



On the construction of integrated vertex in the pure spinor formalism in curved background

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Abstract

We have previously described a way of describing the relation between unintegrated and integrated vertex operators in $AdS_5 \times S^5$ which uses the interpretation of the BRST cohomology as a Lie algebra cohomology and integrability properties of the AdS background. Here we clarify some details of that description, and develop a similar approach for an arbitrary curved background with nondegenerate RR bispinor. For an arbitrary curved background, the sigma-model is not integrable. However, we argue that a similar construction still works using an infinite-dimensional Lie algebroid.

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1. Introduction

The construction of the worldsheet sigma-model for the Type II superstring in the pure spinor formalism is a fundamental problem. It was more or less solved in [1]. However, we feel that some better understanding is possible. First of all, the sigma-model suggested in [1] is technically very special, and it is not clear why this is the most general solution. In particular, the formulation depends crucially on a special choice of fields. Indeed, the theory is not invariant un-

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der field redefinitions mixing matter fields with ghosts. It would be desirable to have an axiomatic formulation of the sigma-model. Something along these lines:

- A sigma-model with two nilpotent symmetries, Q_L and Q_R , such that the current of Q_L is holomorphic and the current of Q_R is antiholomorphic, and there are symmetries $U(1)_L$ and $U(1)_R$, such that Q_L and Q_R are appropriately charged under them.

However, we feel that this is not enough; the axiomatics sketched above is probably too weak, although it is enough to correctly describe small deformations of the flat space. The correct axiomatics should somehow encode the singularity of the pure spinor cone.

Also, we believe that the worldsheet sigma-model should be formulated as a problem in cohomological perturbation theory. A small neighborhood of each point in space–time can be approximated by flat space:

$$S = \int d\tau^+ d\tau^- (\partial_+ X^\mu \partial_- X^\mu + p_+ \partial_- \theta_L + p_- \partial_+ \theta_R + w_+ \partial_- \lambda_L + w_- \partial_+ \lambda_R) \quad (1)$$

with BRST symmetries:

$$Q_L = \lambda_L^\alpha \left(\frac{\partial}{\partial \theta_L^\alpha} + \Gamma_{\alpha\beta}^m \theta_L^\beta \frac{\partial}{\partial x^m} \right) + (\dots) \frac{\partial}{\partial w_+} \quad (2)$$

$$Q_R = \lambda_R^{\hat{\alpha}} \left(\frac{\partial}{\partial \theta_R^{\hat{\alpha}}} + \Gamma_{\hat{\alpha}\hat{\beta}}^m \theta_R^{\hat{\beta}} \frac{\partial}{\partial x^m} \right) + (\dots) \frac{\partial}{\partial w_-} \quad (3)$$

Then we say that a general background is obtained by the deformation of the action accompanied by some deformation² of Q_L and Q_R . The infinitesimal deformations at the linearized level are well known to correspond to the linearized SUGRA waves. They were classified in [3]. However, it was shown also in [3] that there is a potential obstacle to extending the deformations beyond the linearized level. The obstacle is a nonzero cohomology group, namely the ghost number three vertex operators. Without doubt, the obstacle actually vanishes (there is a nonzero cohomology group, but the actual class vanishes). This, however, is not well understood. As we explained in [3], one way to prove the vanishing of the obstacle is to consider the action of the b -ghost in cohomology. The formalism that would allow to do such calculation has not yet been fully developed. The definition of the b -ghost requires including the non-minimal fields which makes the lack of axiomatic formulation even more acute. And the b -ghost is nonpolynomial, opening the possibility that at some order the deformed action will also become non-polynomial.³ Such questions should be addressed together with the problem of axiomatic formulation of the worldsheet theory.

When we study the pure spinor formalism as a cohomological perturbation theory, one important technical aspect is the relation between integrated and unintegrated vertex operators. The deformation of the action is described by *integrated* vertices:

$$S \rightarrow S + \int d\tau^+ d\tau^- U \quad (4)$$

² As we have shown in [2], the very leading effect will be actually the deformation of Q_L and Q_R leaving S undeformed; this corresponds to the linear dilaton.

³ We have no doubt that this does not happen, it is just that we don't know how to see this using the cohomological perturbation theory.

It is very important, that such deformations are in one-to-one correspondence with the *unintegrated* vertices, which correspond to the cohomology of $Q_L + Q_R$. One of the goals of this paper is to better understand the correspondence between integrated and unintegrated vertices.

In [4–6] we have studied the relation between the pure spinor cohomology in $AdS_5 \times S^5$ and the Lie algebra cohomology, and argued that it is useful for understanding the relation between the integrated and unintegrated vertices. The pure spinor cohomology is the cohomology of the operator Q_{BRST} acting on the space of functions $F(g, \lambda_L, \lambda_R)$:

$$(Q_{\text{BRST}}F)(g, \lambda_L, \lambda_R) = \left(\lambda_L^\alpha L_\alpha + \lambda_R^{\hat{\alpha}} L_{\hat{\alpha}} \right) F(g, \lambda_L \lambda_R) \quad (5)$$

Here $g \in G = PSU(2, 2|4)$ and $L_\alpha, L_{\hat{\alpha}}$ are left shifts by some generators of $\mathfrak{psu}(2, 2|4)$. We introduced an infinite-dimensional Lie superalgebra $\mathcal{L}_{\text{tot}}$, and shown that the cohomology of Q_{BRST} is equivalent to some cohomology of $\mathcal{L}_{\text{tot}}$. Unintegrated vertices of the physical states correspond to the elements of the second cohomology group. Moreover, there is a Lax pair J_+, J_- taking values in $\mathcal{L}_{\text{tot}}$. Given a cohomology class represented by a cocycle $\psi : \Lambda^2 \mathcal{L}_{\text{tot}} \rightarrow \text{Fun}(G)$, the corresponding integrated vertex is $\psi(J_+, J_-)$. This construction uses special properties of $AdS_5 \times S^5$.

Here we will describe a similar construction for an arbitrary curved spacetime with the non-degenerate Ramond–Ramond field strength.⁴ Instead of the superalgebra $\mathcal{L}_{\text{tot}}$ we will use some super Lie algebroid. We will conjecture that the cohomology of this algebroid is equal to the BRST cohomology, i.e. unintegrated vertex operators. Moreover, there seems to be an analogue of a Lax pair, which allows to construct integrated vertices. However, this Lax pair takes values in the sections of a Lie algebroid (instead of a fixed Lie algebra), and presumably does not lead to integrability.

Better understanding of the integrated vertices could also help to explain the consistency of the higher orders of the deformation of the action (1). Superficially, this problem looks similar to the PBW theorem of quadratic-linear algebras which (coincidentally?) is also useful in the construction of the integrated vertex.

In eleven dimensional SUGRA, the analogous problem is the membrane worldsheet theory [7]. However, it appears more difficult than string worldsheet theory. But unintegrated vertices are more or less understood. Constructing integrated vertex operators is very close to understanding the worldsheet theory. Maybe some methods which we are developing here could be useful.

Summary of results Section 2 provides a streamlined proof of previously obtained results for the pure spinor superstring in $AdS_5 \times S^5$. Sections 3 and 4 provide a new point of view on the Type II pure spinor superspace, based on the study of the algebra generated by the covariant derivatives. This is somewhat similar to the Yang–Mills algebra, but quite different in details. Understanding this algebra requires careful analysis of the space–time gauge symmetries. The relations become degenerate in the case of zero Ramond–Ramond field; we feel that our approach could be particularly useful in the case of **strong RR field**. Actually, in Section 4.6 we give a new definition of the Ramond–Ramond fields which, in our opinion, is geometrically the most transparent one. (The problem of “reading” the RR field from the superspace action turns out to be nontrivial; we feel that Eqs. (176) and (177) provide the nicest solution.) As an immediate

⁴ Some elements of our construction become degenerate if the Ramond–Ramond field strength is zero. We do use the inverse of the RR bispinor $P^{\alpha\hat{\alpha}}$ in Section 4. See the discussion of the flat space limit in [6].

application, we provide a relation between integrated and unintegrated vertex operators, which is a nontrivial generalization of the similar relation in $AdS_5 \times S^5$. Unfortunately (or perhaps fortunately) this requires the introduction of some infinite-dimensional algebroid, which could be an interesting object in itself. There is even an algebroid analogue of a Lax pair, but it is not clear in what sense it leads to integrability.

Plan of the paper

- In Section 2 we give a streamlined review of [4–6], also simplifying some of the proofs in those references
- Section 3 develops a different point of view on the formalism of [11]; our approach emphasizes the similarity between the constraints of the Type IIB SUGRA and the constraints of the supersymmetric Yang–Mills theory
- In Section 4 we study the worldsheet currents. We construct an object resembling the Lax pair of the AdS theory, but using an algebroid instead of a Lie algebra. We conjecture that the cohomology of this algebroid corresponds to integrated vertex operators

2. Brief review of the case of $AdS_5 \times S^5$

Here we will briefly review the relation between the unintegrated and integrated vertices described in [4–6]. Both types of vertices are obtained from the same relative cocycle of some infinite-dimensional Lie superalgebra which we call $\mathcal{L}_{\text{tot}}$. In this sense, the relative Lie algebra cohomology of $\mathcal{L}_{\text{tot}}$ provides the unified description of both types of vertex operators.

2.1. Definition of $\mathcal{L}_{\text{tot}}$ and the PBW theorem

The infinite-dimensional superalgebra $\mathcal{L}_{\text{tot}}$ is defined in [5] by “gluing together” two copies of the Yang–Mills algebra which we call $\mathcal{L}_L$ and $\mathcal{L}_R$, in the following way. The $\mathcal{L}_L$ is generated by letters ∇_α^L , and $\mathcal{L}_R$ is generated by $\nabla_{\hat{\alpha}}^R$, satisfying the super–Yang–Mills constraints:

$$\{\nabla_\alpha^L, \nabla_\beta^L\} = \Gamma_{\alpha\beta}^m A_m^L, \quad \{\nabla_{\hat{\alpha}}^R, \nabla_{\hat{\beta}}^R\} = \Gamma_{\hat{\alpha}\hat{\beta}}^m A_m^R \quad (6)$$

(The existence of such A^L and A^R are the constraints.) All we need to do is to explain how ∇_α^L anticommutes with $\nabla_{\hat{\alpha}}^R$. For that we add a copy of the finite-dimensional algebra $\mathfrak{g}_0 = so(1, 4) \oplus so(5)$ with the generators denoted $t_{[mn]}^0$. We impose the commutation relations:

$$\{\nabla_\alpha^L, \nabla_{\hat{\alpha}}^R\} = f_{\alpha\hat{\alpha}}^{[mn]} t_{[mn]}^0 \quad (7)$$

$$[t_{[mn]}^0, \nabla_\alpha^L] = f_{[mn]\alpha}{}^\beta \nabla_\beta^L \quad (8)$$

$$[t_{[mn]}^0, \nabla_{\hat{\alpha}}^R] = f_{[mn]\hat{\alpha}}{}^{\hat{\beta}} \nabla_{\hat{\beta}}^R \quad (9)$$

$$[t_{[mn]}^0, t_{[pq]}^0] = f_{[mn][pq]}{}^{[rs]} t_{[rs]}^0 \quad (10)$$

where the coefficients $f_{\bullet\bullet\bullet}$ are the structure constants of $\mathfrak{g} = \mathfrak{psu}(2, 2|4)$.

One can consider the Lie algebra generated by the letters $\nabla_\alpha^L, \nabla_{\hat{\alpha}}^R, t_{[mn]}^0$ with the above relations, or the associative algebra generated by them. The associative algebra is the same as the universal enveloping $U\mathcal{L}_{\text{tot}}$.

The algebra $U\mathcal{L}_{\text{tot}}$ is an example of a quadratic-linear algebra. It is a *filtered* algebra; $F^p U\mathcal{L}_{\text{tot}}$ consists of those elements which can be represented by sums of products of $\leq p$ letters. For example $A_m^L \in F^2 U\mathcal{L}_{\text{tot}}$.

One can also define a *homogeneous* quadratic algebra $qU\mathcal{L}_{\text{tot}}$ as an associative algebra generated by the letters $\nabla_\alpha^L, \nabla_{\hat{\alpha}}^R, t_{[mn]}^0$ with the relations (6) and $\{\nabla_\alpha^L, \nabla_{\hat{\beta}}^R\} = [t_{[mn]}^0, \nabla_\alpha^L] = [t_{[mn]}^0, \nabla_{\hat{\alpha}}^R] = [t_{[mn]}^0, t_{[pq]}^0] = 0$. The algebra $qU\mathcal{L}_{\text{tot}}$ is *graded*; $\mathbf{gr}^p U\mathcal{L}_{\text{tot}}$ consists of those elements which consist of p letters.

Theorem 1 (PBW).

$$\mathbf{gr}^p U\mathcal{L}_{\text{tot}} = \mathbf{gr}^p (qU\mathcal{L}_{\text{tot}}) \quad (11)$$

Proof. Uses the fact that $qU\mathcal{L}_{\text{tot}}$ is a Koszul quadratic algebra. We will give a proof following Section 3.6.8 of [8]. We will need some standard language, which we will now review. Let V be the vector space generated by the letters $\nabla_\alpha^L, \nabla_{\hat{\alpha}}^R, t_{[mn]}^0$. Consider the subspace $R \subset V \otimes V$ generated by the following elements (the relations of $qU\mathcal{L}_{\text{tot}}$):

$$t_{[mn]}^0 \otimes t_{[pq]}^0 - t_{[mn]}^0 \otimes t_{[pq]}^0 \quad (12)$$

$$t_{[mn]}^0 \otimes \nabla_\alpha^L - \nabla_\alpha^L \otimes t_{[mn]}^0 \quad (13)$$

$$t_{[mn]}^0 \otimes \nabla_{\hat{\alpha}}^R - \nabla_{\hat{\alpha}}^R \otimes t_{[mn]}^0 \quad (14)$$

$$\nabla_\alpha^L \otimes \nabla_{\hat{\beta}}^R + \nabla_{\hat{\beta}}^R \otimes \nabla_\alpha^L \quad (15)$$

$$(\Gamma_{m_1 \dots m_5})^{\alpha\beta} \nabla_\alpha^L \otimes \nabla_\beta^L \quad (16)$$

$$(\Gamma_{m_1 \dots m_5})^{\hat{\alpha}\hat{\beta}} \nabla_{\hat{\alpha}}^R \otimes \nabla_{\hat{\beta}}^R \quad (17)$$

Notice that $qU\mathcal{L}_{\text{tot}}$ can be defined as the *factorspace* of the tensor algebra (=free algebra) TV modulo the ideal generated by R .

The dual coalgebra $U\mathcal{L}_{\text{tot}}^i$ is defined as the following *subspace* of TV :

$$U\mathcal{L}_{\text{tot}}^i = \mathbf{C} \oplus V \oplus R \oplus \bigoplus_{p=3}^{\infty} \bigcap_{q=0}^{p-2} (V^{\otimes q} \otimes R \otimes V^{\otimes(p-q-2)}) \quad (18)$$

The coalgebra structure is induced from the standard coalgebra structure of the tensor product:

$$\Delta(a \otimes b \otimes c \otimes \dots) = a|(b \otimes c \otimes \dots) + (a \otimes b)|(c \otimes \dots) + (a \otimes b \otimes c)|(\dots) + \dots \quad (19)$$

Explanation of notations For any coalgebra C , the coproduct Δ acts from C to $C \otimes C$. In our case, it so happens that C is itself defined as a tensor product. In this case it is common to use the notation $C|C$ instead of $C \otimes C$, just to avoid confusion. The spaces $C|C|\dots|C$ form the so-called *cobar complex*, because there is a natural differential:

$$d(x|y|z|\dots) = \Delta(x)|y|z|\dots - x|\Delta(y)|z|\dots + x|y|\Delta(z)|\dots - \dots \quad (20)$$

The nilpotence of this differential is equivalent to the co-associativity of Δ . This complex is denoted $\Omega(C)$. As a vector space $\Omega(C)$ is:

$$\Omega(C) = \bigoplus_{p=0}^{\infty} C^{\otimes p} \quad (21)$$

It is naturally an algebra, just a tensor (=free) algebra over C :

$$\Omega(C) = TC \quad (22)$$

Also notice that d respects the multiplication: $d(X|Y) = d(X)|Y - (-1)^{r_k(X)} X|dY$. This means that $\Omega(C)$ is a differential algebra. Let us consider the cohomology of d .

