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Câmpus de São José do Rio Preto

JOÃO VINICIUS MIRON AQUINO

A STUDY ON SOLUTIONS TO DAMPED FRACTIONAL-ORDER CAUCHY PROBLEMS

São José do Rio Preto
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Tese apresentada, em regime de cotutela, à Universidade Estadual Paulista "Júlio de Mesquita Filho" (UNESP), Câmpus de São José do Rio Preto, e à Universidad de Santiago de Chile (USACH) para a obtenção do título de Doutor em Matemática (UNESP) e em Ciência Mención Matemática (USACH) .

Área de concentração: Análise.

Orientadora: Profa. Dra. Andréa Cristina Prokopczyk (UNESP)
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Área de Concentração: Análise.

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To my family.

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"A vida é a ruína dos nossos planos".
(THE SHAPE of Water, 2018)

RESUMO

Este trabalho apresenta alguns resultados importantes da teoria das famílias resolvente $(a,k)-$regularizadas, como também das, bem conhecidas, famílias cosseno e seno. Juntamente com os conceitos de transformada de Laplace, são apresentados resultados sobre soluções para problemas de Cauchy abstratos fortemente amortecidos de ordem fracionária. Mais especificamente, são apresentados resultados, em sentido específico, sobre a boa colocação dessas equações.

Palavras-chave: Familias resolventes $(a,k)-$regularizadas. Equações fracionárias. Boa colocação.

ABSTRACT

This work presents some important results from the theory of $(a,k)-$regularized resolvent families, as well as from the well-known cosine and sine families. Together with the concepts of Laplace transform, results are presented regarding the solutions to strongly damped abstract Cauchy problems of fractional order. More specifically, results are presented, in a specific sense, on the well-posedness of these equations.

Keywords: $(a,k)-$regularized resolvent families. Fractional equations. Well-posedness.

RESUMEN

Este trabajo presenta algunos resultados importantes de la teoría de familias resolventes $(a,k)-$regularizadas, así como de las conocidas familias cosenos y senos. Junto con los conceptos de la transformada de Laplace, se presentan resultados sobre soluciones a problemas abstractos de Cauchy de orden fraccionario fuertemente amortiguados. Más concretamente, se presentan resultados, en sentido específico, sobre la buena colocación de estas ecuaciones.

Palavras-chave: Familias resolventes $(a,k)-$regularizadas. Equaciones fraccionarias. Buena colocación.

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1 INTRODUCTION

This work studies well-posedness and existence of solutions for a class of inhomogeneous strongly damped abstract Cauchy problem of fractional order described by

$$\begin{cases} \tau D_t^\alpha u(t) - (aB + bI)D_t^\beta u(t) - (cB + dI)u(t) = f(t), \\ u(0) = x, \\ u'(0) = y, \end{cases} \tag{1.1}$$

with $\tau \neq 0$, $a, b, c, d \in \mathbb{R}$ and $0 < \beta \leqslant 1 < \alpha \leqslant 2$, where B is a closed linear operator defined on a complex Banach space X and D_t^γ denotes the Caputo fractional derivative of order $\gamma > 0$.

In the case where $\alpha = 2$ and $\beta = 1$, the well-posedness for the homogeneous Cauchy problem (1.1), it means the existence of a strongly continuous family of bounded linear operators $\{W(t)\}_{t\geq 0}$ that solves (1.1) in case $f \equiv 0$, was settled by Neubrander (2, Theorem 11). In such case, it was proved in (2, Corollary 18) that for $x, y \in \mathcal{D}(B)$ the unique strict solution of (1.1) is given by

$$u(t) = W'(t)x + W(t)[y - \frac{1}{\tau}(aB + bI)x] + \frac{1}{\tau}\int_0^t W(t-s)f(s)ds, \tag{1.2}$$

where $W(t)$ is formally the inverse Laplace transform of the function

$$\lambda \to \frac{\tau}{a\lambda + c}\left(\frac{\tau\lambda^2 - b\lambda - d}{a\lambda + c} - B\right)^{-1}.$$

Several works in the literature have dealt with the investigation of the equation (1.1) in the integer case, that is $(\alpha, \beta) = (2, 1)$. It corresponds to a second order Cauchy problem and is usually named, in applications, as the strongly damped linear Klein-Gordon equation, or strongly damped wave equation, among others, see (2, 3, 4, 5, 6, 7, 8). For example, for $\tau = 1$ and $d = 0$ it is the linear part of the perturbed sine-equation see (9) and for $\tau = 1$, $b = d = 0$ it corresponds to the linear part of the viscous Cahn-Hilliard equation (9) or the Kuznetsov equation (10).

In the fractional case, Kirane and Tatar (11) consider a semilinear model that include (1.1) for $\alpha = 2, a = d = 0, c = 1, b = -1$ and $0 < \beta < 2$. Agarwal et. al. in (12), and more recently Zhou and He in (13), studied the equation (1.1) for $b = d = 0$ and $0 < \beta < 1 < \alpha < 2$. An interesting review can be found for instance in (14). When $\alpha = 2\beta$, equation (1.1) is known as the time-fractional telegraph equation. The special case $\beta = 1/2$ can be interpreted as the heat equation subject to a damping effect represented by the $1/2$-order time-derivative. In all of these cases, B is the Laplacian operator.

In the abstract case, equation (1.1) with $(\alpha, \beta) = (2, 1)$ has been studied by Ikehata, Todorova and Yordanov (15). These authors consider (1.1) with $b = d = 0$, $f \equiv 0$ and B

being a nonnegative self-adjoint operator defined in a Hilbert space with a dense domain. They prove existence and uniqueness of mild solutions based on semigroup theory (15, Proposition 2.1). Several authors have proposed models more general than (1.1) in the integer case, see e.g. (2, 16, 17). Typically, these authors consider the model

$$u''(t) + Bu'(t) + Au(t) = f(t),$$

where A and B are closed linear operators defined in a complex Banach space. However, due to its generality, this approach lacks the possibility of distinguishing its dynamics by means of an eventual combination of the physical parameters of the equation, and it also loses the special features that could be obtained by an explicit description of the solution in terms of a unique strongly continuous family of operators and by means of a kind of variation of parameter formula like (1.2), which is very useful for exploring associated semi linear problems. The idea to search by solutions described by a strongly continuous family of operators that preserves parameters and by unaware any previous studies in this sense for the proposed fractional model were the motivations to develop this work.

Let $\alpha, \beta > 0$ be given and we define

$$W_{\alpha,\beta}(t) := \int_0^t R_{\alpha,\beta}(s)ds, \tag{1.3}$$

where $R_{\alpha,\beta}(t)$ is the inverse Laplace transform of

$$\lambda \to \frac{\tau\lambda^{\alpha-1}}{a\lambda^\beta + c}\left(\frac{\tau\lambda^\alpha - b\lambda^\beta - d}{a\lambda^\beta + c} - B\right)^{-1}.$$

Consider $g_\alpha(t) := t^{\alpha-1}/\Gamma(\alpha)$, $t > 0$, the Gelfand-Shilov function. Using Laplace transform methods, we can prove that $\{R_{\alpha,\beta}(t)\}_{t\geq 0}$ is a solution family of (1.1), in the sense that a formal solution of (1.1) must have the form

$$\begin{aligned} u(t) = W'_{\alpha,\beta}(t)x + \int_0^t R_{\alpha,\beta}(t-s)[y - \frac{1}{\tau}g_{\alpha-\beta}(s)(aB + bI)x]ds \\ + \frac{1}{\tau}\int_0^t R_{\alpha,\beta}(t-s)(g_{\alpha-1} * f)(s)]ds, \end{aligned} \tag{1.4}$$

which exactly coincides with (1.2) in case $\alpha = 2$ and $\beta = 1$. However, because of the new terms $g_{\alpha-1}(t)$ and $g_{\alpha-\beta}(t)$ appearing in (1.4) we cannot expect differentiability of $u(t)$ in general, and therefore (1.4) fails to be a strict solution for (1.1). The following natural question arises:

(Q) Is (1.4) the solution of (1.1) in some extent?.

In this thesis we solve this problem proving that, for any $x \in \mathcal{D}(B)$ and $y \in X$, (1.4) is a solution of the following integrated version of (1.1)

$$\tau u(t) - (aB + bI)(g_{\alpha-\beta} * u)(t) - (cB + dI)(g_\alpha * u)(t) \\ = (g_\alpha * f)(t) + \tau(x + ty) - g_{\alpha-\beta+1}(t)(aB + bI)x, \quad (1.5)$$

under appropriate conditions.

We note that solutions for integrated versions of (1.1) in the integer case $\alpha = 2$, $\beta = 1$ were previously considered by Neubrander (2, Proposition 19). The study of this kind of solutions is already known because, in applications, it is useful to find a notion of a weaker solution where x, y can be less regular than when considering strict solutions. See also (14, Section 2.2 and Section 2.3) for an explicit fundamental formula of integrated solutions of (1.1) with $a = d = 0$ and B is the Laplacian operator.

In this work, assuming that B is the generator of a solution family $\{R_{\alpha,\beta}(t)\}_{t\geq 0}$, we show that existence of a unique solution for the integrated problem (1.5) can be assured in the sector $0 < \beta < \alpha$, $1 < \alpha \leq 2$, see Theorem 3.22 in Section 3.3.

In some special subsets of the sector described above we are able to show that the existence of solutions for the integrated problem can be guaranteed, under the stronger hypothesis that B is the generator of a C_0-semigroup or a strongly continuous cosine family. However, some restrictions are needed. More precisely, we first consider the set

$$\Omega_{\alpha,\beta} := \{(\alpha, \beta) : 0 < \beta \leq 1 < \alpha < 2\}, \quad (1.6)$$

and we show that, for $c = 0$ in (1.5), there exists a solution family, and hence we have well-posedness, if B is the generator of a cosine family. See Theorem 3.11 (b). Then, we decompose $\Omega_{\alpha,\beta}$ in two complementary subsets:

$$\Omega^1_{\alpha,\beta} := \{(\alpha, \beta) : 0 < \beta \leq 1 < \alpha < 1+\beta \leq 2\}, \ \Omega^2_{\alpha,\beta} := \{(\alpha, \beta) : 0 < \beta \leq 1 < 1+\beta < \alpha \leq 2\}.$$

For $\Omega^1_{\alpha,\beta}$ we need $c = 0, a \neq 0,\ \tau \neq 0, b, d \in \mathbb{R}$, and B generator of a C_0-semigroup for well-posedness, while for $\Omega^2_{\alpha,\beta}$ we need $b = d = 0, a, c \geq 0,\ \tau > 0$ and B generator of a cosine family. See Theorem 3.11 (a) and Theorem 3.13, respectively.

Our approach consists in the subordination methods, Laplace transform theory, and a strong application of a criterion, due to Prüss (18, Theorem 4.3, p.104), for the existence of resolvent families under the assumption B is the generator of a cosine family, plus certain specific conditions in the associated kernel.

This thesis is organized as follows: In Chapter 2, we give the necessary preliminaries for the development of our study. Section 2.1 presents some notations that will be used throughout the work and Section 2.2 stats some results of the existing theory about $(a, k)-$regularized resolvent family. Chapter 3 contains the main results and is divided into: Section 3.1 we present some sufficient conditions for existence of a solution family and we observe that in the entire case, that is, $(\alpha, \beta) = (2, 1)$, well-posedness follows under the subordination hypothesis that B is the generator of a C_0-semigroup. Then, we show that under certain constraints on the pair (α, β) we can distinguish two situations: $1 < \alpha < 1+\beta \leq 2, 0 < \beta \leq 1$ and $1+\beta < \alpha \leq 2, 0 < \beta \leq 1$. In the first case, if $c = 0$, the leading term in the equation (1.1) is $D_t^\alpha u$ and the solution family is subordinate to B being the generator of a C_0-semigroup. In the second case, if $b = d = 0$, the leading term is $D_t^\beta u$ and the solution family is subordinated to B being a generator of a cosine family. These conclusions provide new insights into how fractional terms influence the class of abstract evolution equations (1.1). Furthermore, intermediate results, which will be used later, are presented. The Section 3.2 deal about the relationship with the best known theory of (a, k)-regularized families and our definition of solution family, proving that solutions families are a particular case of the latter when the parameters satisfy $b \geq 0$ and $\tau > 0$. In the main Section 3.3, we define our notions of mild and well-posed solution for the homogeneous problem, proving that under the hypothesis of existence of a solution family, well-posedness can be guaranteed. Then, the existence and uniqueness of mild solutions for the inhomogeneous problem can be proved. Finally, some examples to illustrate our abstract results are given.

We note that the results developed on this study, and presented in Chapter 3, are accepted and will be published in Proceedings A of the Royal Society of Edinburgh, see (19).

2 PRELIMINARIES

2.1 Notation and important results

In this section we provide some of the fundamental concepts that we will need.

Let X and Y Banach spaces. The space $\mathcal{L}(X;Y)$ is composed by all bounded linear operators from X to Y endowed with the usual norm $\|T\| = \sup\limits_{\substack{x\in X\\ \|x\|\leqslant 1}} \|T(x)\|$, $T \in \mathcal{L}(X;Y)$. If $X = Y$ we simply write $\mathcal{L}(X)$. We denote $\mathbb{R}_+ = [0,\infty)$. Let $T : \mathbb{R}_+ \to \mathcal{L}(X;Y)$, we say that T is strongly continuous if $t \mapsto T(t)x$ is a continuous function, for all $x \in X$.

We also will consider the classical spaces: for a compact metric space M, $C(M;X)$ is the space composed by all continuous functions from M to X with the uniform convergence norm $\|f\| = \sup\limits_{t\in M}\|f(t)\| + \sup\limits_{t\in M}\|f'(t)\|$. In general, $C^m(M;X)$ denotes the space of all functions from M to X that have continuous derivative up to order m, and $C^\infty(M;X)$ is composed by all the functions that have derivatives of all orders. We write $C^m(M)$ when $X = \mathbb{R}$.

For a functions $f : X \to \mathbb{R}$, the support of f, denoted by supp(f),is the set supp$(f) = \overline{\{x \in X; f(x) \neq 0\}}$.

Another important space that we will consider is the set of all Lipschitzs functions from M to X, denoted by $Lip(M;X)$, endowed with the norm

$$\|f\|_{Lip} = \sup_{\substack{x,y\in M\\ x\neq y}} \left\{\frac{\|f(x) - f(y)\|}{d(x,y)}\right\},$$

where $d(\cdot,\cdot)$ is the metric of M.

Definition 2.1. *(20, Definition 1, p. 5) Let X be a complex Banach space. Let $f : \mathbb{R}_+ \longrightarrow X$ be an integrable function (as a Bochner integral) and let $T : \mathbb{R}_+ \longrightarrow \mathcal{L}(X,Y)$ be strongly continuous. Then the convolution of T and f is defined by*

$$(T * f)(t) = \int_0^t T(t-s)f(s)\,ds,\ t \in \mathbb{R}_+.$$

Theorem 2.2. *(21, Theorem 1.1.4, p. 9) Let $I \subset \mathbb{R}$ an interval (bounded or unbounded). A function $f : I \longrightarrow X$ is Bochner integrable if and only if f is measurable and $\|f\|$ is integrable. If f is Bochner integrable, then*

$$\left\| \int_I f(t)dt \right\| \leqslant \int_I \|f(t)\|dt.$$

Theorem 2.3. *(21, Theorem 1.1.8, p. 11) Let $f_n : I \to X (n \in \mathbb{N})$ be Bochner integrable functions. Assume that $f(t) := \lim\limits_{n\to\infty} f_n(t)$ exists a. e. for all $t \in I$. Then f is Bochner* *integrable and* $\int_I f(t)dt = \lim_{n\to\infty} \int_I f_n(t)dt$. *Furthermore,*

$$\int_I \|f(t) - f_n(t)\|dt \to 0 \text{ as } n \to \infty.$$

Let $L^1(I;X)$ denote the space of all Bochner integrable functions from I to X with the usual identification, that is, considering the class of functions that differ on a set of measure zero, endowed with the norm

$$\|f\|_1 = \int_I \|f(t)\|dt.$$

The space $L^1_{loc}(I;X)$ is composed by functions that are Bochner integrable on $[0,\delta]$ for every $\delta > 0$. We write $L^1(I)$ and $L^1_{loc}(I)$ when $X = \mathbb{R}$.

Proposition 2.4. *(21, Proposition 1.3.4, p. 24) Let* $f \in L^1_{loc}(\mathbb{R}_+;X)$ *and let* $T : \mathbb{R}_+ \longrightarrow \mathcal{L}(X;Y)$ *be strongly continuous. Then the convolution*

$$(T * f)(t) = \int_0^t T(t-s)f(s)ds$$

exists (as a Bochner integral) and defines a continuous function $T * f : \mathbb{R}_+ \longrightarrow Y$.

Proof. Fix $t \geqslant 0$. First, we show that $s \mapsto T(t-s)f(s)$ is mensurable on $[0,t]$. In the case where f is a characteristic function, that is, $f(s) = \chi_\Omega(s)x$ for some measurable set $\Omega \subset \mathbb{R}$ and $x \in X$, then we have

$$T(t-s)f(s) = \chi_\Omega(s)T(t-s)x.$$

This function is measurable, because is a the product of a measurable scalar-valued function and a continuous vector-valued function. In the case where f is a simple function, that is, $f(s) = \sum_{i=1}^{n} x_i\chi_{\Omega_i}(s)$, with $x_i \in X$ and $\Omega_i \subset \mathbb{R}$ mensurable set, for $i = 1, 2, \cdots, n$, the linearity of $T(\cdot)$ ensure that $T(t-\cdot)f(\cdot)$ is measurable. For a general measurable f, there is a sequence of simple functions $(f_n)_{n\in\mathbb{N}}$ such that $f_n \to f$ a. e. and then $T(t-s)f_n(s) \to T(t-s)f(s)$ a. e..Thus $T(t-\cdot)f(\cdot)$ is measurable. Furthermore, since

$$\|T(t-s)f(s)\| \leqslant \|T(t-s)\| \, \|f(s)\|,$$

it follows from Theorem 2.2 that $(T * f)(t)$ exists.

To prove the continuity of the $(T * f)$ we use dominated convergence Theorem (Theorem 2.3), in fact, let $F(t) = T(t-s)f(s)$, for $t > 0$ and $s \in [0,t]$. Since T is strongly continuous and f is continuous, we have $\lim_{h\to 0^+} F(t+h) = F(t)$. Moreover, $\|F(t)\| \leqslant \|T(t-s)\| \, \|f(s)\|$, $t > 0$ and $s \in [0,t]$. Then, by application of Theorem 2.3 we conclude that $(T*f)(t+h) \to (T*f)(t)$, $\forall t > 0$. □

Proposition 2.5. *(21, Proposition 1.3.7, p. 26) Let* $T : [0,\tau] \longrightarrow \mathcal{L}(X;Y)$ *be Lipschitz continuous with* $T(0) = 0$*, and let* $f \in L^1([0,\tau];X)$*. Then* $T * f \in C^1([0,\tau];Y)$.

