Topological Methods in Nonlinear Analysis
Volume 46, No. 2, 2015, 697–715
DOI: 10.12775/TMNA.2015.081

FUNCTIONS AND VECTOR FIELDS ON $C(\mathbb{C}P^n)$-SINGULAR MANIFOLDS

Alice K.M. Libardi — Vladimir V. Sharko

Abstract. In this paper we study functions and vector fields with isolated singularities on a $C(\mathbb{C}P^n)$-singular manifold. In general, a $C(\mathbb{C}P^n)$-singular manifold is obtained from a smooth $(2n+1)$-manifold with boundary which is a disjoint union of complex projective spaces $\mathbb{C}P^n \cup \ldots \cup \mathbb{C}P^n$ and subsequent capture of the cone over each component $\mathbb{C}P^n$ of the boundary. We calculate the Euler characteristic of a compact $C(\mathbb{C}P^n)$-singular manifold M^{2n+1} with finite isolated singular points. We also prove a version of the Poincaré–Hopf Index Theorem for an almost smooth vector field with finite number of zeros on a $C(\mathbb{C}P^n)$-singular manifold.

1. Introduction

A manifold with isolated singularities is a topological space M which has the structure of a smooth (C^∞) manifold in $M \setminus S$, where S is the discrete set of singular points of M. A diffeomorphism between two such manifolds M and N is a homeomorphism from M into N such that sends the set of singular points of M onto the set of singular points of N and is a diffeomorphism outside of them. We say that M has a cone-like singularity at a (singular) point $P \in S$ if there exists a neighbourhood of the point P diffeomorphic to a cone over

2010 *Mathematics Subject Classification.* Primary: 05C10; Secondary: 57R45.
Key words and phrases. Semi-free circle action, manifold, S^1-invariant Bott function, Morse number, Poincaré–Hopf index.
Research partially supported by FUNDUNESP (PROPG–UNESP) and FAPESP – Fundação de Amparo a Pesquisa do Estado de São Paulo, Projeto Temático Topologia Algébrica, Geométrica e Diferencial – 2012/24454-8

a smooth manifold W_P (W_P is called the link at the point P). In what follows we assume all manifolds have only isolated cone-like singularities, more precisely $C(\mathbb{C}P^n)$-singular manifolds.

In this work it is considered almost Morse functions on $C(\mathbb{C}P^n)$-singular manifolds and it is given an answer for a particular case of the following unsolved problem: for any $C(\mathbb{C}P^n)$-singular manifold M^{2n+1} with singular points $m_1, \ldots, m_k$ and any collection of almost Morse functions $St = \pi_*(f_1), \ldots, \pi_*(f_k)$ in the neighbourhoods $U(m_1), \ldots, U(m_k)$ *find exact values* of Morse number $\mathcal{M}_\lambda(M^{2n+1}, St)$ of index λ. We point out that the notion of almost Morse function is close related to the notion of a stratified Morse function. (See the classical book of Goresky–MacPherson [7].)

In this setting, one has the following result:

Theorem 3.10. *Let M^{2n+1}, $(2n \geq 5)$, be a compact simply connected $C(\mathbb{C}P^n)$-singular manifold with singular points $m_1, \ldots, m_k$. Let σ be a permutation of $(1, \ldots, k)$ and let A (with s points) and B (with $k-s$ points) be the split of the singular points $m_1, \ldots, m_k$ into two disjoint sets*:

$$A = m_{\sigma(1)}, \ldots, m_{\sigma(s)}, \qquad B = m_{\sigma(s+1)}, \ldots, m_{\sigma(k)}.$$

We fix a collection of almost Morse functions

$$St = \underbrace{\pi_*(f_1), \ldots, \pi_*(f_1)}_{s}, \underbrace{\pi_*(f_2), \ldots, \pi_*(f_2)}_{k-s}$$

in the neighbourhoods $U(m_{\sigma(1)}), \ldots, U(m_{\sigma(s)}), U(m_{\sigma(s+1)}), \ldots, U(m_{\sigma(k)})$ respectively, where

$$f_1 = \sum_{i=1}^{2n} |z_i|^2, \qquad f_2 = 1 - \sum_{i=1}^{2n} |z_i|^2.$$

Then

$$\mathcal{M}_\lambda(M^{2n+1}, St) = \mu(H_\lambda(M^{2n+1} \setminus B, A, \mathbb{Z})) + \mu(\operatorname{Tors} H_{\lambda-1}(M^{2n+1} \setminus B, A, \mathbb{Z})),$$

where $\mu(H)$ is the minimal number of generators of the group H.

A (smooth) vector field on a manifold M with isolated singularities is a (smooth) vector field on the set of regular points of M. The set of singular points S_X of a vector field X on a (singular) manifold M is the union of the set of usual singular points of X on M (i.e. points at which X tends to zero) and of the set S of singular points of M itself. For an isolated usual singular point P of a vector field X there is defined its index $\mathrm{ind}_P X$.

Inspired on the book "Vector Fields on Singular Varieties" [5] we study vector fields on manifolds with isolated cone-like singularities and present a proof of a version of the Poincaré–Hopf Theorem. We recall that M.-H. Schwartz was the first to consider the index of vector fields on singular varieties. For her purposes, it was considered a special class of vector fields, called "radial" in [14], [15], [16]. In [5] it is defined an extension of this index for arbitrary stratified vector fields varieties, which is referred as "Schwartz index". This index was first defined by King and Trotman [8], [9] and later independently in [2], [6], [17].

In our case, we consider (M, P) a cone-like singularity (i.e. a germ of a manifold with such a singular point) and let X be a vector field defined on an open neighbourhood U of the point P. Suppose that X has no singular points on $U \setminus P$. Let V be a closed cone-like neighbourhood of the point P in U ($V = CW_P$, $V \subset U$). On the cone

$$CW_P = (W_p \times I)/(W_P \times 0)(I = [0,1]), \tag{1.1}$$

there is defined a natural vector field d/dt (t is out coordinate on I). Let X_{rad} be the corresponding vector field on V. Let $\widetilde{X}$ be a (continuous) vector field on U which coincides with X near the boundary ∂U of the neighbourhood U and with X_{rad} on V and has only isolated singular points.

Then we define the radial index $\text{Ind}_{\text{rad}}(X; M, P)$ of the vector field X at the point P to be equal to

$$1 + \sum \text{Ind}_{\widetilde{P}} \widetilde{X}_{\widetilde{P} \in S_{\widetilde{X}} \setminus \{P\}}$$

(the sum is over all singular points $\widetilde{P}$ of the vector field $\widetilde{X}$ except P itself).

This definition is compatible with the Definition 2.1.1 in [5] of the Schwartz index of an arbitrary vector field on a variety V, both with isolated singularity at 0. Using the sum of these indices in the singular points of the manifold and the Poincaré–Hopf index for the rest of zeros of the vector field it is given a proof for the general case (Theorem 2.1.1. [5]). We prove the version of the Theorem, step by step, adapting the requirements and using the results of the previous chapter of the paper.

