SCIENTIFIC REPORT

POSTDOCTORAL RESEARCHER - FAPESP

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RESEARCH PROJECT

VARIOUS ASPECTS OF DISCRETE DYNAMICS

POSTDOCTORAL RESEARCHER

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PERIOD

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1 SUMMARY

During the period of the project, we studied several problems related to the theory of discrete dynamical systems and its connections with number theory, probability theory, and operator theory. More precisely, the project was focused on dynamical properties of p-adic and linear systems - that is dynamical systems over the space of p-adic integers/numbers and Banach spaces -, on the characterization of stochastic adding machines and its connections with Julia sets and on elimination problems in combinatorial algorithms. This study involved the following thematics:

- Dynamical properties as shadowing and stability of non-invertible dynamics in the space of p-adic integers $\mathbb{Z}_p$ and p-adic numbers $\mathbb{Q}_p$, where $p \geq 2$ is a prime integer.
- Lipschitz structural stability and shadowing of a class of homeomorphisms in complete metric spaces as Banach spaces of infinite dimension.
- Stochastic adding machines and exotic Julia sets.
- The introduction of a randomized version of the Josephus problem and the characterization of the limit distribution of the winning probabilities for the players in this randomized context.

As a consequence of these investigations, we obtained various results which are organized in the form of the papers [7] and [1]. The papers [16] and [8] are in the final stage of writing and soon will be submitted for publication.

2 DESCRIPTION OF THE RESULTS

2.1 Shadowing and Stability Properties of Non-Invertible p-adic Dynamics

Summary: In this work, we provide a family of continuous, non-invertible, maps in the space of p-adic integers $\mathbb{Z}_p$ and p-adic number $\mathbb{Q}_p$ that enjoys stability and shadowing properties.

Details: The classical stability theory is well developed over compact manifolds (see [2, 18]). Let $(\mathcal{X}, d)$ be a compact metric space. The map $f : \mathcal{X} \mapsto \mathcal{X}$ is said to be shadowing if for every $\epsilon > 0$ there exists $\delta = \delta(\epsilon) > 0$ such that for every δ-pseudo-orbit $(x_n)_{n\in\mathbb{N}}$ of f (i.e. $d(f(x_n), x_{n+1}) \leq \delta$ for every $n \in \mathbb{N}$) there exists a point $\xi \in \mathcal{X}$ so that $d(f^n(\xi), x_n) \leq \epsilon$ for every $n \in \mathbb{N}$. The map f is *topologically stable* (respectively, *Lipschitz structurally stable*) if there exists $\delta_0 > 0$ such that for every continuous map $g : \mathcal{X} \mapsto \mathcal{X}$ (respectively, for continuous map g such that $g - f$ is δ_0-Lipschitz map), with $\sup\limits_{x\in\mathcal{X}} d(f(x), g(x)) \leq \delta_0$, there exists a continuous map (respectively, a homeomorphism) $h : \mathcal{X} \mapsto \mathcal{X}$ so that $f \circ h = h \circ g$.

A fundamental theorem due to Walters [20] ensures that if $f : \mathcal{X} \mapsto \mathcal{X}$ is a shadowing and expansive map, then f is topologically stable. In the opposite direction, every topologically stable homeomorphism f is shadowing.

A stability theory in Cantor spaces (i.e. compact spaces with zero topological dimension) has started being developed only during the last decade (see, for instance, [3, 11]).

The spaces of p-adic integers $\mathbb{Z}_p$ and p-adic numbers $\mathbb{Q}_p$ are characteristic examples of zero-dimensional spaces. The main result of [7] provides a large family of non-invertible dynamics in $\mathbb{Z}_p$ and $\mathbb{Q}_p$ in the form of a suitable generalization of the notion of a dilation in these two spaces.

Theorem 2.1 *Let $f : \mathbb{X}_p \mapsto \mathbb{X}_p$ be a continuous map, where $\mathbb{X}_p \in \{\mathbb{Z}_p, \mathbb{Q}_p\}$. Assume that the map f admits as right inverses a family of contractions $R_i : \mathbb{X}_p \mapsto \mathbb{X}_p$, $i \in \mathcal{I}$; i.e. $f \circ R_i = id$ and $Lip\,(R_i) < 1$, for every $i \in \mathcal{I}$. If*

$$\bigcup_{i \in \mathcal{I}} R_i\,(\mathbb{X}_p) \;=\; \mathbb{X}_p, \tag{1}$$

then the map f has the shadowing property. Furthermore, if the images $R_i\,(\mathbb{X}_p)$, $i \in \mathcal{I}$, are open and disjoint sets, then the map f is strongly Lipschitz structurally stable and topologically stable.

