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The Dynamics of a One-Dimensional Parabolic Problem versus the Dynamics of Its Discretization

Simone M. Bruschi[1]

Departamento de Matemática, IGCE-UNESP, Cx. Postal 178, 13506-700 Rio Claro, SP, Brazil
E-mail: sbruschi@ms.rc.unesp.br

Alexandre N. Carvalho[2]

Departamento de Matemática, ICMC-USP, Cx. Postal 668, 13560-970 São Carlos, SP, Brazil
E-mail: andcarva@icmc.sc.usp.br

and

José G. Ruas-Filho[3]

Departamento de Matemática, ICMC-USP, Cx. Postal 668, 13560-970 São Carlos, SP, Brazil
E-mail: jgrfilho@icmc.sc.usp.br

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In this paper we prove that the spatial discretization of a one dimensional system of parabolic equations, with suitably small step size, contains exactly the same asymptotic dynamics as the continuous problem.

Key Words: parabolic equations; spatial discretization; topological equivalence of attractors.

INTRODUCTION

In order to keep the presentation simple we will consider the scalar case and later point out that the changes needed when considering systems of one dimensional parabolic equations. Consider the following one dimensional scalar parabolic problem

$$\begin{aligned} &u_t = a\, u_{xx} + f(u), \qquad 0 < x < 1, \qquad t > 0 \\ &u_x(0) = u_x(1) = 0, \qquad t > 0, \end{aligned} \tag{0.1}$$

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where $a>0$ and $f: \mathbb{R} \to \mathbb{R}$ is a C^2 function satisfying the dissipativeness condition

$$f(u)\,u<0, \qquad |u|>\xi,$$

for some $\xi>0$. Also, consider the semi-implicit discretization of (0.1) with p equally spaced steps

$$\dot{U} = -aLU + \mathrm{f}(U), \tag{0.2}$$

where L is the $p \times p$ matrix given by

$$L = p^2 \begin{bmatrix} 1 & -1 & 0 & \cdots & 0 & 0 & 0 \\ -1 & 2 & -1 & \cdots & 0 & 0 & 0 \\ 0 & -1 & 2 & \cdots & 0 & 0 & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots & \vdots \\ 0 & 0 & 0 & \cdots & 2 & -1 & 0 \\ 0 & 0 & 0 & \cdots & -1 & 2 & -1 \\ 0 & 0 & 0 & \cdots & 0 & -1 & 1 \end{bmatrix}, \tag{0.3}$$

$$\mathrm{f}(U) = (f(u_1), ..., f(u_p))^\top \qquad \text{and} \qquad U = (u_1, \cdots, u_p)^\top$$

Under the above assumptions on f we have the existence of a global attractor $\mathscr{A}$ for (0.1) and a global attractor $\mathscr{A}_p$ for (0.2).

The aim of this work is to show that the asymptotic dynamics of the two equations above are topologically equivalent for a sufficiently large p; that is, for sufficiently small step size.

In order to illustrate the differences that may arise between the dynamics of (0.1) and (0.2) we consider the case $p=2$ in (0.2); that is if we write, $x_1 = 1/4$, $x_2 = 3/4$ and denote by $u_1(t) = u(x_1, t)$ and $u_2(t) = u(x_2, t)$, then we have (already with the boundary conditions incorporated) the following equation:

$$\begin{aligned} \dot{u}_1 &= -4a(u_1 - u_2) + f(u_1), \\ \dot{u}_2 &= 4a(u_1 - u_2) + f(u_2). \end{aligned} \tag{0.4}$$

Take $f(u) = u - u^3$. We observe that for any value of a the equation (0.4) has at most nine equilibrium points whereas the problem (0.1) for small values of a may have any number of equilibrium points (see, [CI]). Besides this, for $4a < 1/3$ we have the existence of equilibrium points for (0.4) which are stable and of the form $U = (u_1, u_2)$ where $u_1 \neq u_2$. If the dynamics of (0.4) were equivalent to the dynamics of (0.1) the equilibrium point U would correspond to a stable, nonconstant equilibrium point for (0.1); that is, a pattern. It is well known (see [Ch, CH]) that patterns do not exist for the problem (0.1). That way even for values of a not so small the dynamics of the discretized equation may differ significantly from that of the continuous problem. This has been pointed out previously in [Ro]. Of course a similar reasoning could be carried out for larger values of p the advantage of $p=2$ is the possibility of computing all the equilibrium points of (0.4) which gives a complete picture of its attractor.

It has been shown in ([CP]) that, for any p given, there is a function $a(\cdot)$ such that the dynamics of (0.2) is equivalent to the dynamics of the problem

$$\begin{aligned} u_t &= (a(x)\,u_x)_x + f(u), \qquad 0<x<1, \qquad t>0 \\ u_x(0) &= u_x(1)=0, \qquad t>0 \end{aligned} \tag{0.5}$$

After having presented the problems that may arise when comparing the dynamics of (0.1) and (0.2) we are ready to state the main result of this paper.

THEOREM 0.1. *For p large enough, there is a homeomorphism* $H\colon \mathscr{A} \to \mathscr{A}_p$ *which maps orbits onto orbits preserving time direction.*

The proof of this result requires us to embed the discrete problem into the setting of the continuous problem. Since the continuous problem is infinite dimensional the first task is to reduce it to a problem on a finite dimensional space. That is accomplished through the invariant manifold theorem. Unfortunately if we consider the continuous problem on a fixed finite dimensional invariant manifold of dimension n and consider the discretization with stepsize n^{-1}, we are not able to prove that the vector fields of the continuous and discrete problem with same dimension are close (due to the fact that the eigenvalues of the L and of the 1-d Neumann Laplacian are not uniformly close). Keeping the continuous problem on a fixed manifold, the proximity of the vector fields (on the part that concerns L and the projected one dimensional Neumann Laplacian) will come when the step size is very small and therefore the dimension of the discrete problem will now exceed the dimension for the continuous problem. We could now project the discrete problem onto an invariant manifold with same dimension as that of the continuous problem. That takes care of the convergence of the part of the vector field coming from the projection of the Laplacian and the projection of L but then we need to study the convergence of the nonlinearities projected on the invariant manifolds. For that we need the convergence of the invariant manifolds which leads to technical complications.

Our approach is to allow the dimensions of both invariant manifolds to increase in such a way that the invariant manifolds are both very flat and therefore very close to one another in the C^1 topology. Then since we already know that the attractors are all containded in submanifolds of fixed dimension we have that vector fields are also C^1 close in these submanifolds and the topological equivalence is a consequence of the structural stablility for the continuous problem.

Let us now consider the slightly more general situation (0.5) with a being a strictly positive $C^1((0,1),\mathbb{R})\cap C([0,1],\mathbb{R})$ function. With a change of variables (0.5) can be converted into

$$\begin{aligned} &u_s = u_{\xi\xi} + \tilde{a}(\xi)\, f(u), \qquad 0<\xi<L, \qquad s>0 \\ &u_\xi(0) = u_\xi(1) = 0, \qquad s>0, \end{aligned} \tag{0.6}$$

where

$$s = (a(x))^{-1}\, t, \qquad \xi = \int_0^x \frac{1}{a(s)}\, ds$$

and $\tilde{a}(\xi) = a(x(\xi))$, $L = \int_0^1 \frac{1}{a(s)}\, ds$. This leads us to study only the case (0.1) possibly with f also depending on the space variable.

All the results proved here are for the case when the nonlinearity f depends only upon the unknown u. More general situations like the case when the function f also depend on the space variable and on the dispersion can be obtained in a similar fashion. The assumptions required for these more general situations and the Dirichlet boundary condition case can be found in Section 5.

There has been several works in the literature where part of the results presented here have been announced. Among them we cite [Ha, FR]. To our knowledgement there is no rigorous proof of such results in the literature.

This paper is organized as follows. In Section 1 we present the discretized problem obtaining uniform bounds for the attractors $\mathscr{A}_p$ and the existence of an exponentially attracting invariant manifold for it. In Section 2 present the continuous problem obtaining uniform bounds for the attractor $\mathscr{A}$ and the existence of an exponentially attracting invariant manifold for it. In Section 3 it is proved that a certain set of eigenvalues and eigenfunctions of the discrete problem converges uniformly to the eigenvalues and eigenfunctions of the continuous problem and use these facts to compare the vector fields of the discrete and continuous problem on the invariant manifolds. In Section 5 we make several comments on possible extensions of the results. Finally, in the Appendix, we prove a theorem on existence of exponentially attracting invariant manifolds that deals with changing spaces and dimensions.

1. DISCRETIZATION

Firstly we discuss the spatial discretization of (0.1), for that consider the points $x_j = \frac{j-1/2}{p}$, $j=1, ..., p$ and denote $u_j(t) = u(x_j, t)$. Then, we have

$$
\begin{aligned}
\dot{u}_1 &= p^2(u_2 - u_1) + f(u_1), \\
\dot{u}_j &= p^2(u_{j-1} - 2u_j + u_{j+1}) + f(u_j), \quad j = 2, ..., p-1 \\
\dot{u}_p &= p^2(u_{p-1} - u_p) + f(u_p)
\end{aligned}
\tag{1.1}
$$

Observe that the boundary conditions have changed to $u_1 = u_0$, $u_{p+1} = u_p$ and have been incorporated to the linear operator L.

