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Classic tests of General Relativity described by brane-based spherically symmetric solutions

R. R. Cuzinatto[1,a], **P. J. Pompeia**[2,3,b], **M. de Montigny**[4,5,c], **F. C. Khanna**[4,6,7,d], **J. M. Hoff da Silva**[8,e]

[1] Instituto de Ciência e Tecnologia, Universidade Federal de Alfenas, Rodovia José Aurélio Vilela, 11999, Cidade Universitária, Poços de Caldas, MG CEP 37715-400, Brazil
[2] Instituto de Fomento e Coordenação Industrial, Departamento de Ciência e Tecnologia Aeroespacial, Praça Mal. Eduardo Gomes 50, São José dos Campos, SP 12228-901, Brazil
[3] Instituto Tecnológico de Aeronáutica, Departamento de Ciência e Tecnologia Aeroespacial, Praça Mal. Eduardo Gomes 50, São José dos Campos, SP 12228-900, Brazil
[4] Theoretical Physics Institute, University of Alberta, Edmonton, AB T6G 2J1, Canada
[5] Campus Saint-Jean, University of Alberta, Edmonton, AB T6C 4G9, Canada
[6] TRIUMF, 4004, Westbrook Mall, Vancouver, BC V6T 2A3, Canada
[7] Department of Physics and Astronomy, University of Victoria, PO box 1700, STN CSC, Victoria, BC V8W 2Y2, Canada
[8] Departamento de Física e Química, Universidade Estadual Paulista, Av. Ariberto Pereira da Cunha 333, Guaratinguetá, SP 12.516-410, Brazil

Abstract We discuss a way to obtain information about higher dimensions from observations by studying a brane-based spherically symmetric solution. The three classic tests of General Relativity are analyzed in detail: the perihelion shift of the planet Mercury, the deflection of light by the Sun, and the gravitational redshift of atomic spectral lines. The braneworld version of these tests exhibits an additional parameter b related to the fifth-coordinate. This constant b can be constrained by comparison with observational data for massive and massless particles.

1 Introduction

Braneworld models have attracted considerable attention of the scientific community since the outstanding works by Randall and Sundrum [1,2]. The possibilities raised in such a framework have been extensively explored since then. In fact, from particle physics to cosmology, a plethora of braneworld models were investigated. In particular, the idea of standard model fields living only on the brane, a necessity in [1], was rapidly overcome [3].

At least from the gravitational point of view, the very idea behind braneworld models rests upon our belief that at high

[a] e-mail: rodrigo.cuzinatto@unifal-mg.edu.br

[b] e-mail: pedropjp@ifi.cta.br

[c] e-mail: mdemonti@ualberta.ca

[d] e-mail: fkhanna@ualberta.ca; khannaf@uvic.ca

[e] e-mail: hoff@feg.unesp.br

enough energies, General Relativity (GR) shall be at least corrected. In this vein, the new scenario provided by the braneworld picture has served as an interesting framework to cosmologists [4–8]. Again, this time within cosmology, from inflation to large scale consequences, the new possibilities for phenomenology provided by the braneworld paradigm have been extensively investigated (for a broad review, see [9]). An important point to be stressed, however, is that even far below high energy scales (which points to the transition between classical and quantum gravity) at Solar System size, there are interesting gravitational effects whose eventual modifications arising from the braneworld that can be compared with experiments.

The aim of this work is to explore classic tests of GR in a spherically symmetric four-dimensional solution embedded into a five-dimensional space. We use the metric

$$ds^2 = -\left(1 - \frac{2m}{\bar{r}}\right)(d\bar{x}^4)^2 + \frac{d\bar{r}^2}{\left(1 - \frac{2m}{\bar{r}}\right)} + \bar{r}^2 d\bar{\theta}^2 + \bar{r}^2 \sin^2\bar{\theta}\, d\bar{\varphi}^2 + (d\bar{x}^5)^2, \tag{1}$$

where $\bar{x}^5$ stands for the extra dimension. Let us make a few remarks about this expression. As usual, this line element is obtained by the Schwarzschild four-dimensional solution embedded into the extra dimension in the sense that at each $\bar{x}^5$ fixed slice we have the standard spherically symmetric solution. Obviously, this line element can be related to the so-called black-string [10]. Nevertheless it should be stressed that the solution presented in Eq. (1) is not necessarily related to the black-string, i.e., the four-dimensional

spherically symmetric metric need not be related to a black hole. In fact, here we shall set up the mass parameter to be far below the value necessary for a black-hole solution, and investigate how the embedding of the solution into the extra-dimensional scenario can be related to the classic tests performed in GR.

Let us point out that the classical tests of General Relativity have been examined for various spherically symmetric static vacuum solutions of braneworld models in Ref. [11]. Therein, the authors exploit the Gauss–Codazzi approach in order to find corrections of the GR results by embedding the brane into the bulk. This must be accomplished by means of the Israel–Darmois junction conditions, which are valid only for singular branes, that is, if the brane is infinitely thin. This can be observed from Eq. (7) in Ref. [11]. The constraints are imposed over the terms ($E_{\mu\nu}$ in Eq. (17) of [12]) of the five-dimensional Weyl tensor that carry information about the gravitational field outside the brane. Thus the main difference between the present study and Ref. [11] is that they deal with singular branes whereas we consider non-singular, or thick, branes [13]. Throughout the paper, branes will be understood in the sense that they are not necessarily singular. Note also that in Section 4.1 of Ref. [11], the authors claim that they obtain a DMPR-type solution (see Ref. [14]), which is the simplest solution for a spherically symmetric vacuum solution; this, too, does not contradict our results, since their results were found for singular branes, whereas our solution is a five-dimensional non-singular, non-spherically symmetric brane solution (only the four-dimensional section of our solution is spherically symmetric).

As we shall see, an additional parameter related to the extra dimension can be constrained by these tests in a quite compatible way for both, massive and massless test particles cases. We shall emphasize that in the context of universal extra dimensions [3], the fields must not be trapped on the brane but, instead, they are allowed to travel along the hole bulk. However, the extra dimension experienced by the fields shall be small (not to contradict $1/r^2$ deviations of Newton's law [15] in the case of gravitational experiments, and some key collider experiments [16]). Therefore the experimental boundaries applied in this work are used in order to make feasible an universal extra dimension from the point of view of classic Solar System tests.

