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Few-Body Techniques Using Momentum Space for Bound and Continuum States

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Abstract This article is based on the notes (arxiv:1710.11228) written for a set of three lectures given in a school at the Max Planck Institute for the Physics of Complex Systems in October/2017 before the workshop "Critical Stability of Quantum Few-Body Systems". The last part of the article includes the specific topic presented in the workshop related to the dimensional effects in three-body systems. These notes are primarily dedicated to the students and are only a tentative to show a technique, among many others, to solve problems in a very rich area of the contemporary physics—the Few-Body Physics.

1 Introduction

The first question we probably think by reading "Few-Body Physics" along the title[1] is what is the meaning of the word "Few-Body". This term may seem a little vague and use it in order to define an area of the physics also seems incautious, mainly by the use of a so subjective word like "few". In our case, the word "body" corresponds to any particle that may be present in several contexts of the physics like, e.g., quarks, protons, neutrons, atoms or molecules. Each of these different constituents should have a common characteristic: they should be a very well-defined object. They should not be treated as an approximation like in a mean-field theory, for example. Thus the word "few" should be understood taking into account the technical and computational difficulties that appear as the number of particles increase. A few-body system may represent 3, 4, 5, etc. particles since the individuality of each object is respected.

This article is based on my notes (arxiv:1710.11228) [2] without the introductory part related to the formal scattering theory and development of the Faddeev equations. Section 2 describes in detail how to calculate three-body bound and scattering states using the subtracted Faddeev equations with a pairwise zero-range interaction [3]. In Sect. 3 we show how the effective dimension generated inside atomic traps may affect the Efimov states. The summary and conclusions are presented in Sect. 4.

This article belongs to the Topical Collection "Critical Stability of Quantum Few-Body Systems".

[1] A curiosity to mention here is that we usually hear that the classical few-body problem doesn't have a solution for three or more particles, as demonstrated by Henri Poincaré when he was 35 years old. In fact, the N-body classical problem was already solved in 1991 by Qiudong Wang, at that time, a student in the beginning of his Ph.D. [1].

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2 Bound and Scattering States

The following formalism will be developed in momentum space. We will focus only in universal situations in which the range of the potential is much smaller than the typical sizes of the system in a manner that the observables do not depend on the form of the short-range potential.

2.1 Two-Body T-Operator for a Separable Potential

A potential, V, is called separable if

$$\langle \mathbf{p}|V|\mathbf{p}'\rangle = \lambda g(\mathbf{p}) g^\star(\mathbf{p}'), \tag{1}$$

where $\mathbf{p}$ is the relative momentum between two particles. We can write generically a rank-1 separable potential in the form of a projection operator as

$$V \equiv \lambda|\chi\rangle\langle\chi|, \tag{2}$$

where λ is the strength of the potential and $\langle \mathbf{p}|\chi\rangle = g(\mathbf{p})$ and $\langle\chi|\mathbf{p}\rangle = g^*(\mathbf{p})$ are the form factors.

Let us start replacing the separable potential, written in the form of Eq. (2), in the T-operator equation calculated at a given energy E. $G_0(E)$ is the free Green operator.

$$t(E) = V + VG_0(E)t(E), \tag{3}$$

$$t(E) = \lambda|\chi\rangle\langle\chi| + \lambda|\chi\rangle\langle\chi|G_0(E)t(E), \tag{4}$$

multiplying by $\langle\chi|G_0(E)$ from the left side and isolating $\langle\chi|G_0(E)t(E)$ we have:

$$\langle\chi|G_0(E)t(E) = \lambda\langle\chi|G_0(E)|\chi\rangle\langle\chi| + \lambda\langle\chi|G_0(E)|\chi\rangle\langle\chi|G_0(E)t(E),$$

$$(1 - \lambda\langle\chi|G_0(E)|\chi\rangle)\langle\chi|G_0(E)t(E) = \lambda\langle\chi|G_0(E)|\chi\rangle\langle\chi|,$$

$$\langle\chi|G_0(E)t(E) = \frac{\lambda\langle\chi|G_0(E)|\chi\rangle\langle\chi|}{1 - \lambda\langle\chi|G_0(E)|\chi\rangle}. \tag{5}$$

Replacing Eq. (5) in (4) we get the two-body T-operator:

$$t(E) = \lambda|\chi\rangle\langle\chi| + \frac{\lambda^2|\chi\rangle\langle\chi|G_0(E)|\chi\rangle\langle\chi|}{1 - \lambda\langle\chi|G_0(E)|\chi\rangle},$$

$$t(E) = \lambda|\chi\rangle\left(1 + \frac{\lambda\langle\chi|G_0(E)|\chi\rangle}{1 - \lambda\langle\chi|G_0(E)|\chi\rangle}\right)\langle\chi|. \tag{6}$$

Then, we can write it as:

$$t(E) = |\chi\rangle\tau(E)\langle\chi|, \tag{7}$$

note here that a separable potential results in a separable T-operator. The function τ is given by:

$$\tau(E) = \frac{1}{\lambda^{-1} - \langle\chi|G_0(E)|\chi\rangle}. \tag{8}$$

Writing explicitly the matrix element of Eq. (7), we have that:

$$\tau(E) = \left(\lambda^{-1} - \int d^3p \frac{|g(\mathbf{p})|^2}{E - \frac{p^2}{2M} + i\epsilon}\right)^{-1}, \tag{9}$$

where M is the reduced mass of the two-body system.