Denoting $U = (u_1, ..., u_p)^\top$ and rewriting the above equation in a matrix form, we obtain

$$
\dot{U} = -LU + \mathrm{f}(U), \tag{1.2}
$$

where L is a $p \times p$ matrix given by (0.2) and $\mathrm{f}(U) = (f(u_1), ..., f(u_p))^\top$.

We observe that the system (1.2) is generically Morse–Smale (see [FO]).

By the conditions imposed on f, the above problem has a global attractor $\mathscr{A}_p$ that satisfies

$$
\mathscr{A}_p \subset R^p_\xi, \tag{1.3}
$$

where $R^p_\xi = \{v \in \mathbb{R}^p;\ |v_i| \leqslant \xi,\ 1 \leqslant i \leqslant p\}$, see [CDR].

Since we are interested on studying the solutions of the above problem in the attractor only we may cut the nonlinearity in such a way that f is bounded with bounded first and second derivatives.

Theorem 1.1. *The eigenvalues of L are given by* $\lambda^p_k = 4p^2 \sin^2 \frac{k\pi}{2p}$ *and the associated eigenvectors are* $w^p_k = (\cos k\pi x_1, ..., \cos k\pi x_p)$ *for* $k = 0, ..., p-1$. *Besides that,* $\lim_{p \to \infty} \lambda^p_k = \lambda_k$, *where* $\lambda_k = (k\pi)^2$ *is the* $(k+1)$*th eigenvector of the operator* $-\Delta$ *with homogeneous Neumann boundary condition.*

To be able to compare the dynamics of the discrete problem with the dynamics of the continuous problem we must assign to $\mathbb{R}^p$ a norm which is compatible with the norm adopted for the continuous problem. That leads us to define in $\mathbb{R}^p$ the inner product: $\langle x, y \rangle = \frac{1}{p} \sum_{i=1}^p x_i y_i$, which is inherited from the L^2 inner product and will be referred as discretized L^2 inner product. Normalizing w^p_k according to this inner product we obtain:

$$
v^p_k = \frac{w^p_k}{\|w^p_k\|} = \frac{(\cos k\pi x_1, ..., \cos k\pi x_p)}{\sqrt{\frac{1}{p} \sum_{i=1}^p \cos^2 k\pi x_i}}.
$$

If we write

$$v_k^p(x) = \frac{\sum_{i=1}^{p} \cos k\pi x_i \chi_{I_i}(x)}{\sqrt{\frac{1}{p} \sum_{i=1}^{p} \cos^2 k\pi x_i}};$$

where we denote by I_i the interval $[\frac{i-1}{p}, \frac{i}{p}]$, we obtain that $v_k^p(x) \in L^\infty(0, 1)$ and $\|v_k^p(x) - \sqrt{2} \cos k\pi x\|_\infty \to 0$ when $p \to \infty$.

Consider a base of eigenvectors v_k^p, $0 \leqslant k \leqslant p-1$, in $\mathbb{R}^p$. This basis is orthonormal with respect to the inner product previously described. We consider the discretized equation in this new coordinates, that is, if we write: $v_1 = \langle U, v_0^p \rangle, ..., v_p = \langle U, v_{p-1}^p \rangle$ and $v = (v_1, ..., v_p)$ we obtain:

$$\dot{v} = -\tilde{L}v + F(v), \tag{1.4}$$

where $\tilde{L}$ is the $p \times p$ matrix given by $\tilde{L} = \operatorname{diag}(\lambda_0^p, ..., \lambda_{p-1}^p)$ and $F(v) = (F_1(v), ..., F_p(v))^\top$ with each $F_j(v)$ given by

$$F_j(v) = \langle \mathrm{f}(U), v_{j-1}^p \rangle = \sum_{k=1}^{p} \frac{1}{p} v_{j-1\,k}^p f(v_{0k}^p v_1 + \cdots + v_{(p-1)\,k}^p v_p), \tag{1.5}$$

where v_{jk}^p denotes the k-th coordinate of v_j^p. We denote the matrix of change of basis by Z; it is given by $z_{kj} = v_{(j-1)\,k}^p$ and the matrix Z^{-1} is given by $(1/p)\, Z^\top$.

Now, we consider a discretization with $p = n^3$ points and consider the following decomposition of $\mathbb{R}^{n^3} = \mathbb{R}^n \oplus \mathbb{R}^{n^3 - n}$ where $\mathbb{R}^n = \operatorname{span}[v_0^{n^3}, ..., v_{n-1}^{n^3}]$ and $\mathbb{R}^{n^3 - n} = \operatorname{span}[v_n^{n^3}, ..., v_{n^3-1}^{n^3}]$, with this decomposition we obtain the following weakly coupled system

$$\begin{aligned} \dot{v}_n + \tilde{B}_n v_n &= \tilde{g}_n(v_n, w_n) \\ \dot{w}_n + \tilde{A}_n w_n &= \tilde{f}_n(v_n, w_n), \end{aligned} \tag{1.6}$$

where $\tilde{B}_n$ is the $n \times n$ diagonal matrix given by $\tilde{B}_n = \operatorname{diag}(\lambda_0^{n^3}, ..., \lambda_{n-1}^{n^3})$, $\tilde{A}_n$ is the $(n^3 - n) \times (n^3 - n)$ diagonal matrix given by $\tilde{A}_n = \operatorname{diag}(\lambda_n^{n^3}, ..., \lambda_{n^3-1}^{n^3})$, $\tilde{g}_n(v_n, w_n) = (F_1(v_n, w_n), ..., F_n(v_n, w_n))^\top$ and $\tilde{f}_n(v_n, w_n) = (F_{(n+1)}(v_n, w_n), ..., F_{n^3}(v_n, w_n))^\top$.

For the weakly coupled system (1.8) we show that there exists an exponentially attracting n-dimensional invariant manifold; that is, the following holds:

THEOREM 1.2. *Let f be twice continuously differentiable, bounded with bounded first and second derivatives; then, the problem* (1.6) *for n sufficiently large, possess a invariant manifold*

$$M_n = \{(v_n, w_n) \in \mathbb{R}^{n^3} \mid w_n = \tilde{\sigma}_n(v_n)\},$$

which is exponentially attracting, where $\tilde{\sigma}_n$ is a smooth function, $\tilde{\sigma}_n\colon \mathbb{R}^n \to \mathbb{R}^{n^3-n}$ and the flux on M_n is given by $u(t) = v_n(t) + \tilde{\sigma}_n(v_n(t))$ where $v_n(t)$ is solution of

$$\dot{v}_n + \tilde{B}_n v_n = \tilde{g}_n(v_n, \tilde{\sigma}_n(v_n)). \tag{1.7}$$

To prove this theorem we use the following result. This result is also used to prove the proximity of the vector fields of the continuous and discrete problems after they are projected on their invariant manifolds.

LEMMA 1.3. *Let and X_n, Y_n be a sequence Banach spaces, $A_n\colon D(A_n) \subset X_n \to X_n$ be a sequence of sectorial operators and $B_n\colon D(B_n) \subset Y_n \to Y_n$ be a sequence of generators of C^0-groups of bounded linear operators. Suppose that $f_n\colon X_n^\alpha \times Y_n^\alpha \to X_n$ and $g_n\colon X_n^\alpha \times Y_n^\alpha \to Y_n$ are a sequence of functions satisfying*:

$$\|f_n(x, y) - f_n(z, w)\|_{X_n} \leqslant L_f(\|x - z\|_{X_n^\alpha} + \|y - w\|_{Y_n^\alpha}),$$
$$\|f_n(x, y)\|_{X_n} \leqslant N_f,$$

for every $(x, y), (z, w)$ in $X_n^\alpha \times Y_n^\alpha$ and

$$\|g_n(x, y) - g_n(z, w)\|_{Y_n} \leqslant L_g(\|x - z\|_{X_n^\alpha} + \|y - w\|_{Y_n^\alpha}),$$
$$\|g_n(x, y)\|_{Y_n} \leqslant N_g,$$

for every $(x, y), (z, w)$ in $X_n^\alpha \times Y_n^\alpha$. Assume that

$$\|e^{-A_n t} w\|_{X_n^\alpha} \leqslant M_a e^{-\beta(n)\, t}\, \|w\|_{X_n^\alpha}, \qquad t \geqslant 0$$
$$\|e^{-A_n t} w\|_{X_n^\alpha} \leqslant M_a t^{-\alpha} e^{-\beta(n)\, t}\, \|w\|_{X_n}, \qquad t > 0,$$
$$\|e^{-B_n t} z\|_{Y_n^\alpha} = \|e^{B_n(-t)} z\|_{Y_n^\alpha} \leqslant M_b e^{-\rho(n)\, t}\, \|z\|_{Y_n^\alpha}, \qquad t \leqslant 0,$$
$$\|e^{-B_n t} z\|_{Y_n^\alpha} \leqslant M_b (-t)^{-\alpha}\, e^{-\rho(n)\, t}\, \|z\|_{Y_n}, \qquad t < 0,$$

for any $w \in X_n^\alpha$ and $z \in Y_n$, where $\beta(n) - \rho(n) \to +\infty$ as $n \to \infty$. Consider the weakly coupled system

$$\begin{cases} \dot{x} = -A_n x + f_n(x, y), \\ \dot{y} = -B_n y + g_n(x, y). \end{cases} \tag{1.8}$$