Theorem 2.1 provides a uniform context that encompasses the majority of examples of Lipschitz stable maps in $\mathbb{Z}_p$ (see for instance [3]) while, at the same time, yields also new examples.

The proof of the shadowing property in Theorem 2.1 is based on applying a fixed point theorem in complete metric spaces on properly constructed operators. Then, once the shadowing property has been established, the claims about the stability follow by constructing conjugations between the map f and its sufficiently small perturbations, upon exploiting the representation of the shadowing points as fixed points in their respective operators.

In view of Theorem 2.1, we also prove that the covering condition (1) is crucial for guaranteeing the shadowing property for the map $f : \mathbb{X}_p \mapsto \mathbb{X}_p$. Indeed, as the following proposition shows, condition (1) cannot be replaced by the sole existence of a single contraction as a right inverse.

Theorem 2.2 (Continuous non-shadowing maps with a right inverse) *There exists a continuous map $f : \mathbb{Z}_p \mapsto \mathbb{Z}_p$ which admits a contraction $R : \mathbb{Z}_p \mapsto \mathbb{Z}_p$ as a right inverse but f is neither shadowing nor topologically stable.*

Kawaguchi [11] proved that every topologically stable homeomorphism of a Cantor space is shadowing. The proof of Theorem 2.2 exploits the following result, also proved in [7], which generalizes that of Kawaguchi for the class of nowhere locally constant topologically stable maps.

Proposition 2.1 *Let $f : \mathbb{X}_p \mapsto \mathbb{X}_p$, $\mathbb{X}_p \in \{\mathbb{Q}_p, \mathbb{Z}_p\}$, be a uniformly continuous map. If f is topologically stable and nowhere locally constant, then f has the shadowing property.*

The second main result of [7] complements Theorem 2.1 and describes stability properties of contractions in $\mathbb{X}_p \in \{\mathbb{Z}_p, \mathbb{Q}_p\}$.

Theorem 2.3 *Let $f : \mathbb{X}_p \mapsto \mathbb{X}_p$ be a contraction, where $\mathbb{X}_p \in \{\mathbb{Z}_p, \mathbb{Q}_p\}$. The following two statements hold:*

1. *If $\mathbb{X}_p = \mathbb{Q}_p$ and $f(\mathbb{Q}_p) = \mathbb{Q}_p$, then f is strongly Lipschitz structurally stable.*
2. *If $\mathbb{X}_p = \mathbb{Z}_p$ and $f(\mathbb{Z}_p)$ is open, then f is strongly Lipschitz structurally stable.*

Theorem 2.3 generalizes a result about affine contractions in [3, Theorem 30]. The proof of the result in $\mathbb{Q}_p$ is analogous to Theorem 2.3. In the case of $\mathbb{Z}_p$, the proof is based on constructing conjugations between f and its perturbations, upon exploiting the unique representation of the points of $\mathbb{Z}_p$ as $f^n(u)$, where $n \in \mathbb{N}$ and $u \in \mathbb{Z}_p \backslash f(\mathbb{Z}_p)$.

2.2 Stability and Shadowing of Dilations and Contractions in Complete Metric Spaces

Summary: In this work, we establish shadowing and stability properties for a large family of dilations and contractions in complete metric spaces. As a consequence, we derive results for the linear dynamics in Banach spaces.

Details: During the last decade, the study of linear dynamics in Banach spaces has developed significantly. The main results focus on drawing the connections between the concepts of stability, shadowing, expansivity, and (generalized) hyperbolicity (see, for instance, [4, 5, 6]).

The first of our main results in [16] describes the stability and shadowing properties of dilations in complete metric spaces.

