



Path integral approach to the full Dicke model

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ABSTRACT

The full Dicke model describes a system of N identical two level-atoms coupled to a single mode quantized bosonic field. The model considers rotating and counter-rotating coupling terms between the atoms and the bosonic field, with coupling constants g_1 and g_2 , for each one of the coupling terms, respectively. We study finite temperature properties of the model using the path integral approach and functional methods. In the thermodynamic limit, $N \rightarrow \infty$, the system exhibits phase transition from normal to superradiant phase, at some critical values of temperature and coupling constants. We distinguish between three particular cases, the first one corresponds to the case of rotating wave approximation, where $g_1 \neq 0$ and $g_2 = 0$, the second one corresponds to the case of $g_1 = 0$ and $g_2 \neq 0$, in these two cases the model has a continuous symmetry. The last one, corresponds to the case of $g_1 \neq 0$ and $g_2 \neq 0$, where the model has a discrete symmetry. The phase transition in each case is related to the spontaneous breaking of its respective symmetry. For each one of these three particular cases, we find the asymptotic behaviour of the partition function in the thermodynamic limit, and the collective spectrum of the system in the normal and the superradiant phase. For the case of rotating wave approximation, and also the case of $g_1 = 0$ and $g_2 \neq 0$, in the superradiant phase, the collective spectrum has a zero energy value, corresponding to the Goldstone mode associated to the continuous symmetry breaking of the model. Our analysis and results are valid in the limit of zero temperature, $\beta \rightarrow \infty$, in which, the model exhibits a quantum phase transition.

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1. Introduction

The Dicke model is an interesting spin-boson model because, being a simple model, it exhibits the superradiance effect [1]. This model describes a system of N identical two level-atoms coupled to a single mode radiation field, simplified according to the rotating wave approximation. In this context, the superradiance is characterized as the coherent spontaneous radiation emission with intensity proportional to N^2 . Thermodynamic properties of the Dicke model were studied in the thermodynamic limit, $N \rightarrow \infty$. It is found that, the model exhibits a second order phase transition from normal to superradiant phase at certain critical temperature and sufficiently larger value of the coupling constant between the atoms and the field [2,3]. The influence of the counter-rotating term on the thermodynamics of the Dicke model also was studied in the literature, [4,5]. Using different coupling constants between the rotating and the counter-rotating coupling, the critical temperature and the free energy of the model are calculated [6,7]. We call this the generalization of full Dicke model. Path integral approach and functional methods were used for studying spin-boson problems, finding critical temperature, free energy and collective spectrum of the models, in the thermodynamic limit, [8,9]. With this approach, Popov and Fedotov [10,11], rigorously calculated the partition function and collective spectrum for the Dicke model in the normal and superradiant phase. Relation between the phase transition and continuous symmetry breaking in the Dicke

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model was pointed out in Ref. [12]. The full Dicke model was studied using the path integral approach [13], here the authors find the asymptotic behaviour of the partition function and collective spectrum in the normal phase. Using the same approach, thermodynamic properties of some other spin-boson models were also studied [14,15].

In this paper, using the path integral approach and functional methods, we find the asymptotic behaviour of the partition function and collective spectrum of the full Dicke model in the thermodynamic limit, $N \rightarrow \infty$, in the normal and superradiant phase. The full Dicke model exhibits phase transition from normal to superradiant phase, at some critical values of temperature and coupling constants. In our study we distinguish three particular cases. The first one corresponds to the case of rotating wave approximation, $g_1 \neq 0$ and $g_2 = 0$, in this case the model has a continuous symmetry, which is associated to the conservation of the sum of the number excitation of the N atoms with the number excitation of the boson field. The second case corresponds to the model with $g_1 = 0$ and $g_2 \neq 0$, in this case the model also has a continuous symmetry, which is associated to the conservation of the difference between the number excitation of the N atoms and the number excitation of the boson field. The last one corresponds to the case of $g_1 \neq 0$ and $g_2 \neq 0$, which the model has a discrete symmetry. The phase transition in each case is related to the spontaneous breaking of their respective symmetry. For the case of rotating wave approximation, and also for the case of $g_1 = 0$ and $g_2 \neq 0$, in the superradiant phase, the collective spectrum has a zero energy value, corresponding to the Goldstone mode associated to the breaking of their respective continuous symmetry. The collective spectrum obtained in this paper is valid for the zero temperature limit, corresponding to the case of quantum phase transition.

Practical realization of the full Dicke model in the laboratory was discussed by Dimer et al. [16]. Since the radiation frequency and energy separation between the two levels of the atoms exceed the coupling constant strength by many orders of magnitude the counter-rotating terms have a little effect on the dynamics. These authors proposed that in cavities with the N qubits, only one mode of quantized field and classical fields (lasers), it is possible to obtain an effective Hamiltonian equal to the full Dicke Hamiltonian. It is possible to control the parameters in this effective Hamiltonian, and it is possible to operate in the phase transition regime. Other authors stressed the importance for quantum information technology of experimental realization of generalizations of the Dicke model in cavity quantum electrodynamics [17,18].

Quantum phase transition of the Dicke model, in the thermodynamic limit, is studied by diagonalizing the Hamiltonian [19]. For this purpose it is applied the Holstein–Primakoff map, which represents the total angular momentum of the N atoms by a single bosonic field. These authors find the collective spectrum in the normal phase. Similar method was used by Emery and Brandes to study the connection between the quantum phase transition and the quantum chaos in the Dicke model without using the rotating wave approximation [20]. They find the collective spectrum of the model in the normal and superradiant phase, as another properties of quantum chaos. The relationship between entanglement and quantum phase transition in the Dicke model was also studied [21,22], the authors find that the atom-field entanglement entropy diverges at the critical point of the phase transition. Studies of this relationship between entanglement and quantum phase transition for others collective models exist in the literature [23].

This paper is organized as follows. In Section 2, we introduce the full Dicke Hamiltonian and study its symmetries. In Section 3, we introduce a map between the spin momentum operators of each atom, with bilinear forms of fermionic operators, defining the fermion full Dicke model. In Section 4, we are able to introduce the path integral approach for the full Dicke model, using functional methods we obtain the critical temperature and the asymptotic behaviour of the partition function in some particular cases of the model. In Section 5, partition function and collective spectrum of the model are presented in the normal phase. In Section 6, partition function and collective spectrum of the model are presented in the superradiant phase. In Section 7 we discuss our conclusions. In the paper we use $k_B = c = \hbar = 1$.

2. The full Dicke Hamiltonian and symmetries

The full Dicke model describes a system of N identical two level-atoms coupled to a single mode quantized bosonic field. The model considers rotating and counter-rotating coupling terms between the atoms and the bosonic field in the Hamiltonian, with coupling constants g_1 and g_2 , for each one of the coupling terms, respectively. Consequently, the Hamiltonian of the full Dicke model can be written as

$$H = \frac{\Omega}{2} \sum_{j=1}^N \sigma_{(j)}^z + \omega_0 b^\dagger b + \frac{g_1}{\sqrt{N}} \sum_{j=1}^N (b \sigma_{(j)}^+ + b^\dagger \sigma_{(j)}^-) + \frac{g_2}{\sqrt{N}} \sum_{j=1}^N (b \sigma_{(j)}^- + b^\dagger \sigma_{(j)}^+). \quad (1)$$

In above equation we define the operators $\sigma_{(j)}^\pm = \frac{1}{2} (\sigma_{(j)}^1 \pm i \sigma_{(j)}^2)$, which the operators $\sigma_{(j)}^1$, $\sigma_{(j)}^2$ and $\sigma_{(j)}^z = \sigma_{(j)}^3$ satisfy the commutation relations $[\sigma_{(j)}^p, \sigma_{(j)}^q] = 2 \epsilon^{pqr} \sigma_{(j)}^r$ with $p, q, r = 1, 2, 3$. Therefore, $[\sigma_{(j)}^+, \sigma_{(j)}^-] = \sigma_{(j)}^z$ and $[\sigma_{(j)}^z, \sigma_{(j)}^\pm] = \pm 2 \sigma_{(j)}^\pm$. The b and b^\dagger are the boson annihilation and creation operators of mode excitations that satisfy the usual commutation relation rules.