Theorem 5.1 Let M^{2n+1} be a $C(\mathbb{C}P^n)$-singular manifold with singular points $m_1, \ldots, m_k$. Suppose that on M^{2n+1} there exists an almost smooth vector field $V(x)$ with finite number of zeros $m_1, \ldots, m_k, x_1, \ldots, x_l$. Then

$$\chi(M^{2n+1}) = \sum_{i=1}^{l} \text{ind}(x_i) + \sum_{i=1}^{k} \text{ind}(m_i).$$

This paper is organized into 4 sections. In Section 2 we give some results on manifolds with semi-free S^1-action which has only isolated fixed points. In Section 3 we study functions on $C(\mathbb{C}P^n)$-singular manifolds. The main result of this section is Theorem 3.8. In Section 4 we study vector fields on $C(\mathbb{C}P^n)$-singular manifolds and finally in Section 5 we present the proof of the main result of this work, which is Theorem 5.1.

2. Manifold with semi-free S^1-action which has only isolated fixed points

Let M^{2n+2} be a closed smooth manifold with semi-free S^1-action

$$\theta\colon S^1 \times M^{2n+2} \to M^{2n+2}$$

which has only isolated fixed points. It is known that every isolated fixed point m of a semi-free S^1-action has the following important property: near such a point the action is equivalent to a certain linear $S^1 = SO(2)$-action on $\mathbb{R}^{2n+2}$. More precisely, for every isolated fixed point m there exists an open invariant neighbourhood U of m and a diffeomorphism h from U to an open unit disk D^{2n+2} in $\mathbb{C}^{n+1}$ centered at origin such that h is conjugate to the given S^1-action on U to the S^1-action on $\mathbb{C}^n$ with weight $(1,\ldots,1)$. We will use both complex $(z_1,\ldots,z_{n+1})$ and real coordinates $(x_1,y_1,\ldots,x_{n+1},y_{n+1})$ on $\mathbb{C}^n = \mathbb{R}^{2n+2}$ with $z_j = x_j + \sqrt{-1}y_j$. The pair (U,h) will be called a *standard chart* at the point m.

The number of fixed points of any smooth semi-free circle action on M^{2n+2} with isolated fixed points is always even and equals to the Euler characteristic of the manifold M^{2n+2} ([12]).

Let M^{2n+2} be a manifold with finite many fixed points $m_1,\ldots,m_{2k}$. Denote by

$$\pi\colon M^{2n+2} \to M^{2n+2}/S^1$$

the canonical map. The set of orbits $N^{2n+1} = M^{2n+2}/S^1$ is a manifold with singular points $\pi(m_1),\ldots,\pi(m_{2k})$. It is clear that a neighbourhood of any singular point is a cone over $\mathbb{C}P^n$.

In general, a $C(\mathbb{C}P^n)$-singular manifold is obtained from a smooth $(2n+1)$-manifold with boundary which is a disjoint union of complex projective spaces $\mathbb{C}P^n \cup \ldots \cup \mathbb{C}P^n$ and subsequent capture of the cone over each component of the boundary. For this type of $C(\mathbb{C}P^n)$-singular manifold parity of the number of singular points depends on parity of the number n.

Proposition 2.1. *Let M^{2n+1} be a compact $C(\mathbb{C}P^n)$-singular manifold with k singular points. The Euler characteristic of M^{2n+1} is equal $\chi(M^{2n+1}) = k(1-n)/2$.*

Proof. To prove the proposition we consider the formula of the Euler characteristic: $\chi(X \cup Y) = \chi(X) + \chi(Y) - \chi(X \cap Y)$, X and Y being simplicial complexes.

Let $m_1,\ldots,m_k$ be singular points of M^{2n+1} and $U(m_1),\ldots,U(m_k)$ respective closed neighbourhoods each one homeomorphic to the cone over $\mathbb{C}P^n$. Consider the smooth manifold with boundary

$$N^{2n+1} = \overline{M^{2n+1} \setminus \bigcup_{i=1}^{k} U(m_i)}.$$

Denote by L^{2n+1} the double of the manifold N^{2n+1} along the boundary ∂N^{2n+1}

$$L^{2n+1} = N^{2n+1} \bigcup_{\partial N^{2n+1}} N^{2n+1}.$$

From the equation

$$0 = \chi(L^{2n+1}) = 2\chi(N^{2n+1}) - k\chi(\mathbb{C}P^n)$$

follows that the Euler characteristic of the manifold N^{2n+1} is equal to

$$\chi(N^{2n+1}) = \frac{k(n+1)}{2}.$$

Further $M^{2n+1} = N^{2n+1} \bigcup_{C(\mathbb{C}P^n)} U(m_1) \cup \ldots \cup \bigcup_{C(\mathbb{C}P^n)} U(m_k)$ and hence

$$\chi(M^{2n+1}) = \chi(N^{2n+1}) + k - k\chi(\mathbb{C}P^n).$$

Therefore $\chi(M^{2n+1}) = k(1-n)/2$. □

From the formula obtained in Proposition 1.1 we give a simple proof of the following results without the use of characteristic classes.

PROPOSITION 2.2. (a) *For n even, the complex projective space $\mathbb{C}P^n$ can not be the boundary of a smooth compact manifold X^{2n+1};*

(b) *For n odd, the complex projective space $\mathbb{C}P^n$ is the boundary of a compact smooth manifold.*

PROOF. (a) Suppose that n is an even number and that the complex projective space $\mathbb{C}P^n$ is the boundary of a smooth compact manifold X^{2n+1}. Consider the compact $C(\mathbb{C}P^n)$-singular manifold

$$M^{2n+1} = C(\mathbb{C}P^n) \cup_{\mathbb{C}P^n} X^{2n+1}.$$

From Proposition 2.1 it should be that the $\chi(M^{2n+1}) = (1-n)/2$. Since the Euler characteristic is an integer this is a contradiction.

(b) The manifold $\mathbb{C}P^n$ has homology in dimensions $0, 2, \ldots, (n-1), (n+1), \ldots, 2n$. Consider on the manifold $\mathbb{C}P^n$ a proper Morse function $f\colon \mathbb{C}P^n \to [0, 2n]$, such that all critical points of index λ lie on $f^{-1}(2(\lambda))$. We choose a submanifold $N_1 = f^{-1}[0, n-1/2]$. It is known ([1]) that N_1 is diffeomorphic to $\overline{\mathbb{C}P^n \setminus N_1}$. Consider the manifold $\mathbb{C}P^n \times [0,1]$ and identify means of the identity mapping submanifolds of $N_1^1 = N_1$ and $N_1^2 = N_1$. The manifold $N_1^1 = N_1$ belong to $\mathbb{C}P^n \times 0$ and the manifold $N_1^2 = N_1$ belong to $\mathbb{C}P^n \times 1$. After smoothing the corners along the submanifold ∂N_1 obtain that $\mathbb{C}P^n$ is boundary. See also [13].□

PROPOSITION 2.3. *Let M^{2n+1} be a compact $C(\mathbb{C}P^n)$-singular manifold with k singular points. If n is an odd number then the number k of singular points can be any number. If n is an even number the number k of singular points is an even number.*

Proof. If n is an odd number, then $\mathbb{C}P^n$ is the boundary of a compact smooth manifold N^{2n+1}.

We will denote by $\sharp$ the connected sum performed in the interior of the manifold. For any integer k consider the manifold

$$X^{2n+1} = \underbrace{N^{2n+1}\sharp \dots \sharp N^{2n+1}}_{k}$$

with boundary $\partial X^{2n+1} = \underbrace{\mathbb{C}P^n \cup \dots \cup \mathbb{C}P^n}_{k}$. Taking the cone over each component of the boundary of X^{2n+1} we get a $C(\mathbb{C}P^n)$-singular manifold with k singular points.