Since now, we just assumed that the two-body potential is separable. From this point we will restrict it a little more. We will focus here only on the universal characteristics of the system. The universal characteristics appear in systems that present an important property called *universality*. In these systems the calculated observables do not depend on the details of the short-range potential. This peculiar situation occurs when the size of the system, represented by a typical scale (the two-body scattering length, for example), is much larger than the range of the potential. This situation can be achieved by construction using a Dirac-delta potential. As this potential has a range equal to zero, any size of the system is infinitely larger than the range of the potential, then any quantity calculated is universal by definition. For a zero-range potential $V(\mathbf{r}) = \lambda\delta(\mathbf{r})$ and the form factor $g(\mathbf{p}) = \langle \mathbf{p} | \chi\rangle = 1$.

The consequence of replacing $g(\mathbf{p}) = 1$ is that the integral in

$$\tau^{-1}(E) = \lambda^{-1} - \int d^3p \frac{1}{E - \frac{p^2}{2M} + i\epsilon} \tag{10}$$

diverges for large momenta. In order to transform Eq. (10) into a finite value we need that λ^{-1} should also be infinite. Calculating $\tau^{-1}(-|E_2|)$ at the two-body binding energy, E_2, and remembering that a bound state is a pole in the T-operator we have that:

$$\tau^{-1}(-|E_2|) = 0 = \lambda^{-1} - \int d^3p \frac{1}{-|E_2| - \frac{p^2}{2M}} \tag{11}$$

$$\lambda^{-1} = \int d^3p \frac{1}{-|E_2| - \frac{p^2}{2M}}, \tag{12}$$

then we can replace λ^{-1} in Eq. (10) (note that λ^{-1} also diverges) obtaining:

$$\tau^{-1}(E) = -\int d^3p \left(\frac{1}{|E_2| + \frac{p^2}{2M}} + \frac{1}{E - \frac{p^2}{2M} + i\epsilon} \right). \tag{13}$$

Now the integral is finite and can be calculated using the residue theorem giving:

$$\tau^{-1}(E) = 2\pi^2 (2M)^{3/2} \left(\sqrt{|E_2|} - \sqrt{|E|} \right). \tag{14}$$

Note that the introduction of the physical scale E_2 not only regularizes the integral but also renormalizes it.

2.2 Three-Body T-Operator for a Separable Potential

We saw that the regularization (and renormalization) of Eq. (10) appears naturally after calculating τ at a bound state energy and noting that such energy is a pole in the T-operator. However, the choice of the two-body energy $|E_2|$ as a scale is arbitrary—it could be another observable or we could just put a cuttoff for high momenta and further relate it with some observable.

The three-body T-operator may also be regularized in a similar way. Let us start defining the three-body T-operator in a given energy $E = -\mu^2$

$$\begin{aligned} T(-\mu^2) &= \left[1 + T(-\mu^2) G_0(-\mu^2)\right] V, \\ V &= \left[(1 + T(-\mu^2) G_0(-\mu^2))\right]^{-1} T(-\mu^2), \end{aligned} \tag{15}$$

replacing the above V into $T(E) = V + VG_0^{(+)}(E)T(E)$ we get:

$$T_R(E, \mu^2) = T_R(-\mu^2) + T_R(-\mu^2) \left(G_0^{(+)}(E) - G_0(-\mu^2) \right) T_R(E), \tag{16}$$

where we inserted the subscript R to indicate a regularized (renormalized) T-operator.

We can show that $T(E, \mu^2)$ does not depends on the subtraction point μ^2 in such a way we will write the T-operator simply as

$$T_R(E) = T_R(-\mu^2) + T_R(-\mu^2) \left(G_0^{(+)}(E) - G_0(-\mu^2) \right) T_R(E). \tag{17}$$

Note that Eq. (17) has the same operatorial form as the original equation for the T-operator (this can be seen making the following replacements $T_R(-\mu^2) \equiv V(-\mu^2)$ and $\left(G_0^{(+)}(E) - G_0(-\mu^2) \right) \equiv G_0(E; -\mu^2)$).

Let us assume an ansatz and define the three-body T-operator in the subtraction point as the sum of all pairs of the renormalized two-body T-operators (t_{R_α})

$$T_R(-\mu_{(3)}^2) = \sum_{\alpha,\beta,\gamma} t_{R_\alpha}\left(-\mu_{(3)}^2 - \frac{q_\alpha^2}{2m_{\beta\gamma,\alpha}}\right), \tag{18}$$

where $\alpha, \beta, \gamma = 1, 2, 3$ ($\alpha \neq \beta \neq \gamma$) and $m_{\beta\gamma,\alpha}$ is the reduced mass of the pair $\beta\gamma$ and α. Note that the argument of t_R is the energy of the center of mass of the pair. From this point the two and three-body T-operators will be represented, respectively, by t and T. The counterpart of regularized/renormalized two and three-body T-operators is the addition of two physical scales which will be generically represented by $\mu_{(i)}^2$, where the subscript $i = 2, 3$ distinguishes between the two and three-body scales. The original T-operator is recovered in the limit $\mu \to \infty$.

