

Flavor mixing in a Lee-type modelC. C. Nishi^{1,2,*} and M. M. Guzzo^{1,+}¹*Instituto de Física “Gleb Wataghin” Universidade Estadual de Campinas, Unicamp 13083-970, Campinas, SP, Brasil*²*Instituto de Física Teórica, UNESP-São Paulo State University Rua Pamplona, 145, 01405-900-São Paulo, Brasil*

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An exactly solvable quantum field theory (QFT) model of Lee type is constructed to study how neutrino flavor eigenstates are created through interactions and how the localization properties of neutrinos follows from the parent particle that decays. The two-particle states formed by the neutrino and the accompanying charged lepton can be calculated exactly as well as their creation probabilities. We can show that the coherent creation of neutrino flavor eigenstates follows from the common negligible contribution of neutrino masses to their creation probabilities. On the other hand, it is shown that it is not possible to associate a well-defined “flavor” to coherent superpositions of charged leptons.

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I. INTRODUCTION

In recent times the neutrino oscillation phenomenon was established as an example where the coherent creation of a superposition of mass eigenstates, which is called a “flavor” eigenstate, tagged by the accompanying charged lepton, leads to an interference among those mass eigenstates inducing the flavor oscillation. We will choose to refer to those specific superpositions of mass eigenstates as neutrino flavor states because there is no well-defined flavor charge operator to which we can associate flavor quantum numbers. Moreover, we will try to show that the concept of flavor for neutrinos is an approximate notion.

In this article a Lee-type model is devised to investigate the neutrino flavor creation problem. The Lee model [1,2] is an example of a simple quantum field theory (QFT) which is exactly soluble but it still requires wave function and charge renormalization to make the model meaningful. Such features allied to strong conservation laws enable us to solve the model completely for each sector invariant by the conservation law. Another unique feature of this model, which will be explored here, is that it permits the study of unstable particles in an exact way [3]. The modifications performed here consist of replacing the fields N , θ , V by l_i , ν_j , π , where the unstable fermion V in the original model is replaced by the scalar boson π and other fields are replaced accordingly. Thus this model will be used to mimic the decay $\pi^\pm \rightarrow l_\alpha^\pm \nu_\alpha$ where $\alpha = e, \mu$.

The decay process $\pi \rightarrow l_i \nu_j$, $i, j = 1, 2$, has four kinematical decay channels available for two neutrino families. The channel $\pi \rightarrow l_1 \nu$ ($l_1 = e$), with arbitrary neutrino content, is, however, very suppressed with respect to the dominant $\pi \rightarrow l_2 \nu$ ($l_2 = \mu$) because of helicity suppression in weak decays. The model devised here does not account for this helicity suppression, giving comparable branching ratios for the production of both charged lep-

tons. However, comparable branching ratios for different charged leptons accompanied by neutrinos exist in nature [4]: (a) $K_L^0 \rightarrow \pi^\pm e^\mp \nu_e$ (40.53%) and $K_L^0 \rightarrow \pi^\pm \mu^\mp \nu_\mu$ (27.02%), (b) $K^+ \rightarrow \pi^0 e^+ \nu_e$ (4.98%) and $K^+ \rightarrow \pi^0 \mu^+ \nu_\mu$ (3.32%), and (c) $W^+ \rightarrow l_\alpha \nu_\alpha$. The last process is also effectively a two body decay kinematically analogous to the pion decay.

In the aforementioned processes we could in principle observe the interference among neutrinos produced jointly with different charged leptons such as $\pi \rightarrow l_1 \nu_1$ and $\pi \rightarrow l_2 \nu_1$. Why is this kind of interference not observed? We will try to address this question within this Lee-type model keeping in mind a fictitious pion decay without helicity suppression, with comparable branching ratios to decay into different charged leptons. Previous discussions about the possibility of charged lepton flavor oscillations exist in the literature [5,6] but we should remark that the question raised previously is more general in the sense that we could observe the interference of the channels by observing the neutrinos and not the charged leptons. On the other hand, there is also the interesting possibility that neutrino oscillations could be suppressed by small momentum or energy uncertainties [7,8].

A thorough study of the mixing phenomenon, either in the bosonic or fermionic case, was already performed in QFT by Blasone and Vitiello (BV) [9]. Despite that, the role played by the interaction which generates the mixing was not completely carried out, although an analysis of flavor states emerging from pion decay is shown in Ref. [10]. In an almost free QFT, BV analyzes the effects of mixing and seeks what should be the flavor states for neutrinos. What emerges from that study is the possibility to define two types of unitarily inequivalent states constructed from two distinct vacua: a usual vacuum and a mixed vacuum. They use the mixed vacuum and obtain a different neutrino oscillation phenomenon. The choice of the vacuum, however, does not seem to be unique. The study of a Lee-type model presents two advantages with respect to the BV approach: (i) the interaction responsible

*ccnishi@ifi.unicamp.br

+guzzo@ifi.unicamp.br

by flavor mixing can be taken into account exactly and (ii) the free states are unitarily equivalent to exact states of the total Hamiltonian. The point (ii) means we do not have to choose between two unitarily inequivalent sets of states but at the same time we cannot study in this type of model the consequences of such phenomenon, which might be present in more realistic QFTs.

Another distinctive approach to treat neutrino oscillations in a rigorous manner was extensively reviewed in Ref. [11], where the so-called external wave packet approach was analyzed. Although, the first QFT treatment of neutrino oscillations was given in Ref. [12]. In the external wave packet approach, the nonobservability of neutrinos through direct detection was translated into the formalism by treating them as internal Feynman propagators connecting the creation and detection processes. Some unclear aspects of such an approach were clarified in Ref. [13]. One of them concerned the possibility that some nonphysical contributions, such as below threshold neutrino detection, could be present in the formalism and would have to be removed by hand [13]. Such an approach could account for the localization aspects of the creation and detection processes [14–16] but the meaning of what would be the intermediate neutrino flavor states could not be investigated. Another attempt to define neutrino flavor states in QFT can be found in Ref. [17]. One can also find attempts to describe neutrino oscillations in relativistic quantum mechanics [13,18].

The two goals of this article consist of providing an exactly solvable model to study (a) how neutrinos are created through interactions, and (b) how the localization of neutrinos follows from the parent pion properties. As is usually adopted in rough estimates, the wave packet size of the daughter particles is considered to be $1/\Gamma$ [5,7], where Γ is the decay width of the parent particle. This model provides exactly such a relation. We can also calculate the two-particle state composed by the charged lepton and the neutrino that is created from pion decay. This calculation is an attempt to properly define a flavor state for neutrinos. We will see that the coherent creation of neutrino flavor states is a consequence of the common negligible kinematical contribution of the different neutrino mass eigenstates to their creation probabilities (amplitudes). We will also study the meaning of defining a different flavor state to charged leptons as the coherent superposition that accompanies a neutrino mass eigenstate such as ν_1 , for instance. We will conclude that the concept of flavor associated with such superposition states cannot be properly defined.

The outline of the paper is as follows. In Sec. II we introduce the model and calculate the relevant eigenstates of the total Hamiltonian. In Sec. III we study how the concept of neutrino flavor emerges from the calculations and derive the localization properties of the decaying states. The discussions are made in Sec. IV and some detailed or additional calculations are shown in the appendices.

II. THE MODEL

The free Hamiltonian of the model is defined by

$$H_0 \equiv \mu \int d^3k A^\dagger(\mathbf{k})A(\mathbf{k}) + M_i \int d^3p b_i^\dagger(\mathbf{p})b_i(\mathbf{p}) + \int d^3p E_j(\mathbf{p})a_j^\dagger(\mathbf{p})a_j(\mathbf{p}), \quad (1)$$

where

$$E_j(\mathbf{p}) = \sqrt{\mathbf{p}^2 + m_j^2}, \quad (2)$$

and A , b_i , and a_j are, respectively, the annihilation operators of the fields π , l_i , and ν_j modelling (unrealistically) the decay $\pi \rightarrow l_i \nu_j$. The summation over repeated indices are implicit. We will restrict ourselves to two families which means $l_i = l_1, l_2$ ($l_1 \equiv e, l_2 \equiv \mu$) and $\nu_j = \nu_1, \nu_2$ correspond to the two neutrino mass eigenstates. Extension to three neutrino families is straightforward. The fixed bare energies of the fields π and l_i corresponding to their masses μ , M_i mean this model treats the static and recoil-less limit of π and l_i . The momentum though is conserved. The neutrinos, on the other hand, obey the relativistic energy dispersion relation (2). The creation and annihilation operators satisfy

$$[A(\mathbf{k}), A^\dagger(\mathbf{k}')] = \delta^3(\mathbf{k} - \mathbf{k}'), \quad (3)$$

$$\{b_i(\mathbf{p}), b_j^\dagger(\mathbf{p}')\} = \delta_{ij} \delta^3(\mathbf{p} - \mathbf{p}'), \quad (4)$$

$$\{a_i(\mathbf{p}), a_j^\dagger(\mathbf{p}')\} = \delta_{ij} \delta^3(\mathbf{p} - \mathbf{p}'). \quad (5)$$

The spin indices are suppressed.