Theorem 2.6 (Titchmarsh's Theorem)**.** *(22, Theorem VII, p. 286), (21, Corollary 2.8.4) Let* $k \in L^1([0,T])$ *with* $0 \in supp(k)$ *and* $f \in L^1([0,T];X)$*. If*

$$(k * f)(t) = \int_0^t k(t-s)f(s)ds = 0$$

on $[0,T]$, *then* $f \equiv 0$.

Let $u \in C(\mathbb{R}_+;X)$ and $v \in C^1(\mathbb{R}_;,X)$. Then for every $t \geqslant 0$, we can prove that

$$\frac{d}{dt}[(u * v)(t)] = u(t)v(0) + (u * v')(t). \tag{2.1}$$

The Laplace transform of a function $f \in L^1(\mathbb{R}_+;X)$ is defined by

$$\mathcal{L}(f)(\lambda) = \widehat{f}(\lambda) := \int_0^\infty e^{-\lambda t}f(t)dt = \lim_{T\to\infty}\int_0^T e^{-\lambda t}f(t)dt,$$

for $\lambda \in \mathbb{C}$ for which his integral converges.

Theorem 2.7. *(18, Theorem 0.2, p. 6) Let* $f : (0,\infty) \longrightarrow X$*. The following statements are equivalent:*

(*i*) *There exixts* $u \in Lip(\mathbb{R}_+;X)$, $u(0) = 0$, *such that* $f(\lambda) = \widehat{u'}(\lambda)$ *for all* $\lambda > 0$*;*

(*ii*) $f \in C^\infty\big((0,\infty);X\big)$ *and*

$$m_\infty(f) = \sup\left\{\frac{\lambda^{n+1}\|f^{(n)}(\lambda)\|}{n!}; \lambda > 0,\ n \in \mathbb{Z}_+\right\} < \infty.$$

In this case $\|u\|_{Lip} = m_\infty(f)$.

Let $\alpha > 0$, $m = \lceil\alpha\rceil$, that is, $m = \lceil\alpha\rceil = \min\{n \in \mathbb{N}; \alpha \leqslant n\}$, and $u : [0,\infty) \longrightarrow C^m(\mathbb{R}_+;X)$ be a function. The Caputo fractional derivative of u of order α is defined by

$$D_t^\alpha u(t) := \int_0^t g_{m-\alpha}(t-s)u^{(m)}(s)ds,\ t > 0,$$

where

$$g_\beta(t) := \frac{t^{\beta-1}}{\Gamma(\beta)},\ t > 0,\ \beta > 0,$$

is the Gelfand-Shilov function. If $\alpha = 0$ we denote $D_t^0 u(t) := u(t)$. In particular,

$$\widehat{D_t^\alpha f}(\lambda) = \lambda^\alpha \widehat{f}(\lambda) - \sum_{k=0}^{m-1} f^{(k)}(0)\lambda^{\alpha-1-k}. \tag{2.2}$$

The Riemann Liouville fractional integral of order $\alpha > 0$ is defined as follows

$$J_t^\alpha f(t) := (g_\alpha * f)(t),\ f \in L^1_{loc}(\mathbb{R}_+),\ t > 0.$$

Moreover, the Riemann Liouville fractional derivative of order $\alpha > 0$ is given by

$$\mathbf{D}_t^\alpha f(t) := D_t^m (g_{m-\alpha} * f)(t) = D_t^m J_t^{m-\alpha} f(t).$$

2.2 The Mittag-Leffler function

Recently the Mittag-Leffler function and its numerous generalizations and relevant properties have acquired great interest due to the connection of the Mittag-Leffler function with Fractional Calculus and its application to the study of Differential and Integral Equations, in particular, of fractional order. In this section we present definitions and some interesting properties of this functions.

The one-parametric Mittag-Leffler functions is defined by

$$E_\alpha(z) := \sum_{k=0}^{\infty} \frac{z^k}{\Gamma(\alpha k + 1)}, \quad \mathrm{Re}(\alpha) > 0,\ z \in \mathbb{C}.$$

This function can be seen as a generalization of the exponential function because when $\alpha = 1$ we obtain

$$E_1(z) = \sum_{k=0}^{\infty} \frac{z^k}{\Gamma(k+1)} = \sum_{k=0}^{\infty} \frac{z^k}{k!} = e^z.$$

The two-parametric Mittag-Leffler function (23, Section 4.1, p. 64) is given by

$$E_{\alpha,\beta}(z) := \sum_{k=0}^{\infty} \frac{z^k}{\Gamma(\alpha k + \beta)}, \ \mathrm{Re}(\alpha) > 0,\ \beta \in \mathbb{C},\ z \in \mathbb{C}.$$

Its clear that when $\beta = 1$, $E_{\alpha,1}(z) = E_\alpha(z)$, so $E_{1,1}(z) = e^z$.

Just like one-parametric Mittag-Leffler functions we can also see well-known particular cases:

(1) $E_{2,1}(z^2) = \sum_{k=0}^{\infty} \frac{z^{2k}}{\Gamma(2k+1)} = \sum_{k=0}^{\infty} \frac{z^{2k}}{2k!} = \cosh(z)$;

(2) (23, p. 65) $E_{2,1}(z) = \cosh(\sqrt{z})$;

(3) (23, p. 65) $E_{1,2}(z) = \dfrac{e^z - 1}{z}$.

Some others interesting properties of these functions can be found in the book by Bateman and Erdelyi (24, Section 18.1, p. 206).

Theorem 2.8. *(25, Theorem 1.6, p. 35) If $\alpha < 2$, β is an arbitrary real number, μ is such that $\frac{\pi\alpha}{2} < \mu < \min\{\pi, \pi\alpha\}$ and C is a real constant, then*

$$|E_{\alpha,\beta}(z)| \leq \frac{C}{1+|z|}, \quad \mu \leq |\boldsymbol{arg}(z)| \leq \pi,\ |z| \geq 0.$$

The Laplace transform of the Mittag-Leffler function (25, Section 1.2.2, p. 21), is obtained by

$$\int_0^\infty e^{-pt}t^{\alpha k+\beta-1}E_{\alpha,\beta}^{(k)}(\pm at^\alpha)dt = \frac{k!p^{\alpha-\beta}}{(p^\alpha \mp a)^{k+1}}, \quad \alpha > 0,\ \beta > 0,\ \mathrm{Re}(p) > |a|^{\frac{1}{\alpha}}. \tag{2.3}$$

About the derivative of the Mittag-Leffler function, according to (25, Section 1,2,3, p. 21–22),

$$\mathbf{D}_t^\gamma\left(t^{\alpha k+\beta-1}E_{\alpha,\beta}^{(k)}(\lambda t^\alpha)\right) = t^{\alpha k+\beta-\gamma-1}E_{\alpha,\beta-\gamma}^{(k)}(\lambda t^\alpha), \tag{2.4}$$

where $\mathbf{D}_t^\gamma$ is the Riemann-Liouville fractional derivative, with order $\gamma \in \mathbb{R}$. In particular, for $\lambda = 1$, $k = 0$ and $\gamma \in \mathbb{N}$,

$$\left(\frac{d}{dt}\right)^m\left(t^{\beta-1}E_{\alpha,\beta}(t^\alpha)\right) = t^{\beta-m-1}E_{\alpha,\beta}(t^\alpha), \qquad (m = 1, 2, 3, \cdots). \tag{2.5}$$

Even more, by (20, Lemma 2, p. 4), for $\alpha > 0$, $a > 0$ and $m \in \mathbb{Z}_+$ we have

$$\frac{d^m}{dt^m}E_{\alpha,1}(-at^\alpha) = -at^{\alpha-m}E_{\alpha,\alpha-m+1}(-at^\alpha). \tag{2.6}$$

2.3 $(a, k)-$regularized resolvent family

In this section, we provide the important results and definitions about $(a, k)-$regularized resolvent families. We recall the following definition that corresponds to a slight modification of (26, Definition 2.1).

Definition 2.9. *Let $k \in C(\mathbb{R}_+)$, $k \neq 0$, and let $a \in L^1_{loc}(\mathbb{R}_+)$, $a \neq 0$. Let A be a linear operator with domain $\mathcal{D}(A) \subset X$ where X is a Banach space. A strongly continuous family $\{R(t)\}_{t\geqslant 0} \subset \mathcal{L}(X)$ is called an (a, k)-regularized resolvent family on X (or simply (a, k)-regularized family) having A as a generator if the following properties hold:*

- (i) $\lim\limits_{t\to 0}\dfrac{R(t)}{k(t)} = I$ *if* $k(0) \in \overline{\mathbb{C}}\backslash\{0\}$ *and* $R(0) = 0$ *if* $k(0) = 0$*;*
- (ii) $R(t)x \in \mathcal{D}(A)$ *and* $R(t)Ax = AR(t)x$*, for all* $x \in \mathcal{D}(A)$ *and* $t \geqslant 0$*;*
- (iii) $R(t)x = k(t)x + \int_0^t a(t-s)AR(s)x\,ds$*,* $t \geqslant 0$*,* $x \in \mathcal{D}(A)$*.*

Remark 2.10. We observe that the choice of the pair (a, k) generates different families.

- (i) If $k \equiv 1$ and $a \equiv 1$, $\{R(t)\}_{t\geq 0}$ is a C_0- semigroup denoted by $\{T(t)\}_{t\geq 0}$. See more in (27, Section 3.1, p. 108).
- (ii) If $k \equiv 1$ and $a(t) = t$, $\{R(t)\}_{t\geq 0}$ is a strongly continuous cosine family, denoted as $\{C(t)\}_{t\geq 0}$. See more in (27, Section 3.14, p. 202).

(iii) If $k(t) = t$ and $a(t) = t$, $\{R(t)\}_{t\geq 0}$ is a sine family. See more in (27, Section 3.14, p. 206).

(iv) In case $k(t) = g_\beta(t)$, the above definition coincides with (28, Definition 1.9);

(v) In case $k \equiv 1$, Definition 2.9 coincides with the definition of a resolvent family (18, Definition 1.3, p. 32).

Lemma 2.11. *(29, Lema 2.2, p. 281) Let $\{R(t)\}_{t\geq 0}$ be an (a, k)-regularized resolvent family having A as a generator. Then, $\int_0^t a(t-s)R(s)xds \in \mathcal{D}(A)$, for all $x \in X$ and $t \geqslant 0$, and*

$$R(t)x = k(t)x + A\int_0^t a(t-s)R(s)xds,\ x \in X,\ t \geqslant 0. \tag{2.7}$$

Proof. Let $x \in X$, $\lambda \in \rho(A)$ and $y = (\lambda - A)^{-1}x \in \mathcal{D}(A)$. Let $z = (a * R)(t)x$, $t \geqslant 0$. First, we will show that $z \in \mathcal{D}(A)$. By (ii) and (iii) of Definition 2.9 we have

$$\begin{aligned} z &= (\lambda - A)(a * R)(t)y = \lambda(a * R)(t)y - (a * AR)(t)y \\ &= \lambda(a * R)(t)y - (R(t)y - k(t)y). \end{aligned}$$

Since $y \in \mathcal{D}(A)$, we obtain $z \in \mathcal{D}(A)$.

Moreover, from the above equality its follows that

$$(\lambda - A)z = \lambda(a * R)(t)x - (R(t)x - k(t)x) = \lambda z - (R(t)x - k(t)x),$$

which implies (2.7). □

Remark 2.12. (29, Remark 2.4 (4), p. 282) If $R(t)$ is an $(a, k)-$regularized resolvent family and $b \in L^1_{loc}(\mathbb{R}_+)$, then $(b * R)(t)$ is an $(a, b * k)-$regularized resolvent family.

Definition 2.13. *A family $\{R(t)\}_{t\geqslant 0}$ of bounded linear operators on a Banach space X is called exponentially bounded if there are constants $M \geqslant 1$ and $\omega \in \mathbb{R}$ such that*

$$\|R(t)\| \leqslant Me^{\omega t},\ \forall t \geqslant 0.$$

In this case, we say that $\{R(t)\}_{t\geqslant 0}$ is of type (M, ω).

Proposition 2.14. *(29, Proposition 3.1, p. 287) Let A be a closed linear operator and $\{R(t)\}_{t\geq 0}$ be an exponentially bounded an strongly continuous operator family in $\mathcal{L}(X)$ of type (M, ω) such that the Laplace transform $\widehat{R}(\lambda)$ exists for $Re(\lambda) > \omega$. Then, $\{R(t)\}_{t\geq 0}$ is an $(a, k)-$regularized resolvent family of type (M, ω) with generator A if and only if, for every $Re(\lambda) > \omega$, $(I - \widehat{a}(\lambda)A)^{-1}$ exists in $\mathcal{L}(X)$ and*

$$\widehat{k}(\lambda)(I - \widehat{a}(\lambda)A)^{-1}x = \frac{\widehat{k}(\lambda)}{\widehat{a}(\lambda)}\left(\frac{1}{\widehat{a}(\lambda)} - A\right)^{-1} = \int_0^\infty e^{-\lambda s}R(s)x\,ds,\ x \in X.$$

Proof. We start this proof by suppose that $\{R(t)\}_{t\geq 0}$ is an $(a,k)-$regularized resolvent family. By assumption, the Laplace transform $H(\lambda) = \widehat{R}(\lambda)$ of the $(a,k)-$regularized resolvent family exists for $\lambda > \omega$. Next, we will prove that $\dfrac{1}{\widehat{a}(\lambda)} \in \rho(A)$ and $H(\lambda) = \widehat{k}(\lambda)(I - \widehat{a}(\lambda)A)^{-1}$.

From (ii) and (iii) of Definition 2.9, Lemma 2.11 and the Laplace transform of the convolution for $\mathrm{Re}(\lambda) > \omega$ it follows that

$$H(\lambda)x = \widehat{k}(\lambda)x + \widehat{a}(\lambda)H(\lambda)Ax,$$

for each $x \in \mathcal{D}(A)$ and

$$H(\lambda)x = \widehat{k}(\lambda)x + A\widehat{a}(\lambda)H(\lambda)x,$$

for each $x \in X$. So,

$$\begin{aligned}
&H(\lambda)[I - \widehat{a}(\lambda)A]x = \widehat{k}(\lambda)x, \text{ for each } x \in \mathcal{D}(A) \text{ and} \\
&[I - \widehat{a}(\lambda)A]H(\lambda)x = \widehat{k}(\lambda)x \text{ for each } x \in X.
\end{aligned}$$

Thus, the operator $(I - \widehat{a}(\lambda)A)$ is invertible for all $\lambda \in \mathbb{C}$, with $\mathrm{Re}(\lambda) > \omega$, and

$$H(\lambda) = \widehat{k}(\lambda)(I - \widehat{a}(\lambda)A)^{-1}, \text{ for } \mathrm{Re}(\lambda) > \omega.$$

Even more, $\dfrac{1}{\widehat{a}(\lambda)} \in \rho(A)$ for all such $\lambda > \omega$ and $\widehat{a}(\lambda) \neq 0$. In fact, assume that $\widehat{a}(\lambda_0) = 0$ for some λ_0 with $\mathrm{Re}(\lambda_0) > \omega$. Then, since $\widehat{a}(\lambda)$ is holomorphic, λ_0 is an isolated zero of finite multiplicity, and $H(\lambda_0) = \widehat{k}(\lambda_0)$. Now, choose a small circle Γ around λ_0 which is entirely contained in the half plane $\mathrm{Re}(\lambda) > \omega$ such that $\widehat{a}(\lambda) \neq 0$ on Γ.

Since $\dfrac{H(\lambda)}{\widehat{k}(\lambda)}$ is holomorphic on Γ then

$$\frac{H(\lambda_0)}{\widehat{k}(\lambda_0)} = \frac{1}{2\pi i}\int_\Gamma \frac{H(\lambda)}{(\lambda - \lambda_0)\widehat{k}(\lambda)}d\lambda \tag{2.8}$$

and, by assumption, A is a closed operator, which implies that

$$\begin{aligned}
A = A\frac{H(\lambda_0)}{\widehat{k}(\lambda_0)} &= A\frac{1}{2\pi i}\int_\Gamma \frac{H(\lambda)}{(\lambda - \lambda_0)\widehat{k}(\lambda)}d\lambda \\
&= \frac{1}{2\pi i}\int_\Gamma \frac{AH(\lambda)}{(\lambda - \lambda_0)\widehat{k}(\lambda)}d\lambda.
\end{aligned}$$

Moreover, the function $AH(\lambda) = \dfrac{H(\lambda) - \widehat{k}(\lambda)}{\widehat{a}(\lambda)}$ is well defined and holomorphic on Γ. Then, by Cauchy Integral Formula, we obtain

$$A = \frac{1}{2\pi i}\int_\Gamma \frac{H(\lambda) - \widehat{k}(\lambda)}{(\lambda - \lambda_0)\widehat{a}(\lambda)\widehat{k}(\lambda)}d\lambda,$$

which allows us to conclude that A is bounded, contradicting the hypothesis. Thus, $\widehat{a}(\lambda) \neq 0$ for all $\mathrm{Re}(\lambda) > \omega$ and, since $(I - \widehat{a}(\lambda)A)$ is invertible, $\frac{1}{\widehat{a}(\lambda)} \in \rho(A)$ for all $\mathrm{Re}(\lambda) > \omega$.

Conversely, let $\mu, \lambda > \omega$ and $x \in \mathcal{D}(A)$. Then, $x = (I - \widehat{a}(\mu)A)^{-1}y$ for some $y \in X$. Basically, since A is closed and $(I - \widehat{a}(\mu)A)^{-1}$ and $(I - \widehat{a}(\lambda)A)^{-1}$ commute, we have that

$$\begin{aligned}\int_0^\infty e^{-\lambda t} R(t)x\,dt &= \widehat{k}(\lambda)(I - \widehat{a}(\lambda)A)^{-1}(I - \widehat{a}(\mu)A)^{-1}y \\ &= (I - \widehat{a}(\mu)A)^{-1}\widehat{k}(\lambda)(I - \widehat{a}(\lambda)A)^{-1}y \\ &= (I - \widehat{a}(\mu)A)^{-1}\widehat{R}(\lambda)y \\ &= \int_0^\infty e^{-\lambda t}(I - \widehat{a}(\mu)A)^{-1}R(t)y\,dt.\end{aligned}$$

So, by uniqueness of the Laplace transform, it follows that

$$R(t)x = (I - \widehat{a}(\mu)A)^{-1}R(t)(I - \widehat{a}(\mu)A)x, \ \forall t \geqslant 0,$$

and then $R(t)x \in \mathcal{D}(A)$. Further, for $\lambda > \omega$, $\widehat{a}(\lambda) \in \rho(A)$ and arguing as at beginning of this proof, we have $\widehat{a}(\lambda) \neq 0$ for $\lambda > \omega$. Thus, from the above equality we conclude that $AR(t)x = R(t)Ax$ for every $t \geqslant 0$ and $x \in \mathcal{D}(A)$.