This paper is structured as follows: after expressing the brane-based spherically symmetric solution in light-cone coordinates, predictions of the solution are confronted with three classic tests of GR in Sect. 2. In Sect. 2.1, we find that the solution describes the perihelion shift of Mercury as does four-dimensional Schwarzschild solution. In Sect. 2.2, we observe that the solution predicts the deflection of light rays by massive bodies like our Sun. In Sect. 2.3, we obtain a similar result for the gravitational shift of atomic spectral lines.

The results of these tests depend on an additional parameter b (see Eq. (15)), which is related to the fifth-coordinate. This constant b can be constrained by comparing with observational data and in Sect. 2.3, the result is interpreted via the uncertainty relations along the extra dimension. Section 3 contains some concluding remarks.

2 Classic tests

We consider a light-cone type transformation in spherical coordinates,

$$r = \bar{r}, \quad \theta = \bar{\theta}, \quad \varphi = \bar{\varphi}, \quad x^4 = \frac{\bar{x}^4 + \bar{x}^5}{\sqrt{2}}, \quad x^5 = \frac{\bar{x}^4 - \bar{x}^5}{\sqrt{2}}. \tag{2}$$

We shall recast the brane-based spherically symmetric solution in an appropriate way that suits our purpose. Hence when expressed in these coordinates, Eq. (1) becomes

$$ds^2 = \frac{dr^2}{1 - \frac{2m}{r}} + r^2 d\theta^2 + r^2 \sin^2\theta d\varphi^2 + \frac{m}{r}\left(dx^4\right)^2$$
$$+ \frac{m}{r}\left(dx^5\right)^2 + 2\left(-1 + \frac{m}{r}\right)dx^4 dx^5. \tag{3}$$

This line element describes the invariant interval on the curved manifold in which the motion of particles and light rays will take place. In the next section, we shall study these aspects in order to test the brane-based spherically symmetric solution.

2.1 Planetary motion

The motion of test particles is described by the geodesic equations,

$$\frac{d^2 x^\mu}{ds^2} + \Gamma^\mu_{\rho\nu} \frac{dx^\rho}{ds} \frac{dx^\nu}{ds} = 0.$$

In order to find solutions, this equation is written as

$$\frac{d^2 r}{ds^2} - \frac{m}{r^2} \frac{1}{\left(1 - \frac{2m}{r}\right)} \left(\frac{dr}{ds}\right)^2$$
$$- \left(1 - \frac{2m}{r}\right) r \left(\frac{d\theta}{ds}\right)^2 - \left(1 - \frac{2m}{r}\right) r \sin^2\theta \left(\frac{d\varphi}{ds}\right)^2$$
$$+ \frac{m}{2r^2}\left(1 - \frac{2m}{r}\right)\left(\frac{dx^4}{ds} + \frac{dx^5}{ds}\right)^2 = 0, \tag{4}$$

$$\frac{d^2\theta}{ds^2} + \frac{2}{r}\frac{dr}{ds}\frac{d\theta}{ds} - \sin\theta\cos\theta\left(\frac{d\varphi}{ds}\right)^2 = 0, \tag{5}$$

$$\frac{d^2\varphi}{ds^2} + \frac{2}{r}\frac{dr}{ds}\frac{d\varphi}{ds} + 2\cot\theta\frac{d\theta}{ds}\frac{d\varphi}{ds} = 0, \tag{6}$$

$$\frac{\mathrm{d}^2 x^4}{\mathrm{d}s^2} + \frac{m}{r^2} \frac{1}{\left(1 - \frac{2m}{r}\right)} \frac{\mathrm{d}r}{\mathrm{d}s} \left(\frac{\mathrm{d}x^4}{\mathrm{d}s} + \frac{\mathrm{d}x^5}{\mathrm{d}s}\right) = 0, \tag{7}$$

$$\frac{\mathrm{d}^2 x^5}{\mathrm{d}s^2} + \frac{m}{r^2} \frac{1}{\left(1 - \frac{2m}{r}\right)} \frac{\mathrm{d}r}{\mathrm{d}s} \left(\frac{\mathrm{d}x^4}{\mathrm{d}s} + \frac{\mathrm{d}x^5}{\mathrm{d}s}\right) = 0. \tag{8}$$

We show that the motion lies in a plane as it happens in classical mechanics for a central force [17]. With an appropriate orientation of the axis, we can choose the initial conditions to be

$$\theta_0 = \frac{\pi}{2}, \qquad \left(\frac{\mathrm{d}\theta}{\mathrm{d}s}\right)_0 = 0,$$

for some initial value of s. This choice implies that the motion of the test particle starts at the ecliptic plane with zero initial azimuthal velocity. It means also that Eq. (5) gives a zero initial elevation acceleration, $\left(\frac{\mathrm{d}^2\theta}{\mathrm{d}s^2}\right)_0 = 0$. Thus, at an infinitesimal proper instant later, $\theta_{\Delta s} = \frac{\pi}{2}$ and $\left(\frac{\mathrm{d}\theta}{\mathrm{d}s}\right)_{\Delta s} = 0$, and similarly after another Δs, and so on. As a result, the motion is confined to the plane $\theta = \frac{\pi}{2}$.