We define the Fourier transform of the fields to be

$$l_i(\mathbf{x}) \equiv \int d^3p b_i(\mathbf{p}, r) u_0^r \frac{e^{i\mathbf{p}\cdot\mathbf{x}}}{(2\pi)^{3/2}}, \quad (6)$$

$$\pi(\mathbf{x}) \equiv \int d^3k A(\mathbf{k}) \frac{e^{i\mathbf{k}\cdot\mathbf{x}}}{(2\pi)^{3/2}}, \quad (7)$$

$$\nu_i(\mathbf{x}) \equiv \int \frac{d^3p}{\sqrt{2E_i}} \left[a_i(\mathbf{p}, r) u_i^r(\mathbf{p}) \frac{e^{i\mathbf{p}\cdot\mathbf{x}}}{(2\pi)^{3/2}} + a_i^\dagger(\mathbf{p}, r) \eta_C \bar{C} u_i^{rT}(\mathbf{p}) \frac{e^{-i\mathbf{p}\cdot\mathbf{x}}}{(2\pi)^{3/2}} \right], \quad (8)$$

where $u_i^r(\mathbf{p})$, $r = 1, 2$, are Dirac spinors normalized as $u_i^{r\dagger}(\mathbf{p})u_i^s(\mathbf{p}) = 2E_i(\mathbf{p})\delta_{rs}$ while u_0^r , $r = 1, 2$, are the corresponding spinors for fermions at rest properly normalized to $u_0^{r\dagger}u_0^s = \delta_{rs}$. Expression (8) corresponds to the usual Majorana type fermion expansion where η_C is a phase factor appearing in the charge conjugation transformation.

The interaction Hamiltonian is chosen to be

$$H_1 = g_0 \int d^3x d^3y [l_i^\dagger(\mathbf{x}) U_{ij} \nu_j^{(-)}(\mathbf{x}) \pi(\mathbf{y}) f(\mathbf{x} - \mathbf{y}) + \text{H.c.}] \quad (9)$$

$$= g_0 U_{ij} \int d^3p d^3p' b_i^\dagger(\mathbf{p}, r) \eta_j^{rs}(\mathbf{p}') a_j^\dagger(\mathbf{p}', s) A(\mathbf{p} + \mathbf{p}') + \text{H.c.}, \quad (10)$$

where $\{U_{ij}\}$ corresponds to the mixing matrix, $\nu_j^{(-)}$ refers to the creation part of expansion (8), and

$$\eta_j^{rs}(\mathbf{p}') = \frac{1}{(2\pi)^{3/2}} \frac{f(\mathbf{p}')}{\sqrt{2E_j(\mathbf{p}')}} \eta_C u_0^{r\dagger} C \bar{u}_j^{sT}(\mathbf{p}'). \quad (11)$$

Using the Dirac representation for gamma matrices, $C = i\gamma_2\gamma_0$, $u_0^r = e_r$ [19], and $u_j^s(\mathbf{p}) = \frac{E_j\gamma_0 - \mathbf{p}\cdot\boldsymbol{\gamma} + m_j}{\sqrt{m_j + E_j}} u_0^s$, we obtain

$$\tilde{\eta}_{rs}(\mathbf{p}) \equiv \eta_C u_0^{r\dagger} C \bar{u}_j^{sT}(\mathbf{p}) = \frac{-i\eta_C(\boldsymbol{\sigma} \cdot \mathbf{p}\sigma_2)_{rs}}{\sqrt{m_j + E_j}}. \quad (12)$$

The function f represents a ‘‘form factor’’ necessary to regularize the expressions in the Lee model [2]. For consistency reasons to be discussed later it also has to be smooth, analytic, approximately flat for energies much lower than the cutoff scale, but falling off not so rapidly above the cutoff scale. In addition, the Lee model may violate unitarity if the coupling constant is above a critical value [2]. We assume the coupling constant is below the critical value thus avoiding such a situation. Nevertheless, the case of an unstable parent particle can be properly described in the unitary regime [3].

There are two charges Q_1 , Q_2 that are conserved and within those sectors of fixed charges we can find the explicit eigenstates of the whole Hamiltonian

$$H = H_0 + H_1. \quad (13)$$

The two charges are

$$Q_1 = \int d^3k A^\dagger(\mathbf{k}) A(\mathbf{k}) + \sum_i \int d^3p b_i^\dagger(\mathbf{p}) b_i(\mathbf{p}), \quad (14)$$

$$Q_2 = \sum_i \int d^3p [b_i^\dagger(\mathbf{p}) b_i(\mathbf{p}) - a_i^\dagger(\mathbf{p}) a_i(\mathbf{p})]. \quad (15)$$

In the sector containing one π [$(Q_1, Q_2) = (1, 0)$] and one pair of l_i, ν_j [$(Q_1, Q_2) = (1, 0)$] we can calculate the exact eigenstates of H in terms of the following eigenstates of H_0 :

$$|\pi\rangle_0 \equiv \int d^3k \psi_\pi(\mathbf{k}) |\pi(\mathbf{k})\rangle_0, \quad (16)$$

$$|l_i(r), \nu_j(\mathbf{p}', s)\rangle_0 \equiv \int d^3p \psi_\pi(\mathbf{p} + \mathbf{p}') |l_i(\mathbf{p}, r), \nu_j(\mathbf{p}', s)\rangle_0, \quad (17)$$

where

$$|\pi(\mathbf{k})\rangle_0 \equiv A^\dagger(\mathbf{k})|0\rangle, \quad (18)$$

$$|l_i(\mathbf{p}, r), \nu_j(\mathbf{p}', s)\rangle_0 \equiv b_i^\dagger(\mathbf{p}, r) a_j^\dagger(\mathbf{p}', s) |0\rangle, \quad (19)$$

and $|0\rangle$ is the vacuum state of H_0 and of H as well [20]. The function ψ_π is an arbitrary function controlling the pion momentum distribution. The subscript 0 indicates the states correspond to eigenstates of H_0 , i.e., bare states. Then we find the eigenvalue equations

$$H_1 |\pi\rangle_0 = g_0 U_{ij} \int d^3p' \eta_j^{rs}(\mathbf{p}') |l_i(r), \nu_j(\mathbf{p}', s)\rangle_0, \quad (20)$$

$$H_1 |l_i(r), \nu_j(\mathbf{p}', s)\rangle_0 = g_0 U_{ij}^* \eta_j^{rs*}(\mathbf{p}') |\pi\rangle_0. \quad (21)$$

The eigenstate of H with energy (mass) E_π corresponding to a dressed state of $|\pi\rangle_0$ can be found if we discover the function χ in

$$|\pi\rangle = |\pi\rangle_0 + \sum_{ij,rs} \int d^3p' \chi_{ij}^{rs}(\mathbf{p}') |l_i(r), \nu_j(\mathbf{p}', s)\rangle_0. \quad (22)$$

From the eigenvalue equation

$$H |\pi\rangle = E_\pi |\pi\rangle, \quad (23)$$

we obtain

$$\chi_{ij}^{rs}(\mathbf{p}') = -\frac{g_0 U_{ij} \eta_j^{rs}(\mathbf{p}')}{M_i + E_j(\mathbf{p}') - E_\pi} \quad (24)$$

and $h_0(E_\pi) = 0$ where

$$h_0(E_\pi) \equiv E_\pi - \mu + \phi_0(E_\pi). \quad (25)$$

The function ϕ_0 is defined as

$$\phi_0(E) \equiv g_0^2 \sum_{ij,rs} \int d^3p' \frac{|\eta_j^{rs}(\mathbf{p}')|^2 |U_{ij}|^2}{M_i + E_j' - E}. \quad (26)$$

We assume there is only one root for Eq. (25) which defines the pion mass M_π , i.e., the energy of the dressed state $|\pi\rangle$. Therefore we could have written the bare pion mass μ as

$$\mu = M_\pi + \delta\mu, \quad (27)$$

where

$$\delta\mu = \phi_0(M_\pi) \quad (28)$$

is the mass counterterm.

We can simplify

$$\phi_0(E) = \gamma_0 \sum_{ij} |U_{ij}|^2 \tilde{\phi}_j(E - M_i), \quad (29)$$

where

$$\gamma_0 \equiv \frac{g_0^2}{2\pi^2} \quad (30)$$

and

$$\tilde{\phi}_j(x) \equiv \int_{m_j}^{\infty} dE \sqrt{E^2 - m_j^2} (E - m_j) \frac{f^2(E)}{E - x}. \quad (31)$$

The intermediate steps to reach Eq. (29) can be found in Appendix A. We should emphasize, however, that the principal value is understood in Eq. (31) if $x > m_j$ [3].