Then, for n large enough, there is an exponentially attracting invariant manifold for (1.8)

$$S = \{(x, y) : x = \sigma_n(y),\ y \in Y_n^\alpha\},$$

where $\sigma_n\colon Y_n^\alpha \to X_n^\alpha$ *satisfies*

$$s(n) = \sup_{\{y \in Y_n^\alpha\}} \|\sigma_n(y)\|_{X_n^\alpha},$$

$$\|\sigma_n(y) - \sigma_n(z)\|_{X_n^\alpha} \leqslant l(n)\ \|y - z\|_{Y_n^\alpha},$$

with $s(n), l(n) \to 0$ *when* $n \to \infty$. *If* f_n, g_n *are smooth; then,* σ_n *is smooth and its derivative* $D\sigma_n$ *satisfy*

$$\sup_{y \in Y_n} \|D\sigma_n(y)\|_{L(Y_n^\alpha, X_n^\alpha)} \leqslant l(n).$$

The proof of this result can be found in the appendix.

Proof of Theorem 1.2. Making $\alpha = 0$ in the previous lemma we have: $Y_n \times X_n$ where $Y_n = \mathbb{R}^n$ and $X_n = \mathbb{R}^{n^3 - n}$, $\tilde{g}_n\colon Y_n \times X_n \to Y_n$ and $\tilde{f}_n\colon Y_n \times X_n \to X_n$. We make the following distinction relatively to the several norms used here, when the index of the norm is X_n or Y_n we are using the base of eigenvectors and that way the norm is given by $\|\cdot\| = (\sum_i x_i^2)^{1/2}$, when the index of the norm is $\mathbb{R}^k$ we are using the canonical basis and the norm given by the L^2 discretized inner product. Firstly we compute the needed estimates on $\tilde{f}_n$ and $\tilde{g}_n$

$$\begin{aligned}
\|\tilde{g}_n(v_n, w_n)\|_{Y_n} &= \left(\sum_{i=1}^{n} (F_i(v_n, w_n))^2\right)^{1/2} \\
&\leqslant \|\mathrm{f}(Z(v_n, w_n)^\top)\|_{\mathbb{R}^{n^3}} = \left(\sum_{i=1}^{n^3} \frac{1}{n^3} (\mathrm{f}_i(Z(v_n, w_n)^\top))^2\right)^{1/2} \\
&= \left(\sum_{i=1}^{n^3} \frac{1}{n^3} (f([Z(v_n, w_n)^\top]_i))^2\right)^{1/2} \\
&\leqslant \left(\sum_{i=1}^{n^3} \frac{1}{n^3} \|f\|_\infty{}^2\right)^{1/2} = \|f\|_\infty.
\end{aligned}$$

Similarly, we obtain the estimate

$$\|\tilde{f}_n(v_n, w_n)\|_{X_n} \leqslant \|f\|_\infty. \tag{1.9}$$

For the Lipschitz constants we have

$$
\begin{aligned}
&\|\tilde{g}_n(v_n, w_n) - \tilde{g}_n(z_n, u_n)\|_{Y_n} \\
&\quad = \left(\sum_{i=1}^{n} (F_i(v_n, w_n) - F_i(z_n, u_n))^2 \right)^{1/2} \\
&\quad \leqslant \|\mathrm{f}(Z(v_n, w_n)^\top) - \mathrm{f}(Z(z_n, u_n)^\top)\|_{\mathbb{R}^{n3}} \\
&\quad = \left(\sum_{i=1}^{n^3} \frac{1}{n^3} (f([Z(v_n, w_n)^\top]_i) - f([Z(z_n, u_n)^\top]_i))^2 \right)^{1/2} \\
&\quad \leqslant \left(\sum_{i=1}^{n^3} \frac{1}{n^3} L_f^2([Z(v_n, w_n)^\top]_i - [Z(z_n, u_n)^\top]_i)^2 \right)^{1/2} \\
&\quad = \left(\sum_{i=1}^{n^3} \frac{1}{n^3} L_f^2([Z(v_n - z_n, w_n - u_n)^\top]_i)^2 \right)^{1/2} \\
&\quad = L_f \|Z((v_n - z_n), (w_n - u_n))^\top\|_{\mathbb{R}^{n3}} \\
&\quad \leqslant \sqrt{2}\, L_f(\|v_n - z_n\|_{Y_n} + \|w_n - u_n\|_{X_n}),
\end{aligned}
$$

where L_f is the Lipschitz constant of the function f.

In the same way we obtain the estimative for the Lipschitz constant of f_n. All the constants are uniform in n.

The constants $\beta(n)$ and $\rho(n)$ are: $\beta(n) = \lambda_n^{n^3}$ and $\rho(n) = \lambda_{n-1}^{n^3}$. That gives us that $\beta(n) \sim n^2$ and $\rho(n) \sim (n-1)^2$ as $n \to \infty$ and this gives us that $\beta(n) - \rho(n) \to \infty$ as $n \to \infty$.

2. THE CONTINUOUS PROBLEM

We now turn to the problem (0.1). Let $X = L^2(0, 1)$, we define f^e: $H^1(0, 1) \subset X \to X$ by $f^e(\phi)(x) = f(\phi(x))$ and we define:

$$
\begin{aligned}
&A\colon D(A) \subset L^2(0, 1) \to L^2(0, 1) \\
&\quad D(A) = H_N^2(0, 1) = \{\phi \in H^2(0, 1)\colon \phi'(0) = \phi'(1) = 0\} \\
&\quad A\phi = -\phi''.
\end{aligned}
$$

That way, we rewrite the problem (0.1) as:

$$
\begin{aligned}
&\frac{d}{dt} u + Au = f^e(u), \\
&u(0) = u_0
\end{aligned}
\tag{2.1}
$$

By the conditions imposed on f we obtain that f^e is Lipschitz continuous in bounded subsets of $H^1(0, 1)$. Then, the above problem has a global attractor $\mathscr{A}$ that satisfies

$$\sup_{u \in \mathscr{A}} \sup_{x \in [0,1]} |u(x)| \leqslant \xi. \tag{2.2}$$

The above bound allow us to cut (without changing the attractor) the nonlinearity f in such a way that it becomes bounded and has bounded first and second derivative. Also after cutting the nonlinearity we may pose the problem in $L^2(0, 1)$ keeping the same attractor. Here after we assume that f is bounded and has first and second derivative we also assume that the problem is posed in $L^2(0, 1)$.

Let $\lambda_0 < \lambda_1 < \lambda_2 < \cdots$ be the sequence of eigenvalues of A, where $\lambda_k = (k\pi)^2$ and $\phi_0, \phi_1, \phi_2, \ldots$ a corresponding sequence of normalized eigenfunctions, $\phi_k(x) = \sqrt{2}\cos(k\pi x)$.

Now consider the following decomposition of $X = W \oplus W^\perp$ where

$$\begin{aligned} W &= \operatorname{span}[\phi_0, \phi_1, \ldots, \phi_{n-1}] \\ W^\perp &= \{\phi \in X \colon \langle \phi, w \rangle = 0, \forall w \in W\}, \end{aligned} \tag{2.3}$$

where $\langle \cdot, \cdot \rangle$ is the inner product of $L^2(0, 1)$.

Then, $u \in L^2(0, 1)$ can be written as

$$u = v_1 \phi_0 + v_2 \phi_1 + \cdots + v_n \phi_{n-1} + w,$$

where

$$v_i = \int_0^1 u(x)\, \phi_{i-1}(x)\, dx, \qquad i = 1, \ldots, n$$

$$w = u - \sum_{i=1}^{n} v_i \phi_{i-1}$$

Let u be a solution of (2.1); then, for each t, we can write

$$u(t, x) = v_1(t)\, \phi_0(x) + v_2(t)\, \phi_1(x) + \cdots + v_n(t)\, \phi_{n-1}(x) + w(t, x) \tag{2.4}$$

and

$$\begin{aligned} \dot{v}_i &= -\lambda_{i-1} v_i + \langle f(u), \phi_{i-1} \rangle \\ w_t + A_n w &= f(u) - \sum_{i=0}^{n-1} \langle f(u), \phi_i \rangle\, \phi_i, \end{aligned}$$

where A_n denotes $A|_{D(A) \cap W^\perp}$.