Let us define three different operators. The first one, the operator N , which is defined by

$$N = b^\dagger b + \frac{1}{2} \sum_{i=1}^N \sigma_{(i)}^z. \quad (2)$$

The second one, the operator N_- , defined by

$$N_- = b^\dagger b - \frac{1}{2} \sum_{i=1}^N \sigma_{(i)}^z. \quad (3)$$

Finally, we define the parity operator Π by

$$\Pi = e^{i\pi N}, \quad (4)$$

with operator N defined in Eq. (2). In particular case of $g_1 \neq 0$ and $g_2 = 0$, which corresponds to the rotating wave approximation case, it is possible to show that $[H, N] = 0$. In particular case of $g_1 = 0$ and $g_2 \neq 0$, it is possible to show that $[H, N_-] = 0$. And it is possible to show that $[H, \Pi] = 0$ for arbitrary non-negative values of g_1 and g_2 . These commutation relations of the Hamiltonian with each operator defined above, correspond to symmetries of the model for each case. It is interesting to see that, for the case of $g_1 \neq 0$ and $g_2 \neq 0$, we only have that $[H, \Pi] = 0$, it means that the system only has parity symmetry. The operators defined by $J^p = \frac{1}{2} \sum_{i=1}^N \sigma_{(i)}^p$ with $p = 1, 2, 3$, satisfy the usual angular momentum commutation relations. The Hilbert space corresponding to the atoms states can be generated by the basis $\{|j m\rangle\}$ with $j = N/2$ and $m = -j, -j+1, \dots, j-1, j$; each basis state satisfies $J^3|j m\rangle = m|j m\rangle$ and $J^2|j m\rangle = j(j+1)|j m\rangle$. The Hilbert space, which the photon states are defined, can be generated by the basis $\{|n\rangle\}$, with their elements satisfying $b^\dagger b|n\rangle = n|n\rangle$, in this case, n is the number of photons. Now we are able to construct a basis for the total system as a tensor product of the above basis introduced, i.e., the set $\{|n\rangle \otimes |j m\rangle\}$. The symmetries mentioned above, are related with conserved quantities. In the case of $g_1 \neq 0$ and $g_2 = 0$, with $[H, N] = 0$, the excitation number of the system, $n + m$, is conserved. It means that the temporal evolution of a state given by $|n\rangle \otimes |j m\rangle$ only evolves towards another states $|n'\rangle \otimes |j m'\rangle$ which $n' + m' = n + m$. In similar fashion, for the case of $g_1 = 0$ and $g_2 \neq 0$, with $[H, N_-] = 0$, the difference of excitation numbers, $n - m$, is conserved. When $g_1 \neq 0$ and $g_2 \neq 0$, which $[H, \Pi] = 0$, the value $e^{i\pi(n+m)}$ is conserved. It means that the temporal evolution of a state given by $|n\rangle \otimes |j m\rangle$ only evolves towards another states $|n'\rangle \otimes |j m'\rangle$ with both, $n + m$ and $n' + m'$ being even or $n + m$ and $n' + m'$ being odd. In all mentioned cases, the phase transition is related to the spontaneous breaking of their respective symmetries. In further analysis we shall see that, the symmetry associated to the commutation relation $[H, \Pi] = 0$ is discrete, and the symmetries associated to the commutation relations $[H, N] = 0$ and $[H, N_-] = 0$ are continuous symmetries. In cases of continuous symmetry breaking the Goldstone theorem is valid, with the appearing of zero energy value in the phase with the symmetry broken.

3. The fermion full Dicke model

Let us define the fermion full Dicke model. For this purpose, let us define the raising and lowering Fermi operators α_i^\dagger , α_i , β_i^\dagger and β_i , that satisfy the anti-commutator relations $\alpha_i \alpha_j^\dagger + \alpha_j^\dagger \alpha_i = \delta_{ij}$ and $\beta_i \beta_j^\dagger + \beta_j^\dagger \beta_i = \delta_{ij}$. In this analysis, we use a representation of the operators $\sigma_{(i)}^z$, $\sigma_{(i)}^+$ and $\sigma_{(i)}^-$ by the following bilinear combination of Fermi operators, $\alpha_i^\dagger \alpha_i - \beta_i^\dagger \beta_i$, $\alpha_i^\dagger \beta_i$ and $\beta_i^\dagger \alpha_i$, the correspondence is given by

$$\sigma_{(i)}^z \longrightarrow \alpha_i^\dagger \alpha_i - \beta_i^\dagger \beta_i, \quad (5)$$

$$\sigma_{(i)}^+ \longrightarrow \alpha_i^\dagger \beta_i, \quad (6)$$

and

$$\sigma_{(i)}^- \longrightarrow \beta_i^\dagger \alpha_i. \quad (7)$$

Using this representation given in Eqs. (5)–(7) in the full Dicke Hamiltonian given by Eq. (1), we define the Hamiltonian of the fermion full Dicke model H_F . So that, we have

$$H_F = \omega_0 b^\dagger b + \frac{\Omega}{2} \sum_{i=1}^N (\alpha_i^\dagger \alpha_i - \beta_i^\dagger \beta_i) + \frac{g_1}{\sqrt{N}} \sum_{i=1}^N (b \alpha_i^\dagger \beta_i + b^\dagger \beta_i^\dagger \alpha_i) + \frac{g_2}{\sqrt{N}} \sum_{i=1}^N (b^\dagger \alpha_i^\dagger \beta_i + b \beta_i^\dagger \alpha_i). \quad (8)$$

We are interested in studying thermodynamic properties of the system, therefore we must find the partition function Z . It is important to note that Hamiltonians H and H_F are defined in different spaces. Each operator σ_i^α appearing in the Hamiltonian H acts on two-dimensional Hilbert space, notwithstanding, Fermi operators α_i^\dagger , α_i , β_i^\dagger and β_i , appearing in the Hamiltonian H_F act on four-dimensional Fock space. The following property relates the partition function of the full Dicke model with the partition function of the fermion full Dicke model:

$$Z = \text{Tr}(\exp(-\beta H)) = i^N \text{Tr}\left(\exp\left(-\beta H_F - \frac{i\pi}{2} N_F\right)\right). \quad (9)$$

In this last relation H is given by the Eq. (1), H_F is given by Eq. (8) and the operator N_F is defined by

$$N_F = \sum_{i=1}^N (\alpha_i^\dagger \alpha_i + \beta_i^\dagger \beta_i). \quad (10)$$

The traces used in Eq. (9) for each Hamiltonians are carried over their respective spaces. The relation given by Eq. (9) let us express the partition function of the full Dicke model Z using the fermion full Dicke Hamiltonian given by Eq. (8).

