UNIVERSIDADE ESTADUAL PAULISTA
"JÚLIO DE MESQUITA FILHO"
CAMPUS DE GUARATINGUETÁ

FELIPE AUGUSTO DA SILVA BARBOSA

Higher rank fields for spin-1 particles

Guaratinguetá
2022

Felipe Augusto da Silva Barbosa

Higher rank fields for spin-1 particles

Dissertação apresentada ao Conselho de Curso de Pós Graduação em Física da Faculdade de Engenharia do Campus de Guaratinguetá, Universidade Estadual Paulista, como parte dos requisitos para obtenção do título de Mestre em Física na área de partículas e campos.

Orientador: Profº Dr. Denis Dalmazi
Coorientador: Dr. Alessandro Luiz Ribeiro dos Santos

Guaratinguetá
2022

Barbosa, Felipe Augusto da Silva
B238h Higher rank fields for spin-1 particles / Felipe Augusto da Silva Barbosa – Guaratinguetá, 2022.
64 f : il.
Bibliografia: f. 57-60

Dissertação (Mestrado) – Universidade Estadual Paulista, Faculdade de Engenharia de Guaratinguetá, 2022.
Orientador: Prof. Dr. Denis Dalmazi
Coorientador: Prof. Dr. Alessandro Luiz Ribeiro dos Santos

1. Teoria de campos (Física). 2. Cosmologia. 3. Relatividade geral (Física).
I. Título.

CDU 530.145(043)

Luciana Máximo
Bibliotecária CRB-8/3595

unesp
UNIVERSIDADE ESTADUAL PAULISTA
CAMPUS DE GUARATINGUETÁ

FELIPE AUGUSTO DA SILVA BARBOSA

ESTA DISSERTAÇÃO FOI JULGADA ADEQUADA PARA A OBTENÇÃO DO TÍTULO DE
"MESTRADO EM FÍSICA"

PROGRAMA: FÍSICA
CURSO: MESTRADO

APROVADA EM SUA FORMA FINAL PELO PROGRAMA DE PÓS-GRADUAÇÃO

Prof. Dr. Ernesto Vieira Neto
Coordenador

BANCA EXAMINADORA:

Prof. Dr. DENIS DALMAZI
Orientador - UNESP

Prof. Dr. ELIAS LEITE MENDONÇA
UNESP

p/ **Prof. Dr. FABRÍCIO AUGUSTO BARONE RANGEL**
UNIFEI
participou por videconferência

Maio *de 2022*

DADOS CURRICULARES

FELIPE AUGUSTO DA SILVA BARBOSA

NASCIMENTO	22/04/1996 - Guaratinguetá / SP
FILIAÇÃO	José Marcelo Barbosa Rosemary Aparecida da Silva
2020 / 2022	Mestrado em Física. UNESP-FEG
2016 / 2021	Bacharelado em Física UNESP-FEG

Sua dedicatoria deve ser digitada aqui.

AGRADECIMENTOS

Acredito que ao fim desta etapa eu devo apenas agradecer ao meu orientador Denis Dalmazi, com o qual estou aprendendo desde meus primeiros anos na graduação. Sou eternamente grato por sua paciência ao me orientar durante todos estes anos, pela sua preocupação com o aprendizado, e pelos seus inúmeros conselhos. Agradeço também pelo seu auxílio em questões técnicas que vão além dos ensinamentos de Física, que foram cruciais para que eu pudesse completar esta dissertação especialmente nestes conturbados últimos dois anos. Irei certamente levar este aprendizado para a vida.

Gostaria de agradecer também aos Professores Júlio Marny Hoff da Silva e Elias Leite Mendonça. Por sua disposição para me aconselhar, ensinar e discutir mesmo não sendo oficialmente meus orientadores. O ambiente proporcionado por estas interações agregou positivamente para a minha formação.

É meu dever agradecer também à Rafael Robson, o famoso R R, e à Gustavo Pazzini de Brito. Seus conselhos foram essenciais para que eu pudesse começar minha inserção no mundo da Física, assim como as discussões educativas das quais tirei grande proveito. Espero que a amizade continue pelos anos que virão.