Theorem 2.4 *Let $(\mathcal{X}, d)$ be a complete metric space and assume that the map $f : \mathcal{X} \mapsto \mathcal{X}$ is a homeomorphism. If f is a dilation or a contraction in $\mathcal{X}$, then f is shadowing and topologically stable. Moreover, if $(\mathcal{X}, ||*||, +)$ is a group equipped with an absolute value, then the map f is strongly Lipschitz structurally stable.*

The arguments developed for the proof of Theorem 2.4, when specialized in the context of Banach spaces, are used to derive the following sufficient condition for a homeomorphism to have hyperbolic coordinates. To this end, we define the *ϵ-stable set* and the *ϵ-unstable set* of f at x by

$$W^s_\epsilon(x, f) = \{y \in \mathcal{X} : ||f^n(x) - f^n(y)|| \le \epsilon \text{ for all } n \ge 0\}$$

and

$$W^u_\epsilon(x, f) = \left\{y \in \mathcal{X} : \left|\left|f^{-n}(x) - f^{-n}(y)\right|\right| \le \epsilon \text{ for all } n \ge 0\right\},$$

respectively. The homeomorphism f is said to have *canonical coordinates* if for each $\epsilon > 0$ there exists $\delta > 0$ such that $||x - y|| \le \delta$ implies $W^s_\epsilon(x, f) \cap W^u_\epsilon(x, f) \ne \emptyset$. We say that the map f has *hyperbolic coordinates* if f has canonical coordinates and there exist constants $c \ge 1$, $0 < \beta < 1$ and $\gamma > 0$ such that the following properties hold for each x [2]:

- $y \in W^s_\gamma(x,f) \implies ||f^n(x) - f^n(y)|| \leq c \cdot \beta^n \cdot ||x-y||$ for all $n \in \mathbb{N}$,
- $y \in W^u_\gamma(x,f) \implies ||f^{-n}(x) - f^{-n}(y)|| \leq c \cdot \beta^n \cdot ||x-y||$ for all $n \in \mathbb{N}$.

In view of this definition, our result reads as follows.

Theorem 2.5 *Let $\mathcal{B}$ be a Banach space. If $f : \mathcal{B} \mapsto \mathcal{B}$ is a shadowing and uniformly expansive homeomorphism (not necessarily linear), then the map f has hyperbolic coordinates.*

In view of Theorem 2.5, we work on the conjecture we state below.

Conjecture 2.1 *Let $(\mathcal{B}, ||*||)$ be a Banach space. If $f : \mathcal{B} \mapsto \mathcal{B}$ is a shadowing and uniformly expansive homeomorphism, then the map f is Lipschitz structurally stable.*

This conjecture was proved independently, for the case of linear homeomorphic maps, in [6] and [10].

2.3 Randomization in the Josephus Problem

Summary: In this work, we introduce a randomized version of the Josephus problem from combinatorial algorithms and we characterize the limit distribution for the survival probabilities of players when their number N tends to infinity. This work has already been accepted for publication (see [1]).

Details: Josephus problem refers to an old elimination process where N players, enumerated from 0 to $N-1$, are standing on a circle and are "eliminated" in the following way: Player 1, whom we can imagine holding a knife, eliminates the player on his right (i.e. Player 2) and passes the knife to the next person alive on his right (i.e. Player 3). Each time, the player holding the knife repeats the process until there is only one survivor left.

Josephus problem is a characteristic example of a counting-out game in the theory of combinatorial algorithms. Indeed, deterministic variations of the problem have been extensively studied in the literature (see, for instance, [14, 17, 19]).

In our work [1], we introduce a natural probabilistic variant of this process. More precisely, in this variant, the survivor is determined after performing a succession of Bernoulli trials with parameter p designating each time the person to be removed. More precisely, with probability p, the direction of "stabbing" is the same as that of the previous round and, with probability $1-p$, the direction of the stabbing is the opposite of that of the previous round.

In this setup, given a fixed parameter $p \in (0,1)$ and the number of players $N \in \mathbb{N}$, denote by $g_N(n,p)$ the survival of the n-th player in a N-players game. Upon assuming that the players are standing on the unit circle $\mathbb{T} = \mathbb{R}/\mathbb{Z}$, we define the probability measure of the survival probabilities as

$$\mu^{(p)}_N(*) := \sum_{n=0}^{N-1} g_N(n,p) \cdot \delta_{n/N}(*),$$

where $\delta_{n/N}$ denotes the *Dirac mass* at the point $n/N \in \mathbb{T}$.

