilde{\theta}_1^{\dot{2}} & \tilde{\theta}_2^{\dot{2}} & 0 & 0 \end{pmatrix}, \text{ etc.} \tag{2.14}$$

The condition $Z \in L$, is given by

$$\mu^{\dot{\alpha}} + x^{\alpha\dot{\alpha}}\lambda_\alpha + \tilde{\theta}_I^{\dot{\alpha}}\zeta^I \ = \ 0 \ \text{ and } \ \theta^{\alpha m}\lambda_\alpha + u_I^m\zeta^I \ = \ 0 \tag{2.15}$$

meaning that each set of vectors in Eq.(2.12) has zero scalar product with the vector Z in Eq.(2.1). The harmonic superspace $(4, 2, 2)$ can be seen as the supergrassmannian $\mathrm{Gr}(2|2, \mathbf{R}^{4|4})$. Following[1] [7], we will think about the four-dimensional $\mathcal{N} = 4$ harmonic superspace as a Grassmannian manifold $M = \mathrm{Gr}(2|2, \mathbf{R}^{4|4})$, which

[1]although with our choice of signature we cannot directly use complex geometry; we will replace complex analyticity with some polynomiality condition.

parametrizes subspaces $\mathbf{R}^{2|2} \subset \mathbf{R}^{4|4}$.

$\mathrm{Gr}(2|2, \mathbf{R}^{4|4})$ generalize the idea of classical planes and provides a geometric structure for studying supersymmetric theories

There are some vector fields whose flows do not change L, for example:

$$A^{i\alpha}\left(\frac{\partial}{\partial\tilde{\theta}_{i\alpha}} + x^{\alpha\dot{1}}\frac{\partial}{\partial\tilde{\theta}_i^{\dot{1}}} + x^{\alpha\dot{2}}\frac{\partial}{\partial\tilde{\theta}_i^{\dot{2}}} + \theta^{\alpha I}\frac{\partial}{\partial u_i^I}\right) \tag{2.16}$$

$$A_L^K\left(\tilde{\theta}_{KA}\frac{\partial}{\partial\tilde{\theta}_{LA}} + \tilde{\theta}_K^{\dot{A}}\frac{\partial}{\partial\tilde{\theta}_L^{\dot{A}}} + u_K^I\frac{\partial}{\partial u_L^I}\right) \tag{2.17}$$

for any constant $A^{i\alpha}$ and A_L^K. This can be used to put $\tilde{\theta}_{i\alpha} = 0$:

$$\tilde{\theta}_{11} = \tilde{\theta}_{12} = \tilde{\theta}_{21} = \tilde{\theta}_{22} = 0 \tag{2.18}$$

To summarize:

$$\texttt{twistors} = \mathbf{T} = \mathbf{R}^{4|4} \tag{2.19}$$

$$\texttt{spacetime} = M = \mathrm{Gr}(2|2, \mathbf{T}) \tag{2.20}$$

The fiber of $\mathcal{S}_M$ over $L \subset M$ is L itself. This means that $\mathcal{S}_M$ is a $2|2$ -dimensional tautological vector bundle over M.

Also, **PT** will denote the projective supertwistor space, *i.e.* Z modulo rescaling: $Z \simeq \kappa Z$.

2.2.1 The Berezinian of a Vector Bundle $\mathcal{V}$

For any vector bundle $\mathcal{V}$, we define the line bundles $\mathrm{Ber}\,\mathcal{V}$ and $|\mathrm{Ber}|\,\mathcal{V}$.Sections of $\mathrm{Ber}\,\mathcal{V}$ are functions of the bases of the fiber V, satisfying the property:

$$\sigma(\{m_I^J e_J\}) = \mathrm{SDet}\, m \;\; \sigma(\{e_I\}) \tag{2.21}$$

where for a supermatrix $m = \begin{pmatrix} A & B \\ C & D \end{pmatrix}$:

$$\mathrm{SDet}\; m \;=\; \det(A - BD^{-1}C)\;(\det\, D)^{-1} \tag{2.22}$$

Similarly, section of $|\mathrm{Ber}|\, V$ satisfy:

$$\sigma(\{m_I^J e_J\}) = \ \mathrm{sign}(\det A)\ \mathrm{SDet}\, m \ \ \sigma(\{e_I\}) \tag{2.23}$$

The characteristic property of $|\mathrm{Ber}|\, \mathcal{V}$ is that a section of $|\mathrm{Ber}|\, \mathcal{V}$ defines an operation of integration along the fiber:

$$\begin{bmatrix} \text{functions on } \mathcal{V} \\ \\ \text{with compact support} \\ \\ \text{on each fiber } V \end{bmatrix} \xrightarrow{\int} \begin{bmatrix} \text{functions} \\ \\ \text{on the base} \end{bmatrix} \tag{2.24}$$

It seems logical to call sections of $\mathrm{Ber}\, \mathcal{V}$ "integral forms on the fiber", and section of $|\mathrm{Ber}|\, \mathcal{V}$ "volume elements on the fiber". Notice that $|\mathrm{Ber}|\mathcal{S}_M = (|\mathrm{Ber}|(\mathbf{T}/\mathcal{S}_M))^{-1} = |\mathrm{Ber}|(\mathbf{T}/\mathcal{S}_M)' = |\mathrm{Ber}|\mathcal{S}_M^{\perp}$. Those four vectors listed in Eq. (2.12) are the basis of $\mathcal{S}_M^{\perp}$.

2.3 The $\mathcal{N} = 4$ SYM field theory

The gauge field theory with the maximal amount of supersymmetry in four dimensions is the $\mathcal{N} = 4$ SYM field theory[16]. Its field theory content are two degrees of freedom for the gauge field $A_\mu^{(a)}$, six for the scalar fields $\phi_{AB}^{(a)}$, in total eight bosonic degrees of freedom. It matches with the eight fermionic degrees of freedom $\psi_B^{\beta(c)}$ and $\bar{\psi}_{\dot{\beta}A}^{(a)}$. All are in the adjoint representation of the gauge group.

From the gauge field $A_\mu^{(a)}$ are constructed the covariant derivative D_μ and the field strength $F_{\mu\nu}^{(a)}$. The action of this field theory is given by

$$\begin{aligned} S \ = \ & -\int d^4x \Big(\frac{1}{4} F_{\mu\nu}^{(a)} F^{\mu\nu(a)} + \frac{1}{4} D_\mu \phi^{AB(a)} D^\mu \phi_{AB}^{(a)} + i\bar{\psi}_A^{(a)} \bar{\sigma}^\mu D_\mu \psi^{(a)A} \\ & + \frac{\sqrt{2}}{2} g_{YM} f_{abc} \psi^{\alpha A(a)} \phi_{AB}^{(b)} \psi_\alpha^{B(c)} + \frac{g_{YM}^2}{16} f_{abc} f_{ade} \phi^{AB(b)} \phi^{CD(c)} \phi_{AB}^{(d)} \phi_{CD}^{(e)} \\ & - \frac{\sqrt{2}}{2} g_{YM} f_{abc} \bar{\psi}_{\dot{\beta}A}^{(a)} \phi^{AB(b)} \bar{\psi}_B^{\dot{\beta}(c)} \Big), \end{aligned} \tag{2.25}$$

where g_{YM} is a mass dimensionless coupling constant. Each term in the action is mass dimensionless. This can be seen by assigning the standard mass-dimensions to the fields

$$[A_\mu] = [\phi_{AB}] = 1, \;\; [\psi^A] = [\bar{\psi}_A] = \frac{3}{2}. \tag{2.26}$$

Therefore, it is said that the action (2.25) is scale invariant. Moreover, in a gauge theory that possesses scale invariance. The Poincaré group and scale invariance combine into a larger symmetry group: the conformal group. Even more, the conformal group and supersymmetry combine to form the superconformal group. Local operators of $\mathcal{N} = 4$ SYM field theory are arranged in multiplets of the superconformal algebra $psl(4|4,\mathbf{R})$ [17]. The components of the $\mathcal{N} = 4$ SYM field theory are listed in Table 2.1[18]

Table 2.1: Components of the $\mathcal{N} = 4$ SYM field theory.

Field	$sl(2,\mathbf{R}) \oplus sl(2,\mathbf{R})$	$sl(4,\mathbf{R})$
$\partial^k \psi^D_\alpha$	$[k, k+1]$	$[0,0,1]$
$\partial^k \bar{\psi}_{M\dot{\alpha}}$	$[k+1, k]$	$[1,0,0]$
$\partial^k f_{\alpha\beta}$	$[k+2, k]$	$[0,0,0]$
$\partial^k \bar{f}_{\dot{\alpha}\dot{\beta}}$	$[k, k+2]$	$[0,0,0]$
$\partial^k \phi^{AB}$	$[k, k]$	$[0,1,0]$

2.3.1 The $\mathcal{N} = 1$ superfield formalism and its deformation

The $\mathcal{N} = 4$ SYM field theory can not be written in terms of the off-shell formalism of superfield. However, it admits a formulation in terms of $\mathcal{N} = 1$ language. The field content is combined as follow: one of the four fermions is combined with the gauge field to form the vector superfield and the remaining fermions are combined with three complex scalar, to form three chiral superfield $\mathbf{\Phi}^i$. The superpotential $\mathcal{W}$ is given by

$$\mathcal{W}_{\mathcal{N}=1} = \frac{g_{YM}\sqrt{2}}{3!}\epsilon_{ijk} f_{abc} \mathbf{\Phi}^i_a \mathbf{\Phi}^j_b \mathbf{\Phi}^k_c, \tag{2.27}$$

where $\mathbf{\Phi}^i$ with $i = 1, 2, 3$ denotes the three chiral superfields. In this formulation, the original $\mathcal{R}$ symmetry is broken down to $SU(3) \times U(1)$.

In this thesis we are interested in studying deformations of this field theory. To begin with, we start with the deformation studied by Leigh and Strassler [6]. They consider deformations of the superpotential (2.27), the new superpotential is given by

$$\mathcal{W}_{\mathcal{N}=1} \mapsto \mathcal{W}_{\mathcal{N}=1} + \frac{1}{3!} h_{ijk} d_{abc} \Phi^i_a \Phi^j_b \Phi^k_c, \tag{2.28}$$

where $d_{abc} = \mathtt{Tr}(T_a\{T_b, T_c\})$ and h_{ijk} being totally symmetric. The deformation at linear order in h_{ijk} [2], the action is

$$S = S^{\text{free}}_{\mathcal{N}=4} - \frac{1}{4}\int d^4x B^{[MN]}_{(IJ)} d_{abc} \psi^{(a)I} \psi^{(b)J} \phi^{(c)}_{MN} + \text{h.c.} \quad , \tag{2.29}$$

where $S^{\text{free}}_{\mathcal{N}=4}$ denotes the action of $\mathcal{N}=4$ SYM in the free field theory. The coupling $B^{[MN]}_{(IJ)}$ is related to the coupling h_{ijk} in Eq. (2.28) by

$$B^{[mn]}_{(ij)} \;=\; h_{ijl}\epsilon^{lmn}, \tag{2.30}$$

with latin indices running from 1 to 3. The coupling $B^{[MN]}_{(IJ)}$ obeys the tracelessness condition [3]

$$B^{[MN]}_{(IN)} \;=\; 0, \tag{2.31}$$

this condition is crucial for matching two different representations of the $psl(4|4, \mathbf{R})$ algebra.

2.3.2 $\mathcal{N} = 4$ SYM fields in Oscillator Representation

Solutions of free SYM theory can be written in the oscillator representation. Let be $|0\rangle$ an invariant non-physical vacuum under $psl(4|4, \mathbf{R})$. It is annhilated by c_A, $b^{\dot\alpha}$ and a^{α} [15]. This means

$$c_A|0\rangle \;=\; a^{\alpha}|0\rangle \;=\; b^{\dot\alpha}|0\rangle \;=\; 0. \tag{2.32}$$

The scalar fields ϕ^{AB} are defined by

$$\phi^{AB} \;=\; c^{\dagger A} c^{\dagger B}|0\rangle, \tag{2.33}$$

[2] Throughout this thesis, we refer to this deformation as the β-deformation

The spinor fields ψ^D_α and $\bar{\psi}_{M\dot{\alpha}}$ in the oscillator representations are obtained from Eq.(2.33) by applying the supersymmetry generators $q_{A\alpha}$ and $\bar{q}^{A\dot{\alpha}}$, respectively:

$$\psi^D_\alpha = a^\dagger_\alpha c^{\dagger D}|0\rangle, \;\; \bar{\psi}_{M\dot{\alpha}} = \frac{1}{3!}\epsilon_{MNBD} b^\dagger_{\dot{\alpha}} c^{\dagger N} c^{\dagger B} c^{\dagger D}|0\rangle. \tag{2.34}$$

The self-dual and anti-self-dual of the field strength are given by

$$f_{\alpha\beta} = a^\dagger_\alpha a^\dagger_\beta |0\rangle \quad \text{and} \tag{2.35}$$

$$\bar{f}_{\dot{\alpha}\dot{\beta}} = \frac{1}{4!}\epsilon_{ABDE} b^\dagger_{\dot{\alpha}} b^\dagger_{\dot{\beta}} c^{A\dagger} c^{B\dagger} c^{D\dagger} c^{E\dagger}|0\rangle = b^\dagger_{\dot{\alpha}} b^\dagger_{\dot{\beta}} (c^\dagger)^4|0\rangle, \tag{2.36}$$

However, in gauge theories we need to write down composite operators, which are products of solutions of the free theory. Those of interest for us are those who are single trace composite operators.The following notation will help us to write single trace operators in oscillator representation

1. If ψ_1 and ψ_2 are fermions then $\psi_1 \bullet \psi_2 = \frac{1}{2}(\psi_1 \otimes \psi_2 - \psi_2 \otimes \psi_1)$, [3]
2. if ψ_1 and ϕ_2 are bosons then $\phi_1 \bullet \phi_2 = \frac{1}{2}(\phi_1 \otimes \phi_2 + \phi_2 \otimes \phi_2)$, and
3. if ϕ is boson and ψ is fermion then $\phi \bullet \psi = \frac{1}{2}(\phi \otimes \psi + \psi \otimes \phi)$.

The deformation in Eq. (2.29) in the oscillator representation is given by

$$\mathcal{O}[B] = \int d^4x \prod_{j=1}^{3} \mathcal{R}^{(j)} B^{[MN]}_{(IJ)} \Psi^{(IJ)}_{[MN]}, \tag{2.37}$$

where the index j denotes in which singleton representation the $\mathcal{R}^{(j)} = e^{-\frac{i}{2}p^{(j)}_{\alpha\dot{\alpha}}x^{\alpha\dot{\alpha}}}$ translation operator acts, with $\Psi^{(IJ)}_{[MN]}$ given by

$$\Psi^{(IJ)}_{[MN]} = a^\dagger_{[1} c^{\dagger (I}|0\rangle \bullet a^\dagger_{2]} c^{\dagger J)}|0\rangle \bullet \epsilon_{MNKL} c^{\dagger K} c^{\dagger L}|0\rangle. \tag{2.38}$$

Eq. (2.37) is invariant under conformal transformation $k^{\varrho\dot{\varrho}}$. When dealing with signature $1+3$, the hermitian conjugate of (2.37) should be considered, in our notation it reads

$$\mathcal{O}[\tilde{B}] = \int d^4x \prod_{m=1}^{3} \int D\lambda_m \tilde{B}^{(MN)}_{[IJ]} \Sigma^{[IJ]}_{(MN)} \tag{2.39}$$

[3] $\bullet$ means the super-symmetrized tensor product.

where

$$\Sigma^{[IJ]}_{(MN)} = \frac{1}{3!^2}\epsilon_{MUVW}\epsilon_{NU'V'W'}b^\dagger_{[\dot{1}}c^{\dagger U}c^{\dagger V}c^{\dagger W}|0\rangle \bullet b^\dagger_{\dot{2}]}c^{\dagger U'}c^{\dagger V'}c^{\dagger W'}|0\rangle \bullet c^{\dagger I}c^{\dagger J}|0\rangle. \tag{2.40}$$

2.3.3 Oscillator and Twistors

The on-shell $\mathcal{N} = 4$ SYM fields can be assembled in one on-shell superfield $\Upsilon(\lambda, \mu, \eta)$[19]

$$\Upsilon(\mathcal{Z}) = \tilde{f} + \eta_A\tilde{\psi}^A + \frac{1}{2!}\eta_A\eta_B\tilde{\phi}^{AB} + \frac{1}{3!}\eta_A\eta_B\eta_C\epsilon^{ABCD}\tilde{\bar{\psi}}_D + \eta_1\eta_2\eta_3\eta_4\tilde{\bar{f}}. \tag{2.41}$$

Notice that $\tilde{f}$ and $\tilde{\bar{f}}$ are independent of the Grassmann variables η. The on-shell superfield $\Upsilon(\mathcal{Z})$ in twistor languague [20, 21] is given by

$$\Upsilon(\mathcal{Z}) = 2\pi i\int_{\mathbb{C}}\frac{ds}{s}e^{s(\mu^{\dot{\alpha}}\bar{p}_{\dot{\alpha}}+\eta_A\xi^A)}\bar{\delta}^2(s\lambda_\alpha - p_\alpha), \tag{2.42}$$

where $\bar{\delta}^2(\lambda) = \bar{\delta}(\lambda_1)\bar{\delta}(\lambda_2)$ with the $\bar{\delta}^1(y)$ denoting the delta function on the complex plane. For details and a derivation of Eq. (2.42) we refer to Appendix C of Ref. [20].

The spacetime dependence of the on-shell fields is recovered by the Penrose transform, for a review on this topic we refer to Appendix A of Ref. [8]. The scalar field $\phi^{AB}(x)$ is given by

$$\phi^{AB}(x) = \int D\lambda\,\frac{\partial}{\partial\eta_A}\frac{\partial}{\partial\eta_B}\Upsilon(\mathcal{Z})\Big|_{\eta=0}, \tag{2.43}$$

where $D\lambda = \frac{\lambda^\alpha d\lambda_\alpha}{2\pi i}$. We list in the Table 2.2 the on-shell fields of $\mathcal{N} = 4$ SYM in both oscillator representation and supertwistors,

By using supertwistors Eq. (2.37) is given by

$$\mathcal{O}[B] = \int d^4x\prod_{m=1}^{3}\int D\lambda_m B^{[MN]}_{(IJ)}\Psi^{(IJ)}_{[MN]}, \tag{2.44}$$

Table 2.2: On-shell fields of $\mathcal{N}=4$ SYM in both oscillator representation and supertwistors.

Fields	**Oscillators**	**Twistors**
$\psi^D_\alpha(x)$	$\mathcal{R}a^\dagger_\alpha c^{\dagger D}\|0\rangle$	$\int D\lambda\, \lambda_\alpha \frac{\partial}{\partial\eta_D}\Upsilon(\mathcal{Z})\Big\|_{\eta=0}$
$\bar\psi_{M\dot\alpha}(x)$	$\epsilon_{MNBD}\mathcal{R}b^\dagger_{\dot\alpha}c^{\dagger N}c^{\dagger B}c^{\dagger D}\|0\rangle$	$\epsilon_{MNBD}\int D\lambda\, \frac{\partial}{\partial\mu^{\dot\alpha}}\frac{\partial}{\partial\eta_N}\frac{\partial}{\partial\eta_B}\frac{\partial}{\partial\eta_D}\Upsilon(\mathcal{Z})\Big\|_{\eta=0}$
$f_{\alpha\beta}(x)$	$\mathcal{R}a^\dagger_\alpha a^\dagger_\beta\|0\rangle$	$\int D\lambda\, \lambda_\alpha\lambda_\beta\, \Upsilon(\mathcal{Z})\|_{\eta=0}$
$\bar f_{\dot\alpha\dot\beta}(x)$	$\mathcal{R}b^\dagger_{\dot\alpha}b^\dagger_{\dot\beta}\left(c^\dagger\right)^4\|0\rangle$	$\int D\lambda\, \frac{\partial}{\partial\mu^{\dot\alpha}}\frac{\partial}{\partial\mu^{\dot\beta}}\left(\frac{\partial}{\partial\eta}\right)^4\Upsilon(\mathcal{Z})\Big\|_{\eta=0}$

where $\boldsymbol{\Psi}^{(IJ)}_{[MN]}$ reads as

$$\boldsymbol{\Psi}^{(IJ)}_{[MN]} = \epsilon_{MNAB}\cdot \\ \lambda^{(1)}_{[1}\frac{\partial}{\partial\eta^{(1)}_I}\Upsilon^{(1)}(\mathcal{Z})\bigg|_{\eta=0}\lambda^{(2)}_{2]}\frac{\partial}{\partial\eta^{(2)}_J}\Upsilon^{(2)}(\mathcal{Z})\bigg|_{\eta=0}\frac{\partial}{\partial\eta^{(3)}_A}\frac{\partial}{\partial\eta^{(3)}_B}\Upsilon^{(3)}(\mathcal{Z})\bigg|_{\eta=0}. \tag{2.45}$$

We see that Eq. (2.44) is finite on-shell. We can evaluate it on the product of three off-shell fields and the integral is convergent. This means that Eq. (2.44) defines an element of the dual space to the tensor product of three singleton representations.

2.3.4 Solutions of free classical super-Maxwell equations from twistor space

In the free field limit a classical solution of the super-Yang-Mills theory with gauge group $U(N)$ can be obtained as an matrix $\mathbf{A}\in\mathbb{R}^{N\times N}$ whose entries are classical solutions of the super-Maxwell equations. For our purpose it is enough to consider free solutions of the form:

$$\begin{bmatrix} \text{solution of} \\ \\ \text{super-Maxwell equations}\end{bmatrix}\times\begin{bmatrix} N\times N \\ \\ \text{matrix}\end{bmatrix} \tag{2.46}$$

We will explain the correspondence between solutions of free classical super-Maxwell equations on M and sections of $(|\mathrm{Ber}|\,\mathcal{S}_M)^{-1}$ which are polynomial in the coordinates u_i^I defined in Eq. (2.12). The choice of coordinates u_i^I is somewhat arbitrary, but the statement of polynomial dependence does not depend on it.

Let us pick a section

$$\sigma \in \Gamma\left((|\mathrm{Ber}|\,\mathcal{S}_M)^{-1}\right) \tag{2.47}$$

which is a polynomial in u, where Γ(line bundle) denotes the space of sections.

The condition that σ is annihilated by the vector fields defined in Eq. (2.17) with A^K_L satisfying $A^K_K = 0$ implies that it can depend on θ and u only through the expressions $\theta^{\alpha[1}u_I^{2]}$, $\theta^{\alpha[1}\theta^{2]\beta}$ and $u_I^{[1}u_J^{2]}$. Since, we require σ to be a polynomial in u, it must be then a polynomial of these three expressions. Moreover, σ should have charge 2 under rescaling of (θ, u) i.e. $(\theta, u) \to c(\theta, u)$; therefore this polynomial is actually a linear function:

$$\sigma = \theta^{\alpha[1}\theta^{2]\beta}\mathbf{F}^+_{\alpha\beta}(x,\tilde{\theta}) + \theta^{\alpha[1}u_I^{2]}\mathbf{\Psi}^I_\alpha(x,\tilde{\theta}) + u_I^{[1}u_J^{2]}\mathbf{\Phi}^{IJ}(x,\tilde{\theta}) \tag{2.48}$$

Consider the condition that σ does not change when we add to first and second vectors of the basis of Eq. (2.12) a linear combination of the third and fourth vectors. It implies that σ should be annihilated by the following vector fields $V^m_{\dot{\alpha}}$:

$$V^m_{\dot{\alpha}}\sigma = 0 \tag{2.49}$$

$$\texttt{where } V^m_{\dot{\alpha}} = \theta^{\alpha m}\frac{\partial}{\partial x^{\alpha\dot{\alpha}}} + u_I^m\frac{\partial}{\partial\tilde{\theta}^{\dot{\alpha}}_I}, \tag{2.50}$$

the expansion of left hand side of Eq. (2.49) in powers of θ implies constraints on the superfields $\boldsymbol{F}^+_{\alpha\beta}(x,\tilde{\theta})$, $\mathbf{\Psi}^J_\alpha(x,\tilde{\theta})$ and $\mathbf{\Phi}^{IJ}(x,\tilde{\theta})$. Let us see in detail this statement

1. At the zeroth order in θ, the term that contributes is

$$u_P^m u_I^{[1}u_J^{2]}\frac{\partial}{\partial\tilde{\theta}^{\dot{\alpha}}_P}\mathbf{\Phi}^{IJ}(x,\tilde{\theta}) \tag{2.51}$$

   the vanishing of Eq (2.51), requires[4]

$$\frac{\partial}{\partial\tilde{\theta}^{\dot{\alpha}}_P}\mathbf{\Phi}^{IJ}(x,\tilde{\theta}) = \frac{\partial}{\partial\tilde{\theta}^{\dot{\alpha}}_{[P}}\mathbf{\Phi}^{IJ]}(x,\tilde{\theta}) \tag{2.52}$$

[4]A fully antisymmetric tensor T_{abc} satisfies, i.e. $T_{[abc]} = \frac{1}{3}(T_{a[bc]} + T_{b[ca]} + T_{c[ab]})$

2. At the linear order in θ, we have

$$-u_I^m\theta^{\alpha[1}u_J^{2]}\frac{\partial}{\partial\tilde{\theta}_I^{\dot{\alpha}}}\mathbf{\Psi}_\alpha^J(x,\tilde{\theta})+\theta^{\alpha m}u_I^{[1}u_J^{2]}\frac{\partial}{\partial x^{\alpha\dot{\alpha}}}\mathbf{\Phi}^{IJ}(x,\tilde{\theta}) \tag{2.53}$$

Notice that the product of two u_I^m can be decomposed as

$$u_I^m u_J^n \;=\; \epsilon^{mn}u_{[I}^{[1}u_{J]}^{2]}+u_{(I}^{(m}u_{J)}^{n)} \tag{2.54}$$

using Eq. (2.54) in $u_I^m u_J^{2]}$, we yield

$$\theta^{\alpha m}u_I^{[1}u_J^{2]}\left(-\frac{\partial}{\partial\tilde{\theta}_{[I}^{\dot{\alpha}}}\mathbf{\Psi}_\alpha^{J]}(x,\tilde{\theta})+\frac{\partial}{\partial x^{\alpha\dot{\alpha}}}\mathbf{\Phi}^{IJ}(x,\tilde{\theta})\right)-\theta^{\alpha m}u_I^{(1}u_J^{2)}\frac{\partial}{\partial\tilde{\theta}_{(I}^{\dot{\alpha}}}\mathbf{\Psi}_\alpha^{J)}(x,\tilde{\theta}) \tag{2.55}$$

vanishing of Eq. (2.55), implies

$$\frac{\partial}{\partial\tilde{\theta}_{(I}^{\dot{\alpha}}}\mathbf{\Psi}_\alpha^{J)}(x,\tilde{\theta})=0 \tag{2.56}$$

$$\frac{\partial}{\partial\tilde{\theta}_{[I}^{\dot{\alpha}}}\mathbf{\Psi}_\alpha^{J]}(x,\tilde{\theta})=\frac{\partial}{\partial x^{\alpha\dot{\alpha}}}\mathbf{\Phi}^{IJ}(x,\tilde{\theta}) \tag{2.57}$$

3. The terms that contribute at second order in θ are

$$u_I^m\theta^{\alpha[1}\theta^{2]\beta}\frac{\partial}{\partial\tilde{\theta}_I^{\dot{\alpha}}}\mathbf{F}_{\alpha\beta}^{+}(x,\tilde{\theta})+\theta^{\beta m}\theta^{\alpha[1}u_I^{2]}\frac{\partial}{\partial x^{\beta\dot{\alpha}}}\mathbf{\Psi}_\alpha^I(x,\tilde{\theta}), \tag{2.58}$$

Using, the fact that the product of two θ's can be written

$$\theta_\beta^m\theta_\alpha^n=\epsilon_{\beta\alpha}\theta_{[1}^{(m}\theta_{2]}^{n)}+\epsilon^{mn}\theta_{(\beta}^{[1}\theta_{\alpha)}^{2]}, \tag{2.59}$$

Eq. (2.58) can be written as

$$\begin{aligned}&u_I^m\theta^{\alpha[1}\theta^{2]\beta}\left(\frac{\partial}{\partial\tilde{\theta}_I^{\dot{\alpha}}}\boldsymbol{F}_{\alpha\beta}^{+}(x,\tilde{\theta})-\frac{\partial}{\partial x^{\dot{\alpha}(\alpha}}\mathbf{\Psi}_{\beta)}^I(x,\tilde{\theta})\right)\\ &+u_I^m\theta^{[1|(1}\theta^{2)|2]}\left(\epsilon^{\alpha\beta}\frac{\partial}{\partial x^{\alpha\dot{\alpha}}}\mathbf{\Psi}_\beta^I(x,\tilde{\theta})\right)\end{aligned} \tag{2.60}$$

This is only zero when both terms vanish separately,

$$\frac{\partial}{\partial\tilde{\theta}^{\dot{\alpha}}_{I}}\boldsymbol{F}^{+}_{\alpha\beta}(x,\tilde{\theta}) = \frac{\partial}{\partial x^{\dot{\alpha}(\alpha}}\boldsymbol{\Psi}^{I}_{\beta)}(x,\tilde{\theta}) \tag{2.61}$$

$$\epsilon^{\alpha\beta}\frac{\partial}{\partial x^{\alpha\dot{\alpha}}}\boldsymbol{\Psi}^{I}_{\beta}(x,\tilde{\theta}) = 0 \tag{2.62}$$

4. At third order in θ, the term that contributes is

$$\theta^{\alpha m}\theta^{\beta[1}\theta^{2]\rho}\frac{\partial}{\partial x^{\alpha\dot{\alpha}}}\boldsymbol{F}^{+}_{\beta\rho}(x,\tilde{\theta}), \tag{2.63}$$

the vanishing of this expression implies that $\frac{\partial}{\partial x^{\alpha\dot{\alpha}}}\boldsymbol{F}^{+}_{\beta\rho}(x,\tilde{\theta})$ must be totally symmetric in $\alpha,\ \beta,\ \rho$, and therefore

$$\epsilon^{\alpha\beta}\frac{\partial}{\partial x^{\alpha\dot{\alpha}}}\boldsymbol{F}^{+}_{\beta\rho}(x,\tilde{\theta}) = 0 \tag{2.64}$$