If n is an even number consider the following k times interior connected sum

$$Y^{2n+1} = \underbrace{(\mathbb{C}P^n \times I)\sharp \dots \sharp(\mathbb{C}P^n \times I)}_{k}.$$

Taking the cone over each component of the boundary of Y^{2n+1} we get a $C(\mathbb{C}P^n)$-singular manifold with $2k$ singular points.

Now we will show that for an even number n, this is the unique possibility which can occurs. Suppose that M^{2n+1} is a compact $C(\mathbb{C}P^n)$-singular manifold with k singular points, where the number k is odd. Let $m_1, \dots, m_k$ be singular points of M^{2n+1} and $U(m_1), \dots, U(m_k)$ respective closed neighbourhood each one homeomorphic to a cone over $\mathbb{C}P^n$. Consider the following smooth manifold

$$N^{2n+1} = \overline{M^{2n+1} \setminus \bigcup_{i=1}^{k} U(m_i)}.$$

with boundary $\partial N^{2n+1} = \underbrace{\mathbb{C}P^n \cup \dots \cup \mathbb{C}P^n}_{k}$.

We glue $(k-1)/2$ pairs of the component of the boundary of the manifold N^{2n+1}. The result is a smooth compact manifold with boundary $\mathbb{C}P^n$. This is a contradiction, since $\mathbb{C}P^n$ can not be the boundary of a smooth compact manifold of dimension $2n+1$. □

3. Functions on $C(\mathbb{C}P^n)$-singular manifolds

Since $C(\mathbb{C}P^n)$-singular manifolds are topological spaces we can consider continuous functions on them and because of the nature of $C(\mathbb{C}P^n)$-singular manifolds it is appropriate to consider continuous functions which are smooth on the complement of the set of singular points. Also it makes sense to study such functions on a $C(\mathbb{C}P^n)$-singular manifold whose singular points of the manifold are critical points of these functions. More precisely, this means the following.

Let M^{2n+1} be a compact $C(\mathbb{C}P^n)$-singular manifold M^{2n+1} with singular points $m_1, \dots, m_k$ and $U(m_1), \dots, U(m_k)$ respective closed neighbourhood each one homeomorphic to the cone over $\mathbb{C}P^n$. For any neighbourhood $U(m_i)$ there is a disc D_i^{2n+2} and a semi-free action of the circle $\theta\colon D_i^{2n+2}\times S^1\to D_i^{2n+2}$ such that $D_i^{2n+2}\xrightarrow{\pi} D_i^{2n+2}/S^1\approx U(m_i)$, where π is the canonical projection map.

We introduce in the disc D_i^{2n+2} complex coordinates $z_1,\dots,z_n$ and recall that the circle is the set of complex numbers of modulus one. We assume that the action of the circle on the disc is defined by the formula

$$\theta((z_1,\dots,z_n),t)=(e^{it}z_1,\dots,e^{it}z_n).$$

Consider an arbitrary S^1-invariant smooth function $f\colon D_i^{2n+2}\to\mathbb{R}$ with a single critical point in the center of the disc. For example, let f be given by

$$f=-|z_1|^2-\dots-|z_{\lambda_i}|^2+|z_{\lambda_i+1}|^2+\dots+|z_n|^2.$$

Notice that the index of the nondegenerate critical point $0\in D_i^{2n+2}$ of such function f is always even ([10]).

Let $\pi_*(f)\colon U(m_i)\to\mathbb{R}$ be the continuous function induced on $U(m_i)$ by the natural map

$$\pi\colon D_i^{2n+2}\to D_i^{2n+2}/S^1\approx U(m_i).$$

It is clear that the function $\pi_*(f)$ is smooth on the manifold $U(m_i)\setminus m_i$.

DEFINITION 3.1. The function $\pi_*(f)\colon U(m_i)\to\mathbb{R}$ is called almost smooth function on the neighbourhood $U(m_i)$ with a singularity at the point m_i. If f is given by

$$f=-|z_1|^2-\dots-|z_{\lambda_i}|^2+|z_{\lambda_i+1}|^2+\dots+|z_n|^2$$

then the function $\pi_*(f)\colon U(m_i)\to\mathbb{R}$ is called *almost Morse function on the neighbourhood* $U(m_i)$.

LEMMA 3.2. *Assume that M^{2n+1} is a compact $C(\mathbb{C}P^n)$-singular manifold with singular points $m_1,\dots,m_k$ and $U(m_1),\dots,U(m_k)$ respective closed neighbourhood each one homeomorphic to a cone over $\mathbb{C}P^n$. Let $\pi_*(f_i)\colon U(m_i)\to\mathbb{R}$ be an almost smooth function on the neighbourhood $U(m_i)$ with a singularity at the point m_i. Then there exists a continuous function f on M^{2n+1} such that $f=\pi_*(f_i)$ on $\widehat{U}(m_i)\subset U(m_i)$ and f is smooth on $M^{2n+1}\setminus\bigcup_{i=1}^{k} m_i$.*

PROOF. The proof of this assertion follows from the theorem on the extension of a smooth function on a closed set. Consider the partition of unity $W_1,\dots,W_k$, $V_1,\dots,V_l$ such that $m_i\in W_i\subset U(m_i)$ and V_j contains no points $m_1,\dots,m_k$. Let $\pi_*(f_1),\dots,\pi_*(f_k)$, $g_1,\dots,g_l$ be smooth functions on $W_1,\dots,W_k$, $V_1,\dots,V_l$, respectively. Then

$$\pi_*(f_1\mid W_1)+\dots+\pi_*(f_k\mid W_k)+g_1\mid V_1\dots+g_l\mid V_l$$

is the desired function on M^{2n+1}. □

Definition 3.3. A function $f\colon M^{2n+1}\to\mathbb{R}$ is called *almost smooth function* on the $C(\mathbb{C}P^n)$-singular manifold M^{2n+1} if f is almost smooth function on the neighbourhoods $U(m_i)$ of singular points m_i of M^{2n+1} and f is a smooth function on the smooth manifold $M^{2n+1}\setminus\bigcup_i m_i$.

From Lemma 3.2 follows that on any compact $C(\mathbb{C}P^n)$-singular manifold M^{2n+1} with singular points $m_1,\dots,m_k$ there exists an almost smooth function.

Corollary 3.4. *Assume that M^{2n+1} is a compact $C(\mathbb{C}P^n)$-singular manifold with singular points $m_1,\dots,m_k$. Suppose that $U(m_1),\dots,U(m_k)$ are their closed neighbourhood each one homeomorphic to a cone over $\mathbb{C}P^n$. Let $\pi_*(f_i)\colon U(m_i)\to\mathbb{R}$ be an almost Morse function in the neighbourhood $U(m_i)$. Then there exists a continuous function f in M^{2n+1} such that $f=\pi_*(f_i)$ on $U(m_i)$ and f is a Morse function in $M^{2n+1}\setminus\bigcup_{i=1}^{k} m_i$.*

Proof. By Lemma 3.2 there exists a continuous function f on M^{2n+1} such that $f=\pi_*(f_i)$ on $U(m_i)$ and f is smooth on $M^{2n+1}\setminus\bigcup_{i=1}^{k} m_i$. It is known, that Morse functions constitute an open and dense set in the space of smooth functions on a manifold (see [11]). Consider a small perturbation of function f, which is fixed on the sets $\bigcup_{i=1}^{k}\operatorname{Int}(U(m_i))$. The resulting function will satisfy the necessary conditions. □

Definition 3.5. A function $f\colon M^{2n+1}\to\mathbb{R}$ is called *almost Morse function* on the $C(\mathbb{C}P^n)$-singular manifold M^{2n+1} if f is an almost Morse function in the neighbourhoods $U(m_i)$ of singular points m_i of M^{2n+1} and f is a Morse function on the smooth manifold $M^{2n+1}\setminus\bigcup_i m_i$.