Por fim gostaria de agradecer ao meu amor Tatiane, pelo seu apoio e principalmente por sua paciência comigo. Certamente, sua companhia e amor foram essenciais para que eu pudesse completar esta etapa.

Este trabalho contou com o apoio da(s) seguinte(s) entidade(s):
Coordenação de Aperfeiçoamento de Pessoa de Nível Superior (CAPES) - Processo nº 88887.495259/2020-00
Conselho Nacional de Desenvolvimento Científico e Tecnológico (CNPQ) - Processo nº 313559/2021-0

“Frase, citação, epígrafe.“

(Autor)

RESUMO

É feito um estudo de modelos duais sem massa e massivos de spin-1. É proposta uma nova ação mestra para partículas sem massa de spin-1 da qual derivamos uma nova teoria dual à Maxwell em termos de um campo de "rank"3. A redução dimensional de Kaluza-Klein é aplicada às ações mestras sem massa a fim de encontrar modelos duais massivos de spin-1. Nós obtemos os modelos duais à Proca usualmente encontrados na literatura e um novo modelo dual de spin-1 massivo descrito por um tensor de "rank"2 sem simetria. Através do uso de ação mestra na integral de trajetória mostramos que a dualidade do novo modelo com o de Proca é preservada na presença de matéria para interações lineares no campo de spin-1. Estudamos também a dualidade quanto a dedução das regras de Feynman na "picture"de interação e no formalismo de integral de caminho para modelos to tipo Kalb-Ramond.

PALAVRAS-CHAVE: Modelos duais. SPIN-1. Teoria de campos.

ABSTRACT

A study of dual models of massless and massive spin-1 is performed. A new parent action for massless spin-1 particles is proposed from which we derive a new model dual to Maxwell, where the field is a rank-3 tensor. We perform a Kaluza-Klein dimensional reduction of massless parent actions in order to derive massive models of spin-1. We obtain the dual models to Proca found in the literature and a new model of massive spin-1 described by a nonsymmetric rank-2 tensor. We use the parent action method in the path integral to show that the duality between the new model and Proca is maintained in the presence of interactions linear in the spin-1 field. We also investigate the duality of Feynman rules in the interaction picture and path integral for dual models of the Kalb-Ramond type

KEYWORDS: Dual models. SPIN-1. Field theory

SUMÁRIO

1 INTRODUCTION

Cosmology has been in a very exciting era in the last decades with many open problems. We have the problem of dark matter, which is included in the standard ΛCDM cosmological model but has an unknown origin. The accelerated expansion of the universe, (RIESS et al., 1998) and (PERLMUTTER et al., 1999), is still an open problem in physics. More recently we have the Hubble tension (VALENTINO et al., 2021), a disagreement between the measures of the Hubble constant today using early and late time Universe data and other open questions.

A proposal with increasing popularity to address some of these tensions is modified gravity (VALENTINO et al., 2021), the idea that Einstein's theory of general relativity is an approximation. The approaches to modify general relativity can be classified by their physical motivation: high energy and low energy.

The high-energy approach can start with quantum field theory or general relativity. On the QFT side, general relativity is considered an effective field theory ((WEINBERG, 1979); (DONOGHUE, 1994); (WEINBERG, 1996); (BURGESS, 2004)), where the Einstein-Hilbert action[1]

$$S_{EH} = \frac{M_P^2}{2} \int d^4x \sqrt{-g}\, R, \tag{1.0.1}$$

is just the first term in a power series on $1/M_P$, where $M_P \equiv G^{-1/2}$ is the Planck mass, which includes every term compatible with the symmetries of the theory. On the other hand, if we start with general relativity the modifications are motivated by its singularities. For instance, the black hole and early universe singularities. So, one looks for a theory compatible with the available data and that does not have these singularities (JIMENEZ et al., 2018); (MOFFAT, 2015); (LIN, 2019); (QUINTIN; YOSHIDA, 2020)). One could argue that singularities in general relativity occur in high energy where the quantum gravity theory is needed and the two motivations are the same. We keep them separated anyway because QFT provides a systematic approach on how to modify gravity, whereas the general relativity motivation does not seem to follow any well-defined path.