To have an unstable pion we should have [3]

$$h_0(M_1 + m_1) < 0 \quad (32)$$

for $M_1 = \min(M_i)$ and $m_1 = \min(m_j)$. This condition implies that

$$M_\pi > M_1 + m_1. \quad (33)$$

These conclusions follow from $\phi_0(-\infty) = 0 - \epsilon$, $d\phi_0(x)/dx > 0$ for $x < m_1 + M_1$ and $\phi_0(x) > 0$ for $x \rightarrow m_1 + M_1 - \epsilon$, which assures $h_0(x)$ is a monotonically increasing function for $x < M_1 + m_1$. Equation (32) guarantees that the root of $h_0(E) = 0$, the pion mass, is above the threshold $M_1 + m_1$, leading thus to Eq. (33). It can be also seen from Eq. (31) that for $x > M_1 + m_1$, $\phi_0(x)$ may continue increasing but it starts to decrease when x is greater than the cutoff scale of f . In addition we will assume all the decaying channels are open, i.e., $M_\pi > m_2 + M_2$. Intermediate cases can be handled with appropriate care.

If condition (33) is satisfied, Eq. (22) does not define a meaningful expansion although the norm is finite

$$\langle \pi | \pi \rangle = h_0'(M_\pi) = 1 + \phi_0'(M_\pi), \quad (34)$$

with $h_0'(x) = dh_0(x)/dx$. For stable pion, the norm (34) is used to (re)normalize the pion state since the expression in Eq. (34) diverges when there is no cutoff, i.e., $f(E) = 1$. It means there is no stable dressed pion state and the only stable eigenstates of H in this sector (one π or one pair of l_i, ν_j) are the scattering states containing one l_i and one ν_j . These states complete the Hilbert space in this sector. This is proved in Appendix B.

The scattering states, eigenstates of H , can be calculated from the expansion

$$\begin{aligned} |l_i(r), \nu_j(\mathbf{p}, s)\rangle &= |l_i(r), \nu_j(\mathbf{p}, s)\rangle_0 \\ &+ \int d^3 p' \alpha_{ij,i'j'}^{rs,r's'}(\mathbf{p}, \mathbf{p}') |l_{i'}(r'), \nu_{j'}(\mathbf{p}', s')\rangle_0 \\ &+ Z_2^{1/2} \beta_{ij}^{rs}(\mathbf{p}) |\pi\rangle_0, \end{aligned} \quad (35)$$

if we find the functions α and $Z_2^{1/2}\beta$. The eigenvalue equation

$$H|l_i(r), \nu_j(\mathbf{p}, s)\rangle = (M_i + E_j(\mathbf{p})) |l_i(r), \nu_j(\mathbf{p}, s)\rangle, \quad (36)$$

leads to

$$\begin{aligned} (M_i + E_j(\mathbf{p}) - M_{i'} - E_{j'}(\mathbf{p}')) \alpha_{ij,i'j'}^{rs,r's'}(\mathbf{p}, \mathbf{p}') \\ = g_0 Z_2^{1/2} \beta_{ij}^{rs}(\mathbf{p}) \eta_{j'}^{r's'}(\mathbf{p}') U_{i'j'}, \end{aligned} \quad (37)$$

$$\begin{aligned} (M_i + E_j(\mathbf{p}) - M_\pi - \delta\mu) Z_2^{1/2} \beta_{ij}^{rs}(\mathbf{p}) \\ = g_0 \eta_j^{rs*}(\mathbf{p}) U_{ij}^* + g_0 \int d^3 p' \alpha_{ij,i'j'}^{rs,r's'} \eta_{j'}^{r's'*} U_{i'j'}. \end{aligned} \quad (38)$$

Inverting Eq. (37) using the incoming (+) or outgoing (-) wave boundary conditions, we obtain

$$\alpha_{ij,i'j'}^{rs,r's'(\pm)}(\mathbf{p}, \mathbf{p}') = \frac{g_0 Z_2^{1/2} \beta_{ij}^{rs(\pm)}(\mathbf{p}) \eta_{j'}^{r's'}(\mathbf{p}') U_{i'j'}}{M_i - M_{i'} + E_j(\mathbf{p}) - E_{j'}(\mathbf{p}') \pm i\epsilon}. \quad (39)$$

Equation (38) yields

$$Z_2^{1/2} \beta_{ij}^{rs(\pm)}(\mathbf{p}) = \frac{g_0 \eta_j^{rs*}(\mathbf{p}) U_{ij}^*}{h_0(M_i + E_j(\mathbf{p}) \pm i\epsilon)}. \quad (40)$$

Notice that the domain of the function h_0 was extended to the complex numbers, hence satisfying the property

$$h_0(E \pm i\epsilon) = h_0(E) \pm \frac{i}{2} \Gamma_0(E), \quad (41)$$

where

$$\Gamma_0(E) \equiv \gamma_0 \sum_{ij} |U_{ij}|^2 \tilde{\Gamma}_j(E - M_i), \quad (42)$$

$$\tilde{\Gamma}_j(x) \equiv 2\pi\theta(x - m_j)(x - m_j) \sqrt{x^2 - m_j^2} f^2(x). \quad (43)$$

Therefore

$$\begin{aligned} \alpha_{ij,i'j'}^{rs,r's'(\pm)}(\mathbf{p}, \mathbf{p}') &= \frac{g_0^2 U_{ij}^* U_{i'j'}}{h_0(M_i + E_j(\mathbf{p}) \pm i\epsilon)} \\ &\times \frac{\eta_j^{rs*}(\mathbf{p}) \eta_{j'}^{r's'}(\mathbf{p}')}{M_i + E_j(\mathbf{p}) - M_{i'} - E_{j'}(\mathbf{p}') \pm i\epsilon}. \end{aligned} \quad (44)$$

Despite the divergence in Eq. (34), we can still define the approximate pion state

$$\begin{aligned} |\pi_\lambda\rangle &= C[|\pi\rangle_0 - g_0 U_{ij} \int d^3 p' \frac{\eta_j^{rs}(\mathbf{p}')}{M_i + E_j(\mathbf{p}') - M_\pi - i\lambda} \\ &\times |l_i(r), \nu_j(\mathbf{p}', s)\rangle_0], \end{aligned} \quad (45)$$

with M_π being the root of Eq. (25). Such state can be properly normalized if $\lambda \neq 0$. That this state is not an exact eigenstate of H is proved in Appendix D. If we expand the state (45) in terms of the scattering states (35), we can see how this state can be seen as an approximate resonant state corresponding to the pion that decays.

We calculate, for outgoing states, with superscript (-) suppressed,

$$\langle l_i(r), \nu_j(\mathbf{p}, s) | \pi \rangle_0 = Z_2^{1/2} \beta_{ij}^{rs*}(\mathbf{p}), \quad (46)$$

$$\begin{aligned} \langle l_i(r), \nu_j(\mathbf{p}, s) | l_{i'}(r), \nu_{j'}(\mathbf{p}', s) \rangle_0 \\ = \delta^3(\mathbf{p} - \mathbf{p}') \delta_{rr'} \delta_{ss'} \delta_{ii'} \delta_{jj'}, \end{aligned} \quad (47)$$

which yield

$$\begin{aligned} |\pi_\lambda\rangle = C \sum_{ij} \int d^3 p |l_i(r), \nu_j(\mathbf{p}, s)\rangle \\ \times \frac{g_0 U_{ij} \eta_j^{rs}(\mathbf{p})}{h_0(M_i + E_j + i\epsilon)(M_i + E_j - M_\pi - i\lambda)} \\ \times [-i\lambda + \phi_0(M_\pi) - \phi_0(M_\pi + i\lambda)]. \end{aligned} \quad (48)$$

For small λ , i.e., $|\lambda| \ll \Gamma_0(M_\pi)/2\phi'_0(M_\pi)$, we can approximate

$$\begin{aligned} |\pi_\lambda\rangle \approx -i\lambda \left[1 + \frac{\Gamma(M_\pi)}{2|\lambda|} \right]^{1/2} \sum_{ij} \int d^3 p |l_i(r), \nu_j(\mathbf{p}, s)\rangle \\ \times \frac{g U_{ij} \eta_j^{rs}(\mathbf{p})}{h(M_i + E_j + i\epsilon)(M_i + E_j - M_\pi - i\lambda)}. \end{aligned} \quad (49)$$

$$\langle \pi_\lambda | \pi_\lambda(t) \rangle = \lambda^2 \left[1 + \frac{\Gamma(M_\pi)}{2|\lambda|} \right] \int_{m_1+M_1}^{\infty} dE \frac{\Gamma(E)}{2\pi} \frac{e^{-Et}}{[(E - M_\pi + \phi^{(2)}(E))^2 + \frac{\Gamma^2(E)}{4}][(E - M_\pi)^2 + \lambda^2]}, \quad (53)$$

where $|\pi_\lambda(t)\rangle = e^{-iHt} |\pi_\lambda\rangle$, $\phi^{(2)}(E) \equiv \phi(E) - \phi(M_\pi) - \phi'(M_\pi)(E - M_\pi)$ and Eq. (42) were used.