The geodesic equations, Eqs. (4)–(8), are then simplified to

$$\frac{\mathrm{d}^2 r}{\mathrm{d}s^2} - \frac{m}{r^2} \frac{1}{\left(1 - \frac{2m}{r}\right)} \left(\frac{\mathrm{d}r}{\mathrm{d}s}\right)^2 - \left(1 - \frac{2m}{r}\right) r \left(\frac{\mathrm{d}\varphi}{\mathrm{d}s}\right)^2$$
$$+ \frac{1}{2} \frac{m}{r^2} \left(1 - \frac{2m}{r}\right) \left(\frac{\mathrm{d}x^4}{\mathrm{d}s} + \frac{\mathrm{d}x^5}{\mathrm{d}s}\right)^2 = 0, \tag{9}$$

$$\frac{\mathrm{d}^2 \varphi}{\mathrm{d}s^2} + \frac{2}{r} \frac{\mathrm{d}r}{\mathrm{d}s} \frac{\mathrm{d}\varphi}{\mathrm{d}s} = 0, \tag{10}$$

$$\frac{\mathrm{d}^2 x^4}{\mathrm{d}s^2} + \frac{m}{r^2} \frac{1}{\left(1 - \frac{2m}{r}\right)} \frac{\mathrm{d}r}{\mathrm{d}s} \left(\frac{\mathrm{d}x^4}{\mathrm{d}s} + \frac{\mathrm{d}x^5}{\mathrm{d}s}\right) = 0, \tag{11}$$

$$\frac{\mathrm{d}^2 x^5}{\mathrm{d}s^2} + \frac{m}{r^2} \frac{1}{\left(1 - \frac{2m}{r}\right)} \frac{\mathrm{d}r}{\mathrm{d}s} \left(\frac{\mathrm{d}x^4}{\mathrm{d}s} + \frac{\mathrm{d}x^5}{\mathrm{d}s}\right) = 0. \tag{12}$$

If we multiply Eq. (10) by r^2, we find

$$r^2 \frac{\mathrm{d}\varphi}{\mathrm{d}s} = h, \tag{13}$$

where h is a constant related to the conserved angular momentum of the particle. Similarly, by adding Eqs. (11) and (12), and by multiplying the result by $\left(1 - \frac{2m}{r}\right)$, we are led to

$$\frac{\mathrm{d}x^4}{\mathrm{d}s} + \frac{\mathrm{d}x^5}{\mathrm{d}s} = \frac{k}{\left(1 - \frac{2m}{r}\right)}, \tag{14}$$

where k is another constant. By subtracting Eq. (11) from Eq. (12), we find

$$\frac{\mathrm{d}x^4}{\mathrm{d}s} - \frac{\mathrm{d}x^5}{\mathrm{d}s} = b, \tag{15}$$

where b is constant. From Eqs. (14) and (15), we obtain

$$\frac{\mathrm{d}x^4}{\mathrm{d}s} = \frac{1}{2} \left[\frac{k}{\left(1 - \frac{2m}{r}\right)} + b\right], \tag{16}$$

and

$$\frac{\mathrm{d}x^5}{\mathrm{d}s} = \frac{1}{2} \left[\frac{k}{\left(1 - \frac{2m}{r}\right)} - b\right]. \tag{17}$$

In principle, both constants k and b could be associated to the extra dimension. However, as will be seen, the constant k does not contribute to the orbital equation obtained below.

By substituting Eqs. (13) and (14) into Eq. (9), it follows that

$$\frac{\mathrm{d}^2 r}{\mathrm{d}s^2} - \frac{m}{r^2} \frac{1}{\left(1 - \frac{2m}{r}\right)} \left(\frac{\mathrm{d}r}{\mathrm{d}s}\right)^2$$
$$- \left(1 - \frac{2m}{r}\right) \frac{h^2}{r^3} + \frac{1}{2} \frac{m}{r^2} \frac{k^2}{\left(1 - \frac{2m}{r}\right)} = 0.$$

As in the classic Kepler problem, we can simplify the integration processes by considering r as a function of φ instead of s. If we change the variable r to

$$u = \frac{1}{r},$$

it is possible to rewrite the last differential equation as

$$\frac{\mathrm{d}^2 u}{\mathrm{d}\varphi^2} + \frac{m}{(1 - 2mu)} \left(\frac{\mathrm{d}u}{\mathrm{d}\varphi}\right)^2$$
$$+ (1 - 2mu)\, u - \frac{1}{2} \frac{k^2}{h^2} \frac{m}{(1 - 2mu)} = 0. \tag{18}$$

The term proportional to $\left(\frac{\mathrm{d}u}{\mathrm{d}\varphi}\right)^2$ can be expressed in another form. For this, we use the constraint

$$g_{\mu\nu} \frac{\mathrm{d}x^\mu}{\mathrm{d}s} \frac{\mathrm{d}x^\nu}{\mathrm{d}s} = -1, \tag{19}$$

which leads to

$$\left(\frac{\mathrm{d}u}{\mathrm{d}\varphi}\right)^2 + (1 - 2mu) \left[u^2 + \frac{1}{h^2}\left(1 + \frac{b^2}{2}\right)\right] - \frac{1}{2} \frac{k^2}{h^2} = 0. \tag{20}$$

By inserting Eq. (20) into Eq. (18), we find the orbital equation,

$$\frac{\mathrm{d}^2 u}{\mathrm{d}\varphi^2} + u - \frac{m}{h^2} - 3mu^2 - \frac{m}{h^2}\frac{b^2}{2} = 0. \tag{21}$$

The first four terms are the usual ones obtained in the standard four-dimensional GR. The extra term should provide corrections related to the additional dimension.

Let us make an important remark concerning large distances. If this case is considered then the terms proportional to u^2 in Eq. (21) should be neglected, thus

$$\frac{d^2u}{d\varphi^2} + u - \frac{m}{h^2} - \frac{m}{h^2}\frac{b^2}{2} = 0.$$ (22)

The first three terms are the usual terms obtained by Newtonian gravitation. The additional term, proportional to b^2, is open for interpretation. If agreement with measurements are to be obtained, then either b should be negligibly small (which would leave us with the usual GR result) or b should be included in a renormalized value of m. In Sect. 2.2 we shall use $b \ll 1$.

2.1.1 The perihelion shift

Let us rewrite Eq. (21) as follows:

$$\frac{d^2u}{d\varphi^2} + u = \frac{m}{\bar{h}^2} + 3mu^2,$$ (23)

with

$$\frac{1}{\bar{h}^2} = \frac{1}{h^2}\left(1 + \frac{b^2}{2}\right).$$ (24)

Formally the orbital equation is exactly as predicted by GR, the only difference being the redefinition $h \rightarrow \bar{h}$. Then we know beforehand that the solution in question will predict a perihelion shift for the orbit of the planets consistent with GR.