Therefore, Eq.(2.49) implies the set of Eqns

$$\epsilon^{\alpha\beta}\frac{\partial}{\partial x^{\alpha\dot{\alpha}}}\boldsymbol{F}^{+}_{\beta\rho}(x,\tilde{\theta}) = 0 \tag{2.65}$$

$$\frac{\partial}{\partial\tilde{\theta}^{\dot{\alpha}}_{I}}\boldsymbol{F}^{+}_{\alpha\beta}(x,\tilde{\theta}) = \frac{\partial}{\partial x^{\dot{\alpha}(\alpha}}\boldsymbol{\Psi}^{J}_{\beta)}(x,\tilde{\theta}) \tag{2.66}$$

$$\epsilon^{\alpha\beta}\frac{\partial}{\partial x^{\alpha\dot{\alpha}}}\boldsymbol{\Psi}^{J}_{\beta}(x,\tilde{\theta}) = 0 \tag{2.67}$$

$$\frac{\partial}{\partial\tilde{\theta}^{\dot{\alpha}}_{(I}}\boldsymbol{\Psi}^{J)}_{\alpha}(x,\tilde{\theta}) = 0 \tag{2.68}$$

$$\frac{\partial}{\partial\tilde{\theta}^{\dot{\alpha}}_{[I}}\boldsymbol{\Psi}^{J]}_{\alpha}(x,\tilde{\theta}) = \frac{\partial}{\partial x^{\alpha\dot{\alpha}}}\boldsymbol{\Phi}^{IJ}(x,\tilde{\theta}) \tag{2.69}$$

$$\frac{\partial}{\partial\tilde{\theta}^{\dot{\alpha}}_{P}}\boldsymbol{\Phi}^{IJ}(x,\tilde{\theta}) = \frac{\partial}{\partial\tilde{\theta}^{\dot{\alpha}}_{[P}}\boldsymbol{\Phi}^{IJ]}(x,\tilde{\theta}) \tag{2.70}$$

Proposition 1. *Components of the $\tilde{\theta}$-expansion of superfields $\boldsymbol{\Phi}^{IJ}(x,\tilde{\theta})$, $\boldsymbol{\Psi}^{J}_{\alpha}(x,\tilde{\theta})$ and $\boldsymbol{F}^{+}_{\alpha\beta}(x,\tilde{\theta})$ can be expressed through solutions of free super-Maxwell theory.*

Proof. From Eq(2.52) we can see that,

$$\frac{\partial}{\partial\tilde{\theta}^{\dot{\gamma}}_{K}}\frac{\partial}{\partial\tilde{\theta}^{\dot{\alpha}}_{L}}\frac{\partial}{\partial\tilde{\theta}^{\dot{\beta}}_{M}}\boldsymbol{\Phi}^{IJ}(x,\tilde{\theta}) \;=\; \frac{\partial}{\partial\tilde{\theta}^{\dot{\gamma}}_{K}}\frac{\partial}{\partial\tilde{\theta}^{\dot{\alpha}}_{L}}\partial^{[M}_{\dot{\beta}}\boldsymbol{\Phi}^{IJ]}(x,\tilde{\theta}) \;=\; \partial^{[K}_{\dot{\gamma}}\partial^{L}_{\dot{\alpha}}\partial^{M}_{\dot{\beta}}\boldsymbol{\Phi}^{IJ]}(x,\tilde{\theta}) = 0, \tag{2.71}$$

this implies that the expansion of $\mathbf{\Phi}^{IJ}(x,\tilde{\theta})$ in powers of $\tilde{\theta}$ terminates at second order

$$\mathbf{\Phi}^{IJ}(x,\tilde{\theta}) = \varphi^{IJ}(x) + \epsilon^{IJPQ}\tilde{\theta}^{\dot{\alpha}}_P\tilde{\psi}_{Q\dot{\alpha}}(x) + \frac{1}{2}\epsilon^{IJPQ}\tilde{\theta}^{\dot{\alpha}}_P\tilde{\theta}^{\dot{\beta}}_Q\tilde{f}_{\dot{\alpha}\dot{\beta}}(x) \tag{2.72}$$

There is no symmetric combinations of I and J on the $\tilde{\theta}$ expansion, since $\partial_{\alpha\dot{\alpha}}\mathbf{\Phi}^{IJ}(x,\tilde{\theta})$ is given by

$$\partial_{\alpha\dot{\alpha}}\mathbf{\Phi}^{IJ}(x,\tilde{\theta}) = \partial_{\alpha\dot{\alpha}}\varphi^{IJ}(x) + \epsilon^{IJPQ}\tilde{\theta}^{\dot{\beta}}_P\partial_{\alpha\dot{\alpha}}\tilde{\psi}_{Q\dot{\beta}}(x) + \frac{1}{2}\epsilon^{IJPQ}\tilde{\theta}^{\dot{\rho}}_P\tilde{\theta}^{\dot{\beta}}_Q\partial_{\alpha\dot{\alpha}}\tilde{f}_{\dot{\rho}\dot{\beta}}(x), \tag{2.73}$$

therefore Eq (2.56),(2.57) yields to

$$\begin{aligned}\mathbf{\Psi}^J_\alpha(x,\tilde{\theta}) = \psi^J_\alpha(x) + \tilde{\theta}^{\dot{\alpha}}_I\partial_{\alpha\dot{\alpha}}\varphi^{IJ}(x) + \frac{1}{2}\epsilon^{IJPQ}\tilde{\theta}^{\dot{\alpha}}_I\tilde{\theta}^{\dot{\beta}}_P\partial_{\alpha(\dot{\alpha}}\tilde{\psi}_{\dot{\beta})Q}(x)\\ + \frac{1}{3!}\epsilon^{IJPQ}\tilde{\theta}^{\dot{\alpha}}_I\tilde{\theta}^{\dot{\rho}}_P\tilde{\theta}^{\dot{\beta}}_Q\partial_{\alpha(\dot{\alpha}}\tilde{f}_{\dot{\rho}\dot{\beta})}(x).\end{aligned} \tag{2.74}$$

This solution exists when the Maxwell and Dirac equations

$$\partial_{\alpha[\dot{\alpha}}\tilde{\psi}_{\dot{\beta}]Q} = 0 \quad \text{and} \quad \partial_{\alpha[\dot{\alpha}}\tilde{f}_{\dot{\rho}]\dot{\beta}} = 0 \tag{2.75}$$

are satisfied. Taking the $\frac{\partial}{\partial x^{\alpha\dot{\alpha}}}$ of Eq(2.74)

$$\begin{aligned}\partial_{\dot{\beta}\beta}\mathbf{\Psi}^J_\alpha(x,\tilde{\theta}) = \partial_{\dot{\beta}\beta}\psi^J_\alpha(x) + \tilde{\theta}^{\dot{\alpha}}_I\partial_{\dot{\beta}\beta}\partial_{\alpha\dot{\alpha}}\varphi^{IJ}(x) + \frac{1}{2}\epsilon^{IJPQ}\tilde{\theta}^{\dot{\alpha}}_I\tilde{\theta}^{\dot{\beta}}_P\partial_{\dot{\beta}\beta}\partial_{\alpha(\dot{\alpha}}\tilde{\psi}_{\dot{\beta})Q}(x)\\ + \frac{1}{3!}\epsilon^{IJPQ}\tilde{\theta}^{\dot{\alpha}}_I\tilde{\theta}^{\dot{\rho}}_P\tilde{\theta}^{\dot{\varrho}}_Q\partial_{\dot{\beta}\beta}\partial_{\alpha(\dot{\alpha}}\tilde{f}_{\dot{\rho}\dot{\varrho})}(x).\end{aligned} \tag{2.76}$$

This determines the expansion of $\boldsymbol{F}_{\alpha\beta}(x,\tilde{\theta})$

$$\begin{aligned}\boldsymbol{F}_{\alpha\beta}(x,\tilde{\theta}) &= f_{\alpha\beta}(x) + \partial_{\dot{\beta}\beta}\psi^J_\alpha(x) + \tilde{\theta}^{\dot{\alpha}}_I\partial_{\dot{\beta}\beta}\partial_{\alpha\dot{\alpha}}\varphi^{IJ}(x)\\ &+ \frac{1}{2}\epsilon^{IJPQ}\tilde{\theta}^{\dot{\alpha}}_I\tilde{\theta}^{\dot{\beta}}_P\partial_{\dot{\beta}\beta}\partial_{\alpha(\dot{\alpha}}\tilde{\psi}_{\dot{\beta})Q}(x) + \frac{1}{3!}\epsilon^{IJPQ}\tilde{\theta}^{\dot{\alpha}}_I\tilde{\theta}^{\dot{\rho}}_P\tilde{\theta}^{\dot{\varrho}}_Q\partial_{\dot{\beta}\beta}\partial_{\alpha(\dot{\alpha}}\tilde{f}_{\dot{\rho}\dot{\varrho})}(x)\end{aligned} \tag{2.77}$$

In addition to Eqns(2.75), also

$$\Box\varphi^{IJ} = 0 \tag{2.78}$$

must hold. $\square$

To conclude, σ is given by Eqn. (2.48) where $\mathbf{F}^{+}_{\dot{\alpha}\dot{\beta}}(x,\theta)$, $\mathbf{\Psi}_{\dot{\beta}J}(x,\theta)$ and $\mathbf{\Phi}_{IJ}(x,\theta)$ are given by Eqns. (2.77), (2.74) and (2.72), respectively.

Chapter 3

The supermultiplet

The supermultiplet of β-deformation is presented in this chapter from a field theory perspective. We use the oscillator representation to convey these findings. At linear level, it is known that $\mathbf{\Psi}^{(IJ)}_{[MN]}$ and $\mathbf{\Sigma}^{[IJ]}_{(MN)}$ are in the wedge product of two copies of the

$$(\mathbf{g} \wedge \mathbf{g})_0 \tag{3.1}$$

representation of $psl(4|4, \mathbf{R})$ [3]. And, its dual in the string theory side is given by

$$c^{\dagger(\mathbf{I}} c_{[\mathbf{M}} \wedge c^{\dagger \mathbf{J})} c_{\mathbf{N}]}. \tag{3.2}$$

We recall that, at linear order, the $\beta-$deformation is given by

$$\begin{aligned}
S_{\text{deformed}} &= \mathcal{O}[B] + \mathcal{O}[\tilde{B}] \ \text{ where } \ \mathcal{O}[B] \ = \ \int d^4x B^{[MN]}_{(IJ)} \prod_{j=1}^{3} \mathcal{R}^{(j)} \mathbf{\Psi}^{(IJ)}_{[MN]} \ \text{ and} \\
\mathcal{O}[\tilde{B}] &= \int d^4x \tilde{B}^{(MN)}_{[IJ]} \prod_{j=1}^{3} \mathcal{R}^{(j)} \mathbf{\Sigma}^{[IJ]}_{(MN)} \ \text{ with} \\
\mathbf{\Psi}^{(IJ)}_{[MN]} &= \frac{1}{2} a^{\dagger}_{[1} c^{\dagger(I} \left|0\right\rangle \bullet a^{\dagger}_{2]} c^{\dagger J)} \left|0\right\rangle \bullet \epsilon_{MNKL} c^{\dagger K} c^{\dagger L} \left|0\right\rangle \ \text{ and} \qquad (3.3) \\
\mathbf{\Sigma}^{[IJ]}_{(MN)} &= \frac{1}{3!^2} \epsilon_{MUVW} \epsilon_{NU'V'W'} b^{\dagger}_{[1} c^{\dagger U} c^{\dagger V} c^{\dagger W} \left|0\right\rangle \bullet b^{\dagger}_{2]} c^{\dagger U'} c^{\dagger V'} c^{\dagger W'} \left|0\right\rangle \bullet c^{\dagger I} c^{\dagger J} \left|0\right\rangle
\end{aligned} \tag{3.4}$$

3.1 First order at Supersymmetry Generators

Let $\mathcal{Q}$ be any supersymmetric generator in the superconformal algebra, our interest is to compute the action of $\mathcal{Q}$ on S_{deformed}, i.e. $\mathcal{Q}S_{\text{deformed}} = \mathcal{Q}\mathcal{O}[B] + \mathcal{Q}\mathcal{O}[\tilde{B}]$.

$$\mathcal{Q}S_{\text{deformed}} = \int d^4x \prod_{j=1}^{3} \mathcal{R}^{(j)} \left(B^{[MN]}_{(IJ)} \tilde{\mathcal{Q}} \Psi^{(IJ)}_{[MN]} + \tilde{B}^{(MN)}_{[IJ]} \tilde{\mathcal{Q}} \Sigma^{[IJ]}_{(MN)} \right), \tag{3.5}$$

where $\tilde{\mathcal{Q}}$ takes the form

$$\tilde{\mathcal{Q}} = -\frac{i}{2} x^{\varrho\dot{\varrho}} [\mathcal{Q}, p_{\varrho\dot{\varrho}}] + \mathcal{Q}, \tag{3.6}$$

the first term in Eq (3.6) comes from the commutation relation between $\mathcal{Q}$s and $\mathcal{R}^{(j)}$. In Table 3.1 is listed the four supersymmetric generators $\mathcal{Q}$ and $\tilde{\mathcal{Q}}$

Table 3.1: The supersymmetric generators $\mathcal{Q}$ and $\tilde{\mathcal{Q}}$

$\mathcal{Q}$	$\tilde{\mathcal{Q}}$
$q_{A\rho}$	$q_{A\rho}$
$\bar{q}^{A}_{\dot{\rho}}$	$\bar{q}^{A}_{\dot{\rho}}$
$s^{A\rho}$	$-\frac{i}{2} x^{\rho\dot{\varrho}} \bar{q}^{A}_{\dot{\varrho}} + s^{A\rho}$
$\bar{s}^{\dot{\rho}}_{A}$	$-\frac{i}{2} x^{\varrho\dot{\rho}} q_{A\varrho} + \bar{s}^{\dot{\rho}}_{A}$

3.1.1 The action of $\tilde{\mathcal{Q}}$ on $\Psi^{(IJ)}_{[MN]}$

The action of $\tilde{\mathcal{Q}}$ on $\Psi^{(IJ)}_{[MN]}$ is computed using the commutation relations in Eq 2.8 and the following equalities

$$c_B c^{\dagger A} c^{\dagger I} |0\rangle = \delta^{[A}_{B} c^{\dagger I]} |0\rangle \tag{3.7}$$

$$c_B c^{\dagger A} c^{\dagger K} c^{\dagger L} |0\rangle = \delta^{A}_{B} c^{\dagger K} c^{\dagger L} |0\rangle - c^{\dagger A} \delta^{[K}_{B} c^{\dagger L]} |0\rangle\,. \tag{3.8}$$

And, from the Table 3.1 we first need to compute

$$q_{A\rho}\Psi^{(IJ)}_{[MN]}, \quad \bar{q}^{A}_{\dot{\rho}}\Psi^{(IJ)}_{[MN]}, \quad s^{A\rho}\Psi^{(IJ)}_{[MN]}, \quad \text{and} \quad \bar{s}^{\dot{\rho}}_{A}\Psi^{(IJ)}_{[MN]}, \tag{3.9}$$

1. The action of $q_{A\rho}$ on $\Psi^{(IJ)}_{[MN]}$, i.e. $q_{A\rho}\Psi^{(IJ)}_{[MN]}$, reads

$$\begin{aligned} q_{A\rho}\Psi^{(IJ)}_{[MN]} = \frac{1}{2}\Big(& 2\delta^{(I}_{A}\; a^{\dagger}_{\rho}a^{\dagger}_{[1}\,|0\rangle \bullet a^{\dagger}_{2]}c^{\dagger J)}\,|0\rangle \bullet \epsilon_{MNKL}c^{\dagger K}c^{\dagger L}\,|0\rangle \\ & + 2\; a^{\dagger}_{[1}c^{\dagger (I}\,|0\rangle \bullet a^{\dagger}_{2]}c^{\dagger J)}\,|0\rangle \bullet \epsilon_{AMNL}a^{\dagger}_{\rho}c^{\dagger L}\,|0\rangle \Big), \end{aligned} \tag{3.10}$$

   where we have used the oscillator algebra and

$$\epsilon_{MNKL}c_{A}\left(c^{\dagger K}c^{\dagger L}\,|0\rangle\right) = \epsilon_{MNKL}\delta^{[K}_{A}c^{\dagger L]}\,|0\rangle = 2\epsilon_{AMNL}c^{\dagger L}\,|0\rangle \tag{3.11}$$

2. The action of $\bar{q}^{A}_{\dot{\rho}}$ on $\Psi^{(IJ)}_{[MN]}$, i.e. $\bar{q}^{A}_{\dot{\rho}}\Psi^{(IJ)}_{[MN]}$, reads

$$\begin{aligned} \bar{q}^{A}_{\dot{\alpha}}\Psi^{(IJ)}_{[MN]} = \frac{1}{2}\Big(& 2\; b^{\dagger}_{\dot{\alpha}}a^{\dagger}_{[1}c^{\dagger A}c^{\dagger (I}\,|0\rangle \bullet a^{\dagger}_{2]}c^{\dagger J)}\,|0\rangle \bullet \epsilon_{MNKL}c^{\dagger K}c^{\dagger L}\,|0\rangle \\ & + \; a^{\dagger}_{[1}c^{\dagger (I}\,|0\rangle \bullet a^{\dagger}_{2]}c^{\dagger J)}\,|0\rangle \bullet \epsilon_{MNKL}b^{\dagger}_{\dot{\alpha}}c^{\dagger A}c^{\dagger K}c^{\dagger L}\,|0\rangle \Big). \end{aligned} \tag{3.12}$$

3. The action of $\bar{s}^{\dot{\rho}}_{A}$ on $\Psi^{(IJ)}_{[MN]}$, i.e. $\bar{s}^{\dot{\rho}}_{A}\Psi^{(IJ)}_{[MN]}$, is zero. This fact comes from the oscillator algebra, since there is no $b^{\dagger}$ on $\Psi^{(IJ)}_{[MN]}$.

4. The action of $s^{A\rho}$ on $\Psi^{(IJ)}_{[MN]}$, i.e. $s^{A\rho}\Psi^{(IJ)}_{[MN]}$ reads

$$s^{A\rho}\Psi^{(IJ)}_{[MN]} = \frac{1}{2}\Big(2\;\delta^{\rho}_{[1}c^{\dagger A}c^{\dagger (I}\,|0\rangle \bullet a^{\dagger}_{2]}c^{\dagger J)}\,|0\rangle \bullet \epsilon_{MNKL}c^{\dagger K}c^{\dagger L}\,|0\rangle\Big). \tag{3.13}$$

By using equations (2.33),(2.34) and (2.35), properties of ϵ_{ABCD} and following the Table 3.1, the results in equations (3.10), (3.12), and (3.13) in field representation are listed in the Table 3.2

3.1.2 The action of $\tilde{\mathcal{Q}}$ on $\Sigma^{[IJ]}_{(MN)}$

When dealing with the signature $1+3$, the hermitian conjugate must be considered. The results of this section cover this issue. From Table 3.1, we first need to

Table 3.2: The action of $\tilde{\mathcal{Q}}$ on $\Psi^{(IJ)}_{[MN]}$

$\mathcal{Q}$	$\tilde{\mathcal{Q}}\Psi^{(IJ)}_{[MN]}$
$q_{A\rho}$	$2\delta^{(I}_A f_{\rho[1}\psi^{J)}_{2]}\phi_{MN} + \epsilon_{AMNL}\psi^{(I}_{[1}\psi^{J)}_{2]}\psi^L_\rho$
$\bar{q}^A_{\dot\rho}$	$2\partial_{\dot\rho[1}\phi^{A(I}\psi^{J)}_{2]}\phi_{MN} + \delta^{PA}_{MN}\psi^{(I}_{[1}\psi^{J)}_{2]}\bar{\psi}_{\dot\rho P}$
$s^{A\rho}$	$-\frac{i}{2}x^{\rho\dot\rho}\left(2\partial_{\dot\rho[1}\phi^{A(I}\psi^{J)}_{2]}\phi_{MN} + \delta^{PA}_{MN}\psi^{(I}_{[1}\psi^{J)}_{2]}\bar{\psi}_{\dot\rho P}\right) + 2\,\delta^\rho_{[1}\phi^{A(I}\psi^{J)}_{2]}\phi_{MN}$
$\bar{s}^{\dot\rho}_A$	$-\frac{i}{2}x^{\rho\dot\rho}\left(2\delta^{(I}_A f_{\rho[1}\psi^{J)}_{2]}\phi_{MN} + \epsilon_{AMNL}\psi^{(I}_{[1}\psi^{J)}_{2]}\psi^L_\rho\right)$

compute

$$q_{A\rho}\Sigma^{[IJ]}_{(MN)},\quad \bar{q}^A_{\dot\rho}\Sigma^{[IJ]}_{(MN)},\quad \bar{s}^{\dot\varrho}_A\Sigma^{[IJ]}_{(MN)},\quad \text{and}\quad s^{A\rho}\Sigma^{[IJ]}_{(MN)} \tag{3.14}$$

1. To begin with, $q_{A\rho}\Sigma^{[IJ]}_{(MN)}$ is expressed as

$$\begin{aligned}
q_{A\rho}\Sigma^{[IJ]}_{(MN)} &= c_A a^\dagger_\rho\left(\Sigma^{[IJ]}_{(MN)}\right)\\
&= \frac{1}{(3!)^2}\epsilon_{MUVW}\epsilon_{NU'V'W'}\\
&\times\Big(6\delta^U_A a^\dagger_\rho b^\dagger_{[1}c^{\dagger V}c^{\dagger W}|0\rangle \bullet b^\dagger_{2]}c^{\dagger U'}c^{\dagger V'}c^{\dagger W'}|0\rangle \bullet c^{\dagger I}c^{\dagger J}|0\rangle\\
&+ b^\dagger_{[1}c^{\dagger U}c^{\dagger V}c^{\dagger W}|0\rangle \bullet b^\dagger_{2]}c^{\dagger U'}c^{\dagger V'}c^{\dagger W'}|0\rangle \bullet \delta^{[I}_A a^\dagger_\rho c^{\dagger J]}|0\rangle\Big)
\end{aligned}\tag{3.15}$$

2. The action of $\bar{q}^A_{\dot\rho}$ on $\Sigma^{[IJ]}_{(MN)}$, i.e. $\bar{q}^A_{\dot\rho}\Sigma^{[IJ]}_{(MN)}$

$$\begin{aligned}
\bar{q}^A_{\dot\rho}\Sigma^{[IJ]}_{(MN)} &= b^\dagger_{\dot\rho}c^{\dagger A}\left(\Sigma^{[IJ]}_{(MN)}\right)\\
&= \frac{1}{(3!)^2}\epsilon_{MUVW}\epsilon_{NU'V'W'}\\
&\times\Big(2b^\dagger_{\dot\rho}b^\dagger_{[1}c^{\dagger A}c^{\dagger U}c^{\dagger V}c^{\dagger W}|0\rangle \bullet b^\dagger_{2]}c^{\dagger U'}c^{\dagger V'}c^{\dagger W'}|0\rangle \bullet c^{\dagger I}c^{\dagger J}|0\rangle\\
&+ b^\dagger_{[1}c^{\dagger U}c^{\dagger V}c^{\dagger W}|0\rangle \bullet b^\dagger_{2]}c^{\dagger U'}c^{\dagger V'}c^{\dagger W'}|0\rangle \bullet b^\dagger_{\dot\rho}c^{\dagger A}c^{\dagger I}c^{\dagger J}|0\rangle\Big)
\end{aligned}\tag{3.16}$$

3. The action of $\bar{s}^{\dot{\varrho}}_A$ on $\Sigma^{[IJ]}_{(MN)}$, i.e. $\bar{s}^{\dot{\varrho}}_A\Sigma^{[IJ]}_{(MN)}$

$$\begin{aligned}\bar{s}^{\dot{\varrho}}_A\Sigma^{[IJ]}_{(MN)} &= \frac{1}{(3!)^2}6\epsilon_{MUVW}\epsilon_{NU'V'W'} \\ &\times\left(\delta^U_A\delta^{\dot{\varrho}}_{[\dot{1}}c^{\dagger V}c^{\dagger W}\left|0\right\rangle\bullet b^\dagger_{\dot{2}]}c^{\dagger U'}c^{\dagger V'}c^{\dagger W'}\left|0\right\rangle\bullet c^{\dagger I}c^{\dagger J}\left|0\right\rangle\right)\end{aligned}\tag{3.17}$$

4. The action of $s^{A\rho}$ on $\Sigma^{[IJ]}_{(MN)}$ is zero, since there is no $a^\dagger$'s in $\Sigma^{[IJ]}_{(MN)}$.

The results in Eqs. (3.15), (3.16) and (3.17) in field representation are listed in the Table 3.3, where we have used equations (2.33), (2.34), and (2.35)

Table 3.3: The action of $\tilde{\mathcal{Q}}$ on $\Sigma^{[IJ]}_{(MN)}$

$\mathcal{Q}$	$\tilde{\mathcal{Q}}\Sigma^{[IJ]}_{(MN)}$
$q_{A\rho}$	$2\delta^U_A\partial_{\rho[\dot{1}\vert}\phi_{MU}\bar{\psi}_{\vert\dot{2}]N}\phi^{IJ}+\bar{\psi}_{M[\dot{1}}\bar{\psi}_{\dot{2}]N}\delta^{[I}_A\psi^{J]}_\rho$
$\bar{q}^A_{\dot{\rho}}$	$2f_{\dot{\rho}[\dot{1}\vert}\delta^A_{(M}\bar{\psi}_{N)\vert\dot{2}]}\phi^{IJ}+\epsilon^{PAIJ}\bar{\psi}_{M[\dot{1}}\bar{\psi}_{\dot{2}]N}\bar{\psi}_{P\dot{\rho}}$
$s^{A\rho}$	$-\frac{i}{2}x^{\rho\dot{\rho}}\left(2f_{\dot{\rho}[\dot{1}\vert}\delta^A_{(M}\bar{\psi}_{N)\vert\dot{2}]}\phi^{IJ}+\epsilon^{PAIJ}\bar{\psi}_{M[\dot{1}}\bar{\psi}_{\dot{2}]N}\bar{\psi}_{P\dot{\rho}}\right)$
$\bar{s}^{\dot{\rho}}_A$	$-\frac{i}{2}x^{\rho\dot{\rho}}\left(2\delta^U_A\partial_{\rho[\dot{1}\vert}\phi_{MU}\bar{\psi}_{N\vert\dot{2}]}\phi^{IJ}+\bar{\psi}_{M[\dot{1}}\bar{\psi}_{\dot{2}]N}\delta^{[I}_A\psi^{J]}_\rho\right)+2\delta^U_A\phi_{MU}\delta^{\dot{\rho}}_{[\dot{1}}\bar{\psi}_{\dot{2}]N}\phi^{IJ}$

3.2 Second order at supersymmetry generators

The action of two supersymmetric generators $\mathcal{Q}$ on S_{deformed}, i.e. $\mathcal{Q}_2\mathcal{Q}_1S_{\text{deformed}}$, reads

$$\mathcal{Q}_2\mathcal{Q}_1S_{\text{deformed}}=\int d^4x\prod_{j=1}^{3}\mathcal{R}^{(j)}\left(B^{[MN]}_{(IJ)}\tilde{\mathcal{Q}}_2\tilde{\mathcal{Q}}_1\Psi^{(IJ)}_{[MN]}+\tilde{B}^{(MN)}_{[IJ]}\tilde{\mathcal{Q}}_2\tilde{\mathcal{Q}}_1\Sigma^{[IJ]}_{(MN)}\right),\tag{3.18}$$

where $\tilde{\mathcal{Q}}_2\tilde{\mathcal{Q}}_1$ has the form

$$\begin{aligned}\tilde{\mathcal{Q}}_2\tilde{\mathcal{Q}}_1 &= -\frac{1}{4}x^{\varrho\dot{\varrho}}x^{\omega\dot{\omega}}[\mathcal{Q}_2,p_{\varrho\dot{\varrho}}][\mathcal{Q}_1,p_{\omega\dot{\omega}}]-\frac{i}{2}x^{\omega\dot{\omega}}\mathcal{Q}_2[\mathcal{Q}_1,p_{\omega\dot{\omega}}]+\mathcal{Q}_2\mathcal{Q}_1 \\ &-\frac{i}{2}x^{\omega\dot{\omega}}[\mathcal{Q}_2,p_{\omega\dot{\omega}}]\mathcal{Q}_1.\end{aligned}\tag{3.19}$$

There are sixteen products of $\tilde{\mathcal{Q}}_2\tilde{\mathcal{Q}}_1$ in equation (3.19), and are listed in Table 3.4. As we can see, the elements in Table 3.4 are combinations of $\mathcal{Q}_2\mathcal{Q}_1$, i.e.

$$q_{B\varrho}q_{A\rho},\ \bar{q}^B_{\dot\varrho}\bar{q}^A_{\dot\rho},\ s^{B\theta}s^{A\rho},\ \bar{s}^{\dot\rho}_B\bar{s}^{\dot\theta}_A,\ q_{B\varrho}\bar{q}^A_{\dot\rho},\ \bar{q}^B_{\dot\rho}q_{A\rho},\ s^{B\rho}\bar{q}^A_{\dot\rho},\ \bar{s}^{\dot\varrho}_B\bar{q}^A_{\dot\rho},$$
$$s^{B\varrho}q_{A\rho},\ \bar{s}^{\dot\varrho}_B q_{A\rho},\ q_{B\omega}s^{A\rho},\ \bar{s}^{\dot\rho}_B s^{A\rho},\ q_{B\varrho}\bar{s}^{\dot\rho}_A,\ \bar{q}^B_{\dot\rho}s^{A\rho},\ \bar{q}^B_{\dot\varrho}\bar{s}^{\dot\rho}_A,\ s^{B\rho}\bar{s}^{\dot\theta}_A \tag{3.20}$$

3.2.1 The action of $\tilde{\mathcal{Q}}_2\tilde{\mathcal{Q}}_1$ on $\Psi^{(IJ)}_{[MN]}$

To begin with, recall that $\bar{s}^{\dot\rho}_A\Psi^{(IJ)}_{[MN]} = 0$. Therefore, we have the following straightforward results

$$q_{B\rho}\bar{s}^{\dot\rho}_A\Psi^{(IJ)}_{[MN]} = 0 = \bar{q}^B_{\dot\varrho}\bar{s}^{\dot\rho}_A\Psi^{(IJ)}_{[MN]} = s^{B\rho}\bar{s}^{\dot\rho}_A\Psi^{(IJ)}_{[MN]} = \bar{s}^{\dot\varrho}_B\bar{s}^{\dot\rho}_A\Psi^{(IJ)}_{[MN]}. \tag{3.21}$$

The remaining products in Eq.(3.20) are computed below, where the relations of commutations in Eq.-(2.8) and

$$\epsilon_{MNKL}c^{\dagger A}\delta^{[K}_B c^{\dagger L]} = \epsilon_{MNKL}c^{\dagger A}c^{\dagger [L}\delta^{K]}_B = 2\epsilon_{MNBL}c^{\dagger A}c^{\dagger L} \tag{3.22}$$

were used.