Now, for $\lambda > \omega$ and $x \in \mathcal{D}(A)$, from the convolution theorem, we obtain

$$\begin{aligned}\int_0^\infty e^{-\lambda t} k(t)x\,dt = \widehat{R}(\lambda)(I - \widehat{a}(\lambda)A)x &= \widehat{R}(\lambda)x - \widehat{R}(\lambda)\widehat{a}(\lambda)Ax \\ &= \int_0^\infty e^{-\lambda t}\left[R(t)x - \int_0^t a(t-s)R(s)Ax\,ds\right]dt.\end{aligned}$$

By the uniqueness of Laplace transform and the strong continuity of $R(t)$ we conclude that

$$R(t)x = k(t)x + \int_0^t a(t-s)R(s)Ax\,ds, \ \forall t \geqslant 0, \ x \in \mathcal{D}(A).$$

Proceeding as in the proof of Lemma 2.11, for $x \in X$, we have $\int_0^t a(t-s)R(s)xds \in \mathcal{D}(A)$ and $R(t)x = k(t)x + A\int_0^t a(t-s)R(s)xds, \ x \in X, \ t \geqslant 0$. Then,

$$\lim_{t\to 0} \frac{R(t)x}{k(t)} = \frac{R(0)x}{k(0)} = \frac{k(0)x}{k(0)} = x, \ \forall x \in X,$$

providing $k(0) \in \overline{\mathbb{C}} - \{0\}$, and $R(0) = 0$ if $k(0) = 0$, which conclude the proof. □

Next, we will consider the generation Theorem for $(a,k)-$regularized resolvent families. There are several theorems of this type for particular cases of families of bounded linear operators, such as the famous Hille-Yosida Theorem (30, Theorem 3.1, p. 8) for C_0-semigroups. Other examples can be found in the following citations:

(1) For cosine families, (31, Theorem 2.1, p. 28).

(2) For resolvent families, (18, Theorem 1.3, p. 43).

(3) For $\alpha-$resolvents, (32, Theorem 2.8, p. 23).

Theorem 2.15. *(18, Theorem 1.3, p. 43) Let $A : \mathcal{D}(A) \subset X \longrightarrow X$ be a closed linear densely defined operator in a Banach space X, $k \in C(\mathbb{R}_+)$, $k \neq 0$ and $a \in L^1_{loc}(\mathbb{R}_+)$. Then A is the generator of an $(a, k)-$regularized resolvent family $\{R(t)\}_{t\geq 0}$ of type (M, ω) if and only if the following conditions hold:*

(i) $\widehat{a}(\lambda) \neq 0$ *and* $\dfrac{1}{\widehat{a}(\lambda)} \in \rho(A)$ *for all* $\lambda \in \mathbb{R},\ \lambda > \omega$;

(ii) $H(\lambda) := \widehat{R}(\lambda)$ *satisfies* $H(\lambda) = \widehat{k}(\lambda)(I - \widehat{a}(\lambda)A)^{-1}$ *and*

$$\|H^{(n)}(\lambda)\| \leqslant \frac{Mn!}{(\lambda-\omega)^{n+1}},\ \lambda > \omega,\ n \in \mathbb{Z}_+. \tag{2.9}$$

Proof. We start by considering that $\{R(t)\}_{t\geq 0}$ is an $(a, k)-$regularized resolvent family of type (M, ω). Let $H(\lambda)$ its Laplace transform, that is,

$$H(\lambda) = \widehat{R}(\lambda) = \int_0^\infty e^{-\lambda s} R(s)\, ds,\ \mathrm{Re}(\lambda) > \omega.$$

Then $H(\lambda)$ is well-defined, holomorphic for $\mathrm{Re}(\lambda) > \omega$ and satisfies

$$\begin{aligned}\|H(\lambda)\| &\leqslant \int_0^\infty \left\|e^{-\lambda s}R(s)\right\| ds \leqslant M\int_0^\infty |e^{-\lambda s}||e^{\omega s}| ds \\ &\leqslant M\int_0^\infty e^{-(\mathrm{Re}(\lambda)-\omega)s} ds = \frac{M}{\mathrm{Re}(\lambda)-\omega}.\end{aligned}$$

Moreover, since $\mathrm{Re}(\lambda) > \omega$, using integration by parts, we have that

$$\begin{aligned}\|H'(\lambda)\| = \left\|\int_0^\infty -se^{-\lambda s}R(s)ds\right\| &\leqslant \int_0^\infty se^{-\mathrm{Re}(\lambda)s}Me^{\omega s}ds \\ &\leqslant \int_0^\infty \frac{Me^{-(\mathrm{Re}(\lambda)-\omega)s}}{(\mathrm{Re}(\lambda)-\omega)}ds \leqslant \frac{M}{(\mathrm{Re}(\lambda)-\omega)^2}.\end{aligned}$$

Now, to prove the estimate for $\|H^{(n)}(\lambda)\|$ we use mathematical induction to ensure that

$$H^{(n)}(\lambda) = \int_0^\infty (-s)^n e^{-\lambda s} R(s) ds.\ \mathrm{Re}(\lambda) > \omega, \tag{2.10}$$

and

$$\int_0^\infty s^n e^{-(\mathrm{Re}(\lambda)-\omega)s} ds \leqslant \frac{n!}{(\mathrm{Re}(\lambda)-\omega)^{n+1}},\ \mathrm{Re}(\lambda) > \omega \tag{2.11}$$

Then, using (2.11) to estimate (2.10) we obtain

$$\begin{aligned}\|H^n(\lambda)\| &= \left\| \int_0^\infty (-s)^n e^{-\lambda s} R(s)\, ds \right\| \\ &\leqslant \int_0^\infty s^n e^{-\mathrm{Re}(\lambda) s} M e^{\omega s} ds \\ &\leqslant \frac{Mn!}{(\mathrm{Re}(\lambda)-\omega)^{n+1}}, \ \mathrm{Re}(\lambda) > \omega.\end{aligned}$$

Furthermore, from (ii) and (iii) of Definition 2.9, Lemma 2.11 and the Laplace transform of the convolution, for $\mathrm{Re}(\lambda) > \omega$, it follows, for each $x \in \mathcal{D}(A)$, that

$$H(\lambda)x = \widehat{k}(\lambda)x + \widehat{a}(\lambda)H(\lambda)Ax, \tag{2.12}$$

and for each $x \in X$,

$$H(\lambda)x = \widehat{k}(\lambda)x + A\widehat{a}(\lambda)H(\lambda)x. \tag{2.13}$$

So, we obtain that

$$\begin{aligned} H(\lambda)[I - \widehat{a}(\lambda)A]x &= \widehat{k}(\lambda)x, \text{ for each } x \in \mathcal{D}(A), \\ [I - \widehat{a}(\lambda)A]H(\lambda)x &= \widehat{k}(\lambda)x, \text{ for each } x \in X. \end{aligned}$$

Then the operators $(I - \widehat{a}(\lambda)A)$ are invertible for all $\lambda \in \mathbb{C}$, with $\mathrm{Re}(\lambda) > \omega$ and

$$H(\lambda) = \widehat{k}(\lambda)(I - \widehat{a}(\lambda)A)^{-1}, \text{ for } \mathrm{Re}(\lambda) > \omega.$$

In particular, assuming that $\widehat{a}(\lambda_0) = 0$ for some $\lambda_0 \in \mathbb{C}$ such that $\mathrm{Re}(\lambda_0) > \omega$, and proceeding by the proof of Proposition 2.14 we obtain that A is bounded, which is a contradiction. So, we conclude that $\widehat{a}(\lambda) \neq 0,\ \forall \mathrm{Re}(\lambda) > \omega$, and then, $\dfrac{1}{\widehat{a}(\lambda)} \in \rho(A)$.

Now, assuming that the conditions (i) and (ii) by Theorem 2.7 are satisfies, there exists $U_\omega \subset \mathcal{L}(\mathbb{R}_+; X)$ such that $U_\omega(0) = 0$ and $\widehat{U'_\omega}(\lambda) = H(\lambda + \omega)$, $\lambda > 0$.

Then

$$\lambda \widehat{U_\omega}(\lambda) = \lambda \widehat{U_\omega}(\lambda) - U_\omega(0) = \widehat{U'}_\omega(\lambda) = H(\lambda + \omega), \ \lambda > 0,$$

that is,

$$\widehat{U_\omega}(\lambda) = \frac{H(\lambda + \omega)}{\lambda}, \ \lambda > 0.$$

We note that, since $H(\lambda) = \widehat{R}(t)$, where $R(t) \in \mathcal{L}(X),\ \forall t \geqslant 0$, $\widehat{U_\omega} \in \mathcal{L}(X)$ and consequently, $U_\omega \in \mathcal{L}(X),\ \forall t \geqslant 0$.

In the sequence, define $\{U(t)\}_{t\geq 0} \subset \mathcal{L}(X)$ by

$$U(t) = e^{\omega t}U_\omega(t) - \omega \int_0^t e^{\omega s}U_\omega(s)\,ds,\ t \geqslant 0.$$

We will show that $U(t)$ is derivable and $\{U'(t)\}_{t\geqslant 0}$ is an $(a,k)-$regularized resolvent family of type (M,ω) with generator A.

Firstly, since $U_\omega \in Lip(\mathbb{R}_+;X)$, U_ω is derivable a. e. and assuming L as the Lipschitz constant of U_ω, we have

$$\|U'_\omega(t)\| \leqslant L \quad a.\,e. \quad t\in[0,\infty).$$

Thus, for $T>0$ and $t,\ s\in[0,T]$, with $s<t$, follows that

$$\begin{aligned}\|U(t)-U(s)\| &= \left\| e^{\omega\tau}U_\omega(\tau)\Big|_s^t - \omega\int_s^t e^{\omega\tau}U_\omega(\tau)\,d\tau\right\| \\ &= \left\|\int_s^t e^{\omega\tau}U'_\omega(\tau)\,d\tau\right\| \\ &\leqslant \int_s^t e^{\omega T}L\,d\tau \\ &= Le^{\omega T}|t-s|,\end{aligned} \tag{2.14}$$

that is, $U(t)$ is locally Lipschitz.

Basically, using properties of the Laplace transform, we have

$$\begin{aligned}\widehat{U}(\lambda) &= \widehat{U_\omega}(\lambda-\omega) - \frac{\omega}{\lambda}\widehat{U_\omega}(\lambda-\omega) \\ &= \frac{H(\lambda)}{\lambda-\omega} - \frac{\omega H(\lambda)}{(\lambda-\omega)\lambda} \\ &= H(\lambda)\left(\frac{1}{\lambda-\omega} - \frac{\omega}{(\lambda-\omega)\lambda}\right) \\ &= \frac{H(\lambda)}{\lambda}.\end{aligned}$$

By definition of $H(\lambda)$ we have that $H(\lambda)$ commutes with A and then, $\widehat{U}(\lambda)$ commutes with A and we have the identity

$$\widehat{U}(\lambda) = \lambda^{-1}\widehat{k}(\lambda) + \widehat{a}(\lambda)\widehat{U}(\lambda)A,$$

which implies that

$$U(t)x = \int_0^t k(s)x\,ds + (a*U)(t)Ax,\ \text{for all } x\in\mathcal{D}(A) \text{ and } t>0.$$

Next we will show that $t\mapsto U(t)x$ is derivable for all fixed $x\in X$. But first, we will consider the case $x\in\mathcal{D}(A)$.

For $x \in \mathcal{D}(A)$ let $g(t) = \int_0^t k(s)x\,ds$ and $h(t) = (a * U)(t)Ax$, for $t \geqslant 0$. Thus,

$$g'(t) = k(t)x,\ \forall t \geqslant 0,$$

and, since $U(t)$ is locally Lipschitz, $U(t) \in \mathcal{L}(X)$, $\forall t \geqslant 0$, $U(0) = 0$ and $a(\dot{)}A_x \in L^1_{loc}(\mathbb{R}_+; X)$, by Proposition 2.5 we obtain that

$$h = (U * a)Ax \in C^1([0,T], X),\ T > 0.$$

Therefor, for $x \in \mathcal{D}(A)$,

$$U(t)x = g(t) + h(t) \in C^1([0,T], X),\ T > 0, \tag{2.15}$$

because $k \in C(\mathbb{R}_+)$.

Now let us consider the general case where $x \in X$.

For $T > 0$, $t \in [0,T]$ and $h \in \mathbb{R}$ such that $t + h \in [0,T]$ we define

$$T_h := \frac{U(t+h) - U(t)}{h}$$

In this way, $T_h(t)$ is a linear operator from X to X and, by (2.14)

$$\|T_h(t)x\| \leqslant \frac{\left\|U(t+h)x - U(t)x\right\|}{|h|} \leqslant \frac{Le^{\omega T}|h|\|x\|}{|h|} = Le^{\omega T}\|x\|,\ \forall x \in X,$$

i. e., $T_h(t) \in \mathcal{L}(X)$. Taking the restriction $T_h(t) : \mathcal{D}(A) \longrightarrow X$ it follows that $T_h(t) \in \mathcal{L}(\mathcal{D}(A); X)$. Moreover, $U'(t) : \mathcal{D}(A) \longrightarrow X$ is a linear operator and, for $x \in \mathcal{D}(A)$,

$$\|U'(t)x\| = \lim_{h \to 0} \|T_h(t)x\| \leqslant Le^{\omega T}\|x\|$$

which implies that $U'(t) \in \mathcal{L}(\mathcal{D}(A); X)$.

To conclude, since $\overline{\mathcal{D}(A)} = X$, we can extend the operator $U'(t)$ for all X so that $U'(t) \in \mathcal{L}(X)$ and $\|U'(t)\|_{\mathcal{L}(X)} = \|U'(t)\|_{\mathcal{L}(\mathcal{D}(A);X)}$.

Define $R(t) : X \longrightarrow X$ by $R(t) := U'(t)$, $t \geqslant 0$.

Next we will study the properties of $\{R(t)\}_{t \geqslant 0}$ to show that it is an $(a,k)-$regularized family:

(1) From the above, $R(t) \in \mathcal{L}(X)$, $\forall t \geqslant 0$;

(2) $t \mapsto R(t)x$ is continuous for all $x \in X$.

In fact, as seen previously, $t \mapsto U(t)x \in C^1([0,T], X)$ for $x \in \mathcal{D}(A)$, it means that $t \mapsto U'(t)x$ is continuous for $x \in \mathcal{D}(A)$. Since $\overline{\mathcal{D}(A)} = X$ and $\|U'(t)\|_{\mathcal{L}(X)} = \|U'(t)\|_{\mathcal{L}(\mathcal{D}(A);X)} \leqslant Le^{\omega T}$, $t \in [0,T]$, we conclude that $t \mapsto U'(t)x$ is continuous for all $x \in X$.

Then, $\{R(t)\}_{t \geqslant 0}$ is a strongly continuous family. Note that, from (2.15), for $x \in \mathcal{D}(A)$,

$$R(t)x = U'(t)x = k(t)x + (a * U')(t)Ax$$
$$= k(t)x + \int_0^t a(t-s)U'(s)Ax\, ds. \tag{2.16}$$

(3) If $k(0) = 0$, for (2.16), $R(0)x = 0$, $\forall x \in \mathcal{D}(A)$, and them, by the density of $\mathcal{D}(A)$ an X, it follows that $R(0)x = 0$.

(4) If $k(0) \neq 0$, for $x \in \mathcal{D}(A)$,

$$\lim_{t\to 0} \frac{R(t)x}{k(t)} = x + \lim_{t\to 0} \frac{1}{k(t)} \int_0^t a(t-s)U'(s)Ax\, ds = x,$$

because $k \in C(\mathbb{R}_+)$ with $k(0) \neq 0$ and $s \mapsto a(t-s)U(s)Ax$ is integrable.

Using again the density of $\mathcal{D}(A)$, we conclude $\lim_{t\to 0} \dfrac{R(t)}{k(t)} = I$.

We also observe that

$$\widehat{R}(\lambda) = \widehat{U'}(\lambda) = \lambda\widehat{U}(\lambda) + U(0) = H(\lambda) + 0 = \widehat{k}(\lambda)\big(I - \widehat{a}(\lambda)A\big)^{-1}.$$

(5) For $x \in \mathcal{D}(A)$, applying inverse of the Laplace transform, denoted by $\mathcal{L}^{-1}$, on the above equality, we obtain $R(t)x = \mathcal{L}^{-1}\big(\widehat{k}(\lambda)(I - \widehat{a}(\lambda)A)^{-1}x\big)$, with $\big(I - \widehat{a}(\lambda)A\big)^{-1}x \in \mathcal{D}(A)$. Since $\mathcal{L}^{-1}$ is a convergent integral, we conclude that $R(t)x \in \mathcal{D}(A)$.

Furthermore, since A is a closed operator that commute with $H(\lambda)$, we have that

$$AR(t)x = A\mathcal{L}^{-1}\big(H(\lambda)x\big) = \mathcal{L}^{-1}\big(AH(\lambda)x\big) = \mathcal{L}^{-1}\big(H(\lambda)Ax\big) = R(t)Ax, \ \forall x \in \mathcal{D}(A),$$

and $t \geqslant 0$.

(6) To conclude, note bt (2.16) that

$$R(t)x = k(t)x + \int_0^t a(t-s)R(s)Ax\, ds$$
$$= k(t)x + \int_0^t a(t-s)AR(s)x\, ds, \ \forall x \in \mathcal{D}(A), \text{ and } t \geqslant 0.$$

Thus, we show that $\{R(t)\}$ is an $(a,k)-$regularized family with generator A.
Furthermore, by (ii) follows that

$$\|\widehat{R}(\lambda)x\| = \|H(\lambda)x\| \leqslant \frac{M}{\lambda - \omega}\|x\|,$$

which implies that

$$\|R(t)x\| \leqslant Me^{\omega t}\|x\|, \ \forall t \geqslant 0,$$

i. e. $\{R(t)\}_{t\geqslant 0}$ is of type (M, ω), completing the proof of the theorem. □

In the particular case of Definition 2.9 with $a(t) = g_\alpha(t)$ and $k(t) = g_\beta(t)$ we obtain the following equivalent definition.