If the functions $\Gamma(E)$, $\phi(E)$ vary more slowly than E near $E = M_\pi$, we can approximate $\Gamma(E) \approx \Gamma(M_\pi)$ and if $M_\pi \gg M_i + m_j$ we can replace the lower limit of the integral in Eq. (53) by $-\infty$:

$$\langle \pi_\lambda | \pi_\lambda(t) \rangle \approx e^{-iM_\pi t} \left[1 + \frac{\Gamma_\pi}{2|\lambda|} \right]^{-1} \left[e^{-\Gamma_\pi t/2} - \frac{\Gamma_\pi}{2|\lambda|} e^{-|\lambda|t} \right] \quad (54)$$

$$\stackrel{|\lambda| \gg \Gamma_\pi}{\approx} e^{-i(M_\pi - i\Gamma_\pi/2)t}, \quad (55)$$

where

$$\Gamma_\pi \equiv \Gamma(M_\pi). \quad (56)$$

Equation (55) expresses the usual exponential decay law with decay rate Γ_π . This result is an indication that the approximate state $|\pi_\lambda\rangle$ in Eq. (45) represents appropriately a resonant pion state that decays into $|l_i(r), \nu_j(\mathbf{p}, s)\rangle$ states independently of the arbitrary parameter λ . If we had calculated Eq. (54) for a state analogous to Eq. (45) but with approximate energy very different from M_π we would obtain a decaying amplitude in Eq. (55) with decay rate primarily given by $|\lambda| \gg \Gamma_\pi$ (see proper discussion in Ref. [3]). It is important to remark that $|\lambda|$ should be also sufficiently small to allow Eq. (41) when ϵ is replaced by

We have defined the renormalized coupling constant

$$g \equiv Z_2^{1/2} g_0, \quad \text{or} \quad \gamma \equiv Z_2 \gamma_0, \quad (50)$$

which defines the ‘‘renormalized’’ functions Γ , ϕ obtained by replacing γ_0 by γ in Γ_0 , ϕ_0 , thus multiplicatively, while

$$h(E) \equiv Z_2 h_0(E). \quad (51)$$

The renormalization constant is

$$Z_2^{-1} = 1 + \phi'_0(M_\pi) = \frac{1}{1 - \phi'(M_\pi)}. \quad (52)$$

Since the scattering states $|l_i(r), \nu_j(\mathbf{p}, s)\rangle$ are eigenstates of the total Hamiltonian H with energy $E = M_i + E_j(\mathbf{p})$, we can easily compute

λ . Combining $|\lambda| \gg \Gamma_\pi$ and $|\lambda| \ll \Gamma_\pi/2\phi'(M_\pi)$, it is necessary that $\phi'(M_\pi) \ll 1$.

The emission spectrum of l_i, ν_j is given by

$$|\langle l_i(r), \nu_j(\mathbf{p}, s) | \pi_\lambda(t) \rangle|^2 = |C_{ij}^{rs}(\mathbf{p}, \lambda)|^2, \quad (57)$$

where

$$\begin{aligned} C_{ij}^{rs}(\mathbf{p}, \lambda) = -i\lambda \left[1 + \frac{\Gamma_\pi}{2|\lambda|} \right]^{1/2} \\ \times \frac{g U_{ij} \eta_j^{rs}(\mathbf{p})}{h(M_i + E_j + i\epsilon)(M_i + E_j - M_\pi - i\lambda)} \end{aligned} \quad (58)$$

is the expansion coefficient in Eq. (49). We can see

$$\sum_{ij} \int d^3 p |C_{ij}^{rs}(\mathbf{p}, \lambda)|^2 = \sum_{ij} \int_{m_1+M_1}^{\infty} dE W_{ij}(E - M_i, \lambda) = 1. \quad (59)$$

The function $W_{ij}(E, \lambda)$ is the emission probability for $\pi \rightarrow l_i \nu_j$, per unit energy, with neutrino energy E , summed over spins:

$$W_{ij}(E - M_i, \lambda) = \gamma \tilde{\Gamma}_j(E - M_i) |U_{ij}|^2 \frac{W_\lambda(E)}{\Gamma(E)}. \quad (60)$$

The total emission probability is

$$W_\lambda(E) = \frac{1}{2\pi} \frac{\Gamma(E)}{h^2(E) + \Gamma^2(E)/4} \frac{\lambda^2[1 + \Gamma_\pi/2|\lambda|]}{(E - M_\pi)^2 + \lambda^2} \quad (61)$$

which approximates the usual Breit-Wigner distribution in the $\lambda \gg \Gamma_\pi$ limit, within the approximations $h(E) \approx E - M_\pi$ and $\Gamma(E) \approx \Gamma_\pi$,

$$W(E) \equiv \lim_{\lambda \rightarrow \infty} W_\lambda(E) = \frac{1}{2\pi} \frac{\Gamma_\pi}{(E - M_\pi)^2 + \Gamma_\pi^2/4}. \quad (62)$$

From probability conservation we have

$$\sum_{ij} W_{ij}(E - M_i, \lambda) = W_\lambda(E). \quad (63)$$

The energy E in $W_\lambda(E)$ refers to the total energy of the pair $l + \nu$.

We can see the pion state is a resonant state by calculating the scattering cross section of $l_i \nu_j \rightarrow l \nu$, irrespective of the final product flavors and spins and averaged over initial spins:

$$\frac{d\sigma}{d\Omega}(l_i \nu_j \rightarrow l \nu; E) = \frac{|U_{ij}|^2}{(E - M_i)^2 - m_j^2} \times \gamma \tilde{\Gamma}_j(E - M_i) \frac{\pi}{4} W(E), \quad (64)$$

where $E = E_j + M_i$ corresponds to the total energy of the incident particles. The kinematics is totally determined because the energy of l_i and π are fixed. The difference between the initial and final momenta (velocities) of neutrinos has to be taken into account to obtain the appropriate cross section obeying

$$\sum_{ij} \frac{d\sigma}{d\Omega}(ij \rightarrow l \nu; E) = \frac{\pi}{4\mathbf{p}^2} \Gamma(E) W(E) |_{\mathbf{p}^2 = (E - M_i)^2 - m_j^2}. \quad (65)$$

The presence of $W(E)$ guarantees the resonance for $E = M_\pi$.

III. FLAVOR (EIGEN)STATES

In the previous section we have seen, from the QFT point of view, it makes more sense to calculate transition probabilities (amplitudes) with respect to the state $|l_i(r), \nu_j(\mathbf{p}, s)\rangle$ that has definite energy $E_j(\mathbf{p}) + M_i$ and thus are mass eigenstates. What we see in the actual pion decay, however, is not $\pi \rightarrow l_i \nu_j$, $i, j = 1, 2$, but $\pi \rightarrow l_\alpha \nu_\alpha$, with a definite flavor $\alpha = e, \mu$. For the charged leptons the terminology flavor is clear and coincides with the mass eigenstates $l_i = l_\alpha$. For neutrinos what characterizes a flavor state depends on the accompanying charged lepton.

Let us define flavor states as

$$|l_\alpha(r), \nu_\beta(\mathbf{p}, s)\rangle \equiv \delta_{\alpha i} U_{\beta j} |l_i(r), \nu_j(\mathbf{p}, s)\rangle, \quad (66)$$

$$\alpha, \beta = e, \mu.$$

Let us then calculate

$$|\langle l_\alpha, \nu_\beta(\mathbf{p}) | \pi_\lambda(t) \rangle|^2 = \sum_{rs} \left| \sum_j U_{\beta j}^* C_{\alpha j}^{rs}(\mathbf{p}) e^{-i(M_\alpha + E_j(\mathbf{p}))t} \right|^2 \quad (67)$$

$$= g^2 \sum_{rs} \left| \sum_j U_{\alpha j} e^{-iE_j t} (U^\dagger)_{j\beta} \frac{\eta_j^{rs}(\mathbf{p})}{h(E_j + M_\alpha + i\epsilon)} \right|^2, \quad (68)$$

already in the $\lambda \gg \Gamma_\pi$ limit. If we take the $m_j \rightarrow 0$ limit ($E_j \rightarrow E_\nu = |\mathbf{p}|$) in the terms except in the exponent we arrive at

$$|\langle l_\alpha, \nu_\beta(\mathbf{p}) | \pi_\lambda(t) \rangle|^2 = \mathcal{P}_{\alpha\beta}(t) \frac{W_\alpha(E_\nu)}{4\pi |\mathbf{p}| E_\nu}, \quad (69)$$

where

$$\mathcal{P}_{\alpha\beta}(t) = \left| \sum_j U_{\alpha j} e^{-iE_j t} U_{\beta j}^* \right|^2, \quad (70)$$

and $W_\alpha(E)$ is $W_{\alpha j}(E)$ in the $m_j \rightarrow 0$ limit and with $|U_{ij}|^2$ factored out. We see the probability of finding the flavor state (66) from pion decay decouples into the usual oscillation probability times the emission probability $\pi \rightarrow l_\alpha \nu$ irrespective of the neutrino flavor.