Observation 1.

$$H_d^0(\Omega(U\mathcal{L}_{\text{tot}}^i)) = qU\mathcal{L}_{\text{tot}} \quad (23)$$

Proof. This is obvious from the definitions.

So far the definition of $U\mathcal{L}_{\text{tot}}^i$ only used the homogeneous relations of $qU\mathcal{L}_{\text{tot}}$, it is really $(qU\mathcal{L}_{\text{tot}})^i$ rather than $U\mathcal{L}_{\text{tot}}^i$. We have to somehow take into account the nonzero right hand sides of (7), (8), (9), (10). This is done by supplying $U\mathcal{L}_{\text{tot}}^i$ with a differential d_1 , which is defined as follows⁵:

$$d_1(a \otimes b \otimes c \otimes \dots) = ((d_1(a \otimes b)) \otimes c \otimes \dots) - (a \otimes (d_1(b \otimes c)) \otimes \dots) + \dots \quad (24)$$

$$d_1(t_{[kl]}^0 \otimes t_{[mn]}^0 - t_{[mn]}^0 \otimes t_{[kl]}^0) = f_{[kl][mn]}^{[pq]} t_{[pq]}^0 \quad (25)$$

$$d_1(t_{[mn]}^0 \otimes \nabla_\alpha^L - \nabla_\alpha^L \otimes t_{[mn]}^0) = f_{[mn]\alpha}^\beta \nabla_\beta^L \quad (26)$$

$$d_1(t_{[mn]}^0 \otimes \nabla_{\hat{\alpha}}^R - \nabla_{\hat{\alpha}}^R \otimes t_{[mn]}^0) = f_{[mn]\hat{\alpha}}^{\hat{\beta}} \nabla_{\hat{\beta}}^R \quad (27)$$

$$d_1(\nabla_\alpha^L \otimes \nabla_{\hat{\beta}}^R + \nabla_{\hat{\beta}}^R \otimes \nabla_\alpha^L) = f_{\alpha\hat{\beta}}^{[mn]} t_{[mn]}^0 \quad (28)$$

$$d_1((\Gamma_{m_1 \dots m_5})^{\alpha\beta} \nabla_\alpha^L \otimes \nabla_\beta^L) = 0 \quad (29)$$

$$d_1((\Gamma_{m_1 \dots m_5})^{\hat{\alpha}\hat{\beta}} \nabla_{\hat{\alpha}}^R \otimes \nabla_{\hat{\beta}}^R) = 0 \quad (30)$$

The verification of the nilpotence of d_1 is equivalent to the verification of the Jacobi identity of $\mathcal{L}_{\text{tot}}$ in filtration ≤ 3 . There are the following cases to verify:

$$d_1^2(\Gamma_{m_1 \dots m_5}^{\alpha\beta} \nabla_\alpha^L \wedge \nabla_\beta^L \wedge \nabla_{\hat{\alpha}}^R) = 0 \quad (31)$$

$$d_1^2(\Gamma_{m_1 \dots m_5}^{\alpha\beta} \nabla_\alpha^L \wedge \nabla_\beta^L \wedge t_{[mn]}^0) = 0 \quad (32)$$

$$d_1^2(\nabla_\alpha^L \wedge \nabla_{\hat{\beta}}^R \wedge t_{[mn]}^0) = 0 \quad (33)$$

$$d_1^2(\nabla_\alpha^L \wedge t_{[mn]}^0 \wedge t_{[pq]}^0) = 0 \quad (34)$$

$$d_1^2(t_{[mn]}^0 \wedge t_{[pq]}^0 \wedge t_{[rs]}^0) = 0 \quad (35)$$

and similar equations with $L \leftrightarrow R$. Eq. (35) is the Jacobi identity for $\mathfrak{g}_0$. Eq. (34) says that the spinor representation is a representation of $\mathfrak{g}_0$. Eq. (33) is one of the Jacobi identities of the $psl(4|4)$:

⁵ Notice that the signs do not depend on whether $a, b, c, \dots$ are “odd” or “even”, and in fact we do not use such words at this point. The notion of “odd” or “even” elements only becomes useful when we say that our quadratic-linear algebra is in fact a universal enveloping of a *super*-Lie algebra.

$$f_{\alpha\hat{\beta}}^{[pq]} f_{[pq][mn]}^{[rs]} = f_{\alpha\hat{\gamma}}^{[rs]} f_{\hat{\beta}[mn]}^{\hat{\gamma}} + f_{\alpha[mn]}^{\gamma} f_{\gamma\hat{\beta}}^{[rs]} \quad (36)$$

Eq. (32) is derived as follows. After first time applying d_1 we get:

$$\begin{aligned} d_1 \left(\Gamma_{m_1 \dots m_5}^{\alpha\beta} \nabla_{\alpha}^L \wedge \nabla_{\beta}^L \wedge t_{[mn]}^0 \right) &= \\ &= \Gamma_{m_1 \dots m_5}^{\alpha\beta} \left(f_{\alpha[mn]}^{\gamma} \nabla_{\gamma}^L \wedge \nabla_{\beta}^L + f_{\beta[mn]}^{\gamma} \nabla_{\alpha}^L \wedge \nabla_{\gamma}^L \right) = \\ &= [\Gamma_{mn}, \Gamma_{m_1 \dots m_5}]^{\alpha\beta} \nabla_{\alpha}^L \wedge \nabla_{\beta}^L \end{aligned} \quad (37)$$

Since $[\Gamma_{mn}, \Gamma_{m_1 \dots m_5}]$ is a five-form, the second application of d_1 results in zero. Eq. (31) is derived as follows:

$$\begin{aligned} d_1^2 \left(\Gamma_{m_1 \dots m_5}^{\alpha\beta} \nabla_{\alpha}^L \wedge \nabla_{\beta}^L \wedge \nabla_{\hat{\alpha}}^R \right) &= \\ &= 2 \Gamma_{m_1 \dots m_5}^{\alpha\beta} f_{\beta\hat{\alpha}}^{[mn]} f_{\alpha[mn]}^{\gamma} \nabla_{\gamma}^L = \\ &= -\Gamma_{m_1 \dots m_5}^{\alpha\beta} f_{\beta\alpha}^k f_{\hat{\alpha}k}^{\gamma} \nabla_{\gamma}^L = 0 \end{aligned} \quad (38)$$

where we have taken into account that $f_{\beta\alpha}^k = \Gamma_{\beta\alpha}^k$ and therefore the contraction with $\Gamma_{m_1 \dots m_5}^{\alpha\beta}$ is zero. $\square$

Observation 2.

$$H_{d+d_1}^0(\Omega(U\mathcal{L}_{\text{tot}}^i)) = U\mathcal{L}_{\text{tot}} \quad (39)$$

Proof. This is also obvious from the definitions.

Notice that until now we have not done anything nontrivial, just developed a language. But now we are ready to proceed with the proof of the PBW theorem (11). Before the proof, we probably have to explain why the statement is nontrivial. Let us consider, for example, the following element of $\mathcal{L}_{\text{tot}}$:

$$X = [\{\nabla_{\alpha}^L, \nabla_{\beta}^L\}, \nabla_{\gamma}^L] \quad (40)$$

This expression can be represented by the following element of $V|V|V \subset \Omega(U\mathcal{L}_{\text{tot}}^i)$:

$$X = (\nabla_{\alpha}^L |\nabla_{\beta}^L + \nabla_{\beta}^L |\nabla_{\alpha}^L) |\nabla_{\gamma}^L - \nabla_{\gamma}^L |(\nabla_{\alpha}^L |\nabla_{\beta}^L + \nabla_{\beta}^L |\nabla_{\alpha}^L) \quad (41)$$

The question is, how do we know that this element is nonzero? Maybe one can prove that it is zero, using the relations of $\mathcal{L}_{\text{tot}}$? We know however that it is nonzero as an element of $\mathcal{L}_L$. (We are not going to prove it now; in fact this particular expression corresponds to the field strength superfield.) The $\mathcal{L}_L$ is a *homogeneous* quadratic algebra. We want to prove that X it is also nonzero as an element of $\mathcal{L}_{\text{tot}}$, an *inhomogeneous* (quadratic-linear) algebra. The danger is that maybe there is some element Y_0 , for example in $R|V|V \subset \Omega(U\mathcal{L}_{\text{tot}}^i)$, such that $dY_0 = 0$ and $d_1 Y_0 = X$. This would imply that $(d + d_1)Y_0 = X$ and therefore X is actually zero as an element of $\mathcal{L}_{\text{tot}}$. Or, perhaps there are $Y_0 \in V|V|V|R$ and $Y_1 \in V|V|R$ such that $d_1 Y_1 = X$ and $dY_1 = -d_1 Y_0$ and $dY_0 = 0$; then again $(d + d_1)(Y_0 + Y_1) = X$ and therefore X is zero. We deal with such fears in the following manner. Suppose $X = (d + d_1)Y$ and Y_0 be the highest bar-order term of Y (the term with the highest number of $|$). Then $dY_0 = 0$. Because $q\mathcal{L}_{\text{tot}}$ is Koszul,⁶ this

⁶ The Koszul property implies that the cohomology group corresponding to d -closed Y_0 modulo d -exact Y_0 vanishes, see Section 3.6.8 of [8] for details. The algebra of functions of ten-dimensional spinors satisfying $(\lambda\Gamma^m\lambda) = 0$ is Koszul

implies that $Y_0 = dZ_0$. We therefore have $(d + d_1)(Y - (d + d_1)Z_0) = X$, and the bar-order of $Y - (d + d_1)Z_0$ is one less than the bar-order of Y . We repeat this until the bar-order of Y is equal to the bar-order of X . Now $(d + d_1)Y = X$ implies that the leading order term in X is zero in qUL_{tot} . This contradicts the assumption and completes the proof of the PBW theorem. $\square$

Theorem 2. *As a linear space*

$$\mathcal{L}_{\text{tot}} = \mathcal{L}_L \oplus \mathcal{L}_R \oplus \mathfrak{g}_0 \quad (42)$$

Proof. The Lie superalgebra $\mathcal{L}_{\text{tot}}$ can be considered a subspace of UL_{tot} , consisting of those elements which can be represented as nested commutators. Then (42) follows from the PBW theorem. $\square$

Comment A physical interpretation of qUL_{tot} could be the flat space limit of UL_{tot} .

2.2. BRST complex

2.2.1. The structure of the dual coalgebra

Besides the PBW theorem, the Koszulity also implies that the complex $(UL_{\text{tot}})^i \otimes UL_{\text{tot}}$ is acyclic. Notice that $(UL_{\text{tot}})^i$ has the following structure. As a linear space:

$$(UL_{\text{tot}})^i = (UL_L)^i \otimes (UL_R)^i \otimes \Lambda^i \mathfrak{g}_0 \quad (43)$$

Any element $\omega \in (UL_{\text{tot}})^i$ can be presented as a linear sum of the expressions of the “decomposable” elements of the form:

$$\begin{aligned} \omega = & F^{\alpha_1 \dots \alpha_p} \hat{\beta}_1 \dots \hat{\beta}_q [m_1 n_1] \dots [m_r n_r] \\ & \nabla_{\alpha_1}^L \otimes \dots \otimes \nabla_{\alpha_p}^L \otimes \nabla_{\hat{\beta}_1}^R \otimes \dots \otimes \nabla_{\hat{\beta}_q}^R \otimes t_{[m_1 n_1]}^0 \otimes \dots \otimes t_{[m_r n_r]}^0 + \\ & + \dots \end{aligned} \quad (44)$$

where $F^{\alpha_1 \dots \alpha_p} \hat{\beta}_1 \dots \hat{\beta}_q [m_1 n_1] \dots [m_r n_r]$ is symmetric in $\alpha_1, \dots, \alpha_p$ and in $\hat{\beta}_1, \dots, \hat{\beta}_q$ and antisymmetric in $[m_1 n_1], \dots, [m_r n_r]$ and satisfies:

$$\Gamma_{\alpha_1 \alpha_2}^k F^{\alpha_1 \dots \alpha_p} \hat{\beta}_1 \dots \hat{\beta}_q [m_1 n_1] \dots [m_r n_r] = 0 \quad (45)$$

$$\Gamma_{\hat{\beta}_1 \hat{\beta}_2}^k F^{\alpha_1 \dots \alpha_p} \hat{\beta}_1 \dots \hat{\beta}_q [m_1 n_1] \dots [m_r n_r] = 0 \quad (46)$$

and ... in (44) stand for the terms which are obtained from the first term by permutations of the tensor product, which are needed so that the resulting expression belong to the exterior product of $p + q + r$ copies of the linear superspace generated by ∇^L , ∇^R and t^0 . For example, when $p = 2$ and $r = 1$, we get:

$$\omega = F^{\alpha_1 \alpha_2 [mn]} \left(\nabla_{\alpha_1}^L \otimes \nabla_{\alpha_2}^L \otimes t_{[mn]}^0 - \nabla_{\alpha_1}^L \otimes t_{[mn]}^0 \otimes \nabla_{\alpha_2}^L + t_{[mn]}^0 \otimes \nabla_{\alpha_1}^L \otimes \nabla_{\alpha_2}^L \right) \quad (47)$$

The action of d_1 on this ω is:

$$d_1 \omega = F^{\alpha_1 \alpha_2 [mn]} f_{[mn] \alpha_1}{}^\alpha \nabla_\alpha^L \otimes \nabla_{\alpha_2}^L \quad (48)$$

by the results of [9]. The SYM algebras $\mathcal{L}_L$ and $\mathcal{L}_R$ are both Koszul as quadratic duals to the Koszul algebra of pure spinors. The algebra qUL_{tot} is the commutative product of UL_L , UL_R and $\Lambda \mathfrak{g}_0$, and therefore is Koszul by the Corollary 1.2 in Chapter 3 of [10] (where the commutative product is denoted $\otimes^{q=1}$).