1. $q_{B\varrho}q_{A\rho}\Psi^{(IJ)}_{[MN]}$ is obtained by acting with $q_{B\varrho}$ on the right-hand side of Eq. (3.10), and making use of the commutation relations in Eq. (2.8). The result is given by

$$\begin{aligned} q_{B\varrho}q_{A\rho}\Psi^{(IJ)}_{[MN]} &= \frac{1}{2}\Big(2\delta^{(I}_A\ a^\dagger_\rho a^\dagger_{[1}\left|0\right\rangle \bullet a^\dagger_{2]}a^\dagger_\varrho\ \delta^{J)}_B\left|0\right\rangle \bullet \epsilon_{MNKL}c^{\dagger K}c^{\dagger L}\left|0\right\rangle \\ &- 4\delta^{(I}_A\ a^\dagger_\rho a^\dagger_{[1}\left|0\right\rangle \bullet a^\dagger_{2]}c^{\dagger J)}\left|0\right\rangle \bullet \epsilon_{BMNL}a^\dagger_\varrho c^{\dagger L}\left|0\right\rangle \\ &+ 4\ \delta^{(I}_B\ a^\dagger_\varrho a^\dagger_{[1}\left|0\right\rangle \bullet a^\dagger_{2]}c^{\dagger J)}\left|0\right\rangle \bullet \epsilon_{AMNL}a^\dagger_\rho c^{\dagger L}\left|0\right\rangle \\ &+ 2\ a^\dagger_{[1}c^{\dagger (I}\left|0\right\rangle \bullet a^\dagger_{2]}c^{\dagger J)}\left|0\right\rangle \bullet \epsilon_{ABMN}a^\dagger_\rho a^\dagger_\varrho\left|0\right\rangle\Big). \end{aligned} \tag{3.23}$$

The field-theory expression corresponding to Eq. (3.23) is given by

$$\begin{aligned} q_{B\varrho}q_{A\rho}\Psi^{(IJ)}_{[MN]} &= 2\delta^{(I}_A\delta^{J)}_B f_{\rho[1}f_{2]\varrho}\phi_{MN} - 2\epsilon_{MNBL}f_{\rho[1|}\delta^{(I}_A\psi^{J)}_{|2]}\psi^L_\varrho \\ &+ 2\epsilon_{MNAL}f_{\varrho[1|}\delta^{(I}_B\psi^{J)}_{|2]}\psi^L_\rho + \epsilon_{MNAB}\psi^{(I}_{[1}\psi^{J)}_{2]}f_{\rho\varrho}. \end{aligned} \tag{3.24}$$

Table 3.4: Two supersymmetric generators $\mathcal{Q}$ and $\tilde{\mathcal{Q}}_2\tilde{\mathcal{Q}}_1$

$\mathcal{Q}_2$	$\mathcal{Q}_1$	$\tilde{\mathcal{Q}}_2\tilde{\mathcal{Q}}_1$
$q_{B\varrho}$	$q_{A\rho}$	$q_{B\varrho}q_{A\rho}$
$q_{B\varrho}$	$\bar{q}^A_{\dot\rho}$	$q_{B\varrho}\bar{q}^A_{\dot\rho}$
$\bar{q}^B_{\dot\rho}$	$q_{A\rho}$	$\bar{q}^B_{\dot\rho}\,q_{A\rho}$
$\bar{q}^B_{\dot\varrho}$	$\bar{q}^A_{\dot\rho}$	$\bar{q}^B_{\dot\varrho}\,\bar{q}^A_{\dot\rho}$
$s^{B\rho}$	$\bar{q}^A_{\dot\rho}$	$-\frac{i}{2}x^{\rho\dot\omega}\bar{q}^B_{\dot\omega}\,\bar{q}^A_{\dot\rho} + s^{B\rho}\bar{q}^A_{\dot\rho}$
$\bar{s}^{\dot\varrho}_B$	$\bar{q}^A_{\dot\rho}$	$-\frac{i}{2}x^{\omega\dot\varrho}q_{B\omega}\bar{q}^A_{\dot\rho} + \bar{s}^{\dot\varrho}_B\bar{q}^A_{\dot\rho}$
$s^{B\varrho}$	$q_{A\rho}$	$-\frac{i}{2}x^{\varrho\dot\omega}\bar{q}^B_{\dot\omega}\,q_{A\rho} + s^{B\varrho}q_{A\rho}$
$\bar{s}^{\dot\varrho}_B$	$q_{A\rho}$	$-\frac{i}{2}x^{\omega\dot\varrho}q_{B\omega}q_{A\rho} + \bar{s}^{\dot\varrho}_B q_{A\rho}$
$\bar{s}^{\dot\rho}_B$	$s^{A\rho}$	$-\frac{1}{4}x^{\varrho\dot\rho}x^{\rho\dot\omega}q_{B\varrho}\bar{q}^A_{\dot\omega} - \frac{i}{2}x^{\rho\dot\omega}\bar{s}^{\dot\rho}_B\bar{q}^A_{\dot\omega} - \frac{i}{2}x^{\omega\dot\rho}q_{B\omega}s^{A\rho} + \bar{s}^{\dot\rho}_B s^{A\rho}$
$q_{B\varrho}$	$s^{A\rho}$	$-\frac{i}{2}x^{\rho\dot\omega}q_{B\varrho}\bar{q}^A_{\dot\omega} + q_{B\varrho}s^{A\rho}$
$q_{B\varrho}$	$\bar{s}^{\dot\rho}_A$	$-\frac{i}{2}x^{\rho\dot\rho}q_{B\varrho}q_{A\rho} + q_{B\varrho}\bar{s}^{\dot\rho}_A$
$\bar{q}^B_{\dot\rho}$	$s^{A\rho}$	$-\frac{i}{2}x^{\rho\dot\omega}\bar{q}^B_{\dot\rho}\,\bar{q}^A_{\dot\omega} + \bar{q}^B_{\dot\rho}\,s^{A\rho}$
$\bar{q}^B_{\dot\varrho}$	$\bar{s}^{\dot\rho}_A$	$-\frac{i}{2}x^{\varrho\dot\rho}\bar{q}^B_{\dot\varrho}\,q_{A\varrho} + \bar{q}^B_{\dot\varrho}\,\bar{s}^{\dot\rho}_A$
$s^{B\theta}$	$s^{A\rho}$	$-\frac{1}{4}x^{\theta\dot\varrho}x^{\rho\dot\omega}\bar{q}^B_{\dot\varrho}\,\bar{q}^A_{\dot\omega} - \frac{i}{2}x^{\rho\dot\omega}s^{B\theta}\bar{q}^A_{\dot\omega} - \frac{i}{2}x^{\theta\dot\omega}\bar{q}^B_{\dot\omega}\,s^{A\rho} + s^{B\theta}s^{A\rho}$
$s^{B\rho}$	$\bar{s}^{\dot\theta}_A$	$-\frac{1}{4}x^{\rho\dot\varrho}x^{\omega\dot\theta}\bar{q}^B_{\dot\varrho}\,q_{A\omega} - \frac{i}{2}x^{\omega\dot\theta}s^{B\rho}q_{A\omega} - \frac{i}{2}x^{\rho\dot\varrho}\bar{q}^B_{\dot\varrho}\,\bar{s}^{\dot\theta}_A + s^{B\rho}\bar{s}^{\dot\theta}_A$
$\bar{s}^{\dot\rho}_B$	$\bar{s}^{\dot\theta}_A$	$-\frac{1}{4}x^{\varrho\dot\rho}x^{\omega\dot\theta}q_{B\varrho}q_{A\omega} - \frac{i}{2}x^{\omega\dot\theta}\bar{s}^{\dot\rho}_B q_{A\omega} - \frac{i}{2}x^{\omega\dot\rho}q_{B\omega}\bar{s}^{\dot\theta}_A + \bar{s}^{\dot\rho}_B\bar{s}^{\dot\theta}_A$

2. $\bar{q}^B_{\dot\beta}\bar{q}^A_{\dot\alpha}\Psi^{(IJ)}_{[MN]}$ is obtained by acting with $\bar{q}^B_{\dot\beta}$ on the right-hand side of Eq. (3.12), using the commutation relations in Eq. (2.8), and is given by

$$\begin{aligned}\bar{q}^B_{\dot\beta}\bar{q}^A_{\dot\alpha}\Psi^{(IJ)}_{[MN]} &= \frac{1}{2}\Big(2\, b^\dagger_{\dot\beta}b^\dagger_{\dot\alpha}a^\dagger_{[1}c^{\dagger B}c^{\dagger A}c^{\dagger(I}\,|0\rangle \bullet a^\dagger_{2]}c^{\dagger J)}\,|0\rangle \bullet \epsilon_{MNKL}c^{\dagger K}c^{\dagger L}\,|0\rangle \\ &- 2\, b^\dagger_{\dot\alpha}a^\dagger_{[1}c^{\dagger A}c^{\dagger(I}\,|0\rangle \bullet a^\dagger_{2]}b^\dagger_{\dot\beta}c^{\dagger J)}c^{\dagger B}\,|0\rangle \bullet \epsilon_{MNKL}c^{\dagger K}c^{\dagger L}\,|0\rangle \\ &- 2\, b^\dagger_{\dot\alpha}a^\dagger_{[1}c^{\dagger A}c^{\dagger(I}\,|0\rangle \bullet a^\dagger_{2]}c^{\dagger J)}\,|0\rangle \bullet \epsilon_{MNKL}b^\dagger_{\dot\beta}c^{\dagger B}c^{\dagger K}c^{\dagger L}\,|0\rangle \\ &+ 2\, a^\dagger_{[1}b^\dagger_{\dot\beta}c^{\dagger B}c^{\dagger(I}\,|0\rangle \bullet a^\dagger_{2]}c^{\dagger J)}\,|0\rangle \bullet \epsilon_{MNKL}b^\dagger_{\dot\alpha}c^{\dagger A}c^{\dagger K}c^{\dagger L}\,|0\rangle \\ &+ a^\dagger_{[1}c^{\dagger(I}\,|0\rangle \bullet a^\dagger_{2]}c^{\dagger J)}\,|0\rangle \bullet \epsilon_{MNKL}b^\dagger_{\dot\beta}b^\dagger_{\dot\alpha}c^{\dagger B}c^{\dagger A}c^{\dagger K}c^{\dagger L}\,|0\rangle\Big).\end{aligned} \tag{3.25}$$

The field-theoretic form of Eq. (3.25) is

$$\begin{aligned}\bar{q}^B_{\dot\beta}\bar{q}^A_{\dot\alpha}\Psi^{(IJ)}_{[MN]} &= 2\epsilon^{PBA(I}\partial_{[1|\dot\alpha}\bar\psi_{P\dot\beta}\psi^{J)}_{|2]}\phi_{MN} - 2\partial_{\dot\alpha[1}\phi^{A(I}\partial_{2]\dot\beta}\phi^{J)B}\phi_{MN} \\ &- 2\partial_{\dot\alpha[1}\phi^{A(I}\psi^{J)}_{2]}\delta^{PB}_{MN}\bar\psi_{P\dot\beta} + 2\partial_{\dot\beta[1}\phi^{B(I}\psi^{J)}_{2]}\delta^{PA}_{MN}\bar\psi_{P\dot\alpha} \\ &+ \psi^{(I}_{[1}\psi^{J)}_{2]}\delta^{BA}_{MN}\bar f_{\dot\alpha\dot\beta},\end{aligned} \tag{3.26}$$

where δ^{BA}_{MN}, which is antisymmetric in M and N, follows from

$$\delta^{BA}_{[MN]} = \epsilon_{MNKL}\epsilon^{BAKL}. \tag{3.27}$$

3. $s^{B\varrho}s^{A\rho}\Psi^{(IJ)}_{[MN]}$. The action of $s^{B\varrho}$ on the right-hand side of Eq. (3.13), using Eq. (2.8), yields

$$s^{B\varrho}s^{A\rho}\Psi^{(IJ)}_{[MN]} = \frac{1}{2}\Big(2\delta^{\rho}_{[1}c^{\dagger A}c^{\dagger(I}\,|0\rangle \bullet \delta^{\varrho}_{2]}c^{\dagger B}c^{\dagger J)}\,|0\rangle \bullet \epsilon_{MNKL}c^{\dagger K}c^{\dagger L}\,|0\rangle\Big). \tag{3.28}$$

The field-theory expression corresponding to Eq. (3.28) is given by

$$s^{B\varrho}s^{A\rho}\Psi^{(IJ)}_{[MN]} = -2\delta^{\rho}_{[1}\delta^{\varrho}_{2]}\phi^{A(I}\phi^{J)B}\phi_{MN} \tag{3.29}$$

4. $q_{B\varrho}\bar{q}^A_{\dot\rho}\Psi^{(IJ)}_{[MN]}$ is obtained by acting with $q_{B\varrho}$ on Eq. (3.12), using Eq. (2.8)

and Eq. (3.8)

$$\begin{aligned} q_{B\varrho}\bar{q}^{A}_{\dot\alpha}\mathbf{\Psi}^{(IJ)}_{[MN]} &= \frac{1}{2}\Big(2b^{\dagger}_{\dot\alpha}a^{\dagger}_{\varrho}a^{\dagger}_{[1}\delta^{[A}_{B}c^{\dagger(I]}\left|0\right\rangle \bullet a^{\dagger}_{2]}c^{\dagger J)}\left|0\right\rangle \bullet \epsilon_{MNKL}c^{\dagger K}c^{\dagger L}\left|0\right\rangle \\ &+ 2b^{\dagger}_{\dot\alpha}a^{\dagger}_{[1}c^{\dagger A}c^{\dagger(I}\left|0\right\rangle \bullet a^{\dagger}_{2]}a^{\dagger}_{\varrho}\delta^{J)}_{B}\left|0\right\rangle \bullet \epsilon_{MNKL}c^{\dagger K}c^{\dagger L}\left|0\right\rangle \\ &- 2b^{\dagger}_{\dot\alpha}a^{\dagger}_{[1}c^{\dagger A}c^{\dagger(I}\left|0\right\rangle \bullet a^{\dagger}_{2]}c^{\dagger J)}\left|0\right\rangle \bullet \epsilon_{MNKL}\delta^{[K}_{B}a^{\dagger}_{\varrho}c^{\dagger L]}\left|0\right\rangle \\ &+ 2a^{\dagger}_{\varrho}a^{\dagger}_{[1}\delta^{(I}_{B}\left|0\right\rangle \bullet a^{\dagger}_{2]}c^{\dagger J)}\left|0\right\rangle \bullet \epsilon_{MNKL}b^{\dagger}_{\dot\alpha}c^{\dagger A}c^{\dagger K}c^{\dagger L}\left|0\right\rangle \\ &+ a^{\dagger}_{[1}c^{\dagger(I}\left|0\right\rangle \bullet a^{\dagger}_{2]}c^{\dagger J)}\left|0\right\rangle \bullet \epsilon_{MNKL}a^{\dagger}_{\varrho}b^{\dagger}_{\dot\alpha}\delta^{A}_{B}c^{\dagger K}c^{\dagger L}\left|0\right\rangle \\ &- a^{\dagger}_{[1}c^{\dagger(I}\left|0\right\rangle \bullet a^{\dagger}_{2]}c^{\dagger J)}\left|0\right\rangle \bullet \epsilon_{MNKL}a^{\dagger}_{\varrho}b^{\dagger}_{\dot\alpha}c^{\dagger A}\delta^{[K}_{B}c^{\dagger L]}\left|0\right\rangle\Big). \end{aligned} \tag{3.30}$$

The result in Eq. (3.30), expressed in the field representation, is given by

$$\begin{aligned} &= 2\delta^{[A}_{B}\partial_{\varrho\dot\alpha}\psi^{(I]}_{[1}\psi^{J)}_{2]}\phi_{MN} + 2\partial_{\dot\alpha[1}\phi^{A(I}f_{2]\varrho}\delta^{J)}_{B}\phi_{MN} - 2\partial_{\dot\alpha[1}\phi^{A(I}\psi^{J)}_{2]}\epsilon_{MNBL}\psi^{L}_{\varrho} \\ &+ 2f_{\varrho[1}\psi^{(I}_{2]}\delta^{J)}_{B}\delta^{PA}_{MN}\bar\psi_{P\dot\alpha} + \psi^{(I}_{[1}\psi^{J)}_{2]}\delta^{A}_{B}\partial_{\varrho\dot\alpha}\phi_{MN} - \psi^{(I}_{[1}\psi^{J)}_{2]}\epsilon_{MNBL}\partial_{\varrho\dot\alpha}\phi^{AL} \end{aligned} \tag{3.31}$$

5. $\bar{q}^{B}_{\dot\alpha}q_{A\rho}\mathbf{\Psi}^{(IJ)}_{[MN]}$, is computed by acting with $\bar{q}^{B}_{\dot\alpha}$ on Eq. (3.10) and using the commutation relations in Eq. (2.8), is given by

$$\begin{aligned} \bar{q}^{B}_{\dot\alpha}q_{A\rho}\mathbf{\Psi}^{(IJ)}_{[MN]} &= \frac{1}{2}\Big(2\delta^{(I}_{A}a^{\dagger}_{\rho}b^{\dagger}_{\dot\alpha}a^{\dagger}_{[1}c^{\dagger B}\left|0\right\rangle \bullet a^{\dagger}_{2]}c^{\dagger J)}\left|0\right\rangle \bullet \epsilon_{MNKL}c^{\dagger K}c^{\dagger L}\left|0\right\rangle \\ &- 2\delta^{(I}_{A}a^{\dagger}_{\rho}a^{\dagger}_{[1}\left|0\right\rangle \bullet a^{\dagger}_{2]}b^{\dagger}_{\dot\alpha}c^{\dagger J)}c^{\dagger B}\left|0\right\rangle \bullet \epsilon_{MNKL}c^{\dagger K}c^{\dagger L}\left|0\right\rangle \\ &- 2\delta^{(I}_{A}\ a^{\dagger}_{\rho}a^{\dagger}_{[1}\left|0\right\rangle \bullet a^{\dagger}_{2]}c^{\dagger J)}\left|0\right\rangle \bullet \epsilon_{MNKL}b^{\dagger}_{\dot\alpha}c^{\dagger B}c^{\dagger K}c^{\dagger L}\left|0\right\rangle \\ &+ 4\ a^{\dagger}_{[1}b^{\dagger}_{\dot\alpha}c^{\dagger B}c^{\dagger(I}\left|0\right\rangle \bullet a^{\dagger}_{2]}c^{\dagger J)}\left|0\right\rangle \bullet \epsilon_{AMNL}a^{\dagger}_{\rho}c^{\dagger L}\left|0\right\rangle \\ &+ 2\ a^{\dagger}_{[1}c^{\dagger(I}\left|0\right\rangle \bullet a^{\dagger}_{2]}c^{\dagger J)}\left|0\right\rangle \bullet \epsilon_{AMNL}a^{\dagger}_{\rho}b^{\dagger}_{\dot\alpha}c^{\dagger B}c^{\dagger L}\left|0\right\rangle\Big). \end{aligned} \tag{3.32}$$

The result in Eq. (3.32), expressed in the field representation, is

$$\begin{aligned} \bar{q}^{B}_{\dot\alpha}q_{A\rho}\mathbf{\Psi}^{(IJ)}_{[MN]} &= 2\partial_{\rho\dot\alpha}\psi^{B}_{[1|}\delta^{(I}_{A}\psi^{J)}_{|2]}\phi_{MN} - 2\delta^{(I}_{A}f_{\rho[1}\partial_{2]\dot\alpha}\phi^{J)B}\phi_{MN} \\ &- 2\delta^{(I}_{A}f_{\rho[1}\psi^{J)}_{2]}\delta^{PB}_{MN}\bar\psi_{P\dot\alpha} + 2\epsilon_{AMNL}\partial_{\dot\alpha[1}\phi^{B(I}\psi^{J)}_{2]}\psi^{L}_{\rho} \\ &+ \epsilon_{AMNL}\psi^{(I}_{[1}\psi^{J)}_{2]}\partial_{\rho\dot\alpha}\phi^{BL}. \end{aligned} \tag{3.33}$$

6. $s^{B\rho}\bar{q}^{A}_{\dot\alpha}\mathbf{\Psi}^{(IJ)}_{[MN]}$. Applying $s^{B\rho}$ to the right-hand side of Eq. (3.12) and using

relations in Eq. (2.8) yields to

$$s^{B\rho}\bar{q}^{A}_{\dot{\alpha}}\mathbf{\Psi}^{(IJ)}_{[MN]} = \frac{1}{2}\Big(2b^{\dagger}_{\dot{\alpha}}\delta^{\rho}_{[1}c^{\dagger B}c^{\dagger A}c^{\dagger (I}\left|0\right\rangle \bullet a^{\dagger}_{2]}c^{\dagger J)}\left|0\right\rangle \bullet \epsilon_{MNKL}c^{\dagger K}c^{\dagger L}\left|0\right\rangle$$
$$- 2b^{\dagger}_{\dot{\alpha}}a^{\dagger}_{[1}c^{\dagger A}c^{\dagger (I}\left|0\right\rangle \bullet \delta^{\rho}_{2]}c^{\dagger J)}c^{\dagger B}\left|0\right\rangle \bullet \epsilon_{MNKL}c^{\dagger K}c^{\dagger L}\left|0\right\rangle$$
$$+ 2\delta^{\rho}_{[1}c^{\dagger B}c^{\dagger (I}\left|0\right\rangle \bullet a^{\dagger}_{2]}c^{\dagger J)}\left|0\right\rangle \bullet \epsilon_{MNKL}b^{\dagger}_{\dot{\alpha}}c^{\dagger A}c^{\dagger K}c^{\dagger L}\left|0\right\rangle\Big). \tag{3.34}$$

The field-representation form of Eq. (3.34) is

$$s^{B\rho}\bar{q}^{A}_{\dot{\alpha}}\mathbf{\Psi}^{(IJ)}_{[MN]} = 2\delta^{\rho}_{[1|}\epsilon^{PBA(I}\bar{\psi}_{P\dot{\alpha}}\psi^{J)}_{|2]}\phi_{MN} - 2\partial_{\dot{\alpha}[1}\phi^{A(I}\phi^{J)B}\delta^{\rho}_{2]}\phi_{MN}$$
$$+ 2\delta^{\rho}_{[1}\phi^{B(I}\psi^{J)}_{2]}\delta^{PA}_{MN}\bar{\psi}_{P\dot{\alpha}}. \tag{3.35}$$

7. $\bar{s}^{\dot{\varrho}}_{B}\bar{q}^{A}_{\dot{\alpha}}\mathbf{\Psi}^{(IJ)}_{[MN]}$. The action of $\bar{s}^{\dot{\alpha}}_{B}$ on Eq. (3.12), together with the commutation relations in Eq. (2.8) and Eq. (3.22), gives $\bar{s}^{\dot{\varrho}}_{B}\bar{q}^{A}_{\dot{\alpha}}\mathbf{\Psi}^{(IJ)}_{[MN]}$

$$\bar{s}^{\dot{\varrho}}_{B}\bar{q}^{A}_{\dot{\alpha}}\mathbf{\Psi}^{(IJ)}_{[MN]} = \frac{1}{2}\,2\delta^{\dot{\varrho}}_{\dot{\alpha}}\delta^{[A}_{B}a^{\dagger}_{[1}c^{\dagger (I]}\left|0\right\rangle \bullet a^{\dagger}_{2]}c^{\dagger J)}\left|0\right\rangle \bullet \epsilon_{MNKL}c^{\dagger K}c^{\dagger L}\left|0\right\rangle$$
$$+ \frac{1}{2}a^{\dagger}_{[1}c^{\dagger (I}\left|0\right\rangle \bullet a^{\dagger}_{2]}c^{\dagger J)}\left|0\right\rangle \bullet \epsilon_{MNKL}\delta^{\dot{\varrho}}_{\dot{\alpha}}\delta^{A}_{B}c^{\dagger K}c^{\dagger L}\left|0\right\rangle$$
$$+ a^{\dagger}_{[1}c^{\dagger (I}\left|0\right\rangle \bullet a^{\dagger}_{2]}c^{\dagger J)}\left|0\right\rangle \bullet \epsilon_{MNKB}\delta^{\dot{\varrho}}_{\dot{\alpha}}c^{\dagger A}c^{\dagger K}\left|0\right\rangle. \tag{3.36}$$

In field-theoretic form Eq. (3.36) reads

$$\bar{s}^{\dot{\varrho}}_{B}\bar{q}^{A}_{\dot{\alpha}}\mathbf{\Psi}^{(IJ)}_{[MN]} = 2\delta^{\dot{\varrho}}_{\dot{\alpha}}\delta^{[A}_{B}\psi^{(I]}_{[1}\psi^{J)}_{2]}\phi_{MN} + \psi^{(I}_{[1}\psi^{J)}_{2]}\delta^{\dot{\varrho}}_{\dot{\alpha}}\delta^{A}_{B}\phi_{MN}$$
$$+ \psi^{(I}_{[1}\psi^{J)}_{2]}\delta^{\dot{\varrho}}_{\dot{\alpha}}\epsilon_{MNKB}\phi^{AK}. \tag{3.37}$$

8. $s^{B\varrho}q_{A\rho}\mathbf{\Psi}^{(IJ)}_{[MN]}$ proceeds by applying $s^{B\varrho}$ to Eq. (3.10) and using the commutation relations in Eq. (2.8)

$$s^{B\varrho}q_{A\rho}\mathbf{\Psi}^{(IJ)}_{[MN]} = \frac{1}{2}\Big(2\;\delta^{(I}_{A}\delta^{\varrho}_{(\rho}a^{\dagger}_{[1)}c^{\dagger B}\left|0\right\rangle \bullet a^{\dagger}_{2]}c^{\dagger J)}\left|0\right\rangle \bullet \epsilon_{MNKL}c^{\dagger K}c^{\dagger L}\left|0\right\rangle$$
$$- 2\;\delta^{(I}_{A}\;a^{\dagger}_{\rho}a^{\dagger}_{[1}\left|0\right\rangle \bullet \delta^{\varrho}_{2]}c^{\dagger J)}c^{\dagger B}\left|0\right\rangle \bullet \epsilon_{MNKL}c^{\dagger K}c^{\dagger L}\left|0\right\rangle$$
$$+ 4\;\delta^{\varrho}_{[1}c^{\dagger B}c^{\dagger (I}\left|0\right\rangle \bullet a^{\dagger}_{2]}c^{\dagger J)}\left|0\right\rangle \bullet \epsilon_{AMNL}a^{\dagger}_{\rho}c^{\dagger L}\left|0\right\rangle$$
$$+ 2\;a^{\dagger}_{[1}c^{\dagger (I}\left|0\right\rangle \bullet a^{\dagger}_{2]}c^{\dagger J)}\left|0\right\rangle \bullet \epsilon_{AMNL}\delta^{\varrho}_{\rho}c^{\dagger B}c^{\dagger L}\left|0\right\rangle\Big). \tag{3.38}$$

which, in field-theoretic form, is given by

$$\begin{aligned} s^{B\varrho}q_{A\rho}\Psi^{(IJ)}_{[MN]} &= 2\delta^{\varrho}_{(\rho}\psi^{B}_{[1|)}\delta^{(I}_{A}\psi^{J)}_{|2]}\phi_{MN} - 2\delta^{(I}_{A}\phi^{J)B}f_{\rho[1}\delta^{\varrho}_{2]}\phi_{MN} \\ &+ 2\delta^{\varrho}_{[1}\phi^{B(I}\psi^{J)}_{2]}\epsilon_{AMNL}\psi^{L}_{\rho} + \psi^{(I}_{[1}\psi^{J)}_{2]}\epsilon_{AMNL}\delta^{\varrho}_{\rho}\phi^{BL}. \end{aligned} \tag{3.39}$$

9. $\bar{s}^{\dot\beta}_{B}q_{A\rho}\Psi^{(IJ)}_{[MN]}$ starts from Eq. (3.10), we act with $\bar{s}^{\dot\beta}_{B}$ and use Eq. (2.8)

$$\bar{s}^{\dot\beta}_{B}q_{A\rho}\Psi^{(IJ)}_{[MN]} = 0, \tag{3.40}$$

this result arises because (3.10) contains no $b^\dagger$ oscillators.

10. $q_{B\omega}s^{A\rho}\Psi^{(IJ)}_{[MN]}$ follows from acting with $q_{B\omega}$ on Eq. (3.13) and using Eqs. (2.8) and (3.11)

$$\begin{aligned} q_{B\omega}s^{A\rho}\Psi^{(IJ)}_{[MN]} &= \frac{1}{2}\Big(2a^{\dagger}_{\omega}\delta^{[A}_{B}c^{\dagger(I]}\delta^{\rho}_{[1}\left|0\right\rangle \bullet a^{\dagger}_{2]}c^{\dagger J)}\left|0\right\rangle \bullet \epsilon_{MNKL}c^{\dagger K}c^{\dagger L}\left|0\right\rangle \\ &+ 2\delta^{\rho}_{[1}c^{\dagger A}c^{\dagger(I}\left|0\right\rangle \bullet a^{\dagger}_{2]}a^{\dagger}_{\omega}\delta^{J)}_{B}\left|0\right\rangle \bullet \epsilon_{MNKL}c^{\dagger K}c^{\dagger L}\left|0\right\rangle \\ &+ 4\delta^{\rho}_{[1}c^{\dagger A}c^{\dagger(I}\left|0\right\rangle \bullet a^{\dagger}_{2]}c^{\dagger J)}\left|0\right\rangle \bullet \epsilon_{MNKB}a^{\dagger}_{\omega}c^{\dagger K}\left|0\right\rangle\Big). \end{aligned} \tag{3.41}$$

Eq (3.41) when written in the field representation is given by

$$\begin{aligned} q_{B\omega}s^{A\rho}\Psi^{(IJ)}_{[MN]} &= 2\delta^{\rho}_{[1|}\delta^{[A}_{B}\psi^{(I]}_{\omega}\psi^{J)}_{|2]}\phi_{MN} + 2\phi^{A(I}\delta^{J)}_{B}\delta^{\rho}_{[1}f_{2]\omega}\phi_{MN} \\ &- 2\delta^{\rho}_{[1}\phi^{A(I}\psi^{J)}_{2]}\epsilon_{MNBL}\psi^{L}_{\omega}. \end{aligned} \tag{3.42}$$

11. $\bar{s}^{\dot\rho}_{B}s^{A\rho}\Psi^{(IJ)}_{[MN]}$ follows from applying $\bar{s}^{\dot\rho}_{B}$ to Eq. (3.13) and using Eq. (2.8)

$$\bar{s}^{\dot\rho}_{B}s^{A\rho}\Psi^{(IJ)}_{[MN]} = 0, \tag{3.43}$$

this result follows directly from the absence of any $b^\dagger$ term in Eq. (3.13).