The concept of neutrino flavor states emerges from Eq. (69) taking $t = 0$ (or $t \ll$ oscillation period), for which we obtain

$$\mathcal{P}_{\alpha\beta}(t \approx 0) = \delta_{\alpha\beta}. \quad (71)$$

Therefore only a neutrino of flavor α is produced jointly with the charged lepton of flavor α . If we try to calculate the creation probability for the equally possible superposition state, corresponding to a neutrino mass eigenstate accompanied by a superposition state of charged leptons,

$$\left| \sum_i U_{i\beta}^* \langle l_i, \nu_j(\mathbf{p}) | \pi_\lambda(t) \rangle \right|^2, \quad (72)$$

we see it does not decouple and since the masses M_i have different kinematical contributions, there is no way to define a superposition state of charged leptons associated with a neutrino mass eigenstate ν_1 for instance. Stated differently, the transition probability (72) is not equal to $\delta_{\beta j}$ for $t = 0$, not even approximately.

The definition of the neutrino flavor state in Eq. (66) was justified only *a posteriori* as the state that accompanies a definite charged lepton. In this model, however, we could, in principle, calculate the more interesting quantity

$$\langle l_i(r), \nu_j(\mathbf{x}, s) | \pi_\lambda(t) \rangle = \Psi_{ij}^{rs}(\mathbf{x}, t) e^{-iM_i t}, \quad (73)$$

where

$$|l_i(r), \nu_j(\mathbf{x}, s)\rangle \equiv \int d^3 p \frac{e^{-i\mathbf{p}\cdot\mathbf{x}}}{(2\pi)^{3/2}} |l_i(r), \nu_j(\mathbf{p}, s)\rangle. \quad (74)$$

This “wave function” squared ($|\Psi_{ij}^{rs}(\mathbf{x}, t)|^2$) would give us

the probability density of finding the neutrino ν_j in the position \mathbf{x} at time t jointly with a charged lepton l_i . The exact information of $\Psi_{ij}^{rs}(\mathbf{x}, t)$ allow us to expand

$$|\pi_\lambda(t)\rangle = \sum_{ij,rs} \int d^3x \Psi_{ij}^{rs}(\mathbf{x}, t) e^{-iM_i t} |l_i(r), \nu_j(\mathbf{x}, s)\rangle. \quad (75)$$

Equation (75) characterizes completely how the neutrinos and charged leptons arise from pion decay. Although the spatial information of the charged lepton is not determined or calculable in this model.

The exact Fourier transform necessary to compute Eq. (73) is very difficult to be performed analytically but a rough approximate calculation can be performed. We find, in the limit $|\lambda| \gg \Gamma_\pi$,

$$\begin{aligned} \Psi_{ij}^{rs}(\mathbf{x}, t) &= \int d^3p e^{-iE_j(\mathbf{p})t} C_{ij}^{rs}(\mathbf{p}) \frac{e^{i\mathbf{p}\cdot\mathbf{x}}}{(2\pi)^{3/2}} \quad (76) \\ &\approx -\frac{e^{-i\bar{E}_j t}}{\sqrt{4\pi r}} \frac{\gamma^{1/2}}{\sqrt{\bar{v}_j}} U_{ij} \tilde{\Gamma}_j^{1/2}(\bar{E}_j) \frac{(\boldsymbol{\sigma} \cdot \hat{\mathbf{x}} \sigma_2)_{rs}}{\sqrt{2}} \\ &\quad \times [\theta(\tau_-) e^{i\bar{p}_j r} e^{-(\Gamma_\pi/2)\tau_-} + \theta(\tau_+) e^{-i\bar{p}_j r} e^{-(\Gamma_\pi/2)\tau_+}], \end{aligned} \quad (77)$$

where

$$\tau_\pm \equiv t \pm r/\bar{v}_j, \quad (78)$$

$r = |\mathbf{x}|$ and the bar in \bar{v}_j , \bar{p}_j , and \bar{E}_j denote, respectively, the velocity, momentum, and energy of ν_j when $\bar{E}_j = M_\pi - M_i$, which corresponds to the kinematics of $\pi \rightarrow l_i \nu_j$ with π, l_i at rest. The detailed calculation of Eq. (77) is carried out in Appendix C.

We can verify the normalization of Eq. (77):

$$\begin{aligned} \sum_{rs} \int d^3x |\Psi_{ij}^{rs}(\mathbf{x}, t)|^2 &= \frac{\gamma}{\Gamma_\pi} |U_{ij}|^2 \tilde{\Gamma}_j(\bar{E}_j) \\ &\quad \times \left[1 + 4 \frac{\Gamma_\pi}{\bar{p}_j} e^{-\Gamma_\pi t} \sin(\bar{p}_j \bar{v}_j t) \right]. \end{aligned} \quad (79)$$

Equation (79) guarantees that

$$\sum_{ij,rs} \int d^3x |\Psi_{ij}^{rs}(\mathbf{x}, t)|^2 = 1, \quad (80)$$

for $t \gg 1/\Gamma_\pi$ or $\frac{\Gamma_\pi}{\bar{p}_j} \ll 1$. Therefore, the total probability is conserved by the approximation. One cannot, however, recover the flavor oscillation phenomenon from the approximate wave function (77) because the phase is common to all states $l_i \nu_j$ with fixed i . Numerical studies can be performed to test the accuracy of the approximation, although appropriate care should be taken with the enormous difference between the scales of m_j and M_π, M_i .

To visualize the localization aspects of the neutrinos created, we can calculate the radial probability density

for the creation of neutrinos ν_j at radius r and time t jointly with l_i ,

$$\rho_{ij}(r, t) = \sum_{rs} \int d\Omega_{\mathbf{x}} r^2 |\Psi_{ij}^{rs}(\mathbf{x}, t)| \quad (81)$$

$$\begin{aligned} &= \gamma |U_{ij}|^2 \tilde{\Gamma}_j(M_\pi - M_i) \frac{1}{\bar{v}_j} [\theta(\tau_-) e^{-\Gamma_\pi \tau_-} + \theta(\tau_+) e^{-\Gamma_\pi \tau_+} \\ &\quad + 2\theta(\tau_+) \theta(\tau_-) \cos(\bar{p}_j r) e^{-\Gamma_\pi t}]. \end{aligned} \quad (82)$$

For $t \gg 1/\Gamma_\pi$, we see only the first term containing τ_- contributes. Thus $\rho_{ij}(r, t)$ has a triangular shape with an abrupt peak in $r = \bar{v}_j t$ and an exponential tail for $r < \bar{v}_j t$, being negligible for $r > \bar{v}_j t$. The neutrino is then roughly localized in the region $\bar{v}_j(t - 1/\Gamma_\pi) \lesssim r \leq \bar{v}_j t$. The size of the wave packet is roughly \bar{v}_j/Γ_π as is usually assumed in rough estimates [5,7].