The usual procedure is to obtain a solution of Eq. (23) through an iterative procedure taking $u \simeq u^{(0)} + u^{(1)}$ [24]. The zero-order is the unperturbed solution of Eq. (22), or Eq. (23) with $3mu^2 = 0$. It is utilized as a source term for the differential equation of $u^{(1)}$, i.e. we shall write $3mu^2 = 3m\left(u^{(0)}\right)^2$. The integration constants are the eccentricity of the orbit e and an arbitrary initial value φ_0 for the azimuthal angle. The constant e is related to the major axis $a = r_{max}$ by

$$a = \frac{L}{1 - e^2},$$ (25)

where

$$L = \frac{m}{\bar{h}^2}$$ (26)

is the semi-latus rectum of the orbit. We proceed as in GR by considering orbits of small eccentricity (like the ones of Mercury) and find that the perihelion, after a full revolution, shifts by

$$\Delta\varphi_0 = 6\pi\frac{m^2}{\bar{h}^2} = 6\pi\frac{m}{L} = 6\pi\frac{GM}{c^2a\left(1 - e^2\right)}.$$ (27)

2.1.2 Numerical analysis for the planet Mercury

Here, the quantities of interest are expressed in terms of orbital parameters of the planet under consideration and the geometrical mass of the Sun. Let us consider the planet Mercury, with the following orbital data [18]:

$$a = 0.38709893\,\text{AU} = 57.909\,175 \times 10^6\,\text{km},$$
$$e = 0.20563069.$$

The numerical value of the geometrical mass [19] of the Sun is

$$m = \frac{GM_{Sun}}{c^2} = 1.4766250385\,(1)\,\text{km}.$$ (28)

Hence the value of L for Mercury is

$$L = \left(1 - e^2\right)a = 55.460545 \times 10^6\,\text{km}$$

and the perihelion shift, from Eq. (27), is

$$\Delta\varphi_0 = 1.59748705\pi \times 10^{-7} = 0.10351716''.$$

This is an extremely small angle, but this is a secular effect which increases with the number of revolutions. The shift above is observed in a single Mercury-year; which corresponds to 0.24084960 Earth-years [18]. So, the total shift per Earth-year is

$$\Delta\varphi_E = \frac{\Delta\varphi_0}{0.24084960} = 0.4298''.$$

If this effect is accumulated over 100 Earth-years, the total shift is

$$\Delta\varphi = 100\,\Delta\varphi_E = 42.98''.$$

The conclusion is that the solution is quite similar to the one obtained with GR Schwarzschild solution for a prediction of the perihelion shift of Mercury. The difference is that we can calculate the value for the constant $\bar{h}$ while in GR we obtain directly the value of h. The relative difference that would be obtained using GR calculations and the one done here is

$$\left|\frac{\Delta\varphi_{GR} - \Delta\varphi}{\Delta\varphi_{GR}}\right| = \left|1 - \frac{1}{\left(1 + \frac{b^2}{2}\right)}\right| = \frac{b^2}{2}\frac{1}{\left(1 + \frac{b^2}{2}\right)}.$$

If we consider that "the excess shift is known to about 0.1 percent" [20], then this difference can be used to evaluate an upper limit for the values of b for Mercury. In this case,

$|b| < 0.045$. Of course this analysis does not take into account the parametrized post-newtonian (PPN) corrections. If this was done then the upper limit for b would certainly be smaller.

In 1997, Tegmark argued that there exists no stable orbit in a four-dimensional spacetime [21], giving rise to a stability problem for the spacetime described by Eq. (1). Let us remark that the assertion of instability in the (4, 1) space is based on Ref. [22], whose analysis is performed on an n-dimensional spherically symmetric line element (see Eq. (3.1) of Ref. [22]). Equation (1) is spherically symmetric only in four dimensions. Within the context of non-singular branes, the fields are localized around the brane core, but not restricted to a four-dimensional slice of the spacetime. The extra dimension being small (that is, the fields being restricted to a small part of the extra dimension), there is no problem with the motion through the bulk and no stability problems. If the fifth dimension (or the fourth *space-like* dimension) were infinite, then we would face instability problems. In our case, we have a three-dimensional spherical elements plus a *finite* fourth dimension. Then, rather than a $1/r^2$ potential (which would be the case in a four-dimensional manifold with all coordinates with an infinite domain), as mentioned in the last paragraph of Ref. [21], we obtain a Yukawa-type potential, which allows stable orbits (see Section 3.3 of Ref. [23]).

2.2 The deflection of light rays

Having investigated the parameter b for massive particles, let us turn our attention to the massless case. Light rays consist of massless test particles which travel at the speed of light. In special relativity, the photons move along the light-cone, following a null geodesic: $ds^2 = 0$. We will keep $ds^2 = 0$ for photon traveling in our choice of the background. So the framework is a curved manifold described by the brane-based spherically symmetric solution on which the relativistic particles will propagate. We feel justified in doing so because the photons are test particles and, by definition, test particles do not affect the geometry of the background spacetime.

By taking $ds^2 = 0$ in the Schwarzschild-like spacetime, Eq. (3), and dividing the result by $d\sigma^2$ (where σ is an appropriate invariant length), the line element becomes

$$\frac{1}{\left(1 - \frac{2m}{r}\right)}\left(\frac{dr}{d\sigma}\right)^2 + r^2\left(\frac{d\theta}{d\sigma}\right)^2 + r^2\sin^2\theta\left(\frac{d\varphi}{d\sigma}\right)^2$$
$$+ \frac{m}{r}\left(\frac{dx^4}{d\sigma} + \frac{dx^5}{d\sigma}\right)^2 - 2\frac{dx^4}{d\sigma}\frac{dx^5}{d\sigma} = 0. \tag{29}$$