Writing $v=(v_1, v_2, ..., v_n)$, $u=(v, w)$ and B_n a $n \times n$ diagonal matrix $B_n = \mathrm{diag}(\lambda_0, \lambda_1, ..., \lambda_{n-1})$ we obtain the following system

$$\begin{aligned} \dot{v} + B_n v &= g_n(v, w) \\ w_t + A_n w &= f_n(v, w), \end{aligned} \tag{2.5}$$

where $g_n(v, w) = (\langle f(v, w), \phi_0 \rangle, ..., \langle f(v, w), \phi_{n-1} \rangle)^\top$ and $f_n(v, w) = f(v, w) - \sum_{i=0}^{n-1} \langle f(v, w), \phi_i \rangle \phi_i$.

THEOREM 2.1. *Let $f \in C^2(R, R)$ be bounded with bounded first and second derivatives; then, for sufficiently large n there exists an exponentially attracting, smooth invariant manifold S_n for* (2.5). *The flux on S_n is given by: $u(t, x) = (v(t), \sigma_n(v(t)))$ where v is solution of*

$$\dot{v} + B_n v = g_n(v, \sigma_n(v)) \tag{2.6}$$

Proof. Let L_{f^e} be Lipschitz constant of f^e and $N_{f^e} = \|f\|_\infty$. Take $L_f = L_g = L_{f^e}$, $N_f = N_g = N_{f^e}$, $\beta(n) = \lambda_n$, $\rho(n) = \lambda_{n-1}$ and observe that $\beta(n) - \rho(n) = \pi^2(2n+1)$. The theorem follows form Lemma 1.3.

3. UNIFORM SPECTRAL CONVERGENCE

To compare the asymptotic dynamics of the discretized problem with the asymptotic dynamics of the continuous problem we project the first on the invariant manifold $\tilde{\sigma}_n$ and the second on the invariant manifold σ_n, only after that we are able to compare their asymptotic dynamics. This is accomplished comparing the vector fields (now with same finite dimension). To compare the vector fields we need to obtain a way of comparing $\widetilde{B}_n$ and B_n. That is achieved if we prove the uniform (with respect to n) convergence of the eigenvalues and eigenfunctions of $\widetilde{B}_n$ to the eigenvalues and eigenfunctions of B_n as $n \to \infty$.

Another way to compare the vector fields would be projecting both problems on fixed invariant manifolds of same dimension and then to study the convergence of the vector fields. That would involve studying the convergence of the invariant manifolds and would lead to unnecessary technical complications. This approach has the clear property that the eigenvalues and eigenfunctions (a fixed number) converge uniformly. Here we exploit the fact that for large values of n the invariant manifolds have a very small C^1 norm and therefore we can simply neglect them; on the other hand one needs to be careful in order to guarantee the uniform convergence of eigenvalues and eigenfunctions. That is the reason why we make the cut between the $n-$th and $(n+1)$th eigenvalue with the discrete problem having n^3 eigenvalues.

If we consider the matrix L with $p=n^3$ we have n^3 simple eigenvalues and orthonormal eigenfunctions for L. Of course these eigenvalues and eigenfunctions do not converge uniformly to the first n^3 eigenvalues and eigenfunctions of the Neumann Laplacian as $n \to \infty$. It is also clear that any finite subset of eigenvalues of L converge uniformly to the corresponding eigenvalues of the Neumann Laplacian as $n \to \infty$ and in fact more is true. The first n eigenvalues and eigenfunctions of the $n^3 \times n^3$ matrix L will converge uniformly to the first n eigenvalues and eigenfunctions of the Neumann Laplacian as $n \to \infty$. That is what we prove in Subsections 3.1 and 3.2.

3.1. *Uniform Convergence of Eigenvalues*

The eigenvalues of the operators $\widetilde{B}_n$ and B_n are respectively $\lambda_k^{n^3}$ and λ_k, with $k=0, ..., n-1$. In this case we have that

$$|\lambda_k^{n^3} - \lambda_k| = (k\pi)^2 \left| \left(\frac{\sin \frac{k\pi}{2n^3}}{\frac{k\pi}{2n^3}} \right)^2 - 1 \right|.$$

Using the power series expansion of the function sin we obtain

$$|\lambda_k^{n^3} - \lambda_k| \leqslant (k\pi)^2 \left| \frac{2}{3!} \left(\frac{k\pi}{2n^3} \right)^2 + o\left(\left(\frac{k\pi}{2n^3} \right)^3 \right) \right|.$$

That way we have that for k, $k=0, ..., n-1$

$$|\lambda_k^{n^3} - \lambda_k| \leqslant (n\pi)^2 \left| \frac{2}{3!} \left(\frac{\pi}{2n^2} \right)^2 + o\left(\left(\frac{1}{n^2} \right)^3 \right) \right| = o\left(\frac{1}{n} \right)$$

so $|\lambda_k^{n^3} - \lambda_k| \to 0$ for all $k=0, ..., n-1$ uniformly, as $n \to \infty$.

3.2. *Uniform Convergence of Eigenfunctions*

We will show that $\|v_k^{n^3}(x) - \sqrt{2}\cos(k\pi x)\|_\infty \leqslant \varepsilon$ for sufficiently large n and for all $k=0, ..., n-1$. First we consider $|\cos(k\pi x) - \cos(k\pi x_j)|$ for $x \in [\frac{j-1}{n^3}, \frac{j}{n^3}]$.

For $x \in [\frac{j-1}{n^3}, \frac{j}{n^3}]$ we have that

$$|\cos(k\pi x) - \cos(k\pi x_j)| \leqslant k\pi \frac{1}{2n^3}.$$

For all $k=0, \ldots, n-1$ and for all $j=1, \ldots, n^3$ we have

$$|\cos(k\pi x)-\cos(k\pi x_j)| \leqslant \frac{\pi}{2n^2}$$

and hence, $\|v_k^{n^3}(x)-\sqrt{2}\cos(k\pi x)\|_\infty \leqslant \frac{c}{n^2}$, for some constant c.

4. COMPARISON OF THE VECTOR FIELDS

Now we show the proximity of the vector fields. Denote by $\tilde{g}_n^i$ and g_n^i the ith coordinate function of $g_n^{\sim}$ and g_n respectively. Then, we have:

$$
\begin{aligned}
&|g_n^j(v, 0)-\tilde{g}_n^j(v, 0)| \\
&\quad = \left| \sum_{k=1}^{n^3} \frac{1}{n^3} v_{(j-1)k}^{n^3} f\left(\sum_{l=1}^{n} v_{(l-1)k}^{n^3} v_l\right) \right. \\
&\qquad \left. -\int_0^1 f\left(\sum_{l=1}^{n} v_l \sqrt{2}\cos(l-1)\pi x\right) \sqrt{2}\cos(j-1)\pi x\, dx \right| \\
&\quad \leqslant \left| \sum_{k=1}^{n^3} \int_{(k-1)/n^3}^{k/n^3} \left(v_{(j-1)k}^{n^3} f\left(\sum_{l=1}^{n} v_{(l-1)k}^{n^3} v_l\right)\right.\right. \\
&\qquad \left.\left. -f\left(\sum_{l=1}^{n} v_l \sqrt{2}\cos(l-1)\pi x\right) \sqrt{2}\cos(j-1)\pi x\right) dx \right| \\
&\quad \leqslant \left| \sum_{k=1}^{n^3} \int_{(k-1)/n^3}^{k/n^3} f\left(\sum_{l=1}^{n} v_{(l-1)k}^{n^3} v_l\right) (v_{(j-1)k}^{n^3}-\sqrt{2}\cos(j-1)\pi x)\, dx \right| \\
&\qquad + \left| \sum_{k=1}^{n^3} \int_{k-1n^3}^{kn^3} \left(f\left(\sum_{l=1}^{n} v_{(l-1)k}^{n^3} v_l\right)\right.\right. \\
&\qquad \left.\left. -f\left(\sum_{l=1}^{n} v_l \sqrt{2}\cos(l-1)\pi x\right)\right) \sqrt{2}\cos((j-1)\pi x)\, dx \right| \\
&\quad \leqslant \sum_{k=1}^{n^3} \int_{(k-1)/n^3}^{k/n^3} \left| f\left(\sum_{l=1}^{n} v_{(l-1)k}^{n^3} v_l\right)\right| |(v_{(j-1)k}^{n^3}-\sqrt{2}\cos(j-1)\pi x)|\, dx \\
&\qquad + \sum_{k=1}^{n^3} \int_{(k-1)/n^3}^{k/n^3} \left| f\left(\sum_{l=1}^{n} v_{(l-1)k}^{n^3} v_l\right)\right. \\
&\qquad \left. -f\left(\sum_{l=1}^{n} v_l \sqrt{2}\cos(l-1)\pi x\right)\right| |\sqrt{2}\cos((j-1)\pi x)|\, dx \\
&\quad \leqslant \|f\|_\infty \frac{c}{n^2}+L_f \frac{c\sqrt{2}}{n^2} \sum_{l=1}^{n} |v_l|.
\end{aligned}
$$

Therefore,

$$
\begin{aligned}
&\| g_n(v, \sigma_n(v)) - \tilde{g}_n(v, \tilde{\sigma}_n(v)) \|_{Y_n} \\
&\qquad \leqslant \| g_n(v, \sigma_n(v)) - g_n(v, 0) \|_{Y_n} + \| g_n(v, 0) - \tilde{g}_n(v, 0) \|_{Y_n} \\
&\qquad\quad + \| \tilde{g}_n(v, 0) - \tilde{g}_n(v, \tilde{\sigma}_n(v)) \|_{Y_n} \\
&\qquad \leqslant L_f \|\sigma_n(v)\| + L_f \|\tilde{\sigma}_n(v)\| + \frac{1}{\sqrt{n}} \left(\|f\|_\infty \frac{c}{n} + L_f \xi c \sqrt{2} \right),
\end{aligned}
$$

where ξ is as in (1.3).