2.2.2. Computing $H^p(U\mathcal{L}_{\text{tot}}, V)$ as $\text{Ext}_{U\mathcal{L}_{\text{tot}}}(\mathbf{C}, V)$

We have seen that the complex:

$$\dots \rightarrow (U\mathcal{L}_{\text{tot}})_2^i \otimes U\mathcal{L}_{\text{tot}} \rightarrow (U\mathcal{L}_{\text{tot}})_1^i \otimes U\mathcal{L}_{\text{tot}} \rightarrow U\mathcal{L}_{\text{tot}} \rightarrow \mathbf{C} \rightarrow 0 \quad (49)$$

provides a free resolution of $\mathbf{C}$ as a $U\mathcal{L}_{\text{tot}}$ -module. Therefore, for any representation V of $U\mathcal{L}_{\text{tot}}$, we can compute the cohomology group:

$$H^p(U\mathcal{L}_{\text{tot}}, V) = \text{Ext}_{U\mathcal{L}_{\text{tot}}}(\mathbf{C}, V) \quad (50)$$

as the cohomology of the complex $\text{Hom}_{U\mathcal{L}_{\text{tot}}}((U\mathcal{L}_{\text{tot}})_p^i \otimes U\mathcal{L}_{\text{tot}}, V)$:

$$0 \rightarrow V \rightarrow \text{Hom}_{\mathbf{C}}((U\mathcal{L}_{\text{tot}})_1^i, V) \rightarrow \text{Hom}_{\mathbf{C}}((U\mathcal{L}_{\text{tot}})_2^i, V) \rightarrow \dots \quad (51)$$

We will now interpret this complex in terms of ghosts. An element of $(U\mathcal{L}_{\text{tot}})^i$ is the sum of expressions of the form (44). Notice that the dual space $((U\mathcal{L}_{\text{tot}})^i)'$ is the space of functions of commuting variables $c_L^\alpha, c_R^{\hat{\alpha}}$ satisfying the pure spinor constraints $c_L^\alpha \Gamma_{\alpha\beta}^m c_L^\beta = c_R^{\hat{\alpha}} \Gamma_{\hat{\alpha}\hat{\beta}}^m c_R^{\hat{\beta}} = 0$ and anticommuting variables $c_0^{[mn]}$. In this language the BRST operator becomes

$$\begin{aligned} Q_{\text{BRST}} = & c_0^{[mn]} \rho(t_{[mn]}^0) + f^\alpha_{\beta[mn]} c_L^\beta c_0^{[mn]} \frac{\partial}{\partial c_L^\alpha} + f^{\hat{\alpha}}_{\hat{\beta}[mn]} c_R^{\hat{\beta}} c_0^{[mn]} \frac{\partial}{\partial c_R^{\hat{\alpha}}} + \\ & + \frac{1}{2} f^{[mn]}_{[m_1 n_1][m_2 n_2]} c_0^{[m_1 n_1]} c_0^{[m_2 n_2]} \frac{\partial}{\partial c_0^{[mn]}} + \\ & + c_L^\alpha \rho(\nabla_\alpha^L) + c_R^{\hat{\alpha}} \rho(\nabla_{\hat{\alpha}}^R) + \\ & + f^{[mn]}_{\alpha\hat{\alpha}} c_L^\alpha c_R^{\hat{\alpha}} \frac{\partial}{\partial c_0^{[mn]}} \end{aligned} \quad (52)$$

Notice that the ghosts corresponding to $\mathbf{g}_{\bar{0}}$ are non-abelian while the ghosts c_L^α and $c_R^{\hat{\alpha}}$ are pure spinors. We will therefore call (51) the “mixed complex”: it is the pure spinor BRST complex coupled with the Serre–Hochschild complex of the finite-dimensional Lie algebra $\mathbf{g}_{\bar{0}}$.

2.2.3. Decoupling of the c_0 -ghosts

In the mixed complex (51) Let us consider the decreasing filtration by the power of c_L^α plus the power of $c_R^{\hat{\alpha}}$. The leading term is the cohomology of $\mathbf{g}_{\bar{0}}$ with values in the functions of (x, c_0, c_L, c_R) . Let us restrict ourselves to those vertex operators which are polynomial functions of x, c_0, c_L, c_R . The space of such operators splits as an infinite sum of finite-dimensional representations of $\mathbf{g}_{\bar{0}}$. Then the cohomology sits on the functions which do not depend on c_0 and are invariant under the action of $\mathbf{g}_{\bar{0}}$. The resulting complex is the physical BRST complex for the unintegrated vertex operators in $AdS_5 \times S^5$:

In the Serre–Hochschild complex of $\mathcal{L}_{\text{tot}}$ Similarly, the decoupling of the c_0 ghosts in the Serre–Hochschild complex of $\mathcal{L}_{\text{tot}}$ leads to the relative cohomology:

$$H^p(\mathcal{L}_{\text{tot}}, (U\mathbf{g})') = H^p(\mathcal{L}_{\text{tot}}, \mathbf{g}_{\bar{0}}; (U\mathbf{g})') \quad (53)$$

This establishes the relation between the BRST cohomology and the relative Lie algebra cohomology [4].

2.2.4. Cohomology of the ideal

Consider the ideal $I \subset \mathcal{L}_{\text{tot}}$ such that:

$$\mathcal{L}_{\text{tot}}/I = \mathbf{g} \quad (54)$$

By the Shapiro's theorem:

$$H^p(\mathcal{L}_{\text{tot}}, (U\mathbf{g})') = H^p(I) \quad (55)$$

This helps to identify various supergravity field strengths [4].

2.3. Integrated vertex

2.3.1. Generalized Lax operator

Consider a classical string solution in $AdS_5 \times S^5$, i.e. a field configuration in the worldsheet sigma-model solving the classical equations of motion. It was shown in [5,6] that one can construct the *Lax pair*:

$$L_+ = \left(\frac{\partial}{\partial \tau^+} + J_{0+}^{[mn]} t_{[mn]}^0 \right) + J_{3+}^\alpha \nabla_\alpha^L + J_{2+}^m A_m^L + (J_{1+})_\alpha W_L^\alpha + \lambda_L^\alpha w_{\beta+}^L \left(\{ \nabla_\alpha^L, W_L^\beta \} - f_\alpha^{\beta[mn]} t_{[mn]}^0 \right) \quad (56)$$

$$L_- = \left(\frac{\partial}{\partial \tau^-} + J_{0-}^{[mn]} t_{[mn]}^0 \right) + J_{1-}^{\dot{\alpha}} \nabla_{\dot{\alpha}}^R + J_{2-}^m A_m^R + (J_{3-})_{\dot{\alpha}} W_R^{\dot{\alpha}} + \lambda_R^{\dot{\alpha}} w_{\dot{\beta}-}^R \left(\{ \nabla_{\dot{\alpha}}^R, W_R^{\dot{\beta}} \} - f_{\dot{\alpha}}^{\dot{\beta}[mn]} t_{[mn]}^0 \right) \quad (57)$$

where $J_\pm$ and λ, w are worldsheet fields and t_0, ∇, A, W generators of $\mathcal{L}_{\text{tot}}$, satisfying the zero curvature equations:

$$[L_+, L_-] = 0 \quad (58)$$

and having simple BRST transformation laws:

$$Q_{BRST} L_\pm = \left[L_\pm, \left(\lambda_L^\alpha \nabla_\alpha^L + \lambda_R^{\dot{\alpha}} \nabla_{\dot{\alpha}}^R \right) \right] \quad (59)$$

We will denote $J_\pm$ the connections in $L_\pm$:

$$L_\pm = \frac{\partial}{\partial \tau^\pm} + J_\pm \quad (60)$$

2.3.2. Bicomplex $d + Q_{BRST}$

Let us denote: $J = J_+ d\tau^+ + J_- d\tau^-$ – an $\mathcal{L}_{\text{tot}}$ -valued one-form on the worldsheet. For the purpose of calculations, it is convenient to assume that $d\tau^+$ and $d\tau^-$ anticommute with the worldsheet fields θ :

$$d\tau^+ \theta^\alpha = -\theta^\alpha d\tau^+ \quad (61)$$

$$d\tau^- \theta^\alpha = -\theta^\alpha d\tau^- \quad (62)$$

We also introduce arbitrarily many anticommuting parameters $\epsilon_1, \epsilon_2, \dots$, which anticommute among themselves, with θ , and with $d\tau^\pm$. With these notations, we have:

$$(\epsilon_1 d + \epsilon_1 Q_{\text{BRST}}) (\epsilon_2 d \tau^j J_j - \epsilon_2 \lambda) = \quad (63)$$

$$= -\frac{1}{2} [\epsilon_1 d \tau^i J_i - \epsilon_1 \lambda, \epsilon_2 d \tau^i J_i - \epsilon_2 \lambda] \quad (64)$$

Schematically:

$$\epsilon_1 (d + Q_{\text{BRST}}) \epsilon_2 (J - \lambda) = -\frac{1}{2} [\epsilon_1 (J - \lambda), \epsilon_2 (J - \lambda)] \quad (65)$$

Also:

$$\epsilon_1 (d + Q_{\text{BRST}}) g = -\epsilon_1 \pi (J - \lambda) g \quad (66)$$

Given an n -cochain $\psi \in C^n(\mathcal{L}_{\text{tot}}, \mathbf{g}_0; (U\mathbf{g})')$, let us consider an inhomogeneous form $\text{ev}(\psi)$ on the worldsheet which can schematically be defined by the following formula:

$$\text{ev}_{\epsilon_1, \dots, \epsilon_n}(\psi) = \psi (\epsilon_1 (J - \lambda) \otimes \epsilon_2 (J - \lambda) \otimes \dots) (g) \quad (67)$$

This schematic notation is deciphered as follows. Notice that ψ is a function of the type:

$$\Lambda^n \mathcal{L}_{\text{tot}} \rightarrow [U\mathbf{g} \rightarrow \mathbf{C}] \quad (68)$$

We first evaluate it on $\epsilon_1 (J - \lambda) \otimes \epsilon_2 (J - \lambda) \otimes \dots$, which gives us a function of the type $U\mathbf{g} \rightarrow \mathbf{C}$. We then evaluate it on a “group element” $g \in U\mathbf{g}$. The “group elements” are defined as expressions of the form $g = e^\xi$ where $\xi \in \mathbf{g}$. Being infinite series, they strictly speaking do not belong to $U\mathbf{g}$. This rises the question of convergence, which we will ignore.

Then we observe:

$$\epsilon_1 (d + Q_{\text{BRST}}) \text{ev}_{\epsilon_2, \dots, \epsilon_{n+1}}(\psi) = \frac{1}{n+1} \text{ev}_{\epsilon_1, \dots, \epsilon_{n+1}}(Q_{\text{Lie}} \psi) \quad (69)$$

The derivation of this formula, besides (65) and (66), also uses the fact that ψ is a *relative* cocycle, and therefore:

$$\psi (\{\lambda, \lambda\} \otimes \dots) = \psi \left(2\lambda_L^\alpha \lambda_R^{\hat{\alpha}} f_{\alpha\hat{\alpha}}^{[mn]} t_{[mn]}^0 \otimes \dots \right) = 0 \quad (70)$$

In our case ψ is a 2-cocycle. Therefore:

$$(d + Q_{\text{BRST}}) \text{ev}_{\epsilon_1, \epsilon_2}(\psi) = 0 \quad (71)$$

This means that the ghost number two part of $\text{ev}_{\epsilon_1, \epsilon_2}(\psi)$ is an unintegrated vertex, and the ghost number zero part of $\text{ev}_{\epsilon_1, \epsilon_2}(\psi)$ is an integrated vertex.

Therefore our construction provides one way of thinking about the relation between unintegrated and integrated vertices.

3. General curved superspace

The pure spinor description of the Type IIB SUGRA emphasizes the local Lorentz symmetry of the supergravity theory. More specifically, the Type IIB superstring combines left and right sectors, and there are two copies of the local Lorentz group.

We will now describe some structure on the superspace, which we call “SUGRA data”. We first describe it as an abstract geometrical structure, and then explain how it emerges in the sigma-model.

3.1. Weyl superspace

The formulation of the pure spinor sigma-model in [1] uses the so-called Weyl superspace. In this formalism, besides the local Lorentz symmetry, there are also two copies of the local dilatation symmetries:

$$\hat{\mathbf{h}} = \hat{\mathbf{h}}_L \oplus \hat{\mathbf{h}}_R = \text{spin}(1, 9)_L \oplus \mathbf{R}_L \oplus \text{spin}(1, 9)_R \oplus \mathbf{R}_R \quad (72)$$

Let $S_L \oplus S_R$ denote the spinor representation of $\hat{\mathbf{h}}$. We will also denote $\hat{H}$ the Lie group corresponding to $\hat{\mathbf{h}}$. To summarize:

$$\hat{H} = \text{Spin}(1, 9)_L \times \mathbf{R}_L^\times \times \text{Spin}(1, 9)_R \times \mathbf{R}_R^\times \quad (73)$$

$$\hat{\mathbf{h}} = \text{Lie}(H) \quad (74)$$

$$S_L \oplus S_R = \text{spinor representation of } H \quad (75)$$

We will now start describing the SUGRA data.

Let M be a 10|32-dimensional supermanifold, the super-space–time.

The first part of the SUGRA data is:

- a distribution $S_L \oplus S_R \subset TM$
- for every point $x \in M$, an orbit of the action of $\hat{H}$ on some linear map $\mathcal{D} : S_L \oplus S_R \rightarrow S_L(x) \oplus S_R(x)$ (the action of $h \in \hat{H}$ is $\mathcal{D} \mapsto \mathcal{D} \circ h$); notice that the map $\mathcal{D}$ itself does not enter into the SUGRA data, only its orbit (with the action of $\hat{H}$ on it)

Let $\widehat{M} \xrightarrow{\pi} M$ be the principal bundle over M whose fiber over a point $x \in M$ is that orbit. In other words, a point of $\widehat{M}$ is a pair $(x, \mathcal{D})$. Let π denote the natural projection:

$$\begin{aligned} \pi : \widehat{M} &\rightarrow M \\ \pi(x, \mathcal{D}) &= x \end{aligned} \quad (76)$$

More explicitly, any linear map $\mathcal{D}$ is of the form:

$$\mathcal{D}(s_L + s_R) = s_L^\alpha E_\alpha^L + s_R^{\hat{\alpha}} E_{\hat{\alpha}}^R \quad (77)$$

$$E_\alpha^L \in S_L(x) \quad (78)$$

$$E_{\hat{\alpha}}^R \in S_R(x) \quad (79)$$

Sometimes we will simply write E_α and $E_{\hat{\alpha}}$ instead of E_α^L and $E_{\hat{\alpha}}^R$.

Let $\text{Vect}(\widehat{M}) = \Gamma(T\widehat{M})$ denote the infinite-dimensional space of all vector fields on $\widehat{M}$.

The second part of the SUGRA data is a map

$$\mathbf{D} : S_L \oplus S_R \rightarrow \text{Vect}(\widehat{M}) \quad (80)$$

$$\mathbf{D}(s_L + s_R) = s_L^\alpha \mathbf{D}_\alpha^L + s_R^{\hat{\alpha}} \mathbf{D}_{\hat{\alpha}}^R \quad (81)$$

satisfying the following properties:

- $\mathbf{D}$ commutes with the action of $\hat{H}$
- $\mathbf{D}$ is “fixed modulo $\text{Vect}\widehat{M}/M$ ” in the following sense: for any point $(x, \mathcal{D}) \in M$ let $\pi(x)$ be the natural projection $T_{(x, \mathcal{D})}\widehat{M} \rightarrow T_x M$, then

$$\pi(x)\left((\mathbf{D}(s_L + s_R))(x, \mathcal{D})\right) = \mathcal{D}(s_L + s_R) \quad (82)$$

(in other words, only the vertical component of $\mathbf{D}$ is non-obvious; the projection to TM is tautological)

- SUGRA constraints:

$$\begin{aligned} & \{\mathbf{D}(s_L + s_R), \mathbf{D}(s_L + s_R)\} = \\ & = (s_L \Gamma^m s_L) \mathbf{A}_m^L + (s_R \Gamma^m s_R) \mathbf{A}_m^R + \\ & \quad + R_{\alpha\beta}^{LL} s_L^\alpha s_L^\beta + R_{\dot{\alpha}\dot{\beta}}^{RR} s_R^{\dot{\alpha}} s_R^{\dot{\beta}} + R_{\alpha\dot{\beta}}^{LR} s_L^\alpha s_R^{\dot{\beta}} \end{aligned} \quad (83)$$

where:

- $\mathbf{A}_m^L$ and $\mathbf{A}_m^R$ are some sections of $\mathcal{T}\widehat{M}$ and
- $R_{\alpha\beta}^{LL}$, $R_{\dot{\alpha}\dot{\beta}}^{RR}$ and $R_{\alpha\dot{\beta}}^{LR}$ some sections of $\mathcal{T}\widehat{M}/M$ (i.e. vertical vector fields); they are essentially “curvatures”

Notice that satisfying the SUGRA constraints *does* depend on the vertical component of $\mathbf{D}$.