4. The partition function with path integral approach

In this section we perform calculations in order to obtain an asymptotic expression for the partition function Z of the full Dicke model in the limit of $N \rightarrow \infty$. For this purpose we use path integral approach and functional methods. Let us define the Euclidean action S of the full Dicke model in the following form

$$S = \int_0^\beta d\tau \left(b^*(\tau) \partial_\tau b(\tau) + \sum_{i=1}^N \left(\alpha_i^*(\tau) \partial_\tau \alpha_i(\tau) + \beta_i^*(\tau) \partial_\tau \beta_i(\tau) \right) \right) - \int_0^\beta d\tau H_F(\tau), \quad (11)$$

the Hamiltonian $H_F(\tau)$ is the full Hamiltonian for the full fermion Dicke model, which is given by

$$\begin{aligned} H_F(\tau) = & \omega_0 b^*(\tau) b(\tau) + \frac{\Omega}{2} \sum_{i=1}^N \left(\alpha_i^*(\tau) \alpha_i(\tau) - \beta_i^*(\tau) \beta_i(\tau) \right) \\ & + \frac{g_1}{\sqrt{N}} \sum_{i=1}^N \left(\alpha_i^*(\tau) \beta_i(\tau) b(\tau) + \beta_i^*(\tau) \alpha_i(\tau) b^*(\tau) \right) \\ & + \frac{g_2}{\sqrt{N}} \sum_{i=1}^N \left(\beta_i^*(\tau) \alpha_i(\tau) b(\tau) + \alpha_i^*(\tau) \beta_i(\tau) b^*(\tau) \right). \end{aligned} \quad (12)$$

Let us define the formal quotient of the partition function of the full Dicke model and the partition function of the free Dicke model. Therefore we are interested in calculating the following quantity

$$\frac{Z}{Z_0} = \frac{\int [d\eta] \exp \left(S - \frac{i\pi}{2\beta} \int_0^\beta n(\tau) d\tau \right)}{\int [d\eta] \exp \left(S_0 - \frac{i\pi}{2\beta} \int_0^\beta n(\tau) d\tau \right)}, \quad (13)$$

the function $n(\tau)$ is defined by

$$n(\tau) = \sum_{i=1}^N \left(\alpha_i^*(\tau) \alpha_i(\tau) + \beta_i^*(\tau) \beta_i(\tau) \right), \quad (14)$$

$S = S(b, b^*, \alpha, \alpha^\dagger, \beta, \beta^\dagger)$ is the Euclidean action of the full Dicke model given by Eq. (11), $S_0 = S_0(b, b^*, \alpha, \alpha^\dagger, \beta, \beta^\dagger)$ is the free Euclidean action for the free single bosonic mode and the free atoms, i.e., the expression of the complete action S taking $g_1 = g_2 = 0$ and finally $[d\eta]$ is the functional measure. The functional integrals involved in Eq. (13), are functional integrals with respect to the complex functions $b^*(\tau)$ and $b(\tau)$ and Fermi fields $\alpha_i^*(\tau)$, $\alpha_i(\tau)$, $\beta_i^*(\tau)$ and $\beta_i(\tau)$. Since we are using thermal equilibrium boundary conditions, in the imaginary time formalism, the integration variables in Eq. (13) obey periodic boundary conditions for the Bose field, i.e., $b(\beta) = b(0)$ and anti-periodic boundary conditions for Fermi fields i.e., $\alpha_i(\beta) = -\alpha_i(0)$ and $\beta_i(\beta) = -\beta_i(0)$.

In Section 2, we have analysed the symmetry of the model studying commutation relations between the Hamiltonian given by Eq. (1) with some operators defining the symmetry. Now we are able to analyse the symmetry of the model studying the invariance of the action given by Eq. (11) under symmetry transformations. In this way, let us introduce the following field transformation

$$\begin{aligned} b(\tau) & \rightarrow e^{i\gamma} b(\tau), & \alpha(\tau) & \rightarrow e^{i\theta} \alpha(\tau), & \beta(\tau) & \rightarrow e^{i\phi} \beta(\tau), \\ b^*(\tau) & \rightarrow e^{-i\gamma} b^*(\tau), & \alpha^*(\tau) & \rightarrow e^{-i\theta} \alpha^*(\tau), & \beta^*(\tau) & \rightarrow e^{-i\phi} \beta^*(\tau). \end{aligned} \quad (15)$$

In the case of $g_1 \neq 0$ and $g_2 = 0$, corresponding to the case of rotating wave approximation, its respective action is invariant under transformation given by Eq. (15), taking $\gamma = \theta - \phi$. In the case of $g_1 = 0$ and $g_2 \neq 0$, its corresponding action is invariant under transformation given by Eq. (15), taking $\gamma = \phi - \theta$. Finally, in the case of $g_1 \neq 0$ and $g_2 \neq 0$, its corresponding action is invariant under transformation given by Eq. (15), with $\gamma = \theta - \phi = 0$ or $\gamma = \theta - \phi = \pi$. In the two first cases, the case of $g_1 \neq 0$ and $g_2 = 0$, and the case of $g_1 = 0$ and $g_2 \neq 0$, their respective actions are invariant under continuous transformation, $U(1)$, of the boson field $b(\tau)$. In the case of $g_1 \neq 0$ and $g_2 \neq 0$, its action is invariant under discrete transformations, Z_2 , of the boson field $b(\tau)$, i.e., $b(\tau) \rightarrow b(\tau)$ and $b(\tau) \rightarrow -b(\tau)$.

Following with the purpose of calculating the quantity $\frac{Z}{Z_0}$ given by Eq. (13), let us use the following transformation

$$\begin{aligned} \alpha_i(\tau) & \rightarrow e^{\frac{i\pi}{2\beta} \tau} \alpha_i(\tau), & \alpha_i^*(\tau) & \rightarrow e^{-\frac{i\pi}{2\beta} \tau} \alpha_i^*(\tau), \\ \beta_i(\tau) & \rightarrow e^{\frac{i\pi}{2\beta} \tau} \beta_i(\tau), & \beta_i^*(\tau) & \rightarrow e^{-\frac{i\pi}{2\beta} \tau} \beta_i^*(\tau). \end{aligned} \quad (16)$$

With this last transformation, the term $n(\tau)$ appearing in Eq. (13) can be dropped. Therefore, applying the transformation given by Eq. (16) into the expression given by Eq. (13), we obtain that

$$\frac{Z}{Z_0} = \frac{\int [d\eta] e^S}{\int [d\eta] e^{S_0}}. \tag{17}$$

In Eq. (17), the Bose field obeys periodic boundary conditions, i.e., $b(\beta) = b(0)$, and the Fermi fields obey the following boundary conditions:

$$\begin{aligned} \alpha_i(\beta) &= i \alpha_i(0), & \alpha_i^*(\beta) &= -i \alpha_i^*(0), \\ \beta_i(\beta) &= i \beta_i(0), & \beta_i^*(\beta) &= -i \beta_i^*(0). \end{aligned} \tag{18}$$

The free action for the single mode bosonic field $S_{B0}(b)$ is given by

$$S_{B0}(b) = \int_0^\beta d\tau b^*(\tau) (\partial_\tau - \omega_0) b(\tau). \tag{19}$$