The problem arising in this randomized setup is to describe the limit distribution of the survival probabilities $\mu_N^{(p)}$. Our main result, appearing in [1], characterizes this limit with an increasing degree of precision as the probabilistic parameter p approaches the unbiased case $p = 1/2$.

Theorem 2.6 *Assume that $p \in (0,1)$. Then, the sequence of probability measures $\left(\mu_N^{(p)}\right)_{N\geq 1}$ admits a weak limit $\mu^{(p)}$ supported on $\mathbb{T}$. This limit measure $\mu^{(p)}$ can furthermore be described with an increasing degree of precision as the parameter p approaches the unbiased case $p = 1/2$ as follows:*

1. *In the general case when $p \in (0,1)$, the limit measure $\mu^{(p)}$ is a convex combination of Dirac masses concentrated at the origin and at the point $1/2 \pmod 1$; that is,*

$$\mu^{(p)} \; = \; (1 - c(p)) \cdot \delta_0 + c(p) \cdot \delta_{1/2}$$

   for some constant $c(p) \in [0,1]$.

2. *In the case where p lies in the middle interval $(1/3, 2/3)$, the limit measure $\mu^{(p)}$ is the Dirac mass concentrated at the mid-point $1/2 \pmod 1$; that is, it then holds that $c(p) = 1$.*

3. *In the unbiased case where $p = 1/2$, a rate of convergence to the Dirac mass δ_1 can be obtained in the form of a Central-Limit theorem. Indeed, if $(X_N)_{N\geq 1}$ is a sequence of random variables drawn successively and independently, each according to the probability measure $\mu_N^{(p)}$, then the following convergence in law to the standard normal distribution $\mathcal{N}(0,1)$ is verified:*

$$\frac{1}{S_L} \cdot \sum_{N=1}^{L} \left(X_N - \frac{1}{2} \right) \underset{L\to+\infty}{\longrightarrow} \mathcal{N}(0,1),$$

   where $(S_L)_{L\geq 1}$ is a sequence of positive reals such that

$$S_L \; \asymp \; \sqrt{ln(L)}.$$

Theorem 2.6 combined with numerical data gathered from computer simulations give rise to the following conjecture.

Conjecture 2.1 *Following the notation of Theorem 2.6, it holds that $c(p) = 1$ for every $p \in (0,1)$. In other words, $\mu^{(p)} = \delta_{1/2}$ for every $p \in (0,1)$.*

2.4 Operators of Stochastic Adding Machines and Julia Sets

Summary: In this work, we consider a Markov chain associated to the stochastic adding machine based on Cantor Systems of numeration and we prove that the spectrum of the transition operator associated with this Markov chain in some classical Banach space is equal to a fibered Julia set.

Details: The stochastic adding machines were introduced in the work of Killeen and Taylor [12] (see also [9, 15]) in the following way: given a natural number $n \in \mathbb{Z}_+$, we can express n in a unique way in its binary expansion $n = \sum_{i=0}^{k} a_i(n)2^i = a_k \dots a_0$, where $k \geq 0$ and the digits $a_i(n)$ take the values 0 and 1. Through an algorithm, we obtain the addition of 1 with n in base 2 by $n + 1 = a_k \dots (a_l + 1)0 \dots 0$, where $l = \min\{i \geq 0 : a_i(n) = 0\}$. Assuming that the 1 can be added with probability $0 < p < 1$ and not be added with probability $1-p$, we obtain a stochastic process in which every integer number $n = a_k \dots a_0$ transitions to n with probability $p_{n,n} = 1 - p$ (the digit 1 was not added with probability $1 - p$), to $n + 1$ with probability $p_{n,n+1} = p^{k+1}$ (the digit 1 was added k times with independent probability each time equal to p), and to $m = n - 2^r + 1 = a_k \dots a_{l+1}0 \dots 0$ with probability $p_{n,m} = p^r(1-p)$. In this way, Killeen and Taylor obtained a Markov chain the states of which are all positive integers and for which the transition operator $P = (p_{i,j})_{i,j\geq 0}$ is an infinite stochastic matrix whose spectrum $\sigma(P)$ (as an operator in the space l^∞) equals the full Julia set of the quadratic function $f(z) = \left[(z^2 - 1 + p)/p\right]^2$. It is worth mentioning that the stochastic adding machine of Killeen and Taylor has interesting applications in psychology [13].