From Corollary 3.4 follows that on any compact $C(\mathbb{C}P^n)$-singular manifold M^{2n+1} with singular points $m_1,\dots,m_k$ there exists an almost Morse function. The number of critical points of an almost Morse function is dependent of the structure of such function in the neighbourhood of singular points of the $C(\mathbb{C}P^n)$-singular manifold. Let us examine this issue in more detail.

Definition 3.6. Let f be an almost Morse function on the $C(\mathbb{C}P^n)$-singular manifold M^{2n+1} with singular points $m_1,\dots,m_k$. Denote by $\pi_*(f_i)\colon U(m_i)\to\mathbb{R}$ its almost Morse function in the neighbourhood $U(m_i)$ of singular point m_i of M^{2n+1}. The *state* of the almost Morse function f is the collection $\pi_*(f_1),\dots,\pi_*(f_k)$ of all almost Morse functions in the neighbourhood $U(m_i)$ which we will be denoted by $St(f)=\pi_*(f_1),\dots,\pi_*(f_k)$.

Remark 3.7. It follows from Corollary 3.4 that for every $C(\mathbb{C}P^n)$-singular manifold M^{2n+1} with singular points $m_1,\dots,m_k$ and any collection of almost

Morse functions $\pi_*(f_1), \ldots, \pi_*(f_k)$ in the neighbourhoods $U(m_1), \ldots, U(m_k)$ there exists an almost Morse function f on M^{2n+1} with state

$$St(f) = \pi_*(f_1), \ldots, \pi_*(f_k).$$

DEFINITION 3.8. Let M^{2n+1} be a $C(\mathbb{C}P^n)$-singular manifold with singular points $m_1, \ldots, m_k$ and their cone neighbourhoods $U(m_1), \ldots, U(m_k)$. Fix any collection of almost Morse functions $St = \pi_*(f_1), \ldots, \pi_*(f_k)$ in neighbourhoods $U(m_1), \ldots, U(m_k)$. The Morse number $\mathcal{M}_\lambda(M^{2n+1}, St)$ of index λ is the minimum number of critical points of index λ taken over all almost Morse functions f on M^{2n+1} with state $St = \pi_*(f_1), \ldots, \pi_*(f_k)$.

Consider the case where M^{2n+1}, $(2n \geq 5)$, is a compact simply connected $C(\mathbb{C}P^n)$-singular manifold. Recall that for a simply connected smooth manifold we can calculate the Morse number via its homology groups. More precisely, if we consider a closed manifold N^n and Morse functions $f\colon N^n \to \mathbb{R}$ then to count the Morse number for the class of such functions we can use the homology group $H_j(N^n, \mathbb{Z})$.

If we consider a compact manifold N^n with boundary $\partial N^n = \partial_1 N^n \cup \partial_2 N^n$ and Morse functions $f\colon (N^n, \partial_1 N^n, \partial_2 N^n) \to \mathbb{R}$ such that $f^{-1}(0) = \partial_1 N^n$ and $f^{-1}(1) = \partial_2 N^n$ then to calculate the Morse numbers for this class of functions we use the group $H_j(N^n, \partial_1 N^n, \mathbb{Z})$ (see [18]).

Let M^{2n+1} $(2n \geq 5)$ be a compact simply connected $C(\mathbb{C}P^n)$-singular manifold with singular points $m_1, \ldots, m_k$. Let σ be a permutation of (1,...,k). We split the singular point $m_1, \ldots, m_k$ into two disjoint sets A and B consisting of s and $k - s$ points, respectively:

$$A = m_{\sigma(1)}, \ldots, m_{\sigma(s)}, \qquad B = m_{\sigma(s+1)}, \ldots, m_{\sigma(k)}.$$

The case when A or B is empty set is not excluded. Consider the homology groups $H_j(M^{2n+1} \setminus B, A, \mathbb{Z})$.

REMARK 3.9. If τ is another permutation of (1,...,k) and

$$\widetilde{A} = m_{\tau(1)}, \ldots, m_{\tau(s)}, \qquad \widetilde{B} = m_{\tau(s+1)}, \ldots, m_{\tau(k)}$$

is another splitting of the singular points $m_1, \ldots, m_k$ into two disjoint sets $\widetilde{A}$ and $\widetilde{B}$ then, in general,

$$H_j(M^{2n+1} \setminus B, A, \mathbb{Z}) \neq H_j(M^{2n+1} \setminus \widetilde{B}, \widetilde{A}, \mathbb{Z}).$$

Now, we state the main result of this section.

THEOREM 3.10. *Let M^{2n+1}, $(2n \geq 5)$, be a compact simply connected $C(\mathbb{C}P^n)$ -singular manifold with singular points $m_1, \ldots, m_k$. Let σ be a permutation of*

$(1,\dots,k)$ and let A (with s points) and B (with $k-s$ points) be the split of the singular points $m_1,\dots,m_k$ into two disjoint sets:

$$A = m_{\sigma(1)},\dots,m_{\sigma(s)}, \qquad B = m_{\sigma(s+1)},\dots,m_{\sigma(k)}.$$

We fix a collection of almost Morse functions

$$St = \underbrace{\pi_*(f_1),\dots,\pi_*(f_1)}_{s}, \underbrace{\pi_*(f_2),\dots,\pi_*(f_2)}_{k-s}$$

in the neighbourhoods $U(m_{\sigma(1)}),\dots,U(m_{\sigma(s)}),U(m_{\sigma(s+1)}),\dots,U(m_{\sigma(k)})$, respectively, where

$$f_1 = \sum_{i=1}^{2n} |z_i|^2, \qquad f_2 = 1 - \sum_{i=1}^{2n} |z_i|^2.$$

Then

$$\mathcal{M}_\lambda(M^{2n+1}, St) = \mu(H_\lambda(M^{2n+1}\setminus B, A, \mathbb{Z})) + \mu(\operatorname{Tors} H_{\lambda-1}(M^{2n+1}\setminus B, A, \mathbb{Z})),$$

where $\mu(H)$ is the minimal number of generators of the group H.