Then we can write the action S of the full fermion Dicke model, given by Eq. (11), using the free action for the single mode bosonic field $S_{B0}(b)$ defined by Eq. (19), plus an additional term that can be expressed in matrix form. Therefore the total action S can be written as

$$S = S_{B0}(b) + \int_0^\beta d\tau \sum_{i=1}^N \rho_i^\dagger(\tau) M(b^*, b) \rho_i(\tau), \tag{20}$$

the column matrix $\rho_i(\tau)$ is given in terms of Fermi field operators in the following way

$$\begin{aligned} \rho_i(\tau) &= \begin{pmatrix} \beta_i(\tau) \\ \alpha_i(\tau) \end{pmatrix}, \\ \rho_i^\dagger(\tau) &= (\beta_i^*(\tau) \quad \alpha_i^*(\tau)) \end{aligned} \tag{21}$$

and the matrix $M(b^*, b)$ is given by

$$M(b^*, b) = \begin{pmatrix} L & -(N)^{-1/2} (g_1 b^*(\tau) + g_2 b(\tau)) \\ -(N)^{-1/2} (g_1 b(\tau) + g_2 b^*(\tau)) & L_* \end{pmatrix}, \tag{22}$$

the operators L and L_* are defined by $\partial_\tau + \Omega/2$ and $\partial_\tau - \Omega/2$ respectively. Substituting the action S given by Eq. (20) in the functional integral form of the partition function given by Eq. (17) we see that this functional integral is Gaussian in the Fermi fields. Now, let us begin integrating with respect these Fermi fields, therefore we obtain

$$Z = \int [d\eta(b)] e^{S_{B0}} (\det M(b^*, b))^N, \tag{23}$$

in this case, $[d\eta(b)]$ is the functional measure only for the bosonic field. With the help of the following property for matrices with operator components

$$\det \begin{pmatrix} A & B \\ C & D \end{pmatrix} = \det (AD - ACA^{-1}B), \tag{24}$$

and determinant properties, we have that

$$\det M(b^*, b) = \det (LL_*) \det (1 - N^{-1}L_*^{-1} (g_1 b + g_2 b^*) L^{-1} (g_1 b^* + g_2 b)). \tag{25}$$

Substituting Eqs. (23) and (25) in Eq. (17), we have that

$$\frac{Z}{Z_0} = \frac{Z_A}{\int [d\eta(b)] e^{S_{B0}}}, \tag{26}$$

with Z_A defined by

$$Z_A = \int [d\eta(b)] \exp(S_{B0} + N \operatorname{tr} \ln(1 - N^{-1}L_*^{-1}(g_1 b + g_2 b^*)L^{-1}(g_1 b^* + g_2 b))). \tag{27}$$

We are interested in knowing the asymptotic behaviour of the quotient $\frac{Z}{Z_0}$ in the thermodynamic limit, i. e., $N \rightarrow \infty$. With this intention, we analyse the asymptotic behaviour of the last defined expression Z_A . First, let us scale the bosonic

field by $b \rightarrow \sqrt{N} b$ and $b^* \rightarrow \sqrt{N} b^*$, so that we get

$$Z_A = A(N) \int [d\eta(b)] \exp(N \Phi(b^*, b)), \quad (28)$$

with the function $\Phi(b^*, b)$ defined by

$$\Phi(b^*, b) = S_{B0} + tr \ln(1 - L_*^{-1}(g_1 b + g_2 b^*)L^{-1}(g_1 b^* + g_2 b)). \quad (29)$$

The term $A(N)$ in Eq. (28) comes from transforming the functional measure $[d\eta(b)]$ under scaling the bosonic field by $b \rightarrow \sqrt{N} b$ and $b^* \rightarrow \sqrt{N} b^*$. The asymptotic behaviour of the integral functional appearing in Eq. (28) when $N \rightarrow \infty$, can be obtained by using the method of steepest descent [24]. In this method, we expand the function $\Phi(b^*, b)$ around the point $b(\tau) = b_0(\tau)$ and $b^*(\tau) = b_0^*(\tau)$, which can be of two kinds. One kind that makes $Re(\Phi(b^*, b))$ maximum, and the other kind is defined as saddle point. We consider the first terms of the expansion in the integral functional, which are the leading terms for the value of the integral function. We can find the maximum points, or saddle points, finding the stationary points. The stationary points are solution of the following equations $\frac{\delta \Phi(b^*, b)}{\delta b(\tau)} = 0$ and $\frac{\delta \Phi(b^*, b)}{\delta b^*(\tau)} = 0$. For the full Dicke model, the stationary points are constant functions $b(\tau) = b_0$ and $b^*(\tau) = b_0^*$. It is not difficult to show that for $\beta \leq \beta_c$ the stationary point is given by $b_0 = b_0^* = 0$, which is a maximum point. The critical value β_c is obtained by solving the following equation

$$\frac{\omega_0 \Omega}{(g_1 + g_2)^2} = \tanh\left(\frac{\beta_c \Omega}{2}\right). \quad (30)$$

In this last equation, it is possible to find some solution for β_c , in the case of $(g_1 + g_2)^2 > \omega_0 \Omega$. With this condition the system undergoes a phase transition. When the system has $\beta < \beta_c$ we say that the system is in the normal phase. For $\beta > \beta_c$ the stationary points $b(\tau) = b_0$ and $b^*(\tau) = b_0^*$ satisfy the following equation

$$\frac{\omega_0 \Omega_\Delta}{(g_1 + g_2)^2} = \tanh\left(\frac{\beta \Omega_\Delta}{2}\right), \quad (31)$$

with Ω_Δ defined by

$$\Omega_\Delta = \sqrt{\Omega^2 + 4(g_1 + g_2)^2 |b_0|^2}. \quad (32)$$

Phase transition happens if it is possible to find some real solution for $|b_0| \neq 0$ in Eq. (31). It is only possible when $(g_1 + g_2)^2 > \omega_0 \Omega$ and $\beta > \beta_c$. In the case of $g_1 \neq 0$ and $g_2 = 0$, and also in the case of $g_1 = 0$ and $g_2 \neq 0$, the maximum points are a continuous set of values given by the expression $b_0 = \rho e^{i\phi}$ and $b_0^* = \rho e^{-i\phi}$ with $\phi \in [0, 2\pi)$ and $\rho = |b_0|$, with $|b_0|$ defined by Eq. (31). In the case of $g_1 \neq 0$ and $g_2 \neq 0$, we have two maximum points, which are given by $b_0^* = b_0 = \pm |b_0|$, with $|b_0|$ defined by Eq. (31). When the system has $\beta > \beta_c$ we say that the system is in the superradiant phase.

Let us continue, with the computation of the asymptotic behaviour for the integral functional appearing in Eq. (28), for the thermodynamic limit, $N \rightarrow \infty$. In following steps, we shall find this asymptotic behaviour when we only have one maximum point defined by $b_0 = b_0^*$. The resulting expressions will be useful for the normal phase of the full Dicke model, and also for the superradiant phase in the case of $g_1 \neq 0$ and $g_2 \neq 0$. We consider the two first leading terms in the integral function appearing in Eq. (28) coming from the expansion of $\Phi(b^*, b)$ around the maximal value $b_0^* = b_0$, this expansion is given by

$$\Phi(b^*, b) = \Phi(b_0^*, b_0) + \frac{1}{2} \int_0^\beta d\tau_1 d\tau_2 (b^*(\tau_1) - b_0^*, b(\tau_1) - b_0) M_\Phi \begin{pmatrix} b^*(\tau_2) - b_0^* \\ b(\tau_2) - b_0 \end{pmatrix}, \quad (33)$$

the matrix M_Φ , is given by

$$M_\Phi = \begin{pmatrix} \frac{\delta^2 \Phi(b^*, b)}{\delta b^*(\tau_1) \delta b^*(\tau_2)} & \frac{\delta^2 \Phi(b^*, b)}{\delta b^*(\tau_1) \delta b(\tau_2)} \\ \frac{\delta^2 \Phi(b^*, b)}{\delta b(\tau_1) \delta b^*(\tau_2)} & \frac{\delta^2 \Phi(b^*, b)}{\delta b(\tau_1) \delta b(\tau_2)} \end{pmatrix} \Bigg|_{b^*=b=b_0}. \quad (34)$$