Since, by Lemma 1.3 $\|\sigma_n\| \to 0$ and $\|\tilde{\sigma}_n\| \to 0$ as $n \to \infty$ then

$$
\| g_n(v, \sigma_n(v)) - \tilde{g}_n(v, \tilde{\sigma}_n(v)) \| \to 0
$$

as $n \to \infty$.

Similarly, using the fact that f' is globally Lipschitz, $\|D\sigma_n\| \to 0$ and $\|D\tilde{\sigma}_n\| \to 0$ as $n \to \infty$, we show that the functions $g_n(v, \sigma_n(v))$ and $\tilde{g}_n(v, \tilde{\sigma}_n(v))$ are C^1 close. So, we have the following theorem:

THEOREM 4.1. *Let $f \in C^2(R, R)$ be a bounded function with first and second derivatives. Assume that the flow on $\mathscr{A}$ is structurally stable. Then, for n large enough the flow of* (2.5) *on the attractor $\mathscr{A}$ and the flow of* (1.6) *on $\mathscr{A}_n$ are topologically equivalent.*

Proof. We first note that, from [He2], we have that (0.1) is generically Morse–Smale and therefore, our assumption on structural stability is not a strong restriction to the class of maps f under consideration. So, we have that (0.1) is $\mathscr{A}$-structurally stable.

We also have that the vector field of (2.5) is a C^1 small perturbation of the (1.6) and the theorem is proved.

5. FURTHER COMMENTS

Though we have chosen to present the results in the simplest formulation they can be extended to much more general situations. The proofs can be easily adapted for the case when f also depends upon the space variable x. Another simple extension is that for which f depends on x, u and u_x. The later can be done if the function $f(x, u, u_x)$ is a locally Lipschitz function that satisfies

$$
f(x, u, 0)\, u < 0, \qquad |u| > \xi > 0
$$

and

$$|f(x, u, p) - f(x, u, q)| \leqslant L_f \, |p - q|, \qquad \forall x, u.$$

In this case we use Lemma 1.3 for $\alpha = 1/2$ and the comparison of the vector fields need additional care but it can all be accomplished without significant change. We point out the main differences. First note that the continuous problem has a global attractor $\mathscr{A}$ satisfying (2.2) and additionally there is a constant C such that $\sup_{u \in \mathscr{A}} \sup_{s \in [0, 1]} |u_x(s)| \leqslant C$. That ensures that we may assume that the nonlinearity f is globally bounded with globally bounded partial derivatives of first and second order. The discretized equations in this case are

$$\begin{aligned} \dot{u}_1 &= p^2(u_2 - u_1) + f(x_1, u_1, 0), \\ \dot{u}_j &= p^2(u_{j-1} - 2u_j + u_{j+1}) \\ &\qquad + f(x_j, u_j, (p/2)(u_{j+1} - u_{j-1})), \qquad j = 2, \cdots p-1 \\ \dot{u}_p &= p^2(u_{p-1} - u_p) + f(x_p, u_p, 0). \end{aligned} \tag{5.1}$$

It is easy to check that if $p > L$ then each rectangle of the form $[-\eta, \eta]^p$ with $\eta > \xi$ is invariant and the results of [CDR] ensure that the attractor for the discretized problem is contained in in the rectangle $[-\xi, \xi]^p$ for any p.

To efficiently handle the dependence of the nonlinearities on the spatial derivative we need to change from the L^2 setting to the H^1 setting when obtaining the invariant manifolds for the continuous and discrete problems. The continuous problem can be projected on the invariant manifold obtained from Lemma 1.3 with $\alpha = 1/2$. The discrete problem needs more attention. The first remark is that, even though it is a finite dimensional problem, it should be treated as its infinite dimensional counterpart. Denote by $U = (u_1, ..., u_p)^\top$ and consider $\mathrm{f}_\mathrm{p}\colon (\mathbb{R}^p, \langle \cdot, \cdot \rangle_{12}) \to (\mathbb{R}^p, \langle \cdot, \cdot \rangle)$ defined by

$$\mathrm{f}_p(U) = (f(x_1, u_1, 0), f(x_2, u_2, (p/2)(u_3 - u_1)), ..., f(x_p, u_p, 0))^\top,$$

where $\langle u, v \rangle_{12} = \langle u, v \rangle + \langle Lu, v \rangle$ and $\langle \cdot, \cdot \rangle$ is the discrete L^2 inner product. Then, we apply Lemma 1.10 for $\alpha = 1/2$. After projecting both problems on the invariant manifolds we use the L^2 norm to study their proximity. The discrete and continuous spatial derivative require that we consider $p = n^4$ instead of $p = n^3$. As for the splitting of the spectrum, it is still done between the $n-$th and $(n+1)$th eigenvalue. After these additional considerations the proofs will follow as in the case without dispersion.

The case of Dirichlet boundary condition can be treated in a a completely similar way. In this case the matrix L has to be replaced by the matrix

$$L_D = p^2 \begin{bmatrix} 2 & -1 & 0 & \cdots & 0 & 0 & 0 \\ -1 & 2 & -1 & \cdots & 0 & 0 & 0 \\ 0 & -1 & 2 & \cdots & 0 & 0 & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots & \vdots \\ 0 & 0 & 0 & \cdots & 2 & -1 & 0 \\ 0 & 0 & 0 & \cdots & -1 & 2 & -1 \\ 0 & 0 & 0 & \cdots & 0 & -1 & 2 \end{bmatrix}. \tag{0.3}$$

The eigenvalues and eigenfunctions of L_D are given by

$$\lambda_p^k = 4p^2 \sin^2 \frac{k\pi}{2(p+1)},$$

$$\phi_p^k = \left(\sin \frac{k\pi}{p+1}, \sin \frac{2k\pi}{p+1}, ..., \sin \frac{pk\pi}{p+1} \right), \qquad k = 1, ..., p$$

(see [Sm], for example). After normalization of the eigenfunctions they can be used to prove the results for the Dirichlet boundary condition case following the Neumann case step by step.

Finally we observe that if we consider a system of n one-dimensional parabolic equations of the form (0.1), the same results will hold as long as the system is structurally stable. The dissipativeness assumptions on the function $f: \mathbb{R}^n \to \mathbb{R}^n$ can be of the form

$$f_i(u)\, u_i < 0, \qquad |u_i| > \xi, \qquad 1 \leqslant i \leqslant n.$$

There are other dissipativeness assumptions that will also work. They are needed to guarantee that the attractor for the system of parabolic equations remain bounded in the uniform topology. That allow us to cut the nonlinearity in such a way that it has bounded first and second derivatives. For the dissipativeness conditions above the uniform bounds on the attractors are proven in [ACR].

6. APPENDIX

This section is devoted to the proof of Lemma 1.3. This result is reproduced from classical invariant manifold results as in [He2]. Its proof is adapted to encompass the possibility that the space (including space dimension) changes according to a parameter and to track the dependence of the invariant manifold upon the parameter.

In the case for which we apply the abstract invariant manifold result contained in Lemma 1.3, the parameter is a natural number n. It means that we are splitting the phase space into the space generated by the first n eigenfunctions of the problem and its orthogonal complement. After projecting the heat equation onto these spaces we produce the pair of equations that appear in the statement of the lemma.