Proof. Let U_A and U_B be a disjoint union of neighbourhoods of singular points belonging to A and B, respectively. It is clear that the space $M^{2n+1}\setminus (A\cup B)$ is diffeomorphic to $M_1^{2n+1} = M^{2n+1}\setminus(U_A\cup U_B)$. It is known ([18]) that Morse number of index λ for Morse functions on cobordism $(M_1^{2n+1}, \partial A, \partial B)$

$$f\colon (M_1^{2n+1}, \partial A, \partial B) \to ([0,1],0,1)$$

is equal to

$$\begin{aligned}\mathcal{M}_\lambda(M^{2n+1}, \partial A, \partial B) \\ = \mu(H_\lambda(M^{2n+1}\setminus U_B), U_A, \mathbb{Z})) + \mu(\operatorname{Tors} H_{\lambda-1}(M^{2n+1}\setminus U_B), U_A, \mathbb{Z})).\end{aligned}$$

But it is the same with

$$\mathcal{M}_\lambda(M^{2n+1}, St) = \mu(H_\lambda(M^{2n+1}\setminus B, A, \mathbb{Z})) + \mu(\operatorname{Tors} H_{\lambda-1}(M^{2n+1}\setminus B, A, \mathbb{Z})). \quad \square$$

4. Vector fields on $C(\mathbb{C}P^n)$-singular manifolds

A compact $C(\mathbb{C}P^n)$-singular manifold M^{2n+1} can be equipped with a Riemannian metric as follows. Let $m_1,\dots,m_k$ be singular points of manifold M^{2n+1} and $U(m_1),\dots,U(m_k)$ closed neighbourhoods each one homeomorphic to a cone over $\mathbb{C}P^n$, respectively. It is clear that the manifold $V_i = U(m_i)\setminus m_i$ is diffeomorphic to $\mathbb{C}P^n\times\mathbb{R}_+$ ($\mathbb{R}_+$ denotes the set of non-negative real numbers). Let ρ be a Riemannian metric on $\mathbb{C}P^n$ and dx^2 metric on $\mathbb{R}$. On the manifold V_i Riemannian metric is set by using $\eta = \rho\oplus dx^2$. We consider in the manifold $M^{2n+1}\setminus \bigcup_{i=1}^{k} m_i$ the Riemannian metric η of manifold V_i.

Definition 4.1. Let M^{2n+1} be a compact $C(\mathbb{C}P^n)$-singular manifold with k singular points $m_1, \dots, m_k$. A vector field $V(x)$ on M^{2n+1} is called *almost smooth* vector field if $V(x)$ is smooth on the smooth manifold $M^{2n+1} \setminus \bigcup\limits_{i=1}^{k} m_i$ and is zero outside of this manifold. In addition, for each sequence of points $x_1, \dots, x_n, \dots$ tending to singular points $m_i, \dots, m_k$ $(i = 1, \dots, k)$ the sequence of norms of vector $\|V(x_1)\|, \dots, \|V(x_n)\|, \dots$ converges to zero in some Riemannian metric on M^{2n+1}.

Proposition 4.2. *Let M^{2n+1} be a $C(\mathbb{C}P^n)$-singular manifold with singular points $m_1, \dots, m_k$. Then on M^{2n+1} there exists an almost smooth vector field $V(x)$.*

Proof. By Corollary 3.2 on a $C(\mathbb{C}P^n)$-singular manifold M^{2n+1} there exists an almost smooth function f. We fix a Riemannian metric ρ on M^{2n+1} and consider on smooth manifold $M^{2n+1} \setminus \bigcup\limits_{i=1}^{k} m_i$ the vector field $\mathrm{grad}(f)$. We set

$$V(x) = \begin{cases} \mathrm{grad}(f) & \text{on } M^{2n+1} \setminus \bigcup\limits_{i=1}^{k} m_i, \\ 0 & \text{at the points } m_1, \dots, m_k. \end{cases}$$

By construction, the field $V(x)$ will be almost smooth vector field on $C(\mathbb{C}P^n)$-singular manifold M^{2n+1}. □

Remark 4.3. If an almost smooth function f has a finite number of critical points then the vector field $V(x)$ will have a finite number of points where $V(x)$ is zero.

Let $V(x)$ be an almost smooth vector field on $C(\mathbb{C}P^n)$-singular manifold with singular points $m_1, \dots, m_k$. Suppose that the vector field $V(x)$ have points $m_1, \dots, m_k$ as isolated singular points. We want to determine the index of the vector field $V(x)$ in the singular point m_i.

In what follows we will call zero (resp. zeros) for singular point (resp. singular points) of a vector field.

Let $V(x)$ be a smooth vector field on a smooth compact manifold N^{2m+1} with boundary with a finite number of points $C(\mathbb{C}P^n)$ in the interior of the manifold N^{2m+1} where $V(x)$ is zero. Suppose that the restriction of the field $V(x)$ on the boundary ∂N^{2m+1} of the manifold N^{2m+1} is outwardly directed to the manifold N^{2m+1}. Recall the definition of the index of a zero of the vector field $V(x)$ on a smooth manifold N^{2m+1}.

Let $D_\varepsilon^{2m+1}(n_i)$ be the ball of radius ε centered at the point n_i such that the vector field $V(x)$ has no zeros in it except at n_i. Let S_ε^{2m} be the $(2m)$-dimensional sphere $S^{2m}_\varepsilon = \partial D^{2m+1}_\varepsilon(n_i)$. The vector field $V(x)$ defines a map

$$\frac{V(x)}{\|V(x)\|} : S^{2m}_\varepsilon \to S^{2m}_1.$$

The index $\mathrm{ind}(n_i)_{V(x)}$ of the vector field $V(x)$ at the point n_i is defined as the degree of the map $V(x)/\|V(x)\|$. One can see that it is independent of the chosen coordinates around the point n_i.

Theorem 4.4 (Poincare–Hopf [5]).

$$\chi(N^{2m+1}) = \sum_{i=1}^{m}(-1)^i \mathrm{rank}(H_i(N^{2m+1}, \mathbb{Q})) = \sum_{i=1}^{l} \mathrm{ind}(n_i)_{V(x)}$$

where $H_i(N^{2m+1}, \mathbb{Q})$ is i-th homology group of the manifold N^{2m+1}.

This formula implies that the index of the point n_{i_o} is equal to

$$\mathrm{ind}(n_{i_o})_{V(x)} = \chi(N^{2m+1}) - \sum_{i=1, i\neq i_o}^{l} \mathrm{ind}(n_i)_{V(x)}.$$

This formula is used to define the index of the zero m of a vector field on a $C(\mathbb{C}P^n)$-singular manifold M^{2n+1} such that m is a singular point of M^{2n+1}.

Definition 4.5. Let N^{2m+1} be a cone over $C(\mathbb{C}P^n)$ and let $V(x)$ be an almost smooth vector field on N^{2m+1} such that the singular point $n \in N^{2m+1}$ is an isolated zero of $V(x)$, the field $V(x)$ has finite number of zeros $n_1, \dots, n_l$ and such zeros belong to $N^{2m+1} \setminus \partial N^{2m+1}$ and $V(x)$ on the boundary of the manifold N^{2m+1} is pointed out to the manifold N^{2m+1}. The index $ind(n)_{V(x)}$ of the point n of the vector field $V(x)$ is defined as the

$$\mathrm{ind}(n)_{V(x)} = \chi(N^{2m+1}) - \sum_{i=1}^{l} \mathrm{ind}(n_i)_{V(x)}.$$

For definition of the index at a zero of an arbitrary vector field on a cone over $C(\mathbb{C}P^n)$ we will need some additional constructions. First we prove the lemma.