Substituting this expansion given by Eq. (33) in Eq. (28) we obtain

$$Z_A = e^{N\Phi(b_0^*, b_0)} \int [d\eta(b)] \exp\left(\frac{1}{2} \int_0^\beta d\tau_1 d\tau_2 (b^*(\tau_1), b(\tau_1)) M_\Phi \begin{pmatrix} b^*(\tau_2) \\ b(\tau_2) \end{pmatrix}\right), \quad (35)$$

to obtain the last expression, we have applied the transformation $b(\tau) \rightarrow (b(\tau) + b_0)/\sqrt{N}$ and $b^*(\tau) \rightarrow (b^*(\tau) + b_0^*)/\sqrt{N}$ in the functional integral involved. In order to make easier the integration of the functional integral given by Eq. (28), let us use the following transformation

$$\begin{aligned} c(\tau) &= \alpha (g_2 b(\tau) + g_1 b^*(\tau)) \\ c^*(\tau) &= \alpha (g_1 b(\tau) + g_2 b^*(\tau)), \end{aligned} \quad (36)$$

the parameter α defined by the equation $\alpha^2 = (g_2^2 - g_1^2)^{-1}$. It is worth mentioning that, the Jacobian of this transformation is 1. Applying this transformation in Eq. (28) we obtain that

$$Z_A = A(N) \int [d\eta(c)] \exp(N \Phi_I(c^*, c)), \tag{37}$$

the function $\Phi_I(c^*, c)$ is given by

$$\Phi_I(c^*, c) = \alpha^2 \int_0^\beta d\tau (g_1 c(\tau) - g_2 c^*(\tau)) (\partial_\tau - \omega_0) (g_1 c^*(\tau) - g_2 c(\tau)) + \text{tr} \ln (1 - \alpha^{-2} L_*^{-1} c^* L^{-1} c). \tag{38}$$

The maximum point corresponds to $c_0^* = c_0 = \alpha(g_1 + g_2)b_0$, the point $b_0^* = b_0$ corresponds to a maximum for the function $\text{Re}(\Phi(b^*, b))$. Using the same expansion given in Eq. (33) for $\Phi_I(c^*, c)$ and substituting in Eq. (37) we obtain that

$$Z_A = e^{N\Phi(b_0^*, b_0)} \int [d\eta(c)] \exp\left(\frac{1}{2} \int_0^\beta d\tau_1 d\tau_2 (c^*(\tau_1), c(\tau_1)) M_{\Phi_I} \begin{pmatrix} c^*(\tau_2) \\ c(\tau_2) \end{pmatrix}\right), \tag{39}$$

we have used the identity $\Phi_I(c_0^*, c_0) = \Phi(b_0^*, b_0)$, and the matrix M_{Φ_I} is defined by

$$M_{\Phi_I} = \left(\begin{array}{cc} \frac{\delta^2 \Phi_I(c^*, c)}{\delta c^*(\tau_1) \delta c^*(\tau_2)} & \frac{\delta^2 \Phi_I(c^*, c)}{\delta c^*(\tau_1) \delta c(\tau_2)} \\ \frac{\delta^2 \Phi_I(c^*, c)}{\delta c(\tau_1) \delta c^*(\tau_2)} & \frac{\delta^2 \Phi_I(c^*, c)}{\delta c(\tau_1) \delta c(\tau_2)} \end{array} \right) \Big|_{c^*=c=c_0}. \tag{40}$$

At this level, it is convenient to use Fourier representation of the field $c(\tau)$ in the functional integral equation (39). From boundaries conditions of the bosonic field $b(\tau)$ and from Eq. (36), we deduce that $c(\tau)$ and $c^*(\tau)$ satisfies periodic boundary conditions $c(\beta) = c(0)$ and $c^*(\beta) = c^*(0)$ respectively. Therefore Fourier representation of $c(\tau)$ and $c^*(\tau)$ are given by

$$\begin{aligned} c(\tau) &= \frac{1}{\sqrt{\beta}} \sum_{\omega} c(\omega) e^{i\omega\tau}, \\ c^*(\tau) &= \frac{1}{\sqrt{\beta}} \sum_{\omega} c^*(\omega) e^{-i\omega\tau}, \end{aligned} \tag{41}$$

the parameter ω takes the values: $2\pi n/\beta$, with n being all the integers. These values correspond to the Matsubara frequencies for bosonic fields. Substituting this Fourier representation, Eq. (41), in Eq. (39), we obtain that

$$Z_A = e^{N\Phi(b_0^*, b_0)} \int [d\eta(c)] \exp\left(\frac{1}{2} \sum_{\omega_1 \omega_2} (c^*(\omega_1), c(\omega_1)) \delta^2 \Phi(\omega_1, \omega_2) \begin{pmatrix} c^*(\omega_2) \\ c(\omega_2) \end{pmatrix}\right), \tag{42}$$

with $\delta^2 \Phi(\omega_1, \omega_2)$ being defined by

$$\delta^2 \Phi(\omega_1, \omega_2) = \begin{pmatrix} \delta^2 \Phi_{11}(\omega_1, \omega_2) & \delta^2 \Phi_{12}(\omega_1, \omega_2) \\ \delta^2 \Phi_{21}(\omega_1, \omega_2) & \delta^2 \Phi_{22}(\omega_1, \omega_2) \end{pmatrix}, \tag{43}$$

and each component of this matrix satisfies

$$\begin{aligned} \delta^2 \Phi_{11}(\omega_1, \omega_2) &= \frac{1}{\beta} \int_0^\beta d\tau_1 d\tau_2 e^{-i\omega_1 \tau_1} \frac{\delta^2 \Phi_I(c^*, c)}{\delta c^*(\tau_1) \delta c^*(\tau_2)} \Big|_{c=c^*=c_0} e^{-i\omega_2 \tau_2}, \\ \delta^2 \Phi_{12}(\omega_1, \omega_2) &= \frac{1}{\beta} \int_0^\beta d\tau_1 d\tau_2 e^{-i\omega_1 \tau_1} \frac{\delta^2 \Phi_I(c^*, c)}{\delta c^*(\tau_1) \delta c(\tau_2)} \Big|_{c=c^*=c_0} e^{i\omega_2 \tau_2}, \\ \delta^2 \Phi_{21}(\omega_1, \omega_2) &= \frac{1}{\beta} \int_0^\beta d\tau_1 d\tau_2 e^{i\omega_1 \tau_1} \frac{\delta^2 \Phi_I(c^*, c)}{\delta c(\tau_1) \delta c^*(\tau_2)} \Big|_{c=c^*=c_0} e^{-i\omega_2 \tau_2}, \\ \delta^2 \Phi_{22}(\omega_1, \omega_2) &= \frac{1}{\beta} \int_0^\beta d\tau_1 d\tau_2 e^{i\omega_1 \tau_1} \frac{\delta^2 \Phi_I(c^*, c)}{\delta c(\tau_1) \delta c(\tau_2)} \Big|_{c=c^*=c_0} e^{i\omega_2 \tau_2}. \end{aligned} \tag{44}$$