Definition 2.16. *(28, Definition 1.9, p. 4) Let X be a Banach space and $\alpha > 0$, $\beta > 0$. A one parameter family $\{S_{\alpha,\beta}(t)\}_{t\geq 0} \subset \mathcal{L}(X)$ is called an (α,β)-resolvent family if the following conditions are satisfied:*

(a) $\lim_{t\to 0} t^{1-\beta}S_{\alpha,\beta}(t) = \dfrac{1}{\Gamma(\beta)}I$ if $0 < \beta < 1$, $S_{\alpha,1}(0) = I$ and $S_{\alpha,\beta}(0) = 0$ if $\beta > 1$;

(b) $S_{\alpha,\beta}(s)S_{\alpha,\beta}(t) = S_{\alpha,\beta}(t)S_{\alpha,\beta}(s)$, for all $s, t > 0$;

(c) The functional equation

$$S_{\alpha,\beta}(s)J_t^\alpha S_{\alpha,\beta}(t) - J_s^\alpha S_{\alpha,\beta}(s)S_{\alpha,\beta}(t) = g_\beta(s)J_t^\alpha S_{\alpha,\beta}(t) - g_\beta(t)J_s^\alpha S_{\alpha,\beta}(s),$$

holds for all $t, s > 0$.

We recall the following subordination result.

Theorem 2.17. *(28, Theorem 2.5, p. 14) Let $0 < \eta_1 \leq 2$, $0 < \eta_2$. If A generates an exponentially bounded (η_1,η_2)-resolvent family $\{S_{\eta_1,\eta_2}(t)\}_{t>0}$, then for each $\beta' \geq 0$ and $0 < \alpha' < 1$ we have that A generates an exponentially bounded $(\alpha'\eta_1, \alpha'\eta_2 + \beta')$-resolvent family given by*

$$S_{\alpha'\eta_1,\alpha'\eta_2+\beta'}(t) = \int_0^\infty \psi_{\alpha',\beta'}(t,s)S_{\eta_1,\eta_2}(s)ds, \ t > 0,$$

where

$$\psi_{\alpha',\beta'}(t,s) := t^{\beta'-1}\sum_{n=0}^{\infty}\frac{(-st^{-\alpha'})^n}{n!\Gamma(-\alpha' n + \beta')},$$

is called scaled Wright function (33).

We conclude this Section with the definition of creep functions which will be use in the next Section, in an important result of generation given by Pruss (18).

Definition 2.18. *(18, Definition 4.4, p. 94) A function $a : (0,\infty) \to \mathbb{R}$ is called a creep function if $a(t)$ is nonnegative, nondecreasing, and concave. A creep function $a(t)$ has the standard form*

$$a(t) = a_0 + a_\infty t + \int_0^t a_1(\tau)d\tau, \ t > 0, \tag{2.17}$$

where $a_0 = a(0+) \geq 0$, $a_\infty = \lim_{t\to\infty} \dfrac{a(t)}{t} = \inf_{t>0} \dfrac{a(t)}{t} \geq 0$ and $a_1(t) = a'(t) - a_\infty$ is nonnegative, nonincreasing, $\lim_{t\to\infty} a_1(t) = 0$.

3 WELL POSEDNESS FOR STRONGLY DAMPED ABSTRACT CAUCHY PROBLEMS OF FRACTIONAL ORDER

In this chapter, we present some more necessary results and definitions for the development of the objective of this work.We start introducing the following definition of a family of operators which we will are in our study.

Definition 3.1. *Let $\beta > 0$, $\alpha \geq 1$, $a, b, c, d, \tau \in \mathbb{R}$, $\tau \neq 0$, where $(a, c) \neq (0, 0)$. A closed linear operator B is said to generate a solution family $\{R_{\alpha,\beta}(t)\}_{t\geqslant 0} \subseteq \mathcal{L}(X)$ if $R_{\alpha,\beta}(t)$ is strongly continuous of type (M, ω), the set $\left\{\mu \in \mathbb{C} : \mu = \dfrac{\tau\lambda^\alpha - b\lambda^\beta - d}{a\lambda^\beta + c},\ \lambda > \omega\right\} \subset \rho(B)$ and*

$$\widehat{R_{\alpha,\beta}}(\lambda) = \frac{\tau\lambda^{\alpha-1}}{a\lambda^\beta + c}\left(\frac{\tau\lambda^\alpha - b\lambda^\beta - d}{a\lambda^\beta + c} - B\right)^{-1}, \quad Re(\lambda) > \omega.$$

Next some examples of solution families.

Example 3.2. (a) If $\beta = 1$, $a = b = d = 0$ and $c = \tau = 1$, then for each $\alpha \geq 1$, $\{R_{\alpha,1}(t)\}_{t\geq 0}$ is an $(\alpha, 1)$-resolvent family for the equation $D_t^\alpha u(t) = Bu(t)$ because in this case

$$\widehat{R_{\alpha,1}}(\lambda) = \lambda^{\alpha-1}(\lambda^\alpha - B)^{-1}, \quad Re(\lambda) > \omega,$$

see (34, p. 215). If $\alpha = 2$, $\{R_{2,1}(t)\}_{t\geq 0}$ is a strongly continuous cosine family, see (27, Proposition 3.14.4). In the border case $\alpha = 1$, $\{R_{1,1}(t)\}_{t\geq 0}$ is a C_0-semigroup, see (27, Theorem 3.1.7).

(b) If $\alpha \geq 1$, $0 < \beta \leq 1$, $a = d = 0, \tau = c = 1$ and $b \leq 0$ then $\{R_{\alpha,\beta}(t)\}_{t\geq 0}$ is a $(\alpha - 1, \beta)_{-b}$-regularized family as defined in (35, Definition 2.4).

(c) If $\alpha = 1$, $0 < \beta < 1$, $a = \tau = 1$, $b = c = 0$ and $d = -1$ then $\{R_{\alpha,\beta}(t)\}_{t\geq 0}$ is a (a, k) regularized resolvent family with $a(t) = t^{-\beta}E_{1,1-\beta}(t)$ and $k(t) = e^{-t}$. In this case

$$\widehat{R_{1,\beta}}(\lambda) = \frac{1}{\lambda^\beta}\left(\frac{\lambda + 1}{\lambda^\beta} - B\right)^{-1}, \quad Re(\lambda) > \omega,$$

see (36, p. 138).

The following is an important result of generations of resolvent families.

Theorem 3.3. *(18, Theorem 4.3, p. 104) Let A generate a cosine family and assume that $a(t)$ is a creep function with $a_1(t)$ log-convex. Then A generate an $(a, 1)-$resolvent family.*

We finish this introduction with a new interesting property of (a, k)-regularized families, that is a generalization of Lemma 2.11, and that will be repeatedly used.

Lemma 3.4. *Let $\{R(t)\}_{t\geq 0}$ be an (a,k)-regularized family having A as a generator and satisfying with $(1*a)(t)\neq 0$, for all $t>0$. If $b\in L^1_{loc}(\mathbb{R}_+)$, then $(b*R)(t)x\in\mathcal{D}(A)$ for all $x\in X,\ t\geq 0$.*

Proof. Since $\{R(t)\}_{t\geqslant 0}$ is an (a,k)-regularized family, by Remark 2.12, $(b*R)(t)$ is an $(a,b*k)$-regularized family. Define $S(t):=(b*R)(t)$. By Lemma 2.11, $(a*S)(t)x\in\mathcal{D}(A)$ for every $x\in X$. Let $\lambda\in\rho(A)$ and $x\in X$ be fixed. Define $y=(\lambda-A)^{-1}x\in\mathcal{D}(A)$ and let $z:=(b*R)(t)x$. We have

$$z=(b*R)(t)(\lambda-A)y=\lambda(b*R)(t)y-(b*R)(t)Ay$$

then, convolving with the kernel $a(t)$ we obtain

$$(1*a)(t)z=\lambda(a*b*R)(t)y-(a*b*R)(t)Ay=\lambda(a*S)(t)y-(a*S)(t)Ay\in\mathcal{D}(A).$$

This complete the proof. □

3.1 Sufficient conditions for existence of solution families

In this section we want to give conditions on a closed operator B so that it is a generator of a solution family.

Proposition 3.5. *Let $0<\beta<\alpha<2+\beta$ and B generator of a cosine family $\{C(t)\}_{t>0}$. Then B generates an exponentially bounded $(\alpha-\beta,\alpha)$-resolvent family.*

Proof. Recall that B being a generator of a cosine family $\{C(t)\}_{t\geqslant 0}$ is equivalent to say that B is generator of an exponentially bounded $(2,1)$-resolvent family. By Theorem 2.17, choosing $\eta_1=2$, $\eta_2=1$ and $\alpha'=\dfrac{\alpha-\beta}{2}$, $\beta'=\dfrac{\alpha+\beta}{2}$, and using the hypothesis, we obtain $0<\dfrac{\alpha-\beta}{2}<1$. We conclude that B is the generator of an exponentially bounded $(\alpha'\eta_1,\alpha'\eta_2+\beta')=(\alpha-\beta,\alpha)$-resolvent family $\{S_{\alpha-\beta,\alpha}(t)\}_{t\geqslant 0}$. Furthermore,

$$S_{\alpha-\beta,\alpha}(t)x=\int_0^\infty \psi_{\frac{\alpha-\beta}{2},\frac{\alpha+\beta}{2}}(t,s)C(s)x\,ds,\ t>0 \text{ and } x\in X.$$

□

Our next result assumes B as generator of a C_0-semigroup instead a cosine family.

Proposition 3.6. *Let $0<\beta<\alpha<1+\beta$ and B generator of a C_0-semigroup $\{T(t)\}_{t\geqslant 0}$. Then B generates an exponentially bounded $(\alpha-\beta,\alpha)$-resolvent family.*

Proof. Since B generates a C_0-semigoup $\{T(t)\}_{t\geqslant 0}$, then it generates an exponentially bounded $(1,1)$-resolvent family $\{S_{1,1}(t)\}_{t\geqslant 0}$. Define $\eta_1=\eta_2=1$ and $\alpha'=\alpha-\beta,\ \beta'=\beta$. By hypothesis we obtain $0<\alpha-\beta<1$. Therefore, by Theorem 2.17, we obtain that B generates an exponentially bounded $(\alpha'\eta_1,\alpha'\eta_2+\beta')=(\alpha-\beta,\alpha)$-resolvent family $\{S_{\alpha-\beta,\alpha}(t)\}_{t\geqslant 0}$. Furthermore,

$$S_{\alpha-\beta,\alpha}(t)x = \int_0^\infty \psi_{\alpha-\beta,\beta}(t,s)T(s)xds,\ t>0,\ \text{and}\ x\in X.$$

□

Remark 3.7. We recall from (27, Theorem 3.14.17, p. 215) that if an operator B generates a cosine family then it generates a C_0-semigroup.

Corollary 3.8. *Let $0<\beta<\alpha<1+\beta$ and B generator of a cosine family. Then B generates an exponentially bounded $(\alpha-\beta,\alpha)$-resolvent family.*

Remark 3.9. By (33, Theorem 12), the following equality holds:

$$S_{\alpha-\beta,\alpha}(t)x = (g_\beta * S_{\alpha-\beta,\alpha-\beta})(t)x.$$

The next lemma is an auxiliary result for our main theorem of this section.

Lemma 3.10. *Let $\tau,a,b,d\in\mathbb{R}$. Suppose that $a\neq 0,\ \tau\neq 0, \dfrac{a}{\tau}>0,\ c=0,$, and assume any of the following:*

(i) $0<\beta<\alpha<1+\beta$ and B generator of a C_0-semigroup;

(ii) $0<\beta<\alpha<2+\beta$ and B generator of a cosine family.

Then the set $\left\{\mu\in\mathbb{C}\ :\ \mu=\dfrac{\tau\lambda^\alpha-b\lambda^\beta-d}{a\lambda^\beta}\right\}\subseteq\rho(B)$ and there exist a strongly continuous family $\{Q(t)\}_{t\geq 0}\subset\mathcal{L}(X)$, exponentially bounded, such that

$$\widehat{Q(\lambda)} = \tau(\tau\lambda^\alpha-(aB+bI)\lambda^\beta-d)^{-1},\quad Re(\lambda)>\omega,$$

for a given constant $\omega\in\mathbb{R}_+$.

Proof. Since B generates a C_0-semigroup (resp. cosine family) then $\dfrac{a}{\tau}B+\dfrac{b}{\tau}I$ also generates a C_0-semigroup (resp. cosine family), see (27). Then, by Proposition 3.6 (resp. Proposition 3.5), we obtain in any case that $\dfrac{a}{\tau}B+\dfrac{b}{\tau}I$ generates an exponentially bounded $(\alpha-\beta,\alpha)$-resolvent family $\{S_{\alpha-\beta,\alpha}(t)\}_{t\geq 0}$, we say $\|S_{\alpha-\beta,\alpha}(t)\|\leqslant Me^{\omega t}$, $\omega\in\mathbb{R}_+$, $t\geqslant 0$. Then, by Theorem 2.15 with $a(t)=g_{\alpha-\beta}(t)$, we obtain that $\dfrac{1}{\widehat{a}(\lambda)}=\lambda^{\alpha-\beta}\in\rho\left(\dfrac{a}{\tau}B+\dfrac{b}{\tau}I\right)$, $\lambda>\omega$, which allows us to conclude that $\left\{\mu\in\mathbb{C}\ :\ \mu=\dfrac{\tau\lambda^\alpha-b\lambda^\beta}{a\lambda^\beta},\ \lambda>\omega\right\}\subseteq\rho(B)$.

Furthermore, considering the following notation

$$\begin{aligned} S^{*(1)}_{\alpha-\beta,\alpha}(t) &= S_{\alpha-\beta,\alpha}(t),\ t\geqslant 0\\ S^{*(k+1)}_{\alpha-\beta,\alpha}(t) &= \left(S^{*(k)}_{\alpha-\beta,\alpha} * S_{\alpha-\beta,\alpha}\right)(t),\ t\geqslant 0,\ k\in\mathbb{N}, \end{aligned}$$

we can prove that

$$\|S^{*(k+1)}_{\alpha-\beta,\alpha}(t)\| \leq M^{k+1}e^{\omega t}\frac{t^k}{k!},\ k\in\mathbb{N},\ t\geqslant 0.$$

Indeed, by induction on k, note that:

(i) for $k=0$, $\|S^{*(0+1)}_{\alpha-\beta,\alpha}(t)\| = \|S_{\alpha-\beta,\alpha}(t)\| \leq Me^{\omega t} = M^{0+1}e^{\omega t}\dfrac{t^0}{0!}$, $t \geq 0$;

(ii) assuming for $k = j \in \mathbb{N}$ that

$$\|S^{*(j+1)}_{\alpha-\beta,\alpha}(t)\| \leq M^{j+1}e^{\omega t}\frac{t^j}{j!},\ t \geqslant 0, \tag{3.1}$$

we will show that $\|S^{*(j+1+1)}_{\alpha-\beta,\alpha}(t)\| \leq M^{j+1+1}e^{\omega t}\dfrac{t^{j+1}}{(j+1)!}$, $t \geqslant 0$. In fact, by (3.1), we have

$$\begin{aligned}
\|S^{*(j+1+1)}_{\alpha-\beta,\alpha}(t)\| &= \|(S^{*(j+1)}_{\alpha-\beta,\alpha} * S_{\alpha-\beta,\alpha})(t)\| = \left\| \int_0^t S_{\alpha-\beta,\alpha}(t-s)S^{*(j+1)}_{\alpha-\beta,\alpha}(s)ds \right\| \\
&\leq \int_0^t \|S_{\alpha-\beta,\alpha}(t-s)\| M^{j+1}e^{\omega s}\frac{s^j}{j!}ds \leq \int_0^t Me^{\omega(t-s)}M^{j+1}e^{\omega s}\frac{s^j}{j!}ds \\
&= \int_0^t M^{j+1+1}e^{\omega t}\frac{s^j}{j!}ds = M^{j+1+1}e^{\omega t}\int_0^t \frac{s^j}{j!}ds = M^{j+1+1}e^{\omega t}\frac{t^{j+1}}{(j+1)j!} \\
&= M^{j+1+1}e^{\omega t}\frac{t^{j+1}}{(j+1)!},\ t \geqslant 0,
\end{aligned}$$

proving our claim.