Lemma 4.6. *Let N^{2m+1} be a cone over $C(\mathbb{C}P^n)$ and let $V(x)$ be an almost smooth vector field on N^{2m+1} such that the singular point $n \in N^{2m+1}$ is an isolated zero of $V(x)$. Then on singular manifold N^{2m+1} there exists an almost smooth vector field $W(x)$ such that*:

(a) *$W(x)$ coincides with the vector field $V(x)$ in some neighbourhood of singular point $n \in N^{2m+1}$;*
(b) *$W(x)$ has finite number of zeros and such zeros belong to $N^{2m+1} \setminus \partial N^{2m+1}$;*
(c) *$W(x)$ on the boundary of the manifold N^{2m+1} is pointed out to the manifold N^{2m+1}.*

PROOF. Since $n \in N^{2m+1}$ is an isolated zero of $V(x)$ then there is a neighbourhood $U(n)$ of the point n in which there are no other zeros of the vector field $V(x)$. Denote by $U(\partial N^{2m+1})$ some open neighbourhood of boundary of N^{2m+1} such that $U(n) \cap U(\partial N^m) = \emptyset$. We construct on the smooth manifold $\widetilde{N} = N^{2m+1} \setminus n$ a smooth function $\varphi\colon \widetilde{N} \to [0,1]$ such that:

- $\varphi_{|U(n)} = 1$;
- $\varphi_{|U(\partial N^{2m+1})} = 0$.

Consider on N^{2m+1} a smooth vector field $V_1(x)$ such that

- $V_1(x)$ on the boundary of the manifold N^{2m+1} is nonzero and pointed out to N^{2m+1};
- $V_1(x)_{|U(n)} = 0$.

We take on the $C(\mathbb{C}P^n)$-singular manifold N^{2m+1} the almost smooth vector field $W_1(x) = V_1(x) + \varphi(x) \cdot V(x)$. By construction $W_1(x)$ coincides with the vector field $V(x)$ on neighbourhood $U(n)$ of singular point $n \in N^{2m+1}$ and the vector field W_1^{2x+1} on the boundary of the manifold N^{2m+1} is nonzero and pointed out to the manifold N^{2m+1}. With small perturbation of vector field $W_1(x)$ which will be fixed in the neighbourhoods $U(n)$ and $U(\partial N^{2m+1})$ we construct the almost smooth vector field $W(x)$ which will have a finite number of points inside of the $C(\mathbb{C}P^n)$-singular manifold N^{2m+1} where $W(x)$ is zero. □

REMARK 4.7. In fact we can assume that the index of zeros of the vector field $W(x)$ on manifold $N^{2m+1} \setminus U(n)$ is ± 1.

PROOF. Indeed, if all zeros of the vector field $W(x)$ are isolated and this number is finite, then a small perturbation of $W(x)$ on smooth manifold $N^{2m+1} \setminus U(n)$ became non-degenerate. □

DEFINITION 4.8. Let N^{2m+1} be a cone over $C(\mathbb{C}P^n)$ and let $V(x)$ be an almost smooth vector field on N^{2m+1} such that the singular point $n \in N^{2m+1}$ is an isolated zero of $V(x)$. The *index* of the zero n of the vector field $V(x)$ is the number

$$\mathrm{ind}(n)_{V(x)} = \chi(N^{2m+1}) - \sum_{i=1}^{l} \mathrm{ind}(n_i)_{W(x)}.$$

where $n_1, \dots, n_l$ are all zeros belonging to $N^{2m+1} \setminus U(n)$ of the almost smooth vector field $W(x)$ given by Lemma 4.5.

We must show that so defined index at the singular point $n \in N^{2m+1}$ does not depend on the choice of the vector field $W(x)$. We prove the following lemma.

LEMMA 4.9. *Let N^{2n+1} be a smooth closed manifold and let $V(x)$ and $W(x)$ be smooth vector fields on $N^{2n+1} \times [0,1)$ such that:*

(a) *The vector field $W(x)$ coincides with the vector field $V(x)$ on $N^{2n+1} \times [1-\varepsilon, 1)$, where $0 < \varepsilon < 1$;*

(b) *Vector fields $V(x)$ and $W(x)$ have finite number of zeros*;

(c) *Vector fields $V(x)$ and $W(x)$ are not zero on the boundary $N^{2n+1} \times 0$ of the manifold $N^{2n+1} \times [0,1)$ and pointed out to the manifold $N^{2n+1} \times [0,1)$.*

Let $x_1, \dots, x_s$ and $y_1, \dots, y_t$ be zeros of the vector fields $V(x)$ and $W(x)$ respectively. Then

$$\sum_{i=1}^{s} \operatorname{ind}(x_i)_{V(x)} = \sum_{i=1}^{t} \operatorname{ind}(y_i)_{W(x)}.$$

Proof. Let $\delta > 0$ such that on collar $K = N^{2n+1} \times [1-\delta, 1) \subset N^{2n+1} \times [1-\varepsilon, 1)$ there is no zeros of vector fields $V(x)$ and $W(x)$. Consider smaller collar $K_1 = N^{2n+1} \times [1-\delta_1, 1) \subset K$. We construct on the smooth manifold $N^{2n+1} \times [0,1)$ smooth function $\varphi\colon N^{2n+1} \times [0,1) \to [0,1]$ such that:

- $\varphi_{|K_1} = 1$;
- $\varphi_{N^{2n+1} \times [0,1) \setminus N^{2n+1} \times (1-\delta,1)} = 0$.

Consider on manifold $N^{2n+1} \times [0,1)$ smooth unit vector field $G(x) = d/dt$ which is tangent to the lines $y \times [0,1)$ where $y \in N^{2n+1} \times 0$. It is clear that vector field $G(x)$ is transverse to any submanifold $N^{2n+1} \times t_0$, where $t_0 \in [0,1)$. Form the smooth vector fields

$$\begin{aligned} F_1(x) &= \varphi(x) \cdot G(x) + (1-\varphi(x)) \cdot V(x), \\ F_2(x) &= \varphi(x) \cdot G(x) + (1-\varphi(x)) \cdot W(x). \end{aligned}$$

By construction, vector fields $F_1(x)$ and $F_2(x)$ coincide on the set $N^{2n+1} \times [1-\varepsilon, 1) \setminus K$. On the set K_1 vector fields $F_1(x)$ and $F_2(x)$ coincide with $G(x)$. The vector field $F_1(x)$ on the set

$$N^{2n+1} \times [0,1) \setminus N^{2n+1} \times (1-\delta, 1)$$

coincides with vector field $V(x)$ and the vector field $F_2(x)$ on the set

$$N^{2n+1} \times [0,1) \setminus N^{2n+1} \times (1-\delta, 1)$$

coincides with vector field $W(x)$. Consider small perturbation H_1 of vector field F_1 which is fixed on the set $N^{2n+1} \times [0,1) \setminus N^{2n+1} \times (1-\delta,1)$ and on K_1 and such that on the set of $K \setminus K_1$ the vector field H_1 has only isolated zeros $z_1, \dots, z_p$. Similarly, let H_2 a small perturbation of vector field F_2 which is fixed on $N^{2n+1} \times [0,1) \setminus N^{2n+1} \times (1-\delta,1)$ and on K_1 and such that on the set of $K \setminus K_1$ vector field H_2 has only isolated zeros. We can assume that $H_1 = H_2$ on $K \setminus K_1$.