In this Fourier representation of the functional integral given by Eq. (42), the integral measure $[d\eta(c)]$ takes the tractable form $\prod_{\omega} dc^*(\omega) dc(\omega)$. Using the expression for $\Phi_I(c^*, c)$ given in Eq. (38), we can calculate the matrix $\delta^2 \Phi(\omega_1, \omega_2)$ with components given by Eq. (44). Performing these calculations we obtain that

$$\begin{aligned} \delta^2 \Phi_{11}(\omega_1, \omega_2) &= \delta^2 \Phi_{12}(\omega_1, \omega_2) = \delta_{\omega_1, -\omega_2} R(\omega_1), \\ \delta^2 \Phi_{21}(\omega_1, \omega_2) &= \delta^2 \Phi_{22}(\omega_1, \omega_2) = \delta_{\omega_1, \omega_2} S(\omega_1), \end{aligned} \tag{45}$$

with $\delta_{\omega_1, \omega_2}$ being the delta Kronecker and the functions $R(\omega)$ and $S(\omega)$ are given by

$$R(\omega) = 2 \omega_0 g_1 g_2 \alpha^2 - \frac{(\Omega_\Delta^2 - \Omega^2) \alpha^{-2}}{2 \Omega_\Delta (\omega^2 + \Omega_\Delta^2)} \tanh\left(\frac{\beta \Omega_\Delta}{2}\right),$$

$$S(\omega) = i \omega \left(1 - \frac{\Omega \alpha^{-2}}{\Omega_\Delta (\omega^2 + \Omega_\Delta^2)} \tanh\left(\frac{\beta \Omega_\Delta}{2}\right)\right) - \omega_0 (g_1^2 + g_2^2) \alpha^2 + \frac{(\Omega_\Delta^2 + \Omega^2) \alpha^{-2}}{2 \Omega_\Delta (\omega^2 + \Omega_\Delta^2)} \tanh\left(\frac{\beta \Omega_\Delta}{2}\right). \quad (46)$$

The expression for Ω_Δ is given by Eq. (32). Substituting the matrix $\delta^2 \Phi(\omega_1, \omega_2)$, with components given by Eq. (45), in the functional integral appearing in Z_A , given by Eq. (42), we obtain that

$$Z_A = e^{N\Phi(b_0^*, b_0)} \int [d\eta(c)] \exp \sum_{\omega} \left(S(\omega) c(\omega) c^*(\omega) + \frac{1}{2} R(\omega) (c(\omega) c(-\omega) + c^*(\omega) c^*(-\omega)) \right). \quad (47)$$

Performing this Gaussian functional integral, we finally obtain that

$$Z_A = e^{N\Phi(b_0^*, b_0)} \frac{2\pi i}{(S^2(0) - R^2(0))^{1/2}} \prod_{\omega \geq 1} \frac{(2\pi i)^2}{S(\omega) S(-\omega) - R^2(\omega)}. \quad (48)$$

In order to find the asymptotic behaviour of $\frac{Z}{Z_0}$ when $N \rightarrow \infty$, we must calculate $\int [d\eta(b)] e^{S_{B0}}$ appearing in Eq. (26). Using the free bosonic action S_{B0} given by Eq. (19), we obtain that

$$\int [d\eta(b)] e^{S_{B0}} = \prod_{\omega} \frac{2\pi i}{\omega_0 - i\omega}. \quad (49)$$

Substituting Eqs. (48) and (49) in Eq. (26) we have that

$$\frac{Z}{Z_0} = e^{N\Phi(b_0^*, b_0)} \frac{1}{(H(0))^{1/2}} \prod_{\omega \geq 1} \frac{1}{H(\omega)}, \quad (50)$$

in this last equation, the function $H(\omega)$ is given by

$$H(\omega) = \frac{S(\omega) S(-\omega) - R^2(\omega)}{\omega^2 + \omega_0^2}. \quad (51)$$

The Eq. (46) gives the expressions for the functions $S(\omega)$ and $R(\omega)$, substituting these functions in Eq. (51) we obtain that

$$H(\omega) = 1 + \frac{(g_1^2 - g_2^2)^2 \Omega^2}{\Omega_\Delta^2 (\omega^2 + \Omega_\Delta^2) (\omega^2 + \omega_0^2)} \tanh^2\left(\frac{\beta \Omega_\Delta}{2}\right) + \frac{2(g_1^2 - g_2^2) \Omega \omega^2 - (g_1^2 + g_2^2) (\Omega^2 + \Omega_\Delta^2) \omega_0 + 2g_1 g_2 (\Omega_\Delta^2 - \Omega^2) \omega_0}{\Omega_\Delta (\omega^2 + \Omega_\Delta^2) (\omega^2 + \omega_0^2)} \tanh\left(\frac{\beta \Omega_\Delta}{2}\right). \quad (52)$$

The expression, given by Eq. (50), with $H(\omega)$ given by Eq. (52), provides a valid expression for the quotient $\frac{Z}{Z_0}$ in the normal phase, and also in the superradiant phase for the particular case of $g_1 \neq 0$ and $g_2 \neq 0$.

5. Normal phase: $\beta < \beta_c$

In the normal phase, $\beta < \beta_c$, from Eq. (31) we have that $b_0 = b_0^* = 0$, i.e. $\Omega_\Delta = \Omega$. Substituting this equality in Eqs. (50) and (52), we obtain that

$$\frac{Z}{Z_0} = \frac{1}{(H_I(0))^{1/2}} \prod_{\omega \geq 1} \frac{1}{H_I(\omega)}, \quad (53)$$

where

$$H_I(\omega) = 1 + \frac{(g_1^2 - g_2^2)^2}{(\omega^2 + \Omega^2) (\omega^2 + \omega_0^2)} \tanh^2\left(\frac{\beta \Omega}{2}\right) + \frac{2(g_1^2 - g_2^2) \omega^2 - 2(g_1^2 + g_2^2) \Omega \omega_0}{(\omega^2 + \Omega^2) (\omega^2 + \omega_0^2)} \tanh\left(\frac{\beta \Omega}{2}\right). \quad (54)$$

Making the analytic continuation ($i\omega \rightarrow E$) in $H_I(\omega)$, we solve the equation $H_I(-iE) = 0$, which corresponds to the collective spectrum equation. Solving the equation, we have that

$$2E^2 = \omega_0^2 + \Omega^2 + 2(g_1^2 - g_2^2) \tanh\left(\frac{\beta \Omega}{2}\right) \pm \left((\omega_0^2 - \Omega^2)^2 + 4(g_1^2 (\omega_0 + \Omega)^2 - g_2^2 (\omega_0 - \Omega)^2) \tanh\left(\frac{\beta \Omega}{2}\right) \right)^{1/2}. \quad (55)$$

It is interesting to see, when $\beta = \beta_c$ we find the following roots [13]

$$E_1 = 0 \tag{56}$$

and

$$E_2 = \left(\frac{g_1 (\Omega + \omega_0)^2 + g_2 (\Omega - \omega_0)^2}{(g_1 + g_2)} \right)^{1/2}. \tag{57}$$

With Eq. (55) we can obtain the collective spectrum for the two following known cases: the first one, when $g_2 = 0$, corresponds to the Dicke model considering the rotating wave approximation [10]. Here we have that

$$2E = \omega_0 + \Omega \pm \left((\omega_0 - \Omega)^2 + 4g_1^2 \tanh\left(\frac{\beta \Omega}{2}\right) \right)^{1/2}. \tag{58}$$

The second one corresponds to $g_1 = g_2 = g$. Here we have that

$$2E^2 = \omega_0^2 + \Omega^2 \pm \left((\omega_0^2 - \Omega^2)^2 + 16g^2 \omega_0 \Omega \tanh\left(\frac{\beta \Omega}{2}\right) \right)^{1/2}. \tag{59}$$

In the case of quantum phase transition, we are in the particular case where $\beta = \infty$. Here, the collective spectrum corresponds to the Eq. (59) where: $\tanh(\beta \Omega/2) = 1$ [20].