In [8], we extend the above results due to Killeen and Taylor [12] in a more general context. To this end, given a sequence of positive integers $\boldsymbol{d} = (d_j)_{j\geq 0}$ such that $d_0 = 1$ and $d_j \geq 2$ for every $j \geq 1$, set $q_j = \prod_{i=1}^{j} d_i$. The sequence $\boldsymbol{d}$ defines a *Cantor enumeration system* since every natural number $n \in \mathbb{N}$ can be uniquely decomposed in the form $n = \sum_{j=1}^{+\infty} a_j(n) \cdot q_{j-1}$, with the digits $a_j(n)$ taking values in $\{0, \dots, d_j - 1\}$. Fix also a sequence of probabilities $\boldsymbol{p} = (p_j)_{j=1}^{+\infty}$. Then, analogously to the stochastic machine of Killeen and Taylor, one can define a stochastic machine (in base $\boldsymbol{d}$ where the information of the carry on the j-th step of the addition is lost with probability p_j). Denote by $S = S_{\boldsymbol{d},\boldsymbol{p}} = (s(m,n))_{m,n\geq 0}$ be the stochastic map of the *transition probabilities* (from m to n) of the stochastic machine, given by

$$s(m,n) = \begin{cases} (1 - p_{r+1}) \cdot \prod_{j=1}^{r} p_j, & \text{if } m = n - \sum_{j=1}^{r} (d_j - 1) \cdot q_{j-1}, \\ 1 - p_1, & \text{if } m = n, \\ \prod_{j=1}^{\zeta_n} p_j, & \text{if } m = n + 1, \\ 0, & \text{otherwise,} \end{cases}$$

where $\zeta_n = \zeta_{n,\boldsymbol{d}} := \min\{j \geq 1 : \ a_j(n) \neq d_j - 1\}$. Here, the stochastic map S can be viewed as an operator over the space l^∞. Moreover, the *fibered filled Julia set* is defined as

$$E_{\boldsymbol{d},\boldsymbol{p}} := \left\{ z \in \mathbb{C} : \ \left(\left| \bar{f}_j(z) \right| \right)_{j\geq 1} \text{ is bounded} \right\},$$

where $\bar{f}_j = f_j \circ \cdots \circ f_1$ for all $j \geq 1$ and $f_j : \mathbb{C} \mapsto \mathbb{C}$ is the map defined by $f_j(z) = (z - (1 - p_j)/p_j)^{d_j}$.

Consider the set

$$\bar{\Gamma} = \left\{ (a_j)_{j=1}^{+\infty} : \ a_j \in \{0, 1, \dots, d_j - 1\} \right\}.$$

The set $\bar{\Gamma}$ endowed with the product topology of the discrete topology of any $\{0, \dots, d_j - 1\}$, $j \geq 1$, is a metric space by *Tychonoff theorem* for compact spaces. Moreover, the set $\bar{\Gamma}$ can be viewed as the closure of $\mathbb{Z}_+$. Now, consider the Banach space

$$C\left(\bar{\Gamma}\right) \;=\; \left\{g : \bar{\Gamma} \mapsto \mathbb{C} :\; g \;\text{ is continuous}\right\}$$

endowed with the supremum norm. Since l^∞ can be identified with

$$C\left(\mathbb{Z}_+\right) = \left\{g : \mathbb{Z}_+ \mapsto \mathbb{C} :\; g \;\text{ is continuous}\right\},$$

the operator S can be extended to an operator $\bar{S}$ over $C\left(\bar{\Gamma}\right)$. Observe that when $d_j = 2$ for all $j \geq 2$, then $\bar{\Gamma} = \mathbb{Z}_2$, that is, the group of 2-adic integers.