Replacing Eq. (18) in (17) we have:

$$T_R(E) = \sum_{\alpha,\beta,\gamma} t_{R_\alpha}\left(-\mu_{(3)}^2 - \frac{q_\alpha^2}{2m_{\beta\gamma,\alpha}}\right)\left[1 + \left(G_0^{(+)}(E) - G_0(-\mu_{(3)}^2)\right)T_R(E)\right]. \tag{19}$$

Defining the component 1 of the three-body T-operator as

$$T_{R_1}(E) = t_{R_1}\left(-\mu_{(3)}^2 - \frac{q_1^2}{2m_{23,1}}\right)\left[1 + \left(G_0^{(+)}(E) - G_0(-\mu_{(3)}^2)\right)T_R(E)\right], \tag{20}$$

and writing $T_R(E)$ as the sum of the components corresponding to an interacting pair (we are using the odd-man-out notation where the subscript 1 says that the pair (23) interacts, 2 (13) and 3(12)) as

$$T_R(E) = \sum_{\alpha=1,2,3} T_{R_\alpha}(E), \tag{21}$$

we may finally write the Faddeev component 1 as a function of the other two:

$$\begin{aligned} T_{R_1}(E) = t_{R_1}\left(E - \frac{q_1^2}{2m_{23,1}}\right)&\left\{1 + \left(G_0^{(+)}(E) - G_0(-\mu_{(3)}^2)\right)\right. \\ &\left.\times\left[T_{R_2}(E) + T_{R_3}(E)\right]\right\}, \end{aligned} \tag{22}$$

2.3 Bound State Equations

Consider the following completeness relation:

$$\mathbf{1} = \sum_B |\Phi_B\rangle\langle\Phi_B| + \int d^3k|\Psi_k^{(+)}\rangle\langle\Psi_k^{(+)}|, \tag{23}$$

where the first and second terms containing $|\Phi_B\rangle$ and $|\Psi_k^{(+)}\rangle$, decomposes, respectively, a given state into bound and scattering states of initial momentum $\mathbf{k}$. Inserting Eq. (23) in the T-operator equation we have:

$$\begin{aligned} T(E) &= V + \sum_B VG^{(+)}(E)|\Phi_B\rangle\langle\Phi_B|V + \int d^3k VG^{(+)}(E)|\Psi_k^{(+)}\rangle\langle\Psi_k^{(+)}|V; \\ T(E) &= V + \sum_B \frac{V|\Phi_B\rangle\langle\Phi_B|V}{E - E_B + i\epsilon} + \int d^3k \frac{V|\Psi_k^{(+)}\rangle\langle\Psi_k^{(+)}|V}{E - \bar{E}_k + i\epsilon}, \end{aligned} \tag{24}$$

where the complete propagator $G^{(+)}(E)$ was explicitly written in terms of the bound ($E_B < 0$) and continuous ($\bar{E}_k > 0$) state eigenvalues. The T-operator written in the form of Eq. (24) is called Low equation [4]. For a given bound states ($E \approx E_B$) we have that the second term is dominant due to the presence of a pole. Then, we can write:

$$T(E) \approx \frac{V|\Phi_B\rangle\langle\Phi_B|V}{E+|E_B|} = \frac{|\Gamma_B\rangle\langle\Gamma_B|}{E+|E_B|}, \tag{25}$$

where we defined $|\Gamma_B\rangle = V|\Phi_B\rangle$. The same applies to a Faddeev component of the T-operator:

$$T_\alpha = v_\alpha + v_\alpha G V \tag{26}$$

$$T_\alpha(E) \approx \frac{v_\alpha|\Phi_\alpha\rangle\langle\Phi_\alpha|V}{E+|E_\alpha|} = \frac{|\Gamma_\alpha\rangle\langle\Gamma|}{E+|E_\alpha|}, \tag{27}$$

with $\alpha = 1, 2, 3$, $|\Gamma_\alpha\rangle = v_\alpha|\Phi_\alpha\rangle$ and $\langle\Gamma| = \langle\Phi_\alpha|V$. Replacing Eq. (27) in (22), we have:

$$\frac{|\Gamma_1\rangle\langle\Gamma|}{E+|E_1|} \approx t_{R_1}\left(E - \frac{q_1^2}{2m_{23,1}}\right)\left\{1 + \left[G_0^{(+)}(E) - G_0(-\mu_{(3)}^2)\right]\right.$$
$$\left.\times\left(\frac{|\Gamma_2\rangle\langle\Gamma|}{E+|E_1|} + \frac{|\Gamma_3\rangle\langle\Gamma|}{E+|E_1|}\right)\right\}, \tag{28}$$

where E_1 is the energy of the bound pair 23. Cancelling the common terms on both sides we finally have the homogeneous equation

$$|\Gamma_1\rangle = t_{R_1}\left(E - \frac{q_1^2}{2m_{23,1}}\right)\left(G_0^{(+)}(E) - G_0(-\mu_{(3)}^2)\right)(|\Gamma_2\rangle + |\Gamma_3\rangle)\,. \tag{29}$$

Remember that this approximation is as good as closer to the limit $E \to -|E_1|$.