12. $\bar{q}^{B}_{\dot\rho}s^{A\rho}\Psi^{(IJ)}_{[MN]}$ is derived by acting with $\bar{q}^{B}_{\dot\rho}$ on Eq. (3.13) and using commutation relations in Eq. (2.8)

$$\begin{aligned} \bar{q}^{B}_{\dot\rho}s^{A\rho}\Psi^{(IJ)}_{[MN]} &= \frac{1}{2}\Big(2b^{\dagger}_{\dot\rho}\delta^{\rho}_{[1}c^{\dagger B}c^{\dagger A}c^{\dagger(I}\left|0\right\rangle \bullet a^{\dagger}_{2]}c^{\dagger J)}\left|0\right\rangle \bullet \epsilon_{MNKL}c^{\dagger K}c^{\dagger L}\left|0\right\rangle \\ &- 2\delta^{\rho}_{[1}c^{\dagger A}c^{\dagger(I}\left|0\right\rangle \bullet a^{\dagger}_{2]}b^{\dagger}_{\dot\rho}c^{\dagger J)}c^{\dagger B}\left|0\right\rangle \bullet \epsilon_{MNKL}c^{\dagger K}c^{\dagger L}\left|0\right\rangle \\ &- 2\delta^{\rho}_{[1}c^{\dagger A}c^{\dagger(I}\left|0\right\rangle \bullet a^{\dagger}_{2]}c^{\dagger J)}\left|0\right\rangle \bullet \epsilon_{MNKL}b^{\dagger}_{\dot\rho}c^{\dagger B}c^{\dagger K}c^{\dagger L}\left|0\right\rangle\Big) \end{aligned} \tag{3.44}$$

Eq. (3.44) expressed in the field representation, is

$$\begin{aligned}\bar{q}^B_{\dot\rho} s^{A\rho} \Psi^{(IJ)}_{[MN]} &= 2\delta^{\rho}_{[1|}\epsilon^{PBA(I}\bar{\psi}_{P\dot\rho}\psi^{J)}_{|2]}\phi_{MN} - 2\delta^{\rho}_{[1}\phi^{A(I}\partial_{2]\dot\rho}\phi^{J)B}\phi_{MN} \\ &- 2\delta^{\rho}_{[1}\phi^{A(I}\psi^{J)}_{2]}\delta^{PB}_{MN}\bar{\psi}_{P\dot\rho}. \end{aligned} \tag{3.45}$$

Using the operator combinations in Table 3.4, we compute the corresponding actions on $\Psi^{(IJ)}_{[MN]}$. The explicit results, given in Eqs. (3.24), (3.26), (3.29), (3.31), (3.33), (3.35), (3.37), (3.39), (3.40), (3.42), (3.43), and (3.45) are collected and summarized in Table 3.5 below

Table 3.5: The action of $\tilde{\mathcal{Q}}_2\tilde{\mathcal{Q}}_1$ on $\Psi^{(IJ)}_{[MN]}$

$\mathcal{Q}_2$	$\mathcal{Q}_1$	$\tilde{\mathcal{Q}}_2\tilde{\mathcal{Q}}_1\Psi^{(IJ)}_{[MN]}$
$q_{B\varrho}$	$q_{A\rho}$	$2\delta^{(I}_A\delta^{J)}_B f_{\rho[1}f_{2]\varrho}\phi_{MN} - 2\epsilon_{MNBL}f_{\rho[1|}\delta^{(I}_A\psi^{J)}_{\|2]}\psi^L_\varrho + 2\epsilon_{MNAL}f_{\varrho[1|}\delta^{(I}_B\psi^{J)}_{\|2]}\psi^L_\rho + \epsilon_{MNAB}\psi^{(I}_{[1}\psi^{J)}_{2]}f_{\rho\varrho}$
$q_{B\varrho}$	$\bar{q}^A_{\dot\alpha}$	$2\delta^{[A}_B\partial_{\varrho\dot\alpha}\psi^{(I]}_{[1}\psi^{J)}_{2]}\phi_{MN} + 2\partial_{\dot\alpha[1}\phi^{A(I}f_{2]\varrho}\delta^{J)}_B\phi_{MN} - 2\partial_{\dot\alpha[1}\phi^{A(I}\psi^{J)}_{2]}\epsilon_{MNBL}\psi^L_\varrho + 2f_{\varrho[1}\psi^{(I}_{2]}\delta^{J)}_B\delta^{PA}_{MN}\bar{\psi}_{P\dot\alpha} + \psi^{(I}_{[1}\psi^{J)}_{2]}\delta^A_B\partial_{\varrho\dot\alpha}\phi_{MN} - \psi^{(I}_{[1}\psi^{J)}_{2]}\epsilon_{MNBL}\partial_{\varrho\dot\alpha}\phi^{AL}$
$\bar{q}^B_{\dot\alpha}$	$q_{A\rho}$	$2\partial_{\rho\dot\alpha}\psi^B_{[1\|}\delta^{(I}_A\psi^{J)}_{\|2]}\phi_{MN} - 2\delta^{(I}_A f_{\rho[1}\partial_{2]\dot\alpha}\phi^{J)B}\phi_{MN} - 2\delta^{(I}_A f_{\rho[1}\psi^{J)}_{2]}\delta^{PB}_{MN}\bar{\psi}_{P\dot\alpha} + 2\epsilon_{AMNL}\partial_{\dot\alpha[1}\phi^{B(I}\psi^{J)}_{2]}\psi^L_\rho + \epsilon_{AMNL}\psi^{(I}_{[1}\psi^{J)}_{2]}\partial_{\rho\dot\alpha}\phi^{BL}$
$\bar{q}^B_{\dot\beta}$	$\bar{q}^A_{\dot\alpha}$	$2\epsilon^{PBA(I}\partial_{\dot\alpha[1\|}\bar{\psi}_{P\dot\beta}\psi^{J)}_{\|2]}\phi_{MN} - 2\partial_{\dot\alpha[1}\phi^{A(I}\partial_{2]\dot\beta}\phi^{J)B}\phi_{MN} - 2\partial_{\dot\alpha[1}\phi^{A(I}\psi^{J)}_{2]}\delta^{PB}_{MN}\bar{\psi}_{P\dot\beta} + \partial_{\dot\beta[1}\phi^{B(I}\psi^{J)}_{2]}\delta^{PA}_{MN}\bar{\psi}_{P\dot\alpha} + \psi^{(I}_{[1}\psi^{J)}_{2]}\delta^{BA}_{MN}\bar{f}_{\dot\alpha\dot\beta}$

$\mathcal{Q}_2$	$\mathcal{Q}_1$	$\tilde{\mathcal{Q}}_2\tilde{\mathcal{Q}}_1\Psi^{(IJ)}_{[MN]}$
$s^{B\rho}$	$\bar{q}^A_{\dot{\alpha}}$	$-\frac{i}{2}x^{\rho\dot{\beta}}\Big(2\epsilon^{PBA(I}\partial_{\dot{\alpha}[1\vert}\bar{\psi}_{P\dot{\beta}}\psi^{J)}_{\vert 2]}\phi_{MN} - 2\partial_{\dot{\alpha}[1}\phi^{A(I}\partial_{2]\dot{\beta}}\phi^{J)B}\phi_{MN} - 2\partial_{\dot{\alpha}[1}\phi^{A(I}\psi^{J)}_{2]}\delta^{PB}_{MN}\bar{\psi}_{P\dot{\beta}} + \partial_{\dot{\beta}[1}\phi^{B(I}\psi^{J)}_{2]}\delta^{PA}_{MN}\bar{\psi}_{P\dot{\alpha}} + \psi^{(I}_{[1}\psi^{J)}_{2]}\delta^{BA}_{MN}\bar{f}_{\dot{\alpha}\dot{\beta}}\Big) + 2\delta^{\rho}_{[1\vert}\epsilon^{PBA(I}\bar{\psi}_{P\dot{\alpha}}\psi^{J)}_{\vert 2]}\phi_{MN} - 2\partial_{\dot{\alpha}[1}\phi^{A(I}\phi^{J)B}\delta^{\rho}_{2]}\phi_{MN} + 2\delta^{\rho}_{[1}\phi^{B(I}\psi^{J)}_{2]}\delta^{PA}_{MN}\bar{\psi}_{P\dot{\alpha}}$
$\bar{s}^{\dot{\varrho}}_B$	$\bar{q}^A_{\dot{\alpha}}$	$-\frac{i}{2}x^{\varrho\dot{\varrho}}\Big(2\delta^{[A}_B\partial_{\varrho\dot{\alpha}}\psi^{(I]}_{[1}\psi^{J)}_{2]}\phi_{MN} + 2\partial_{\dot{\alpha}[1}\phi^{A(I}f_{2]\varrho}\delta^{J)}_B\phi_{MN} - 2\partial_{\dot{\alpha}[1}\phi^{A(I}\psi^{J)}_{2]}\epsilon_{MNBL}\psi^L_{\varrho} + 2f_{\varrho[1}\psi^{(I}_{2]}\delta^{J)}_B\delta^{PA}_{MN}\bar{\psi}_{P\dot{\alpha}} + \psi^{(I}_{[1}\psi^{J)}_{2]}\delta^A_B\partial_{\varrho\dot{\alpha}}\phi_{MN} - \psi^{(I}_{[1}\psi^{J)}_{2]}\epsilon_{MNBL}\partial_{\varrho\dot{\alpha}}\phi^{AL}\Big) + 2\delta^{\dot{\varrho}}_{\dot{\alpha}}\delta^{[A}_B\psi^{(I]}_{[1}\psi^{J)}_{2]}\phi_{MN} + \psi^{(I}_{[1}\psi^{J)}_{2]}\delta^{\dot{\varrho}}_{\dot{\alpha}}\delta^A_B\phi_{MN} + \psi^{(I}_{[1}\psi^{J)}_{2]}\delta^{\dot{\varrho}}_{\dot{\alpha}}\epsilon_{MNKB}\phi^{AK}$
$s^{B\varrho}$	$q_{A\rho}$	$-\frac{i}{2}x^{\varrho\dot{\alpha}}\Big(2\partial_{\rho\dot{\alpha}}\psi^B_{[1\vert}\delta^{(I}_A\psi^{J)}_{\vert 2]}\phi_{MN} - 2\delta^{(I}_A f_{\rho[1}\partial_{2]\dot{\alpha}}\phi^{J)B}\phi_{MN} - 2\delta^{(I}_A f_{\rho[1}\psi^{J)}_{2]}\delta^{PB}_{MN}\bar{\psi}_{P\dot{\alpha}} + 2\epsilon_{AMNL}\partial_{\dot{\alpha}[1}\phi^{B(I}\psi^{J)}_{2]}\psi^L_{\rho} + \epsilon_{AMNL}\psi^{(I}_{[1}\psi^{J)}_{2]}\partial_{\rho\dot{\alpha}}\phi^{BL}\Big) + 2\delta^{\varrho}_{(\rho}\psi^B_{[1\vert)}\delta^{(I}_A\psi^{J)}_{\vert 2]}\phi_{MN} - 2\delta^{(I}_A\phi^{J)B}f_{\rho[1}\delta^{\varrho}_{2]}\phi_{MN} + 2\delta^{\varrho}_{[1}\phi^{B(I}\psi^{J)}_{2]}\epsilon_{AMNL}\psi^L_{\rho} + \psi^{(I}_{[1}\psi^{J)}_{2]}\epsilon_{AMNL}\delta^{\varrho}_{\rho}\phi^{BL}$
$\bar{s}^{\dot{\varrho}}_B$	$q_{A\rho}$	$-\frac{i}{2}x^{\omega\dot{\varrho}}\Big(2\delta^{(I}_A\delta^{J)}_B f_{\rho[1}f_{2]\varrho}\phi_{MN} - 2\epsilon_{MNBL}f_{\rho[1\vert}\delta^{(I}_A\psi^{J)}_{\vert 2]}\psi^L_{\varrho} + 2\epsilon_{MNAL}f_{\varrho[1\vert}\delta^{(I}_B\psi^{J)}_{\vert 2]}\psi^L_{\rho} + \epsilon_{MNAB}\psi^{(I}_{[1}\psi^{J)}_{2]}f_{\rho\varrho}\Big)$

$\mathcal{Q}_2$	$\mathcal{Q}_1$	$\tilde{\mathcal{Q}}_2\tilde{\mathcal{Q}}_1\Psi^{(IJ)}_{[MN]}$
$\bar{s}^{\dot\rho}_B$	$s^{A\rho}$	$-\frac{1}{4}x^{\varrho\dot\rho}x^{\rho\dot\alpha}\Big(2\delta^{[A}_B\partial_{\varrho\dot\alpha}\psi^{(I]}_{[1}\psi^{J)}_{2]}\phi_{MN} + 2\partial_{\dot\alpha[1}\phi^{A(I}f_{2]\varrho}\delta^{J)}_B\phi_{MN} - 2\partial_{\dot\alpha[1}\phi^{A(I}\psi^{J)}_{2]}\epsilon_{MNBL}\psi^L_\varrho + 2f_{\varrho[1}\psi^{(I}_{2]}\delta^{J)}_B\delta^{PA}_{MN}\bar\psi_{P\dot\alpha} + \psi^{(I}_{[1}\psi^{J)}_{2]}\delta^A_B\partial_{\varrho\dot\alpha}\phi_{MN} - \psi^{(I}_{[1}\psi^{J)}_{2]}\epsilon_{MNBL}\partial_{\varrho\dot\alpha}\phi^{AL}\Big) - \frac{i}{2}x^{\rho\dot\alpha}\Big(2\delta^{\dot\rho}_{\dot\alpha}\delta^{[A}_B\psi^{(I]}_{[1}\psi^{J)}_{2]}\phi_{MN} + \psi^{(I}_{[1}\psi^{J)}_{2]}\delta^{\dot\rho}_{\dot\alpha}\delta^A_B\phi_{MN} + \psi^{(I}_{[1}\psi^{J)}_{2]}\delta^{\dot\rho}_{\dot\alpha}\epsilon_{MNKB}\phi^{AK}\Big) - \frac{i}{2}x^{\omega\dot\rho}\Big(2\delta^\rho_{[1\vert}\delta^{[A}_B\psi^{(I]}_\omega\psi^{J)}_{\vert 2]}\phi_{MN} + 2\phi^{A(I}\delta^{J)}_B\delta^\rho_{[1}f_{2]\omega}\phi_{MN} - 2\delta^\rho_{[1}\phi^{A(I}\psi^{J)}_{2]}\epsilon_{MNBL}\psi^L_\omega\Big)$
$q_{B\varrho}$	$s^{A\rho}$	$-\frac{i}{2}x^{\rho\dot\alpha}\Big(2\delta^{[A}_B\partial_{\varrho\dot\alpha}\psi^{(I]}_{[1}\psi^{J)}_{2]}\phi_{MN} + 2\partial_{\dot\alpha[1}\phi^{A(I}f_{2]\varrho}\delta^{J)}_B\phi_{MN} - 2\partial_{\dot\alpha[1}\phi^{A(I}\psi^{J)}_{2]}\epsilon_{MNBL}\psi^L_\varrho + 2f_{\varrho[1}\psi^{(I}_{2]}\delta^{J)}_B\delta^{PA}_{MN}\bar\psi_{P\dot\alpha} + \psi^{(I}_{[1}\psi^{J)}_{2]}\delta^A_B\partial_{\varrho\dot\alpha}\phi_{MN} - \psi^{(I}_{[1}\psi^{J)}_{2]}\epsilon_{MNBL}\partial_{\varrho\dot\alpha}\phi^{AL}\Big) + 2\delta^\rho_{[1\vert}\delta^{[A}_B\psi^{(I]}_\varrho\psi^{J)}_{\vert 2]}\phi_{MN} + 2\phi^{A(I}\delta^{J)}_B\delta^\rho_{[1}f_{2]\varrho}\phi_{MN} - 2\delta^\rho_{[1}\phi^{A(I}\psi^{J)}_{2]}\epsilon_{MNBL}\psi^L_\varrho$
$q_{B\varrho}$	$\bar{s}^{\dot\rho}_A$	$-\frac{i}{2}x^{\rho\dot\rho}\Big(2\delta^{(I}_A\delta^{J)}_B f_{\rho[1}f_{2]\varrho}\phi_{MN} - 2\epsilon_{MNBL}f_{\rho[1\vert}\delta^{(I}_A\psi^{J)}_{\vert 2]}\psi^L_\varrho + 2\epsilon_{MNAL}f_{\varrho[1\vert}\delta^{(I}_B\psi^{J)}_{\vert 2]}\psi^L_\rho + \epsilon_{MNAB}\psi^{(I}_{[1}\psi^{J)}_{2]}f_{\rho\varrho}\Big)$
$\bar{q}^B_{\dot\beta}$	$s^{A\rho}$	$-\frac{i}{2}x^{\rho\dot\alpha}\Big(2\epsilon^{PBA(I}\partial_{\dot\alpha[1\vert}\bar\psi_{P\dot\beta}\psi^{J)}_{\vert 2]}\phi_{MN} - 2\partial_{\dot\alpha[1}\phi^{A(I}\partial_{2]\dot\beta}\phi^{J)B}\phi_{MN} - 2\partial_{\dot\alpha[1}\phi^{A(I}\psi^{J)}_{2]}\delta^{PB}_{MN}\bar\psi_{P\dot\beta} + \partial_{\dot\beta[1}\phi^{B(I}\psi^{J)}_{2]}\delta^{PA}_{MN}\bar\psi_{P\dot\alpha} + \psi^{(I}_{[1}\psi^{J)}_{2]}\delta^{BA}_{MN}\bar f_{\dot\alpha\dot\beta}\Big) + 2\delta^\rho_{[1\vert}\epsilon^{PBA(I}\bar\psi_{P\dot\beta}\psi^{J)}_{\vert 2]}\phi_{MN} - 2\delta^\rho_{[1}\phi^{A(I}\partial_{2]\dot\beta}\phi^{J)B}\phi_{MN} - 2\delta^\rho_{[1}\phi^{A(I}\psi^{J)}_{2]}\delta^{PB}_{MN}\bar\psi_{P\dot\beta}$
$\bar{q}^B_{\dot\alpha}$	$\bar{s}^{\dot\rho}_A$	$-\frac{i}{2}x^{\rho\dot\rho}\Big(2\partial_{\rho\dot\alpha}\psi^B_{[1\vert}\delta^{(I}_A\psi^{J)}_{\vert 2]}\phi_{MN} - 2\delta^{(I}_A f_{\rho[1}\partial_{2]\dot\alpha}\phi^{J)B}\phi_{MN} - 2\delta^{(I}_A f_{\rho[1}\psi^{J)}_{2]}\delta^{PB}_{MN}\bar\psi_{P\dot\alpha} + 2\epsilon_{AMNL}\partial_{\dot\alpha[1}\phi^{B(I}\psi^{J)}_{2]}\psi^L_\rho + \epsilon_{AMNL}\psi^{(I}_{[1}\psi^{J)}_{2]}\partial_{\rho\dot\alpha}\phi^{BL}\Big)$

$\mathcal{Q}_2$	$\mathcal{Q}_1$	$\tilde{\mathcal{Q}}_2\tilde{\mathcal{Q}}_1\Psi^{(IJ)}_{[MN]}$
$s^{B\theta}$	$s^{A\rho}$	$-\frac{1}{4}x^{\theta\dot\beta}x^{\rho\dot\alpha}\Big(2\epsilon^{PBA(I}\partial_{\dot\alpha[1|}\bar\psi_{P\dot\beta}\psi^{J)}_{\|2]}\phi_{MN} - 2\partial_{\dot\alpha[1}\phi^{A(I}\partial_{2]\dot\beta}\phi^{J)B}\phi_{MN} - 2\partial_{\dot\alpha[1}\phi^{A(I}\psi^{J)}_{2]}\delta^{PB}_{MN}\bar\psi_{P\dot\beta} + \partial_{\dot\beta[1}\phi^{B(I}\psi^{J)}_{2]}\delta^{PA}_{MN}\bar\psi_{P\dot\alpha} + \psi^{(I}_{[1}\psi^{J)}_{2]}\delta^{BA}_{MN}\bar f_{\dot\alpha\dot\beta}\Big) - \frac{i}{2}x^{\rho\dot\alpha}\Big(2\delta^{\theta}_{[1\|}\epsilon^{PBA(I}\bar\psi_{P\dot\alpha}\psi^{J)}_{\|2]}\phi_{MN} - 2\partial_{\dot\alpha[1}\phi^{A(I}\phi^{J)B}\delta^{\theta}_{2]}\phi_{MN} + 2\delta^{\theta}_{[1}\phi^{B(I}\psi^{J)}_{2]}\delta^{PA}_{MN}\bar\psi_{P\dot\alpha}\Big) - \frac{i}{2}x^{\theta\dot\alpha}\Big(2\delta^{\rho}_{[1\|}\epsilon^{PBA(I}\bar\psi_{P\dot\alpha}\psi^{J)}_{\|2]}\phi_{MN} - 2\delta^{\rho}_{[1}\phi^{A(I}\partial_{2]\dot\alpha}\phi^{J)B}\phi_{MN} - 2\delta^{\rho}_{[1}\phi^{A(I}\psi^{J)}_{2]}\delta^{PB}_{MN}\bar\psi_{P\dot\alpha}\Big) - 2\delta^{\rho}_{[1}\delta^{\theta}_{2]}\phi^{A(I}\phi^{J)B}\phi_{MN}$
$s^{B\varrho}$	$\bar s^{\dot\theta}_{A}$	$-\frac{1}{4}x^{\varrho\dot\alpha}x^{\rho\dot\theta}\Big(2\partial_{\rho\dot\alpha}\psi^{B}_{[1\|}\delta^{(I}_{A}\psi^{J)}_{\|2]}\phi_{MN} - 2\delta^{(I}_{A}f_{\rho[1}\partial_{2]\dot\alpha}\phi^{J)B}\phi_{MN} - 2\delta^{(I}_{A}f_{\rho[1}\psi^{J)}_{2]}\delta^{PB}_{MN}\bar\psi_{P\dot\alpha} + 2\epsilon_{AMNL}\partial_{\dot\alpha[1}\phi^{B(I}\psi^{J)}_{2]}\psi^{L}_{\rho} + \epsilon_{AMNL}\psi^{(I}_{[1}\psi^{J)}_{2]}\partial_{\rho\dot\alpha}\phi^{BL}\Big) - \frac{i}{2}x^{\rho\dot\theta}\Big(2\delta^{\varrho}_{(\rho}\psi^{B}_{[1\|)}\delta^{(I}_{A}\psi^{J)}_{\|2]}\phi_{MN} - 2\delta^{(I}_{A}\phi^{J)B}f_{\rho[1}\delta^{\varrho}_{2]}\phi_{MN} + 2\delta^{\varrho}_{[1}\phi^{B(I}\psi^{J)}_{2]}\epsilon_{AMNL}\psi^{L}_{\rho} + \psi^{(I}_{[1}\psi^{J)}_{2]}\epsilon_{AMNL}\delta^{\varrho}_{\rho}\phi^{BL}\Big)$
$\bar s^{\dot\rho}_{B}$	$\bar s^{\dot\theta}_{A}$	$-\frac{1}{4}x^{\varrho\dot\rho}x^{\rho\dot\theta}\Big(2\delta^{(I}_{A}\delta^{J)}_{B}f_{\rho[1}f_{2]\varrho}\phi_{MN} - 2\epsilon_{MNBL}f_{\rho[1\|}\delta^{(I}_{A}\psi^{J)}_{\|2]}\psi^{L}_{\varrho} + 2\epsilon_{MNAL}f_{\varrho[1\|}\delta^{(I}_{B}\psi^{J)}_{\|2]}\psi^{L}_{\rho} + \epsilon_{MNAB}\psi^{(I}_{[1}\psi^{J)}_{2]}f_{\rho\varrho}\Big)$

3.2.2 The action of $\tilde{\mathcal{Q}}_2\tilde{\mathcal{Q}}_1$ on $\Sigma^{[IJ]}_{(MN)}$

We start this section by recalling that $s^{A\rho}\Sigma^{[IJ]}_{(MN)} = 0$. Consequently, one obtains the following straightforward results

$$q_{B\varrho}s^{A\rho}\Sigma^{[IJ]}_{(MN)} = 0 = \bar q^{B}_{\dot\rho}s^{A\rho}\Sigma^{[IJ]}_{(MN)} = s^{B\theta}s^{A\rho}\Sigma^{[IJ]}_{(MN)} = \bar s^{\dot\rho}_{B}s^{A\rho}\Sigma^{[IJ]}_{(MN)}. \quad (3.46)$$

The remaining products in Eq. (3.20) are evaluated below. Besides the relations listed in Eqs. (2.8), (3.7),(3.8), (3.11), and (3.22), the following additional identities are needed in this section

$$\epsilon_{MUVW} c_B \left(c^{\dagger U} c^{\dagger V} c^{\dagger W} |0\rangle \right) = 3\epsilon_{MUVW} \delta^U_B c^{\dagger V} c^{\dagger W} |0\rangle \tag{3.47}$$

$$c_B \left(c^{\dagger A} c^{\dagger U} c^{\dagger V} c^{\dagger W} |0\rangle \right) = \delta^{[A}_B c^{\dagger U]} c^{\dagger V} c^{\dagger W} |0\rangle + c^{\dagger A} c^{\dagger U} c^{\dagger [W} \delta^{V]}_B |0\rangle \tag{3.48}$$

$$\epsilon_{MUVW} c_B \left(c^{\dagger A} c^{\dagger U} c^{\dagger V} c^{\dagger W} |0\rangle \right) = \epsilon_{MUVW} \left(\delta^A_B c^{\dagger U} c^{\dagger V} c^{\dagger W} |0\rangle - 3\delta^U_B c^{\dagger A} c^{\dagger V} c^{\dagger W} |0\rangle \right) \tag{3.49}$$

1. Starting from (3.15), we act with the supersymmetry generator $q_{B\varrho}$. As a result, $q_{B\varrho} q_{A\rho} \Sigma^{[IJ]}_{(MN)}$ in oscillator representation is given by

$$\begin{aligned} q_{B\varrho} q_{A\rho} \Sigma^{[IJ]}_{(MN)} &= \frac{1}{(3!)^2} \epsilon_{MUVW} \epsilon_{NU'V'W'} \\ &\times \Big(6\delta^U_A a^\dagger_\rho b^\dagger_{[\dot{1}} a^\dagger_\varrho \delta^{[V}_B c^{\dagger W]} |0\rangle \bullet b^\dagger_{\dot{2}]} c^{\dagger U'} c^{\dagger V'} c^{\dagger W'} |0\rangle \bullet c^{\dagger I} c^{\dagger J} |0\rangle \\ &+ 6 \cdot 3\delta^U_A a^\dagger_\rho b^\dagger_{[\dot{1}} c^{\dagger V} c^{\dagger W} |0\rangle \bullet a^\dagger_\varrho b^\dagger_{\dot{2}]} \delta^{U'}_B c^{\dagger V'} c^{\dagger W'} |0\rangle \bullet c^{\dagger I} c^{\dagger J} |0\rangle \\ &- 6\delta^U_A a^\dagger_\rho b^\dagger_{[\dot{1}} c^{\dagger V} c^{\dagger W} |0\rangle \bullet b^\dagger_{\dot{2}]} c^{\dagger U'} c^{\dagger V'} c^{\dagger W'} |0\rangle \bullet \delta^{[I}_B a^\dagger_\varrho c^{\dagger J]} |0\rangle \\ &+ 6 a^\dagger_\varrho b^\dagger_{[\dot{1}} \delta^U_B c^{\dagger V} c^{\dagger W} |0\rangle \bullet b^\dagger_{\dot{2}]} c^{\dagger U'} c^{\dagger V'} c^{\dagger W'} |0\rangle \bullet \delta^{[I}_A a^\dagger_\rho c^{\dagger J]} |0\rangle \\ &+ b^\dagger_{[\dot{1}} c^{\dagger U} c^{\dagger V} c^{\dagger W} |0\rangle \bullet b^\dagger_{\dot{2}]} c^{\dagger U'} c^{\dagger V'} c^{\dagger W'} |0\rangle \bullet \delta^{[I}_A \delta^{J]}_B a^\dagger_\varrho a^\dagger_\rho |0\rangle \Big). \end{aligned} \tag{3.50}$$

Translating into the field-theoretic representation, result in Eq. (3.50) reads

$$\begin{aligned} q_{B\varrho} q_{A\rho} \Sigma^{[IJ]}_{(MN)} &= -2\partial_{\rho[\dot{1}|} \psi^W_\varrho \epsilon_{ABW(M} \bar{\psi}_{N)|\dot{2}]} \phi^{IJ} - 2\partial_{\rho[\dot{1}|} \phi_{A(M} \partial_{\varrho|\dot{2}]} \phi_{N)B} \phi^{IJ} \\ &+ 2\partial_{\rho[\dot{1}|} \phi_{A(M} \bar{\psi}_{N)|\dot{2}]} \delta^{[I}_B \psi^{J]}_\varrho - 2\partial_{\varrho[\dot{1}|} \phi_{B(M} \bar{\psi}_{N)|\dot{2}]} \delta^{[I}_A \psi^{J]}_\rho \\ &+ \bar{\psi}_{[\dot{1}|(M} \bar{\psi}_{N)|\dot{2}]} f_{\varrho\rho} \delta^{[I}_A \delta^{J]}_B. \end{aligned} \tag{3.51}$$

2. Starting from the expression in Eq. (3.16), we act with $\bar{q}^B_{\dot{\varrho}}$. This yields $\bar{q}\bar{q}\Sigma$,

which explicitly reads

$$
\begin{aligned}
\bar{q}^B_{\dot{\varrho}}\bar{q}^A_{\dot{\rho}}\Sigma^{[IJ]}_{(MN)} &= \frac{1}{(3!)^2}\epsilon_{MUVW}\epsilon_{NU'V'W'} \\
&\times \Big(2b^\dagger_{\dot{\rho}} b^\dagger_{[\dot{1}} c^{\dagger A} c^{\dagger U} c^{\dagger V} c^{\dagger W}\,|0\rangle \bullet b^\dagger_{\dot{2}]} b^\dagger_{\dot{\varrho}} c^{\dagger B} c^{\dagger U'} c^{\dagger V'} c^{\dagger W'}\,|0\rangle \bullet c^{\dagger I} c^{\dagger J}\,|0\rangle \\
&- 2b^\dagger_{\dot{\rho}} b^\dagger_{[\dot{1}} c^{\dagger A} c^{\dagger U} c^{\dagger V} c^{\dagger W}\,|0\rangle \bullet b^\dagger_{\dot{2}]} c^{\dagger U'} c^{\dagger V'} c^{\dagger W'}\,|0\rangle \bullet b^\dagger_{\dot{\varrho}} c^{\dagger B} c^{\dagger [I} c^{\dagger J]}\,|0\rangle \\
&+ 2b^\dagger_{\dot{\varrho}} b^\dagger_{[\dot{1}} c^{\dagger B} c^{\dagger U} c^{\dagger V} c^{\dagger W}\,|0\rangle \bullet b^\dagger_{\dot{2}]} c^{\dagger U'} c^{\dagger V'} c^{\dagger W'}\,|0\rangle \bullet b^\dagger_{\dot{\rho}} c^{\dagger A} c^{\dagger I} c^{\dagger J}\,|0\rangle \\
&+ b^\dagger_{[\dot{1}} c^{\dagger U} c^{\dagger V} c^{\dagger W}\,|0\rangle \bullet b^\dagger_{\dot{2}]} c^{\dagger U'} c^{\dagger V'} c^{\dagger W'}\,|0\rangle \bullet b^\dagger_{\dot{\rho}} b^\dagger_{\dot{\varrho}} c^{\dagger B} c^{\dagger A} c^{\dagger I} c^{\dagger J}\,|0\rangle \Big).
\end{aligned}
\tag{3.52}
$$