6. Superradiant phase: $\beta > \beta_c$

6.1. Case of $g_1 \neq 0$ and $g_2 \neq 0$

In the superradiant phase $\beta > \beta_c$, in the case of $g_1 \neq 0$ and $g_2 \neq 0$, we have two maximum points. Both maximum points contribute equally to the partition function. Therefore $\frac{Z}{Z_0}$, given by Eq. (50) must be multiplied by a factor 2. In this case $b_0 \neq 0$, i.e. $\Omega_\Delta \neq \Omega$. From Eq. (50), we have that the expression for $\frac{Z}{Z_0}$ is given by

$$\frac{Z}{Z_0} = 2 e^{N\phi} \frac{1}{(H_{II}(0))^{1/2}} \prod_{\omega \geq 1} \frac{1}{H_{II}(\omega)}, \tag{60}$$

where the factor ϕ , is defined by

$$\phi = -\frac{\omega_0 \beta (\Omega_\Delta^2 - \Omega^2)}{4(g_1 + g_2)^2} + \ln \left(\frac{\cosh\left(\frac{\beta \Omega_\Delta}{2}\right)}{\cosh\left(\frac{\beta \Omega}{2}\right)} \right). \tag{61}$$

The function $H_{II}(\omega)$ has the form

$$H_{II}(\omega) = \frac{1}{(\omega^2 + \Omega_\Delta^2)(\omega^2 + \omega_0^2)} \left[\omega^4 + \left(\omega_0^2 + \Omega_\Delta^2 + \frac{2(g_1^2 - g_2^2)}{(g_1 + g_2)^2} \omega_0 \Omega \right) \omega^2 + \frac{4g_1 g_2}{(g_1 + g_2)^2} \omega_0^2 (\Omega_\Delta^2 - \Omega^2) \right], \tag{62}$$

and setting $\omega = 0$ in Eq. (62), we obtain the expression for $H_{II}(0)$, so that

$$H_{II}(0) = \frac{4g_1 g_2 (\Omega_\Delta^2 - \Omega^2)}{(g_1 + g_2)^2 \Omega_\Delta^2}. \tag{63}$$

Making the analytic continuation ($i\omega \rightarrow E$) in $H_{II}(\omega)$ given by Eq. (62), we solve the equation $H_{II}(-iE) = 0$. The set of solutions, E , are the collective spectrum in the superradiant phase for the case of $g_1 \neq 0$ and $g_2 \neq 0$. Therefore, solving the equation we have that

$$2E^2 = \omega_0^2 + \Omega_\Delta^2 + 2 \frac{(g_1^2 - g_2^2)}{(g_1 + g_2)^2} \Omega \omega_0 \pm \left[\left(\omega_0^2 + \Omega_\Delta^2 + 2 \frac{(g_1^2 - g_2^2)}{(g_1 + g_2)^2} \Omega \omega_0 \right)^2 - \frac{16g_1 g_2}{(g_1 + g_2)^2} \omega_0^2 (\Omega_\Delta^2 - \Omega^2) \right]^{1/2}. \tag{64}$$

For the particular case of $g_1 = g_2 = g$, the collective spectrum energy takes the particular form

$$2E^2 = \omega_0^2 + \Omega_\Delta^2 \pm ((\omega_0^2 - \Omega_\Delta^2)^2 + 4\omega_0^2 \Omega^2)^{1/2}. \tag{65}$$

In the limit of zero temperature, $\beta \rightarrow \infty$, from Eq. (31) we have that $\Omega_\Delta = 4g^2/\omega_0$. Consequently, at zero temperature we obtain that [20]

$$2E^2 = \omega_0^2 + \frac{16g^4}{\omega_0^2} \pm \left[\left(\omega_0^2 - \frac{16g^4}{\omega_0^2} \right)^2 + 4\omega_0^2 \Omega^2 \right]^{1/2}. \tag{66}$$

6.2. Case of $g_1 \neq 0$ and $g_2 = 0$

Now let us study, the case of rotating wave approximation, i.e., the case of $g_1 \neq 0$ and $g_2 = 0$, in the superradiant phase. Here, the expression for $\frac{Z}{Z_0}$ is obtained setting $g_2 = 0$ in Eqs. (28) and (29), therefore we have that

$$Z_A = A(N) \int [d\eta(b)] \exp(N \Phi_{g_1}(b^*, b)), \quad (67)$$

the function $\Phi_{g_1}(b^*, b)$ is defined by

$$\Phi_{g_1}(b^*, b) = \int_0^\beta d\tau b^*(\tau) (\partial_\tau - \omega_0) b(\tau) + \text{tr} \ln(1 - g_1^2 L_*^{-1} b L^{-1} b^*). \quad (68)$$

In last equation, Eq. (68), we can see that the function $\Phi_{g_1}(b^*, b)$ is invariant by transformation $b(\tau) \rightarrow \exp(i\theta) b(\tau)$ and $b^*(\tau) \rightarrow \exp(-i\theta) b^*(\tau)$, where θ is an arbitrary factor independent of τ . This continuous invariance is responsible for the appearing of Goldstone mode in the system. In order to perform the functional integral given by Eq. (67), let us separate the function $b(\tau)$ in the following form

$$\begin{aligned} b(\tau) &= b_c + b'(\tau), \\ b^*(\tau) &= b_c^* + b'^*(\tau), \end{aligned} \quad (69)$$

where b_c is a constant function, and the fields $b'(\tau)$ and $b'^*(\tau)$ satisfy the following boundaries conditions $b'(0) = b'(\beta) = 0$ and $b'^*(0) = b'^*(\beta) = 0$. Using the representation $b_c = \rho e^{i\phi}$ and $b_c^* = \rho e^{-i\phi}$ in the functional integral given by Eqs. (67) and (68), and after applying the transformation $b'(\tau) \rightarrow e^{i\phi} b'(\tau)$ and $b'^*(\tau) \rightarrow e^{-i\phi} b'^*(\tau)$, we obtain that

$$Z_A = 2\pi i A(N) \int_0^\infty d\rho^2 \int [d\eta(b')] \exp(N \Phi_{g_1}(\rho, b'^*, b')), \quad (70)$$

the function $\Phi_{g_1}(\rho, b'^*, b')$ is given by

$$\Phi_{g_1}(\rho, b'^*, b') = \int_0^\beta d\tau (\rho + b'^*(\tau)) (\partial_\tau - \omega_0) (\rho + b'(\tau)) + \text{tr} \ln(1 - g_1^2 L_*^{-1} (\rho + b') L^{-1} (\rho + b'^*)). \quad (71)$$

In the integral function appearing in Eq. (70), one variable of integration is ρ^2 . Here we use the steepest descent method in order to analyse the limit $N \rightarrow \infty$, we find the stationary point with respect to the variable ρ^2 . Therefore, the stationary point satisfies the following equation $\left. \frac{\delta \Phi_{g_1}}{\delta (\rho^2)} \right|_{\rho=\rho_0} = 0$ with $b'^*(\tau) = b'(\tau) = 0$. In this case the value for ρ_0 is the same as $|b_0|$ defined by Eq. (31) setting $g_2 = 0$. Let us consider the two first leading terms in the functional integral appearing in Eq. (70), coming from the expansion of $\Phi_{g_1}(\rho, b'^*, b')$ around the point defined by ρ_0 and $b'^*(\tau) = b'(\tau) = 0$, giving the maximum for $\text{Re}(\Phi_{g_1}(\rho, b'^*, b'))$. This expansion is given by

$$\begin{aligned} \Phi_{g_1}(\rho, b'^*, b') &= \Phi_{g_1}(\rho_0, 0, 0) + \frac{1}{2} \left. \frac{\delta^2 \Phi_{g_1}}{\delta (\rho^2)^2} \right|_{\rho=\rho_0, b'=b'^*=0} (\rho^2 - \rho_0^2)^2 \\ &+ \frac{1}{2} \int_0^\beta d\tau_1 d\tau_2 (b'^*(\tau_1), b'(\tau_1)) M_{\Phi_{g_1}} \begin{pmatrix} b'^*(\tau_2) \\ b'(\tau_2) \end{pmatrix}, \end{aligned} \quad (72)$$

here the matrix $M_{\Phi_{g_1}}$ is given by

$$\left(\begin{array}{cc} \frac{\delta^2 \Phi_{g_1}}{\delta b'^*(\tau_1) \delta b'^*(\tau_2)} & \frac{\delta^2 \Phi_{g_1}}{\delta b'^*(\tau_1) \delta b'(\tau_2)} \\ \frac{\delta^2 \Phi_{g_1}}{\delta b'(\tau_1) \delta b'^*(\tau_2)} & \frac{\delta^2 \Phi_{g_1}}{\delta b'(\tau_1) \delta b'(\tau_2)} \end{array} \right) \Bigg|_{\rho=\rho_0, b'=b'^*=0}. \quad (73)$$