Now, for each $t \geq 0$ and $x \in X$, we define

$$Q(t)x := \sum_{k=0}^{\infty} \left(\frac{d}{\tau}\right)^k S^{*(k+1)}_{\alpha-\beta,\alpha}(t)x.$$

It is clear that $Q(t)$ is a strongly continuous operator. Moreover,

$$\begin{aligned}
\|Q(t)\| &= \left\| \sum_{k=0}^{\infty} \left(\frac{d}{\tau}\right)^k S^{*(k+1)}_{\alpha-\beta,\alpha}(t) \right\| \leq \sum_{k=0}^{\infty} \left|\frac{d}{\tau}\right|^k \|S^{*(k+1)}_{\alpha-\beta,\alpha}(t)\| \leq \sum_{k=0}^{\infty} \left|\frac{d}{\tau}\right|^k M^{k+1}e^{\omega t}\frac{t^k}{k!} \\
&= e^{\omega t}\sum_{k=0}^{\infty} \left|\frac{d}{\tau}\right|^k M\, M^k \frac{t^k}{k!} = Me^{\omega t}\sum_{k=0}^{\infty} \frac{\left(\left|\frac{d}{\tau}\right|Mt\right)^k}{k!} = Me^{\omega t}e^{\left|\frac{d}{\tau}\right|Mt} = Me^{(\omega+\left|\frac{d}{\tau}\right|M)t},
\end{aligned}$$

which proves that $Q(t)$ is well defined and it is exponentially bounded. Taking the Laplace transform of $Q(t)$, by continuity of the Laplace transform, we obtain

$$\begin{aligned}
\widehat{Q}(\lambda) &= \mathcal{L}\left(\sum_{k=0}^{\infty} \left(\frac{d}{\tau}\right)^k S^{*(k+1)}_{\alpha-\beta,\alpha}(t) \right)(\lambda) = \sum_{k=0}^{\infty} \left(\frac{d}{\tau}\right)^k \mathcal{L}\big(S^{*(k+1)}_{\alpha-\beta,\alpha}(t)\big)(\lambda) \\
&= \sum_{k=0}^{\infty} \left(\frac{d}{\tau}\right)^k \mathcal{L}\big(S^{*(k)}_{\alpha-\beta,\alpha}(t)\big)(\lambda)\widehat{S_{\alpha-\beta,\alpha}}(\lambda) = \widehat{S_{\alpha-\beta,\alpha}}(\lambda)\sum_{k=0}^{\infty} \left[\frac{d}{\tau}\widehat{S_{\alpha-\beta,\alpha}}(\lambda)\right]^k.
\end{aligned}$$

By Theorem 2.15, we conclude that $\widehat{S_{\alpha-\beta,\alpha}}(\lambda) = \lambda^{-\beta}(\lambda^{\alpha-\beta} - (\frac{a}{\tau}B + \frac{b}{\tau}I))^{-1} = \tau\lambda^{-\beta}(\tau\lambda^{\alpha-\beta} -$

$(aB + bI))^{-1}$, and $\alpha > \beta$, we obtain, for $\mathrm{Re}(\lambda)$ sufficiently large, that

$$\widehat{S_{\alpha-\beta,\alpha}}(\lambda) \sum_{k=0}^{\infty} \left[d\widehat{S_{\alpha-\beta,\alpha}}(\lambda)\right]^k = \tau\lambda^{-\beta}(\tau\lambda^{\alpha-\beta}-(aB+bI))^{-1} \sum_{k=0}^{\infty} \left[\frac{d}{\tau}\tau\lambda^{-\beta}(\tau\lambda^{\alpha-\beta}-(aB+bI))^{-1}\right]^k. \tag{3.2}$$

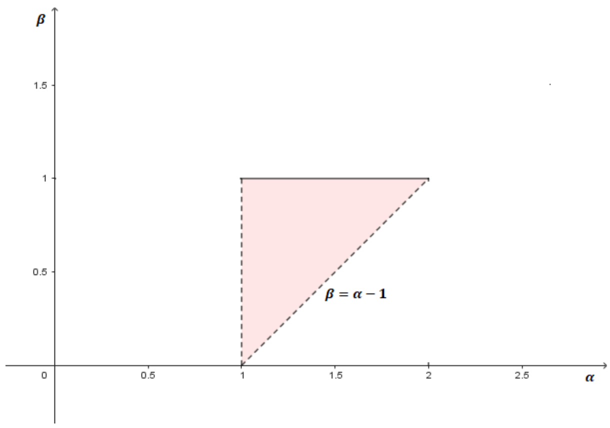
Since $\|d\lambda^{-\beta}(\tau\lambda^{\alpha-\beta} - (aB + bI))^{-1}\| < 1$ for λ large enough we obtain, using Neumann series, that $\left\{\mu \in \mathbb{C}; \mu = \dfrac{\tau\lambda^{\alpha} - b\lambda^{\beta} - d}{a\lambda^{\beta}},\ \lambda > \omega\right\} \subseteq \rho(B)$. Moreover, (3.2) equals to

$$\begin{aligned}
\widehat{Q}(\lambda) &= \tau\lambda^{-\beta}(\tau\lambda^{\alpha-\beta} - (aB + bI))^{-1}\left[1 - d\lambda^{-\beta}\left(\tau\lambda^{\alpha-\beta} - (aB + bI)\right)^{-1}\right]^{-1} \\
&= \tau\frac{1}{\lambda^{\beta}}(\tau\lambda^{\alpha-\beta} - (aB + bI) - d\lambda^{\beta})^{-1} = \tau(\tau\lambda^{\alpha} - (aB + bI)\lambda^{\beta} - d)^{-1},
\end{aligned}$$

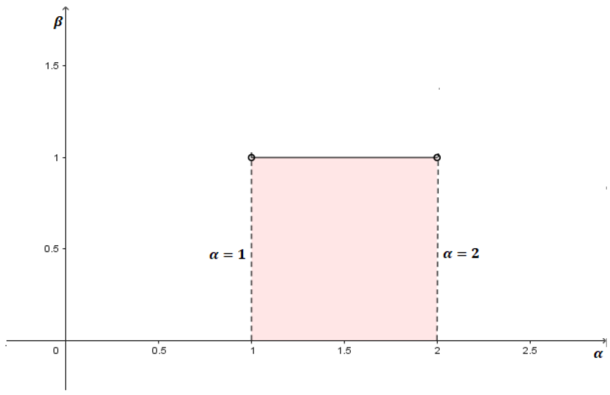
proving the Lemma. $\square$

The following main result give our first practical condition for generation of a solution family. It focuses on two sectors illustrated below:

Figure 3.1 – Sectors of Theorem 3.11



(a) $0 < \beta \leq 1 < \alpha < 1 + \beta \leq 2$.



(b) $0 < \beta \leq 1 < \alpha < 2$.

Source: prepared by the author.

Theorem 3.11. *Let $a, b, c, d, \tau \in \mathbb{R}$ where $a \neq 0,\ \tau \neq 0,\ c = 0$, and assume any of the following conditions.*

(a) $0 < \beta \leq 1 < \alpha < 1 + \beta \leq 2$ and B generator of a C_0-semigroup;

(b) $0 < \beta \leq 1 < \alpha < 2$ and B generator of a cosine family.

Then B generates a solution family $\{R_{\alpha,\beta}(t)\}_{t\geq 0}$.

Proof. By Lemma 3.10 there exists a strongly continuous family $\{Q(t)\}_{t\geqslant 0}$ satisfying $\widehat{Q(\lambda)} = \tau(\tau\lambda^\alpha - (aB + bI)\lambda^\beta - d)^{-1},\ Re(\lambda) > \omega$, for some $\omega \in \mathbb{R}_+$. Note that

$$\widehat{Q}(\lambda) = \tau(\tau\lambda^\alpha - a\lambda^\beta B - b\lambda^\beta - d)^{-1} = \frac{\tau}{a\lambda^\beta}\left(\frac{\tau\lambda^\alpha - b\lambda^\beta - d}{a\lambda^\beta} - B\right)^{-1}.$$

Since $1 < \alpha < 2$ we can define $R_{\alpha,\beta}(t) := \frac{d}{dt}(g_{2-\alpha} * Q)(t)$, which corresponds to the fractional derivative of order $\alpha - 1$ in the sense of Riemann Liouville for $Q(t)$. We obtain that

$$\widehat{R_{\alpha,\beta}}(\lambda) = \lambda\frac{1}{\lambda^{2-\alpha}}\widehat{Q}(\lambda) = \frac{\tau\lambda^{\alpha-1}}{a\lambda^\beta}\left(\frac{\tau\lambda^\alpha - b\lambda^\beta - d}{a\lambda^\beta} - B\right)^{-1}.$$

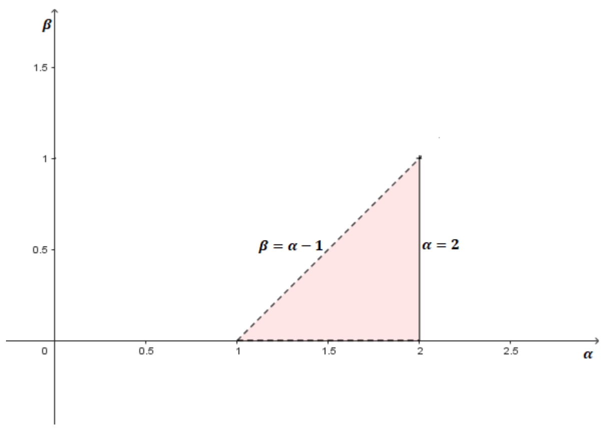
Finally, Definition 3.1 ensures that $\{R_{\alpha,\beta}(t)\}_{t\geqslant 0}$ is a solution family generated by B. $\square$

We will repeatedly use of the following observation.

Remark 3.12. The Gilfand - Shilov function $g_\gamma(t)$ is nonincreasing for $0 < \gamma < 1$ and increasing for $\gamma > 1$. Moreover, $g'_\gamma(t) = \dfrac{\gamma - 1}{t} g_\gamma(t),\ t > 0$.

In case where the parameter $c \neq 0$, we have the next theorem, that aims to the sector illustrated below.

Figure 3.2 – Sector $0 < \beta \leq 1 < 1 + \beta < \alpha \leq 2$.



Source: prepared by the author.

Theorem 3.13. *Let $a, b, c, d, \tau \in \mathbb{R}$ where $b = d = 0,\ a, c \geq 0,\ \tau > 0$ and let B be the generator of a cosine family. Suppose that $0 < \beta \leq 1 < 1 + \beta < \alpha \leq 2$. Then B generates a solution family $\{R_{\alpha,\beta}(t)\}_{t\geq 0}$.*

Proof. First, note that since $b = d = 0$, then if such solution family exists, it must verify

$$\widehat{S_{\alpha,\beta}}(\lambda) = \frac{1}{\lambda\widehat{a}(\lambda)}\left(\frac{1}{\widehat{a}(\lambda)} - B\right)^{-1} = \frac{1}{\lambda\dfrac{(a\lambda^\beta + c)}{\tau\lambda^\alpha}}\left(\frac{\tau\lambda^\alpha}{a\lambda^\beta + c} - B\right)^{-1}$$
$$= \frac{1}{\dfrac{(a\lambda^\beta + c)}{\tau\lambda^{\alpha-1}}}\left(\frac{\tau\lambda^\alpha}{a\lambda^\beta + c} - B\right)^{-1} = \frac{\tau\lambda^{\alpha-1}}{a\lambda^\beta + c}\left(\frac{\tau\lambda^\alpha}{a\lambda^\beta + c} - B\right)^{-1}.$$

Therefore, $\{R_{\alpha,\beta}(t)\}_{t\geq 0}$ is an $(a, 1)$-regularized family, that is, a resolvent family (see Remark 2.10) with $a(t) = \frac{a}{\tau}g_{\alpha-\beta}(t) + \frac{c}{\tau}g_\alpha(t)$. Our strategy for the proof is to use Theorem 3.3.

We claim that $a(t)$ is a creep function (see Definition 2.18). In fact, consider $a_1(t) := \frac{a}{\tau}g_{\alpha-\beta-1}(t) + \frac{c}{\tau}g_{\alpha-1}(t)$, then

$$a(t) = \frac{a}{\tau}g_{\alpha-\beta}(t) + \frac{c}{\tau}g_\alpha(t) = \int_0^t \left[\frac{a}{\tau}(\alpha-\beta-1)\frac{s^{\alpha-\beta-2}}{\Gamma(\alpha-\beta)} + \frac{c}{\tau}(\alpha-1)\frac{s^{\alpha-2}}{\Gamma(\alpha)}\right]ds$$
$$= a_0 + a_\infty t + \int_0^t a_1(s)ds,$$

where $a_0 = 0$, $a_\infty := \lim_{t\to\infty}\frac{a(t)}{t} = \lim_{t\to\infty}\frac{1}{t}\left[\frac{a}{\tau}g_{\alpha-\beta}(t)+\frac{c}{\tau}g_\alpha(t)\right] = \lim_{t\to\infty}\left[\frac{a}{\tau}\frac{t^{\alpha-\beta-2}}{\Gamma(\alpha-\beta)}+\frac{c}{\tau}\frac{t^{\alpha-2}}{\Gamma(\alpha)}\right] = 0$ and $a_1(t) = a'(t) - a_\infty$. Moreover, since $a, c \geq 0$ and $\tau > 0$ it is clear that $a_1(t)$ is non negative, $\lim_{t\to\infty} a_1(t) = 0$ and since by hypothesis $0 < \alpha - \beta - 1 < 1$ and $0 < \alpha - 1 < 1$, by Remark 3.12 we obtain that $a_1(t)$ is nonincreasing. By Definition 2.18, we conclude that $a(t)$ is a creep function, proving the claim.

Next, we claim that $a_1(t)$ is log-convex. Indeed, consider $f(t) = \log(a_1(t))$ then $f''(t) = \dfrac{a_1''(t)a_1(t) - a_1'(t)a_1'(t)}{a_1^2(t)}$ where, using Remark 3.12, we obtain

$$a_1'(t) = \frac{a}{\tau}(\alpha-\beta-2)t^{-1}g_{\alpha-\beta-1}(t) + \frac{c}{\tau}(\alpha-2)t^{-1}g_{\alpha-1}(t)$$

and

$$a_1''(t) = \frac{a}{\tau}(\alpha-\beta-2)(\alpha-\beta-3)t^{-2}g_{\alpha-\beta-1}(t) + \frac{c}{\tau}(\alpha-2)(\alpha-3)t^{-2}g_{\alpha-1}(t).$$

We will to show that $a_1''(t)a_1(t) \geq a_1'(t)a_1'(t)$. We have that

$$a_1''(t)a_1(t) = \left[\frac{a}{\tau}(\alpha-\beta-1)(\alpha-\beta-2)(\alpha-\beta-3)\frac{t^{\alpha-\beta-4}}{\Gamma(\alpha-\beta)}\right.$$
$$\left.+\frac{c}{\tau}(\alpha-1)(\alpha-2)(\alpha-3)\frac{t^{\alpha-4}}{\Gamma(\alpha)}\right]\cdot\left[\frac{a}{\tau}(\alpha-\beta-1)\frac{t^{\alpha-\beta-2}}{\Gamma(\alpha-\beta)} + \frac{c}{\tau}(\alpha-1)\frac{t^{\alpha-2}}{\Gamma(\alpha)}\right]$$
$$= \frac{a^2}{\tau^2}(\alpha-\beta-1)^2(\alpha-\beta-2)(\alpha-\beta-3)\frac{t^{2\alpha-2\beta-6}}{\Gamma(\alpha-\beta)\Gamma(\alpha-\beta)}$$

$$
\begin{aligned}
&+\frac{ac}{\tau^2}(\alpha-1)(\alpha-\beta-1)(\alpha-\beta-2)(\alpha-\beta-3)\frac{t^{2\alpha-\beta-6}}{\Gamma(\alpha-\beta)\Gamma(\alpha)}\\
&+\frac{ac}{\tau^2}(\alpha-1)(\alpha-2)(\alpha-3)(\alpha-\beta-1)\frac{t^{2\alpha-\beta-6}}{\Gamma(\alpha)\Gamma(\alpha-\beta)}\\
&+\frac{c^2}{\tau^2}(\alpha-1)^2(\alpha-2)(\alpha-3)\frac{t^{2\alpha-6}}{\Gamma(\alpha)\Gamma(\alpha)}\\
=&\frac{a^2}{\tau^2}(\alpha-\beta-1)^2(\alpha-\beta-2)(\alpha-\beta-3)\frac{t^{2\alpha-2\beta-6}}{\Gamma(\alpha-\beta)\Gamma(\alpha-\beta)} \qquad (3.3)\\
&+\frac{ac}{\tau^2}(\alpha-1)(\alpha-\beta-1)\frac{t^{2\alpha-\beta-6}}{\Gamma(\alpha)\Gamma(\alpha-\beta)}\Big[(\alpha-\beta-2)(\alpha-\beta-3)+(\alpha-2)(\alpha-3)\Big]\\
&+\frac{c^2}{\tau^2}(\alpha-1)^2(\alpha-2)(\alpha-3)\frac{t^{2\alpha-6}}{\Gamma(\alpha)\Gamma(\alpha)},
\end{aligned}
$$

and

$$
\begin{aligned}
a_1'(t)a_1'(t)=&\left[\frac{a}{\tau}(\alpha-\beta-1)(\alpha-\beta-2)\frac{t^{\alpha-\beta-3}}{\Gamma(\alpha-\beta)}+\frac{c}{\tau}(\alpha-1)(\alpha-2)\frac{t^{\alpha-3}}{\Gamma(\alpha)}\right]^2\\
=&\frac{a^2}{\tau^2}(\alpha-\beta-1)^2(\alpha-\beta-2)^2\frac{t^{2\alpha-2\beta-6}}{\Gamma(\alpha-\beta)\Gamma(\alpha-\beta)}\\
&+2\frac{ac}{\tau^2}(\alpha-\beta-1)(\alpha-\beta-2)(\alpha-1)(\alpha-2)\frac{t^{2\alpha-\beta-6}}{\Gamma(\alpha-\beta)\Gamma(\alpha)}\\
&+\frac{c^2}{\tau^2}(\alpha-1)^2(\alpha-2)^2\frac{t^{2\alpha-6}}{\Gamma(\alpha)\Gamma(\alpha)}\\
=&\frac{a^2}{\tau^2}(\alpha-\beta-1)^2(\alpha-\beta-2)^2\frac{t^{2\alpha-2\beta-6}}{\Gamma(\alpha-\beta)\Gamma(\alpha-\beta)}\\
&+\frac{ac}{\tau^2}(\alpha-1)(\alpha-\beta-1)\frac{t^{2\alpha-\beta-6}}{\Gamma(\alpha)\Gamma(\alpha-\beta)}\Big[(\alpha-\beta-2)(\alpha-2)\\
&\quad+(\alpha-\beta-2)(\alpha-2)\Big]+\frac{c^2}{\tau^2}(\alpha-1)^2(\alpha-2)^2\frac{t^{2\alpha-6}}{\Gamma(\alpha)\Gamma(\alpha)}\\
=&\frac{a^2}{\tau^2}(\alpha-\beta-1)^2(\alpha-\beta-2)^2\frac{t^{2\alpha-2\beta-6}}{\Gamma(\alpha-\beta)\Gamma(\alpha-\beta)}\\
&+\frac{ac}{\tau^2}(\alpha-1)(\alpha-\beta-1)\frac{t^{2\alpha-\beta-6}}{\Gamma(\alpha)\Gamma(\alpha-\beta)}\Big[2(\alpha-2)^2-2\beta(\alpha-2)\Big] \qquad (3.4)\\
&+\frac{c^2}{\tau^2}(\alpha-1)^2(\alpha-2)^2\frac{t^{2\alpha-6}}{\Gamma(\alpha)\Gamma(\alpha)}
\end{aligned}
$$