Consider a restriction of the vector fields H_1 and H_2 on manifold

$$D = N^{2n+1} \times [0,1) \setminus N^{2n+1} \times (1-\delta_1, 1).$$

By construction vector fields H_1 and H_2 are transversal to manifold $N^{2n+1} \times (1-\delta_1)$ and pointed out to the manifold D. On the set $K \setminus K_1$ vector fields H_1

and H_2 have the same zeros $z_1, \dots, z_p$. Hence for vector fields H_1 and H_2 we have that

$$\sum_{i=1}^{s} \operatorname{ind}(x_i)_{H_1(x)} + \sum_{j=1}^{p} \operatorname{ind}(z_j)_{H_1(x)} = \sum_{i=1}^{t} \operatorname{ind}(y_i)_{H_2(x)} + \sum_{j=1}^{p} \operatorname{ind}(z_j)_{H_2(x)},$$

which implies that

$$\sum_{i=1}^{s} \operatorname{ind}(x_i)_{H_1(x)} = \sum_{i=1}^{t} \operatorname{ind}(y_i)_{H_2(x)}.$$

But by the construction

$$\operatorname{ind}(x_i)_{H_1(x)} = \operatorname{ind}(x_i)_{V(x)} \quad \text{and} \quad \operatorname{ind}(y_i)_{H_2(x)} = \operatorname{ind}(y_i)_{W(x)},$$

therefore

$$\sum_{i=1}^{s} \operatorname{ind}(x_i)_{V(x)} = \sum_{i=1}^{t} \operatorname{ind}(y_i)_{W(x)}. \qquad \square$$

PROPOSITION 4.10. *Let N^{2m+1} be a cone over $C(\mathbb{C}P^n)$ and let $V(x)$ be an almost smooth vector field on N^{2m+1} such that the singular point $n \in N^{2m+1}$ is an isolated zero of $V(x)$. The index of the zero n of the vector field $V(x)$ in Definition* 4.6 *does not depend of the almost smooth vector field $W(x)$.*

PROOF. Let $V(x)$ be an almost smooth vector field on N^{2m+1} such that the singular point $n \in N^{2m+1}$ is an isolated zero of $V(x)$. By the Lemma 4.7 we can construct on N^{2m+1} two almost smooth vector fields $W_1(x)$ and $W_2(x)$. It is clear that manifold $N^{2m+1} \setminus n$ is diffeomorphic to $G_I \times [0,1)$, where G_I is diffeomorphic to ∂N^{2m+1}. Consider restriction vector fields $W_1(x)$ and $W_2(x)$ on the manifold $N^{2m+1} \setminus n$. Let $x_1, \dots, x_s$ and $y_1, \dots, y_t$ be zeros of the vector fields $W_1(x)$ and $W_2(x)$ on the manifold $N^{2m+1} \setminus n$ respectively. Then by definition of the index of the zero n of the vector field $V(x)$ is the number

$$\operatorname{ind}(n)_{V(x)} = \chi(N^{2m+1}) - \sum_{i=1}^{s} \operatorname{ind}(x_i)_{W_1(x)}.$$

or the number

$$\operatorname{ind}(n)_{V(x)} = \chi(N^{2m+1}) - \sum_{i=1}^{t} \operatorname{ind}(y_i)_{W_2(x)}.$$

From Lemma 4.7 we have that

$$\sum_{i=1}^{s} \operatorname{ind}(x_i)_{W_1(x)} = \sum_{i=1}^{t} \operatorname{ind}(y_i)_{W_2(x)},$$

which implies the independence of the index of the zero n of the almost smooth vector field on N^{2m+1}. $\square$

5. Proof of the Poincare–Hopf theorem

Theorem 5.1. *Let M^{2n+1} be a $C(\mathbb{C}P^n)$-singular manifold with singular points $m_1,\dots,m_k$. Suppose that on M^{2n+1} there exists an almost smooth vector field $V(x)$ with finite number of zeros $m_1,\dots,m_k$, $x_1,\dots,x_l$. Then*

$$\chi(M^{2n+1})=\sum_{i=1}^{l}\operatorname{ind}(x_i)+\sum_{i=1}^{k}\operatorname{ind}(m_i).$$

Proof. We consider first the simple case. Let $U(m_i)$ be a closed neighbourhood of singular point m_i such that $U(m_i)$ is homeomorphic to a cone over $C(\mathbb{C}P^n)$ and $\partial U(m_i)$ is a smooth submanifold of $M^{2n+1}\setminus\bigcup_{i=1}^{k} m_i$ $(i=1,\dots,k)$. Suppose that an almost smooth vector field $V(x)$ is *transverse* to $\partial U(m_i)$ and pointed out to $U(m_i)$. In addition, assume that the set of zeros of $V(x)$ is divided into disjoint subsets

$$A=\{x_1^0,\dots,x_{s_0}^0\}\quad\text{and}\quad B_i=\{x_1^i,\dots,x_{s_i}^i\}\quad(i=1,\dots,k),$$

where $s_0+\dots+s_k=l$ and $A\subset M^n\setminus\bigcup_{i=1}^{k}U(m_i)$ and $B_i\subset U(m_i)$. Note that A or B_i may be empty. We have that

$$\chi(M^{2n+1})=\chi\bigg(\overline{M^n\setminus\bigcup_{i=1}^{k}U(m_i)}\bigg)+\chi\bigg(\bigcup_{i=1}^{k}U(m_i)\bigg)-\chi\bigg(\bigcup_{i=1}^{k}\partial U(m_i)\bigg).$$

Consider restriction of vector field $V(x)$ on the smooth manifold with boundary

$$N^{2n+1}=\overline{M^n\setminus\bigcup_{i=1}^{k}U(m_i)}$$

and denote this restriction by $\overline{V}(x)$. Of course the subset A of zeros of vector field $V(x)$ coincides with the set of all zeros of vector field $\overline{V}(x)$. For vector field $\overline{V}(x)$ we have from

$$\chi(N^{2n+1})-\chi(\partial N^{2n+1})=\sum_{j=1}^{s_0}\operatorname{ind}(x_j^0)$$

(see [5]). Let $\widetilde{V}_i(x)$ be the restriction of vector field $V(x)$ on the $C(\mathbb{C}P^n)$-singular manifold $U(m_i)$. For vector field $\widetilde{V}_i(x)$ using the Definition 4.4 we have

$$\chi(U(m_i))=\operatorname{ind}(m_i)+\sum_{j=1}^{s_i}\operatorname{ind}(x_j^i).$$

and therefore

$$\chi\bigg(\bigcup_{i=1}^{k}U(m_i)\bigg)=\sum_{i=1}^{k}\operatorname{ind}(m_i)+\sum_{i=1}^{k}\bigg(\sum_{j=1}^{s_i}\operatorname{ind}(x_j^i)\bigg).$$

It is clear that

$$\sum_{j=1}^{s_0} \operatorname{ind}(x_j^0) + \sum_{i=1}^{k} \operatorname{ind}(m_i) + \sum_{i=1}^{k} \left(\sum_{j=1}^{s_i} \operatorname{ind}(x_j^i) \right) = \chi(N^{2n+1}) - \chi(\partial N^{2n+1}) + \chi\left(\bigcup_{i=1}^{k} U(m_i) \right).$$

But by construction

$$\chi(M^{2n+1}) = \chi(N^{2n+1}) - \chi(\partial N^{2n+1}) + \chi\left(\bigcup_{i=1}^{k} U(m_i) \right)$$

and therefore

$$\chi(M^{2n+1}) = \sum_{i=0}^{l} \operatorname{ind}(x_i) + \sum_{i=1}^{k} \operatorname{ind}(m_i).$$