Before we can start the proof of the Lemma 1.3 we need to establish a generalized version of Gronwall's lemma. That requires that we study the convergence of the series

$$E_\beta(z) = \sum_{k=0}^{\infty} \frac{z^{\beta k}}{\Gamma(k\beta+1)}.$$

It is not hard to see that E_β is an entire function and following [E] we may obtain that there is a constant c such that

$$E_\beta(z) \leqslant c e^z.$$

LEMMA 6.1 [Generalized Gronwall's Lemma]. *Let $t<r$, ϕ: $[t, r] \to \mathbb{R}^+$ be a continuous function, a: $[t, r] \to \mathbb{R}^+$ be an integrable function, $b>0$ and $0<\beta\leqslant 1$. Assume that*

$$\phi(t) \leqslant a(t) + b\int_t^r (s-t)^{\beta-1}\,\phi(s)\,ds, \qquad t\leqslant r. \tag{6.1}$$

Then,

$$\phi(t) \leqslant \sum_{k=0}^{\infty} (B^k a)(t) \tag{6.2}$$

with

$$B^k a(t) = \int_t^r \frac{(b\Gamma(\beta))^k}{\Gamma(k\beta)}\,(s-t)^{k\beta-1}\,a(s)\,ds. \tag{6.3}$$

Furthermore, if $a(t)\equiv a = const$ then we have that

$$\phi(t) \leqslant aE_\beta((b\Gamma(\beta))^{1/\beta}\,(r-t)) \leqslant a\,c\,e^{(b\Gamma(\beta))^{1/\beta}\,(r-t)}, \tag{6.4}$$

if $a(t) = c_0 \int_t^r (s-t)^{-\alpha}\,e^{\rho(s-r)}\,ds$, $\rho>0$, then we have that

$$\phi(t) \leqslant \frac{c_0}{b}\,[E_\beta((b\Gamma(\beta))^{1/\beta}\,(r-t))-1] \leqslant \frac{c_0}{b}\,[c\,e^{(b\Gamma(\beta))^{1/\beta}(r-t)}-1] \tag{6.5}$$

and finally, if $\psi: [t, r] \to \mathbb{R}^+$ *is a continuous function and* $a(t) = c_0 \int_t^r (s-t)^{-\alpha} e^{\rho s} \psi(s)\, ds$, $\rho > 0$ *then we have that*

$$\phi(t) \leqslant c_0\, c\, \Gamma(\beta) \int_t^r (s-t)^{\beta-1} e^{\rho s} e^{(b\Gamma(\beta))^{1/\beta}(s-t)} \psi(s)\, ds. \tag{6.6}$$

The proof of this result can be easily adapted from similar results contained in [He2].

Let X and Y be Banach spaces, $-A: D(A) \subset X \to X$ be a sectorial operator such that $\operatorname{Re} \sigma(-A) > 0$ and $B: D(B) \subset Y \to Y$ be the generator of a C^0-group of bounded linear operators $\{S(t), t \geqslant 0\}$ on Y. Let $\{T(t)\ t \geqslant 0\}$ be the analitic semigroup of bounded linear operators generated by A and denote by $(-A)^\alpha$ the α fractional power of A and $X^\alpha = D((-A)^\alpha)$ endowed with the graph norm.

Definition 6.1. Let $f: X^\alpha \times Y^\alpha \to X$, $g: X^\alpha \times Y^\alpha \to Y$ be locally Lipschitz continuous functions. A set $S \subset X^\alpha \times Y^\alpha$ is an *invariant manifold* for a differential equation

$$\dot{x} = Ax + f(x, y)$$

$$\dot{y} = By + g(x, y),$$

if there exists $\sigma: Y^\alpha \to X^\alpha$ such that $S = \{(x, y) \in X^\alpha \times Y^\alpha : x = \sigma(y)\}$ and, for any $(x_0, y_0) \in S$, there exists a solution $(x(\cdot), y(\cdot))$ of the differential equation on $\mathbb{R}$ such that $(x(t), y(t)) \in S\ \forall t \in \mathbb{R}$. An invariant manifold S is said exponentially attracting if there are positive constants γ and K such that

$$\|x(t) - \sigma(y(t))\|_{X^\alpha} \leqslant K e^{-\gamma t}\, \|x(0) - \sigma(y(0))\|_{X^\alpha},$$

whenever $(x(t), y(t))$ is a solution of the differential equation.

Proof of Lemma 1.3. The first step is to prove the existence of the invariant manifold. For $D > 0$, $\Delta > 0$ given, let $\sigma_n: Y_n^\alpha \to X_n^\alpha$ satisfying

$$|\|\sigma_n|\| := \sup_{y \in Y_n^\alpha} \|\sigma_n(y)\|_{X_n^\alpha} \leqslant D, \qquad \|\sigma_n(y) - \sigma_n(y')\|_{X_n^\alpha} \leqslant \Delta\, \|y - y'\|_{Y_n^\alpha}. \tag{6.7}$$

Let $y(t) = \psi(t, \tau, \eta, \sigma_n)$ be the solution of

$$\frac{dy}{dt} = -B_n y + g_n(\sigma_n(y), y), \qquad \textit{for} \quad t < \tau, \qquad y(\tau) = \eta, \tag{6.8}$$

and define

$$G(\sigma_n)(\eta) = \int_{-\infty}^{\tau} e^{-A_n(\tau-s)} f_n(\sigma_n(y(s)), y(s))\, ds. \tag{6.9}$$

Note that

$$\|G(\sigma_n)(\cdot)\|_{X_n^\alpha} \leqslant \int_{-\infty}^{\tau} N_f M_a (\tau-s)^{-\alpha} e^{-\beta(n)(\tau-s)}\, ds. \tag{6.10}$$

Let n_0 be such that, for $n \geqslant n_0$, $\|G(\sigma_n)(\cdot)\|_{X_n^\alpha} \leqslant D$. Next, suppose that σ_n and σ_n' are functions satisfying (6.7), $\eta, \eta' \in Y_n^\alpha$ and denote $y(t) = \psi(t, \tau, \eta, \sigma_n)$, $y'(t) = \psi(t, \tau, \eta', \sigma_n')$. Then,

$$y(t) - y'(t) = e^{-B_n(t-\tau)}\eta + \int_t^\tau e^{-B_n(t-s)} g_n(\sigma_n(y), y)\, ds$$
$$- e^{-B_n(t-\tau)}\eta' - \int_t^\tau e^{-B_n(t-s)} g_n(\sigma_n'(y'), y')\, ds.$$

And

$$\begin{aligned}
&\|y_n(t) - y_n'(t)\|_{Y_n^\alpha}\\
&\quad\leqslant M_b e^{\rho(n)(\tau-t)}\, \|\eta-\eta'\|_{Y_n^\alpha} + M_b \int_t^\tau (s-t)^{-\alpha}\, e^{\rho(n)(s-t)}\\
&\qquad\times \|g_n(\sigma_n(y_n), y_n) - g_n(\sigma_n'(y_n'), y_n')\|_{Y_n}\, ds\\
&\quad\leqslant M_b e^{\rho(n)(\tau-t)}\, \|\eta-\eta'\|_{Y_n^\alpha} + M_b L_g \int_t^\tau (s-t)^{-\alpha}\, e^{-\rho(n)(t-s)}\\
&\qquad\times (\|\sigma_n(y_n) - \sigma_n'(y_n')\|_{X_n^\alpha} + \|y_n - y_n'\|_{Y_n^\alpha})\, ds\\
&\quad\leqslant M_b e^{\rho(n)(\tau-t)}\, \|\eta-\eta'\|_{Y_n^\alpha} + M_b L_g \int_t^\tau (s-t)^{-\alpha}\, e^{\rho(n)(s-t)}\\
&\qquad\times (\|\sigma_n(y_n') - \sigma_n'(y_n')\|_{X_n^\alpha} + (1+\Delta)\, \|y_n - y_n'\|_{Y_n^\alpha})\, ds\\
&\quad\leqslant M_b e^{\rho(n)(\tau-t)}\, \|\eta-\eta'\|_{Y_n^\alpha} + M_b L_g \int_t^\tau (s-t)^{-\alpha}\, e^{\rho(n)(s-t)}((1+\Delta)\\
&\qquad\times \|y_n - y_n'\|_{Y_n^\alpha} + |\|\sigma_n - \sigma_n'|\|_{X_n^\alpha})\, ds\\
&\quad\leqslant M_b e^{\rho(n)(\tau-t)}\, \|\eta-\eta'\|_{Y_n^\alpha}\\
&\qquad+ M_b L_g (1+\Delta) \int_t^\tau (s-t)^{-\alpha}\, e^{\rho(n)(s-t)}\, \|y_n - y_n'\|_{Y_n^\alpha}\, ds\\
&\qquad+ M_b L_g\, |\|\sigma_n - \sigma_n'|\|_{X_n^\alpha} \int_t^\tau (s-t)^{-\alpha}\, e^{\rho(n)(s-t)}\, ds.
\end{aligned}$$

Let

$$\phi(t)=e^{\rho(n)\,(t-\tau)}\,\|y_n(t)-y'_n(t)\|_{Y_n^\alpha}.$$

Then,

$$\phi(t)\leqslant M_b[\,\|\eta-\eta'\|_{Y_n^\alpha}+L_g\int_t^\tau (s-t)^{-\alpha}\,e^{\rho(n)(s-\tau)}\,ds\,|\|\sigma_n-\sigma'_n|\|_{X_n^\alpha}]$$

$$+M_b\,L_g\,(1+\Delta)\int_t^\tau (s-t)^{-\alpha}\,\phi(s)\,ds.$$

By Generalized Gronwall's Lemma

$$\|y_n(t)-y'_n(t)\|_{Y_n^\alpha}\leqslant[c_1\,\|\eta-\eta'\|_{Y_n^\alpha}+c_2\,|\|\sigma_n-\sigma'_n|\|_{X_n^\alpha}]\,e^{[\rho(n)+c_\Gamma](\tau-t)}.$$