Our main result reads as follows:

Theorem 2.7 *Let $\boldsymbol{p} = (p_i)_{i\geq 1} \in (0,1]^{\mathbb{N}}$ and $\boldsymbol{d} = (d_i)_{i\geq 1}$ be a sequence of positive integers where $d_i \geq 2$ for all $i \geq 1$. Then $\sigma\left(\bar{S}\right) = E_{d,\boldsymbol{p}}$ if $\prod_{i=1}^{+\infty} p_i = 0$ and $\sigma\left(\bar{S}\right) = \partial E_{d,\boldsymbol{p}}$ otherwise. In particular, in both cases, the set of eigenvalues $\sigma_{pt}\left(\bar{S}\right)$ is $\sigma_{pt}\left(\bar{S}\right) = \bigcup_{n=1}^{+\infty} \bar{f}_n^{-1}\{1\}$ and the residual spectrum $\sigma_r\left(\bar{S}\right)$ is $\sigma_r\left(\bar{S}\right) = \emptyset$.*

In particular, when $d_j = 2$, $p_j = p$, for every $j \in \mathbb{N}$, that is, the case corresponding to Killeen and Taylor [12] we obtain that $\sigma\left(\tilde{S}\right)$ is the filled Julia set of f given by $f(z) = \left[(z-1+p)/p\right]^2$.

The proof of Theorem 2.7 is very technical; it uses methods from spectral operator theory and exploits the topological properties of Julia sets.

3 OTHER ACTIVITIES OF THE RESEARCHER

3.1 Research Involving Other Researchers in the Project

Together with the supervising professor, we interacted with the researchers Danilo Caprio and Fernardo Lenarduzzi to exchange ideas about stable maps in zero-dimensional spaces and stochastic adding machines. In particular, at the one hand, we were able to successfully provide results concerning the existence of non-invertible stable dynamics in the space of p-adic integers $\mathbb{Z}_p$ and p-adic numbers $\mathbb{Q}_p$ and, on the other hand, to describe the Julia set of stochastic adding machines over $\mathbb{Z}_p$.

In joint work with professors Danilo Caprio (UNESP - Ilha Solteira) and Fernando Lenarduzzi (UNESP - São José do Rio Preto), we have already submitted for publication the following paper:

- D. Caprio, F. Lenarduzzi, A. Messaoudi, I. Tsokanos,
  Shadowing and Stability Properties of Non-invertible p-adic Dynamics,
  submitted for publication, p. 24, 2024, (see attachment).

In joint work with professors Danilo Caprio (UNESP - Ilha Solteira) and Glauco Valle (Universidade Federal do Rio de Janeiro), we have developed the following paper which is in the final stage of preparation and will be submitted for publication during the next months:

- D. Caprio, A. Messaoudi, I. Tsokanos, G. Valle, *Operators of Stochastic Machines and Julia Sets*, to be submitted for publication, 2024.

3.2 Giving a Mini-Course

The Postdoctoral Researcher gave a series of seminar-type lectures in the form of a mini-course titled

Diophantine Approximation and Applications

at the host university during the months of January and February of 2024. In the course, various undergraduate, master's, and PhD students and professors from our department participated. Bellow follows the summary of the course (see attachment):

The theory of *Diophantine Approximation* is the field of mathematics that deals with the quantitative analysis of the distribution of rational numbers in the real line. The fundamental theorem of *Dirichlet in Diophantine approximation* guarantees that for every irrational number $x \in \mathbb{R} \backslash \mathbb{Q}$ there exists infinitely many pairs of coprime integers $p \in \mathbb{Z}$ and $q \in \mathbb{N}$ such that

$$\left| x - \frac{p}{q} \right| \leq \frac{1}{q^2}.$$

The course will be divided into two parts. The first part of the course will cover fundamental aspects of the theory such as the introductory notions of continued fractions and bad approximability, the metric theory of Diophantine approximation, and generalizations of these concepts in higher dimensions. The second part of the talk focuses on more advanced topics in currently active research fields. To this end, Cassels' method of enclosed interval will be introduced to study its applications to the theory of Approximation by Algebraic Numbers and the Theory of Topological Games. The dimensional aspects of this study will be completed by introducing tools and concepts from the dimensional theory such as Rajchman measures and Salem sets.