Suppose now that the almost smooth vector field $V(x)$ is *nontransverse* to $\partial U(m_i)$. Without loss of generality, we can assume that the vector field $V(x)$ has no zeros in the neighbourhood $U(m_i)$, with the exception of the singular point m_i. This may be achieved by reducing neighbourhood $U(m_i)$. We will construct on $U(m_i)$ a new almost smooth vector field $V_{i,1}(x)$ such that:

- $V_{i,1}(x)$ coincides with the vector field $V(x)$ on the neighbourhood of $\partial U(m_i)$ and at some neighbourhood of the point m_i;
- $V_{i,1}(x)$ will be transverse to the boundary of the smallest neighbourhood $U_1(m_i) \subset U(m_i)$ of the point m_i;
- $V_{i,1}(x)$ has finite number of zeros $y_1^i, \dots, y_{r_i}^i, m_i$ on $U(m_i)$;
- The sum $\sum_{i=1}^{r} \operatorname{ind}(y_i) = 0$.

As before, let $\widetilde{V}_i(x)$ be the restriction of vector field $V(x)$ on $C(\mathbb{C}P^n)$-singular manifold $U(m_i)$. It is clear that the smooth manifold $W(m_i) = U(m_i) \setminus m_i$ is diffeomorphic to $G_i \times [0,1)$ (where G_i is diffeomorphic to $\partial U(m_i)$). Consider in $W(m_i)$ smooth submanifold $T_{i,1} = G_i \times t_1$, where $t_1 \in (0,1/2)$ and let $\widetilde{V}_{T_{i,1}}(x)$ denote the restriction of the vector field $\widetilde{V}_i(x)$ on submanifold $T_{i,1}$. Cut the manifold $W(m_i)$ along the submanifold $T_{i,1}$. As a result, we obtain

$$W(m_i) \setminus T_{i,1} = W_{i,1} \cup W_{i,2}.$$

By construction the smooth manifold $W_{i,1}$ is diffeomorphic to $G_i \times [0, t_1)$ and the smooth manifold $W_{i,2}$ is diffeomorphic to $G_i \times (t_1, 1)$. Denote by

$$Z_{i,1} = \overline{W}_{i,1} = G_i \times [0, t_1] \quad \text{and} \quad Z_{i,2} = \overline{W}_{i,2} \setminus G_i \times 1 = G_i \times [t_1, 1).$$

It is clear, that on manifolds $Z_{i,1}$ and $Z_{i,2}$ there are vector fields $\widetilde{V}_{Z_{i,1}}(x)$ and $\widetilde{V}_{Z_{i,2}}(x)$ respectively, obtained by restrictions on $Z_{i,1}$ and $Z_{i,2}$ of the vector field $\widetilde{V}_i(x)$. Consider manifold $Q_i = G_i \times [t_1, 2t_1]$ and we will construct on Q_i the vector field $\widetilde{V}_{Q_i}(x)$ putting $\widetilde{V}_{Q_i}(x)_{|G_i\times t} = \widetilde{V}_{T_{i,1}}(x)$, $(t \in [t_1, 2t_1])$. Glue together the components of the boundary of manifold Q_i using the identity map. As result we obtain a smooth manifold $Q_{i,1} = G_i \times S^1$ and a non-singular vector field $\widetilde{V}_{Q_{i,1}}(x)$ on it.

Let points $a, b \in S^1$ and consider the fibres $F_a = G_I \times a$ and $F_b = G_I \times b$. Denote by $U(F_a)$ some open neighbourhood of F_a in manifold $Q_{i,1}$ such that $U(F_a) \cap F_b = \emptyset$. With small perturbation of vector field $\widetilde{V}_{Q_{i,1}}(x)$ on the manifold $Q_{i,1}$ new vector field $\widehat{V}_{Q_{i,1}}(x)$ is obtained such that:

- $\widehat{V}_{Q_{i,1}}(x)$ coincides with the vector field $\widetilde{V}_{Q_{i,1}}(x)$ on the neighbourhood $U(F_a)$;
- $\widehat{V}_{Q_{i,1}}(x)$ is transverse to F_b in the desired direction;
- $\widehat{V}_{Q_{i,1}}(x)$ has finite number of zeros $y_1^i, \dots, y_{r_i}^i$.

Since the Euler characteristic of the manifold $Q_{i,1}$ is zero then

$$\sum_{j=1}^{r} \operatorname{ind}(y_j) = 0.$$

If we cut the manifold $Q_{i,1}$ along the fibre F_a then we obtain an open manifold P_i diffeomorphic to $G_i \times (t_1, 2t_1)$. Denote by $\widehat{P}_i = \overline{P}_i \approx G_i \times [t_1, 2t_1]$ the closure of the manifold P_i. It is clear that vector field $\widehat{V}_{Q_{i,1}}(x)$ on the manifold $Q_{i,1}$ determines a vector field $\widehat{V}_{\widehat{P}_i}(x)$ on the manifold $\widehat{P}_i$. Glue together the manifolds $Z_{i,1}$ and $Z_{i,2}$ with help of the manifold $\widehat{P}_i$ which we denote by $\widehat{U}(m_i)$. It is clear that $G_1, \dots, G_k$-singular manifold $U(m_i)$ is homeomorphic to $\widehat{U}(m_i)$. By construction, vector fields $\widetilde{V}_{Z_{i,1}}(x)$, $\widetilde{V}_{Z_{i,2}}(x)$ and $\widehat{V}_{\widehat{P}_i}(x)$ "stapled" into one almost smooth vector field $V_{i,1}(x)$ on the manifold $\widehat{U}(m_i)$. For any index $i = 1, \dots, k$ do a surgery on $U(m_i)$ of vector field $V(x)$, in another words, using the vector fields $V_{i,1}(x)$ change vector field $V(x)$ on manifold M^n to a vector field $V_1(x)$. By construction the vector field $V_1(x)$ have zeros $m_1, \dots, m_k$, $x_1, \dots, x_l$, $y_1^i, \dots, y_{r_i}^i$ on the $C(\mathbb{C}P^n)$-singular manifold M^n and satisfies the transversality condition in the desired direction. Therefore as in the beginning of the proof we have that

$$\chi(M^{2n+1}) = \sum_{i=1}^{l} \operatorname{ind}(x_i) + \sum_{i=1}^{k} \left(\sum_{j=1}^{r_i} \operatorname{ind}(y_j^i) \right) + \sum_{i=1}^{k} \operatorname{ind}(m_i).$$

But $\sum_{j=1}^{r_i} \operatorname{ind}(y_j^i) = 0$ for any i and the theorem is proved. □

Acknowledgements. The authors are grateful to the referee for his/her attention and advice expressed in his review.

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Manuscript received February 20, 2014

accepted March 13, 2015

ALICE KIMIE MIWA LIBARDI
Departamento de Matemática
I.G.C.E – Unesp
Univeristy Estadual Paulista
Caixa Postal 178
Rio Claro 13500-230, BRAZIL

E-mail address: alicekml@rc.unesp.br

VLADIMIR V. SHARKO
Institute of Mathematics
National Academy of Sciences of Ukraine
UKRAINE