The point of view adopted here, comes from an attempt to solve the tensions in cosmology and general relativity without postulating any dark sector ((SARIDAKIS et al., 2021); (NOJIRI; ODINTSOV; OIKONOMOU, 2017)). There is by now a big variety of modified gravity theories, some of them provide self-accelerating solutions but up to now, none of them have succeeded in replacing general relativity.

One motivation to modify gravity comes from looking at general relativity from the field theory perspective. It is well known that Einstein's equations are equivalent to the wave equations of a self-interacting field associated with massless spin-2 particles[2]. Then, one obvious modification comes from adding a mass to this particle. However, this idea has been abandoned for a long time, since

[1] In natural units $\hbar = c = 1$

[2] See for example (WEINBERG, 1972) section 7.6. Also, it has been shown that the equivalence principle, a central aspect of general relativity, is an inevitable consequence of any Lorentz invariant long-range interaction mediated by massless spin-2 particles (WEINBERG, 1964)

(BOULWARE; DESER, 1972) seemed to rule out this possibility claiming that a non-linear massive spin-2 could not be physically consistent. Recently this has changed and apparently, a non-linear theory of massive gravity is possible, see (RHAM; GABADADZE; TOLLEY, 2011) and (RHAM, 2014) for a review. There is now even the so-called "bi-gravity"theory whose content is a massless and a massive spin-2 particle, see (SCHMIDT-MAY; STRAUSS, 2016).

The introduction of a mass usually breaks the gauge invariance present in the massless theory. In general relativity, gauge symmetry translates into invariance under general coordinate transformations. In the recently proposed massive theory, (RHAM, 2014) the symmetry by general coordinate transformations is recovered by the introduction of Stueckelberg fields. However, this version of massive gravity does not have stable homogeneous and isotropic solutions which are relevant in cosmology (KENNA-ALLISON, 2021). In the case of bi-gravity, the symmetry by general coordinate transformations is guaranteed by the massless part of the theory, but one has to pay the price of having two spin-2 particles. Also, the stable cosmological solutions of bi-gravity seem to have problems in describing the late-time acceleration of the Universe.

Considering these points one can ask whether it's possible to introduce gauge symmetry in a non-trivial way while keeping only one massive spin-2 particle, and whether a new massive gravity theory can follow from this linear free theory.

In the search of an answer to this question we are naturally led to the concept of duality, one can have different descriptions for the same physics. Research in duality has been very fruitful in the last decades in condensed matter physics, field theory, string theory and gravitational physics ((HARTNOLL, 2009), (SEN, 1994), (GIVEON; PORRATI; RABINOVICI, 1994)). The duality relation between two theories can be used to do practical calculations in one theory, that are impossible in the other. Regarding the introduction of gauge symmetry in a nontrivial way, in the context of massive spin-1 our previous question has a positive answer: the Cremmer-Sherk model, (CREMMER; SCHERK, 1974). This theory is dual to Maxwell-Proca and has gauge symmetry without Stueckelberg fields. The Cremmer-Scherk model is of second order in derivatives, but it can be obtained from a first-order Lagrangian for the Maxwell theory via dimensional reduction. Therefore, in the search for a Cremmer-Scherk of spin-2, one would look for dimensional reductions of first-order versions of massless spin-2 theories. There has been some progress in this approach recently (DALMAZI; SANTOS, 2020).