Passing to the field-theoretic representation, the result obtained in (3.52) takes the form

$$
\begin{aligned}
\bar{q}^B_{\dot{\varrho}}\bar{q}^A_{\dot{\rho}}\Sigma^{[IJ]}_{(MN)} &= 2\delta^A_{(M}\delta^B_{N)}\bar{f}_{\dot{\rho}[\dot{1}}\bar{f}_{\dot{2}]\dot{\varrho}}\phi^{IJ} \;-\; 2\epsilon^{PBIJ}\bar{f}_{\dot{\rho}[\dot{1}|}\delta^A_{(M}\bar{\psi}_{N)|\dot{2}]}\bar{\psi}_{P\dot{\varrho}} \\
&+ 2\epsilon^{PAIJ}\bar{f}_{\dot{\varrho}[\dot{1}|}\delta^B_{(M}\bar{\psi}_{N)|\dot{2}]}\bar{\psi}_{\dot{\rho}P} \;+\; \bar{\psi}_{[\dot{1}|(M}\bar{\psi}_{N)|\dot{2}]}\epsilon^{BAIJ}\bar{f}_{\dot{\rho}\dot{\varrho}}.
\end{aligned}
\tag{3.53}
$$

3. Acting with generator $\bar{s}^{\dot{\rho}}_B$ on (3.17). The resulting expression $\bar{s}^{\dot{\rho}}_B\bar{s}^{\dot{\varrho}}_A\Sigma^{[IJ]}_{(MN)}$ reads

$$
\begin{aligned}
\bar{s}^{\dot{\rho}}_B\bar{s}^{\dot{\varrho}}_A\Sigma^{[IJ]}_{(MN)} &= \frac{1}{2}\epsilon_{MAVW}\epsilon_{NBV'W'} \\
&\times \Big(\delta^{\dot{\varrho}}_{[\dot{1}} c^{\dagger V} c^{\dagger W}\,|0\rangle \bullet \delta^{\dot{\rho}}_{\dot{2}]} c^{\dagger V'} c^{\dagger W'}\,|0\rangle \bullet c^{\dagger I} c^{\dagger J}\,|0\rangle \Big).
\end{aligned}
\tag{3.54}
$$

The field-theoretic form of Eq. (3.54) is

$$
\bar{s}^{\dot{\rho}}_B\bar{s}^{\dot{\varrho}}_A\Sigma^{[IJ]}_{(MN)} \;=\; -2\delta^{\dot{\varrho}}_{[\dot{1}}\delta^{\dot{\rho}}_{\dot{2}]}\phi_{A(M}\phi_{N)B}\phi^{IJ}. \tag{3.55}
$$

4. We start from the relation in Eq. (3.16) and act with $q_{B\varrho}$. This leads to the

expression $q_{B\varrho}\bar{q}^A_{\dot\rho}\Sigma^{[IJ]}_{(MN)}$, explicitly

$$\begin{aligned}
q_{B\varrho}\bar{q}^A_{\dot\rho}\Sigma^{[IJ]}_{(MN)} &= \frac{1}{(3!)^2}\epsilon_{MUVW}\epsilon_{NU'V'W'} \\
&\times \Big(2\delta^A_B a^\dagger_\varrho b^\dagger_{\dot\rho} b^\dagger_{[\dot 1} c^{\dagger U} c^{\dagger V} c^{\dagger W}\,|0\rangle \bullet b^\dagger_{\dot 2]} c^{\dagger U'} c^{\dagger V'} c^{\dagger W'}\,|0\rangle \bullet c^{\dagger I} c^{\dagger J}\,|0\rangle \\
&- 6\delta^U_B a^\dagger_\varrho b^\dagger_{\dot\rho} b^\dagger_{[\dot 1} c^{\dagger A} c^{\dagger V} c^{\dagger W}\,|0\rangle \bullet b^\dagger_{\dot 2]} c^{\dagger U'} c^{\dagger V'} c^{\dagger W'}\,|0\rangle \bullet c^{\dagger I} c^{\dagger J}\,|0\rangle \\
&+ 6 b^\dagger_{\dot\rho} b^\dagger_{[\dot 1} c^{\dagger A} c^{\dagger U} c^{\dagger V} c^{\dagger W}\,|0\rangle \bullet b^\dagger_{\dot 2]} a^\dagger_\varrho \delta^{U'}_B c^{\dagger V'} c^{\dagger W'}\,|0\rangle \bullet c^{\dagger I} c^{\dagger J}\,|0\rangle \\
&- 2 b^\dagger_{\dot\rho} b^\dagger_{[\dot 1} c^{\dagger A} c^{\dagger U} c^{\dagger V} c^{\dagger W}\,|0\rangle \bullet b^\dagger_{\dot 2]} c^{\dagger U'} c^{\dagger V'} c^{\dagger W'}\,|0\rangle \bullet a^\dagger_\varrho \delta^{[I}_B c^{\dagger J]}\,|0\rangle \\
&+ 6 a^\dagger_\varrho b^\dagger_{[\dot 1}\delta^U_B c^{\dagger V} c^{\dagger W}\,|0\rangle \bullet b^\dagger_{\dot 2]} c^{\dagger U'} c^{\dagger V'} c^{\dagger W'}\,|0\rangle \bullet b^\dagger_{\dot\rho} c^{\dagger A} c^{\dagger I} c^{\dagger J}\,|0\rangle \\
&+ b^\dagger_{[\dot 1} c^{\dagger U} c^{\dagger V} c^{\dagger W}\,|0\rangle \bullet b^\dagger_{\dot 2]} c^{\dagger U'} c^{\dagger V'} c^{\dagger W'}\,|0\rangle \bullet a^\dagger_\varrho b^\dagger_{\dot\rho}\delta^A_B c^{\dagger I} c^{\dagger J}\,|0\rangle \\
&+ b^\dagger_{[\dot 1} c^{\dagger U} c^{\dagger V} c^{\dagger W}\,|0\rangle \bullet b^\dagger_{\dot 2]} c^{\dagger U'} c^{\dagger V'} c^{\dagger W'}\,|0\rangle \bullet a^\dagger_\varrho b^\dagger_{\dot\rho} c^{\dagger A} c^{\dagger [I}\delta^{J]}_B\,|0\rangle\Big).
\end{aligned} \tag{3.56}$$

In the field-theoretic formulation, the oscillator expression in Eq. (3.56) reads

$$\begin{aligned}
q_{B\varrho}\bar{q}^A_{\dot\rho}\Sigma^{[IJ]}_{(MN)} &= 2\delta^A_B\partial_{\varrho\dot\rho}\bar\psi_{[\dot 1|(M}\bar\psi_{N)|\dot 2]}\phi^{IJ} + 2\partial_{\varrho\dot\rho}\bar\psi_{[\dot 1|P}\delta^{PA}_{B(M}\bar\psi_{N)|\dot 2]}\phi^{IJ} \\
&+ 2\delta^A_{(M}\bar f_{\dot\rho[\dot 1}\partial_{\dot 2]\varrho}\phi_{N)B}\phi^{IJ} - 2\delta^A_{(M}\bar f_{\dot\rho[\dot 1|}\bar\psi_{N)|\dot 2]}\delta^{[I}_B\psi^{J]}_\varrho \\
&- 2\partial_{\varrho[\dot 1|}\phi_{B(M}\bar\psi_{N)|\dot 2]}\epsilon^{PAIJ}\bar\psi_{P\dot\rho} + \bar\psi_{[\dot 1|(M}\bar\psi_{N)|\dot 2]}\delta^A_B\partial_{\varrho\dot\rho}\phi^{IJ} \\
&+ \bar\psi_{[\dot 1|(M}\bar\psi_{N)|\dot 2]}\partial_{\varrho\dot\rho}\phi^{A[I}\delta^{J]}_B.
\end{aligned} \tag{3.57}$$

5. We proceed by acting with $\bar{q}^B_{\dot\rho}$ on Eq. (3.15). This operation produces $\bar{q}^B_{\dot\rho} q_{A\rho}\Sigma^{[IJ]}_{(MN)}$, whose oscillator representation is

$$\begin{aligned}
\bar{q}^B_{\dot\rho} q_{A\rho}\Sigma^{[IJ]}_{(MN)} &= \frac{1}{(3!)^2}\epsilon_{MUVW}\epsilon_{NU'V'W'} \\
&\times \Big(6\delta^U_A a^\dagger_\rho b^\dagger_{\dot\rho} b^\dagger_{[\dot 1} c^{\dagger B} c^{\dagger V} c^{\dagger W}\,|0\rangle \bullet b^\dagger_{\dot 2]} c^{\dagger U'} c^{\dagger V'} c^{\dagger W'}\,|0\rangle \bullet c^{\dagger I} c^{\dagger J}\,|0\rangle \\
&+ 6\delta^U_A a^\dagger_\rho b^\dagger_{[\dot 1} c^{\dagger V} c^{\dagger W}\,|0\rangle \bullet b^\dagger_{\dot 2]} b^\dagger_{\dot\rho} c^{\dagger B} c^{\dagger U'} c^{\dagger V'} c^{\dagger W'}\,|0\rangle \bullet c^{\dagger I} c^{\dagger J}\,|0\rangle \\
&- 6\delta^U_A a^\dagger_\rho b^\dagger_{[\dot 1} c^{\dagger V} c^{\dagger W}\,|0\rangle \bullet b^\dagger_{\dot 2]} c^{\dagger U'} c^{\dagger V'} c^{\dagger W'}\,|0\rangle \bullet b^\dagger_{\dot\rho} c^{\dagger B} c^{\dagger I} c^{\dagger J}\,|0\rangle \\
&+ 2 b^\dagger_{\dot\rho} b^\dagger_{[\dot 1} c^{\dagger B} c^{\dagger U} c^{\dagger V} c^{\dagger W}\,|0\rangle \bullet b^\dagger_{\dot 2]} c^{\dagger U'} c^{\dagger V'} c^{\dagger W'}\,|0\rangle \bullet \delta^{[I}_A a^\dagger_\rho c^{\dagger J]}\,|0\rangle \\
&+ b^\dagger_{[\dot 1} c^{\dagger U} c^{\dagger V} c^{\dagger W}\,|0\rangle \bullet b^\dagger_{\dot 2]} c^{\dagger U'} c^{\dagger V'} c^{\dagger W'}\,|0\rangle \bullet a^\dagger_\rho b^\dagger_{\dot\rho} c^{\dagger B}\delta^{[I}_A c^{\dagger J]}\,|0\rangle\Big).
\end{aligned} \tag{3.58}$$

When rewritten in the field-theory language, the expression in Eq. (3.58) becomes

$$\begin{aligned}\bar{q}^{B}_{\dot\rho} q_{A\rho}\Sigma^{[IJ]}_{(MN)} =& -2\partial_{\rho\dot\rho}\bar\psi_{P[\dot 1|}\delta^{PB}_{A(M}\bar\psi_{N)|\dot 2]}\phi^{IJ} - 2\partial_{\rho[\dot 1|}\phi_{A(M}\bar f_{|\dot 2]\dot\rho}\delta^{B}_{N)}\phi^{IJ}\\ &+ 2\partial_{\rho[\dot 1|}\phi_{A(M}\bar\psi_{N)|\dot 2]}\bar\psi_{Q\dot\rho}\epsilon^{QBIJ} + 2\bar f_{\dot\rho[\dot 1|}\delta^{B}_{(M}\bar\psi_{N)|\dot 2]}\delta^{[I}_{A}\psi^{J]}_{\rho}\\ &+ \bar\psi_{[\dot 1|(M}\bar\psi_{N)|\dot 2]}\partial_{\rho\dot\rho}\phi^{B[J}\delta^{I]}_{A}. \end{aligned} \tag{3.59}$$

6. We proceed by acting with $s^{B\rho}$ on Eq. (3.16). This operation produces $s^{B\rho}\bar{q}^{A}_{\dot\rho}\Sigma^{[IJ]}_{(MN)}$, whose oscillator representation is

$$s^{B\varrho}\bar{q}^{A}_{\dot\rho}\Sigma^{[IJ]}_{(MN)} = 0, \tag{3.60}$$

this result arises because (3.16) contains no $a^\dagger$ oscillators.

7. Starting from Eq. (3.16), the action of the generator $\bar{s}^{\dot\varrho}_{B}$ yields $\bar{s}^{\dot\varrho}_{B}\bar{q}^{A}_{\dot\rho}\Sigma^{[IJ]}_{(MN)}$, whose oscillator representation is given by

$$\begin{aligned}\bar{s}^{\dot\varrho}_{B}\bar{q}^{A}_{\dot\rho}\Sigma^{[IJ]}_{(MN)} =& \frac{1}{(3!)^2}\epsilon_{MUVW}\epsilon_{NU'V'W'}\\ &\times\Big(2\delta^{\dot\varrho}_{(\dot\rho}b^{\dagger}_{[\dot 1)}\delta^{A}_{B}c^{\dagger U}c^{\dagger V}c^{\dagger W}|0\rangle \bullet b^{\dagger}_{\dot 2]}c^{\dagger U'}c^{\dagger V'}c^{\dagger W'}|0\rangle \bullet c^{\dagger I}c^{\dagger J}|0\rangle\\ &- 6\delta^{\dot\varrho}_{(\dot\rho}b^{\dagger}_{\dot 1)}\delta^{U}_{B}c^{\dagger A}c^{\dagger V}c^{\dagger W}|0\rangle \bullet b^{\dagger}_{\dot 2]}c^{\dagger U'}c^{\dagger V'}c^{\dagger W'}|0\rangle \bullet c^{\dagger I}c^{\dagger J}|0\rangle\\ &+ 6b^{\dagger}_{\dot\rho}b^{\dagger}_{[\dot 1}c^{\dagger A}c^{\dagger U}c^{\dagger V}c^{\dagger W}|0\rangle \bullet \delta^{\dot\varrho}_{\dot 2]}\delta^{U'}_{B}c^{\dagger V'}c^{\dagger W'}|0\rangle \bullet c^{\dagger I}c^{\dagger J}|0\rangle\\ &+ 6\delta^{\dot\varrho}_{[\dot 1}\delta^{U}_{B}c^{\dagger V}c^{\dagger W}|0\rangle \bullet b^{\dagger}_{\dot 2]}c^{\dagger U'}c^{\dagger V'}c^{\dagger W'}|0\rangle \bullet b^{\dagger}_{\dot\rho}c^{\dagger A}c^{\dagger I}c^{\dagger J}|0\rangle\\ &+ b^{\dagger}_{[\dot 1}c^{\dagger U}c^{\dagger V}c^{\dagger W}|0\rangle \bullet b^{\dagger}_{\dot 2]}c^{\dagger U'}c^{\dagger V'}c^{\dagger W'}|0\rangle \bullet \delta^{\dot\varrho}_{\dot\rho}\delta^{A}_{B}c^{\dagger I}c^{\dagger J}|0\rangle\\ &+ b^{\dagger}_{[\dot 1}c^{\dagger U}c^{\dagger V}c^{\dagger W}|0\rangle \bullet b^{\dagger}_{\dot 2]}c^{\dagger U'}c^{\dagger V'}c^{\dagger W'}|0\rangle \bullet \delta^{\dot\varrho}_{\dot\rho}c^{\dagger A}c^{\dagger[I}\delta^{J]}_{B}|0\rangle\Big), \end{aligned} \tag{3.61}$$

where the factor $\delta^{\dot\varrho}_{(\dot\rho}b^{\dagger}_{[\dot 1)}$ originates from the action

$$b^{\dot\varrho}\left(b^{\dagger}_{\dot\rho}b^{\dagger}_{[\dot 1}|0\rangle\right) = \delta^{\dot\varrho}_{(\dot\rho}b^{\dagger}_{[\dot 1)}|0\rangle. \tag{3.62}$$

The field-theoretic representation of Eq. (3.61) is

$$\begin{aligned}
\bar{s}^{\dot{\varrho}}_{B}\bar{q}^{A}_{\dot{\rho}}\Sigma^{[IJ]}_{(MN)} &= 2\delta^{A}_{B}\delta^{\dot{\varrho}}_{(\dot{\rho}}\bar{\psi}_{[\dot{1})|(M}\bar{\psi}_{N)|\dot{2}]}\phi^{IJ} + 2\delta^{\dot{\varrho}}_{(\dot{\rho}}\bar{\psi}_{[\dot{1})|P}\delta^{PA}_{B(M}\bar{\psi}_{N)|\dot{2}]}\phi^{IJ} \\
&+ 2\bar{f}_{\dot{\rho}[\dot{1}}\delta^{\dot{\varrho}}_{\dot{2}]}\delta^{A}_{(M}\phi_{N)B}\phi^{IJ} \;-\; 2\delta^{\dot{\varrho}}_{[\dot{1}|}\phi_{B(M}\bar{\psi}_{N)|\dot{2}]}\epsilon^{PAIJ}\bar{\psi}_{\dot{\rho}P} \\
&+ \bar{\psi}_{[\dot{1}|(M}\bar{\psi}_{N)|\dot{2}]}\delta^{\dot{\varrho}}_{\dot{\rho}}\delta^{A}_{B}\phi^{IJ} \;+\; \bar{\psi}_{[\dot{1}|(M}\bar{\psi}_{N)|\dot{2}]}\delta^{\dot{\varrho}}_{\dot{\rho}}\phi^{A[I}\delta^{J]}_{B}. \qquad (3.63)
\end{aligned}$$

8. From Eq. (3.15), the action of the generator $s^{B\varrho}$ leads to $s^{B\varrho}q_{A\rho}\Sigma^{[IJ]}_{(MN)}$, whose oscillator realization is

$$\begin{aligned}
s^{B\varrho}q_{A\rho}\Sigma^{[IJ]}_{(MN)} &= \frac{1}{(3!)^2}\epsilon_{MUVW}\epsilon_{NU'V'W'} \\
&\times \Big(6\delta^{U}_{A}\delta^{\varrho}_{\rho}b^{\dagger}_{[1}c^{\dagger B}c^{\dagger V}c^{\dagger W}\left|0\right\rangle \bullet b^{\dagger}_{2]}c^{\dagger U'}c^{\dagger V'}c^{\dagger W'}\left|0\right\rangle \bullet c^{\dagger I}c^{\dagger J}\left|0\right\rangle \\
&+ b^{\dagger}_{[1}c^{\dagger U}c^{\dagger V}c^{\dagger W}\left|0\right\rangle \bullet b^{\dagger}_{2]}c^{\dagger U'}c^{\dagger V'}c^{\dagger W'}\left|0\right\rangle \bullet \delta^{\varrho}_{\rho}c^{\dagger B}\delta^{[I}_{A}c^{\dagger J]}\left|0\right\rangle\Big).
\end{aligned} \qquad (3.64)$$

The field-theoretic realization of the result in Eq. (3.64) is

$$s^{B\varrho}q_{A\rho}\Sigma^{[IJ]}_{(MN)} = -\,2\delta^{\varrho}_{\rho}\bar{\psi}_{P[\dot{1}|}\delta^{PB}_{A(M}\bar{\psi}_{N)|\dot{2}]}\phi^{IJ} - \delta^{\varrho}_{\rho}\bar{\psi}_{[\dot{1}|(M}\bar{\psi}_{N)|\dot{2}]}\phi^{B[I}\delta^{J]}_{A}. \quad (3.65)$$

9. Considering the expression in (3.15), acting with $\bar{s}^{\dot{\varrho}}_{B}$ produces $\bar{s}^{\dot{\varrho}}_{B}q_{A\rho}\Sigma^{[IJ]}_{(MN)}$. Its oscillator realization is

$$\begin{aligned}
\bar{s}^{\dot{\varrho}}_{B}q_{A\rho}\Sigma^{[IJ]}_{(MN)} &= \frac{1}{(3!)^2}\epsilon_{MUVW}\epsilon_{NU'V'W'} \\
&\times \Big(12\delta^{U}_{A}a^{\dagger}_{\rho}\delta^{\dot{\varrho}}_{[\dot{1}}\delta^{V}_{B}c^{\dagger W}\left|0\right\rangle \bullet b^{\dagger}_{\dot{2}]}c^{\dagger U'}c^{\dagger V'}c^{\dagger W'}\left|0\right\rangle \bullet c^{\dagger I}c^{\dagger J}\left|0\right\rangle \\
&+ 18\delta^{U}_{A}a^{\dagger}_{\rho}b^{\dagger}_{[\dot{1}}c^{\dagger V}c^{\dagger W}\left|0\right\rangle \bullet \delta^{\dot{\varrho}}_{\dot{2}]}\delta^{U'}_{B}c^{\dagger V'}c^{\dagger W'}\left|0\right\rangle \bullet c^{\dagger I}c^{\dagger J}\left|0\right\rangle \\
&+ 6\delta^{\dot{\varrho}}_{[\dot{1}}\delta^{U}_{B}c^{\dagger V}c^{\dagger W}\left|0\right\rangle \bullet b^{\dagger}_{\dot{2}]}c^{\dagger U'}c^{\dagger V'}c^{\dagger W'}\left|0\right\rangle \bullet \delta^{[I}_{A}a^{\dagger}_{\rho}c^{\dagger J]}\left|0\right\rangle\Big) \qquad (3.66)
\end{aligned}$$

The field-theoretic realization of the result in Eq. (3.66) reads

$$\begin{aligned}
\bar{s}^{\dot{\varrho}}_{B}q_{A\rho}\Sigma^{[IJ]}_{(MN)} &= 2\delta^{\dot{\varrho}}_{[\dot{1}|}\psi^{W}_{\rho}\epsilon_{(M|ABW}\bar{\psi}_{|N)|\dot{2}]}\phi^{IJ} \;-\; 2\partial_{\rho[\dot{1}|}\phi_{A(M}\phi_{N)B}\delta^{\dot{\varrho}}_{|\dot{2}]}\phi^{IJ} \\
&- 2\delta^{\dot{\varrho}}_{[\dot{1}|}\phi_{B(M}\bar{\psi}_{N)|\dot{2}]}\delta^{[I}_{A}\psi^{J]}_{\rho}
\end{aligned} \qquad (3.67)$$

10. Considering the expression in (3.17), acting with $q_{B\varrho}$ produces $q_{B\varrho}\bar{s}^{\dot{\varrho}}_{A}\Sigma^{[IJ]}_{(MN)}$.

Its oscillator realization is

$$\begin{aligned} q_{B\varrho}\bar{s}^{\dot{\varrho}}_{A}\Sigma^{[IJ]}_{(MN)} &= \frac{1}{3!}\epsilon_{MUVW}\epsilon_{NU'V'W'} \\ &\times \Big(2\delta^{U}_{A}\delta^{\dot{\varrho}}_{[\dot{1}}a^{\dagger}_{\varrho}\delta^{V}_{B}c^{\dagger W}\,|0\rangle \bullet b^{\dagger}_{\dot{2}]}c^{\dagger U'}c^{\dagger V'}c^{\dagger W'}\,|0\rangle \bullet c^{\dagger I}c^{\dagger J}\,|0\rangle \\ &+ 3\delta^{U}_{A}\delta^{\dot{\varrho}}_{[\dot{1}}c^{\dagger V}c^{\dagger W}\,|0\rangle \bullet b^{\dagger}_{\dot{2}]}a^{\dagger}_{\varrho}\delta^{U'}_{B}c^{\dagger V'}c^{\dagger W'}\,|0\rangle \bullet c^{\dagger I}c^{\dagger J}\,|0\rangle \\ &- \delta^{U}_{A}\delta^{\dot{\varrho}}_{[\dot{1}}c^{\dagger V}c^{\dagger W}\,|0\rangle \bullet b^{\dagger}_{\dot{2}]}c^{\dagger U'}c^{\dagger V'}c^{\dagger W'}\,|0\rangle \bullet a^{\dagger}_{\varrho}\delta^{[I}_{B}c^{\dagger J]}\,|0\rangle \Big). \end{aligned} \tag{3.68}$$

In the field-theoretic formulation, Eq. (3.68) reads

$$\begin{aligned} q_{B\varrho}\bar{s}^{\dot{\varrho}}_{A}\Sigma^{[IJ]}_{(MN)} &= 2\psi^{W}_{\varrho}\delta^{\dot{\varrho}}_{[\dot{1}}\bar{\psi}_{\dot{2}](N}\epsilon_{M)ABW}\phi^{IJ} \;-\; 2\delta^{\dot{\varrho}}_{[\dot{1}|}\phi_{A(M|}\partial_{|\dot{2}]\varrho}\phi_{|N)B}\phi^{IJ} \\ &+ 2\delta^{\dot{\varrho}}_{[\dot{1}|}\phi_{A(M}\bar{\psi}_{N)|\dot{2}]}\delta^{[I}_{B}\psi^{J]}_{\varrho}. \end{aligned} \tag{3.69}$$

11. From the expression given in Eq. (3.17), acting with $\bar{q}^{B}_{\dot{\rho}}$ yields $\bar{q}^{B}_{\dot{\rho}}\bar{s}^{\dot{\varrho}}_{A}\Sigma^{[IJ]}_{(MN)}$. In the oscillator representation, this is

$$\begin{aligned} \bar{q}^{B}_{\dot{\rho}}\bar{s}^{\dot{\varrho}}_{A}\Sigma^{[IJ]}_{(MN)} &= \frac{1}{(3!)^2}6\epsilon_{MUVW}\epsilon_{NU'V'W'} \\ &\times \Big(\delta^{U}_{A}b^{\dagger}_{\dot{\rho}}\delta^{\dot{\varrho}}_{[\dot{1}}c^{\dagger B}c^{\dagger V}c^{\dagger W}\,|0\rangle \bullet b^{\dagger}_{\dot{2}]}c^{\dagger U'}c^{\dagger V'}c^{\dagger W'}\,|0\rangle \bullet c^{\dagger I}c^{\dagger J}\,|0\rangle \\ &+ \delta^{U}_{A}\delta^{\dot{\varrho}}_{[\dot{1}}c^{\dagger V}c^{\dagger W}\,|0\rangle \bullet b^{\dagger}_{\dot{2}]}b^{\dagger}_{\dot{\rho}}c^{\dagger B}c^{\dagger U'}c^{\dagger V'}c^{\dagger W'}\,|0\rangle \bullet c^{\dagger I}c^{\dagger J}\,|0\rangle \\ &- \delta^{U}_{A}\delta^{\dot{\varrho}}_{[\dot{1}}c^{\dagger V}c^{\dagger W}\,|0\rangle \bullet b^{\dagger}_{\dot{2}]}c^{\dagger U'}c^{\dagger V'}c^{\dagger W'}\,|0\rangle \bullet b^{\dagger}_{\dot{\rho}}c^{\dagger B}c^{\dagger I}c^{\dagger J}\,|0\rangle \Big) \end{aligned} \tag{3.70}$$

The corresponding field-theoretic expression associated with Eq. (3.70) is given by

$$\begin{aligned} \bar{q}^{B}_{\dot{\rho}}\bar{s}^{\dot{\varrho}}_{A}\Sigma^{[IJ]}_{(MN)} &= -\,2\bar{\psi}_{P\dot{\rho}}\delta^{PB}_{A(M}\bar{\psi}_{N)[\dot{2}}\delta^{\dot{\varrho}}_{\dot{1}]}\phi^{IJ} - 2\phi_{A(M}\delta^{B}_{N)}\delta^{\dot{\varrho}}_{[\dot{1}}f_{\dot{2}]\dot{\rho}}\phi^{IJ} \\ &+ 2\phi_{A(M}\bar{\psi}_{N)[\dot{2}}\delta^{\dot{\varrho}}_{\dot{1}]}\bar{\psi}_{P\dot{\rho}}\epsilon^{PBIJ} \end{aligned} \tag{3.71}$$

12. We conclude by actingr with $s^{B\rho}$ on Eq. (3.17), obtaining $s^{B\rho}\bar{s}^{\dot{\varrho}}_{A}\Sigma^{[IJ]}_{(MN)}$, which in oscillator representation is given by

$$s^{B\rho}\bar{s}^{\dot{\varrho}}_{A}\Sigma^{[IJ]}_{(MN)} = 0, \tag{3.72}$$

this result arises since no $a^{\dagger}$ oscillators appear in the expression (3.17).

Employing the operator combinations of Table 3.4, we evaluate their action on $\Sigma^{[IJ]}_{(MN)}$. The resulting expressions, given in Eqs. (3.51), (3.53), (3.55), (3.57), (3.59), (3.60), (3.63), (3.65), (3.67), (3.69), (3.71), and (3.72), are compiled in Table 3.6.