The most faithful state that we can construct to describe the neutrino flavor states created from pion decay jointly with a charged lepton l_i with momentum \mathbf{q} and spin r is

$$\begin{aligned} e^{iM_i t} \langle l_i(\mathbf{q}, r) | \pi(t) \rangle &= \sum_{j,s} \int d^3p \psi_\pi(\mathbf{q} + \mathbf{p}) e^{-iE_j(\mathbf{p})t} | \nu_j(\mathbf{p}, s) \rangle_0 \\ &\quad \times \frac{g U_{ij} \eta_j^{rs}(\mathbf{p})}{h(M_i + E_j(\mathbf{p}) + i\epsilon)} \\ &\quad \times [1 - e^{-i(M_\pi - M_i - E_j)t} e^{-\Gamma_\pi t/2}], \end{aligned} \quad (83)$$

where ψ_π is the function appearing in the definitions (16) and (17), corresponding to the pion momentum wave function. In the limit $t \gg \Gamma_\pi$, we can approximate $m_j \rightarrow 0$ in all factors except in the exponent since $m_j \ll |\mathbf{p}|, M_i, M_\pi$. Equation (83) becomes

$$\begin{aligned} e^{iM_i t} \langle l_i(\mathbf{q}, r) | \pi(t) \rangle &\approx \sum_{j,s} \int d^3p \psi_\pi(\mathbf{q} + \mathbf{p}) \\ &\quad \times e^{-iE_j(\mathbf{p})t} U_{ij} | \nu_j(\mathbf{p}, s) \rangle_0 \\ &\quad \times \frac{g \eta_j^{rs}(\mathbf{p})}{h(M_i + |\mathbf{p}| + i\epsilon)}. \end{aligned} \quad (84)$$

We could then write, for $t = 0$,

$$\begin{aligned} \langle l_\alpha(\mathbf{q}, r) | \pi(t) \rangle &\approx \sum_s \int d^3p \psi_\pi(\mathbf{q} + \mathbf{p}) | \nu_\alpha(\mathbf{p}, s) \rangle_0 \\ &\quad \times \frac{g \eta^{rs}(\mathbf{p})}{h(M_\alpha + |\mathbf{p}| + i\epsilon)}, \end{aligned} \quad (85)$$

where $l_i = l_\alpha, M_i = M_\alpha$, and

$$| \nu_\alpha(\mathbf{p}, s) \rangle_0 \equiv \sum_j U_{\alpha j} | \nu_j(\mathbf{p}, s) \rangle_0. \quad (86)$$

If we impose $\psi_\pi(\mathbf{p}) = \delta^3(\mathbf{p})$ we obtain exactly $\mathbf{q} = -\mathbf{p}$ for the momenta of the charged leptons (\mathbf{q}) and neutrinos (\mathbf{p}). In general, the presence of the function ψ_π ensures momentum conservation and forces the amplitude (83) to be appreciable only around $\mathbf{q} = -\mathbf{p}$ where \mathbf{p} has magni-

tude satisfying $E_j(\mathbf{p}) \approx M_\pi - M_i$. Equation (83) also clarifies the contribution of the intrinsic momentum uncertainty Δp (intrinsic of ψ_π) of the parent particle that is inherited by the daughter particles, apart from the contribution of the decay width Γ_π : the smallest between Γ_π and Δp dominates. Other transition amplitudes of $|\pi(t)\rangle$ with respect to the free states $|\pi\rangle_0$ and $|l_i(r), \nu_j(\mathbf{p}, s)\rangle_0$ can be found in Appendix E. One can identify some similarity with the Wigner-Weisskopf approximation used in systems with couplings between discrete and continuum energy levels.

IV. DISCUSSIONS

One knows neutrino flavor oscillation ceases when quantum coherence is lost due to the lack of spatial overlap among the mass eigenstates that compounds the flavor state neutrinos ν_α [21,22]. The characteristic time (distance) scale for such a phenomenon to occur is usually very large for neutrinos and is given by $\delta x/\Delta v_{ij}$, where δx is the characteristic spatial size of the neutrino wave packets and Δv_{ij} is the velocity difference of the ν_i and ν_j that compound the flavor state. In this model, in the pion rest frame,

$$\delta x_j \approx \bar{v}_j/\Gamma_\pi. \quad (87)$$

Neutrino flavor oscillation occurs because the neutrinos produced in channels $\pi \rightarrow l_2 \nu_1$ and $\pi \rightarrow l_2 \nu_2$ interfere coherently and remain interfering in the same region of space as long as $t \ll \delta x_j/\Delta v_{ij}$. But what prevents the channels $\pi \rightarrow l_1 \nu_1$ and $\pi \rightarrow l_2 \nu_1$ from interfering? For real pion decays such interference is not observable even in principle since these two channels have very different probabilities to occur because of helicity suppression in weak decays. In this model, however, the branching ratios are comparable and may mimic other real decays.

One should notice that the neutrinos created in channels $\pi \rightarrow l_1 \nu_1$ and $\pi \rightarrow l_2 \nu_1$, for example, may have significant spatial overlap. We see that $\Delta_M v(m_j)$ defined as

$$\Delta_M v(m_1) \equiv \bar{v}_1(M_1) - \bar{v}_1(M_2), \quad (88)$$

may be of order of

$$\Delta_m v(M_i) \equiv \bar{v}_1(M_i) - \bar{v}_2(M_i), \quad (89)$$

unless the neutrino masses are nearly degenerate $\Delta m_{ij}^2 \ll m_i^2, m_j^2$. We use the notation $\bar{v}_j(M_i)$ instead of \bar{v}_j throughout this discussion to avoid ambiguities. It means that the noninterference of these channels is probably due to the charged leptons, but its detailed account deserves further study. Simple calculations show [5] that the big mass difference between charged leptons make them lose spatial coherence over a distance of atomic length or less for most decaying processes. Is this loss of spatial coherence responsible for the incoherence of the channels with definite flavor $l_1 = e$ or $l_2 = \mu$? Another possibility is that the detection of charged leptons is what makes the coherent

superposition of the four channels $\pi \rightarrow l_i \nu_j$, $i, j = 1, 2$, reduce into two incoherent $\pi \rightarrow l_1 \nu$ and $\pi \rightarrow l_2 \nu$ quantum processes. These two explanations are, however, very distinct since the former is detection independent while the latter is detection dependent.

Concerning the absence of the interference between the channels $\pi \rightarrow l_1 \nu$ and $\pi \rightarrow l_2 \nu$, there is one ingredient that could play an important role, i.e., the spatial entanglement. This question, however, cannot be discussed within this model due to the lack of spatial entanglement between l_i , in the static limit, and ν_j .

Another known situation when neutrino oscillations can be suppressed is the case when it is possible to probe the neutrino kinematical quantities with enough precision to single out one neutrino mass eigenstate that is being created in the process [7]. In the language of this article it means all the four pion decay channels proceed incoherently. Such suppression, of course, can be induced by localization properties of the source or detector that produces and detects the neutrinos [14]. Conditions necessary for neutrino oscillation can be analyzed, for instance, within a realistic QFT description [16]. In that case, the condition for neutrino oscillation requires [Eq. (4.7) of Ref. [16]]

$$\frac{\Delta m^2}{2E_\nu} \lesssim \Gamma, \quad (90)$$

if the intrinsic momentum spread of the decaying particle, apart from the decay width Γ , is small compared to its mass. Condition (90) is exactly the condition necessary to prevent one from knowing which neutrino mass eigenstate is being created [7], when the momentum uncertainty is of the order of Γ , which is also the case of this work. Notice, however, that in this work we do not treat the propagation of neutrinos but rather focus the attention on the creation process. Surprisingly, condition (90), with \ll instead of \lesssim , is exactly what is needed to guarantee, initially, approximate neutrino flavor definition. Such analysis involves estimating the first flavor violating term of Eq. (68) that follows after Eq. (69).

Therefore, the study of this Lee-type model sheds light on the question of what is a superposition state of definite flavor. We could confirm, within this model, the result that once the channels with charged leptons with definite masses are singled out, coherent creation of neutrino flavor states is allowed because the kinematical contributions from each neutrino mass eigenstate to their creation probabilities are negligible. Therefore the creation probability amplitudes of neutrinos are mostly independent of neutrino masses (mass differences to be precise), enabling the decoupling of Eq. (69) into the creation probability times the flavor conversion probability. The conversion probability, however, depends crucially on the tiny neutrino mass differences. The neutrino flavor states ν_α can be created jointly with the charged lepton $l_i = l_\alpha$ and the wrong flavor

is initially absent. On the other hand, if we try to calculate, as in Eq. (72), the creation probability of the coherent superposition of charged leptons jointly with the neutrino ν_1 , for instance, there is no meaning to attribute the name flavor for that superposition of charged leptons because such flavor is not created in an unambiguous manner but both the correct and the wrong flavors would be created initially.

To be extremely precise, the notion of neutrino flavor states rely on the approximation of negligible neutrino masses, as could be seen in Eq. (69). For that reason, there is a negligible probability to create the wrong flavor ν_e , for example, jointly with the charged lepton $l_2 = \mu$. Such probability is calculable but negligible because of the proportionality to the neutrino masses. Such flavor indefiniteness due to the neutrino mass differences can be also seen in other attempts to exactly define what is a neutrino flavor state such as in Refs. [9] or [17]. The exact definition is always blurred by the differences in the kinematical or dynamical factors that depend, strictly speaking, on the neutrino masses, that in turn, cannot be factored out as a common factor. For example, the spinorial character of neutrinos prevent even a description as simple as a mixed free second quantized theory to provide exact initial neutrino flavor definition [13]. Another simple treatment that leads to initial flavor violation is the adoption of general kinematical conditions when scalar wave packets are used [23].