Now observe that

$$
\begin{aligned}
(\alpha-\beta-2)(\alpha-\beta-3)&=(\alpha-\beta-2)(\alpha-\beta-2-1)\\
&=(\alpha-\beta-2)(\alpha-\beta-2)-(\alpha-\beta-2)
\end{aligned}
$$

and since $0<\beta\leq 1$ and $1<\alpha\leq 2$ we obtain that $(\alpha-\beta-2)\leq 0$ and $-(\alpha-\beta-2)\geq 0$

proving that

$$(\alpha-\beta-2)(\alpha-\beta-3) \geq (\alpha-\beta-2)^2. \tag{3.5}$$

Moreover, not that

$$\begin{aligned}(\alpha-2)(\alpha-3) &= (\alpha-2)(\alpha-2-1)\\ &= (\alpha-2)(\alpha-2)-(\alpha-2)\end{aligned}$$

and since $(\alpha-2)\leq 0$, then $-(\alpha-2)\geq 0$, proving that

$$(\alpha-2)(\alpha-3) > (\alpha-2)^2. \tag{3.6}$$

Even more observe that

$$\begin{aligned}&(\alpha-\beta-2)(\alpha-\beta-3)+(\alpha-2)(\alpha-3) = \big((\alpha-2)-\beta\big)(\alpha-\beta-3)+(\alpha-2)(\alpha-3)\\ &= (\alpha-2)(\alpha-\beta-3)-\beta(\alpha-\beta-3)+(\alpha-2)(\alpha-3)\\ &= (\alpha-2)\big((\alpha-2)-\beta-1\big)-\beta\big((\alpha-2)-\beta-1\big)+(\alpha-2)\big((\alpha-2)-1\big)\\ &= (\alpha-2)(\alpha-2)-\beta(\alpha-2)-(\alpha-2)-\beta(\alpha-2)+\beta^2+\beta+(\alpha-2)(\alpha-2)-(\alpha-2)\\ &= 2(\alpha-2)^2-2\beta(\alpha-2)-2(\alpha-2)+\beta^2+\beta\end{aligned}$$

and since $(\alpha-2)\leq 0, \beta>0, -2(\alpha-2)\geq 0$, follow that

$$2(\alpha-2)^2-2\beta(\alpha-2)-2(\alpha-2)+\beta^2+\beta \geq 2(\alpha-2)^2-2\beta(\alpha-2). \tag{3.7}$$

Then, by (3.3), (3.4), (3.5), (3.6) and (3.7) we obtain that

$$\begin{aligned}a_1''(t)a_1(t) &= \frac{a^2}{\tau^2}(\alpha-\beta-1)^2(\alpha-\beta-2)(\alpha-\beta-3)\frac{t^{2\alpha-2\beta-6}}{\Gamma(\alpha-\beta)\Gamma(\alpha-\beta)}\\ &+\frac{ac}{\tau^2}(\alpha-1)(\alpha-\beta-1)\frac{t^{2\alpha-\beta-6}}{\Gamma(\alpha)\Gamma(\alpha-\beta)}\Big[(\alpha-\beta-2)(\alpha-\beta-3)+(\alpha-2)(\alpha-3)\Big]\\ &+\frac{c^2}{\tau^2}(\alpha-1)^2(\alpha-2)(\alpha-3)\frac{t^{2\alpha-6}}{\Gamma(\alpha)\Gamma(\alpha)}\\ &\geq \frac{a^2}{\tau^2}(\alpha-\beta-1)^2(\alpha-\beta-2)^2\frac{t^{2\alpha-2\beta-6}}{\Gamma(\alpha-\beta)\Gamma(\alpha-\beta)}\\ &+\frac{ac}{\tau^2}(\alpha-1)(\alpha-\beta-1)\frac{t^{2\alpha-\beta-6}}{\Gamma(\alpha)\Gamma(\alpha-\beta)}\Big[2(\alpha-2)^2-2\beta(\alpha-2)\Big]\\ &+\frac{c^2}{\tau^2}(\alpha-1)^2(\alpha-2)^2\frac{t^{2\alpha-6}}{\Gamma(\alpha)\Gamma(\alpha)} = a_1'(t)a_1'(t).\end{aligned}$$

Once $a_1''(t)a_1(t)\geq a_1'(t)a_1'(t)$, $f''(t)\geq 0$ for all $t\geq 0$, and then $a_1(t)$ is log-convex, proving the claim and conclude the theorem □

3.2 A particular (a,k)-regularized family

The following main theorem show that under some conditions on the parameters, a solution family $\{R_{\alpha,\beta}(t)\}_{t\geq 0}$ is a particular case of an (a,k)-regularized family, and therefore many properties can be available from the general theory, see e.g. (37, 29, 26).

Theorem 3.14. *Let $a,b,c,d,\tau \in \mathbb{R}$ be given with $b \geq 0,\ \tau > 0,\ (a,c) \neq (0,0)$ and let B be the generator of a solution family $\{R_{\alpha,\beta}(t)\}_{t\geq 0}$. Then $\{R_{\alpha,\beta}(t)\}_{t\geq 0}$ is an (a,k)-regularized family with*

(I) $a(t) = aD_t^{\beta} p_{\alpha,\beta}(t) + cp_{\alpha,\beta}(t)$ *and* $k(t) = \tau D_t^{\alpha-1} p_{\alpha,\beta}(t)$, *if* $0 < \beta \leq 1 < \alpha < 2$;

(II) $a(t) = aD_t^{\beta} p_{1,\beta}(t) + \dfrac{a}{\tau} g_{1-\beta}(t) + cp_{1,\beta}(t)$ *and* $k(t) = \tau p_{1,\beta}(t)$, *if* $0 < \beta \leq 1 = \alpha$,

where

$$p_{\alpha,\beta}(t) = \sum_{k=0}^{\infty} \frac{d^k}{\tau^{k+1}} \left[t^{\alpha-1} E_{\alpha-\beta,\alpha}\left(\frac{b}{\tau} t^{\alpha-\beta}\right)\right]^{*(k+1)} \tag{3.8}$$

and $*(k+1)$ represents the $(k+1)$ convolutions.

Proof. From Definition 3.1, $\{R_{\alpha,\beta}(t)\}_{t\geqslant 0}$ is an exponentially bounded family of strongly continuous operators of type (M,ω). From Proposition 2.14, it follows that $\{R_{\alpha,\beta}(t)\}_{t\geqslant 0}$ is a (a,k)-regularized family with generator B if we can find $a(t)$ and $k(t)$ Laplace transformable functions such that

$$\widehat{R_{\alpha,\beta}}(\lambda) = \frac{\tau\lambda^{\alpha-1}}{a\lambda^{\beta}+c}\left(\frac{\tau\lambda^{\alpha} - b\lambda^{\beta} - d}{a\lambda^{\beta}+c} - B\right)^{-1} = \frac{\widehat{k}(\lambda)}{\widehat{a}(\lambda)}\left(\frac{1}{\widehat{a}(\lambda)} - B\right)^{-1}, \tag{3.9}$$

for all $\lambda \in \mathbb{C}$, $\mathrm{Re}(\lambda) > \omega$.

For $0 < \beta \leq 1 < \alpha < 2$, we define $p_{\alpha,\beta}(t)$ as in (3.8). We first prove that $p_{\alpha,\beta}(t)$ is well defined and exponentially bounded. In fact, by Theorem 2.8, since $b \geq 0,\ \tau > 0$, for all $t > 0$, we have

$$\left| t^{\alpha-1} E_{\alpha-\beta,\alpha}\left(\frac{b}{\tau} t^{\alpha-\beta}\right)\right| \leq t^{\alpha-1} \frac{C}{1 + \dfrac{b}{\tau} t^{\alpha-\beta}} = t^{\alpha-1} C \frac{\tau}{\tau + bt^{\alpha-\beta}} \leq t^{\alpha-1} C.$$

Note that $t^{\alpha-1} = g_{\alpha}(t)\Gamma(\alpha)$ and since $1 < \alpha < 2$, $\Gamma(\alpha) \leq 1$. Then

$$\left| t^{\alpha-1} E_{\alpha-\beta,\alpha}\left(\frac{b}{\tau} t^{\alpha-\beta}\right)\right| \leq g_{\alpha}(t) C.$$

Using induction, we obtain

$$\left| \frac{d^k}{\tau^{k+1}} \left[t^{\alpha-1} E_{\alpha-\beta}\left(\frac{b}{\tau} t^{\alpha-\beta}\right)\right]^{*(k+1)} \right| \leq \frac{|d|^k C^{k+1}}{\tau^{k+1}} g_{(k+1)\alpha}(t), \text{ for all } k \in \mathbb{N}_0.$$

Hence,

$$
\begin{aligned}
|p_{\alpha,\beta}(t)| &= \left| \sum_{k=0}^{\infty} \frac{d^k}{\tau^{k+1}} \left[t^{\alpha-1} E_{\alpha-\beta,\alpha}\left(\frac{b}{\tau} t^{\alpha-\beta} \right) \right]^{*(k+1)} \right| \\
&\leq \frac{C}{\tau} \sum_{k=0}^{\infty} \frac{|d|^k C^k}{\tau^k} \frac{t^{\alpha k+\alpha-1}}{\Gamma(\alpha k+\alpha)} = \frac{C}{\tau} t^{\alpha-1} \sum_{k=0}^{\infty} \frac{\left(\frac{|d|C}{\tau} t^{\alpha} \right)^k}{\Gamma(\alpha k+\alpha)} = \frac{C}{\tau} t^{\alpha-1} E_{\alpha,\alpha}\left(\frac{|d|C}{\tau} t^{\alpha} \right).
\end{aligned}
$$

It proves that $p_{\alpha,\beta}(t)$ is well defined for each $t > 0$, and it is Laplace transformable. Thus, taking the Laplace transform of $p_{\alpha,\beta}(t)$ we have that

$$
\begin{aligned}
\widehat{p_{\alpha,\beta}}(\lambda) &= \mathcal{L}\left(\sum_{k=0}^{\infty} \frac{d^k}{\tau^{k+1}} \left[t^{\alpha-1} E_{\alpha-\beta,\alpha}\left(\frac{b}{\tau} t^{\alpha-\beta} \right) \right]^{*(k+1)} \right)(\lambda) \\
&= \sum_{k=0}^{\infty} \frac{d^k}{\tau^{k+1}} \left[\mathcal{L}\left(t^{\alpha-1} E_{\alpha-\beta,\alpha}\left(\frac{b}{\tau} t^{\alpha-\beta} \right) \right)(\lambda) \right]^{k+1} \\
&= \sum_{k=0}^{\infty} d^k \left[\frac{1}{\tau} \mathcal{L}\left(t^{\alpha-1} E_{\alpha-\beta,\alpha}\left(\frac{b}{\tau} t^{\alpha-\beta} \right) \right)(\lambda) \right]^{k+1} = \sum_{k=0}^{\infty} d^k \left[\frac{\lambda^{-\beta}}{\tau\left(\lambda^{\alpha-\beta} - \frac{b}{\tau} \right)} \right]^{k+1} \\
&= \sum_{k=0}^{\infty} d^k \left[\frac{\lambda^{-\beta}}{\tau\lambda^{\alpha-\beta} - b} \right]^{k+1} = \sum_{k=0}^{\infty} \frac{d^k \lambda^{-\beta k-\beta}}{(\tau\lambda^{\alpha-\beta} - b)^{k+1}} = \frac{\lambda^{-\beta}}{(\tau\lambda^{\alpha-\beta} - b)} \sum_{k=0}^{\infty} \frac{d^k \lambda^{-\beta k}}{(\tau\lambda^{\alpha-\beta} - b)^{k}}.
\end{aligned}
$$

Since $\frac{d\lambda^{-\beta}}{\tau\lambda^{\alpha-\beta}-b} \to 0$ as $\lambda \to \infty$, for λ sufficiently large, we obtain

$$
\widehat{p_{\alpha,\beta}}(\lambda) = \frac{\lambda^{-\beta}}{(\tau\lambda^{\alpha-\beta} - b)} \frac{1}{\left[1 - \frac{d\lambda^{-\beta}}{\tau\lambda^{\alpha-\beta} - b} \right]} = \frac{1}{\tau\lambda^{\alpha} - b\lambda^{\beta} - d}. \tag{3.10}
$$

An application of (18, Theorem 0.4) with $k = 1$ and $g(\lambda) = \dfrac{1}{\tau\lambda^{\alpha} - b\lambda^{\beta} - d}$, together with the uniqueness of the Laplace transform, shows that $p'_{\alpha,\beta}(t)$ exists. We distinguish the following cases:

(I) $0 < \beta \leq 1 < \alpha < 2$. We define $a(t) = a(g_{1-\beta} * p'_{\alpha,\beta})(t) + cp_{\alpha,\beta}(t)$ and $k(t) = \tau(g_{2-\alpha} * p'_{\alpha,\beta})(t)$. From (3.10) and using the initial value Theorem for the Laplace transform, we obtain

$$
p_{\alpha,\beta}(0) = \lim_{|\lambda|\to\infty} \lambda\widehat{p_{\alpha,\beta}}(\lambda) = \lim_{|\lambda|\to\infty} \frac{1}{\lambda^{\alpha-1}\left(\tau - \frac{b}{\lambda^{\alpha-\beta}} - \frac{d}{\lambda^{\alpha}} \right)} = 0.
$$

Therefore, taking the Laplace transform of $a(t)$ we have

$$
\widehat{a}(\lambda) = a\lambda^{\beta}\widehat{p_{\alpha,\beta}}(\lambda) - a\lambda^{\beta-1}p_{\alpha,\beta}(0) + c\widehat{p_{\alpha,\beta}}(\lambda) = \frac{a\lambda^{\beta} + c}{\tau\lambda^{\alpha} - b\lambda^{\beta} - d}.
$$

Analogously, taking the Laplace transform of $k(t)$ we obtain

$$\widehat{k}(\lambda) = \tau\lambda^{\alpha-1}\widehat{p_{\alpha,\beta}}(\lambda) - \lambda^{\alpha-2}p_{\alpha,\beta}(0) = \frac{\tau\lambda^{\alpha-1}}{\tau\lambda^{\alpha} - b\lambda^{\beta} - d}.$$

By (3.9) we conclude that $\{R_{\alpha,\beta}(t)\}_{t\geq 0}$ is an (a,k)-regularized family.

(II) $0 < \beta < \alpha = 1$. We define $a(t) = a(g_{1-\beta} * p'_{1,\beta})(t) + \frac{a}{\tau}g_{1-\beta}(t) + cp_{1,\beta}(t)$ and $k(t) = \tau p_{1,\beta}(t)$. In this case, using the initial value Theorem, we obtain

$$p_{1,\beta}(0) = \lim_{|\lambda|\to\infty} \lambda\widehat{p_{1,\beta}}(\lambda) = \lim_{|\lambda|\to\infty} \frac{1}{\left(\tau - \frac{b}{\lambda^{1-\beta}} - \frac{d}{\lambda}\right)} = \frac{1}{\tau}.$$

Therefore, taking the Laplace transform of $a(t)$ we have

$$\begin{aligned}\widehat{a}(\lambda) &= a\lambda^{\beta}\widehat{p_{1,\beta}}(\lambda) - a\lambda^{\beta-1}p_{1,\beta}(0) + \frac{a}{\tau}\frac{1}{\lambda^{1-\beta}} + c\widehat{p_{1,\beta}}(\lambda)\\ &= a\lambda^{\beta}\widehat{p_{1,\beta}}(\lambda) + c\widehat{p_{1,\beta}}(\lambda) = \frac{a\lambda^{\beta} + c}{\tau\lambda - b\lambda^{\beta} - d}.\end{aligned}$$

Furthermore, taking the Laplace transform of $k(t)$, we have

$$\widehat{k}(\lambda) = \tau\widehat{p_{1,\beta}}(\lambda) = \tau\frac{1}{\tau\lambda - b\lambda^{\beta} - d} = \frac{\tau}{\tau\lambda - b\lambda^{\beta} - d}.$$

It proves the claim and conclude the proof. □

We finish this section with the following lemmas that will be useful in the next section.

Lemma 3.15. *Let $0 < \beta \leq 1 \leq \alpha < 2, a, b, c, d, \tau \in \mathbb{R}$, $\tau \neq 0$, where $(a,c) \neq (0,0)$. Let B be the generator of a solution family $\{R_{\alpha,\beta}(t)\}_{t\geq 0}$. Then for each $\gamma > 0$, the family $\{(g_\gamma * R_{\alpha,\beta})(t)\}_{t\geq 0}$ is exponentially bounded and, for each $x \in X$, $(g_\gamma * R_{\alpha,\beta})(t)x \in \mathcal{D}(B)$.*

Proof. By Definition 3.1, the family $\{R_{\alpha,\beta}(t)\}_{t\geq 0}$ is exponentially bounded, we say of type (M,ω), and then for $t > 0$ and $x \in X$ we have

$$\begin{aligned}\|(g_\gamma * R_{\alpha,\beta})(t)x\| &= \left\|\int_0^t g_\gamma(t-s)R_{\alpha,\beta}(s)x ds\right\| \leq \int_0^t |g_\gamma(t-s)|\, \|R_{\alpha,\beta}(s)\| \|x\| ds\\ &\leq \int_0^t g_\gamma(t-s)Me^{\omega s}ds\|x\| = M\int_0^t g_\gamma(s)e^{\omega(t-s)}ds\|x\|\\ &\leq Me^{\omega t}\int_0^\infty g_\gamma(s)e^{-\omega s}ds\|x\| = \frac{M}{\omega^\gamma}e^{\omega t}\|x\|.\end{aligned}$$

Since by Theorem 3.14, $\{R_{\alpha,\beta}(t)\}_{t\geq 0}$ is an (a,k)-regularized family, then by Lemma 3.4 we have that $(g_\gamma * R_{\alpha,\beta})(t)x \in \mathcal{D}(B)$ for each $x \in X$ and $\gamma > 0$. □

We continue with the following result.