Table 3.6: The action of $\tilde{\mathcal{Q}}_2\tilde{\mathcal{Q}}_1$ on $\Sigma^{[IJ]}_{(MN)}$

$\mathcal{Q}_2$	$\mathcal{Q}_1$	$\tilde{\mathcal{Q}}_2\tilde{\mathcal{Q}}_1\Sigma^{[IJ]}_{(MN)}$
$q_{B\varrho}$	$q_{A\rho}$	$-2\partial_{\rho[\dot{1}\vert}\psi^W_\varrho\epsilon_{ABW(M}\bar{\psi}_{N)\vert\dot{2}]}\phi^{IJ} - 2\partial_{\rho[\dot{1}\vert}\phi_{A(M}\partial_{\varrho\vert\dot{2}]}\phi_{N)B}\phi^{IJ} + 2\partial_{\rho[\dot{1}\vert}\phi_{A(M}\bar{\psi}_{N)\vert\dot{2}]}\delta^{[I}_B\psi^{J]}_\varrho - 2\partial_{\varrho[\dot{1}\vert}\phi_{B(M}\bar{\psi}_{N)\vert\dot{2}]}\delta^{[I}_A\psi^{J]}_\rho + \bar{\psi}_{[\dot{1}\vert(M}\bar{\psi}_{N)\vert\dot{2}]}f_{\varrho\rho}\delta^{[I}_A\delta^{J]}_B$
$q_{B\varrho}$	$\bar{q}^A_{\dot{\rho}}$	$2\delta^A_B\partial_{\varrho\dot{\rho}}\bar{\psi}_{[\dot{1}\vert(M}\bar{\psi}_{N)\vert\dot{2}]}\phi^{IJ} + 2\partial_{\varrho\dot{\rho}}\bar{\psi}_{[\dot{1}\vert P}\delta^{PA}_{B(M}\bar{\psi}_{N)\vert\dot{2}]}\phi^{IJ} + 2\delta^A_{(M}\bar{f}_{\dot{\rho}[\dot{1}}\partial_{\dot{2}]\varrho}\phi_{N)B}\phi^{IJ} - 2\delta^A_{(M}\bar{f}_{\dot{\rho}[\dot{1}\vert}\bar{\psi}_{N)\vert\dot{2}]}\delta^{[I}_B\psi^{J]}_\varrho - 2\partial_{\varrho[\dot{1}\vert}\phi_{B(M}\bar{\psi}_{N)\vert\dot{2}]}\epsilon^{PAIJ}\bar{\psi}_{P\dot{\rho}} + \bar{\psi}_{[\dot{1}\vert(M}\bar{\psi}_{N)\vert\dot{2}]}\delta^A_B\partial_{\varrho\dot{\rho}}\phi^{IJ} + \bar{\psi}_{[\dot{1}\vert(M}\bar{\psi}_{N)\vert\dot{2}]}\partial_{\varrho\dot{\rho}}\phi^{A[I}\delta^{J]}_B$
$\bar{q}^B_{\dot{\rho}}$	$q_{A\rho}$	$-2\partial_{\rho\dot{\rho}}\bar{\psi}_{P[\dot{1}\vert}\delta^{PB}_{A(M}\bar{\psi}_{N)\vert\dot{2}]}\phi^{IJ} - 2\partial_{\rho[\dot{1}\vert}\phi_{A(M}\bar{f}_{\vert\dot{2}]\dot{\rho}}\delta^B_{N)}\phi^{IJ} + 2\partial_{\rho[\dot{1}\vert}\phi_{A(M}\bar{\psi}_{N)\vert\dot{2}]}\bar{\psi}_{Q\dot{\rho}}\epsilon^{QBIJ} + 2\bar{f}_{\dot{\rho}[\dot{1}\vert}\delta^B_{(M}\bar{\psi}_{N)\vert\dot{2}]}\delta^{[I}_A\psi^{J]}_\rho + \bar{\psi}_{[\dot{1}\vert(M}\bar{\psi}_{N)\vert\dot{2}]}\partial_{\rho\dot{\rho}}\phi^{B[J}\delta^{I]}_A$
$\bar{q}^B_{\dot{\varrho}}$	$\bar{q}^A_{\dot{\rho}}$	$2\delta^A_{(M}\delta^B_{N)}\bar{f}_{\dot{\rho}[\dot{1}}\bar{f}_{\dot{2}]\dot{\varrho}}\phi^{IJ} - 2\epsilon^{PBIJ}\bar{f}_{\dot{\rho}[\dot{1}\vert}\delta^A_{(M}\bar{\psi}_{N)\vert\dot{2}]}\bar{\psi}_{P\dot{\varrho}} + 2\epsilon^{PAIJ}\bar{f}_{\dot{\varrho}[\dot{1}\vert}\delta^B_{(M}\bar{\psi}_{N)\vert\dot{2}]}\bar{\psi}_{\dot{\rho}P} + \bar{\psi}_{[\dot{1}\vert(M}\bar{\psi}_{N)\vert\dot{2}]}\epsilon^{BAIJ}\bar{f}_{\dot{\rho}\dot{\varrho}}$
$s^{B\rho}$	$\bar{q}^A_{\dot{\rho}}$	$-\frac{i}{2}x^{\rho\dot{\varrho}}\Big(2\delta^A_{(M}\delta^B_{N)}\bar{f}_{\dot{\rho}[\dot{1}}\bar{f}_{\dot{2}]\dot{\varrho}}\phi^{IJ} - 2\epsilon^{PBIJ}\bar{f}_{\dot{\rho}[\dot{1}\vert}\delta^A_{(M}\bar{\psi}_{N)\vert\dot{2}]}\bar{\psi}_{P\dot{\varrho}} + 2\epsilon^{PAIJ}\bar{f}_{\dot{\varrho}[\dot{1}\vert}\delta^B_{(M}\bar{\psi}_{N)\vert\dot{2}]}\bar{\psi}_{\dot{\rho}P} + \bar{\psi}_{[\dot{1}\vert(M}\bar{\psi}_{N)\vert\dot{2}]}\epsilon^{BAIJ}\bar{f}_{\dot{\rho}\dot{\varrho}}\Big)$

$\mathcal{Q}_2$	$\mathcal{Q}_1$	$\tilde{\mathcal{Q}}_2\tilde{\mathcal{Q}}_1\Sigma^{[IJ]}_{(MN)}$
$\bar{s}^{\dot\varrho}_B$	$\bar{q}^A_{\dot\rho}$	$-\frac{i}{2}x^{\varrho\dot\varrho}\Big(2\delta^A_B\partial_{\varrho\dot\rho}\bar\psi_{[\dot1\|(M}\bar\psi_{N)\|\dot2]}\phi^{IJ} +$ $2\partial_{\varrho\dot\rho}\bar\psi_{[\dot1\|P}\delta^{PA}_{B(M}\bar\psi_{N)\|\dot2]}\phi^{IJ} + 2\delta^A_{(M}\bar f_{\dot\rho[\dot1}\partial_{\dot2]\varrho}\phi_{N)B}\phi^{IJ} -$ $2\delta^A_{(M}\bar f_{\dot\rho[\dot1\|}\bar\psi_{N)\|\dot2]}\delta^{[I}_B\psi^{J]}_\varrho - 2\partial_{\varrho[\dot1\|}\phi_{B(M}\bar\psi_{N)\|\dot2]}\epsilon^{PAIJ}\bar\psi_{P\dot\rho} +$ $\bar\psi_{[\dot1\|(M}\bar\psi_{N)\|\dot2]}\delta^A_B\partial_{\varrho\dot\rho}\phi^{IJ} + \bar\psi_{[\dot1\|(M}\bar\psi_{N)\|\dot2]}\partial_{\varrho\dot\rho}\phi^{A[I}\delta^{J]}_B\Big) +$ $2\delta^A_B\delta^{\dot\varrho}_{(\dot\rho}\bar\psi_{[\dot1)\|(M}\bar\psi_{N)\|\dot2]}\phi^{IJ} + 2\delta^{\dot\varrho}_{(\dot\rho}\bar\psi_{[\dot1)\|P}\delta^{PA}_{B(M}\bar\psi_{N)\|\dot2]}\phi^{IJ} +$ $2\bar f_{\dot\rho[\dot1}\delta^{\dot\varrho}_{\dot2]}\delta^A_{(M}\phi_{N)B}\phi^{IJ} - 2\delta^{\dot\varrho}_{[\dot1\|}\phi_{B(M}\bar\psi_{N)\|\dot2]}\epsilon^{PAIJ}\bar\psi_{\dot\rho P} +$ $\bar\psi_{[\dot1\|(M}\bar\psi_{N)\|\dot2]}\delta^{\dot\varrho}_{\dot\rho}\delta^A_B\phi^{IJ} + \bar\psi_{[\dot1\|(M}\bar\psi_{N)\|\dot2]}\delta^{\dot\varrho}_{\dot\rho}\phi^{A[I}\delta^{J]}_B$
$s^{B\varrho}$	$q_{A\rho}$	$-\frac{i}{2}x^{\varrho\dot\rho}\Big(- 2\partial_{\rho\dot\rho}\bar\psi_{P[\dot1\|}\delta^{PB}_{A(M}\bar\psi_{N)\|\dot2]}\phi^{IJ} -$ $2\partial_{\rho[\dot1\|}\phi_{A(M}\bar f_{\|\dot2]\dot\rho}\delta^B_{N)}\phi^{IJ} + 2\partial_{\rho[\dot1\|}\phi_{A(M}\bar\psi_{N)\|\dot2]}\bar\psi_{Q\dot\rho}\epsilon^{QBIJ} +$ $2\bar f_{\dot\rho[\dot1\|}\delta^B_{(M}\bar\psi_{N)\|\dot2]}\delta^{[I}_A\psi^{J]}_\rho + \bar\psi_{[\dot1\|(M}\bar\psi_{N)\|\dot2]}\partial_{\rho\dot\rho}\phi^{B[J}\delta^{I]}_A\Big) -$ $2\delta^\varrho_\rho\bar\psi_{P[\dot1\|}\delta^{PB}_{A(M}\bar\psi_{N)\|\dot2]}\phi^{IJ} - \delta^\varrho_\rho\bar\psi_{[\dot1\|(M}\bar\psi_{N)\|\dot2]}\phi^{B[I}\delta^{J]}_A$
$\bar{s}^{\dot\varrho}_B$	$q_{A\rho}$	$-\frac{i}{2}x^{\varrho\dot\varrho}\Big(- 2\partial_{\rho[\dot1\|}\psi^W_\varrho\epsilon_{ABW(M}\bar\psi_{N)\|\dot2]}\phi^{IJ} -$ $2\partial_{\rho[\dot1\|}\phi_{A(M}\partial_{\varrho\|\dot2]}\phi_{N)B}\phi^{IJ} + 2\partial_{\rho[\dot1\|}\phi_{A(M}\bar\psi_{N)\|\dot2]}\delta^{[I}_B\psi^{J]}_\varrho -$ $2\partial_{\varrho[\dot1\|}\phi_{B(M}\bar\psi_{N)\|\dot2]}\delta^{[I}_A\psi^{J]}_\rho + \bar\psi_{[\dot1\|(M}\bar\psi_{N)\|\dot2]}f_{\varrho\rho}\delta^{[I}_A\delta^{J]}_B\Big) +$ $2\delta^{\dot\varrho}_{[\dot1\|}\psi^W_\rho\epsilon_{(M\|ABW}\bar\psi_{\|N)\|\dot2]}\phi^{IJ} - 2\partial_{\rho[\dot1\|}\phi_{A(M}\phi_{N)B}\delta^{\dot\varrho}_{\|\dot2]}\phi^{IJ} -$ $2\delta^{\dot\varrho}_{[\dot1\|}\phi_{B(M}\bar\psi_{N)\|\dot2]}\delta^{[I}_A\psi^{J]}_\rho$
$\bar{s}^{\dot\rho}_B$	$s^{A\rho}$	$-\frac{1}{4}x^{\varrho\dot\rho}x^{\rho\dot\omega}\Big(2\delta^A_B\partial_{\varrho\dot\omega}\bar\psi_{[\dot1\|(M}\bar\psi_{N)\|\dot2]}\phi^{IJ} +$ $2\partial_{\varrho\dot\omega}\bar\psi_{[\dot1\|P}\delta^{PA}_{B(M}\bar\psi_{N)\|\dot2]}\phi^{IJ} + 2\delta^A_{(M}\bar f_{\dot\omega[\dot1}\partial_{\dot2]\varrho}\phi_{N)B}\phi^{IJ} -$ $2\delta^A_{(M}\bar f_{\dot\omega[\dot1\|}\bar\psi_{N)\|\dot2]}\delta^{[I}_B\psi^{J]}_\varrho - 2\partial_{\varrho[\dot1\|}\phi_{B(M}\bar\psi_{N)\|\dot2]}\epsilon^{PAIJ}\bar\psi_{P\dot\omega} +$ $\bar\psi_{[\dot1\|(M}\bar\psi_{N)\|\dot2]}\delta^A_B\partial_{\varrho\dot\omega}\phi^{IJ} + \bar\psi_{[\dot1\|(M}\bar\psi_{N)\|\dot2]}\partial_{\varrho\dot\omega}\phi^{A[I}\delta^{J]}_B\Big) -$ $\frac{i}{2}x^{\rho\dot\omega}\Big(2\delta^A_B\delta^{\dot\rho}_{(\dot\omega}\bar\psi_{[\dot1)\|(M}\bar\psi_{N)\|\dot2]}\phi^{IJ} +$ $2\delta^{\dot\rho}_{(\dot\omega}\bar\psi_{[\dot1)\|P}\delta^{PA}_{B(M}\bar\psi_{N)\|\dot2]}\phi^{IJ} + 2\bar f_{\dot\omega[\dot1}\delta^{\dot\rho}_{\dot2]}\delta^A_{(M}\phi_{N)B}\phi^{IJ} -$ $2\delta^{\dot\rho}_{[\dot1\|}\phi_{B(M}\bar\psi_{N)\|\dot2]}\epsilon^{PAIJ}\bar\psi_{\dot\omega P} + \bar\psi_{[\dot1\|(M}\bar\psi_{N)\|\dot2]}\delta^{\dot\rho}_{\dot\omega}\delta^A_B\phi^{IJ} +$ $\bar\psi_{[\dot1\|(M}\bar\psi_{N)\|\dot2]}\delta^{\dot\rho}_{\dot\omega}\phi^{A[I}\delta^{J]}_B\Big)$

$\mathcal{Q}_2$	$\mathcal{Q}_1$	$\tilde{\mathcal{Q}}_2\tilde{\mathcal{Q}}_1\Sigma^{[IJ]}_{(MN)}$
$q_{B\varrho}$	$s^{A\rho}$	$-\frac{i}{2}x^{\rho\dot\rho}\Big(2\delta^A_B\partial_{\varrho\dot\rho}\bar\psi_{[\dot1\vert(M}\bar\psi_{N)\vert\dot2]}\phi^{IJ} + 2\partial_{\varrho\dot\rho}\bar\psi_{[\dot1\vert P}\delta^{PA}_{B(M}\bar\psi_{N)\vert\dot2]}\phi^{IJ} + 2\delta^A_{(M}\bar f_{\dot\rho[\dot1}\partial_{\dot2]\varrho}\phi_{N)B}\phi^{IJ} - 2\delta^A_{(M}\bar f_{\dot\rho[\dot1\vert}\bar\psi_{N)\vert\dot2]}\delta^{[I}_B\psi^{J]}_\varrho - 2\partial_{\varrho[\dot1\vert}\phi_{B(M}\bar\psi_{N)\vert\dot2]}\epsilon^{PAIJ}\bar\psi_{P\dot\rho} + \bar\psi_{[\dot1\vert(M}\bar\psi_{N)\vert\dot2]}\delta^A_B\partial_{\varrho\dot\rho}\phi^{IJ} + \bar\psi_{[\dot1\vert(M}\bar\psi_{N)\vert\dot2]}\partial_{\varrho\dot\rho}\phi^{A[I}\delta^{J]}_B\Big)$
$q_{B\varrho}$	$\bar s^{\dot\rho}_A$	$-\frac{i}{2}x^{\rho\dot\rho}\Big(- 2\partial_{\rho[\dot1\vert}\psi^W_\varrho\epsilon_{ABW(M}\bar\psi_{N)\vert\dot2]}\phi^{IJ} - 2\partial_{\rho[\dot1\vert}\phi_{A(M}\partial_{\varrho\vert\dot2]}\phi_{N)B}\phi^{IJ} + 2\partial_{\rho[\dot1\vert}\phi_{A(M}\bar\psi_{N)\vert\dot2]}\delta^{[I}_B\psi^{J]}_\varrho - 2\partial_{\varrho[\dot1\vert}\phi_{B(M}\bar\psi_{N)\vert\dot2]}\delta^{[I}_A\psi^{J]}_\rho + \bar\psi_{[\dot1\vert(M}\bar\psi_{N)\vert\dot2]}f_{\varrho\rho}\delta^{[I}_A\delta^{J]}_B\Big) + 2\psi^W_\varrho\delta^{\dot\rho}_{[\dot1}\bar\psi_{\dot2](N}\epsilon_{M)ABW}\phi^{IJ} - 2\delta^{\dot\rho}_{[\dot1\vert}\phi_{A(M\vert}\partial_{\vert\dot2]\varrho}\phi_{\vert N)B}\phi^{IJ} + 2\delta^{\dot\rho}_{[\dot1\vert}\phi_{A(M}\bar\psi_{N)\vert\dot2]}\delta^{[I}_B\psi^{J]}_\varrho$
$\bar q^B_{\dot\varrho}$	$s^{A\rho}$	$-\frac{i}{2}x^{\rho\dot\rho}\Big(2\delta^A_{(M}\delta^B_{N)}\bar f_{\dot\rho[\dot1}\bar f_{\dot2]\dot\varrho}\phi^{IJ} - 2\epsilon^{PBIJ}\bar f_{\dot\rho[\dot1\vert}\delta^A_{(M}\bar\psi_{N)\vert\dot2]}\bar\psi_{P\dot\varrho} + 2\epsilon^{PAIJ}\bar f_{\dot\varrho[\dot1\vert}\delta^B_{(M}\bar\psi_{N)\vert\dot2]}\bar\psi_{\dot\rho P} + \bar\psi_{[\dot1\vert(M}\bar\psi_{N)\vert\dot2]}\epsilon^{BAIJ}\bar f_{\dot\rho\dot\varrho}\Big)$
$\bar q^B_{\dot\varrho}$	$\bar s^{\dot\rho}_A$	$-\frac{i}{2}x^{\rho\dot\rho}\Big(- 2\partial_{\rho\dot\varrho}\bar\psi_{P[\dot1\vert}\delta^{PB}_{A(M}\bar\psi_{N)\vert\dot2]}\phi^{IJ} - 2\partial_{\rho[\dot1\vert}\phi_{A(M}\bar f_{\vert\dot2]\dot\varrho}\delta^B_{N)}\phi^{IJ} + 2\partial_{\rho[\dot1\vert}\phi_{A(M}\bar\psi_{N)\vert\dot2]}\bar\psi_{P\dot\varrho}\epsilon^{PBIJ} + 2\bar f_{\dot\varrho[\dot1\vert}\delta^B_{(M}\bar\psi_{N)\vert\dot2]}\delta^{[I}_A\psi^{J]}_\rho + \bar\psi_{[\dot1\vert(M}\bar\psi_{N)\vert\dot2]}\partial_{\rho\dot\varrho}\phi^{B[J}\delta^{I]}_A\Big) + -2\bar\psi_{P\dot\varrho}\delta^{PB}_{A(M}\bar\psi_{N)[\dot2}\delta^{\dot\rho}_{\dot1]}\phi^{IJ} - 2\phi_{A(M}\delta^B_{N)}\delta^{\dot\rho}_{[\dot1}f_{\dot2]\dot\varrho}\phi^{IJ} + 2\phi_{A(M}\bar\psi_{N)[\dot2}\delta^{\dot\rho}_{\dot1]}\bar\psi_{P\dot\varrho}\epsilon^{PBIJ}$
$s^{B\theta}$	$s^{A\rho}$	$-\frac{1}{4}x^{\theta\dot\varrho}x^{\rho\dot\rho}\Big(2\delta^A_{(M}\delta^B_{N)}\bar f_{\dot\rho[\dot1}\bar f_{\dot2]\dot\varrho}\phi^{IJ} - 2\epsilon^{PBIJ}\bar f_{\dot\rho[\dot1\vert}\delta^A_{(M}\bar\psi_{N)\vert\dot2]}\bar\psi_{P\dot\varrho} + 2\epsilon^{PAIJ}\bar f_{\dot\varrho[\dot1\vert}\delta^B_{(M}\bar\psi_{N)\vert\dot2]}\bar\psi_{\dot\rho P} + \bar\psi_{[\dot1\vert(M}\bar\psi_{N)\vert\dot2]}\epsilon^{BAIJ}\bar f_{\dot\rho\dot\varrho}\Big)$

$\mathcal{Q}_2$	$\mathcal{Q}_1$	$\tilde{\mathcal{Q}}_2\tilde{\mathcal{Q}}_1\Sigma^{[IJ]}_{(MN)}$
$s^{B\rho}$	$\bar{s}^{\dot\theta}_A$	$-\frac{1}{4}x^{\rho\dot\rho}x^{\omega\dot\theta}\Big(- 2\partial_{\omega\dot\rho}\bar\psi_{P[\dot 1|}\delta^{PB}_{A(M}\bar\psi_{N)|\dot 2]}\phi^{IJ} - 2\partial_{\omega[\dot 1|}\phi_{A(M}\bar f_{|\dot 2]\dot\rho}\delta^B_{N)}\phi^{IJ} + 2\partial_{\omega[\dot 1|}\phi_{A(M}\bar\psi_{N)|\dot 2]}\bar\psi_{P\dot\rho}\epsilon^{PBIJ} + 2\bar f_{\dot\rho[\dot 1|}\delta^B_{(M}\bar\psi_{N)|\dot 2]}\delta^{[I}_A\psi^{J]}_\omega + \bar\psi_{[\dot 1|(M}\bar\psi_{N)|\dot 2]}\partial_{\omega\dot\rho}\phi^{B[J}\delta^{I]}_A\Big) - \frac{i}{2}x^{\omega\dot\theta}\Big(- 2\delta^\rho_\omega\bar\psi_{P[\dot 1|}\delta^{PB}_{A(M}\bar\psi_{N)|\dot 2]}\phi^{IJ} - \delta^\rho_\omega\bar\psi_{[\dot 1|(M}\bar\psi_{N)|\dot 2]}\phi^{B[I}\delta^{J]}_A\Big) - \frac{i}{2}x^{\rho\dot\rho}\Big(- 2\bar\psi_{P\dot\rho}\delta^{PB}_{A(M}\bar\psi_{N)[\dot 2}\delta^{\dot\theta}_{\dot 1]}\phi^{IJ} - 2\phi_{A(M}\delta^B_{N)}\delta^{\dot\theta}_{[\dot 1}f_{\dot 2]\dot\rho}\phi^{IJ} + 2\phi_{A(M}\bar\psi_{N)[\dot 2}\delta^{\dot\theta}_{\dot 1]}\bar\psi_{P\dot\rho}\epsilon^{PBIJ}\Big)$
$\bar{s}^{\dot\rho}_B$	$\bar{s}^{\dot\theta}_A$	$-\frac{1}{4}x^{\varrho\dot\rho}x^{\rho\dot\theta}\Big(- 2\partial_{\rho[\dot 1|}\psi^W_\varrho\epsilon_{ABW(M}\bar\psi_{N)|\dot 2]}\phi^{IJ} - 2\partial_{\rho[\dot 1|}\phi_{A(M}\partial_{\varrho|\dot 2]}\phi_{N)B}\phi^{IJ} + 2\partial_{\rho[\dot 1|}\phi_{A(M}\bar\psi_{N)|\dot 2]}\delta^{[I}_B\psi^{J]}_\varrho - 2\partial_{\varrho[\dot 1|}\phi_{B(M}\bar\psi_{N)|\dot 2]}\delta^{[I}_A\psi^{J]}_\rho + \bar\psi_{[\dot 1|(M}\bar\psi_{N)|\dot 2]}f_{\varrho\rho}\delta^{[I}_A\delta^{J]}_B\Big) - \frac{i}{2}x^{\rho\dot\theta}\Big(2\delta^{\dot\rho}_{[\dot 1|}\psi^W_\rho\epsilon_{(M\|ABW}\bar\psi_{\|N)\|\dot 2]}\phi^{IJ} - 2\partial_{\rho[\dot 1|}\phi_{A(M}\phi_{N)B}\delta^{\dot\rho}_{\|\dot 2]}\phi^{IJ} - 2\delta^{\dot\rho}_{[\dot 1|}\phi_{B(M}\bar\psi_{N)\|\dot 2]}\delta^{[I}_A\psi^{J]}_\rho\Big) - \frac{i}{2}x^{\varrho\dot\rho}\Big(2\psi^W_\varrho\delta^{\dot\theta}_{[\dot 1}\bar\psi_{\dot 2](N}\epsilon_{M)ABW}\phi^{IJ} - 2\delta^{\dot\theta}_{[\dot 1\|}\phi_{A(M\|}\partial_{\|\dot 2]\varrho}\phi_{\|N)B}\phi^{IJ} + 2\delta^{\dot\theta}_{[\dot 1\|}\phi_{A(M}\bar\psi_{N)\|\dot 2]}\delta^{[I}_B\psi^{J]}_\varrho\Big) + -2\delta^{\dot\theta}_{[\dot 1}\delta^{\dot\rho}_{\dot 2]}\phi_{A(M}\phi_{N)B}\phi^{IJ}$

3.3 Third order at supersymmetry generators

The results of the previous section show that the computation of descendants of $\mathcal{O}[B]$ and $\mathcal{O}[\tilde{B}]$ can be systematically extended to higher orders in the supersymmetry generators. However, the commutation relations in Eq. (2.8) enforce specific symmetries already at second order in the supersymmetry generators. Let us explain this fact below

3.3.1 Symmetry of $\tilde{\mathcal{Q}}\tilde{\mathcal{Q}}\Psi^{(IJ)}_{[MN]}$

1. **Symmetry in $q_{B\varrho}q_{A\rho}\Psi^{(IJ)}_{[MN]}$.** Recall the result given in Eq. (3.24)

$$\begin{aligned} q_{B\varrho} q_{A\rho} \mathbf{\Psi}^{(IJ)}_{[MN]} &= 2\delta^{(I}_{A}\delta^{J)}_{B} f_{\rho[1} f_{2]\varrho} \phi_{MN} - 2\epsilon_{MNBL} f_{\rho[1|} \delta^{(I}_{A} \psi^{J)}_{|2]} \psi^{L}_{\varrho} \\ &+ 2\epsilon_{MNAL} f_{\varrho[1|} \delta^{(I}_{B} \psi^{J)}_{|2]} \psi^{L}_{\rho} + \epsilon_{MNAB} \psi^{(I}_{[1} \psi^{J)}_{2]} f_{\rho\varrho} \end{aligned} \tag{3.73}$$

Antisymmetrization in A and B is achieved by applying ϵ^{RSBA} to both sides. And using the bellow equalities

$$\begin{aligned} \epsilon^{RSBA}\delta^{(I}_{A}\delta^{J)}_{B} &= 0 \\ \epsilon^{RSBA}\epsilon_{MNBL}\delta^{(I}_{A} &= -\delta^{RSA}_{MNL}\delta^{(I}_{A} = -(\delta^{R}_{[M}\delta^{S}_{N]}\delta^{A}_{L} + \delta^{R}_{[N}\delta^{S}_{L]}\delta^{A}_{M} + \delta^{R}_{[L}\delta^{S}_{M]}\delta^{A}_{N})\delta^{(I}_{A} \\ \epsilon^{RSBA}\epsilon_{MNAL}\delta^{(I}_{B} &= \delta^{RSB}_{MNL}\delta^{(I}_{B} = (\delta^{R}_{[M}\delta^{S}_{N]}\delta^{B}_{L} + \delta^{R}_{[N}\delta^{S}_{L]}\delta^{B}_{M} + \delta^{R}_{[L}\delta^{S}_{M]}\delta^{B}_{N})\delta^{(I}_{B} \\ \psi^{J)}_{\beta}\psi^{L}_{\rho} &= \frac{1}{2}\epsilon_{\beta\rho}\psi^{J)}\psi^{L} \\ \epsilon^{RSBA}\epsilon_{MNAB} &= -2\delta^{RS}_{MN} = -2\delta^{R}_{[M}\delta^{S}_{N]} \end{aligned} \tag{3.74}$$

We conclude that

$$\epsilon^{RSBA} q_{B\varrho} q_{A\rho} \mathbf{\Psi}^{(IJ)}_{[MN]} = 0 + \text{traces}, \tag{3.75}$$

where traces encodes terms proportional to δ^{I}_{M}, which leads to $B^{[MN]}_{(MJ)} = 0$

2. **Symmetry in $\bar{q}^{B}_{\dot\beta}\bar{q}^{A}_{\dot\alpha}\mathbf{\Psi}^{(IJ)}_{[MN]}$** Recall the result obtained in Eq. (3.26)

$$\begin{aligned} \bar{q}^{B}_{\dot\beta}\bar{q}^{A}_{\dot\alpha}\mathbf{\Psi}^{(IJ)}_{[MN]} &= 2\epsilon^{PBA(I}\partial_{[1|\dot\alpha}\bar{\psi}_{P\dot\beta}\psi^{J)}_{|2]}\phi_{MN} - 2\partial_{\dot\alpha[1}\phi^{A(I}\partial_{2]\dot\beta}\phi^{J)B}\phi_{MN} \\ &- 2\partial_{\dot\alpha[1}\phi^{A(I}\psi^{J)}_{2]}\delta^{PB}_{MN}\bar{\psi}_{P\dot\beta} + 2\partial_{\dot\beta[1}\phi^{B(I}\psi^{J)}_{2]}\delta^{PA}_{MN}\bar{\psi}_{P\dot\alpha} \\ &+ \psi^{(I}_{[1}\psi^{J)}_{2]}\delta^{BA}_{MN}\bar{f}_{\dot\alpha\dot\beta}. \end{aligned} \tag{3.76}$$

The following relations must be taken into account when antisymmetrizing with respect to A and B on both sides of the above equation

$$\begin{aligned} \epsilon_{RSBA}\epsilon^{PBAI} &= \delta^{PI}_{RS} \\ \epsilon_{RSBA}\delta^{PA}_{MN} &= \epsilon_{RSBA}\delta^{P}_{[M}\delta^{A}_{N]} = \epsilon_{RSB[N}\delta^{P}_{M]} = -\epsilon_{RSB[M}\delta^{P}_{N]} \\ \epsilon_{RSBA}\delta^{PB}_{MN} &= \epsilon_{RSBA}\delta^{P}_{[M}\delta^{B}_{N]} = -\epsilon_{RSA[N}\delta^{P}_{M]} = \epsilon_{RSA[M}\delta^{P}_{N]} \end{aligned} \tag{3.77}$$

the first term in (3.26) reduces to

$$2\delta^{P(I}_{RS}\partial_{\dot\alpha[1}\bar\psi_{P\dot\beta}\psi^{J)}_{|2]}\phi_{MN}, \tag{3.78}$$

the second term in (3.26) vanishes, since $\partial_{\dot\alpha[1}\phi^{A(I}\partial_{2]\dot\beta}\phi^{J)B}$ is symmetric in A and B. The third and fourth term in (3.26) gives

$$2\epsilon_{RSA[M}\bar\psi_{N](\dot\beta}\partial_{\dot\alpha)[1}\phi^{A(I}\psi^{J)}_{2]}. \tag{3.79}$$

This leads to

$$\epsilon_{RSBA}\bar q^B_{\dot\beta}\bar q^A_{\dot\alpha}\Psi^{(IJ)}_{[MN]} \;=\; 0 + \text{traces}. \tag{3.80}$$

3. **Symmetry of $s^{B\varrho}s^{A\rho}\Psi^{(IJ)}_{[MN]}$.** We recall the result presented in Eq. (3.29)

$$s^{B\varrho}s^{A\rho}\Psi^{(IJ)}_{[MN]} \;=\; -2\delta^{\rho}_{[1}\delta^{\varrho}_{2]}\phi^{A(I}\phi^{J)B}\phi_{MN} \tag{3.81}$$