Moreover:

- there is an equivalence relation, which we will describe in Section 3.2.4

3.2. Relation to the formalism of [1]

3.2.1. SUGRA constraints, oversimplified

Let M be the super-space–time. In supergravity, M comes equipped with the distribution $\mathcal{S} \subset TM$. The SUGRA constraints are conditions on the Frobenius form of $\mathcal{S}$, which go roughly speaking as follows. We choose some vector fields ∇_α , $\alpha \in \{1, \dots, \dim \mathcal{S}\}$ tangent to $\mathcal{S}$ and say that:

$$\{\nabla_\alpha, \nabla_\beta\} = \Gamma_{\alpha\beta}^m A_m \text{ mod } \mathcal{S} \quad (84)$$

where A_m are some other vector fields. (The point of the constraint being that the RHS is proportional to $\Gamma_{\alpha\beta}^m$.) It is important to remember that when we write such conditions, we need to fix a basis of $\mathcal{S}$, i.e. a set of ∇_α . If we choose some linear combination:

$$\nabla'_\alpha = X_\alpha^\beta \nabla_\beta \quad (85)$$

then, generally speaking, ∇'_α will not satisfy the constraint (84). If we want ∇'_α to satisfy the constraint, we should require that $X \in so(1, 9) \oplus \mathbf{R}$ – an antisymmetric matrix plus a scalar. This means that $\mathcal{S}$ actually comes with an additional structure, namely an orbit of the action of $SO(1, 9) \times \mathbf{R}_\times$ on some linear map $\mathcal{D} : S \rightarrow S$ where S is the spinor representation of $so(1, 9) \oplus \mathbf{R}$. As we said in Section 3.1, the map $\mathcal{D}$ itself does not enter into the SUGRA data, only its orbit (with the action of $SO(1, 9) \times \mathbf{R}_\times$ on it). Given a point $x \in M$, and an orbit of $SO(1, 9) \times \mathbf{R}_\times$ in $\mathcal{S}(x)$, we can choose a point $\mathcal{D}$ in this orbit, then choose *any* set of vector fields ∇_α such that $\nabla_\alpha(x) = \mathcal{D}_\alpha$, and verify Eq. (84).

This means that it is useful instead of M to consider $\widehat{M}$, which is the $SO(1, 9) \times \mathbf{R}_\times$ -bundle over M whose fiber over $x \in M$ is that $SO(1, 9) \times \mathbf{R}_\times$ -orbit in $\text{Hom}_{\mathbf{C}}(S, \mathcal{S}(x))$ which we should have received as part of our SUGRA data. It is natural to think that the matter fields live in $\widehat{M}$ rather than M , except that *the fiber is a gauge degree of freedom*. The fiber can be gauged away because, as we said, the map $\mathcal{D} \in \text{Hom}_{\mathbf{C}}(S, \mathcal{S}(x))$ itself does not enter into the SUGRA data, only its orbit. This is how the $AdS_5 \times S^5$ sigma-model is formulated [11]. In that case M is $PSU(2, 2|4)/(SO(1, 4) \times SO(5))$ and $\widehat{M}$ is $PSU(2, 2|4)$. The sigma model has the $SO(1, 4) \times SO(5)$ gauge symmetry which gauges away the fiber. It is $SO(1, 4) \times SO(5)$ rather than $SO(1, 9) \times \mathbf{R}_\times$ because in that particular case some of the gauge symmetry can be canonically fixed.

In the sigma-model we couple matter fields with the ghosts λ which belong to the *pure spinor cone* $C \subset S$. As $\mathcal{D} \in \text{Hom}_{\mathbf{C}}(S, \mathcal{S})$ can be thought of as linear functions from S to $\mathcal{S}$, it make sense to apply it to $\lambda \in S$. The resulting vector field $\mathcal{D}(\lambda)$ describes the action of the BRST operator on the matter fields:

$$Q_{\text{matter}} = \mathcal{D}(\lambda) \quad (86)$$

3.2.2. Sigma-model

The target space of the sigma-model is $\widehat{M}$, but as we explained there is a gauge symmetry which reduces $\widehat{M} \rightarrow M$. The action, copied from [11], is:

$$\begin{aligned} S = \frac{1}{2\pi\alpha'} \int d^2z \bigg(& \frac{1}{2} (G_{MN}(Z) + B_{MN}(Z)) \partial Z^M \bar{\partial} Z^N + E_M^\alpha(Z) d_\alpha \bar{\partial} Z^M + \\ & + E_M^{\hat{\alpha}}(Z) \tilde{d}_{\hat{\alpha}} \partial Z^M + \Omega_{M\alpha}{}^\beta(Z) \lambda^\alpha w_\beta^L \bar{\partial} Z^M + \hat{\Omega}_{M\hat{\alpha}}{}^{\hat{\beta}}(Z) \tilde{\lambda}^{\hat{\alpha}} w_{\hat{\beta}}^R \partial Z^M + \\ & + P^{\hat{\alpha}\beta}(Z) d_\alpha \tilde{d}_{\hat{\beta}} + C_\alpha^{\beta\hat{\gamma}}(Z) \lambda^\alpha w_\beta^L \tilde{d}_{\hat{\gamma}} + \hat{C}_{\hat{\alpha}}^{\hat{\beta}\gamma}(Z) \tilde{\lambda}^{\hat{\alpha}} w_{\hat{\beta}}^R d_\gamma + \\ & + S_{\alpha\hat{\gamma}}^{\beta\hat{\delta}}(Z) \lambda^\alpha w_\beta^L \tilde{\lambda}^{\hat{\gamma}} w_{\hat{\delta}}^R + \frac{1}{2} \alpha' \Phi(Z) r + w_{\alpha+}^L \partial_- \lambda_\alpha^L + w_{\hat{\alpha}-}^R \partial_+ \lambda_{\hat{\alpha}}^R \bigg) \quad (87) \end{aligned}$$

In a generic background, one can integrate out $d, \tilde{d}$ and get a simpler-looking action. It is postulated that the field d should be the same as the density of the BRST charge. This is, essentially, a restriction on the choice of fields. Notice that the form of the Lagrangian (87) is not invariant under the field redefinitions, specifically under those redefinitions which mix the ghosts λ with the matter fields Z^M . (And this, in our opinion, is a defect of the formalism in its current form.)

The phase space of this sigma-model will be denoted $\mathcal{X}$. It can be identified with the moduli space of all classical solutions:

$$\mathcal{X} = \text{the space of classical solutions of the string } \sigma\text{-model}$$

We can also consider the space of off-shell field configurations:

$$\mathcal{X}_{\text{OS}} = \text{the space of off-shell field configurations}$$

3.2.3. From pure spinor Q to SUGRA constraints

We just said that the target space of the sigma-model is $\widehat{M}$. This, however, is not the full truth, because there are also ghosts. With ghosts, the target space is a cone in the associated vector bundle of the principal bundle $\widehat{M}$ corresponding to the spinor representation of $\hat{H}$:

$$\text{Target space with ghosts} = \widehat{M} \times_{\hat{H}} (C_L \times C_R) \quad (88)$$

where C_L is the pure spinor cone in S_L and C_R the pure spinor cone in S_R . The BRST operator of the sigma-model is a nilpotent odd vector field:

$$Q \in \text{Vect}(\widehat{M} \times_{\hat{H}} (C_L \times C_R)) \quad (89)$$

Generally speaking, consider a coset space X/G , where the action of G on X is free and transitive. Then vector fields on X/G can be described as follows. Let us start by considering the subalgebra $A \subset \text{Vect}(X)$ which consists of those vector fields which are invariant under the action of G , i.e. $g_*v = v$ for any $g \in G$ (a.k.a “Atiyah algebroid”). One can check that the vertical vector fields (those which are tangent to the orbits of G) are an ideal $I \subset A$. The factoralgebra is isomorphic to the algebra of vector fields on X/G :

$$\text{Vect}(X/G) \simeq A/I \quad (90)$$

Let us see how this description works in the particular case:

$$X = \widehat{M} \times (C_L \times C_R), \quad G = \hat{H} \text{ and } X/G = \widehat{M} \times_{\hat{H}} (C_L \times C_R)$$

Let us fix *some lift*

$$\mathbf{Lift} : \widehat{M} \times_{\hat{H}} (C_L \times C_R) \rightarrow \widehat{M} \times (C_L \times C_R) \quad (91)$$

Consider $\mathbf{Lift}_*Q$ – a vector field on the image of $\mathbf{Lift}$. Notice that $(\mathbf{Lift}_*Q)^2 = 0$, but don't forget that $\mathbf{Lift}_*Q$ is not a vector field on the whole $\widehat{M} \times (C_L \times C_R)$, but only on a submanifold – the image of $\mathbf{Lift}$. However, we can define an $\hat{H}$ -invariant vector field on the whole $\widehat{M} \times (C_L \times C_R)$ using the fact that $\widehat{M} \times (C_L \times C_R)$ is foliated by the translations of the image of $\mathbf{Lift}$ by elements of $\hat{H}$:

$$\widehat{M} \times (C_L \times C_R) = \bigcup_{h \in \hat{H}} h(\text{im}(\mathbf{Lift})) \quad (92)$$

This means that we can extend $\mathbf{Lift}_*Q$ to the whole $\widehat{M} \times (C_L \times C_R)$ in an h -invariant matter, simply by translating. In other words, let $Q^\uparrow$ be the nilpotent vector field on $\widehat{M} \times (C_L \times C_R)$ such that:

$$Q^\uparrow|_{\text{im}(\mathbf{Lift})} = \mathbf{Lift}_*Q \quad (93)$$

$$\text{for any } h \in \hat{H} : h_*Q^\uparrow = Q^\uparrow \quad (94)$$

A different choice of $\mathbf{Lift}$ will result in another $Q^\uparrow$, but the difference in $Q^\uparrow$ will be in adding a vertical vector field, i.e. and element of I . This is precisely (90).

Notice that the vertical component of $Q^\uparrow$ is $\hat{H}$ -invariant. Moreover, $C_L \times C_R$ is an orbit of the action of $\hat{H}$ on $S_L \times S_R$. In other words, any pair of pure spinors (λ_L, λ_R) can be obtained from a fixed pair $(\lambda_L^{(0)}, \lambda_R^{(0)})$ by the action of some element $h \in \hat{H}$. Therefore exists a vertical vector field ω such that the following vector field on $\widehat{M} \times (C_L \times C_R)$:

$$\widehat{Q} = Q^\uparrow + \omega \quad (95)$$

acts trivially on $C_L \times C_R$. In other words, $\widehat{Q}$ is a vector field on $\widehat{M}$.

To clarify the construction, let us describe it in coordinates. A point of $\widehat{M} \times (C_L \times C_R)$ is described in coordinates as follows: $(Z, (E_\alpha^L), (E_{\hat{\alpha}}^R), \lambda_L, \lambda_R)$. A point of $\widehat{M} \times_{\hat{H}} (C_L \times C_R)$ is described in the same way but with the equivalence relation:

$$(Z, (E_\alpha^L), (E_{\hat{\alpha}}^R), \lambda_L, \lambda_R) \sim (Z, ((h_L^{-1})_\alpha^{\alpha'} E_{\alpha'}^L), ((h_R^{-1})_{\hat{\alpha}}^{\hat{\alpha}'} E_{\hat{\alpha}'}^R), h_L \lambda_L, h_R \lambda_R) \quad (96)$$

Our **Lift** is essentially gauge fixing. It is described by specifying the functions:

$$E_\alpha^L = E_\alpha^{L0}(Z) \text{ and } E_{\hat{\alpha}}^R = E_{\hat{\alpha}}^{R0}(Z) \quad (97)$$

The way it works, for every point in $\widehat{M} \times_{\hat{H}} (C_L \times C_R)$, to calculate its lift we use the equivalence relations (96) to bring its coordinates $(Z, (E_\alpha^L), (E_{\hat{\alpha}}^R), \lambda_L, \lambda_R)$ to the form satisfying (97). The resulting $(Z, (E_\alpha^{L0}(Z)), (E_{\hat{\alpha}}^{R0}(Z)), \lambda_L^{\text{new}}, \lambda_R^{\text{new}})$ specifies a point in $\widehat{M} \times (C_L \times C_R)$, which is the lift. The lift of the BRST field **Lift** $_*Q$ is of the form:

$$\mathbf{Lift}_*Q_L = \lambda_L^\alpha \left(E_\alpha^{L0M}(Z) \frac{\partial}{\partial Z^M} + X_{\alpha\gamma}^\beta(Z) \lambda_L^\gamma \frac{\partial}{\partial \lambda_L^\beta} \right) \quad (98)$$

We must stress that **Lift** $_*Q$ is only defined on the image of **Lift**. To extend this vector field to the whole $\widehat{M} \times (C_L \times C_R)$, we must relax the gauge fixing (97). We observe that any E_α^L and $E_{\hat{\alpha}}^R$ can be presented in the form:

$$E_\alpha^L = (g^L)_\alpha^{\alpha'} E_{\alpha'}^{L0}(Z) \text{ and } E_{\hat{\alpha}}^R = (g^R)_{\hat{\alpha}}^{\hat{\alpha}'} E_{\hat{\alpha}'}^{R0}(Z) \quad (99)$$

Let us use $(Z, g^L, g^R, \lambda_L, \lambda_R)$ as coordinates on $\widehat{M} \times (C_L \times C_R)$. Then we have:

$$Q_L^\uparrow = \lambda_L^\alpha \left((g^L)_\alpha^{\alpha'} E_{\alpha'}^{L0M}(Z) \frac{\partial}{\partial Z^M} + (g^L)_\alpha^{\alpha'} X_{\alpha'\gamma'}^{\beta'}(Z) ((g^L)^{-1})_{\beta'}^\beta (g^L)_{\gamma'}^{\gamma} \lambda_L^\gamma \frac{\partial}{\partial \lambda_L^\beta} \right) \quad (100)$$

Finally:

$$\widehat{Q}_L = \lambda_L^\alpha \left((g^L)_\alpha^{\alpha'} E_{\alpha'}^{L0M}(Z) \frac{\partial}{\partial Z^M} + (g^L)_\alpha^{\alpha'} X_{\alpha'\gamma'}^{\beta'}(Z) (g^L)_\delta^{\gamma'} \frac{\partial}{\partial (g^L)_\delta^{\beta'}} \right) \quad (101)$$

– a vector field on $\widehat{M}$.

Notice that $\widehat{Q}_L$ depends linearly on λ_L . Therefore, $\widehat{Q}$ defines sixteen vector fields $\mathbf{D}_\alpha^L$:

$$\widehat{Q}_L = \lambda_L^\alpha \mathbf{D}_\alpha^L \quad (102)$$

These are the vector fields which were postulated in Section 3.1.

Ambiguity However, the definition of $\widehat{Q}_L$, and therefore of $\mathbf{D}_\alpha^L$, contains an ambiguity. It is possible to add to $\widehat{Q}_L$ a vertical vector field:

$$\widehat{Q}_{L,\text{new}} = \widehat{Q}_L + \lambda_L^\alpha \omega_\alpha^L \quad (103)$$

such that $\lambda_L^\alpha \omega_\alpha^L \in \text{St}(\lambda_L) \subset \hat{\mathbf{h}}_L$. This corresponds to the “shift gauge transformations” of [1]. We will now describe such ω_α .