This constraint replaces the one given by Eq. (19) for massive particles. All the equations before Eq. (19) remain valid for the propagation of light, provided that we replace the invariant length s by σ; that is,

$$\frac{d^2r}{d\sigma^2} - \frac{m}{r^2}\frac{1}{\left(1 - \frac{2m}{r}\right)}\left(\frac{dr}{d\sigma}\right)^2 - \left(1 - \frac{2m}{r}\right)r\sin^2\theta\left(\frac{d\varphi}{d\sigma}\right)^2$$
$$+ \frac{1}{2}\frac{m}{r^2}\left(1 - \frac{2m}{r}\right)\left(\frac{dx^4}{d\sigma} + \frac{dx^5}{d\sigma}\right)^2 = 0, \tag{30}$$

$$\frac{d\varphi}{d\sigma} = \frac{h}{r^2}, \tag{31}$$

$$\frac{dx^4}{d\sigma} = \frac{1}{2}\left[\frac{k}{\left(1 - \frac{2m}{r}\right)} + b\right], \tag{32}$$

$$\frac{dx^5}{d\sigma} = \frac{1}{2}\left[\frac{k}{\left(1 - \frac{2m}{r}\right)} - b\right], \tag{33}$$

where the initial conditions are $\theta_0 = \pi/2$ and $\left(\frac{d\theta}{d\sigma}\right)_0 = 0$. These conditions imply $\left(\frac{d^2\theta}{d\sigma^2}\right)_0 = 0$ and restrict our study to the plane $\theta = \pi/2$. Therefore, Eq. (18) is still valid for the light rays. However, Eq. (20) must be modified, since it was obtained using $ds^2 \neq 0$ and $g_{\mu\nu}u^\mu u^\nu = -1$, whereas here we have $ds^2 = 0$ and $g_{\mu\nu}u^\mu u^\nu = 0$.

Let us rewrite the new constraint, Eq. (29), by substituting $\theta = \pi/2$, $d\theta/d\sigma = 0$, together with Eqs. (31), (32), and (33):

$$\left(\frac{dr}{d\sigma}\right)^2 + \left(1 - \frac{2m}{r}\right)\left(\frac{b^2}{2} + \frac{h^2}{r^2}\right) - \frac{k^2}{2} = 0.$$

Since

$$\frac{dr}{d\sigma} = \frac{dr}{d\varphi}\frac{d\varphi}{d\sigma} = \frac{dr}{d\varphi}\frac{h}{r^2},$$

we have

$$\left(\frac{dr}{d\varphi}\right)^2 + \left[\left(1 - \frac{2m}{r}\right)\left(\frac{b^2}{2} + \frac{h^2}{r^2}\right) - \frac{k^2}{2}\right]\frac{r^4}{h^2} = 0.$$

Now this equation is written as a function of $u = 1/r$:

$$\left(\frac{du}{d\varphi}\right)^2 + (1 - 2mu)\left[u^2 + \frac{1}{h^2}\frac{b^2}{2}\right] - \frac{1}{2}\frac{k^2}{h^2} = 0, \tag{34}$$

which differs only slightly from our previous constraint, Eq. (20). By substituting Eq. (18) into Eq. (34), one obtains

$$\frac{d^2u}{d\varphi^2} + u - 3mu^2 + \frac{m}{h^2}\frac{b^2}{2} = 0. \tag{35}$$

If $b \simeq 0$, we observe that Eq. (35) reduces to the equation obtained in the standard four-dimensional GR leading to the deflection of light rays. In analogy with the previous subsection, we first set $3mu^2 = 0$, in order to get an approximate solution of Eq. (35) by an iterative procedure, starting with a zero-order solution,

$$u^{(0)} = \frac{1}{R}\cos(\varphi - \varphi_0) - \frac{m}{h^2}\frac{b^2}{2}, \tag{36}$$

where φ_0 and R are integration constants. The interpretation of R becomes clear when we set $\varphi_0 = 0$, and introduce a Cartesian coordinate system $x = r \cos \varphi$, $y = r \sin \varphi$, with origin O at the center of the massive body, which is the source of the field.

With $b = 0$, we find that Eq. (36) reduces to $R = r \cos \varphi = x$. This is a straight line parallel to the y-axis, and R is the minimum distance between the light ray and the origin O. In this case, $u^{(0)}$ does not bend the straight trajectory of the photon and there is no deflection of light. However, this is not the best possible approximation, and there is also a $b \neq 0$ contribution to be taken into account.

The first approximation to Eq. (35) leads to

$$\frac{d^2 u^{(1)}}{d\varphi^2} + u^{(1)} = -\frac{m}{h^2} \frac{b^2}{2} + 3m \left(u^{(0)} \right)^2,$$

which becomes

$$\frac{d^2 u^{(1)}}{d\varphi^2} + u^{(1)} = -\frac{m}{h^2} \frac{b^2}{2} + \frac{3m}{R^2} \cos^2 \varphi$$
$$+ \frac{3m^3}{h^4} \left(\frac{b^4}{4} \right) - 6 \frac{m^2}{h^2} \left(\frac{b^2}{2} \right) \frac{1}{R} \cos \varphi.$$

It is to be noted that if we consider $b \ll 1$, then the factor of b^4 can be neglected, and we get

$$\frac{d^2 u^{(1)}}{d\varphi^2} + u^{(1)} = \frac{3m}{R^2} \cos^2 \varphi - \frac{m}{h^2} \frac{b^2}{2} - 6m \frac{m}{h^2} \frac{b^2}{2} \frac{1}{R} \cos \varphi.$$

A particular solution of this differential equation is

$$u^{(1)} = \frac{m}{R^2} \left(\cos^2 \varphi + 2 \sin^2 \varphi \right)$$
$$- \frac{m}{h^2} \frac{b^2}{2} - 3m \frac{m}{h^2} \frac{b^2}{2} \frac{1}{R} \varphi \sin \varphi.$$

Therefore,

$$u \simeq u^{(0)} + u^{(1)},$$
$$= \frac{1}{R} \cos \varphi + \frac{m}{R^2} \left(\cos^2 \varphi + 2 \sin^2 \varphi \right)$$
$$- \frac{m}{h^2} \frac{b^2}{2} \left(2 + \frac{3m}{R} \varphi \sin \varphi \right). \tag{37}$$

Notice that if $b = 0$ this equation reduces to the expression derived in GR [24]. This also means that all possible modifications predicted by the braneworld picture for the deflection of light are present in the last term of Eq. (37).