To summarize, we could confirm two aspects, within a Lee-type model: (a) the spatial size of the neutrinos produced through a decaying particle with decay rate Γ is ν/Γ for the parent particle at rest. This relation is commonly used in estimates of the neutrino wave packet sizes but this model provides an explicit example satisfying such relation. (b) The coherent creation of neutrino flavor states is possible because the kinematical contribution of neutrinos to their creation probabilities are negligible, allowing one to define the flavor state as a superposition of mass eigenstates in the expected way. We should remark that the second conclusion is, in principle, applicable to more realistic decays because the main ingredient is universal: neutrino mass differences, compared to the charged lepton mass differences, contribute negligibly to their creation probabilities.

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APPENDIX A: THE FUNCTION ϕ_0

The function ϕ_0 appearing in Eq. (26) can be written in the following simple form:

$$\begin{aligned}\phi_0(E_\pi) &= 2 \sum_{ij} \int d^3p \frac{\tilde{f}_j^2(\mathbf{p}) |U_{ij}|^2}{M_i + E_j(\mathbf{p}) - E_\pi} \\ &= \frac{g_0^2}{2\pi^2} \sum_{ij} \int_0^\infty dp \frac{p^4 f^2(p)}{E_j(E_j + m_j)} \frac{|U_{ij}|^2}{M_i + E_j - E_\pi}\end{aligned}\quad (\text{A1})$$

$$\begin{aligned}&= \gamma_0 \sum_{ij} \int_{m_j}^\infty dE \theta(E - m_j) \sqrt{E^2 - m_j^2} (E - m_j) f^2(E) \\ &\quad \times \frac{|U_{ij}|^2}{M_i + E - E_\pi}\end{aligned}\quad (\text{A2})$$

$$= \gamma_0 \sum_{ij} |U_{ij}|^2 \tilde{\phi}_j(E_\pi - M_i). \quad (\text{A3})$$

In Eq. (A1) we have assumed the cutoff function is isotropic $f(\mathbf{p}) = f(|\mathbf{p}|)$, $|\mathbf{p}| = p$. In addition, it is assumed in Eq. (A2) that f is a function of E only, i.e., $f(\mathbf{p}) \rightarrow f(E_j(\mathbf{p}))$. We used Eqs. (30) and (31) and

$$\eta_j^{rs}(\mathbf{p}) = \frac{p}{(2\pi)^{3/2}} \frac{f(|\mathbf{p}|)}{\sqrt{2E_j}\sqrt{E_j + m_j}} (-i\eta_C)(\hat{\mathbf{p}} \cdot \boldsymbol{\sigma}\sigma_2)_{rs}, \quad (\text{A4})$$

$$\sum_{rs} |\eta_j^{rs}(\mathbf{p})|^2 = 2(\tilde{f}_j(\mathbf{p}))^2, \quad (\text{A5})$$

where

$$\tilde{f}_j(\mathbf{p}) = \frac{1}{(2\pi)^{3/2}} \frac{f(\mathbf{p})}{\sqrt{2E_j}}. \quad (\text{A6})$$

If we extend the variables to small complex values, we obtain

$$\tilde{\phi}_j(x \pm i\epsilon) = \tilde{\phi}_j(x) \pm \frac{i}{2} \tilde{\Gamma}_j(x), \quad (\text{A7})$$

where

$$\tilde{\Gamma}_j(x) \equiv 2\pi\theta(x - m_j)(x - m_j)\sqrt{x^2 - m_j^2}f^2(x). \quad (\text{A8})$$

We see $\tilde{\Gamma}_j(x) \geq 0$ for all x while $\tilde{\Gamma}_j'(x) \geq 0$ for $x > m_j$ and x much lower than the cutoff scale of f , assuming f is a smooth function on that region.

APPENDIX B: COMPLETENESS OF THE STATES

$$|l_i(\mathbf{r}), \nu_j(\mathbf{p}, s)\rangle^{(\pm)}$$

We prove here the states $|l_i(\mathbf{r}), \nu_j(\mathbf{p}, s)\rangle^{(\pm)}$ in Eq. (35) complete the Hilbert space in the sector of one π or $l_i\nu_j$. We calculate

$$\begin{aligned}
& \sum_{ij} \int d^3 p |l_i, \nu_j(\mathbf{p})\rangle \langle l_i, \nu_j(\mathbf{p})| \\
&= C |\pi\rangle_{00} \langle \pi| + \sum_{ij} \int d^3 p |l_i, \nu_j(\mathbf{p})\rangle_{00} \langle l_i, \nu_j(\mathbf{p})| \\
&+ \sum_{ij, i'j'} \int d^3 q d^3 q' |l_{i'}, \nu_{j'}(\mathbf{q})\rangle_0 A_{i'j', ij}(\mathbf{q}, \mathbf{q}')_0 \langle l_i, \nu_j(\mathbf{q}')| \\
&+ \sum_{ij} \int d^3 q B_{ij}(\mathbf{q}) |l_i, \nu_j(\mathbf{q})\rangle_{00} \langle \pi| + \text{H.c.} \quad (\text{B1})
\end{aligned}$$

The spin indices are suppressed. We get

$$C = Z_2 \sum_{ij} \int d^3 p |\beta_{ij}(\mathbf{p})|^2, \quad (\text{B2})$$

$$\begin{aligned}
A_{i'j', i''j''}(\mathbf{q}, \mathbf{q}') &= \left[\alpha_{i''j'', i'j'}(\mathbf{q}', \mathbf{q}) + \alpha_{i'j', i''j''}^*(\mathbf{q}, \mathbf{q}') \right. \\
&\left. + \sum_{ij} \int d^3 p \alpha_{ij, i'j'}(\mathbf{p}, \mathbf{q}) \alpha_{ij, i''j''}^*(\mathbf{p}, \mathbf{q}') \right], \quad (\text{B3})
\end{aligned}$$

$$B_{i'j'}(\mathbf{q}) = Z_2^{1/2} \beta_{i'j'}^*(\mathbf{q}) + \sum_{ij} \int d^3 p Z_2^{1/2} \beta_{ij}^*(\mathbf{p}) \alpha_{ij, i'j'}(\mathbf{p}, \mathbf{q}). \quad (\text{B4})$$

The first coefficient yields

$$C = Z_2 \int d^3 p |\beta_{ij}(\mathbf{p})|^2 \quad (\text{B5})$$

$$= \sum_{ij, rs} \int d^3 p \frac{|g_0 U_{ij} \eta_j^{rs}(\mathbf{p})|^2}{|h_0(M_i + E_j(\mathbf{p}) + i\epsilon)|^2} \quad (\text{B6})$$

$$= \frac{\gamma_0}{\pi} \int_{M_1+m_1}^{\infty} dE \sum_{ij} \frac{\tilde{\Gamma}_j(E - M_i) |U_{ij}|^2}{|h_0(E + i\epsilon)|^2} \quad (\text{B7})$$

$$= \frac{\text{Im}}{\pi} \int_{M_1+m_1}^{\infty} dE \frac{h_0(E + i\epsilon)}{|h_0(E + i\epsilon)|^2} \quad (\text{B8})$$

$$= -\frac{1}{2\pi i} \int_{M_1+m_1}^{\infty} dE \left[\frac{1}{h_0(E + i\epsilon)} - \frac{1}{h_0(E - i\epsilon)} \right] \quad (\text{B9})$$

$$= -\frac{1}{2\pi i} \int_P \frac{dz}{h_0(z)} \quad (\text{B10})$$

$$= 1. \quad (\text{B11})$$

The contour P is a path along the real axis coming from ∞ to $m_1 + M_1$ below the real axis then going from $m_1 + M_1$ to ∞ over the real axis. If the contour is closed with the aid of a very large circle, the integral over the closed contour is zero due to the absence of poles. The integral over the circle is equal to 1 due to $h_0(z) \sim z$.