3.2.4. Shift gauge transformations

Let us modify $\mathbf{D}_\alpha^L$ by adding to it a vector field in $T\widehat{M}/M$ (i.e. tangent to the fiber) of the form (cf. Eq. (61) of [1]):

$$(\omega_\alpha^L)^\beta_\gamma = (\Gamma^n \Gamma^m)_\gamma^\beta \Gamma_{\alpha\bullet}^m h_L^{\bullet n} \quad (104)$$

The characteristic property of such ω_α^L is that $\lambda_L^\alpha \omega_\alpha^L \in \text{St}(\lambda_L) \subset \hat{\mathbf{h}}_L$; in other words:

$$\lambda_L^\alpha (\omega_\alpha^L)^\beta_\gamma \lambda_L^\gamma = 0 \quad (105)$$

Similarly, we can modify $\mathbf{D}_{\hat{\alpha}}^R$ by adding to it some $\omega_{\hat{\alpha}}^R$ defined in a similar way; we stress that ω^L takes values in $\hat{\mathbf{h}}_L$ and ω_R takes values in $\hat{\mathbf{h}}_R$. Obviously, these “shift transformations” depend on two parameters: $h_L^{\alpha n}$ and $h_R^{\hat{\alpha} n}$. In terms of Section 3.1 this modifies $\mathbf{D}$:

$$\mathbf{D}_{\text{new}}(s_L + s_R) = \mathbf{D}(s_L + s_R) + s_L^\alpha \omega_\alpha^L + s_R^{\hat{\alpha}} \omega_{\hat{\alpha}}^R \quad (106)$$

3.2.5. SUGRA fields

The action (87) involves various SUGRA fields, which are either sections of associated vector bundles over M , or connections on them. They enter the action through their pullback on the string worldsheet.

Sections For example, $P^{\alpha\hat{\alpha}}$ is a section of $\hat{M} \times_{\hat{H}} (S_L \otimes S_R) \xrightarrow{\pi} M$. Such sections can be interpreted as $\hat{H}$ -invariant maps⁷ $\hat{M} \rightarrow S_L \otimes S_R$. From this point of view we consider $P^{\alpha\hat{\alpha}}$ as a function $P^{\alpha\hat{\alpha}}(Z, (E_\beta^L), (E_{\hat{\beta}}^R))$ such that:

$$P^{\alpha\hat{\alpha}}(Z, ((h_L)_{\beta}^{\beta'} E_{\beta'}^L), ((h_R)_{\hat{\beta}}^{\hat{\beta}'} E_{\hat{\beta}'}^R)) = (h_L^{-1})_{\alpha'}^{\alpha} (h_R^{-1})_{\hat{\alpha}'}^{\hat{\alpha}} P^{\alpha'\hat{\alpha}'}(Z, (E_{\beta}^L), (E_{\hat{\beta}}^R)) \quad (107)$$

Connections Connections are needed to define the kinetic terms for the ghost fields. A connection on the associated vector bundle $\hat{M} \times_{\hat{H}} (S_L \otimes S_R) \xrightarrow{\pi} M$ is constructed from a connection on the principal bundle $\hat{M} \xrightarrow{\pi} M$. We will now remind how this works. For any vector field $\xi \in \text{Vect}(M)$, a connection in the principal bundle defines a lift $\xi' \in \text{Vect}(\hat{M})$, which is $\hat{H}$ -invariant in the sense that for any $\chi \in \hat{\mathbf{h}}$ the corresponding vector field $v(\chi)$ commutes with ξ' :

$$[v(\chi), \xi'] = 0 \text{ for any } \chi \in \hat{\mathbf{h}} \quad (108)$$

For any representation $\rho : \hat{\mathbf{h}} \rightarrow \text{End}(V)$, sections of the associated bundle $\hat{M} \times_{\hat{H}} V \xrightarrow{\pi} M$ can be understood as maps $\sigma : \hat{M} \rightarrow V$, invariant under $\hat{H}$ in the following sense:

$$\mathcal{L}_{v(\chi)}\sigma = \rho(\chi)\sigma \text{ for any } \chi \in \hat{\mathbf{h}} \quad (109)$$

where $\mathcal{L}$ is the Lie derivative. Eq. (108) implies that for any σ satisfying (109), $\mathcal{L}_{\xi'}\sigma$ also satisfies (109). This means that the lift $\xi \mapsto \xi'$ consistently defines the action of ξ on the sections of the associated vector bundle.

Let us explain how a connection in the principal bundle $\hat{M} \rightarrow M$ defines a kinetic term for the ghosts. Consider the ghost λ_L ; in the flat space limit it is a left-moving field. In the general curved space, the kinetic term for λ_L should involve the derivative $\partial_- \lambda_L$. A point of the target space is $(Z, (E_\alpha^L), (E_{\hat{\alpha}}^R), \lambda_L, \lambda_R)$. The worldsheet is foliated by the characteristics. Let us consider the right-moving characteristic $\tau^+ = \text{const}$. It is parametrized by the τ^- :

$$(Z(\tau^-), (E_\alpha^L(\tau^-)), (E_{\hat{\alpha}}^R(\tau^-)), \lambda_L(\tau^-), \lambda_R(\tau^-)) \quad (110)$$

Let us choose a representative so that $\left(\frac{dZ(\tau^-)}{d\tau^-}, \left(\frac{dE_\alpha^L(\tau^-)}{d\tau^-}, \left(\frac{dE_{\hat{\alpha}}^R(\tau^-)}{d\tau^-} \right) \right) \right)$ is a horizontal vector, in the sense defined by the principal bundle connection in $\hat{M}$. Then the kinetic term is:

⁷ Indeed, every such map defines $\sigma : M \rightarrow \hat{M} \times_{\hat{H}} (S_L \otimes S_R)$ such that $\pi \circ \sigma = \text{id}$.

$$\int d\tau^+ d\tau^- \left(w_+^L, \frac{d\lambda_L}{d\tau^-} \right) \quad (111)$$

where w_+^L is the conjugate momentum to λ_L .

3.3. Lorentz superspace

There is a way to canonically fix $\mathbf{R}_L^\times \times \mathbf{R}_R^\times$. In this paper we will use the variant of the formalism which has $\mathbf{R}_L^\times \times \mathbf{R}_R^\times$ fixed. For us the gauge algebra is:

$$\mathbf{h} = \mathbf{h}_L \oplus \mathbf{h}_R = \text{spin}(1, 9)_L \oplus \text{spin}(1, 9)_R \quad (112)$$

This version of the formalism is called “Lorentz superspace”. We will now review how the Lorentz superspace is derived, as much as we understand.

Consider the sigma-model (87) and let us **integrate out** d_α and $\tilde{d}_{\hat{\beta}}$. It turns out that it is always possible to choose the gauge so that the coupling to the ghosts is only through the **traceless** currents⁸:

$$(w_+^L \Gamma_{mn} \lambda_L) \text{ and } (w_-^R \Gamma_{mn} \lambda_R) \quad (113)$$

The $u(1)$ combinations $(w_+^L \lambda_L)$ and $(w_-^R \lambda_R)$ appear only in the kinetic terms $(w_+^L \partial_- \lambda_L)$ and $(w_-^R \partial_+ \lambda_R)$. This fixes the gauge from $\hat{\mathbf{h}}_L \oplus \hat{\mathbf{h}}_R$ to $\mathbf{h}_L \oplus \mathbf{h}_R$. In the language of the present paper this simply means that we can use a slightly simpler $\hat{M}$. A point of this simplified $\hat{M}$ is a point $x \in M$ and a point in the orbit of H in $S(x) \subset T_x M$; the simplification is in replacing the orbit of $\hat{H} = \text{Spin}(1, 9)_L \times \mathbf{R}_L^\times \times \text{Spin}(1, 9)_R \times \mathbf{R}_R^\times$ with the orbit of $H = \text{Spin}(1, 9)_L \times \text{Spin}(1, 9)_R$. As in Section 3.2.3, we can still trade the BRST operator for an H -invariant vector field on $\hat{M}$. This statement is somewhat nontrivial, because what if the BRST operator Q involves a rescaling of λ_L and λ_R ? Let us consider the action of Q_R on λ_L :

$$Q_R \lambda_L^\alpha = \lambda_R^{\hat{\alpha}} X_{\hat{\alpha}\beta}^\alpha \lambda_L^\beta \quad (114)$$

In particular, the Q_R variation of the kinetic term $w_{\alpha+}^L \partial_- \lambda_L^\alpha$ gives the term $w_{\alpha+}^L X_{\hat{\alpha}\beta}^\alpha \lambda_L^\beta \partial_- \lambda_R^{\hat{\alpha}}$ which has nothing to cancel unless if $X_{\hat{\alpha}}$ is traceless, i.e. if $X_{\hat{\alpha}\alpha}^\alpha \neq 0$. (In this case it is canceled by the variation of the connection on which $w_{\alpha+}^L \partial_- \lambda_L^\alpha$ depends, implicitly in our language.) Now consider the action of Q_L on λ_L :

$$Q_L \lambda_L^\alpha = \lambda_L^\alpha X_{\alpha\gamma}^\beta \lambda_L^\gamma \quad (115)$$

Now it is even meaningless to ask if X_α is traceless or not, because X_α is only defined by (115) up to a shift transformation of Section 3.2.4. We therefore use these shift transformations to remove the trace of X_α . Then we have to remember that when we work in the Lorentz superspace formalism, the shift transformations have their parameter restricted to:

$$\Gamma_{\alpha\bullet}^n h_L^{\bullet n} = 0 \quad (116)$$

⁸ N. Berkovits, private communication; notice that we semiautomatically arrived at this gauge in our study of linearized excitations of $AdS_5 \times S^5$ in [6].

4. Worldsheet currents, quadratic-linear algebroid

Let us consider the algebroid $\mathcal{A}$ over $\widehat{M}$ freely generated by $\mathbf{D}$ satisfying Eq. (83) where $\mathbf{A}_m^L$ and $\mathbf{A}_m^R$ are free and $R_{\alpha\beta}^{LL}$, $R_{\dot{\alpha}\dot{\beta}}^{RR}$ and $R_{\alpha\dot{\beta}}^{LR}$ are *same* sections of $\Gamma(T\widehat{M}/M)$ as in Eq. (83). The definition of $\mathcal{A}$ is a direct generalization of the definition of $\mathcal{L}_{\text{tot}}$ in [4]. We “leave alone” the vertical generators $R_{\alpha\beta}^{LL}$, $R_{\dot{\alpha}\dot{\beta}}^{RR}$ and $R_{\alpha\dot{\beta}}^{LR}$ in the sense that their commutation relations are postulated as the commutation relation in $\Gamma(T\widehat{M}/M)$. But we consider $\mathbf{D}_\alpha^L$ and $\mathbf{D}_{\dot{\alpha}}^R$ and $\mathbf{A}_m^L$ and $\mathbf{A}_m^R$ as *free* generators modulo the relations (83).

(Open question: Does $\mathcal{A}$ satisfy a PBW theorem?)

From now on we will use letters $\mathbf{D}_\alpha^L$ and $\mathbf{D}_{\dot{\alpha}}^R$ to denote the generators of the algebroid. The vector fields defined in Eq. (81) will now be interpreted as the corresponding values of the anchor and therefore denoted $a(\mathbf{D}_\alpha^L)$ and $a(\mathbf{D}_{\dot{\alpha}}^R)$ (instead of simply $\mathbf{D}_\alpha^L$ and $\mathbf{D}_{\dot{\alpha}}^R$):

$$\begin{aligned} \mathbf{D}_\alpha^L, \mathbf{D}_{\dot{\alpha}}^R, \mathbf{A}_m^L, \mathbf{A}_m^R &: \text{generators of the algebroid } \mathcal{A} \\ a(\mathbf{D}_\alpha^L), a(\mathbf{D}_{\dot{\alpha}}^R), a(\mathbf{A}_m^L), a(\mathbf{A}_m^R) &: \text{vector fields on } \widehat{M} \end{aligned}$$

We introduce a bidirectional filtration on $\mathcal{A}$ in the following sense. For $n > 0$, we will say that $\xi \in \mathcal{A}_{\leq n}$ if ξ can be represented as a nested supercommutator of $\leq n$ generators $\mathbf{D}_\alpha^L$. For $n < 0$, we will say that $\xi \in \mathcal{A}_{\geq n}$ if ξ can be represented as a nested supercommutator of $\leq |n|$ generators $\mathbf{D}_{\dot{\alpha}}^R$. Notice that the expression containing nested supercommutators of both $\mathbf{D}_\alpha^L$ and $\mathbf{D}_{\dot{\alpha}}^R$ can be reduced to expressions containing either all $\mathbf{D}_\alpha^L$ or all $\mathbf{D}_{\dot{\alpha}}^R$.

4.1. Basic consequences of the defining relations

We observe that the basic commutation relations of (83) imply the existence of $\mathbf{W}_L^\alpha$ such that:

$$[\mathbf{D}_\alpha^L, \mathbf{A}_m^L] = \Gamma_{m\alpha\beta} \mathbf{W}_L^\beta \bmod \mathcal{A}_{\leq 1} \quad (117)$$

Furthermore, notice the existence of $\mathbf{F}_{[mn]}$ such that:

$$\{\mathbf{D}_\alpha^L, \mathbf{W}_L^\beta\} = (\Gamma^{mn})_\alpha^\beta \mathbf{F}_{[mn]} \bmod \mathcal{A}_{\leq 2} \quad (118)$$

Indeed:

$$\Gamma_{m\beta(\gamma)} \{\mathbf{D}_\alpha^L, \mathbf{W}_L^\beta\} = \{\mathbf{D}_{(\alpha}^L, [\mathbf{D}_{\gamma)}^L, \mathbf{A}_m^L]\} = \Gamma_{\alpha\gamma}^n [\mathbf{A}_n^L, \mathbf{A}_m^L] \bmod \mathcal{A}_{\leq 2} \quad (119)$$

and

$$10\{\mathbf{D}_\alpha^L, \mathbf{W}_L^\alpha\} = \Gamma_m^{\gamma\alpha} \Gamma_{\alpha\beta}^m \{\mathbf{D}_\gamma^L, \mathbf{W}_L^\beta\} = \Gamma_m^{\gamma\alpha} \{\mathbf{D}_\gamma^L, [\mathbf{D}_\alpha^L, \mathbf{A}_m^L]\} = 0 \bmod \mathcal{A}_{\leq 2} \quad (120)$$

This implies the existence of $\mathbf{F}_{[mn]}$.

4.2. Worldsheet currents

Remember that the string worldsheet is spanned by the left-moving characteristics $\tau^- = \text{const.}$ Consider an observer moving along a characteristic with the constant velocity $\dot{\tau}^+ = 1$. The velocity vector can be decomposed via the *worldsheet currents*:

$$\partial_+ Z^{\mathbf{M}} = \tilde{J}_{0+}^{LM} + \tilde{J}_{0+}^{RM} + \tilde{J}_{+}^{\alpha\mathbf{M}} a(\mathbf{D}_\alpha^L) + \tilde{\Pi}_{+}^m a(\mathbf{A}_m^L) + \tilde{\psi}_{\alpha+} a^{\mathbf{M}}(\mathbf{W}_L^\alpha) \quad (121)$$

Here we used the abbreviation:

$$\tilde{J}_{0+}^{LM} = \tilde{J}_{0+}^{L[mn]} a^M(t_{[mn]}^{L0}) \quad (122)$$

where $t_{[mn]}^0$ are generators of $\mathbf{h}_L$.