If we multiply Eq. (37) by rR, we have

$$R = r \cos \varphi + \frac{m}{R} \left(r \cos^2 \varphi + 2r \sin^2 \varphi \right)$$
$$- \frac{m}{h^2} \frac{b^2}{2} \left[(2R) r + (3m\varphi) r \sin \varphi \right].$$

Then, in Cartesian coordinates, we have

$$x = R - \frac{m}{R} \left(\frac{x^2 + 2y^2}{\sqrt{x^2 + y^2}} \right)$$
$$+ \frac{m}{h^2} \frac{b^2}{2} \left[2R \sqrt{x^2 + y^2} + \left(3m \arctan \frac{y}{x} \right) y \right].$$

The second term on the r.h.s. gives the GR's deviation of the light ray from the straight line $x = R$. The last term is the contribution arising from the extra dimension. In the limit where $y \gg x$ (which means great distances from the source), we obtain the asymptotic solution:

$$x \simeq R - \frac{m}{R} (\pm 2y) + \frac{m}{h^2} \frac{b^2}{2} \left[2R (\pm y) + \left(3m \frac{\pi}{2} \right) y \right],$$

since $\lim_{\alpha \to +\infty} (\arctan \alpha) = \pi/2$. Thus, the two possible values of x are

$$x_+ = R + \frac{2m}{R} y + \frac{m}{h^2} \frac{b^2}{2} \left(2R + \frac{3\pi}{2} m \right) y,$$

and

$$x_- = R - \frac{2m}{R} y + \frac{m}{h^2} \frac{b^2}{2} \left(-2R + \frac{3\pi}{2} m \right) y,$$

and the deflection is described by the angle

$$\tan \delta \simeq \frac{x_+ - x_-}{y}.$$

With $\tan \delta = \delta + O \left(\delta^3 \right)$, the deflection angle is

$$\delta = \frac{4m}{R} + (2mR) \frac{b^2}{h^2} = \frac{m}{R} \left(4 + 2 \frac{R^2 b^2}{h^2} \right). \tag{38}$$

A comparison with experimental data [25], where the deflection for the case under consideration would be

$$\delta = \frac{m}{R} (3.99966 \pm 0.00090),$$

shows that $2\frac{R^2 b^2}{h^2}$ is constrained to be $2\frac{R^2 b^2}{h^2} < 0.00056 \Rightarrow \left| \frac{b}{h} \right| < \sqrt{\frac{0.00056}{2R^2}}$. Using the value of radius of the Sun [26], $R = (696{,}342 \pm 65)$ km, we find $\left| \frac{b}{h} \right| < 2.403015 \times 10^{-8}$ km^{-1}.

2.3 Gravitational redshift of spectral lines

Next we examine the shift of the atomic spectral lines in the presence of a gravitational field, also called the *gravitational redshift*.

From the line element (3) we define

$$d\tau^2 = -\frac{1}{2} ds^2$$

as the time interval between two events with vanishing spatial separation, $dr = d\theta = d\varphi = 0$. The minus sign is a result of

our choice for the signature of the metric, and the factor $1/2$ is chosen in order to allow agreement with the four-dimensional analog (Schwarzschild solution) when $b = 0$.

The time interval is related to the coordinate time differential, dx^4, and the additional brane differential coordinate, dx^5, by

$$-2d\tau^2 = \frac{m}{r}\left(dx^4\right)^2 + \frac{m}{r}\left(dx^5\right)^2 + 2\left(-1 + \frac{m}{r}\right)dx^4 dx^5,$$
(39)

where the differentials are constrained by Eqs. (16) and (17):

$$\frac{dx^4}{d\tau} = \frac{1}{2}\left[\frac{k}{\left(1 - \frac{2m}{r}\right)} + b\right], \qquad \frac{dx^5}{d\tau} = \frac{1}{2}\left[\frac{k}{\left(1 - \frac{2m}{r}\right)} - b\right],$$

so that

$$dx^5 = dx^4 - b\, d\tau.$$
(40)

It is to be noted that these are general equations; they are valid for massive particles, but they have exactly the same form as for massless particles, such as photons.

Now the time interval in Eq. (39) is expressed as

$$d\tau^2 = \left(1 - \frac{2m}{r}\right)\left(dx^4\right)^2 - \frac{m}{r}\frac{b^2}{2}d\tau^2 - \left(1 - \frac{2m}{r}\right)b\, dx^4 d\tau.$$
(41)

For the special case of $b = 0$, we have

$$d\tau = +\sqrt{1 - \frac{2m}{r}}\, dx^4,$$
(42)

which is the expected Schwarzschild solution of GR (see Ref. [24], Eq. (4.92)). The positive sign follows from the natural assumption that the proper time τ should increase with the time coordinate x^4. With $b \neq 0$, Eq. (41) leads to

$$\left(1 + \frac{m}{r}\frac{b^2}{2}\right)d\tau^2 + \left(1 - \frac{2m}{r}\right)b\, dx^4 d\tau - \left(1 - \frac{2m}{r}\right)\left(dx^4\right)^2 = 0.$$

This is a second-order equation for $d\tau$. By solving for $d\tau = d\tau\left(dx^4\right)$, we find

In order for this expression to agree with Eq. (42) for $b = 0$, we must choose the plus sign in the r.h.s., which leads to

$$d\tau = \left[\frac{\sqrt{1 + \frac{b^2}{4}} - \frac{b}{2}\sqrt{1 - \frac{2m}{r}}}{\left(1 + \frac{2m}{r}\frac{b^2}{4}\right)}\right]\sqrt{1 - \frac{2m}{r}}\, dx^4.$$
(43)

This distinction between proper time and the time coordinate gives rise to a difference between the proper frequency and the coordinate frequency of a periodic phenomenon in the curved spacetime, like the emission of electromagnetic radiation by an atom. Consider the propagation of electromagnetic waves in the limit of geometrical optics. Then the electromagnetic field can be written as $f = a\exp(i\psi)$, with $a = a(\mathbf{x}, t)$ the wave amplitude, and the phase $\psi = \psi(\mathbf{x}, t)$ an eikonal function. Then the frequency of the wave can be expressed as the derivative of ψ with respect to the time, and one has a coordinate frequency, $\omega_0 = \frac{\partial\psi}{\partial t}$, and a proper frequency, $\omega = \frac{\partial\psi}{\partial \tau}$.