3.3 Given Talks

The Postdoctoral Researcher gave various talks in seminars and conferences, as part of disseminating our research in dynamical systems and probability theory. Below, we present the complete list of the talks given by the Postdoctoral Researcher:

- *Shadowing and Stability of Dilations and Contractions in Complete Metric Spaces*, Instituto de Matemática, Estatítica e Computação Científica, Universidade Estadual de Campinas, July 2024.
- *Shadowing and Stability of Non-invertible p-adic Dynamics*, Diophantine Approximation, Fractal Geometry and Related Topics 2024, Université Gustave Eiffel, June 2024.
- *Randomization in the Josephus Problem*, Online Seminar in Diophantine Approximation and Related Topics, 2024. Link: https://perso.math.u-pem.fr/marnat.antoine/online-seminar-in-diophantine-approximation-and-related-topics.html
- *Visibility Problems in Convex Geometry*, Seminar of Mathematics 2024, Universidade Estadual Paulista, IBILCE, March 2024.

- *Randomization in the Josephus Problem*, Seminar of Mathematics 2024, Universidade Estadual Paulista, Câmpus de Ilha Solteira, March 2024.

- *Density of Oscillating Sequences in the Real Line*, XII Workshop on Dynamical Systems 2023, Universidade Estadual Paulista, November 2023.

3.4 Technical Visits

During his employment by FAPESP, the Postdoctoral Researcher made four technical visits at other institutions in the context of giving talks and working on topics of his research project "Various aspects of discrete dynamics". These activities are described in the following list.

- Technical visit at Instituto de Matemática, Universidade Federal do Rio de Janeiro (September 2023). The Postdoctoral Researcher met his collaborator Professor Glauco Valle to discuss their common project on "Operators of Stochastic Machines and Julia Sets".

- Technical visit at Departamento de Matemática, Faculdade de Engenharia - Unesp Câmpus de Ilha Solteira (March 2024). The Postdoctoral Researcher met his collaborator Professor Danilo Caprio to discuss their common project on "Shadowing and Stability Properties of Non-Invertible p-adic Dynamics". In the context of his visit, the Postdoctoral Researcher was invited to give a talk at the Department of Mathematics about "Randomization in the Josephus Problem".

- Conference at Université Gustave Eiffel, Paris, France, and technical visit at Université Paris-Est Créteil (UPEC), France (June 2024). The Postdoctoral Researcher was invited to present his work on "Shadowing and Stability Properties of Non-Invertible p-adic Dynamics" at the conference "Diophantine Approximation, Fractal Geometry and Related Topics", where he gave a talk. Following the event, the Postdoctoral Researcher was invited for a technical visit at Université Paris-Est Créteil (UPEC), France, by his collaborator Professor Faustin Adiceam. During his technical visit, the Postdoctoral Researcher discussed problems and future directions related to his research project "Various aspects of discrete dynamics".

- Technical visit and talk at Instituto de Matemática, Estatítica e Computação Científica, Universidade Estadual de Campinas (July 2024). The Postdoctoral Researcher met his collaborator Professor Régis Verão to discuss future projects related to "Various aspects of discrete dynamics". In the context of his visit, the Postdoctoral Researcher was invited to give a talk to the PhD students of the hosting department on "Shadowing and Stability Properties of Dilations and Contractions in Complete Metric Spaces".

4 BENEFITS OF THE COLLABORATION

During the employment of the Postdoctoral Researcher, various benefits were obtained:

1. We resolved various problems in the stability theory of Dynamical Systems which are written in the form of two papers, one already submitted [7] and the other in the final stage of writing [16].

2. We resolved problems in probability theory, written in the form of the paper [1] which has already been accepted for publication.

3. We resolved problems concerning the connections between the operators of stochastic adding machines and the geometric structure of Julia sets, written in the form of the paper [7] which is in the final stage of writing.

4. We obtained various conjectures that will stimulate the development of research work in the future.

5. The Postdoctoral Researcher interacted with researchers from major Brazilian and international institutes during the presentation of its work in domestic and international seminars and conferences.

6. The Postdoctoral Researcher participated in the series of summer mini-courses of our institution by organizing and teaching a mini-course on Diophantine Approximation.

7. The Postdoctoral Researcher was invited to present his joint work with his supervisor Professor Ali Messaoudi at the international conference *Diophantine Approximation, Fractal Geometry and Related Topics 2024* that took place at Université Gustave Eiffel, Paris, France.

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