Writing explicitly the two-body T-operator in the operatorial form given by Eq. (7):

$$|\Gamma_1\rangle = |\chi\rangle\tau\left(E - \frac{q_1^2}{2m_{23,1}}\right)\langle\chi|\left(G_0^{(+)}(E) - G_0(-\mu_{(3)}^2)\right)(|\Gamma_2\rangle + |\Gamma_3\rangle)\,, \tag{30}$$

where $\tau(E)$ is the function given by Eq. (14). Multiplying Eq. (30) by $\langle\mathbf{p}_1, \mathbf{q}_1|$ from the left, where $\mathbf{p}_1$ is the relative momentum of particles 2 and 3 and $\mathbf{q}_1$ the momentum of particle 1 with respect to the center of mass of particles 2 and 3, we get:

$$\langle\mathbf{p}_1, \mathbf{q}_1|\Gamma_1\rangle = \langle\mathbf{p}_1|\chi\rangle\tau\left(E - \frac{q_1^2}{2m_{23,1}}\right)\langle\chi|\langle\mathbf{q_1}|\left(G_0^{(+)}(E) - G_0(-\mu_{(3)}^2)\right)$$
$$\times(|\Gamma_2\rangle + |\Gamma_3\rangle)\,, \tag{31}$$

using that

$$\langle\mathbf{p}_1, \mathbf{q}_1|\Gamma_1\rangle = \langle\mathbf{p}_1, \mathbf{q}_1|V|\Phi_1\rangle = \int d\mathbf{p}'\langle\mathbf{p}_1|\chi\rangle\langle\chi|\mathbf{p}'\rangle\langle\mathbf{q}_1, \mathbf{p}'|\Phi_1\rangle = f_1(\mathbf{q_1}), \tag{32}$$

where we used that the form factor for a Dirac-delta potential is $g(\mathbf{p}) = \langle\mathbf{p}|\chi\rangle = 1$, we have the homogeneous equation for a three-body bound state for the Faddeev component 1.

$$f_1(\mathbf{q_1}) = \langle\mathbf{p}_1|\chi\rangle\tau\left(E - \frac{q_1^2}{2m_{23,1}}\right)\langle\chi|\langle\mathbf{q_1}|\left(G_0^{(+)}(E) - G_0(-\mu_{(3)}^2)\right)$$
$$\times(|\Gamma_2\rangle + |\Gamma_3\rangle)\,. \tag{33}$$

The function f is called *spectator function* [5]. Note that for the specific case of three identical bosons, the three spectator functions are exactly the same, such that:

$$\langle\mathbf{q}_1|f_1\rangle = \langle\mathbf{q}_2|f_2\rangle = \langle\mathbf{q}_3|f_3\rangle. \tag{34}$$

Then, we have now to calculate the matrix elements on the right side of Eq. (33).

The homogeneous equation for three identical bosons reads:

$$f(\mathbf{q}) = 2\tau \left(-|E_3| - \frac{3}{4}q^2\right) \times \int d^3q' \left(\frac{1}{-|E_3| - q^2 - q'^2 - \mathbf{q}' \cdot \mathbf{q}} - \frac{1}{-\mu_{(3)}^2 - q^2 - q'^2 - \mathbf{q}' \cdot \mathbf{q}}\right) f(\mathbf{q}'), \quad (35)$$

where we removed all indexes. This is the Skorniakov and Ter-Martirosian (STM) equation for the bound state and zero-range potential [6]. It is worth to remind that the three-body scale $\mu_{(3)}$ is arbitrary. All momenta and E_3 can be rescaled with respect to $\mu_{(3)}$ in order to have dimensionless quantities as $|\epsilon_3| = \frac{|E_3|}{\mu_{(3)}^2}$, $y = \frac{q}{\mu_{(3)}}$ and $x = \frac{q'}{\mu_{(3)}}$. Thus, the homogeneous equation (35) for the three-body bound state is rewritten as

$$f(y) = 4\pi\tau \left(-|\epsilon_3| - \frac{3}{4}y^2\right) \int_0^\infty dx x^2 \int_1^{-1} dz \left[\frac{1}{|\epsilon_3| + y^2 + x^2 + xyz} - \frac{1}{1 + y^2 + x^2 + xyz}\right] f(x), \quad (36)$$

with $z \equiv \cos(\mathbf{q} \cdot \mathbf{q}')$.