Notice that the “currents” $\tilde{J}_{0+}^{L[mn]}$, $\tilde{J}_{0+}^L$, $\tilde{J}_+^\alpha$, $\tilde{\Pi}_+^m$, $\tilde{\psi}_{\alpha+}$ are local functions on the phase space. We will denote the space of such functions $\text{Loc}(\mathcal{X})$:

$\mathcal{X}$ = phase space

$\text{Loc}(\mathcal{X})$ = the space of local functions on $\mathcal{X}$

At the same time, $a(t^{L0})$, $a(\mathbf{D}_\alpha^L)$, $a(\mathbf{A}_m^L)$ and $a(\mathbf{W}_L^\alpha)$ are vector fields on $\hat{M}$. Notice that a function $f(Z)$ on $\hat{M}$ and a point (τ^+, τ^-) on the worldsheet define a function on $\mathcal{X}$, namely $f(Z(\tau^+, \tau^-))$. In this sense, we should think of $\partial_+ Z$ as an element of the space:

$$\mathcal{V} = \text{Loc}(\mathcal{X}) \otimes_{\text{Fun}(\hat{M})} \text{Vect}(\hat{M}) \quad (123)$$

This is *not* an algebroid over $\mathcal{X}$, because generally speaking there is no way to lift a vector field on $\hat{M}$ to a vector field on the phase space. But this is possible if the vector field generates a symmetry of the sigma-model. When two elements $X \in \mathcal{V}$ and $Y \in \mathcal{V}$ both correspond to some symmetry of the sigma-model, then it is possible to define the commutator $[X, Y]$. Another way of turning $\mathcal{V}$ into an algebroid is to go off-shell, i.e. replace the $\mathcal{X}$ with the space of off-shell configurations $\mathcal{X}_{\text{OS}}$.

4.3. Tautological Lax pair

4.3.1. The case of $AdS_5 \times S^5$

Consider the sigma model of the classical string in $AdS_5 \times S^5$. It is classically integrable. There is a Lax pair, which depends on the spectral parameter z . At some particular value of z , the Lax pair becomes tautological, the zero curvature equations being just the Maurer–Cartan equation for the worldsheet currents. We will now briefly review how this goes.

The current is $J = -dg g^{-1}$. For any representation of $\mathfrak{g}$ with generators t_a , it is straightforward to verify the Maurer–Cartan equation:

$$\left[\frac{\partial}{\partial \tau^+} + J_+^a t_a, \frac{\partial}{\partial \tau^-} + J_-^b t_b \right] = 0 \quad (124)$$

We will need a slight variation of this construction. Let $\tilde{\mathfrak{g}}$ be the Lie superalgebra obtained from $\mathfrak{g}$ by changing the sign of the anticommutators (all the commutators are the same, but all the anticommutators have the opposite sign). The left regular representation of $\tilde{\mathfrak{g}}$ on the space of functions on the group manifold of G is defined as follows:

$$(L(\xi)f)(g) = \frac{d}{dt} \Big|_{t=0} f(e^{-t\xi} g) \quad (125)$$

This means that:

$$\frac{\partial f(g(\tau^+, \tau^-))}{\partial \tau^\pm} - (L(J)f)(g(\tau^+, \tau^-)) = 0 \quad (126)$$

We get⁹:

⁹ Notice the difference in sign between (126) and (127).

$$\left[\frac{\partial}{\partial \tau^+} + L(J_+), \frac{\partial}{\partial \tau^-} + L(J_-) \right] = 0 \quad (127)$$

Eq. (127) is almost the particular case of (124) corresponding to the left regular representation. The only difference is that the left regular representation, as we defined it, is the representation of $\tilde{\mathfrak{g}}$ and not $\mathfrak{g}$. But at the same time, notice that the odd–odd terms in $L(J_+)$ are of the form: $(-\partial_+ \theta^\alpha + \dots) \left(\frac{\partial}{\partial \theta^\alpha} + \dots \right)$ where $(-\partial_+ \theta^\alpha + \dots)$ is J_+^α and $\left(\frac{\partial}{\partial \theta^\alpha} + \dots \right) = \tilde{t}_\alpha$ is the corresponding generator of $\tilde{\mathfrak{g}}$, let us call it $\tilde{t}_\alpha$. Notice that $\tilde{t}_\alpha$ anti-commutes with J_+^α , while in (124) by definition J^a commute with t_a . In spite of this subtlety, the two definitions are actually equivalent. Given a Lax pair in the sense of (127), let us replace every term of the form $J^\alpha \tilde{t}_\alpha$ with $J^\alpha \tilde{t}_\alpha(-)^F$. Notice that $\tilde{t}_\alpha(-)^F$ are the generators of some representation of $\mathfrak{g}$ (which should also be called “left regular”), and also that $\tilde{t}_\alpha(-)^F$ commutes with J^α . Therefore we obtained the Lax pair in the sense of (124).

We can interpret the operator $\frac{\partial}{\partial \tau^\pm} + L(J_\pm)$ in the following way. Consider the space $\mathcal{X}_{\text{os}}$ of all field configurations (off-shell) in the classical sigma-model. Let $\text{Loc}(\mathcal{X}_{\text{os}})$ denote the space of all local functions on $\mathcal{X}_{\text{os}}$. Let us consider the space:

$$\text{Fun}(\hat{M}) \otimes_{\text{Fun}(\hat{M})} \text{Loc}(\mathcal{X}_{\text{os}}) \quad (128)$$

This is, obviously, the same as $\text{Loc}(\mathcal{X}_{\text{os}})$. Let us, however, define the action of the Lax operator on this space, as follows: $\frac{\partial}{\partial \tau^\pm}$ acts only on $\text{Loc}(\mathcal{X}_{\text{os}})$ and $L(J_\pm)$ acts only on $\text{Fun}(\hat{M})$. Our point here is that this action is correctly defined. For example, the action of $\frac{\partial}{\partial \tau^+} + \alpha L(J_+)$ with $\alpha \neq 1$ would not be correctly defined on (128), because it would act differently on $f \otimes \phi$ and $1 \otimes f\phi$.

4.3.2. General case

Consider the velocity of the coordinate function $Z^{\hat{M}}$ pulled back on the string worldsheet:

$$\begin{aligned} \frac{\partial Z^{\hat{M}}(\tau^+, \tau^-)}{\partial \tau^+} &= \tilde{J}_{0+}^{L[mn]} a^{\hat{M}}(t_{[mn]}^{L0}) + \tilde{J}_{0+}^{R[mn]} a^{\hat{M}}(t_{[mn]}^{R0}) + \\ &+ \tilde{J}_{+}^{\alpha} a^{\hat{M}}(\mathbf{D}_\alpha^L) + \tilde{\Pi}_{+}^m a^{\hat{M}}(\mathbf{A}_m^L) + \tilde{\psi}_{\alpha+} a^{\hat{M}}(\mathbf{W}_L^\alpha) \end{aligned} \quad (129)$$

We write $Z^{\hat{M}}$ instead of simply Z^M , to stress that the coordinates include also the fiber. In the $AdS_5 \times S^5$ language, $Z^{\hat{M}}$ would parametrize $PSU(2, 2|4)$ rather than AdS . The terms $\tilde{J}_{0+}^{L[mn]} a^{\hat{M}}(t_{[mn]}^{L0})$ and $\tilde{J}_{0+}^{R[mn]} a^{\hat{M}}(t_{[mn]}^{R0})$ are vertical (along the fiber).

Then $\left[\frac{\partial}{\partial \tau^+}, \frac{\partial}{\partial \tau^-} \right] Z^{\hat{M}} = 0$ leads to the tautological zero curvature equation:

$$\begin{aligned} &\frac{\partial}{\partial \tau^+} \left(\tilde{J}_{0-}^{LM} + \tilde{J}_{0-}^{RM} + \tilde{J}_{-}^{\alpha} a(\mathbf{D}_\alpha^L) + \tilde{\Pi}_{-}^m a(\mathbf{A}_m^L) + \tilde{\psi}_{\alpha-} a(\mathbf{W}_L^\alpha) \right) - \\ &- \frac{\partial}{\partial \tau^-} \left(\tilde{J}_{0+}^{LM} + \tilde{J}_{0+}^{RM} + \tilde{J}_{+}^{\alpha} a(\mathbf{D}_\alpha^L) + \tilde{\Pi}_{+}^m a(\mathbf{A}_m^L) + \tilde{\psi}_{\alpha+} a(\mathbf{W}_L^\alpha) \right) + \\ &+ \left[\tilde{J}_{0+}^{LM} + \tilde{J}_{0+}^{RM} + \tilde{J}_{+}^{\alpha} a(\mathbf{D}_\alpha^L) + \tilde{\Pi}_{+}^m a(\mathbf{A}_m^L) + \tilde{\psi}_{\alpha+} a(\mathbf{W}_L^\alpha), \right. \\ &\left. \tilde{J}_{0-}^{LM} + \tilde{J}_{0-}^{RM} + \tilde{J}_{-}^{\alpha} a(\mathbf{D}_\alpha^L) + \tilde{\Pi}_{-}^m a(\mathbf{A}_m^L) + \tilde{\psi}_{\alpha-} a(\mathbf{W}_L^\alpha) \right] = 0 \end{aligned} \quad (130)$$

In this formula $\frac{\partial}{\partial \tau^+}$ in the first line and $\frac{\partial}{\partial \tau^-}$ in the second line only act on the currents $\tilde{J}$, $\tilde{\Pi}$, $\tilde{\psi}$ and do not act on $a(\mathbf{D})$, $a(\mathbf{A})$, $a(\mathbf{W})$. The commutator is the commutator of the vector fields, e.g.:

$$\left[\tilde{\Pi}_{+}^m a(\mathbf{A}_m^L), \tilde{\Pi}_{-}^n a(\mathbf{A}_n^L) \right] = \tilde{\Pi}_{+}^m \tilde{\Pi}_{-}^n \left[a(\mathbf{A}_m^L), a(\mathbf{A}_n^L) \right] \quad (131)$$

Consequences of $[Q_L, \partial_+] = 0$ Let us consider the BRST variation:

$$\epsilon Q_L Z^{\hat{M}} = \epsilon \lambda_L^\alpha a^{\hat{M}}(\mathbf{D}_\alpha^L) \quad (132)$$

We have two vector fields on the phase space, Q_L and $\frac{\partial}{\partial \tau^+}$. They commute:

$$\begin{aligned} & (\epsilon Q_L \tilde{J}_{0+}^{L[mn]})a(t_{[mn]}^L) + (\epsilon Q_L \tilde{J}_{0+}^{R[mn]})a(t_{[mn]}^R) + \\ & + (\epsilon Q_L \tilde{J}_+^\alpha)a(\mathbf{D}_\alpha^L) + (\epsilon Q_L \tilde{\Pi}_+^m)a(\mathbf{A}_m^L) + (\epsilon Q_L \tilde{\psi}_{\alpha+})a(\mathbf{W}_L^\alpha) - \\ & - \partial_+(\epsilon \lambda_L^\beta)a(\mathbf{D}_\beta^L) + \\ & + \tilde{J}_{0+}^{L[mn]}a\left(\epsilon \lambda_L^\beta[\mathbf{D}_\beta^L, t_{[mn]}^0]\right) + \tilde{J}_{0+}^{R[mn]}a\left(\epsilon \lambda_L^\beta[\mathbf{D}_\beta^L, t_{[mn]}^0]\right) + \\ & + \tilde{J}_+^\alpha a\left(\epsilon \lambda_L^\beta\{\mathbf{D}_\beta^L, \mathbf{D}_\alpha^L\}\right) + \\ & + \tilde{\Pi}_+^m a\left(\epsilon \lambda_L^\beta[\mathbf{D}_\beta^L, \mathbf{A}_m^L]\right) + \\ & + \tilde{\psi}_{\alpha+} a\left(\epsilon \lambda_L^\beta\{\mathbf{D}_\beta^L, \mathbf{W}_L^\alpha\}\right) = 0 \end{aligned} \quad (133)$$

In particular, we can say something about $Q_L \tilde{\psi}_{\alpha+}$. Let us define the superfield $C_{\beta\gamma}^\alpha(Z)$ by the following formula:

$$\{a(\mathbf{D}_\beta^L), a(\mathbf{W}_L^\alpha)\} = C_{\beta\gamma}^\alpha a(\mathbf{W}_L^\gamma) \bmod \mathcal{A}_{[0,2]} \quad (134)$$

where $\bmod \mathcal{A}_{[0,2]}$ stands for a linear combination of $a(\mathbf{D}_\alpha^L)$, $a(\mathbf{A}_m^L)$, $a(t_{[mn]}^L)$ and $a(t_{[mn]}^R)$. From (133) we read:

$$(\epsilon Q_L \tilde{\psi}_{\alpha+})a(\mathbf{W}_L^\alpha) + a\left([\epsilon \lambda_L^\gamma \mathbf{D}_\gamma^L, \tilde{\psi}_{\alpha+} \mathbf{W}_L^\alpha]\right) + a\left([\epsilon \lambda_L^\gamma \mathbf{D}_\gamma^L, \tilde{\Pi}_+^m \mathbf{A}_m^L]\right) \in \mathcal{A}_{[0,2]} \quad (135)$$

and therefore:

$$Q_L \tilde{\psi}_{\alpha+} - (\tilde{\psi}_+ C_\alpha \lambda_L) + \lambda_L^\gamma \tilde{\Pi}_+^m \Gamma_{\gamma\alpha}^m = 0 \quad (136)$$

Similarly we have:

$$Q_L \tilde{\Pi}_+^m + \tilde{J}_+^\alpha \Gamma_{\alpha\beta}^m \lambda_L^\beta + \tilde{\psi}_{\alpha+} F_\beta^{\alpha m} \lambda_L^\beta = 0 \quad (137)$$

with some $F_\beta^{\alpha m}$ originating from $[\epsilon \lambda_L^\gamma \mathbf{D}_\gamma^L, \tilde{\psi}_{\alpha+} \mathbf{W}_L^\alpha]$. The nilpotence of Q_L implies:

$$\begin{aligned} Q_L^2 \tilde{\psi}_{\alpha+} &= -(\tilde{\psi}_+(Q_L C_\alpha) \lambda_L) + ((\tilde{\psi}_+ C \lambda_L) C_\alpha \lambda_L) - (\lambda_L Q_L (\tilde{\Pi}_+^m) \Gamma_\alpha^m) = \\ &= -R_{\alpha_1 \alpha_2 \alpha}^{\alpha'} \tilde{\psi}_{\alpha'+} \end{aligned} \quad (138)$$

Happily, the only part of $Q_L \tilde{\Pi}_+^m$ which gives a nonvanishing contribution is proportional to $\tilde{\psi}_{\alpha+}$; let us extract its coefficient:

$$\begin{aligned} & \lambda_L^{\alpha_1} \lambda_L^{\alpha_2} a(\mathbf{D}_{\alpha_1}^L) C_{\alpha_2 \alpha}^\beta = \\ &= \lambda_L^{\alpha_1} \lambda_L^{\alpha_2} C_{\alpha_1 \delta}^\beta C_{\alpha_2 \alpha}^\delta + \lambda_L^{\alpha_1} \lambda_L^{\alpha_2} R_{\alpha_1 \alpha_2 \alpha}^\beta + \lambda_L^{\alpha_1} \lambda_L^{\alpha_2} F_{\alpha_1}^{\beta m} \Gamma_{\alpha_2 \alpha}^m \end{aligned} \quad (139)$$

4.4. Identification of $\tilde{\psi}_{\alpha+}$

We will now show that $\tilde{\psi}_{\alpha+}$ can be identified as the matter part of the BRST charge density.