where $c_\Gamma=(M_bL_g(1+\Delta)\,\Gamma(1-\alpha))^{1/(1-\alpha)}$. Thus,

$$\begin{aligned}
&\|G(\sigma_n)(\eta)-G(\sigma'_n)(\eta')\|_{X_n^\alpha}\\
&\quad\leqslant M_a\int_{-\infty}^\tau (\tau-s)^{-\alpha}\,e^{-\beta(n)(\tau-s)}\,\|f_n(\sigma_n(y),\,y)-f_n(\sigma'_n(y'),\,y')\|_{X_n}\,ds\\
&\quad\leqslant M_a\int_{-\infty}^\tau (\tau-s)^{-\alpha}\,e^{-\beta(n)(\tau-s)}\\
&\qquad\times(L_f\,\|\sigma_n(y)-\sigma'_n(y')\|_{X_n^\alpha}+L_f\,\|y-y'\|_{Y_n^\alpha})\,ds\\
&\quad\leqslant M_a\int_{-\infty}^\tau (\tau-s)^{-\alpha}\,e^{-\beta(n)(\tau-s)}L_f((1+\Delta)\,\|y-y'\|_{Y_n^\alpha}+|\|\sigma_n-\sigma'_n|\|)\,ds\\
&\quad\leqslant M_a\int_{-\infty}^\tau (\tau-s)^{-\alpha}\,e^{-\beta(n)(\tau-s)}L_f(1+c_2(1+\Delta)\,e^{[\rho(n)+c_\Gamma](\tau-s)})\,ds\\
&\qquad\times|\|\sigma_n-\sigma'_n|\|\\
&\qquad+c_1M_aL_f(1+\Delta)\int_{-\infty}^\tau (\tau-s)^{-\alpha}\,e^{-[\beta(n)-\rho(n)-c_\Gamma](\tau-s)}\,ds\,\|\eta-\eta'\|_{Y_n^\alpha}.
\end{aligned}$$

Let

$$I_\sigma(n)=M_aL_f\int_{-\infty}^\tau (\tau-s)^{-\alpha}\,e^{-\beta(n)(\tau-s)}(1+c_2(1+\Delta)\,e^{[\rho(n)+c_\Gamma](\tau-s)})\,ds$$

and

$$I_\eta(n) = c_1 M_a L_f (1+\Delta) \int_{-\infty}^{\tau} (\tau - s)^{-\alpha} e^{-[\beta(n) - \rho(n) - c_\Gamma](\tau - s)} \, ds.$$

It is easy to see that, given $\theta < 1$, there exists a n_0 such that, for $n \geqslant n_0$, $I_\sigma(n) \leqslant \theta$ and $I_\eta(n) \leqslant \Delta$ and

$$\|G(\sigma_n)(\eta) - G(\sigma'_n)(\eta')\|_{X_n^\alpha} \leqslant I_\eta(n) \, \|\eta - \eta'\|_{Y_n^\alpha} + I_\sigma(n) \, |\|\sigma_n - \sigma'_n|\|. \qquad (6.11)$$

The inequalities (6.10) and (6.11) imply that G is a contraction map from the class of functions that satisfy (6.7) into itself. Therefore, it has a unique fixed point $\sigma_n^* = G(\sigma_n^*)$ in this class.

It remains to prove that $S = \{(y, \sigma_n^*(y)) : y \in Y_n^\alpha\}$ is an invariant manifold for (1.10). Let $(x_0, y_0) \in S$, $x_0 = \sigma_n^*(y_0)$. Denote by $y_n^*(t)$ the solution of the following initial value problem

$$\frac{dy}{dt} = -B_n \, y + g_n(\sigma_n^*(y), y), \qquad y(0) = y_0.$$

This defines a curve $(\sigma_n^*(y_n^*(t)), y_n^*(t)) \in S$, $\quad t \in \mathbb{R}$. But the only solution of

$$\dot{x} = -A_n x + f_n(\sigma_n^*(y_n^*(t)), y_n^*(t)),$$

which remains bounded as $t \to -\infty$ is

$$x^*(t) = \int_{-\infty}^{t} e^{-A_n(t-s)} f(\sigma_n^*(y_n^*(s), y_n^*(s)) \, ds = \sigma_n^*(y_n^*(t)).$$

Therefore, $(\sigma_n^*(y_n^*(t)), y_n^*(t))$ is a solution of (1.10) through (x_0, y_0) and the invariance is proved.

From (6.10) it is clear that $s(n) \to 0$ as $n \to \infty$ and from (6.11) that $l(n) \to 0$ as $n \to \infty$.

The next step is to prove that, for n large enough, the invariant manifold S is exponentially attracting. Specifically, if $(x_n(t), y_n(t))$ is a solution of (1.10), there are positive constants γ and K such that

$$\|x_n(t) - \sigma_n^*(y_n(t))\|_{X_n^\alpha} \leqslant K e^{-\gamma t} \, \|x_n(t_0) - \sigma_n^*(y_n(t_0))\|_{X_n^\alpha}.$$

Let $\xi(t) = x_n(t) - \sigma_n^*(y_n(t))$ and $y_n^*(s, t)$, $s \leqslant t$ be the solution of

$$\frac{dy_n^*}{ds} = -B_n \, y_n^* + g_n(\sigma_n^*(y_n^*), y_n^*), \qquad s \leqslant t,$$

$$y_n^*(t, t) = y_n(t), \qquad s = t.$$

Then,

$$
\begin{aligned}
&\| y_n^*(s,t) - y_n(s)\|_{Y_n^\alpha} \\
&\quad = \| e^{-B_n(s-t)} y_n^*(t,t) + \int_t^s e^{-B_n(s-\theta)} g_n(\sigma_n^*(y_n^*(\theta,t)), y_n^*(\theta,t))\, d\theta \\
&\qquad - e^{-B_n(s-t)} y_n(t) - \int_t^s e^{-B_n(\theta-s)} g_n(x_n(\theta), y_n(\theta))\, d\theta \|_{Y_n^\alpha} \\
&\quad \leqslant M_b \int_s^t (\theta-s)^{-\alpha}\, e^{\rho(n)(\theta-s)} \\
&\qquad \times \| g_n(\sigma_n^*(y_n^*(\theta,t)), y_n^*(\theta,t)) - g_n(x_n(\theta), y_n(\theta))\|_{Y_n}\, d\theta \\
&\quad \leqslant M_b L_g \int_s^t (\theta-s)^{-\alpha}\, e^{\rho(n)(\theta-s)} \\
&\qquad \times (\|\sigma_n^*(y_n^*(\theta,t)) - x_n(\theta)\|_{X_n^\alpha} + \| y_n^*(\theta,t) - y_n(\theta)\|_{Y_n^\alpha})\, d\theta \\
&\quad \leqslant M_b L_g \int_s^t (\theta-s)^{-\alpha}\, e^{\rho(n)(\theta-s)} \\
&\qquad \times (\|\sigma_n^*(y_n(\theta)) - x_n(\theta)\|_{X_n^\alpha} + (1+\Delta)\| y_n^*(\theta,t) - y_n(\theta)\|_{Y_n^\alpha})\, d\theta \\
&\quad \leqslant M_b L_g \int_s^t (\theta-s)^{-\alpha}\, e^{\rho(n)(\theta-s)} \\
&\qquad \times ((1+\Delta)\, \| y_n^*(\theta,t) - y_n(\theta)\|_{Y_n^\alpha} + \|\xi(\theta)\|_{X_n^\alpha})\, d\theta.
\end{aligned}
$$

Therefore,

$$
\begin{aligned}
z(s) \leqslant M_b L_g (1+\Delta) &\int_s^t (\theta-s)^{-\alpha} z(\theta)\, d\theta \\
&+ M_b L_g \int_s^t (\theta-s)^{-\alpha}\, e^{\rho(n)\theta}\, \|\xi(\theta)\|_{X_n^\alpha}\, d\theta,
\end{aligned}
$$

where $z(s) = e^{\rho(n)s}\, \| y_n^*(s,t) - y_n(t)\|_{Y_n^\alpha}$. By Generalized Gronwall's Lemma,

$$
\| y_n^*(s,t) - y_n(s)\|_{Y_n^\alpha} \leqslant c_3 \int_s^t (\theta-s)^{-\alpha}\, e^{[\rho(n)+c_\Gamma](\theta-s)}\, \|\xi(\theta)\|_{X_n^\alpha}\, d\theta, \qquad s \leqslant t.
$$

Let $s \leqslant t_0 \leqslant t$. In what follows estimates for $\| y_n^*(s, t) - y_n^*(s, t_0)\|_{Y_n^\alpha}$ are obtained.