Here, we might be tempted to use the solution coming from the Fredholm theory, however, we have here two problems: we don't know neither f nor ϵ_3. Among several numerical methods that we can use to solve this problem, we will focus in only one. Generically, the structure of this integral equation reads (after integrating out the angular part)

$$f(y) = \int dx K(y, x; E) f(x). \quad (37)$$

In order to calculate this equation numerically we should discretize it. Let us call by $f_i \equiv f(y_i)$ the value of f calculated in a given mesh point y_i $(i = 1, \ldots, N)$. We then have

$$f(y_i) = \sum_{j=1}^{N} (wx)_j K(y_i, x_j; E) f(x_j) \quad (38)$$

$$f_i = \sum_{j=1}^{N} w_j K_{ij}(E) f_j \quad (39)$$

$$\sum_{j=1}^{N} \left(\delta_{ij} - w_j K_{ij}(E)\right) f_j = 0 \rightarrow MF = 0, \quad (40)$$

where $K(y_i, x_j; E) \equiv K_{ij}(E)$, $(wx)_i \equiv w_i$ is a given weight associated to the mesh point x_i (if you are not familiar with these terms, search for Gauss-Legendre quadrature), and δ_{ij} is a Kronecker delta. We then have a homogeneous equation with matrices M and F given by:

$$M = \begin{pmatrix} 1 - w_1K_{11}(E) & -w_2K_{12}(E) & \cdots & -w_NK_{1N}(E) \\ -w_1K_{21}(E) & 1 - w_2K_{22}(E) & \cdots & -w_NK_{2N}(E) \\ \vdots & \vdots & \ddots & \vdots \\ -w_1K_{N1}(E) & -w_2K_{N2}(E) & \cdots & -w_NK_{NN}(E) \end{pmatrix} \text{ and } F = \begin{pmatrix} f_1 \\ f_2 \\ \vdots \\ f_N \end{pmatrix}. \quad (41)$$

Now, if we want a different solution from the trivial one we have to find an $E = E_3$ that gives a determinant $\det(M(E_3)) = 0$. Once a bound state energy is determined, we can now discuss how to determine the three-body wave function.

The three-body wave function may be written as a function of the spectator function. So, let us first consider how to determine f. As the determinant is equal to zero, one of the equations of our homogeneous system is redundant and it can be eliminated. Then, f_1, for example (it could be another position than 1), can be set arbitrarily as $f_1 \equiv 1$ and the other f's are calculated with respect to this f_1. This is not a problem at all as the wave function will be further normalized and this arbitrariness will be washed out. Thus, we have now that

$$\sum_{j=1}^{N} M_{ij} f_j = 0 \iff \sum_{j=2}^{N} M_{ij} f_j = -M_{i1} f_1 = -M_{i1}, \tag{42}$$

with $i = 2, \ldots, N$. Representing by $\bar{M}$ the remaining matrix after eliminating the first column and line of M, and by C the first column of M, M_{i1} $(i = 2, \ldots, N)$, without the element M_{11}, we have that

$$\sum_{j=2}^{N} f_j = \bar{F} = -\bar{M}^{-1} C. \tag{43}$$

Remember that we are considering only the case where the particles are identical. For two or three different particles we will also have, respectively, two or three different spectator functions. For a general three-body system with three distinct spectator functions, the three-body wave function emerges directly from the Schroedinger equation as:

$$\left(H_0 + \sum_{\alpha=1,2,3} \lambda_\alpha |\chi_\alpha\rangle\langle\chi_\alpha| \right) |\Psi\rangle = E|\Psi\rangle \tag{44}$$

$$(E - H_0)|\Psi\rangle = \sum_{\alpha=1,2,3} \lambda_\alpha |\chi_\alpha\rangle\langle\chi_\alpha|\Psi\rangle, \tag{45}$$

where the two-body separable potential was replaced by $v_\alpha = \lambda_\alpha |\chi_\alpha\rangle\langle\chi_\alpha|$ and H_0 is the free Hamiltonian. Multiplying Eq. (45) by $\langle \mathbf{q}_1, \mathbf{p}_1|$ from the left we may write the three-body wave function in terms of the spectator functions in the coordinates $\mathbf{q}_1, \mathbf{p}_1$ as:

$$\langle \mathbf{q}_1, \mathbf{p}_1|\Psi\rangle = \frac{f_1(|\mathbf{q}_1|) + f_2(|\mathbf{p}_1 - \frac{\mathbf{q}_1}{2}|) + f_3(|\mathbf{p}_1 + \frac{\mathbf{q}_1}{2}|)}{|E_3| + H_0}. \tag{46}$$

We have now the full picture to calculate the three-body binding energies and wave function.

2.4 Scattering States Equation

In order to write the Lippmann-Schwinger equation in momentum space, we have to insert in Eq. (35) the inhomogeneous term coming from the solution of the free problem. In configuration space this term is given by a plane wave $e^{i\mathbf{q}\cdot\mathbf{r}}$ and in momentum space it can be written as $(2\pi)^{3/2}\delta(\mathbf{q} - \mathbf{k}_i)$. Here, the momentum $\mathbf{q}$ represents the relative momentum between the free particle and the center of mass of the bound pair. The in and outcoming momenta given, respectively, by k_i and k_f are related to the total energy of a free particle and a bound pair as $E_3 = -E_2 + k_i^2/2m_{23,1} = -E_2 + k_f^2/2m_{23,1}$. Thus, the full equation with the inhomogeneous term reads:

$$f(\mathbf{q}) = (2\pi)^{3/2}\delta(\mathbf{q} - \mathbf{k}_i) + 2\tau\left(E_3 - \frac{3}{4}q^2\right) \times \int d^3q' \left(\frac{1}{E_3 - q^2 - q'^2 - \mathbf{q}'\cdot\mathbf{q}} - \frac{1}{-\mu_{(3)}^2 - q^2 - q'^2 - \mathbf{q}'\cdot\mathbf{q}} \right) f(\mathbf{q}'), \tag{47}$$

with $E_3 > 0$. Now, we have to insert in Eq. (47) the boundary condition for the elastic scattering given by:

$$f(\mathbf{q}) \to (2\pi)^{3/2}\delta(\mathbf{q} - \mathbf{k}_i) + \frac{h(\mathbf{q}, k_i)}{E_3 \pm i\epsilon - q^2}, \tag{48}$$

where $h(\mathbf{q}\,, k_i)$ is the scattering amplitude. After replacing Eq. (48) in (47) we have that

$$h(\mathbf{q}\,, k_i) = \mathcal{V}(q, k_i; E_3) + \int d^3q' \frac{\mathcal{V}(q, q'; E_3)}{E_3 \pm i\epsilon - q'^2} h(\mathbf{q}'\,, k_i), \tag{49}$$

where

$$\mathcal{V}(q, k_i; E_3) = 2\tau \left(E_3 - \frac{3}{4}q^2\right)\left(E_3 - q^2\right) \times \left(\frac{1}{E_3 - q^2 - k_i^2 - \mathbf{q}\cdot\mathbf{k}_i} - \frac{1}{-\mu^2 - q^2 - k_i^2 - \mathbf{q}\cdot\mathbf{k}_i}\right). \tag{50}$$

Note that we are not being very precise here as there is a missing factor $(2\pi)^{3/2}$ dividing the function h. However, this is meaningless as we could in principle redefine another function as $\bar{h} \equiv h/(2\pi)^{3/2}$. Now, after integrating out the angular part we will arrive in a equation very similar to Eq. (37), but with an inhomogeneous term

$$h(x, y) = g(x, y; E) + \int dx' K(x, x'; E) h(x', y). \tag{51}$$

The numerical method we used to solve this problem is very close to the one used for bound states. The difference is that now the spectrum is continuous and the energy E enters as an input. Let us call $h_{ij} = h(y_i, x_j)$, $g_{ij} = g(y_i, x_j)$ and $K_{ij}(E) = K(y_i, x_j; E)$ the values of h, g and K calculated in the mesh points $y_i, x_i \;\; (i = 1, \ldots, N)$. Then, the discretization reads:

$$\begin{aligned} hij &= g_{ij}(E) + \sum_{k=1}^{M} w_k K_{ik}(E) h_{kj} \\ \sum_{k=1}^{M} (\delta_{ik} - w_k K_{ik})\, h_{kj} &= g_{ij} \rightarrow D\mathcal{H} = g \end{aligned} \tag{52}$$

where w_k is a given weight associated to the Gauss point x_k and δ_{ik} is the Kronecker delta. Now, the matrix $\mathcal{H}$ returns the function h which is directly associated with the differential cross section $\frac{d\sigma}{d\Omega} = |h(\mathbf{q}, k_i)|^2$.

3 Dimensional Effects in Efimov States

In 1970, Vitaly Efimov published a paper [7] where he studied the energy spectrum of a system formed by bound states of three-identical bosons, interacting by a two-body short-range potential. He observed a very curious behaviour: the number of three-body bound states increased to infinity if the two-body binding energy tends to zero. It took more than 30 years to have an experimental evidence of this counter-intuitive phenomenon in the context of ultracold atoms [8]. These bound states still present a very interesting scaling: the ratio between two consecutive bound states is given by $E_3^{(N)}/E_3^{(N+1)} = e^{2\pi/s} \;\; (N = 0, 1, \ldots)$, where $s = 1.006$ for three identical bosons. Also, the ratio of two consecutive root-mean-square hyperradius, is exactly the square root of the energy ratios $e^{\pi/s}$.

For a system AAB, formed by two identical bosons of masses m_A and a different particle with mass m_B, s depends on the mass ratio $A = m_B/m_A$ and also on the dimension D where the system is embedded. Nielsen and collaborators [9] showed that for three identical bosons the Efimov effect ceases to exist in the interval $2.3 < D < 3.8$. It is possible to reproduce and extend the previous result for an AAB system noting that the wave function which results in the characteristic Efimov energy spectrum is scale invariant for large relative momenta. The following equation extends the Skorniakov and Ter-Martirosian equation (STM) [6] for an AAB system and generalizes it to an arbitrary dimension

$$f_A(q) = \tau_{AB}\left(E_3 - \frac{\mathcal{A}+2}{2(\mathcal{A}+1)}q^2\right) \times \int d^D k \left(\frac{f_B(k)}{E_3 - q^2 - \frac{\mathcal{A}+1}{2\mathcal{A}}k^2 - \mathbf{k}\cdot\mathbf{q}} + \frac{f_A(k)}{E_3 - \frac{\mathcal{A}+1}{2\mathcal{A}}(k^2+q^2) - \frac{1}{\mathcal{A}}\mathbf{k}\cdot\mathbf{q}}\right), \tag{53}$$

$$f_B(q) = 2\,\tau_{AA}\left(E_3 - \frac{\mathcal{A}+2}{4\mathcal{A}}q^2\right)\int d^D k\,\frac{f_A(k)}{E_3 - \frac{\mathcal{A}+1}{2\mathcal{A}}q^2 - k^2 - \mathbf{k}\cdot\mathbf{q}}, \tag{54}$$