We call ω_1 the proper frequency of the wave emitted by an atom at a point P_1. At another point, P_2, the observed proper frequency will be different, say ω_2, once the gravitational field is not the same. They are related so that

$$\frac{\omega_2}{\omega_1} = \frac{\left(\frac{\partial\psi}{\partial\tau}\right)_2}{\left(\frac{\partial\psi}{\partial\tau}\right)_1} = \frac{\frac{\partial\psi}{\partial t}\left(\frac{\partial t}{\partial\tau}\right)_2}{\frac{\partial\psi}{\partial t}\left(\frac{\partial t}{\partial\tau}\right)_1} = \frac{\omega_0\left(\frac{\partial x^4}{\partial\tau}\right)_2}{\omega_0\left(\frac{\partial x^4}{\partial\tau}\right)_1},$$

with ω_0 constant and where x^4 is the time coordinate t. From Eq. (43), we find

$$\left(\frac{dx^4}{d\tau}\right)_1 = \frac{\left(1 + \frac{2m}{r_1}\frac{b^2}{4}\right)}{\sqrt{1 + \frac{b^2}{4}} - \frac{b}{2}\sqrt{1 - \frac{2m}{r_1}}}\frac{1}{\sqrt{1 - \frac{2m}{r_1}}}$$

and similarly for $\left(dx^4/d\tau\right)_2$. Then the ratio of the frequencies is

$$\frac{\omega_2}{\omega_1} = \frac{\left(\frac{\partial x^4}{\partial\tau}\right)_2}{\left(\frac{\partial x^4}{\partial\tau}\right)_1} = \frac{\left(1 + \frac{2m}{r_2}\frac{b^2}{4}\right)\left(\sqrt{1 + \frac{b^2}{4}} - \frac{b}{2}\sqrt{1 - \frac{2m}{r_1}}\right)}{\left(1 + \frac{2m}{r_1}\frac{b^2}{4}\right)\left(\sqrt{1 + \frac{b^2}{4}} - \frac{b}{2}\sqrt{1 - \frac{2m}{r_2}}\right)}$$

$$\times \frac{\sqrt{1 - \frac{2m}{r_1}}}{\sqrt{1 - \frac{2m}{r_2}}}.$$
(44)

This exact result can be approximated by taking into account the fact that regular estimates assume $r_1 \ll r_2$, and

$$d\tau = \frac{-\left(1 - \frac{2m}{r}\right)b\, dx^4 \pm \sqrt{\left(1 - \frac{2m}{r}\right)^2 b^2\left(dx^4\right)^2 + 4\left(1 + \frac{m}{r}\frac{b^2}{2}\right)\left(1 - \frac{2m}{r}\right)\left(dx^4\right)^2}}{2\left(1 + \frac{m}{r}\frac{b^2}{2}\right)}.$$

that b is small. For instance, let us assume that the radiation observed on Earth, at $r = r_2 \approx 149.6 \times 10^6$ km, was emitted at the surface of the Sun, at $r = r_1 = R \approx 696$ km. Then the approximation $r_1 \ll r_2$ is certainly valid. Moreover, the quantity m/r_1 is very small, since $m = 1.4766250385\,(1)$ km, and the functions in Eq. (44) can be approximated accordingly. Therefore, for $m/r_1 \ll 1$ and $m/r_2 \ll 1$, the ratio of the frequencies is

$$\frac{\omega_2}{\omega_1} \simeq \left(1 + \frac{2m}{r_2}\frac{b^2}{4}\right)\left(1 - \frac{2m}{r_1}\frac{b^2}{4}\right)\frac{\left[\sqrt{1 + \frac{b^2}{4}} - \frac{b}{2}\left(1 - \frac{m}{r_1}\right)\right]}{\left[\sqrt{1 + \frac{b^2}{4}} - \frac{b}{2}\left(1 - \frac{m}{r_2}\right)\right]}$$
$$\times \left(1 - \frac{m}{r_1}\right)\left(1 + \frac{m}{r_2}\right).$$

If m/r_2 is assumed near zero, then this ratio simplifies to

$$\frac{\omega_2}{\omega_1} \simeq \left(1 - \frac{2m}{r_1}\frac{b^2}{4}\right)\frac{\left[\sqrt{1 + \frac{b^2}{4}} - \frac{b}{2}\left(1 - \frac{m}{r_1}\right)\right]}{\left[\sqrt{1 + \frac{b^2}{4}} - \frac{b}{2}\right]}$$
$$\times \left(1 - \frac{m}{r_1}\right).$$

Furthermore, if b is also assumed to be very small, $b \ll 1$, this equation becomes

$$\frac{\omega_2}{\omega_1} \simeq 1 - \frac{m}{r_1}\left(1 - \frac{b}{2}\right), \tag{45}$$

so that

$$\frac{\Delta\omega}{\omega} = \frac{\omega_2 - \omega_1}{\omega_1} = -\frac{m}{R}\left(1 - \frac{b}{2}\right).$$

When $b = 0$, this result agrees with GR. The value predicted by GR using the solar geometrical mass and radius used earlier leads to a redshift of $\left|\frac{\Delta\omega}{\omega}\right| = 2.120546 \times 10^{-6}$. In a velocity scale, i.e. converting this redshift as if it was due to a Doppler effect, this redshift corresponds to $v \approx 636$ m s^{-1}. The value estimated in Ref. [27] is 633 m s^{-1} with an uncertainty $\lesssim 100$ m s^{-1}. If we consider this value we can fix an upper limit for $|b|$ using the lower experimental value 533 m s^{-1}. We obtain $|b| < 0.323$. If instead of considering the lower limit we had taken the estimated value we would have obtained $|b| < 0.00857$. This would be the case if the uncertainty of measurements can be considerably reduced and if the estimated value remains the same.

Now if reconsider the case of light deflection and take this former value as a limit for $|b|$ for light rays then we are able to estimate for the values of

$$\left|\frac{b}{h}\right| < 2.403015 \times 10^{-8}\,\text{km}^{-1} < \frac{0.323}{|h|}$$
$$\Rightarrow |h| < 1.34 \times 10^8\,\text{km}.$$

We conclude this section by interpreting the obtained upper limit to the b parameter in terms of the fifth dimension by using the uncertainty relations. Therefore we find that the b parameter can be directly related to the extra dimension. Therefore, its upper limit also means a limit on the velocity along x^5.