Using this expansion given by Eq. (72) to perform functional integral given by Eq. (70), we have that

$$\begin{aligned} Z_A &= 2\pi i \sqrt{N} e^{N\Phi_{g_1}} \int_{-\sqrt{N}\rho_0^2}^\infty dy e^{\frac{1}{2} \left. \frac{\delta^2 \Phi_{g_1}}{\delta (\rho^2)^2} \right|_{\rho=\rho_0, b'=b'^*=0} y^2} \\ &\times \int [d\eta(b')] \exp\left(\frac{1}{2} \int_0^\beta d\tau_1 d\tau_2 (b'^*(\tau_1), b'(\tau_1)) M_{\Phi_{g_1}} \begin{pmatrix} b'^*(\tau_2) \\ b'(\tau_2) \end{pmatrix}\right), \end{aligned} \quad (74)$$

the expression Φ_{g_1} corresponds to the expression of ϕ defined in Eq. (61) taking $g_2 = 0$. The factor \sqrt{N} appearing in Eq. (74) come from the scaling $\rho^2 \rightarrow \rho^2/\sqrt{N}$. For $N \rightarrow \infty$, integrals appearing in Eq. (74) are Gaussians. We represent the functions

$b'(\tau)$ and $b'^*(\tau)$ in Fourier series, which do not possess the zero mode, since they satisfy the boundary conditions given by $b'(0) = b'(\beta) = 0$ and $b'^*(0) = b'^*(\beta) = 0$. Therefore, performing the functional integral and substituting in Eq. (26), we obtain that

$$\frac{Z}{Z_0} = \sqrt{N} e^{N\phi_{g_1}} \frac{1}{A_0} \prod_{\omega \geq 1} \frac{1}{H_{II}(\omega)}, \tag{75}$$

functions ϕ_{g_1} and $H_{II}(\omega)$ are given respectively by Eqs. (61) and (62) setting $g_2 = 0$, and A_0 is given by

$$A_0 = \frac{g_1}{\Omega_\Delta \sqrt{\pi \beta \omega_0}} \left(1 - \frac{\beta \Omega_\Delta}{\sinh(\beta \Omega_\Delta)} \right)^{\frac{1}{2}}. \tag{76}$$

Making the analytic continuation ($i\omega \rightarrow E$) in $H_{II}(\omega)$ given by Eq. (62) setting $g_2 = 0$, the collective spectrum is obtained by solving the equation $H_{II}(-iE) = 0$. So that, we obtain the following spectrum

$$E_1 = 0, \tag{77}$$

and

$$E_2^2 = \omega_0^2 + \Omega_\Delta^2 + 2 \omega_0 \Omega. \tag{78}$$

The particular value of the spectrum given by $E_1 = 0$ in Eq. (77) corresponds to the Goldstone mode [10].

6.3. Case of $g_1 = 0$ and $g_2 \neq 0$

Now let us study the case of $g_1 = 0$ and $g_2 \neq 0$, in the superradiant phase. Here, the expression for $\frac{Z}{Z_0}$ is obtained setting $g_1 = 0$ in Eqs. (28) and (29). For this case we have

$$Z_A = A(N) \int [d\eta(b)] \exp(N \Phi_{g_2}(b^*, b)), \tag{79}$$

the function $\Phi_{g_2}(b^*, b)$ is defined by

$$\Phi_{g_2}(b^*, b) = \int_0^\beta d\tau b^*(\tau) (\partial_\tau - \omega_0) b(\tau) + tr \ln(1 - g_2^2 L_*^{-1} b^* L^{-1} b). \tag{80}$$

In last equation, Eq. (80), we can see that the function $\Phi_{g_2}(b^*, b)$ is invariant by transformation $b(\tau) \rightarrow \exp(i\theta) b(\tau)$ and $b^*(\tau) \rightarrow \exp(-i\theta) b^*(\tau)$, where θ is an arbitrary factor independent of τ . This continuous invariance is responsible for the appearing of Goldstone mode in the system. Since Eq. (80) is very similar to Eq. (68), we can see that, the calculation to obtain $\frac{Z}{Z_0}$ in the case of $g_1 = 0$ follows the same steps as the calculation performed to obtain $\frac{Z}{Z_0}$ in the case of rotating wave approximation. Consequently, we have that

$$\frac{Z}{Z_0} = \sqrt{N} e^{N\phi_{g_2}} \frac{1}{A_0} \prod_{\omega \geq 1} \frac{1}{H_{II}(\omega)}, \tag{81}$$

where $H_{II}(\omega)$ is given by Eq. (62) setting $g_1 = 0$, the value ϕ_{g_2} corresponds to the expression for ϕ defined in Eq. (61) taking $g_1 = 0$, and A_0 is given by

$$A_0 = \frac{g_2}{\Omega_\Delta \sqrt{\pi \beta \omega_0}} \left(1 - \frac{\beta \Omega_\Delta}{\sinh(\beta \Omega_\Delta)} \right)^{\frac{1}{2}}. \tag{82}$$

Making the analytic continuation ($i\omega \rightarrow E$) in $H_{II}(\omega)$ given by Eq. (62) setting $g_1 = 0$, the collective spectrum is obtained by solving the equation $H_{II}(-iE) = 0$. So that, we obtain the following spectrum

$$E_1 = 0, \tag{83}$$

and

$$E_2^2 = \omega_0^2 + \Omega_\Delta^2 - 2 \omega_0 \Omega. \tag{84}$$

The particular value of the spectrum given by $E_1 = 0$ in Eq. (83) corresponds to the Goldstone mode.

7. Summary

In this paper, using the path integral approach and functional methods, in the thermodynamic limit $N \rightarrow \infty$, we find the asymptotic behaviour of the partition function and collective spectrum of the full Dicke model in the normal and superradiant phase. In our study we distinguish three particular cases. The first one corresponds to the case of rotating

wave approximation, $g_1 \neq 0$ and $g_2 = 0$, in this case the model has a continuous symmetry, which is associated to the conservation of the sum of the number excitation of the N atoms with the number excitation of the boson field. The second case corresponds to the model with $g_1 = 0$ and $g_2 \neq 0$, in this case the model has a continuous symmetry, which is associated to the conservation of the difference between the number excitation of the N atoms and the number excitation of the boson field. The last one, corresponds to the case of $g_1 \neq 0$ and $g_2 \neq 0$, which the model has a discrete symmetry. The phase transition in each case is related to the spontaneous breaking of their respective symmetry. In the case of rotating wave approximation, and also in the case of $g_1 = 0$ and $g_2 \neq 0$, the collective spectrum has a zero energy value, corresponding to the Goldstone mode associated to the continuous symmetry breaking for these cases.

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