The subindexes distinguish between the two different spectator function. The two-body transition amplitudes τ_{AB} and τ_{AA} are given by

$$\tau^{-1}_{AB}\left(E_3 - \frac{\mathcal{A}+2}{2(\mathcal{A}+1)}q^2\right) = \int d^D k\left(\frac{1}{-|E_2^{AB}| - \frac{\mathcal{A}+1}{2\mathcal{A}}k^2} - \frac{1}{E_3 - \frac{\mathcal{A}+2}{2(\mathcal{A}+1)}q^2 - \frac{\mathcal{A}+1}{2\mathcal{A}}k^2}\right), \tag{55}$$

$$\tau^{-1}_{AA}\left(E_3 - \frac{\mathcal{A}+2}{4\mathcal{A}}q^2\right) = \int d^D k\left(\frac{1}{-|E_2^{AA}| - k^2} - \frac{1}{E_3 - \frac{\mathcal{A}+2}{4\mathcal{A}}q^2 - k^2}\right), \tag{56}$$

where E_2^{AB} and E_2^{AA} are the two-body energies of the bound AB and AA systems, respectively. For large values of q, relevant for exploring the Efimov effect, the integrals in Eqs. (55) and (56) are determined by the region of large values of k [10] and, therefore, the energies E_3, E_2^{AA} and E_2^{AB} can be set to zero in those integrals. In this situation, one can obtain closed forms for the amplitudes τ_{AB} and τ_{AA}:

$$\tau^{-1}_{AB}\left(-\frac{\mathcal{A}+2}{2(\mathcal{A}+1)}q^2\right) = -q^{D-2}\left(\frac{\mathcal{A}+2}{2(\mathcal{A}+1)}\right)^{D/2-1}\left(\frac{2\mathcal{A}}{\mathcal{A}+1}\right)^{D/2}\frac{\pi^{D/2}}{\Gamma(D/2)} \times\Gamma\left(D/2-1\right)\Gamma\left(2-D/2\right), \tag{57}$$

$$\tau^{-1}_{AA}\left(-\frac{\mathcal{A}+2}{4\mathcal{A}}q^2\right) = -q^{D-2}\left(\frac{\mathcal{A}+2}{4\mathcal{A}}\right)^{D/2-1}\frac{\pi^{D/2}}{\Gamma\left(D/2\right)} \times\Gamma\left(D/2-1\right)\Gamma\left(2-D/2\right), \tag{58}$$

where $\Gamma(z)$ is the gamma function, defined for all complex numbers z except for the non-positive integers. This restricts the validity of our results to the interval $2 < D < 4$. Their solutions are homogeneous functions, that is, the amplitudes $f_A(q)$ and $f_B(q)$ are given by

$$f_A(q) = C_A\,q^{r+is} \;\text{ and }\; f_B(q) = C_B\,q^{r+is}, \tag{59}$$

where r and s are real numbers. These solutions are the well-known log-periodic functions, associated with the infinitely many three-body bound states in the Efimov limit. Using Eq. (53) in Eqs. (54) and (59) leads to a complex homogeneous linear matrix equation for the coefficients C_A and C_B. The parameters r and s are found by solving the corresponding characteristic equation. We have determined numerically that $r = 1 - D$ for all D, which removes any ultraviolet divergence. Once s and r have been found numerically, we have than made a consistency check, in that we replaced $r = 1 - D + \epsilon$ in the characteristic equation and expanded it in powers of ϵ and checked analytically that the only possible solution for epsilon is $\epsilon = 0$ when we insert the value of s found numerically for all D and mass ratios. The characteristic equation is a transcendental equation in r and s and is given by

$$\left(\frac{\mathcal{A}+2}{2(\mathcal{A}+1)}\right)^{D/2-1}\left(\frac{2\mathcal{A}}{\mathcal{A}+1}\right)^{D/2} = \mathcal{F}_D\left[\mathcal{A}\,I_1(\mathcal{A},s) + 2\left(\frac{4\mathcal{A}}{\mathcal{A}+2}\right)^{D/2-1} \times\mathcal{F}_D\,I_2(\mathcal{A},s)I_3(\mathcal{A},s)\right], \tag{60}$$