Let us assume for the sake of argument that the dispersion relation for a photon in a four-dimensional space is

$$p^2 = p_\mu p^\mu = 0 \Rightarrow E^2 = \mathbf{p}^2 c^2 \Rightarrow E = |\mathbf{p}|\,c = h\nu,$$

where E is the energy, $\mathbf{p}$ is the 3-momentum, c is the velocity of light, h is the Planck constant, and ν is the frequency of the photon.

If we assume a quantum behavior for the photon, then the following uncertainty relations are expected [28]:

$$\Delta E \Delta t \gtrsim \frac{\hbar}{2}$$
$$\Delta x_i \Delta p_i \gtrsim \frac{\hbar}{2}, \quad i = 1, 2, 3.$$

Hence, if the fifth dimension is supposed to exist, we would expect the uncertainty relations to be generalized as

$$\Delta E \Delta t \gtrsim \frac{\hbar}{2}$$
$$\Delta x_i \Delta p_i \gtrsim \frac{\hbar}{2}, \quad i = 1, 2, 3, 5,$$

and the dispersion relation as

$$p^2 = 0 \Rightarrow E^2 = \left(\mathbf{p}^2 + p_5^2\right)c^2 \Rightarrow E^2 = \mathbf{p}^2 c^2 \left(1 + \frac{p_5^2}{\mathbf{p}^2}\right)$$
$$\Rightarrow E = |\mathbf{p}|\,c \sqrt{1 + \frac{p_5^2}{\mathbf{p}^2}}.$$

If $\frac{|p_5|}{|\mathbf{p}|} \ll 1$, then we can approximate the relation as

$$E \approx |\mathbf{p}|\,c\left(1 + \frac{1}{2}\frac{p_5^2}{\mathbf{p}^2}\right).$$

When comparing this result with the four-dimensional case, it is possible to interpret the extra term as a fluctuation of energy (which is supposed to be small since all influences of the fifth-dimensional quantities are not directly observed):

$$\Delta E \approx \frac{1}{2}c\frac{p_5^2}{|\mathbf{p}|}.$$

If this fluctuation of energy is supposed to be consistent with uncertainty relations, then we would expect, with the best accuracy:

$$\Delta E \Delta t \approx \frac{\hbar}{2} \Rightarrow \frac{1}{2}c\frac{p_5^2}{|\mathbf{p}|}\Delta t \approx \frac{\hbar}{2}.$$

During this time interval Δt, the distance traveled by light in the fifth dimension is

$$\Delta x^5 = bc\Delta t.$$

If $p_5 \approx \Delta p_5$, which means that the magnitude of p_5 is comparable to its uncertainty, then we obtain

$$\frac{1}{2}\frac{(\Delta p_5)^2}{|\mathbf{p}|}\frac{\Delta x^5}{b} \approx \frac{\hbar}{2} \Rightarrow \frac{1}{2}\frac{(\Delta p_5)^2}{|\mathbf{p}|\,b}\left(\Delta x^5\right)^2 \approx \frac{\hbar}{2}\Delta x^5.$$

As stated earlier, if we assume that the accuracy of measurements is the best possible, then $\Delta p_5 \Delta x^5 \approx \frac{\hbar}{2}$, so that

$$\frac{1}{2}\left(\frac{\hbar}{2}\right)^2\frac{1}{|\mathbf{p}|\,b} \approx \frac{\hbar}{2}\Delta x^5 \Rightarrow \frac{1}{8\pi}\frac{1}{b}\frac{h}{|\mathbf{p}|} \approx \Delta x^5.$$

From the de Broglie relation, $\frac{h}{|\mathbf{p}|} = \lambda$, we find

$$\frac{1}{8\pi}\frac{1}{b}\lambda \approx \Delta x^5.$$

If we assume that Δx^5 represents an estimate of the size of the fifth dimension, and keeping in mind that we have obtained an upper limit for b, then we conclude that our results lead to

$$b < b_0 \Rightarrow \Delta x^5 \gtrsim \frac{1}{8\pi}\frac{1}{b_0}\lambda.$$

In our case, the value of b_0 is estimated with results from experiments that used a wavelength in the range of 5,188–5,212 Å. We use the average value to estimate Δx^5 as follows:

$$\Delta x^5 \gtrsim \frac{1}{8\pi}\frac{1}{b_0}\lambda = \frac{1}{8\pi}\frac{1}{0.323}5{,}200 \times 10^{-10}\text{ m}$$
$$= 6.4 \times 10^{-8}\text{ m}.$$

Therefore, we find that the uncertainty on the measured position along the fifth dimension is larger than the above value. The point to be stressed, however, is that this argument can now be read backwards: the very existence of the fifth dimension, when scrutinized by quantum particles, has an associated uncertainty related to its size implying an upper limit in the velocity, here parameterized by b.

Let us note that if we consider Δx^5 as the uncertainty of measurement of lengths along the fifth dimension and if the size of the fifth dimension is smaller than the value estimated above, then we will not be able to distinguish its existence because there will not be sufficient accuracy for that. If the size of the fifth dimension is larger than shown by our calculations, then it would be possible to notice it. Therefore, no matter the size of the fifth dimension, our calculation just established our capability of measuring it. In the absence of any other evidence, then we conclude that the size of the fifth dimension must be smaller than what we have estimated.

3 Concluding remarks

In this paper, we have investigated the embedding of a spherically symmetric gravitational solution into the five-dimensional braneworld scenario. The classic tests of GR are considered in order to study the possible influence of the extra dimension in low energy experiments. In fact, we have examined three GR classic tests: the perihelion shift of Mercury, the deflection of light by the Sun, and the gravitational redshift of atomic spectral lines. The investigated solution gives results similar to the 4D Schwarzschild line element predictions. More precisely, a new parameter is related to the extra dimension that brings subtle but important corrections to the usual GR case can be constrained in order not to contradict any experimental results. Moreover, at least in the massless (photon) case, it is possible to interpret the obtained results with fundamental concepts as the uncertainty principle.

Finally we wish to stress the relevance of the aforementioned analysis. Firstly it can be used to refine the extra-dimensional models of General Relativity, by constraining the new parameters using experimental observations. In addition the present study can help in explaining small differences observed in gravitational experiments. The exploration of GR's classical tests, with the possible departures, may serve as an important tool in connecting high energy models to low energy experiments.

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