$$\begin{aligned}
&\| y_n^*(s, t) - y_n^*(s, t_0)\|_{Y_n^\alpha} \\
&\quad \leqslant \| e^{-B_n(s-t_0)}[y_n^*(t_0, t) - y_n(t_0)]\|_{Y_n^\alpha} \\
&\qquad + \Big\| \int_{t_0}^{s} e^{-B_n(s-\theta)}[g_n(\sigma_n^*(y_n^*(\theta, t)), y_n^*(\theta, t)) \\
&\qquad - g_n(\sigma_n^*(y_n^*(\theta, t_0)), y_n^*(\theta, t_0))]\, d\theta \Big\|_{Y_n^\alpha} \\
&\quad \leqslant c_3 M_b e^{\rho(n)(t_0-s)} \int_{t_0}^{t} (\theta - t_0)^{-\alpha}\, e^{[\rho(n)+c_\Gamma](\theta-t_0)}\, \|\xi(\theta)\|_{X_n^\alpha}\, d\theta \\
&\qquad + M_b L_g(1+\Delta) \int_{s}^{t_0} (\theta - s)^{-\alpha}\, e^{\rho(n)(\theta-s)}\, \| y_n^*(\theta, t) - y_n^*(\theta, t_0)\|_{Y_n^\alpha}\, d\theta,
\end{aligned}$$

and by Generalized Gronwall's Lemma

$$\| y_n^*(s, t) - y_n^*(s, t_0)\|_{Y_n^\alpha} \leqslant c_4 \int_{t_0}^{t} (\theta - t_0)^{-\alpha}\, e^{[\rho(n)+c_\Gamma](\theta-s)}\, \|\xi(\theta)\|_{X_n^\alpha}.$$

Next, the estimates above are used to estimate $\|\xi(t)\|_{X_n^\alpha}$.

$$\begin{aligned}
&\xi(t) - e^{-A_n(t-t_0)}\xi(t_0) \\
&\quad = x_n(t) - \sigma_n^*(y_n(t)) - e^{-A_n(t-t_0)}(x_n(t_0) - \sigma_n^*(y_n(t_0))) \\
&\quad = \int_{t_0}^{t} e^{-A_n(t-s)} f_n(x_n(s), y_n(s))\, ds - \sigma_n^*(y_n(t)) + e^{-A_n(t-t_0)}\sigma_n^*(y_n(t_0)) \\
&\quad = \int_{t_0}^{t} e^{-A_n(t-s)} f_n(x_n(s), y_n(s))\, ds \\
&\qquad - \int_{-\infty}^{t} e^{-A_n(t-s)} f_n(\sigma_n^*(y_n^*(s, t), y_n^*(s, t))\, ds \\
&\qquad + e^{-A_n(t-t_0)} \int_{-\infty}^{t_0} e^{-A_n(t_0-s)} f_n(\sigma_n^*(y_n^*(s, t_0), y_n^*(s, t_0))\, ds \\
&\quad = \int_{t_0}^{t} e^{-A_n(t-s)}[f_n(x_n(s), y_n(s)) - f_n(\sigma_n^*(y_n^*(s, t), y_n^*(s, t))]\, ds \\
&\qquad - \int_{-\infty}^{t_0} e^{-A_n(t-s)}[f_n(\sigma_n^*(y_n^*(s, t), y_n^*(s, t)) \\
&\qquad - f_n(\sigma_n^*(y_n^*(s, t_0), y_n^*(s, t_0))]\, ds
\end{aligned}$$

and

$$
\begin{aligned}
z(t) &= \|\xi(t) - e^{-A_n(t-t_0)}\xi(t_0)\|_{X_n^\alpha} \\
&\leqslant M_a L_f \int_{t_0}^{t} (t-s)^{-\alpha}\, e^{-\beta(n)(t-s)} (\|x_n(s) - \sigma_n^*(y_n^*(s,t))\|_{X_n^\alpha} \\
&\quad + \|y_n(s) - y_n^*(s,t)\|_{Y_n^\alpha})\, ds \\
&\quad + M_a L_f (1+\Delta) \int_{-\infty}^{t_0} (t-s)^{-\alpha}\, e^{-\beta(n)(t-s)} \\
&\quad \times \|y_n^*(s,t_0) - y_n^*(s,t)\|_{Y_n^\alpha}\, ds \\
&\leqslant M_a L_f \int_{t_0}^{t} (t-s)^{-\alpha}\, e^{-\beta(n)(t-s)}\, \|\xi(s)\|_{X_n^\alpha}\, ds \\
&\quad + c_3 M_a L_f (1+\Delta) \int_{t_0}^{t} (t-s)^{-\alpha}\, e^{-\beta(n)(t-s)} \\
&\quad \times \int_{s}^{t} (\theta - s)^{-\alpha}\, e^{[\rho(n)+c_\Gamma](\theta - s)}\, \|\xi(\theta)\|_{X_n^\alpha}\, d\theta\, ds \\
&\quad + c_4 M_a L_f (1+\Delta) \int_{-\infty}^{t_0} (t-s)^{-\alpha}\, e^{-\beta(n)(t-s)} \\
&\quad \times \int_{t_0}^{t} (\theta - t_0)^{-\alpha}\, e^{[\rho(n)+c_\Gamma](\theta - s)}\, \|\xi(\theta)\|_{X_n^\alpha}\, d\theta\, ds \\
&\leqslant M_a L_f \int_{t_0}^{t} (t-s)^{-\alpha}\, e^{-\beta(n)(t-s)}\, \|\xi(s)\|_{X_n^\alpha}\, ds \\
&\quad + c_5 \int_{t_0}^{t} (t-\theta)^{-\alpha}\, e^{-\beta(n)(t-\theta)}\, \|\xi(\theta)\|_{X_n^\alpha} \\
&\quad \times \int_{t_0}^{\theta} (\theta - s)^{-\alpha}\, e^{-[\beta(n) - (\rho(n)+c_\Gamma)](\theta - s)}\, ds\, d\theta \\
&\quad + c_6 \int_{t_0}^{t} (\theta - t_0)^{-\alpha}\, e^{[\rho(n)+c_\Gamma](\theta - t)}\, \|\xi(\theta)\|_{X_n^\alpha} \\
&\quad \times \int_{-\infty}^{t} (t-s)^{-\alpha}\, e^{-[\beta(n) - [\rho(n)+c_\Gamma]](t-s)}\, ds\, d\theta \\
&\leqslant \left[M_a L_f + \frac{c_5 \Gamma(1-\alpha)}{[\beta(n) - \rho(n) - c_\Gamma]^{1-\alpha}} \right] \int_{t_0}^{t} (t-s)^{-\alpha}\, e^{-\beta(n)(t-s)}\, \|\xi(s)\|_{X_n^\alpha}\, ds \\
&\quad + \frac{c_6 \Gamma(1-\alpha)}{[\beta(n) - \rho(n) - c_\Gamma]^{1-\alpha}} \int_{t_0}^{t} (\theta - t_0)^{-\alpha}\, e^{[\rho(n)+c_\Gamma](\theta - t)}\, \|\xi(\theta)\|_{X_n^\alpha}\, d\theta.
\end{aligned}
$$

Thus,

$$\begin{aligned}\|\xi(t)\|_{X_n^\alpha} \leqslant & M_a e^{-\beta(n)(t-t_0)} \|\xi(t_0)\|_{X_n^\alpha} \\ & + [M_a L_f + \frac{c_5 \Gamma(1-\alpha)}{[\beta(n)-\rho(n)-c_\Gamma]^{1-\alpha}}] \\ & \times \int_{t_0}^t (t-s)^{-\alpha} e^{-\beta(n)(t-s)} \|\xi(s)\|_{X_n^\alpha} ds \\ & + \frac{c_6 \Gamma(1-\alpha)}{[\beta(n)-\rho(n)-c_\Gamma]^{1-\alpha}} \int_{t_0}^t (s-t_0)^{-\alpha} e^{[\rho(n)+c_\Gamma](s-t)} \|\xi(s)\|_{X_n^\alpha} ds\end{aligned}$$

and, if $w(t) = \sup\{\|\xi(s)\|_{X_n^\alpha},\ t_0 \leqslant s \leqslant t\}$, then

$$e^{\beta(n)(t-t_0)} \|\xi(t)\|_{X_n^\alpha} \leqslant M_a \|\xi(t_0)\|_{X_n^\alpha} + \gamma(n)\, e^{\beta(n)(t-t_0)} w(t)$$

where

$$\gamma(n) = \frac{\Gamma(1-\alpha)}{\beta(n)^{1-\alpha}} [M_a L_f + \frac{c_5 \Gamma(1-\alpha)}{[\beta(n)-\rho(n)-c_\Gamma]^{1-\alpha}}] + \frac{c_6 \Gamma(1-\alpha)\, K}{[\beta(n)-\rho(n)-c_\Gamma]^{1-\alpha}}$$

where $K = \sup_{\eta \geqslant 0} (\int_0^\eta u^{-\alpha} e^{[\rho(n)+c_\Gamma](u-\eta)} du)$. Choose $n_0 > 0$ such that $\gamma(n) \leqslant \frac{1}{2}$ for every $n \geqslant n_0$. Therefore

$$e^{\beta(n)(t-t_0)} \|\xi(t)\|_{X_n^\alpha} \leqslant e^{\beta(n)(t-t_0)} w(t) \leqslant M_a \|\xi(t_0)\|_{X_n^\alpha} + \gamma(n)\, e^{\beta(n)(t-t_0)} w(t)$$

and

$$\|\xi(t)\|_{X_n^\alpha} \leqslant 2M_a \|\xi(t_0)\|_{X_n^\alpha} e^{-\beta(n)(t-t_0)}.$$

The smoothness of σ_n^* is proved in the same way as in [He2] and the estimate for the derivative follows from the estimate for its Lipschitz constant. This concludes the proof.

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