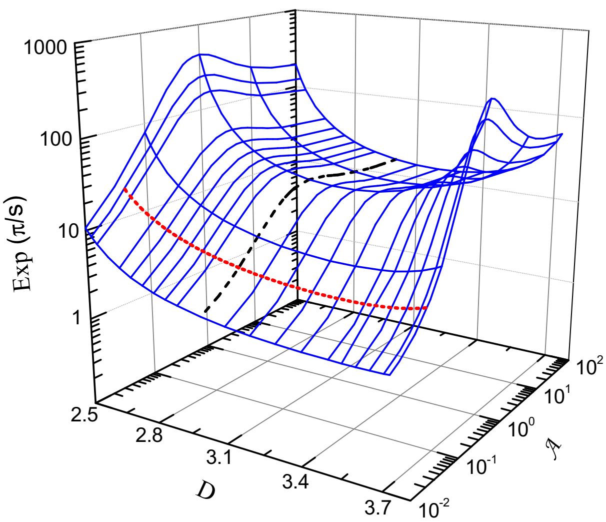

Fig. 1 Discrete scaling factor as a function of the mass ratio $\mathcal{A} = m_B/m_A$, and dimension D. The black dashed line shows the well-known situation of $D = 3$ and the red dotted line the values for the heteronuclear ^{6}Li-^{133}Cs mixture, $\mathcal{A} = 6/133$

where

$$\mathcal{F}_D = \frac{1}{\Gamma\left(D/2-1\right)\Gamma\left(2-D/2\right)}, \tag{61}$$

and

$$I_1(\mathcal{A}, s) = \int_0^\infty dz \frac{z^{is}}{z} \log\left[\frac{(z^2+1)(\mathcal{A}+1)+2z}{(z^2+1)(\mathcal{A}+1)-2z}\right], \tag{62}$$

$$I_2(\mathcal{A}, s) = \int_0^\infty dz \frac{z^{is}}{z} \log\left[\frac{2\mathcal{A}(z^2+z)+\mathcal{A}+1}{2\mathcal{A}(z^2-z)+\mathcal{A}+1}\right], \tag{63}$$

$$I_3(\mathcal{A}, s) = \int_0^\infty dz \frac{z^{is}}{z} \log\left[\frac{2\mathcal{A}(1+z)+(\mathcal{A}+1)z^2}{2\mathcal{A}(1-z)+(\mathcal{A}+1)z^2}\right], \tag{64}$$

which are the same integrals found in Ref. [11] for the $D = 3$ problem.

Figure 1 shows the value of the discrete scaling factor $\exp(\pi/s)$ as a function of the mass ratio $\mathcal{A}$. The black dashed line indicates the well-known result for $D = 3$ [12]. The most symmetrical case, where $\mathcal{A} = 1$, presents the worst situation to observe consecutive Efimov excited states as for any D the scaling factor presents a maximum for this mass ratio. The red dotted line shows a more favourable case of a heteronuclear ^{6}Li-^{133}Cs system, $\mathcal{A} = 6/133$, where the gap between the energy levels is decreased comparing to $\mathcal{A} = 1$.

The experimental connection of the present result for ^{6}Li-^{133}Cs can be made through the Fig. 2 where it is plotted the effective dimension, D, as a function of an oscillator length in the squeezed direction, b_ω. The inset shows the ratio of the first to second excited state three-body energies calculated by solving the Faddeev equations in momentum space with a compactified dimension as detailed in Ref. [13]. With this ratio we can extract s and then Fig. 1 returns D [14]. a_3 is the AB scattering length for $D = 3$, for the situation in which E_2^{AB} is kept fixed to its value for $D = 3$ (dashed curves in Fig. 2 of Ref. [13]).

4 Conclusions

Excluding the Sect. 3, the text in this article was written focusing the students. We tried to include the concepts which we judged to be important to start a study in momentum space techniques in Few-Body Physics. Section 3 shows our prediction of the Efimov discrete scaling factor, $\exp(\pi/s)$, as a function of a wide range of values

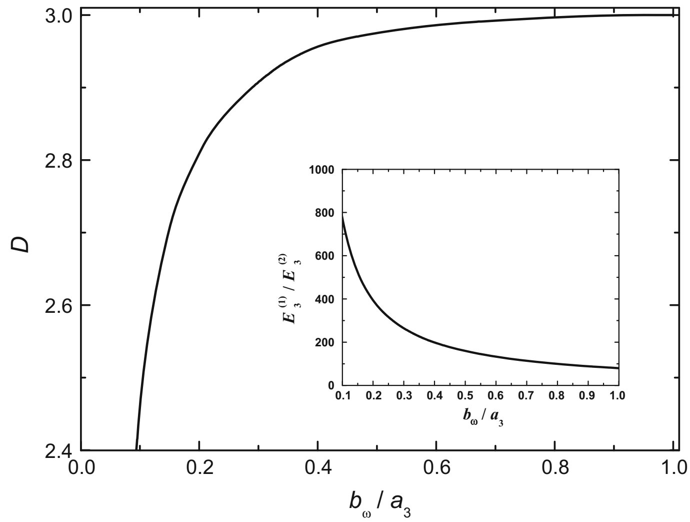

Fig. 2 Effective dimension D for $\mathcal{A} = 6/133$ as a function of b_ω/a_3, where a_3 is the AB scattering length for $D = 3$ and b_ω is the oscillator length in the squeezed direction. The inset shows the ratio of the first to second excited state three-body energies [13]

of $\mathcal{A}$ and D, which can be tested in experiments through the measurements of consecutive peaks associated to the loss of the atoms of the ultracold trap due to three-body recombination processes. A direct connection of the effective dimension with the oscillator length in the squeezed direction is showed for the realistic case of a ^{6}Li-^{133}Cs system.

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