We obtain $A_{i'j', i''j''}(\mathbf{q}, \mathbf{q}') = 0$ from

$$\begin{aligned}
& \sum_{ij} \int d^3 p \alpha_{ij, i'j'}(\mathbf{p}, \mathbf{q}) \alpha_{ij, i''j''}^*(\mathbf{p}, \mathbf{q}') \\
&= \sum_{ij} \int d^3 p \frac{Z_2 |\beta_{ij}(\mathbf{p})|^2 g_0^2 \eta_j(\mathbf{q}) \eta_j^*(\mathbf{q}') U_{i'j'} U_{i''j''}^*}{M_i + E_j(\mathbf{p}) - M_{i'} - E_{j'}(\mathbf{q}) + i\epsilon} \\
&\quad \times \frac{1}{M_i + E_j(\mathbf{p}) - M_{i''} - E_{j''}(\mathbf{q}) - i\epsilon} \quad (\text{B12})
\end{aligned}$$

$$\begin{aligned}
&= g_0^2 \eta_j(\mathbf{q}) \eta_j^*(\mathbf{q}') U_{i'j'} U_{i''j''}^* \frac{(-1)}{2\pi i} \int_P \frac{dz}{h_0(z)} \\
&\quad \times \frac{1}{z - M_{i'} - E_{j'}(\mathbf{q}) + i\epsilon} \frac{1}{z - M_{i''} - E_{j''}(\mathbf{q}) - i\epsilon}, \quad (\text{B13})
\end{aligned}$$

$$\begin{aligned}
&= \frac{-g_0^2 \eta_j(\mathbf{q}) \eta_j^*(\mathbf{q}') U_{i'j'} U_{i''j''}^*}{(M_{i'} + E_{j'}(\mathbf{q}) - M_{i''} - E_{j''}(\mathbf{q}) - i\epsilon)} \\
&\quad \times \left[\frac{1}{h_0(M_{i'} + E_{j'}(\mathbf{q}) - i\epsilon)} - \frac{1}{h_0(M_{i''} + E_{j''}(\mathbf{q}) + i\epsilon)} \right] \quad (\text{B14})
\end{aligned}$$

$$= -\alpha_{i''j'', i'j'}^*(\mathbf{q}, \mathbf{q}') - \alpha_{i'j', i''j''}(\mathbf{q}', \mathbf{q}). \quad (\text{B15})$$

Analogously one can prove

$$B_{ij}(\mathbf{q}) = 0. \quad (\text{B16})$$

APPENDIX C: CALCULATION OF EQ. (77)

The gap between Eqs. (76) and (77) is filled by

$$\begin{aligned}
\Psi_{ij}^{rs}(\mathbf{x}, t) &= \frac{g U_{ij}}{(2\pi)^{3/2}} \int_0^{\infty} dp \eta_j^{rs}(p \hat{\mathbf{x}}) \frac{4\pi}{i} \frac{\partial}{\partial r} \left(\frac{\sin(pr)}{r} \right) \\
&\quad \times \frac{e^{-i(M_i + E_j(\mathbf{p}))t}}{h(E_j + M_i + i\epsilon)}, \quad (\text{C1})
\end{aligned}$$

$$\begin{aligned}
&\approx \frac{4\pi}{ir} \frac{g U_{ij}}{(2\pi)^{3/2}} \int_{m_j+M_i}^{\infty} \frac{dE}{v_j} \eta_j^{rs}(p_j \hat{\mathbf{x}}) \cos p_j r \frac{e^{-iEt}}{h(E) + \frac{i}{2}\Gamma(E)}, \quad (\text{C2})
\end{aligned}$$

$$\begin{aligned}
&\approx \frac{2\sqrt{\pi}}{ir} \gamma^{1/2} U_{ij} \int_{m_j+M_i}^{\infty} \frac{dE}{v_j} \eta_j^{rs}(p_j \hat{\mathbf{x}}) \\
&\quad \times \cos p_j r \frac{e^{-iEt}}{E - M_\pi + \frac{i}{2}\Gamma_\pi}, \quad (\text{C3})
\end{aligned}$$

$$\begin{aligned}
&\approx \frac{2\sqrt{\pi}}{ir} \gamma^{1/2} U_{ij} \eta_j^{rs}(\bar{p}_j \hat{\mathbf{x}}) \frac{e^{-iM_\pi t}}{\bar{v}_j} \\
&\quad \times \int_{-\infty}^{\infty} dE \cos \left(\left(\bar{p}_j + \frac{E}{\bar{v}_j} \right) r \right) \frac{e^{-iEt}}{E + \frac{i}{2}\Gamma_\pi}. \quad (\text{C4})
\end{aligned}$$

To get to Eq. (C1) we made use of

$$\int d^3 p \eta_j^{rs}(\mathbf{p}) e^{i\mathbf{p}\cdot\mathbf{x}} = \int_0^\infty dp \eta_j^{rs}(p\hat{\mathbf{x}}) \frac{4\pi}{i} \frac{\partial}{\partial r} \left(\frac{\sin(pr)}{r} \right). \quad (\text{C5})$$

To get to Eq. (C2) we neglected the nonradiative term containing r^{-2} and used the change of variables $p = |\mathbf{p}| \rightarrow E = E_j(\mathbf{p}) + M_i$. We also used the shorthands $p_j \equiv p_j(E_j) = \sqrt{E_j^2 - m_j^2}$, $\nu_j \equiv \nu_j(E_j) = \frac{p_j(E_j)}{E_j}$ for $E_j = E - M_i$. To get to Eq. (C3) we approximate

$$h(E + i\epsilon) \approx E - M_\pi + \frac{i}{2}\Gamma_\pi, \quad (\text{C6})$$

by Taylor expanding around $E = M_\pi$. Notice that $h(M_\pi) = 0$ and $h'(M_\pi) = 1$. The use of the approximation (C6) requires

$$|\Gamma'(M_\pi)| \ll 1, \quad |h''(M_\pi)\Gamma_\pi| \ll 1. \quad (\text{C7})$$

In Eq. (C4) we approximated $E = M_\pi - M_i$ in the whole integrand, except in the exponentials and in the term h^{-1} . It is assumed that these functions vary slowly in the interval $E \approx M_\pi \pm \Gamma_\pi$. For p_j in the cosine we approximated

$$p_j \approx \bar{p}_j + \frac{E}{\bar{v}_j}, \quad (\text{C8})$$

where E is obtained from the shift $E \rightarrow E + M_\pi$ compared to Eq. (C3). It is also assumed that

$$M_\pi - \Gamma_\pi \gtrsim m_j + M_i, \quad (\text{C9})$$

which allows us to replace the lower limit of the integral by $-\infty$ without changing the integral appreciably. At last, Eq. (77) is recovered by using the integral

$$\int_{-\infty}^{\infty} dE \frac{e^{-iEt}}{E + i\lambda} = -2\pi i \theta(t) e^{-\lambda t}, \quad \lambda > 0, \quad (\text{C10})$$

computed in the complex plane closing the contour on the upper or lower half plane.

APPENDIX D: APPROXIMATE EIGENSTATE

We can see the state in Eq. (45) cannot be an exact eigenstate of the total Hamiltonian H by calculating the eigenvalue equation, analogous to Eq. (25), with the com-

plex eigenvalue

$$E_\pi = M_\pi + i\lambda. \quad (\text{D1})$$

We thus obtain

$$M_\pi + i\lambda + \phi_0(M_\pi + i\lambda) = 0. \quad (\text{D2})$$

The real part is just Eq. (25). The imaginary part yields

$$|\lambda| = -\frac{1}{2} \frac{\Gamma_0(M_\pi)}{1 + \phi_0'(M_\pi)} = -\frac{\Gamma_\pi}{2}, \quad (\text{D3})$$

which can never be satisfied because of the minus sign. Therefore, if both $|\lambda|$ and Γ_π are small, the state (45) represents an approximate eigenstate of H .

APPENDIX E: WIGNER-WEISSKOPF APPROXIMATION

We can compare the results of Eq. (55) with the transition probabilities with respect to the free (bare) states. Such transition amplitudes resemble the Wigner-Weisskopf approximation [24] employed in atomic physics.

For the transition amplitude of the approximate pion state to the free pion we obtain

$$Z_2^{-(1/2)} \langle \pi | \pi(t) \rangle = \langle \pi(0) | \pi(t) \rangle \quad (\text{E1})$$

$$\approx e^{-i(M_\pi - i\Gamma_\pi/2)t}, \quad (\text{E2})$$

where the factor $Z_2^{-(1/2)}$ is necessary from renormalization.

For the transition amplitude of the approximate pion state to the free $l_i \nu_j$ states we obtain

$$\begin{aligned} \langle l_i(r), \nu_j(\mathbf{p}, s) | \pi(t) \rangle &= C_{ij}^{rs}(\mathbf{p}) \left[e^{-i(M_i + E_j)t} + h(M_i + E_j + i\epsilon) \right. \\ &\quad \times \left. \sum_{r's', i'j'} \int d^3 p' \frac{|C_{ij}^{r's'}(\mathbf{p}')|^2 e^{-i(M_{i'} + E_{j'})t}}{M_{i'} + E_{j'} - M_i - E_j - i\epsilon} \right] \quad (\text{E3}) \end{aligned}$$

$$\approx C_{ij}^{rs}(\mathbf{p}) e^{-i(M_i + E_j)t} [1 - e^{-i(M_\pi - M_i - E_j)t} e^{-(\Gamma_\pi/2)t}]. \quad (\text{E4})$$

Equations (E2) and (E4) illustrate clearly the irreversible probability flow from the free state $|\pi\rangle_0 Z^{-(1/2)}$ to the free states $|l_i(r), \nu_j(\mathbf{p}, s)\rangle_0$.

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