Remember that $w_{\perp}^L$ is the momentum conjugate to λ_L – see Eq. (87). Let us define¹⁰ $\psi_{\alpha+}$ and $d_{\alpha+}$ as follows:

$$\psi_{\alpha+} = \left(\tilde{\psi}_{\alpha+} - w_{\circ+}^L C_{\bullet\alpha}^{\circ} \lambda_L^{\bullet} \right) \quad (140)$$

$$d_{\alpha+} = Q_L(w_{\alpha+}^L) \quad (141)$$

It follows:

$$Q_L d_{\alpha+} = -\lambda_L^{\alpha_1} \lambda_L^{\alpha_2} R_{\alpha_1 \alpha_2 \alpha}^{\beta} w_{\beta+}^L \bmod (_)_{m+} \Gamma_{\alpha\gamma}^m \lambda_L^{\gamma} \quad (142)$$

Here “ $\bmod (_)_{m+} \Gamma_{\alpha\gamma}^m \lambda_L^{\gamma}$ ” means “up to adding $u_{m+} \Gamma_{\alpha\gamma}^m \lambda_L^{\gamma}$ with some arbitrary u_{m+} ”. Notice that $\lambda_L^{\alpha} d_{\alpha+}$ is the left BRST current. This follows from the fundamental property of the formalism: the $U(1)_L$ charge of the left BRST current is +1. We have:

$$\begin{aligned} Q_L \psi_{\alpha+} &= Q_L \tilde{\psi}_{\alpha+} - (d_{\circ+} C_{\bullet\alpha}^{\circ} \lambda_L^{\bullet}) - (w_{\circ+} (Q C_{\bullet\alpha}^{\circ}) \lambda_L^{\bullet}) = \\ &= (\tilde{\psi}_{\triangleright+} C_{\bullet\alpha}^{\triangleright} \lambda_L^{\bullet}) - \lambda_L^{\bullet} \tilde{\Pi}_{\triangleright+}^m \Gamma_{\bullet\alpha}^m - (d_{\triangleright+} C_{\bullet\alpha}^{\triangleright} \lambda_L^{\bullet}) - (w_{\triangleright+} (Q C_{\bullet\alpha}^{\triangleright}) \lambda_L^{\bullet}) = \\ &= ((w_{\triangleleft+} C_{\bullet\alpha}^{\triangleleft} \lambda_L^{\bullet}) C_{\triangleright\alpha}^{\bullet} \lambda_L^{\triangleright}) - (w_{\triangleleft+} (Q C_{\bullet\alpha}^{\triangleleft}) \lambda_L^{\bullet}) - \\ &\quad - \lambda_L^{\bullet} \tilde{\Pi}_{\triangleright+}^m \Gamma_{\bullet\alpha}^m + ((\psi_{\perp+} - d_{\perp+}) C_{\alpha} \lambda_L) \end{aligned} \quad (143)$$

It follows from Eq. (139) that:

$$((w_{\perp+} C_{\alpha} \lambda_L) C_{\alpha} \lambda_L) - (w_{\perp+} (Q C_{\alpha}) \lambda_L) = -w_{\beta+} \lambda_L^{\alpha_1} \lambda_L^{\alpha_2} R_{\alpha_1 \alpha_2 \alpha}^{\beta} \bmod (_)_{m+} \Gamma_{\alpha\gamma}^m \lambda_L^{\gamma} \quad (144)$$

Therefore:

$$Q_L \psi_{\alpha+} = -\lambda_L^{\alpha_1} \lambda_L^{\alpha_2} R_{\alpha_1 \alpha_2 \alpha}^{\beta} w_{\beta+}^L + ((\psi_{\circ+} - d_{\circ+}) C_{\bullet\alpha}^{\circ} \lambda_L^{\bullet}) \bmod (_)_{m+} \Gamma_{\alpha\gamma}^m \lambda_L^{\gamma} \quad (145)$$

Let us denote $\zeta_{\alpha+} = \psi_{\alpha+} - d_{\alpha+} \bmod (_)_{m+} \Gamma_{\alpha\gamma}^m \lambda_L^{\gamma}$.

Theorem 3.

$$\zeta_{\alpha+} = 0 \bmod (_)_{m+} \Gamma_{\alpha\bullet}^m \lambda_L^{\bullet} \quad (146)$$

Proof. Comparing Eqs. (145) and (142) we get:

$$Q_L \zeta_{\alpha+} = \zeta_{\circ+} C_{\bullet\alpha}^{\circ} \lambda_L^{\bullet} \bmod (_)_{m+} \Gamma_{\alpha\bullet}^m \lambda_L^{\bullet} \quad (147)$$

(the same equation as (136)).

It follows from the analysis of Eq. (147) in the flat space limit that any $\zeta_{\alpha+}$ satisfying (147) is of the form:

$$\zeta_{\alpha+} = \phi (d_{\alpha+} + C_{\bullet\alpha}^{\circ} w_{\circ+} \lambda_L^{\bullet}) + B_{\bullet\alpha}^{\circ} w_{\circ+} \lambda_L^{\bullet} \quad (148)$$

¹⁰ One could define $d_{\alpha+}$ through the density of the BRST charge Q_L , which is $\lambda_L^{\alpha} d_{\alpha+}$. Such a definition would only specify $d_{\alpha+}$ up to an addition of the terms of the form $X^m \Gamma_{\alpha\beta}^m \lambda_L^{\beta}$ and $X^{klm} \Gamma_{\alpha\beta}^{klm} \lambda_L^{\beta}$. It is possible to reduce this ambiguity by defining $d_{\alpha+}$ from Eq. (141); this leaves only the ambiguity of the form $X^m \Gamma_{\alpha\beta}^m \lambda_L^{\beta}$. In this sense, $d_{\alpha+}$ is “better-defined” than one might think.

where $\phi = \phi(Z)$ and $B_{\gamma\alpha}^\beta = B_{\gamma\alpha}^\beta(Z)$ are some functions. Indeed, in the flat space limit, in the neighborhood of *any* point of M , if θ and λ scale as $R^{-1/2}$ and x as R^{-1} and $w_\pm$ as $R^{-3/2}$, then $\zeta_{\alpha+}$ should be of the order $R^{-3/2}$; this means that the coefficients of $\tilde{J}_+$ and $\tilde{\Pi}_+$ in $\zeta_{\alpha+}$ are zero. The leading term in the flat space expansion of $\zeta_{\alpha+}$ is then of the form $\phi_\alpha^\beta d_{\beta+}$, and its BRST variation is in the leading order $(Q_L \phi_\alpha^\beta) d_{\beta+} + \phi_\alpha^\beta \Gamma_{\beta\gamma}^m \Pi_+^m \lambda_L^\gamma$. Therefore the vanishing of the leading term in Eq. (147) up to $(_)_{m+} \Gamma_{\alpha\bullet}^m \lambda^\bullet$ implies that ϕ_α^β is proportional to δ_α^β .

Notice that Eq. (147) is satisfied when $\phi = \text{const}$ and $B_{\gamma\alpha}^\beta = 0$. When ϕ is not constant, the vanishing of the coefficient of d_+ in (147) implies:

$$B_{\gamma\alpha}^\beta = \delta_\alpha^\beta D_\gamma^L \phi - \frac{1}{2} \Gamma_{\alpha\gamma}^m \Gamma_m^{\beta\bullet} D_\bullet^L \phi \quad (149)$$

and the vanishing of the coefficient of $w_+ \lambda_L \lambda_L$ implies:

$$\lambda_L^\bullet \lambda_L^\bullet \left(D_\bullet^L B_{\alpha\bullet}^\beta + C_{\bullet\alpha}^\circ B_{\bullet\circ}^\beta \right) w_{\beta+} = (_)_{m+} \Gamma_{\alpha\bullet}^m \lambda_L^\bullet \quad (150)$$

This is equivalent to the following equation being satisfied for any pure spinor λ_L :

$$\lambda_L^\bullet \lambda_L^\bullet \lambda_L^\bullet \left(D_\bullet^L B_{\bullet\bullet}^\beta + C_{\bullet\bullet}^\circ B_{\bullet\circ}^\beta \right) = 0 \quad (151)$$

Substitution of (149) gives:

$$C_{\bullet\bullet}^\beta \lambda_L^\bullet \lambda_L^\bullet \lambda_L^\bullet D_\bullet^L \phi = 0 \quad (152)$$

This implies that either $C_{\bullet\bullet}^\beta \lambda_L^\bullet \lambda_L^\bullet = 0$ for any λ_L , which is generally speaking not the case, or $\lambda_L^\bullet D_\bullet^L \phi = 0$, which implies that $\phi = \text{const}$. In the case of $AdS_5 \times S^5$ we know $\zeta_{\alpha+}$ is zero. Therefore $\phi = 0$ and Eq. (146) follows. $\square$

This means that in terms of the sigma-model (87):

$$\tilde{\psi}_{\alpha+} = P_{\alpha\hat{\alpha}}^{-1} E_M^{\hat{\alpha}} \partial_+ Z^M \quad (153)$$

$$d_{\alpha+} = \tilde{\psi}_{\alpha+} - C_{\beta\alpha}^\gamma \lambda^\beta w_{\gamma+} \bmod (_)_{m+} \Gamma_{\alpha\beta}^m \lambda_L^\beta \quad (154)$$

4.5. Identification of $C_{\alpha\gamma}^\beta$

Let us consider the SUGRA superfields $C_{\gamma}^{\beta\hat{\alpha}}$ and $P_{\alpha\hat{\alpha}}$ defined in Eq. (87). (Notice that we use the same letter C as for $C_{\alpha\gamma}^\beta$, but with a different set of indices; we hope this will not lead to confusion.) Eq. (154) implies that:

$$C_{\beta}^{\alpha\hat{\gamma}} P_{\hat{\gamma}\gamma}^{-1} = -C_{\beta\gamma}^\alpha \quad (155)$$

In particular, this implies that in the Lorentz superspace formalism (112):

$$C_{\alpha}^{\alpha\hat{\gamma}} = 0 \quad (156)$$

(This is not stated in [1].) One difference of our approach with [1] is that we do not require that $T_{\beta\gamma}^\alpha = 0$. In fact, it is difficult to define $T_{\beta\gamma}^\alpha$ in our language.

We will now confirm this by comparing the “shift” gauge transformations defined in Eq. (61) of [1]. They correspond to the following variation of ∇_α^L :

$$\delta_h \nabla_\alpha^L = \omega_\alpha \quad (157)$$

$$\omega_{\alpha\gamma}^\beta = (\Gamma_{\alpha\bullet}^k h^{\bullet n})(\Gamma_n^{\beta\circ} \Gamma_{\circ\gamma}^k) \quad (158)$$

where $h^{\alpha n}$ is a gauge parameter. In the Lorentz superspace formalism (112) the shift parameter satisfies:

$$\Gamma_{\alpha\beta}^n h^{\beta n} = 0 \quad (159)$$

Let us determine the transformation of $\psi_{\alpha+}$ and $C_{\beta\gamma}^\alpha$.

$$\{\delta_h \nabla_{(\alpha_1}^L, \nabla_{\alpha_2)}^L\} = -(\Gamma_{(\alpha_1|\bullet}^k h^{\bullet n}) \nabla_{\circ}^L (\Gamma_n^{\circ\bullet} \Gamma_{\bullet|\alpha_2)}^k) = \quad (160)$$

$$= \frac{1}{2} \Gamma_{\alpha_1\alpha_2}^p h^{n\triangleright} \Gamma_{\triangleright\bullet}^p \Gamma_n^{\bullet\circ} \nabla_\alpha^L \quad (161)$$

In other words:

$$\delta_h \mathbf{A}^p = h^{n\triangleright} \Gamma_{\triangleright\bullet}^p \Gamma_n^{\bullet\circ} \nabla_\alpha^L \quad (162)$$

$$\begin{aligned} [\nabla_\alpha, \delta_h \mathbf{A}^p] &= -h^{n\triangleright} \Gamma_{\triangleright\bullet}^p \Gamma_n^{\bullet\circ} \Gamma_{\circ\alpha}^k \mathbf{A}_k = \\ &= h^{n\triangleright} \Gamma_{\triangleright\bullet}^p \Gamma_p^{\circ\circ} \Gamma_{\circ\alpha}^k \mathbf{A}_k - 2h^{p\bullet} \Gamma_{\bullet\alpha}^k \mathbf{A}_k = \\ &= -h^{n\triangleright} \Gamma_{\triangleright\bullet}^n \Gamma_k^{\circ\circ} \Gamma_{\circ\alpha}^p \mathbf{A}_k - 2h^{p\bullet} \Gamma_{\bullet\alpha}^k \mathbf{A}_k + 2h^{n\bullet} \Gamma_{\bullet\alpha}^n \mathbf{A}_p \end{aligned} \quad (163)$$

$$[\delta_h \nabla_\alpha, \mathbf{A}^p] \Gamma_{\alpha_1\alpha_2}^p = -2\omega_{\alpha(\alpha_1}^\bullet \mathbf{A}^p \Gamma_{\alpha_2)}^\bullet = \quad (164)$$

$$= -2(\Gamma_{\alpha\circ}^k h^{\circ n})(\Gamma_n^{\bullet\circ} \Gamma_{\triangleleft(\alpha_1)}^k) \mathbf{A}^p \Gamma_{\alpha_2)}^\bullet = \quad (165)$$

$$= 2(\Gamma_{\alpha\circ}^k h^{\circ n}) \Gamma_{\alpha_1\alpha_2}^n \mathbf{A}^k - 4(\Gamma_{\alpha\circ}^k h^{\circ(k)} \Gamma_{\alpha_1\alpha_2}^p) \mathbf{A}^p \quad (166)$$

where we used the gamma-matrix identity:

$$\Gamma_{(\alpha_1|\bullet}^p \Gamma_n^{\bullet\circ} \Gamma_{\triangleleft(\alpha_2)}^k) = (\Gamma^p \Gamma_n \Gamma^k)_{(\alpha_1\alpha_2)} = (\Gamma^p \Gamma_n \Gamma^k)_{(\alpha_1\alpha_2)} = -\delta^{pk} \Gamma_{\alpha_1\alpha_2}^n + 2\delta^{n(k} \Gamma_{\alpha_1\alpha_2}^p) \quad (167)$$

and therefore:

$$[\delta_h \nabla_\alpha, \mathbf{A}^p] = 2\Gamma_{\alpha\circ}^k h^{\circ p} \mathbf{A}^k - 2(\Gamma_{\alpha\circ}^k h^{\circ k}) \mathbf{A}^p - 2(\Gamma_{\alpha\circ}^p h^{\circ k}) \mathbf{A}^k \quad (168)$$

This implies:

$$\delta_h \mathbf{W}_L^\alpha = -h^{n\bullet} \Gamma_{\bullet\circ}^n \Gamma_k^{\circ\alpha} \mathbf{A}_k - 2h^{\alpha k} \mathbf{A}^k \quad (169)$$

$$\{\nabla_\beta^L, \delta_h \mathbf{W}_L^\alpha\} = h^{n\bullet} \Gamma_{\bullet\circ}^n \Gamma_k^{\circ\alpha} [\nabla_\beta^L, \mathbf{A}_k] + 2h^{\alpha k} [\nabla_\beta^L, \mathbf{A}^k] = \quad (170)$$

$$= h^{n\bullet} \Gamma_{\bullet\circ}^n \Gamma_k^{\circ\alpha} \Gamma_{\beta\gamma}^k \mathbf{W}_L^\gamma + 2h^{\alpha k} \Gamma_{\beta\gamma}^k \mathbf{W}_L^\gamma \quad (171)$$

At the same time:

$$\{\delta_h \nabla_\beta^L, \mathbf{W}_L^\alpha\} = (h^{n\circ} \Gamma_{\circ\beta}^k)(\Gamma_n^{\alpha\bullet} \Gamma_{\bullet\gamma}^k) \mathbf{W}_L^\gamma - 4h^{k\circ} \Gamma_{\circ\beta}^k \mathbf{W}_L^\alpha \quad (172)$$

where the term $-4h^{k\circ} \Gamma_{\circ\beta}^k \mathbf{W}_L^\alpha$ corresponds to the trace part of ω . Therefore:

$$\begin{aligned} \delta_h \{\nabla_\beta^L, \mathbf{W}_L^\alpha\} &= h^{n\bullet} \Gamma_{\bullet\circ}^n \Gamma_k^{\circ\alpha} \Gamma_{\beta\gamma}^k \mathbf{W}_L^\gamma + 2h^{\alpha k} \Gamma_{\beta\gamma}^k \mathbf{W}_L^\gamma + \\ &+ (h^{n\bullet} \Gamma_{\bullet\beta}^k)(\Gamma_n^{\alpha\circ} \Gamma_{\circ\gamma}^k) \mathbf{W}_L^\gamma - 4h^{k\circ} \Gamma_{\circ\beta}^k \mathbf{W}_L^\alpha \end{aligned} \quad (173)$$

This implies that¹¹:

$$\begin{aligned}\delta_h C_{\beta\gamma}^\alpha &= h^{n\bullet} \Gamma_{\bullet\circ}^n \Gamma_k^{\circ\alpha} \Gamma_{\beta\gamma}^k + 2h^{n\alpha} \Gamma_{\beta\gamma}^n + \\ &+ h^{n\bullet} \Gamma_{\bullet\beta}^k \Gamma_n^{\alpha\circ} \Gamma_{\circ\gamma}^k - 4h^{n\bullet} \Gamma_{\bullet\beta}^n \delta_\gamma^\alpha\end{aligned}\quad (174)$$

Therefore:

$$\begin{aligned}\delta_h C_{\beta\gamma}^\alpha + \delta_h (C_{\beta\gamma}^{\alpha\hat{\alpha}} P_{\hat{\alpha}\gamma}^{-1}) &= h^{n\bullet} \Gamma_{\bullet\circ}^n \Gamma_k^{\circ\alpha} \Gamma_{\beta\gamma}^k + 2h^{n\alpha} \Gamma_{\beta\gamma}^n + \\ &+ 2h^{n\bullet} \Gamma_{\bullet(\beta}^k \Gamma_{|\gamma)}^{\alpha\circ} \Gamma_{\circ|\gamma)}^k - 4h^{n\bullet} \Gamma_{\bullet\beta}^n \delta_\gamma^\alpha = \\ &= -4h^{n\bullet} \Gamma_{\bullet\beta}^n \delta_\gamma^\alpha \bmod (_)_m \Gamma_{\alpha\beta}^m \lambda_L^\beta\end{aligned}\quad (175)$$

Given (159), this is in agreement with (155).