$\epsilon_{RSBA}s^{B\varrho}s^{A\rho}\Psi^{(IJ)}_{[MN]} \;=\; -2\delta^{\rho}_{[1}\delta^{\varrho}_{2]}\epsilon_{RSBA}\phi^{A(I}\phi^{J)B}\phi_{MN} \;=\; 0$, the last equality follows from $\phi^{A(I}\phi^{J)B} = \phi^{B(J}\phi^{I)A}$. Therefore,

$$\epsilon_{RSBA}s^{B\varrho}s^{A\rho}\Psi^{(IJ)}_{[MN]} \;=\; 0. \tag{3.82}$$

3.3.2 Symmetry of $\tilde{\mathcal{Q}}\tilde{\mathcal{Q}}\Sigma^{[IJ]}_{(MN)}$

1. **Symmetry in $q_{B\varrho}q_{A\rho}\Sigma^{[IJ]}_{(MN)}$.** Recall the result given in Eq. (3.51)

$$\begin{aligned} q_{B\varrho}q_{A\rho}\Sigma^{[IJ]}_{(MN)} \;=\; & -2\epsilon_{ABW(M}\partial_{\rho[\dot 1|}\psi^W_{\varrho}\bar\psi_{N)|\dot 2]}\phi^{IJ} - 2\partial_{\rho[\dot 1}\phi_{A(M}\partial_{\varrho|\dot 2]}\phi_{N)B}\phi^{IJ} \\ & + \partial_{\rho[\dot 1}\phi_{A(M}\bar\psi_{N)|\dot 2]}\delta^{[I}_{B}\psi^{J]}_{\varrho} - \partial_{\varrho[\dot 1}\phi_{B(M}\bar\psi_{N)\dot 2]}\delta^{[I}_{A}\psi^{J]}_{\rho} \\ & + \bar\psi_{[\dot 1|(M}\bar\psi_{N)|\dot 2]}f_{\varrho\rho}\delta^{[I}_{A}\delta^{J]}_{B} \end{aligned} \tag{3.83}$$

Upon antisymmetrization in A and B, the second term in Eq. (3.51) vanishes since the expression $\partial_{\rho[\dot 1}\phi_{A(M}\partial_{\varrho|\dot 2]}\phi_{N)B}\phi^{IJ}$ is symmetric under the exchange of A and B. The third and fourth term leads to

$$-\epsilon^{PTB[I}\psi^{J]}_{(\rho}\partial_{\varrho)\dot\alpha}\phi_{B(M}\bar\psi_{N)\dot\beta}. \tag{3.84}$$

Therefore,

$$\epsilon^{PTAB} q_{B\varrho} q_{A\rho} \Sigma^{[IJ]}_{(MN)} \;=\; 0 + \text{traces} \tag{3.85}$$

2. **Symmetry in $\bar{q}^{B}_{\dot\varrho}\bar{q}^{A}_{\dot\rho}\Sigma^{[IJ]}_{(MN)}$.** We now recall the result in Eq. (3.53), which serves as the starting point for the following analysis.

$$\begin{aligned}
\bar{q}^{B}_{\dot\varrho}\bar{q}^{A}_{\dot\rho}\Sigma^{[IJ]}_{(MN)} &= 2\delta^{A}_{(M}\delta^{B}_{N)} \bar{f}_{\dot\rho[\dot 1}\bar{f}_{\dot 2]\dot\varrho}\phi^{IJ} \;-\; 2\epsilon^{PBIJ}\bar{f}_{\dot\rho[\dot 1}\delta^{A}_{(M}\bar\psi_{N)|\dot 2]}\bar\psi_{P\dot\varrho} \\
&\quad + 2\epsilon^{PAIJ}\bar{f}_{\dot\varrho[\dot 1}\delta^{B}_{(M}\bar\psi_{N)|\dot 2]}\bar\psi_{\dot\rho P} \;+\; \bar\psi_{[\dot 1|M}\bar\psi_{\dot 2]N}\epsilon^{BAIJ}\bar{f}_{\dot\rho\dot\varrho}
\end{aligned} \tag{3.86}$$

The following relations must be considered when antisymmetrizing in A and B on both sides of the equation above

$$\begin{aligned}
\epsilon_{RSAB}\epsilon^{PBIJ}\delta^{A}_{M} &= -\,\delta^{PIJ}_{RSA}\delta^{A}_{M} \;=\; -\delta^{P}_{M}\delta^{[I}_{R}\delta^{J]}_{S} - \delta^{P}_{R}\delta^{[I}_{S}\delta^{J]}_{M} - \delta^{P}_{S}\delta^{[I}_{M}\delta^{J]}_{R} \\
\epsilon_{RSAB}\epsilon^{PAIJ}\delta^{B}_{M} &= \delta^{P}_{M}\delta^{[I}_{R}\delta^{J]}_{S} + \delta^{P}_{S}\delta^{[I}_{M}\delta^{J]}_{R} + \delta^{P}_{R}\delta^{[I}_{S}\delta^{J]}_{M} \\
\epsilon_{RSAB}\epsilon^{BAIJ} &= -\,2\delta^{IJ}_{RS} = -2\delta^{[I}_{R}\delta^{J]}_{S}
\end{aligned} \tag{3.87}$$

So, we get the preliminary result

$$\begin{aligned}
\epsilon_{RSAB}\bar{q}^{B}_{\dot\varrho}\bar{q}^{A}_{\dot\rho}\Sigma^{[IJ]}_{(MN)} &= 2\delta^{[I}_{R}\delta^{J]}_{S}\bar{f}_{\dot\rho[\dot 1}\bar\psi_{\dot 2](N}\bar\psi_{M)\dot\varrho} + 2\delta^{[I}_{R}\delta^{J]}_{S}\bar{f}_{\dot\varrho[\dot 1}\bar\psi_{\dot 2](N}\bar\psi_{M)\dot\rho} \\
&\quad - 2\delta^{[I}_{R}\delta^{J]}_{S}\bar\psi_{[\dot 1|M}\bar\psi_{|\dot 2]N}\bar{f}_{\dot\rho\dot\varrho} + \texttt{traces}
\end{aligned} \tag{3.88}$$

and using $\epsilon^{\dot\beta\dot\alpha}\bar\psi_{\dot\beta(N}\bar\psi_{M)\dot\varrho} = \frac{1}{2}\delta^{\dot\alpha}_{\dot\varrho}\bar\psi_{(N}\bar\psi_{M)}$. We conclude that,

$$\epsilon_{RSAB}\bar{q}^{B}_{\dot\varrho}\bar{q}^{A}_{\dot\rho}\Sigma^{[IJ]}_{(MN)} = 0 \tag{3.89}$$

3. **Symmetry in $\bar{s}^{\dot\rho}_{B}\bar{s}^{\dot\varrho}_{A}\Sigma^{[IJ]}_{(MN)}$.** Recall the result obtained in Eq. (3.55)

$$\begin{aligned}
\bar{s}^{\dot\rho}_{B}\bar{s}^{\dot\varrho}_{A}\Sigma^{[IJ]}_{(MN)} &= \frac{1}{2}\epsilon_{MAVW}\epsilon_{NBV'W'}\epsilon^{\dot\beta\dot\alpha}\delta^{\dot\varrho}_{\dot\alpha}\delta^{\dot\rho}_{\dot\beta}\phi^{VW}\phi^{V'W'}\phi^{IJ} \\
&= 2\delta^{\dot\varrho}_{[\dot 1}\delta^{\dot\rho}_{\dot 2]}\phi_{MA}\phi_{NB}\phi^{IJ}
\end{aligned} \tag{3.90}$$

Upon antisymmetrization in A and B, we conclude that

$$\epsilon^{RSBA}\bar{s}^{\dot\rho}_{B}\bar{s}^{\dot\varrho}_{A}\Sigma^{[IJ]}_{(MN)} \;=\; =\; 0, \tag{3.91}$$

since $\phi_{MA}\phi_{NB}$ is symmetric under the exchange in A and B.

Chapter 4

The geometrical point of view

The aim of this chapter is to present results appeared in Ref. [5]. In Section 4.1, we introduce coherent wavefunctions solutions to free theory. In Section 4.2, we define $\delta^{\xi}_{Z_{(1)},Z_{(2)}}(Z)$ as cornerstone for constructing the cubic deformation operator evaluated on free solutions; its linearity in $\xi \otimes \xi$ and its symmetry under the exchange of arguments are demonstrated. Finally, in section 4.3 we discuss extensions to other finite-dimensional representations.

4.1 Coherent wavefunctions in free theory

We will now discuss a special family of solutions of free equations, which form an orbit of $PSL(4|4)$.

For every point $Z \in \mathbf{T}$ we define a generalized function section of $(|\mathrm{Ber}|\, \mathcal{S}_M)^{-1}$ (see Eq. (2.47)), which we call δ_Z. By definition $\delta_Z(X)$, for a fixed 2|2-dimensional $X \subset \mathbf{T}$, is the delta-function of the condition that $Z \in X$. More precisely (here Γ(`line bundle`) denotes the space of sections):

$$
\begin{aligned}
\delta_Z \;& \in \Gamma\left(|\mathrm{Ber}|\, \mathcal{S}_M\right)^{-1} \\
& \qquad \texttt{such that}\ \forall\ s\ \in\ \Gamma\left(|\mathrm{Ber}|\, \mathcal{S}_M\right)\ : \\
& \qquad \int_{Z\in\mathbf{T}} f(Z)\langle s(X), \delta_Z(X)\rangle \;=\; \int_{Z\in X} s(X) f(Z),
\end{aligned} \tag{4.1}
$$

Here the integration on the left hand side uses the canonical measure on $\mathbf{T}$,

$$
\Omega \;=\; d^2\lambda d^2\mu d^4\zeta \tag{4.2}
$$

and the integration measure on the right hand side is the integration along the fiber of $\mathcal{S}_M$ defined by the volume element s (Section 2.2.1). In coordinates (2.1), (2.12):

$$
\delta_Z(X) \;=\; \left(\theta^{\alpha m}\lambda_\alpha + u^m_I \zeta^I\right)\delta(x^{\alpha\dot\alpha}\lambda_\alpha + \mu^{\dot\alpha} + \tilde\theta^{\dot\alpha}_I \zeta^I) \tag{4.3}
$$

So defined $\delta_Z(X)$, as a function of $X \in M$ (or rather a section of the line bundle $(|\mathrm{Ber}|\,\mathcal{S})^{-1}$ over M), encodes a solution of $\mathcal{N} = 4$ super-Maxwell equations of motion as described in Chapter 2

To determine the corresponding scalar, electromagnetic and spinor fields, we write $\delta_Z(X)$ as a sum of expressions with definite values of the R charge, which corresponds to twice the power of ζ:

$$\theta^{\alpha 1}\theta^{\beta 2} f_{\alpha\beta} \;:\; \theta^{\alpha 1}\theta^{\beta 2}\lambda_\alpha\lambda_\beta\delta^2(\mu + x\lambda) \tag{4.4}$$

$$\theta^{\alpha[1} u_J^{2]}\psi^J_\alpha \;:\; \theta^{\alpha[1} u_J^{2]}\lambda_\alpha\zeta^J\delta^2(\mu + x\lambda) \tag{4.5}$$

$$u_I^1 u_J^2 \varphi^{IJ} \;:\; u_I^1 u_J^2 \zeta^I\zeta^J\delta^2(\mu + x\lambda) \tag{4.6}$$

$$\epsilon^{IJKP} u_I^1 u_J^2 \tilde\theta^{\dot\alpha}_K \tilde\psi_{P\dot\alpha} \;:\; u^1_{[I} u^2_J \tilde\theta^{\dot\alpha}_{K]}\zeta^I\zeta^J\zeta^K\partial_{\dot\alpha}\delta^2(\mu + x\lambda) \tag{4.7}$$

$$\epsilon^{IJKL} u_I^1 u_J^2 \tilde\theta^{\dot\alpha}_K \tilde\theta^{\dot\beta}_L \tilde f_{\dot\alpha\dot\beta} \;:\; u_I^1 u_J^2 \tilde\theta^{\dot\alpha}_K \tilde\theta^{\dot\beta}_L \zeta^I\zeta^J\zeta^K\zeta^L\partial_{\dot\alpha}\partial_{\dot\beta}\delta^2(\mu + x\lambda) \tag{4.8}$$

4.2 Beta deformation of the free Yang-Mills action

We want to characterize a deforming operator $\int d^4x\, U$ in Eq. (1.4) by its value on free solutions. In particular, we are interested in the case of β−deformation, which corresponds to deforming the action by an expression cubic in the elementary fields [4]. Let us evaluate the corresponding U on the formal sum of three coherent states:

$$\left(\varepsilon_1\delta_{Z_{(1)}}(X) + \varepsilon_2\delta_{Z_{(2)}}(X) + \varepsilon_3\delta_{Z_{(3)}}(X)\right) M \tag{4.9}$$

with bosonic nilpotent coefficients[1] ε_1, ε_2, ε_3, where M is some $N \times N$-matrix (see Eq. (2.46)). As we explained in Chapter 1, linearized β−deformations are parametrized by $B \in (\mathbf{g} \wedge \mathbf{g})_0/\mathbf{g}$. We consider the case when B is a decomposable tensor, *i.e.* is of the form $B = \xi \wedge \xi$ where ξ is an odd element of $\mathbf{g}$. The condition of zero internal commutator (the subindex 0 in $(\mathbf{g} \wedge \mathbf{g})_0$) means that ξ should be nilpotent. We claim

[1] "Nilpotent" means satisfying $\varepsilon_1^2 = \varepsilon_2^2 = \varepsilon_3^2 = 0$. To the best of our knowledge, the idea to use nilpotent coefficient was first suggested in [22]. The authors of [22] constructed a solution of classical field equations, which they called "perturbiner". Our use of nilpotent coefficients is slightly different; we take a solution of the free field equations in the form of Eq. (4.9) and evaluate some $\int d^4x\, U$ on it. We need the nilpotence of coefficients to avoid considering the square of delta-function.

that the result of evaluation of $\int d^4x\, U$ on the free solution given by Eq. (4.9) is:

$$\int d^4x\, U \;=\; \varepsilon_1\varepsilon_2\varepsilon_3\, \delta^{\xi}_{Z_{(1)},Z_{(2)}}(Z_{(3)})\, \mathrm{tr} M^3 \tag{4.10}$$

where $\delta^{\xi}_{Z_{(1)},Z_{(2)}}(Z_{(3)})$ is defined by Eq. (4.15). In fact, our Eq. (4.15) works for any nilpotent $\xi \in \Pi T_e PSL(4|4,\mathbf{R})$; this includes odd elements of $psl(4|4,\mathbf{R})$ as well as linear combinations of even elements of $psl(4|4,\mathbf{R})$ with Grassmann odd coefficients[2].

(At the same time, on the string theory side the integrated vertex operator of the worldsheet sigma-model for this particular B is [3]:

$$U_{\mathrm{AdS}} = \int j_a\xi^a \wedge j_b\xi^b \tag{4.11}$$

Here j_a is the Noether current on the worldsheet, a one-form.)

4.2.1 Definition of $\delta^{\xi}_{Z_{(1)},Z_{(2)}}(Z)$

Let us pick two points[3] in $\mathbf{PT}$: $[Z_{(1)}]$ and $[Z_{(2)}]$. Let $\xi \in \mathbf{g}$ be an **odd nilpotent** element of $\mathbf{g} = psl(4|4,\mathbf{R})$. We can represent ξ as a $4|4 \times 4|4$-supermatrix, which we also denote ξ.

$$\xi \;=\; \begin{pmatrix} 0 & B \\ C & 0 \end{pmatrix}, \quad \xi^2 \;=\; \begin{pmatrix} BC & 0 \\ 0 & CB \end{pmatrix} \tag{4.12}$$

Since ξ is nilpotent as an element of $psl(4|4,\mathbf{R})$, the square of this matrix is proportional to the unit matrix:

$$\xi^2 = c\mathbf{1} \tag{4.13}$$

where c is some number. We assume that ξ is nondegenerate, in the following sense: either $c \neq 0$, or if $c = 0$ then $\mathrm{Ker}\,\xi = \mathrm{Im}\,\xi$. Let us consider a $2|2$-dimensional plane

[2] Since we work with coherent states, we are into supergeometry/supermanifolds; we have to use a "pool" of constant Grassmann odd parameters [23].

[3] Strictly speaking, it is not appropriate to think about "points of a supermanifold". It is better to say that, for an arbitrary supermanifold S, we pick two arbitary morphisms $Z_{(1)} : S \to \mathbf{PT}$ and $Z_{(2)} : S \to \mathbf{PT}$ [24]. For our purposes, it is enough to take $S = \mathbf{R}^{0|K}$ for large enough K. Technically, we just allow all our twistors to depend on K constant Grassmann odd parameters.

L generated by $Z_{(1)}$, $Z_{(2)}$, $\xi Z_{(1)}$ and $\xi Z_{(2)}$:

$$L \;=\; \mathbf{R}Z_{(1)} + \mathbf{R}Z_{(2)} + \mathbf{R}\xi Z_{(1)} + \mathbf{R}\xi Z_{(2)} \tag{4.14}$$

Let us define $\delta^{\xi}_{Z_{(1)},Z_{(2)}}(Z)$, essentially a delta-function of Z, in the following way. For any test function $f \in C^{\infty}(\mathbf{T})$:

$$\begin{aligned} &\int_{Z\in\mathbf{R}^{4|4}} \delta^{\xi}_{Z_{(1)},Z_{(2)}}(Z)\, f(Z) \\ := &\int_{\mathbf{R}^2} da_1\wedge da_2\, \frac{\partial}{\partial\psi^1}\frac{\partial}{\partial\psi^2}\, f(a_1Z_{(1)} + a_2Z_{(2)} + \psi^1\xi Z_{(1)} + \psi^2\xi Z_{(2)}) \end{aligned} \tag{4.15}$$

The integration on the left hand side uses the canonical measure $\mathbf{\Omega} \;=\; d^2\lambda d^2\mu d^4\zeta$ on $\mathbf{R}^{4|4}$ (the "S" in "$PSL(4|4)$"). The integral on the right hand side uses the integral form on L canonically defined by ξ

Proposition 2. *Eq. (4.15) implies that* $\delta^{\xi}_{Z_{(1)},Z_{(2)}}(Z)$ *is a linear function of* $\xi\otimes\xi$*. It has weight zero in* Z *and also in both* $Z_{(1)}$ *and* $Z_{(2)}$*; in other words, for* $\kappa\in\mathbf{R}$*:*

$$\delta^{\xi}_{Z_{(1)},Z_{(2)}}(\kappa Z) = \delta^{\xi}_{\kappa Z_{(1)},Z_{(2)}}(Z) = \delta^{\xi}_{Z_{(1)},\kappa Z_{(2)}}(Z) = \delta^{\xi}_{Z_{(1)},Z_{(2)}}(Z) \tag{4.16}$$

Proof. First equality follows from the fact

$$\begin{aligned} &\int_{Z\in\mathbf{R}^{4|4}} \delta^{\xi}_{Z_{(1)},Z_{(2)}}(\kappa Z)\, f(Z) \;=\; \int_{\tilde Z\in\mathbf{R}^{4|4}} \delta^{\xi}_{Z_{(1)},Z_{(2)}}(\tilde Z)\, f(\frac{\tilde Z}{\kappa}) \\ = &\int_{\mathbf{R}^2} da_1\wedge da_2\, \frac{\partial}{\partial\psi^1}\frac{\partial}{\partial\psi^2}\, f(\frac{a_1Z_{(1)} + a_2Z_{(2)} + \psi^1\xi Z_{(1)} + \psi^2\xi Z_{(2)}}{\kappa}) \\ = &\int_{\mathbf{R}^2} da_1\wedge da_2\, \frac{\partial}{\partial\psi^1}\frac{\partial}{\partial\psi^2}\, f(\tilde a_1Z_{(1)} + \tilde a_2Z_{(2)} + \tilde\psi^1\xi Z_{(1)} + \tilde\psi^2\xi Z_{(2)}), \end{aligned} \tag{4.17}$$

where $\tilde a_1 \;=\; \frac{a_1}{\kappa}$, $\tilde a_2 \;=\; \frac{a_2}{\kappa}$, $\tilde\psi^1 \;=\; \frac{\psi^1}{\kappa}$ and $\tilde\psi^2 \;=\; \frac{\psi^2}{\kappa}$. Since, the Ber of the Jacobian of these transformations is equal to one. We can write the above equality as

$$\begin{aligned} &= \int_{\mathbf{R}^2} d\tilde a_1\wedge d\tilde a_2\, \frac{\partial}{\partial\tilde\psi^1}\frac{\partial}{\partial\tilde\psi^2}\, f(\tilde a_1Z_{(1)} + \tilde a_2Z_{(2)} + \tilde\psi^1\xi Z_{(1)} + \tilde\psi^2\xi Z_{(2)}) \\ &= \int_{Z\in\mathbf{R}^{4|4}} \delta^{\xi}_{Z_{(1)},Z_{(2)}}(\kappa Z)\, f(Z), \end{aligned} \tag{4.18}$$

Therefore, $\delta^{\xi}_{Z_{(1)},Z_{(2)}}(\kappa Z) = \delta^{\xi}_{Z_{(1)},Z_{(2)}}(Z)$. Second and third equality, follows from

$$\begin{aligned}
&\int_{Z\in\mathbf{R}^{4|4}} \delta^{\xi}_{\alpha Z_{(1)},\beta Z_{(2)}}(Z)\, f(Z) \\
&= \int_{\mathbf{R}^2} da_1 \wedge da_2\, \frac{\partial}{\partial\psi^1}\frac{\partial}{\partial\psi^2}\, f(a_1\alpha Z_{(1)} + a_2\beta Z_{(2)} + \psi^1\alpha\xi Z_{(1)} + \psi^2\beta\xi Z_{(2)}) \\
&= \int_{\mathbf{R}^2} \alpha da_1 \wedge \beta da_2\, \frac{\partial}{\partial\tilde\psi^1}\frac{\partial}{\partial\tilde\psi^2}\, f(\tilde a_1 Z_{(1)} + \tilde a_2 Z_{(2)} + \tilde\psi^1\xi Z_{(1)} + \tilde\psi^2\xi Z_{(2)}) \\
&= \int_{Z\in\mathbf{R}^{4|4}} \delta^{\xi}_{Z_{(1)},Z_{(2)}}(Z)\, f(Z)
\end{aligned} \tag{4.19}$$

Therefore, $\delta^{\xi}_{\kappa Z_{(1)},Z_{(2)}}(Z) = \delta^{\xi}_{Z_{(1)},\kappa Z_{(2)}}(Z) = \delta^{\xi}_{Z_{(1)},Z_{(2)}}(Z)$. □

Moreover, the equalities in Eq(4.16) of Proposition 2 can be generalize by doing the following transformations $\tilde Z_{(1)} = \alpha Z_{(1)} + \beta Z_{(2)}$ and $\tilde Z_{(2)} = \gamma Z_{(1)} + \delta Z_{(2)}$

$$\begin{aligned}
&\int_{Z\in\mathbf{R}^{4|4}} \delta^{\xi}_{\tilde Z_{(1)},\tilde Z_{(2)}}(Z)\, f(Z) = \\
&= \int_{\mathbf{R}^2} da_1 \wedge da_2\, \frac{\partial}{\partial\psi^1}\frac{\partial}{\partial\psi^2}\, f\left(a_1\tilde Z_{(1)} + a_2\tilde Z_{(2)} + \psi^1\xi\tilde Z_{(1)} + \psi^2\xi\tilde Z_{(2)}\right) \\
&= \int_{\mathbf{R}^2} da_1 \wedge da_2\, \frac{\partial}{\partial\psi^1}\frac{\partial}{\partial\psi^2}\, f\Big(a_1(\alpha Z_{(1)} + \beta Z_{(2)}) + a_2(\gamma Z_{(1)} + \delta Z_{(2)}) \\
&+\ \psi^1\xi(\alpha Z_{(1)} + \beta Z_{(2)}) + \psi^2\xi(\gamma Z_{(1)} + \delta Z_{(2)})\Big) \\
&= \int_{\mathbf{R}^2} da_1 \wedge da_2\, \frac{\partial}{\partial\psi^1}\frac{\partial}{\partial\psi^2}\, f\Big((a_1\alpha + a_2\gamma)Z_{(1)} + (a_1\beta + a_2\delta)Z_{(2)} \\
&+\ (\alpha\psi^1 + \gamma\psi^2)\xi Z_{(1)} + (\beta\psi^1 + \delta\psi^2)\xi Z_{(2)}\Big) \\
&= \int_{\mathbf{R}^2} \mathrm{Ber}(Jac)^{-1} d\tilde a_1 \wedge d\tilde a_2\, \frac{\partial}{\partial\tilde\psi^1}\frac{\partial}{\partial\tilde\psi^2}\, f\Big(\tilde a_1 Z_{(1)} + \tilde a_2 Z_{(2)} + \tilde\psi^1\xi Z_{(1)} + \tilde\psi^2\xi Z_{(2)}\Big)
\end{aligned} \tag{4.20}$$

where $\tilde a_1 = \alpha a_1 + \gamma a_2,\ \tilde a_2 = \beta a_1 + \delta a_2,\ \tilde\psi_1 = \alpha\psi^1 + \gamma\psi^2,\ \tilde\psi_2 = \beta\psi^1 + \delta\psi^2$

and $\mathrm{Ber}(Jac)^{-1} = 1$ since

$$\mathrm{Ber}(Jac) = \mathrm{Ber}\begin{pmatrix} \alpha & \gamma & 0 & 0 \\ \beta & \delta & 0 & 0 \\ 0 & 0 & \alpha & \gamma \\ 0 & 0 & \beta & \delta \end{pmatrix} = 1. \tag{4.21}$$

Therefore, $\delta^{\xi}_{\tilde{Z}_{(1)},\tilde{Z}_{(2)}} = \delta^{\xi}_{Z_{(1)},Z_{(2)}}$.

Equivalently, we can define $\delta^{\xi}_{Z_{(1)},Z_{(2)}}(Z)$ as follows:

$$\begin{aligned} \delta^{\xi}_{Z_{(1)},Z_{(2)}}(Z) &= \qquad\qquad (4.22) \\ := \int_{\mathbf{R}^2} da_1 \wedge da_2 \, \frac{\partial}{\partial\psi^1}\frac{\partial}{\partial\psi^2}\, &\delta^{(4|4)}(Z - a_1 Z_{(1)} - a_2 Z_{(2)} - \psi^1\xi Z_{(1)} - \psi^2\xi Z_{(2)}). \end{aligned}$$

4.2.2 Orientation of L_{rd} and discontinuity of $\delta^{\xi}_{Z_{(1)},Z_{(2)}}$

In order to integrate an integral form:

$$da_1 \wedge da_2 \, \frac{\partial}{\partial\psi^1}\frac{\partial}{\partial\psi^2}\, f(a_1 Z_{(1)} + a_2 Z_{(2)} + \psi^1\xi Z_{(1)} + \psi^2\xi Z_{(2)}) \tag{4.23}$$

over a supermanifold L (Eq. (4.14)), we need an orientation of its body $L_{\mathrm{rd}} = \mathbf{R}^2$. This $\mathbf{R}^2$ is generated by $Z_{(1)}$ and $Z_{(2)}$. Therefore, we need to know which basis has positive orientation: $\{Z_{(1)}, Z_{(2)}\}$ or $\{Z_{(2)}, Z_{(1)}\}$? A comparison with explicit calculation in Section 4.2.5 shows that the orientation is determined by the sign of $\epsilon^{\alpha\beta}\lambda_{(1)\alpha}\lambda_{(2)\beta}$. This sounds like a contradiction, because $\epsilon^{\alpha\beta}\lambda_{(1)\alpha}\lambda_{(2)\beta}$ is not conformally invariant. However, the sign of $\epsilon^{\alpha\beta}\lambda_{(1)\alpha}\lambda_{(2)\beta}$ is invariant under infinitesimal conformal transformations. Explicit calculation in Section 4.2.5 shows that $\delta^{\xi}_{Z_{(1)},Z_{(2)}}(Z)$, as a function of $Z_{(1)}$ and $Z_{(2)}$, has a discontinuity when $\epsilon^{\alpha\beta}\lambda_{(1)\alpha}\lambda_{(2)\beta} = 0$. The limit at the discontinuity is proportional to $\delta^{(2)}(\lambda - a_1\lambda_{(1)} - a_2\lambda_{(2)})$, but the sign in front of $\delta^{(2)}(\lambda - a_1\lambda_{(1)} - a_2\lambda_{(2)})$ depends on whether we are approaching from $\epsilon^{\alpha\beta}\lambda_{(1)\alpha}\lambda_{(2)\beta} > 0$ or from $\epsilon^{\alpha\beta}\lambda_{(1)\alpha}\lambda_{(2)\beta} < 0$.

This can be explained as follows. We interpret the four-dimensional spacetime as the real Grassmannian: $\mathbf{R}^{2,2} \cup \infty = \mathrm{Gr}(2,4)$ (Section 2.2). For the Yang-Mills action to be conformally invariant, the scalar field should transform as a section of $(\mathrm{Ber} S)^{-1}$, and spinors as sections of $S \otimes (\mathrm{Ber}\, S)^{-1}$ (chiral) and $S^{\perp} \otimes (\mathrm{Ber}\, S)^{-1}$ (antichiral), where S is the tautological vector bundle of $\mathrm{Gr}(2,4)$ (like $\mathcal{S}_M$ of Section 2.2 but without "super-"). However, the δ_Z of Section 4.1 gives sections of $(\ldots) \otimes |\mathrm{Ber}|^{-1}$ instead of $(\ldots) \otimes \mathrm{Ber}^{-1}$ (Section 2.2.1). We would be able to cast sections of $|\mathrm{Ber}|^{-1}$ as sections of Ber^{-1}, if we provided an orientation of S. But this is only possible locally. In fact, we can provide orientation of the fiber of S over all x except at infinity. This is acceptable for our computation, because the integral $\int d^4x\; U$ in Eq. (1.4) is supported on a single point $x \in \mathbf{R}^{2,2}$ determined by the intersection of the planes $\mu_{(1)} - \lambda_{(1)} x = 0$ and $\mu_{(2)} - \lambda_{(2)} x = 0$. But when we consider the case when $\lambda_{(1)} \to \lambda_{(2)}$, the intersection point x goes to ∞, where the orientation is undefined, essentially because $||x||^2 = \det x$ can be positive or negative depending on how $x \to \infty$. This leads to the discontinuity of $\delta^{\xi}_{Z_{(1)},Z_{(2)}}(Z)$.