Lemma 3.16. *Let* $0 < \beta \leq 1 \leq \alpha < 2$, $a, b, c, d, \tau \in \mathbb{R}$, $\tau \neq 0$, *where* $(a, c) \neq (0, 0)$, *and let* B *be the generator of a solution family* $\{R_{\alpha,\beta}(t)\}_{t\geqslant 0}$. *Then for all* $x \in X$,

$$(g_{\alpha-1} * R_{\alpha,\beta})(t)x = \frac{(aB + bI)}{\tau}(g_{2\alpha-\beta-1} * R_{\alpha,\beta})(t)x + \frac{(cB + dI)}{\tau}(g_{2\alpha-1} * R_{\alpha,\beta})(t)x + g_\alpha(t)x. \tag{3.11}$$

Proof. By Theorem 3.14, $\{R_{\alpha,\beta}(t)\}_{t\geq 0}$ is an (a, k)-regularized family then by Lemma 3.15, $(g_\gamma * R_{\alpha,\beta})(t)x \in \mathcal{D}(B)$ for each $x \in X$ and $\gamma > 0$. In addition, since $R_{\alpha,\beta}(t)$ and $(g_\gamma * R_{\alpha,\beta})(t)$, $\gamma > 0$ are exponentially bounded by Definition 3.1 and Lemma 3.15, taking the Laplace transform we obtain for any $x \in X$ and λ sufficiently large:

$$\begin{aligned}
&\widehat{R_{\alpha,\beta}}(\lambda)\widehat{g_{\alpha-1}}(\lambda)x - \frac{1}{\tau}(aB + bI)\widehat{g_{2\alpha-\beta-1}}(\lambda)\widehat{R_{\alpha,\beta}}(\lambda)x - \frac{1}{\tau}(cB + dI)\widehat{g_{2\alpha-1}}(\lambda)\widehat{R_{\alpha,\beta}}(\lambda)x \\
&= \frac{1}{\lambda^{\alpha-1}}\widehat{R_{\alpha,\beta}}(\lambda)x - \frac{(aB + bI)}{\tau}\frac{1}{\lambda^{2\alpha-\beta-1}}\widehat{R_{\alpha,\beta}}(\lambda)x - \frac{(cB + dI)}{\tau}\frac{1}{\lambda^{2\alpha-1}}\widehat{R_{\alpha,\beta}}(\lambda)x \\
&= \frac{\tau\lambda^\alpha}{\tau\lambda^\alpha}\frac{1}{\lambda^{\alpha-1}}\widehat{R_{\alpha,\beta}}(\lambda)x - (aB + bI)\frac{\lambda^\beta}{\tau\lambda^\alpha\lambda^{\alpha-1}}\widehat{R_{\alpha,\beta}}(\lambda)x - (cB + dI)\frac{1}{\tau\lambda^\alpha\lambda^{\alpha-1}}\widehat{R_{\alpha,\beta}}(\lambda)x \\
&= \frac{1}{\lambda^\alpha}\frac{1}{\tau\lambda^{\alpha-1}}\left[\tau\lambda^\alpha - \lambda^\beta(aB + bI) - (cB + dI)\right]\widehat{R_{\alpha,\beta}}(\lambda)x \\
&= \frac{1}{\lambda^\alpha}\frac{1}{\tau\lambda^{\alpha-1}}\left[\tau\lambda^\alpha - b\lambda^\beta - d - a\lambda^\beta B - cB\right]\widehat{R_{\alpha,\beta}}(\lambda)x \\
&= \frac{1}{\lambda^\alpha}\frac{1}{\tau\lambda^{\alpha-1}}\Big(\tau\lambda^\alpha - b\lambda^\beta - d - (a\lambda^\beta + c)B\Big)\widehat{R_{\alpha,\beta}}(\lambda)x \\
&= \frac{1}{\lambda^\alpha}\frac{(a\lambda^\beta + c)}{\tau\lambda^{\alpha-1}}\left(\frac{\tau\lambda^\alpha - b\lambda^\beta - d}{a\lambda^\beta + c} - B\right)\widehat{R_{\alpha,\beta}}(\lambda)x = \frac{x}{\lambda^\alpha} = \widehat{g_\alpha}(\lambda)x.
\end{aligned}$$

Hence, by uniqueness of the Laplace transform, we conclude (3.11). □

We have the next lema.

Lemma 3.17. *Let* $0 < \beta \leq 1 \leq \alpha < 2$, $a, b, c, d, \tau \in \mathbb{R}$, $\tau \neq 0$, *where* $(a, c) \neq (0, 0)$, *and let* B *be the generator of a solution family* $\{R_{\alpha,\beta}(t)\}_{t\geqslant 0}$. *Then for all* $x \in \mathcal{D}(B)$,

$$\begin{aligned}
&\tau R_{\alpha,\beta}(t)x - (aB + bI)\big(g_{\alpha-\beta} * R_{\alpha,\beta}\big)(t)x - \tau x - (aB + bI)\big(g_{\alpha-\beta} * R_{\alpha,\beta}\big)(t)x \\
&\quad + \frac{1}{\tau}(aB + bI)\big(g_{2\alpha-2\beta} * R_{\alpha,\beta}\big)(t)(aB + bI)x + g_{\alpha-\beta+1}(t)(aB + bI)x \\
&\quad - (cB + dI)\big(g_\alpha * R_{\alpha,\beta}\big)(t)x + \frac{1}{\tau}(cB + dI)\big(g_{2\alpha-\beta} * R_{\alpha,\beta}\big)(t)(aB + bI)x = 0.
\end{aligned} \tag{3.12}$$

Proof. Note that by Lemma 3.15, for each $x \in X$, $(g_\gamma * R_{\alpha,\beta})(t)x \in \mathcal{D}(B)$, $\gamma > 0$, and since $x \in \mathcal{D}(B)$ we have that $(aB + bI)x \in X$ and $(g_\gamma * R_{\alpha,\beta})(t)(aB + bI)x \in \mathcal{D}(B)$, for each $x \in \mathcal{D}(B)$. Proceeding as in the above lemma, and taking the Laplace transform of (3.12) we obtain

$$
\begin{aligned}
&\tau\widehat{R_{\alpha,\beta}}(\lambda)x-(aB+bI)\widehat{R_{\alpha,\beta}}(\lambda)\widehat{g_{\alpha-\beta}}(\lambda)x-\frac{\tau}{\lambda}x-(aB+bI)\widehat{R_{\alpha,\beta}}(\lambda)\widehat{g_{\alpha-\beta}}(\lambda)x\\
&\quad+\frac{1}{\tau}(aB+bI)\widehat{R_{\alpha,\beta}}(\lambda)\widehat{g_{2\alpha-2\beta}}(\lambda)(aB+bI)x+(aB+bI)\widehat{g_{\alpha-\beta+1}}(\lambda)x\\
&\quad-(cB+dI)\widehat{g_{\alpha}}(\lambda)\widehat{R_{\alpha,\beta}}(\lambda)x+\frac{1}{\tau}(cB+dI)\widehat{g_{2\alpha-\beta}}(\lambda)\widehat{R_{\alpha,\beta}}(\lambda)(aB+bI)x\\
&=\tau\widehat{R_{\alpha,\beta}}(\lambda)x-\frac{(aB+bI)}{\lambda^{\alpha-\beta}}\widehat{R_{\alpha,\beta}}(\lambda)x-\frac{\tau}{\lambda}x-\frac{1}{\lambda^{\alpha-\beta}}\widehat{R_{\alpha,\beta}}(\lambda)(aB+bI)x\\
&\quad+\frac{1}{\tau}(aB+bI)\frac{1}{\lambda^{2\alpha-2\beta}}\widehat{R_{\alpha,\beta}}(\lambda)(aB+bI)x+(aB+bI)\frac{1}{\lambda^{\alpha-\beta+1}}x\\
&\quad-(cB+dI)\frac{1}{\lambda^{\alpha}}\widehat{R_{\alpha,\beta}}(\lambda)x+\frac{1}{\tau}\frac{(cB+dI)}{\lambda^{2\alpha-\beta}}\widehat{R_{\alpha,\beta}}(\lambda)(aB+bI)x\\
&=\left[\tau-\frac{1}{\lambda^{\alpha-\beta}}(aB+bI)-(cB+dI)\frac{1}{\lambda^{\alpha}}\right]\widehat{R_{\alpha,\beta}}(\lambda)x-\frac{\tau}{\lambda}x+\frac{(aB+bI)}{\lambda^{\alpha-\beta+1}}x\\
&\quad+\left[-\frac{1}{\lambda^{\alpha-\beta}}+\frac{1}{\tau}\frac{1}{\lambda^{2\alpha-2\beta}}(aB+bI)+\frac{1}{\tau\lambda^{2\alpha-\beta}}(cB+dI)\right]\widehat{R_{\alpha,\beta}}(\lambda)(aB+bI)x\\
&=\left[\tau-\frac{\lambda^{\beta}}{\lambda^{\alpha}}(aB+bI)-(cB+dI)\frac{1}{\lambda^{\alpha}}\right]\widehat{R_{\alpha,\beta}}(\lambda)x-\frac{\tau}{\lambda}x+\frac{(aB+bI)}{\lambda^{\alpha-\beta+1}}x\\
&\quad+\left[-\frac{1}{\lambda^{\alpha-\beta}}+\frac{1}{\tau}\frac{\lambda^{\beta}}{\lambda^{2\alpha-\beta}}(aB+bI)+\frac{1}{\tau\lambda^{2\alpha-\beta}}(cB+dI)\right]\widehat{R_{\alpha,\beta}}(\lambda)(aB+bI)x\\
&=\frac{1}{\lambda^{\alpha}}\left[\tau\lambda^{\alpha}-\lambda^{\beta}(aB+bI)-(cB+dI)\right]\widehat{R_{\alpha,\beta}}(\lambda)x-\frac{\tau}{\lambda}x+(aB+bI)\frac{1}{\lambda^{\alpha-\beta+1}}x\\
&\quad+\Bigg[-\frac{1}{\lambda^{\alpha-\beta}}\frac{\tau\lambda^{\alpha-1}}{\tau\lambda^{\alpha-1}}+\frac{1}{\tau}\frac{\lambda^{\beta}\lambda^{-1}}{\lambda^{\alpha-1}\lambda^{\alpha-\beta}}(aB+bI)\\
&\qquad+\frac{\lambda^{-1}}{\tau\lambda^{\alpha-1}\lambda^{\alpha-\beta}}(cB+dI)\Bigg]\widehat{R_{\alpha,\beta}}(\lambda)(aB+bI)x\\
&=\frac{1}{\lambda^{\alpha}}\left[\tau\lambda^{\alpha}-b\lambda^{\beta}-d-a\lambda^{\beta}B-cB\right]\widehat{R_{\alpha,\beta}}(\lambda)x-\frac{\tau}{\lambda}x+(aB+bI)\frac{1}{\lambda^{\alpha-\beta+1}}x\\
&\quad-\frac{1}{\lambda^{\alpha-\beta+1}}\frac{1}{\tau\lambda^{\alpha-1}}\left[\tau\lambda^{\alpha}-\lambda^{\beta}(aB+bI)-(cB+dI)\right]\widehat{R_{\alpha,\beta}}(\lambda)(aB+bI)x\\
&=\frac{1}{\lambda^{\alpha}}\left[\tau\lambda^{\alpha}-b\lambda^{\beta}-d-a\lambda^{\beta}B-cB\right]\widehat{R_{\alpha,\beta}}(\lambda)x-\frac{\tau}{\lambda}x+(aB+bI)\frac{1}{\lambda^{\alpha-\beta+1}}x\\
&\quad-\frac{1}{\lambda^{\alpha-\beta+1}}\frac{1}{\tau\lambda^{\alpha-1}}\left[\tau\lambda^{\alpha}-b\lambda^{\beta}-d-a\lambda^{\beta}B-cB\right]\widehat{R_{\alpha,\beta}}(\lambda)(aB+bI)x\\
&=\frac{1}{\lambda^{\alpha}}\left[\tau\lambda^{\alpha}-b\lambda^{\beta}-d-(a\lambda^{\beta}+c)B\right]\widehat{R_{\alpha,\beta}}(\lambda)x-\frac{\tau}{\lambda}x+(aB+bI)\frac{1}{\lambda^{\alpha-\beta+1}}x\\
&\quad-\frac{1}{\lambda^{\alpha-\beta+1}}\frac{1}{\tau\lambda^{\alpha-1}}\left[\tau\lambda^{\alpha}-b\lambda^{\beta}-d-(a\lambda^{\beta}+c)B\right]\widehat{R_{\alpha,\beta}}(\lambda)(aB+bI)x\\
&=\frac{\tau\lambda^{-1}}{\tau\lambda^{-1}}\frac{(a\lambda^{\beta}+c)}{\lambda^{\alpha}}\left[\frac{\tau\lambda^{\alpha}-b\lambda^{\beta}-d}{(a\lambda^{\beta}+c)}-B\right]\widehat{R_{\alpha,\beta}}(\lambda)x-\frac{\tau}{\lambda}x+(aB+bI)\frac{1}{\lambda^{\alpha-\beta+1}}x\\
&\quad-\frac{1}{\lambda^{\alpha-\beta+1}}\frac{(a\lambda^{\beta}+c)}{\tau\lambda^{\alpha-1}}\left[\frac{\tau\lambda^{\alpha}-b\lambda^{\beta}-d}{(a\lambda^{\beta}+c)}-B\right]\widehat{R_{\alpha,\beta}}(\lambda)(aB+bI)x\\
&=\tau\lambda^{-1}x-\frac{\tau}{\lambda}x+(aB+bI)\frac{1}{\lambda^{\alpha-\beta+1}}x-\frac{1}{\lambda^{\alpha-\beta+1}}(aB+bI)x=0.
\end{aligned}
$$

Hence, the claim follows by uniqueness of the Laplace transform. □

We finally prove the following result.

Lemma 3.18. *Let* $0 < \beta \le 1 \le \alpha < 2$, $a, b, c, d, \tau \in \mathbb{R}$, $\tau \neq 0$, *where* $(a, c) \neq (0, 0)$, *and let* B *be the generator of a solution family* $\{R_{\alpha,\beta}(t)\}_{t\geqslant 0}$. *Then for all* $x \in X$ *we have*

$$\tau(g_1 * R_{\alpha,\beta})(t)x - t\tau x - (aB + bI)(g_{\alpha-\beta} * g_1 * R_{\alpha,\beta})(t)x \\ - (cB + dI)(g_\alpha * g_1 * R_{\alpha,\beta})(t)x = 0. \tag{3.13}$$

Proof. Proceeding as in the proof of the previous lemmas, we have

$$\begin{aligned}
&\frac{\tau}{\lambda}\widehat{R_{\alpha,\beta}}(\lambda)x - \frac{\tau}{\lambda^2}x - (aB + bI)\widehat{g_{\alpha-\beta}}(\lambda)\frac{1}{\lambda}\widehat{R_{\alpha,\beta}}(\lambda)x - (cB + dI)\widehat{g_\alpha}(\lambda)\frac{1}{\lambda}\widehat{R_{\alpha,\beta}}(\lambda)x \\
&= \frac{\tau}{\lambda}\widehat{R_{\alpha,\beta}}(\lambda)x - \frac{\tau}{\lambda^2}x - (aB + bI)\frac{1}{\lambda^{\alpha-\beta}}\frac{1}{\lambda}\widehat{R_{\alpha,\beta}}(\lambda)x - (cB + dI)\frac{1}{\lambda^\alpha}\frac{1}{\lambda}\widehat{R_{\alpha,\beta}}(\lambda)x \\
&= \left[\frac{\tau}{\lambda} - \frac{\lambda^\beta}{\lambda^\alpha}\frac{1}{\lambda}(aB + bI) - \frac{1}{\lambda^\alpha}\frac{1}{\lambda}(cB + dI)\right]\widehat{R_{\alpha,\beta}}(\lambda)x - \frac{\tau}{\lambda^2}x \\
&= \frac{1}{\lambda^\alpha}\frac{1}{\lambda}\left[\tau\lambda^\alpha - \lambda^\beta(aB + bI) - (cB + dI)\right]\widehat{R_{\alpha,\beta}}(\lambda)x - \frac{\tau}{\lambda^2}x \\
&= \frac{1}{\lambda^\alpha}\frac{1}{\lambda}\frac{\tau\lambda^{-1}}{\tau\lambda^{-1}}\left[\tau\lambda^\alpha - b\lambda^\beta - d - a\lambda^\beta B - cB\right]\widehat{R_{\alpha,\beta}}(\lambda)x - \frac{\tau}{\lambda^2}x \\
&= \frac{\tau\lambda^{-1}}{\lambda\tau\lambda^{\alpha-1}}\left[\tau\lambda^\alpha - b\lambda^\beta - d - (a\lambda^\beta + c)B\right]\widehat{R_{\alpha,\beta}}(\lambda)x - \frac{\tau}{\lambda^2}x \\
&= \frac{\tau\lambda^{-1}}{\lambda}\frac{(a\lambda^\beta + c)}{\tau\lambda^{\alpha-1}}\left[\frac{\tau\lambda^\alpha - b\lambda^\beta - d}{(a\lambda^\beta + c)} - B\right]\widehat{R_{\alpha,\beta}}(\lambda)x - \frac{\tau}{\lambda^2}x \\
&= \frac{\tau}{\lambda^2}x - \frac{\tau}{\lambda^2}x = 0,
\end{aligned}$$

which guarantees the result. □

3.3 Well-posedness and sufficient conditions

Let $B : \mathcal{D}(B) \subset X \longrightarrow X$ be a closed linear operator on a complex Banach space X. We consider the abstract Cauchy problem

$$\begin{cases} \tau D_t^\alpha u(t) - (aB + bI)D_t^\beta u(t) - (cB + dI)u(t) = 0, \\ u(0) = x, \\ u'(0) = y, \end{cases} \tag{3.14}$$

with $\tau \neq 0$, $a, b, c, d \in \mathbb{R}$ and $0 < \beta \leqslant 1 < \alpha \leqslant 2$, where D_t^γ denotes the Caputo derivative of order $\gamma > 0$.

By a strict solution of (3.14) we understand a function $u \in C^2(\mathbb{R}_+; X) \cap C^1(\mathbb{R}_+; \mathcal{D}(B))$ such that $u(t) \in \mathcal{D}(B)$ for all $t \geq 0$ and (3.14) holds. If a strict solution exists, then it follows that

$x, y \in \mathcal{D}(B)$. In applications, is useful to find a weaker notion of solution where x, y may be arbitrary. This can be done by integrating the equation. Assume that u is a strict solution of (3.14). Since B is closed, it follows from (27, Proposition 1.1.7) that $(g_{\alpha-\beta} * u)(t) \in \mathcal{D}(B)$, $(g_\alpha * u)(t) \in \mathcal{D}(B)$ and

$$\tau u(t) - (aB + bI)(g_{\alpha-\beta} * u)(t) - (cB + dI)(g_\alpha * u)(t) = \tau(x + ty) - g_{\alpha-\beta+1}(t)(aB + bI)x, \tag{3.15}$$

$t > 0$.

Now, we introduce the following definition.

Definition 3.19. *Let $0 < \beta < \alpha$. A function $u \in C(\mathbb{R}_+; X)$ is called a mild solution of (3.14) if $(g_{\alpha-\beta} * u)(t) \in \mathcal{D}(B)$, $(g_\alpha * u)(t) \in \mathcal{D}(B)$ for all $t > 0$ and (3.15) holds.*

It is clear that mild and strict solutions differ merely by regularity.

Definition 3.20. *We say that (3.14) is well-posed if for each $x \in \mathcal{D}(B)$ and each $y \in X$ there exists a unique mild solution.*

We observe that this notion of well-posedness has been considered by other authors, see e.g. references (27, 38, 39) where it is named mildly well-posedness. Our first main result in this section is the following.