4.6. Ramond–Ramond fields

The Ramond–Ramond bispinor $P^{\alpha\hat{\alpha}}$ must have a similar interpretation. Let us expand $\mathbf{D}_\alpha^R$ in terms of $\mathbf{W}_L^\alpha$, $\mathbf{A}_L^m$, $\mathbf{D}_\alpha^L$ and $T\hat{M}/M$. We will get:

$$\mathbf{D}_{\hat{\alpha}}^R = P_{\hat{\alpha}\alpha} \mathbf{W}_L^\alpha + \dots \quad (176)$$

where ... stand for the terms proportional to $\mathbf{A}_L^m$ and $\mathbf{D}_\alpha^L$ and $T\hat{M}/M$. (Again, the coefficients of $\mathbf{A}_L^m$ and $\mathbf{D}_\alpha^L$ are not defined unambiguously, but the coefficient of $\mathbf{W}_L^\alpha$ is well-defined.) We conjecture that $P_{\hat{\alpha}\alpha}$ is the inverse of the $P^{\alpha\hat{\alpha}}$ of (87), however we do not have a proof. Similarly:

$$\mathbf{D}_\alpha^L = P_{\alpha\hat{\alpha}} \mathbf{W}_R^{\hat{\alpha}} + \dots \quad (177)$$

This is probably the most concise definition of the Ramond–Ramond bispinor in the framework of the pure spinor sigma-model.

4.7. Weighing anchor

Consider a linear map κ :

$$\kappa : \mathcal{T}\hat{M} \rightarrow \mathcal{A} \quad (178)$$

such that:

$$\text{im } \kappa = \mathcal{A}_{[0,3]} \quad (179)$$

$$a \circ \kappa = \text{id} : \mathcal{T}\hat{M} \rightarrow \mathcal{T}\hat{M} \quad (180)$$

Notice that the following operator:

$$a^\perp = \text{id} - \kappa \circ a \quad (181)$$

is the projection to $\ker(a)$ along $\mathcal{A}_{[0,3]}$.

Let us unapply the anchor from the RHS of (121):

$$\tilde{\mathbf{L}}_+ = \partial_+ + \tilde{J}_{0+}^{L[mn]} t_{L[mn]}^0 + \tilde{J}_{0+}^{R[mn]} t_{R[mn]}^0 + \tilde{J}_+^\alpha \mathbf{D}_\alpha^L + \tilde{\Pi}_+^m \mathbf{A}_m^L + \tilde{\psi}_{\alpha+} \mathbf{W}_L^\alpha \quad (182)$$

¹¹ As a consistency check, $\delta_h C_{\alpha\gamma}^\alpha = 0$ and $\delta(\Gamma^{klmn})_\alpha^\beta C_{\beta\gamma}^\alpha = 0$.

Notice that:

$$(\mathbf{Q}_L + \mathbf{Q}_R)^2 \tilde{\mathbf{L}}_+ = 0 \quad (183)$$

Indeed, let us for example look at the $\lambda_L \lambda_L$ part:

$$\mathbf{Q}_L^2 = Q_L^2 + \frac{1}{2} \lambda_L^\alpha \lambda_L^\beta \{\mathbf{D}_\alpha^L, \mathbf{D}_\beta^L\} = Q_L^2 + \frac{1}{2} \lambda_L^\alpha \lambda_L^\beta R_{\alpha\beta} \quad (184)$$

This implies that the calculation of the action of $\mathbf{Q}_L^2$ on $\tilde{\mathbf{L}}_+$ does not lead out of $\mathcal{A}_{[0,3]}$. Therefore the calculation is the same as it would be under the anchor, and the result is zero.

Also notice that:

$$a((\mathbf{Q}_L + \mathbf{Q}_R) \tilde{\mathbf{L}}_+) = 0 \quad (185)$$

However it is not true that $(\mathbf{Q}_L + \mathbf{Q}_R) \tilde{\mathbf{L}}_+ = 0$; we will therefore correct $\tilde{\mathbf{L}}_+$ by adding to it some expression with zero anchor.

4.8. Correction $\tilde{\mathbf{L}}_+ \rightarrow \mathbf{L}_+$

4.8.1. General theory

Let us consider deforming:

$$\tilde{\mathbf{L}}_+ \mapsto \tilde{\mathbf{L}}_+ + \Delta \mathbf{L}_+ \quad (186)$$

where $\Delta \mathbf{L}_+$ does not contain the derivative ∂_+ and is an anchorless element of $\mathcal{A}$ such that:

$$(\mathbf{Q}_L + \mathbf{Q}_R)^2 \Delta \mathbf{L}_+ = 0 \quad (187)$$

We also require that $\Delta \mathbf{L}_+$ be $\mathbf{h}$ -invariant. Let us denote $\mathcal{Y}_+$ the linear space of all expressions $\mathbf{X}_+ \in \mathcal{A}$ satisfying the following properties:

1. $\mathbf{X}_+$ is $\mathbf{h}$ -invariant
2. $\mathbf{X}_+$ has conformal dimension $(1, 0)$
3. $(\mathbf{Q}_L + \mathbf{Q}_R)^2 \mathbf{X}_+ = 0$

By definition $\Delta \mathbf{L}_+$ belongs to $\mathcal{Y}_+$.

Observation 3. *The cohomology of the operator $\mathbf{Q}_L + \mathbf{Q}_R$ acting in $\mathcal{Y}_+$ is zero.*

Proof. Let us prove that the cohomology of $\mathbf{Q}_L$ is zero. First of all let us prove this statement in flat space. In flat space the algebroid is homogeneous, it is defined by the same relations as $qU\mathcal{L}_{\text{tot}}$. Given an expression annihilated by $\mathbf{Q}_L$, let us consider the term with the lowest number of the letters ∇ . It is Q_L -closed. Since the cohomology of Q_L in the expressions of the conformal dimension $(1, 0)$ is trivial, this means that this lowest order term is exact. This completes the proof that the cohomology of $\mathbf{Q}_L$ is zero in flat space.

In a general curved space, let us use the near-flat-space expansion (see [2] for details). For an element $\phi \in \mathcal{A}$ let us define its degree $\deg(\phi)$ so that $\deg(\theta) = \deg(\lambda) = 1$, $\deg(w) = 3$, $\deg(x) = 2$ and $\deg(\mathbf{D}^L) = \deg(\mathbf{D}^R) = -1$. The proof follows from the following observations:

- $\deg((\mathbf{Q}_L + \mathbf{Q}_R)\phi) \geq \deg(\phi)$
- the action in the associated graded space is the same as in flat space,
- we have just proven that the cohomology in flat space is zero. $\square$

Observation 3 implies the existence of such a $\mathbf{Y}_+ \in \mathcal{Y}_+$ that:

$$(\mathbf{Q}_L + \mathbf{Q}_R)(\tilde{\mathbf{L}}_+ + \mathbf{Y}_+) = 0 \quad (188)$$

We therefore denote:

$$\mathbf{L}_+ = \tilde{\mathbf{L}}_+ + \mathbf{Y}_+ \quad (189)$$

4.8.2. Explicit construction

(If the reader is not familiar with the construction of the Lax operator for $AdS_5 \times S^5$ [12] we would recommend to first look at [13].)

We will now show that the leading term of $\mathbf{Y}_+$ is in degree four, i.e. $\mathbf{L}_+$ is of the form:

$$\begin{aligned} \mathbf{L}_+ = & \partial_+ + J_{0+}^{L[mn]} t_{L[mn]}^0 + J_{0+}^{R[mn]} t_{R[mn]}^0 + \\ & + J_+^\alpha \mathbf{D}_\alpha^L + \Pi_+^m \mathbf{A}_m^L + \psi_{\alpha+} \mathbf{W}_L^\alpha + \lambda_L^\alpha w_{\beta+}^L P_{\alpha\beta'}^{\alpha'\beta} \{ \mathbf{D}_{\alpha'}^L, \mathbf{W}_L^{\beta'} \} \end{aligned} \quad (190)$$

where $P_{\gamma\delta}^{\alpha\beta}$ is the projector on the zero-form plus two-form. In other words,

$$\mathbf{Y}_+ = \lambda_L^\alpha w_{\beta+}^L P_{\alpha\beta'}^{\alpha'\beta} a^\perp \{ \mathbf{D}_{\alpha'}^L, \mathbf{W}_L^{\beta'} \} \quad (191)$$

where $a^\perp$ is defined in (181). We have to verify that $\mathbf{Y}_+$ satisfies (188). First of all, notice that $\psi_{\alpha+}$ in Eq. (190) is by its definition¹² the same as $\psi_{\alpha+}$ defined in Eq. (140). This implies that $\mathbf{Q}_L \mathbf{L}_+$ falls into $\mathcal{A}_{[0,3]}$. But at the same time, the anchor of $\mathbf{Q}_L \mathbf{L}_+$ is zero. This implies that $\mathbf{Q}_L \mathbf{L}_+ = 0$.

Observation 4.

$$\mathbf{Q}_R \mathbf{L}_+ = 0 \quad (192)$$

Proof. Unfortunately we did not manage to prove it directly, but we have an indirect argument. Consider the action of $\mathbf{Q}_R$ on $\mathbf{L}_+$. Let us look at the leading term (which is in $\mathcal{A}_4$):

$$\mathbf{Q}_R \{ \lambda_L^\gamma \mathbf{D}_\gamma^L, w_{\alpha+}^L \mathbf{W}_L^\alpha \} = \{ \lambda_L^\gamma \mathbf{D}_\gamma^L, (Q_R w_{\alpha+}^L) \mathbf{W}_L^\alpha \} \bmod \mathcal{A}_{\leq 3} \quad (193)$$

The direct examination of the action shows that $Q_R w_{\alpha+}^L = 0$. (If $Q_R w_{\alpha+}^L$ were nonzero, the variation of the kinetic term $w_+ \partial_- \lambda_L$ would result in the term with the structure $\lambda_R w_+^L \partial_- \lambda_L$ which would have nothing to cancel with.) Therefore $\mathbf{Q}_R \mathbf{L}_+$ falls into $\mathcal{A}_{[-1,3]}$. Since the anchor is automatically zero, it remains to prove that $\mathbf{Q}_R \mathbf{L}_+$ actually falls into $\mathcal{A}_{[0,3]}$. Let us look at the component of $\mathbf{Q}_R \mathbf{L}_+$ in grading -1 . It is of the form:

$$\mathbf{X}_+ = \phi_+^{\hat{\alpha}} \mathbf{D}_{\hat{\alpha}}^R \quad (194)$$

We know that $\mathbf{Q}_L \mathbf{Q}_R \mathbf{L}_+ = 0$. This implies that $Q_L \phi_+^{\hat{\alpha}} = 0$. We conclude that $\phi_+^{\hat{\alpha}}$ has conformal dimension $(1, 0)$, ghost number $(0, 1)$ and is Q_L -closed. But there are not such operators, therefore $\phi_+^{\hat{\alpha}} = 0$. $\square$

¹² We also use the fact that the projector $P_{\alpha\beta'}^{\alpha'\beta}$ only affects things in $\mathcal{A}_{[0,2]}$ – see Section 4.1.

4.8.3. Zero curvature

Theorem 4.

$$[\mathbf{L}_+, \mathbf{L}_-] = 0 \quad (195)$$

Unfortunately we did not manage to prove it directly, but we have an indirect argument. We know that $[\mathbf{L}_+, \mathbf{L}_-]$ is a dimension $(1, 1)$ operator with components in $\mathcal{A}_{[-4,4]}$, annihilated by both $\mathbf{Q}_L$ and $\mathbf{Q}_R$. Let us consider the highest component:

$$X = [\mathbf{L}_+, \mathbf{L}_-] \bmod \mathcal{A}_{\leq 3} \quad (196)$$

We know that X is of the conformal dimension $(1, 1)$ and ghost number zero. It follows that $\mathbf{Q}_R X = 0$. But there are no operators with such properties (as can be seen from the flat space limit). Therefore the components of $[\mathbf{L}_+, \mathbf{L}_-]$ span $\mathcal{A}_{-4,3}$. Then we can consider $[\mathbf{L}_+, \mathbf{L}_-] \bmod \mathcal{A}_{\leq 2}$ and so on.

4.9. Relation between integrated and unintegrated vertex

There must be some analogue of the Koszul duality for algebroids, which should imply that the cohomology of the BRST operator $\lambda_L^\alpha a(\mathbf{D}_\alpha^L) + \lambda_R^{\check{\alpha}} a(\mathbf{D}_{\check{\alpha}}^R)$ is equivalent to the Lie algebroid cohomology of $\mathcal{A}$. Let us define $\mathbf{J}_+$ and $\mathbf{J}_-$ from the Lax pair:

$$\mathbf{L}_\pm = \frac{\partial}{\partial \tau^\pm} + \mathbf{J}_\pm \quad (197)$$

Then, given a 2-cocycle ψ representing the Lie algebroid cohomology, we can construct the corresponding integrated vertex as in [6]:

$$U = \psi(\mathbf{J}_+, \mathbf{J}_-) \quad (198)$$

Moreover, the Koszul duality must also imply the consistency of the definition of the algebroid $\mathcal{A}$ (PBW).

We leave the details for future work.

Is $\mathcal{A}$ an overkill? Notice that $\mathbf{J}_\pm$ only requires a small part of the $\mathcal{A}$; indeed, $\mathbf{J}_+$ belongs to $\mathcal{A}_{[0,4]}$ and $\mathbf{J}_-$ belongs to $\mathcal{A}_{[-4,0]}$. This suggests that our definition of $\mathcal{A}$ is quite an overkill.

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