4.2.3 Symmetry of $\delta^{\xi}_{Z_{(1)},Z_{(2)}}(Z)$

The defining Eq. (4.15) implies that $\delta^{\xi}_{Z_{(1)},Z_{(2)}}(Z)$ is symmetric under the exchange $Z_{(1)} \leftrightarrow Z_{(2)}$. In fact it is symmetric under arbitrary permutations of $Z, Z_{(1)}, Z_{(2)}$. This can be proven as follows:

$$\delta^{(4|4)}(Z - a_1 Z_{(1)} - a_2 Z_{(2)} - \psi_1 \xi Z_{(1)} - \psi_2 \xi Z_{(2)}) \tag{4.24}$$

$$\begin{aligned}
&= \mathrm{sign}(a_1)\delta^{(4|4)}\left(\frac{1}{a_1 + \psi_1\xi} Z - Z_{(1)} - \frac{a_2}{a_1 + \psi_1\xi} Z_{(2)} - \frac{1}{a_1 + \psi_1\xi}\psi_2\xi Z_{(2)}\right)\\
&= \mathrm{sign}(a_1)\times\\
&\quad \times \delta^{(4|4)}\Big(- Z_{(1)} + a_1^{-1} Z - (a_1^{-1} a_2 - a_1^{-2}\psi_1\psi_2 c) Z_{(2)} - a_1^{-2}\psi_1\xi Z -\\
&\qquad - \; (a_1^{-1}\psi_2 - a_1^{-2} a_2 \psi_1)\xi Z_{(2)}\Big).
\end{aligned} \tag{4.25}$$

In Eq. (4.25) above, we have used the equalities below:

$$a_1 + \psi_1\xi \;=\; \begin{pmatrix} a_1\mathbb{I}_4 & \psi_1 B \\ \\ \psi_1 C & a_1\mathbb{I}_4 \end{pmatrix}, \;\; \mathrm{SDet}(a_1 + \psi_1\xi) \;=\; 1 \tag{4.26}$$

$$(\mathbb{I}_8 + a_1^{-1}\psi_1\xi)^{-1} \;=\; \begin{pmatrix} \mathbb{I}_4 & -a_1^{-1}\psi_1 B \\ \\ -a_1^{-1}\psi_1 C & \mathbb{I}_4 \end{pmatrix} \;=\; \mathbb{I}_8 - a_1^{-1}\psi_1\xi \tag{4.27}$$

In Eq. (4.25), we obtain the following change of variables:

$$\tilde{a}_1 \;=\; a_1^{-1}, \tag{4.28}$$

$$\tilde{a}_2 \;=\; -\,a_1^{-1}a_2 + a_1^{-2}\psi_1\psi_2 c, \tag{4.29}$$

$$\tilde{\psi}_1 \;=\; -\,a_1^{-2}\psi_1, \tag{4.30}$$

$$\tilde{\psi}_2 \;=\; a_1^{-2}a_2\psi_1 - a_1^{-1}\psi_2. \tag{4.31}$$

So, the Jacobian associated with the above change of variables is given in the supermatrix $\tilde{m}$:

$$\tilde{m} \;=\; \begin{pmatrix} A & B \\ \\ C & D \end{pmatrix}, \;\; \text{where} \;\; A \;=\; \begin{pmatrix} -a_1^{-2} & 0 \\ \\ a_1^{-2}a_2 - 2a_1^{-3}\psi_1\psi_2 c & -a_1^{-1} \end{pmatrix},$$

$$B \;=\; \begin{pmatrix} 0 & 0 \\ \\ -a_1^{-2}\psi_2 c & -a_1^{-2}\psi_1 c \end{pmatrix}, \;\; C \;=\; \begin{pmatrix} 2a_1^{-3}\psi_1 & 0 \\ \\ -2a_1^{-3}a_2\psi_1 + a_1^{-2}\psi_2 & a_1^{-2}\psi_1 \end{pmatrix}$$

$$\text{and} \;\; D \;=\; \begin{pmatrix} -a_1^{-2} & 0 \\ \\ a_1^{-2}a_2 & -a_1^{-1} \end{pmatrix}. \tag{4.32}$$

We can see that $\mathrm{SDet}(\tilde{m}) = 1$, since

$$BD^{-1}C = \frac{1}{a_1^{-3}}\begin{pmatrix} 0 & 0 \\ 2a_1^{-6}\psi_2\psi_1 c + a_1^{-6}\psi_1\psi_2 c & 0 \end{pmatrix}. \tag{4.33}$$

And, using Eq. (2.22). Therefore,

$$\mathrm{SDet}(\tilde{m}) = \frac{1}{a_1^{-3}}\det\begin{pmatrix} -a_1^{-2} & 0 \\ a_1^{-2}a_2 - a_1^{-3}\psi_1\psi_2 c & -a_1^{-1} \end{pmatrix} = 1. \tag{4.34}$$

In the integral $\int_{\mathbf{R}^2} da_1 \wedge da_2 \frac{\partial}{\partial\psi^1}\frac{\partial}{\partial\psi^2}$ is equivalent to the exchange[4] $Z \leftrightarrow Z_1$.

4.2.4 The case when $\xi \wedge \xi$ corresponds to zero deformation

Suppose that $\xi \wedge \xi$ belongs to the denominator of Eq. (1.7):

$$\xi \wedge \xi = \Delta(\zeta) = \sum C^{ab} t_a \wedge [t_b, \zeta] \tag{4.35}$$

for some $\zeta \in \mathbf{psl(4|4)}$, and C^{ab} the inverse of the Killing metric. We know from [3] that such $\xi \wedge \xi$ should correspond to zero deformations. We will now prove that for such ξ Eq. (4.15) indeed evaluates to zero.

Let us enumerate the basis vectors of $\mathbf{T}$: $\{e_1, e_2, e_3, e_4, f_1, f_2, f_3, f_4\}$ where e are even and f are odd. Without loss of generality, let us assume that $Z_{(1)} = e_1$ and $Z_{(2)} = e_2$. (In other words, we choose a basis so that the first two elements are $Z_{(1)}$ and $Z_{(2)}$.)

Let us first consider the case when ζ is even:

$$\zeta = e_a \otimes e^{\vee b} - \frac{1}{4}\delta_a^b \mathbf{1} \tag{4.36}$$

[4] The sign factor $\mathrm{sign}(a_1)$ is compensated by the possible difference of sign between $\epsilon^{\alpha\beta}\lambda_{(1)\alpha}\lambda_{(2)\beta}$ and $\epsilon^{\alpha\beta}\lambda_{(1)\alpha}\lambda_{(2)\beta}$. This sign determines the orientation of $\mathbf{R}^2$ which is needed to integrate $da_1 \wedge da_2$.

where $e^\vee, f^\vee$ are elements of the dual basis and $\mathbf{1} = \sum_c e_c \otimes e^{\vee c}$. In this case:

$$\Delta(\zeta) \;=\; \sum_{c=1}^{4} \left(e_a \otimes e^{\vee c} - \frac{1}{4}\delta_a^c \mathbf{1}\right) \wedge \left(e_c \otimes e^{\vee b} - \frac{1}{4}\delta_c^b \mathbf{1}\right) + \text{odd} \wedge \text{odd} \tag{4.37}$$

The odd $\wedge$ odd terms do not contribute to the computation. Also, the subtractions do not contribute to the computation, because the integral of Eq. (4.22):

$$\int_{\mathbf{R}^2} da^1 \wedge da^2 \frac{\partial}{\partial \psi^1}\frac{\partial}{\partial \psi^2} f(a^1 Z_{(1)} + a^2 Z_{(2)} + \psi^1 \xi_1 Z_{(1)} + \psi^2 \xi_2 Z_{(2)}) \tag{4.38}$$

is zero when either ξ_1 or ξ_2 is proportional to $\mathbf{1}$ (for example, when $\xi_1 = \alpha \mathbf{1}$, the dependence on ψ^1 can be removed by shifting $a^1 \mapsto a^1 - \psi^1 \alpha$). Let us pick a pair of fermionic constants ϵ and η from the pool[23]; we are evaluating:

$$\int_{\mathbf{R}^2} da_1 \wedge da_2 \frac{\partial}{\partial \psi^1}\frac{\partial}{\partial \psi^2} \sum_{c=1}^{4} f\Big(a^1 e_1 + a^2 e_2 + \\ + \psi^1(\epsilon \delta_1^c e_a + \eta \delta_1^b e_c) + \psi^2(\epsilon \delta_2^c e_a + \eta \delta_2^b e_c)\Big) \tag{4.39}$$

This integral is equal to zero. In fact, each term in $\sum_{c=1}^4$ vanishes separately. Indeed, when c is 3 or 4 it is proportional to $\eta^2 = 0$. If c is 1 or 2, we can shift the integration variable a^1 or a^2 to eliminate the dependence on η; then the integral becomes proportional to $\epsilon^2 = 0$.

Now consider the case when ζ is odd:

$$\zeta \;= f_a \otimes e^{\vee b} \tag{4.40}$$

$$\Delta(\zeta) \;= (f_a \otimes e^{\vee c}) \wedge \left(e_c \otimes e^{\vee b} - \frac{1}{4}\delta_c^b \mathbf{1}\right) + \left(f_a \otimes f^{\vee c} - \frac{1}{4}\delta_a^c \mathbf{1}\right) \wedge (f_c \otimes e^{\vee b}) \tag{4.41}$$

Again, the second term does not contribute to the computation. The contribution of the first term is:

$$\int_{\mathbf{R}^2} da_1 \wedge da_2 \frac{\partial}{\partial \psi^1}\frac{\partial}{\partial \psi^2} \sum_c f\Big(a^1 e_1 + a^2 e_2 + \\ + \psi^1(\delta_1^c f_a + \eta \delta_1^b e_c) + \psi^2(\delta_2^c f_a + \eta \delta_2^b e_c)\Big) \tag{4.42}$$

The terms with c equal 3 or 4 are zero, being proportional to $\eta^2 = 0$. When c is 1 or 2, we can eliminate the dependence on η by shifting a, and then we are left with, for example, when $c = 1$:

$$\int_{\mathbf{R}^2} da_1 \wedge da_2 \frac{\partial}{\partial \psi^1} \frac{\partial}{\partial \psi^2} \sum_c f\left(a^1 e_1 + a^2 e_2 + \psi^1 f_a\right) \tag{4.43}$$

This integral is zero, because the integrand does not depend on ψ^2.

4.2.5 Deformation of the form $\int \phi\psi\psi$

To prove our claim that the evaluation of the deforming operator on the free solution given by Eq. (4.9) is indeed $\delta^{\xi}_{Z_{(1)},Z_{(2)}}(Z)$, we will consider a particular case when the test function $f(Z)$ of Eq. (4.15) is:

$$f(Z) \; = \; A_{IK}(\lambda,\mu)\epsilon^{IKPQ}\zeta_P\zeta_Q \tag{4.44}$$

Eq. (4.10) implies that evaluation of $\int_{Z\in\mathbf{R}^{4|4}} \delta^{\xi}_{Z_{(1)},Z_{(2)}}(Z)\, f(Z)$ with such $f(Z)$ corresponds to the substitution instead of the term proportional to ε_3 in Eq. (4.9) of the following solution of the linearized equations of motion (*cp.* Eq. (4.6)):

$$\varphi_{IK}(x) \; = \; \int d^2\lambda d^2\mu \; A_{IK}(\lambda,\mu)\delta^2(\mu + x\lambda) \tag{4.45}$$

$$\psi = \tilde{\psi} = F^+ = F^- \; = \; 0 \tag{4.46}$$

Let us choose both ξ_1 and ξ_2 as "rotations of S^5", *i.e.*:

$$(\xi_1 \wedge \xi_2)^{I\;J}_{K\;L} = B^{IJ}_{KL} \tag{4.47}$$

with other components of $\xi_1 \wedge \xi_2$ all zero.

Eq. (4.15) implies that $\int_{Z\in\mathbf{R}^{4|4}} \delta^{\xi}_{Z_{(1)},Z_{(2)}}(Z)\, f(Z)$ equals to:

$$\int_{\mathbf{R}^2} A_{PQ}(a^{(1)}\lambda_{(1)} + a^{(2)}\lambda_{(2)}, a^{(1)}\mu_{(1)} + a^{(2)}\mu_{(2)})\epsilon^{PQRS} B^{IJ}_{RS}\zeta_{(1)I}\zeta_{(2)J} da^{(1)} \wedge da^{(2)} \tag{4.48}$$

The same result is obtained by evaluating the deformation term in the Lagrangian [4],

$$\int d^4x \epsilon^{IJKL}\varphi_{IJ}(x) B^{PQ}_{KL}\psi_{(1)P\dot\alpha}(x)\psi_{(2)Q\dot\beta}(x)\epsilon^{\dot\alpha\dot\beta} \tag{4.49}$$

$$\texttt{where } \psi_{(A)\dot\alpha K}(x) = \lambda_{(A)\dot\alpha}\zeta_{(A)K}\delta^2(\mu_{(A)} + x\lambda_{(A)}). \tag{4.50}$$

This establishes the contact with the description of the supermultiplet on the field theory side obtained in [4]; matching of other states follows from applying supersymmetry[5].

Fourier transform into the usual momentum space We will use now the notations of [8]. Following the prescription in [8], let us substitute in Eq. (4.45):

$$A_{IK}(\lambda,\mu) = \delta^2(\lambda-\lambda_{(0)})\exp\left(i\mu\tilde{\lambda}_{(0)}\right)\zeta_{(0)I}\zeta_{(0)K} \tag{4.51}$$

and then multiply by the Fourier transform factor $\exp\left(-i\mu_{(1)}\tilde{\lambda}_{(1)} - i\mu_{(2)}\tilde{\lambda}_{(2)}\right)$ and integrate over $\mu_{(1)}$ and $\mu_{(2)}$. Then Eq. (4.48) gives:

$$\begin{aligned}\int da^1\wedge da^2\,\delta^2\left(\lambda_{(0)} - a^{(1)}\lambda_{(1)} - a^{(2)}\lambda_{(2)}\right)\times \\ \times\delta^2\left(\tilde{\lambda}_{(1)} - a^{(1)}\tilde{\lambda}_{(0)}\right)\delta^2\left(\tilde{\lambda}_{(2)} - a^{(2)}\tilde{\lambda}_{(0)}\right)\,\zeta_{(0)P}\zeta_{(0)Q}\epsilon^{PQRS}B^{IJ}_{RS}\zeta_{(1)I}\zeta_{(2)J}\end{aligned} \tag{4.52}$$

This is equal to:

$$\begin{aligned}&\frac{1}{\langle\lambda_{(1)},\lambda_{(2)}\rangle}\,\delta^2\left(\tilde{\lambda}_{(1)} - \frac{\langle\lambda_{(2)},\lambda_{(0)}\rangle}{\langle\lambda_{(2)},\lambda_{(1)}\rangle}\tilde{\lambda}_{(0)}\right)\delta^2\left(\tilde{\lambda}_{(2)} - \frac{\langle\lambda_{(1)},\lambda_{(0)}\rangle}{\langle\lambda_{(1)},\lambda_{(2)}\rangle}\tilde{\lambda}_{(0)}\right)\times \\ &\times\zeta_{(0)P}\zeta_{(0)Q}\epsilon^{PQRS}B^{IJ}_{RS}\zeta_{(1)I}\zeta_{(2)J}\end{aligned} \tag{4.53}$$

Remember [8] that the momenta of the scattering particles are $p_{(i)\alpha\dot{\alpha}} = \tilde{\lambda}_{(i)\alpha}\lambda_{(i)\dot{\alpha}}$. Eq. (4.53) can be interpreted as a deformation of the three-point scattering amplitude. It is a generalized function with support on $\tilde{\lambda}_{(1)}$, $\tilde{\lambda}_{(2)}$ and $\tilde{\lambda}_{(0)}$ being all collinear to each other.

4.3 Other finite-dimensional representations

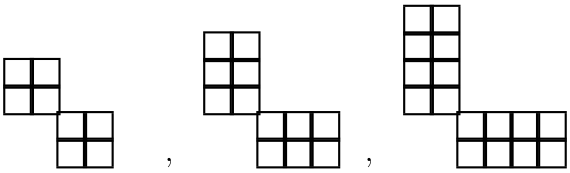
It is likely that the expressions $\delta^{\xi}_{Z_1,Z_2}(Z)$ will serve as building block for evaluation of deformations corresponding to other finite-dimensional representations. It was

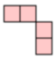
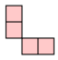
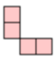
[5]Vanishing of deformations corresponding to the denominator in Eq. (1.7) was not proven in [4]. It follows from Section 4.2.4 that such deformations would vanish when evaluated on (4.9). This should imply that the deformation is acually zero, although we do not have a rigorous proof.

conjectured in [2] that deformations of the form:

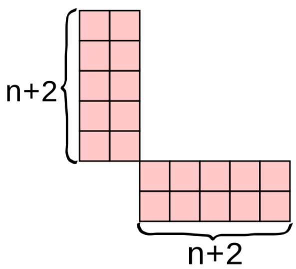
$$\int d^4x\, \rho_0(x)\, \mathrm{tr}(\Phi_1 + i\Phi_2)^4(x)\ , \int d^4x\, \rho_1(x)\, \mathrm{tr}(\Phi_1 + i\Phi_2)^5(x)\ ,$$
$$\int d^4x\, \rho_2(x)\, \mathrm{tr}(\Phi_1 + i\Phi_2)^6(x)\ , \quad \ldots \tag{4.54}$$

where ρ_n are some special polynomial functions described in [2] generate finite-dimensional representations of $\mathbf{g}$ corresponding to supersymmetric Young diagrams:

 , , , ... (4.55)

and their transposed. An element of such a representation is a super-traceless tensor B with $2(n+2)$ lower indices and $2(n+2)$ upper indices. The symmetry type of the lower indices is determined by the lower portion of the Young diagrams, and of the upper indices by the upper portion. For example, the Young diagram  ("case $n = -1$") corresponds to the tensors of the form B^{ab}_{cd} symmetric in ab and antysymmetric in cd, while  corresponds to the tensors of the form B^{ab}_{cd} antisymmetric in ab and symmetric in cd. When all indices a, b, c, d are fermionic,  gives $B^{IJ}_{KL} = B^{JI}_{KL} = -B^{IJ}_{LK}$ (see Eq. (4.49)).

Consider the tensor $B = \xi^{\otimes 2(n+2)}$ where ξ is an odd nilpotent element of $\mathbf{g}$. Let us apply the Young symmetrizer corresponding to:



(56)

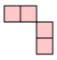
We conjecture that the evaluation of the deformation on the coherent state (as in Eq. (2.46)):

$$\left(\varepsilon_1 \delta_{Z_1}(X) + \varepsilon_2 \delta_{Z_2}(X) + \ldots + \varepsilon_{n+4} \delta_{Z_{n+4}}(X)\right) M \tag{4.57}$$

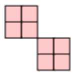
gives $\delta^{\xi}_{Z_1,\ldots,Z_{n+4}}\mathrm{tr}M^{n+4}$ where:

$$\delta^{\xi}_{Z_1,\ldots,Z_{n+4}} = \delta^{\xi}_{Z_1,Z_2}(Z_3)\delta^{\xi}_{Z_2,Z_3}(Z_4)\cdots\delta^{\xi}_{Z_{n+2},Z_{n+3}}(Z_{n+4}) \tag{4.58}$$

We leave the verification of this statement for future work. The symmetry of $\delta^{\xi}_{Z_1,\ldots,Z_n}$ under permutations of Z_i can be proven as in Section 4.2.3.

Let us prove that Eq. (4.58) depends on $\xi\wedge\cdots\wedge\xi$ only through its Young projector. To start, notice that each $\delta^{\xi}_{Z_n,Z_{n+1}}(Z_{n+2})$ depends on $\xi\wedge\xi$ via the projector  $\xi\otimes\xi$. (In other words, the lower through $\xi^{(a}_{[c}\xi^{b)}_{d]}$.) This follows from the fact that $\delta^{\xi}_{Z_n,Z_{n+1}}(Z_{n+2})=0$ when ξ is degenerate, dim im $\xi=1$ (this follows from the definition Eq. (4.15)). This implies that $\delta^{\xi}_{Z_1,\ldots,Z_{n+4}}$ depends on $\xi\wedge\cdots\wedge\xi$ only through:

$$\xi^{(a_1}_{[c_1}\xi^{b_1)}_{d_1]}\,\xi^{(a_2}_{[c_2}\xi^{b_2)}_{d_2]}\cdots\xi^{(a_{n+2}}_{[c_{n+2}}\xi^{b_{n+2})}_{d_{n+2}]} \tag{4.59}$$

But this is not all, there are more projectors. Let us consider the special case $n=0$. Let us write $\delta^{\xi}_{Z_1,Z_2,Z_3,Z_4}$ as $\delta^{\xi}_{Z_2,Z_3}(Z_1)\delta^{\xi}_{Z_2,Z_3}(Z_4)$. The indices c_1 and c_2 both contract with Z_2, therefore they enter symmetrized. The indices d_1 and d_2 also enter symmetrized, because they both contract with Z_3. Therefore, the $\xi\wedge\xi\wedge\xi\wedge\xi$ enters only through the projector to . In general, $\xi^{\wedge 2(n+2)}$ enters only through the projector of Eq. (4.56).

Chapter 5

Conclusion

In this thesis, we studied the $\beta-$ deformations of $\mathcal{N} = 4$ SYM from two perspectives: representation theory and supergeometry. Our analysis was motivated by understanding finite- dimensional representations of the superconformal algebra, which play a central role in AdS/CFT.

The first part of the thesis focuses on the supermultiplet structure of the $\beta-$deformation. We started from the known cubic interaction Eq. (2.29), we systematically generated its descendants by explicit action of the supersymmetry generators by working within the oscillator representation of the superconformal algebra $psl(4|4, \mathbf{R})$. While, the realization of $\beta-$deformation is associated with the wedge product of two adjoint representations of the superconformal algebra is manifest on the string-theory side. It was far from obvious in the field-theory side, we have shown that the field-theory deformation indeed organizes itself into this representation.

In the second part of the thesis, we turned to a geometric interpretation of linearized $\beta-$deformations, using the language of supertwistors, supergrassmannians, and harmonic superspace. Within this framework, four-dimensional spacetime is reinterpreted as a Grassmannian of $(2|2)-$planes inside the supertwistor space $\mathbf{T} = \mathbf{R}^{\mathbf{4}|\mathbf{4}}$ and the basic fields of $\mathcal{N} = 4$ SYM appear as sections of vector bundles over this manifold.

One of the main achievements of this geometric analysis is the identification of linearized $\beta-$deformations with specific distributional Dirac delta $\delta^{\xi}_{Z_{(1)},Z_{(2)}}(Z)$. By evaluating the deformation interaction on solutions of the free super-Maxwell equations, we obtained explicit expressions that encode the deformation as a generalized function on the product of projective supertwistor spaces and the space of deformation parameters.

One immediate extension for future research would be to study whether the geometric picture developed here can be promoted beyond the linearized regime. Another direction of research is the systematic exploration of other finite-dimensional superconformal representations and their associated spacetime-dependent deformations. Finally, the understanding whether the present results can be connected with twistor-string theory would be desirable.

Appendix A

The algebra $psl(4|4,\mathbf{R})$

The relations of commutation of the generators of Lorentz rotations and internal R-symmetry $l^{\alpha}_{\ \beta}$, $\dot{l}^{\dot\alpha}_{\ \dot\beta}$, $r^{A}_{\ B}$ with any generator $\mathcal{J}$ is given by

$$[l^{\alpha}_{\ \beta}, \mathcal{J}_{\varrho}] \ = \ \delta^{\alpha}_{\varrho}\mathcal{J}_{\beta} - \frac{1}{2}\delta^{\ \alpha}_{\beta}\mathcal{J}_{\varrho}, \quad [l^{\alpha}_{\ \beta}, \mathcal{J}^{\varrho}] \ = \ -\delta^{\varrho}_{\beta}\mathcal{J}^{\alpha} + \frac{1}{2}\delta^{\ \alpha}_{\beta}\mathcal{J}^{\varrho}, \tag{A.1}$$

$$[\dot{l}^{\dot\alpha}_{\ \dot\beta}, \mathcal{J}_{\dot\varrho}] \ = \ \delta^{\dot\alpha}_{\dot\varrho}\mathcal{J}_{\dot\beta} - \frac{1}{2}\delta^{\ \dot\alpha}_{\dot\beta}\mathcal{J}_{\dot\varrho}, \quad [\dot{l}^{\dot\alpha}_{\ \dot\beta}, \mathcal{J}^{\dot\varrho}] \ = \ -\delta^{\dot\varrho}_{\ \dot\beta}\mathcal{J}^{\dot\alpha} + \frac{1}{2}\delta^{\ \dot\alpha}_{\dot\beta}\mathcal{J}^{\dot\varrho}, \tag{A.2}$$

$$[r^{A}_{\ B}, \mathcal{J}_{D}] \ = \ -\delta^{\ A}_{D}\mathcal{J}_{B} + \frac{1}{4}\delta^{A}_{\ B}\mathcal{J}_{D}, \quad [r^{A}_{\ B}, \mathcal{J}^{D}] \ = \ \delta^{\ D}_{B}\mathcal{J}^{A} - \frac{1}{4}\delta^{A}_{\ B}\mathcal{J}^{D} \tag{A.3}$$

The another commutation relations are given by

$$[s^{A\varrho}, p_{\alpha\dot\alpha}] \ = \ \delta^{\varrho}_{\alpha}q^{A}_{\dot\alpha}, \quad [\dot{s}^{\dot\varrho}_{A}, p_{\alpha\dot\alpha}] \ = \ \delta^{\dot\varrho}_{\dot\alpha}q_{A\alpha}, \tag{A.4}$$

$$[k^{\dot\alpha\alpha}, \dot{q}^{A}_{\dot\alpha}] \ = \ \delta^{\dot\alpha}_{\dot\varrho}s^{A\alpha}, \quad [k^{\dot\alpha\alpha}, q_{A\varrho}] \ = \ \delta^{\alpha}_{\varrho}\dot{s}^{\dot\alpha}_{A} \tag{A.5}$$

$$\{q_{A\alpha}, \dot{q}^{B}_{\dot\alpha}\} \ = \ \delta^{B}_{A}p_{\alpha\dot\alpha}, \quad \{s^{A\alpha}, \dot{s}^{\dot\alpha}_{B}\} \ = \ \delta^{A}_{B}k^{\alpha\dot\alpha} \tag{A.6}$$

The non-vanishing relations of comutations reads as follows

$$[k^{\dot\alpha\alpha}, p_{\varrho\dot\varrho}] \ = \ \delta^{\dot\alpha}_{\dot\varrho}l^{\ \alpha}_{\varrho} + \delta^{\alpha}_{\ \varrho}\dot{l}_{\dot\varrho} + \delta^{\dot\alpha}_{\dot\varrho}\delta^{\alpha}_{\varrho}D \tag{A.7}$$

$$\{s^{A\alpha}, q_{B\varrho}\} \ = \ \delta^{\alpha}_{\varrho}r^{A}_{\ B} + \delta^{A}_{B} + \frac{1}{2}\delta^{A}_{B}\delta^{\alpha}_{\varrho}\,(D - C) \tag{A.8}$$

$$\{\dot{s}^{\dot\alpha}_{A}, \dot{q}^{B}_{\dot\beta}\} \ = \ \delta^{B}_{A}\dot{l}^{\ \dot\alpha}_{\dot\beta} - \delta^{\dot\alpha}_{\dot\beta}r^{\ B}_{A} + \frac{1}{2}\delta^{\dot\alpha}_{\dot\beta}\delta^{B}_{A}\,(D + C) \tag{A.9}$$

A.1 Oscillator representation of $psl(4|4,\mathbf{R})$ superalgebra

The generator of translations $p_{\alpha\dot\alpha}$ and conformal transformations $k^{\alpha\dot\alpha}$ are:

$$p_{\alpha\dot\alpha} \ = \ a^{\dagger}_{\alpha}b^{\dagger}_{\dot\alpha}, \qquad k^{\dot\alpha\alpha} \ = \ b^{\dot\alpha}a^{\alpha}. \tag{A.10}$$

The dilatation generator reads as follows

$$D \;=\; 1+\frac{1}{2}a^{\dagger}_{\gamma}a^{\gamma}+\frac{1}{2}b^{\dagger}_{\dot\gamma}b^{\dot\gamma}, \tag{A.11}$$

and the $sl(2)\times sl(2)$ and $sl(4)$ rotation generators are

$$\begin{aligned} l_{\beta}{}^{\alpha} &= a^{\dagger}_{\beta}a^{\alpha}-\frac{1}{2}\delta_{\beta}^{\alpha}a^{\dagger}_{\gamma}a^{\gamma}, \\ \bar{l}_{\dot\beta}{}^{\dot\alpha} &= b^{\dagger}_{\dot\beta}b^{\dot\alpha}-\frac{1}{2}\delta_{\dot\beta}^{\dot\alpha}b^{\dagger}_{\dot\gamma}b^{\dot\gamma}, \\ r^{A}{}_{B} &= c^{\dagger A}c_{B}-\frac{1}{4}\delta^{A}_{B}c^{\dagger D}c_{D}. \end{aligned} \tag{A.12}$$

Also, there are two $gl(1)$ generators

$$\begin{aligned} C &= 1-\frac{1}{2}a^{\dagger}_{\gamma}a^{\gamma}+\frac{1}{2}b^{\dagger}_{\dot\gamma}b^{\dot\gamma}-\frac{1}{2}c^{\dagger D}c_{D} \\ B &= \frac{1}{2}a^{\dagger}_{\gamma}a^{\gamma}-\frac{1}{2}b^{\dagger}_{\dot\gamma}b^{\dot\gamma}, \end{aligned} \tag{A.13}$$

which are the central charge and outer derivation, respectively. All fields in $\mathcal{N}=4$ SYM are uncharged with respect to the central charge C, therefore it can be set to zero. This procedure leads to $pl(4|4,\mathbf{R})$. Removing the generator B leads to the $sl(4|4,\mathbf{R})$ algebra. Dropping both B and C generators out lead to the $psl(4|4,\mathbf{R})$ algebra.

A.2 Twistor representation of $psl(4|4,\mathbf{R})$ superalgebra

The $sl(2)$, $sl(2)$, and $sl(4)$ rotation generators are given by

$$l^{\beta}_{\alpha} = \lambda_{\alpha}\frac{\partial}{\partial\lambda_{\beta}}-\frac{1}{2}\delta^{\beta}_{\alpha}\lambda_{\rho}\frac{\partial}{\partial\lambda_{\rho}} \tag{A.14}$$

$$\bar{l}^{\dot\alpha}_{\dot\beta} = \mu^{\dot\alpha}\frac{\partial}{\partial\mu^{\dot\beta}}-\frac{1}{2}\delta^{\dot\alpha}_{\dot\beta}\mu^{\dot\rho}\frac{\partial}{\partial\mu^{\dot\rho}} \tag{A.15}$$

$$r^{A}_{B} = \zeta^{A}\frac{\partial}{\partial\zeta^{B}}-\frac{1}{4}\delta^{A}_{B}\zeta^{D}\frac{\partial}{\partial\zeta^{D}} \tag{A.16}$$

The generator of translations $p_{\alpha\dot{\alpha}}$, conformal transformations $k^{\alpha\dot{\alpha}}$ and the supercharges are given by

$$p_{\alpha\dot{\alpha}} = \lambda_\alpha \frac{\partial}{\partial\mu^{\dot{\alpha}}}, \quad k^{\dot{\alpha}\alpha} = \mu^{\dot{\alpha}} \frac{\partial}{\partial\lambda_\alpha} \tag{A.17}$$

$$q_{A\alpha} = \lambda_\alpha \frac{\partial}{\partial\zeta^A}, \quad s^{A\alpha} = \zeta^A \frac{\partial}{\partial\lambda_\alpha} \tag{A.18}$$

$$\bar{q}^A_{\dot{\alpha}} = \zeta^A \frac{\partial}{\partial\mu^{\dot{\alpha}}}, \quad \bar{s}^{\dot{\alpha}}_A = \mu^{\dot{\alpha}} \frac{\partial}{\partial\zeta^A} \tag{A.19}$$

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