Theorem 3.21. *Let $0 < \beta < \alpha$, $(a, c) \neq (0, 0)$. Let B the generator of a solution family $\{R_{\alpha,\beta}(t)\}_{t\geqslant 0}$ on X. Then (3.14) is well-posed.*

Proof. *Uniqueness:* Let $u_1, u_2 \in C(\mathbb{R}_+; X)$ be two mild solutions of (3.14). Then $u := u_1 - u_2 \in C(\mathbb{R}_+; X)$ and $\tau u(t) - (aB + bI)(g_{\alpha-\beta} * u)(t) - (cB + dI)(g_\alpha * u)(t) = 0$, for all $t \geq 0$. Hence

$$(\tau g_\beta - (aB + bI)g_\alpha - (cB + dI)g_{\alpha+\beta}) * u(t) = 0.$$

Therefore, by Titchmarch's Theorem, it follows that $u \equiv 0$.

Existence: Let $x \in \mathcal{D}(B)$ and $y \in X$. We define

$$u(t) = R_{\alpha,\beta}(t)x + \int_0^t R_{\alpha,\beta}(t-s)\Big[y - \frac{1}{\tau}g_{\alpha-\beta}(s)(aB + bI)x\Big]ds, \quad t \geq 0. \tag{3.16}$$

We divide the proof in three steps.

Step 1: When $y = 0$ we define

$$u_1(t) = R_{\alpha,\beta}(t)x - \frac{1}{\tau}(g_{\alpha-\beta} * R_{\alpha,\beta})(t)(aB + bI)x. \tag{3.17}$$

Now, we will to show that (3.17) satisfies the expression (3.15). In fact, it is enough to show that

$$\tau\big(u_1(t)-x\big)-(aB+bI)\big[(g_{\alpha-\beta}*u_1)(t)-g_{\alpha-\beta+1}(t)x\big]-(cB+dI)(g_\alpha*u_1)(t)=0.$$

By Lemma 3.15, for each $x\in X$, $(g_\gamma*R_{\alpha,\beta})(t)x\in\mathcal{D}(B)$, $\gamma>0$, and since $x\in\mathcal{D}(B)$, for $a\neq 0$, we have that $(aB+bI)x\in X$ and $(g_\gamma*R_{\alpha,\beta})(t)(aB+bI)x\in\mathcal{D}(B)$, for each $x\in\mathcal{D}(B)$. Hence $(g_{\alpha-\beta}*u_1)(t)\in\mathcal{D}(B)$ and $(g_\alpha*u_1)(t)\in\mathcal{D}(B)$. Using (3.17) and Lemma 3.17, we obtain that

$$\begin{aligned}
&\tau R_{\alpha,\beta}(t)x-\big(g_{\alpha-\beta}*R_{\alpha,\beta}\big)(t)(aB+bI)x-\tau x-(aB+bI)\big(g_{\alpha-\beta}*R_{\alpha,\beta}\big)(t)x\\
&\quad+\frac{1}{\tau}(aB+bI)\big(g_{2\alpha-2\beta}*R_{\alpha,\beta}\big)(t)(aB+bI)x+(aB+bI)g_{\alpha-\beta+1}(t)x\\
&\quad-(cB+dI)\big(g_\alpha*R_{\alpha,\beta}\big)(t)x+\frac{1}{\tau}(cB+dI)\big(g_{2\alpha-\beta}*R_{\alpha,\beta}\big)(t)(aB+bI)x\\
&=\Big[\tau R_{\alpha,\beta}(t)-\big(g_{\alpha-\beta}*R_{\alpha,\beta}\big)(t)(aB+bI)-\tau-(aB+bI)\big(g_{\alpha-\beta}*R_{\alpha,\beta}\big)(t)\\
&\quad+\frac{1}{\tau}(aB+bI)\big(g_{2\alpha-2\beta}*R_{\alpha,\beta}\big)(t)(aB+bI)+(aB+bI)g_{\alpha-\beta+1}(t)\\
&\quad-(cB+dI)\big(g_\alpha*R_{\alpha,\beta}\big)(t)+\frac{1}{\tau}(cB+dI)\big(g_{2\alpha-\beta}*R_{\alpha,\beta}\big)(t)(aB+bI)\Big]x=0,
\end{aligned}$$

proving that $u_1(t)$ satisfies (3.15).

Step 2: When $x=0$ we define

$$u_2(t)=(g_1*R_{\alpha,\beta})(t)y. \tag{3.18}$$

By Lemma 3.15 it is clear that $(g_{\alpha-\beta}*u_2)(t)\in\mathcal{D}(B)$ and $(g_\alpha*u_2)(t)\in\mathcal{D}(B)$. We will show that

$$\tau\big(u_2(t)-ty\big)-(aB+bI)\big[(g_{\alpha-\beta}*u_2)(t)\big]-(cB+dI)(g_\alpha*u_2)(t)=0.$$

In fact, using (3.18), we obtain that the left hand side of the above identity equals to

$$\tau(g_1*R_{\alpha,\beta})(t)y-t\tau y-(aB+bI)(g_{\alpha-\beta}*g_1*R_{\alpha,\beta})(t)y-(cB+dI)(g_\alpha*g_1*R_{\alpha,\beta})(t)y$$

being this expression zero by Lemma 3.18, as desired.

Step 3: By steps 1 and 2, we obtain that

$$u(t)=u_1(t)+u_2(t)$$

satisfy (3.15). This proves the theorem.

□

Now we consider the non homogeneous abstract Cauchy problem. Let B be a closed linear operator and let $f \in L^1([0,T];X)$, where $T > 0$. For $0 < \beta \leq 1 < \alpha \leq 2$ we consider the problem

$$\begin{cases} \tau D_t^\alpha u(t) - (aB + bI)D_t^\beta u(t) - (cB + dI)u(t) = f(t), \quad t \in [0,T], \\ u(0) = x, \\ u'(0) = y, \end{cases} \tag{3.19}$$

where $x \in \mathcal{D}(B)$ and $y \in X$. A function $u \in C([0,T];X)$ is called a *mild solution* of (3.19) if $(g_{\alpha-\beta} * u)(t) \in \mathcal{D}(B)$, $(g_\alpha * u)(t) \in \mathcal{D}(B)$ and

$$\begin{aligned} \tau u(t) - (aB + bI)(g_{\alpha-\beta} * u)(t) - (cB + dI)(g_\alpha * u)(t) = (g_\alpha * f)(t) + \tau(x + ty) \\ - g_{\alpha-\beta+1}(t)(aB + bI)x, \quad t \in [0,T]. \end{aligned} \tag{3.20}$$

Our main theorem in this section show that in the case when B generates a solution family there always exists a mild solution.

Theorem 3.22. *Let $0 < \beta < \alpha$, $\alpha > 1$, $a, b, c, d, \tau \in \mathbb{R}$, $\tau \neq 0$, where $(a,c) \neq (0,0)$. Let B be the generator of a solution family $\{R_{\alpha,\beta}(t)\}_{t\geqslant 0}$ on X. Then for every $f \in L^1([0,T];X)$ the problem (3.19) has a unique mild solution u given by*

$$\begin{aligned} u(t) = R_{\alpha,\beta}(t)x + \int_0^t R_{\alpha,\beta}(t-s)\Big[y - \frac{1}{\tau}g_{\alpha-\beta}(s)(aB + bI)x\Big]ds \\ + \frac{1}{\tau}\int_0^t R_{\alpha,\beta}(t-s)(g_{\alpha-1} * f)(s)]ds. \end{aligned} \tag{3.21}$$

Proof. Uniqueness is proven as in Theorem 3.21. For existence, we have seen that

$$R_{\alpha,\beta}(t)x + \int_0^t R_{\alpha,\beta}(t-s)\Big[y - \frac{1}{\tau}g_{\alpha-\beta}(s)(aB + bI)x\Big]ds$$

is a mild solution of the homogeneous problem. It remains to show that

$$u_1(t) = \frac{1}{\tau}(g_{\alpha-1} * R_{\alpha,\beta} * f)(t) \tag{3.22}$$

is a mild solution of (3.19) with initial value $x = y = 0$. Extending f by 0 to $\mathbb{R}_+$ we have that $u_1 \in C([0,T];X)$. Note that $(g_{\alpha-\beta} * u_1)(t) \in \mathcal{D}(B)$ and $(g_\alpha * u_1)(t) \in \mathcal{D}(B)$ thanks to Lemma 3.15. Using (3.22) and Lemma 3.16, we obtain that

$$
\begin{aligned}
&\tau u_1(t) - (aB+bI)(g_{\alpha-\beta} * u_1)(t) - (cB+dI)(g_\alpha * u_1)(t) - (g_\alpha * f)(t) \\
&= \tau\big(\frac{1}{\tau}(g_{\alpha-1} * R_{\alpha,\beta} * f)(t)\big) - (aB+bI)\big[(g_{\alpha-\beta} * \frac{1}{\tau} g_{\alpha-1} * R_{\alpha,\beta} * f)(t)\big] \\
&\quad - (cB+dI)(g_\alpha * \frac{1}{\tau} g_{\alpha-1} * R_{\alpha,\beta} * f)(t) - (g_\alpha * f)(t) \\
&= (g_{\alpha-1} * R_{\alpha,\beta} * f)(t) - \frac{(aB+bI)}{\tau}(g_{2\alpha-\beta-1} * R_{\alpha,\beta} * f)(t) \\
&\quad - \frac{(cB+dI)}{\tau}(g_{2\alpha-1} * R_{\alpha,\beta} * f)(t) - (g_\alpha * f)(t) \\
&= \Bigg(\bigg[(g_{\alpha-1} * R_{\alpha,\beta})(t) - \frac{(aB+bI)}{\tau}(g_{2\alpha-\beta-1} * R_{\alpha,\beta})(t) \\
&\quad - \frac{(cB+dI)}{\tau}(g_{2\alpha-1} * R_{\alpha,\beta})(t)\bigg] * f\Bigg)(t) - (g_\alpha * f)(t) \\
&= (g_\alpha * f)(t) - (g_\alpha * f)(t) = 0,
\end{aligned}
$$

proving that $u_1(t)$ satisfies (3.20). This proves the claim. □

3.4 Examples

We finish this work illustrating some special cases where our abstract results apply.

3.4.1 Linear Kuznetsov equation

The Kuznetsov equation (10) models propagation of non-linear acoustic waves in thermo-viscous elastic media. This equation is treated in different works, for example see (40, 41).

We consider the following linear version of de Kuznetsov equation, with fractional order in time, and Dirichlet boundary conditions

$$
\begin{cases}
u_{tt}(t,x) - c^2\Delta u(t,x) - \dfrac{\delta}{p_0}\Delta D_t^\beta u(t,x) = f(t,x), \;\; t > 0, \;\; x \in (0,1), \\
u(t,0) = u(t,1) = 0, \;\; u(0,x) = u_0(x), \;\; u_t(0,x) = u_1(x),
\end{cases}
\tag{3.23}
$$

where $0 < \beta < 1$, $u(x,0) = u_0(x)$, $u_t(x,0) = u_1(x)$ and c, p_0, γ, δ are the velocity of the sound, the density, the ratio of the specific heats and the viscosity of the medium, respectively.

Choose $X = L^p(0,1)$, $1 \le p < \infty$. Consider $B = \Delta$ be the Dirichlet Laplacian on $L^p(0,1)$, with

$$
\mathcal{D}(B) := W_0^{2,p}(0,1) = \{v \in W^{2,p}(0,1);\; v(0) = v(1) = 0\}.
$$

Then, by (42, Lemma 2.3), for every $1 \le p < \infty$, B generates a bounded cosine family on X. Therefore, using Theorem 3.13 and Theorem 3.22 we obtain the following result.

Theorem 3.23. *Let* $0 < \beta < 1$, $\le p < \infty$, $f \in L^1_{loc}(\mathbb{R}_+; L^p(0,1))$ *be such that* $\int_0^\infty e^{-\omega t}\|f(t)\|dt$

$< \infty$ *for some* $\omega > 0$. *Then for all* $u_0 \in W_0^{2,p}(0,1)$ *and* $u_1 \in L^p(0,1)$, *there exist a unique* $u \in C(\mathbb{R}_+; L^p(0,1))$ *satisfying*

$$\tau\big(u(t,x) - u_0(x) - tu_1(x)\big) - \frac{\delta}{p_0\Gamma(2-\beta)}\Delta\int_0^t (t-s)^{1-\beta}u(s,x)ds + \frac{\delta}{p_0\Gamma(3-\beta)}t^{2-\beta}\Delta u_0(x)$$
$$- c^2\Delta\int_0^t (t-s)u(s,x)ds = \int_0^t (t-s)f(s,x)ds,\ t \geq 0,\ x \in (0,1).$$

Proof. Note that equation (3.23) is a particular case of (1.1) when we consider $\tau = 1,\ \alpha = 2,\ 0 < \beta < 1,\ a = \dfrac{\delta}{p_0} > 0,\ c = c^2 > 0,\ B = \Delta,\ b = d = 0$ and $f \in L^1_{loc}(\mathbb{R}_+; L^p(0,1))$. □

Consider the abstract model

$$u''(t) - aBD_t^\beta u(t) - cBu(t) = f(t), \qquad t \geq 0, \tag{3.24}$$

where $0 < \beta < 1$, $a, c > 0$, $u(0) \in \mathcal{D}(B)$ and $u'(0) \in X$. Using Theorem 3.13 and Theorem 3.22 we obtain the following general result.

Theorem 3.24. *Let B be the generator of a cosine family, then, for each $f \in L^1_{loc}(\mathbb{R}_+; X)$ such that $\int_0^\infty e^{-\omega t}\|f(t)\|dt < \infty$, $\omega > 0$, the model (3.24) has a unique mild solution.*

3.4.2 Linear Klein-Gordon equation

Let $\Omega \subset \mathbb{R}^N$ be an open bounded set with sufficiently smooth boundary $\partial\Omega$. We consider the following fractional version of the Klein-Gordon equation

$$\begin{cases} D_t^\alpha \varphi(t,x) - a\Delta D_t^\beta\varphi(t,x) - bD_t^\beta\varphi(t,x) - \Delta\varphi(t,x) - d\varphi(t,x) = f(t,x),\ (t,x) \in \mathbb{R}_+ \times \Omega, \\ \varphi(t,x) = 0,\ (t,x) \in \mathbb{R}_+ \times \partial\Omega, \\ \varphi(0,x) = \varphi_0(x),\ \varphi_t(0,x) = \varphi_1(x),\ x \in \Omega, \end{cases} \tag{3.25}$$

where $a > 0,\ b < 0$ and $d \in \mathbb{R}$, see (3, Section 1) in case $\alpha = 2$ and $\beta = 1$.

Let $X = L^2(\Omega)$. We define $(Bv)(x) = (\Delta v)(x)$, $x \in \Omega$, $v \in \mathcal{D}(B)$ and

$$\mathcal{D}(B) := \{v \in H_0^1(\Omega); Bv \in L^2(\Omega)\}. \tag{3.26}$$

Using Theorem 3.22 we obtain the following result.

Theorem 3.25. *Let $a > 0,\ b < 0,\ 0 < \beta < \alpha,\ \ \alpha > 1$ and $f \in L^1([0,T]; L^2(\Omega))$. Suppose that the Laplacian operator $B := \Delta$ generates a solution family $\{R_{\alpha,\beta}(t)\}_{t\geq 0}$ on $L^2(\Omega)$. Then,* *for all* $\varphi_0 \in \mathcal{D}(B)$ *and* $\varphi_1 \in L^2(\Omega)$*, there exists a unique* $u \in C([0,T];L^2(\Omega))$ *such that*

$$\varphi(t,x) - \varphi_0(x) - t\varphi_1(x) - (a\Delta + bI)\left[\int_0^t g_{\alpha-\beta}(t-s)u(s,x)ds - g_{\alpha-\beta+1}(t)x\right]$$
$$- (\Delta + dI)\int_0^t g_\alpha(t-s)u(s,x)ds = \int_0^t g_\alpha(t-s)f(s,x)ds,\ t \geq 0.$$

For example, in case $\alpha = 2$ and $\beta = 1$ we know by (27, Example 7.2.1, p. 424) that B is the generator of a C_0-semigroup on $L^2(\Omega)$ and therefore, by (2, Corollary 13), a solution family $\{R_{2,1}(t)\}_{t\geq 0}$ on $L^2(\Omega)$ always exists and is given by

$$R_{2,1}(t)v := W'(t)v, \quad v \in \mathcal{D}(B), \tag{3.27}$$

where $\{W(t)\}_{t\geq 0} \subset \mathcal{L}(L^2(\Omega))$ is a strongly differentiable family, see (2, Corollary 13).

4 CONCLUSIONS

As presented in this text, it was possible to present justifications regarding well-posedness and existence of solutions for a class of inhomogeneous strongly damped abstract Cauchy problem of fractional order described by

$$\begin{cases} \tau D_t^\alpha u(t) - (aB + bI)D_t^\beta u(t) - (cB + dI)u(t) = f(t), \\ u(0) = x, \\ u'(0) = y, \end{cases} \tag{4.1}$$

with $\tau \neq 0$, $a, b,\ c,\ d \in \mathbb{R}$ and $0 < \beta \leqslant 1 < \alpha \leqslant 2$, where B is a closed linear operator defined on a complex Banach space X and D_t^γ denotes the Caputo fractional derivative of order $\gamma > 0$.

More specifically, in Chapter 3 were presented the main results: Section 3.1 was presented some sufficient conditions for existence of a solution family and we observed that in the entire case, that is, $(\alpha, \beta) = (2, 1)$, well-posedness follows under the subordination hypothesis that B is the generator of a C_0-semigroup. Then, we show that under certain constraints on the pair (α, β) we can distinguish two situations: $1 < \alpha < 1+\beta \leq 2, 0 < \beta \leq 1$ and $1+\beta < \alpha \leq 2, 0 < \beta \leq 1$. In the first case, if $c = 0$, the leading term in the equation (1.1) is $D_t^\alpha u$ and the solution family is subordinate to B being the generator of a C_0-semigroup. In the second case, if $b = d = 0$, the leading term is $D_t^\beta u$ and the solution family is subordinated to B being a generator of a cosine family. These conclusions provide new insights into how fractional terms influence the class of abstract evolution equations (1.1). The Section 3.2 dealt about the relationship with the best known theory of (a, k)-regularized families and our definition of solution family, proving that solutions families are a particular case of the latter when the parameters satisfy $b \geq 0$ and $\tau > 0$. In the main Section 3.3, we defined our notions of mild and well-posed solution for the homogeneous problem, proving that under the hypothesis of existence of a solution family, well-posedness can be guaranteed. Then, the existence and uniqueness of mild solutions for the inhomogeneous problem were proved.

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