For massless gravity it is possible to get the non linear theory from a first order linear action with the help of an auxiliary partially symmetric field ($\Gamma^{\mu}_{\alpha\beta} = \Gamma^{\mu}_{\beta\alpha}$), as in (DESER, 1970) [3] [4]

$$S_{as} = \int d^4x \left[h^{\mu\nu}\left(\partial_\alpha \Gamma^{\alpha}_{\mu\nu} - \partial_\nu \Gamma^{\alpha}_{\mu\alpha}\right) + \eta^{\mu\nu}\left(\Gamma^{\alpha}_{\mu\nu}\Gamma^{\beta}_{\beta\alpha} - \Gamma^{\alpha}_{\beta\mu}\Gamma^{\beta}_{\alpha\nu}\right)\right] \tag{1.0.2}$$

where $h^{\mu\nu}$ is symmetric. It is also possible to get general relativity also from the first order lagrangian

[3] This action can also be understood as the linearized Palatini action (FERRARIS; FRANCAVIGLIA; REINA, 1982)

[4] S_{as} stands for "auxiliary symmetric"

of Hilbert-Palatini (PELDÁN, 1994)[5]

$$S_{aa} = -2 \int d^4x \left[Y^{[\mu\nu]\alpha} \partial_{[\mu} e_{\nu]\beta} - \frac{1}{2} Y_{[\mu\nu]\alpha} Y^{[\mu\alpha]\nu} + \frac{Y_\alpha Y^\alpha}{4} \right] \tag{1.0.3}$$

where, $e_{\mu\nu}$ is a rank-2 non symmetric tensor, $Y^{[\mu\nu]\alpha}$ is a partially antisymmetric field[6]. Therefore, we have two first order approaches to free massless spin-2 particles, based on partially symmetric and partially antisymmetric auxiliary fields. The work (DALMAZI; SANTOS, 2020) is based on a dimensional reduction of a first order description of massless spin-2 particles dual to (1.0.3). The symmetric and antisymmetric first order formulations have geometric meaning on their own. The third case of a non-symmetric auxiliary rank-3 tensor $\Gamma^\mu_{\alpha\beta} \neq \Gamma^\mu_{\alpha\beta}$ correspond to a non-vanishing torsion, see (OLMO, 2011). Its dimensional reduction to a massive spin-2 model has not been considered in the literature. We will see that there are analogues of the antisymmetric and of the symmetric formulations for massless spin-1 as well, namely the actions (2.1.4). As a step towards the dimensional reduction of the nonsymmetric massless spin-2 case ($\Gamma^\mu_{\alpha\beta} \neq \Gamma^\mu_{\beta\alpha}$) we explore here a nonsymmetric first order theory for spin-1 particles.

At first, since any non-symmetric field can be decomposed into a symmetric plus an antisymmetric part, it may seem that the non-symmetric case doesn't add anything. We will see that a non-symmetric auxiliary field yields new possibilities even in the spin-1 case. The procedure consists of the generalization of the first order actions (1.0.2) and (1.0.3) to one based on a non-symmetric auxiliary tensor in a higher dimension and then making a Kaluza-Klein dimensional reduction in order to find a first-order massive action in the lower dimension. So far we have only worked out the spin-1 case, which will be presented here in this dissertation. Here we show that this is indeed a new possibility and the results obtained in this way are different from those of the symmetric and antisymmetric cases known in the literature.

Our motivation to work with spin-1 is twofold. At first, the work with spin-1 can serve to develop an intuition for future work in spin-2. On the other hand, there are important astrophysical and cosmological applications of massive and massless spin-1 models. There has been a revival in non-linear electrodynamics, the coupling of these models with general relativity, or even modified theories of gravity involving non-linear electrodynamics have been successful in eliminating the singularities of general relativity (see (BANDOS et al., 2020), (SOROKIN, 2021), (BOKULIĆ; JURIĆ; SMOLIĆ, 2021) and references therein). Self-interacting theories of massive spin-1 have also received great attention recently ((TASINATO, 2014), (HEISENBERG, 2014), (RHAM; POZSGAY, 2020), (RHAM et al., 2021)) in the context of Galileon modified gravity theories. These models have all been built up out of the free theories of Maxwell and Maxwell-Proca. One can wonder if any new physics can be introduced by working with the dual models of spin-1, which has been only partially investigated in (HEISENBERG; TRENKLER, 2020). This motivates us to investigate the free dual models of spin-1 and how they interact with matter and gravity (not considered here)

In chapter 2 we will be dealing with free theories, massless and massive. We start by introducing

[5] S_{aa} stands for auxiliary antisymmetric

[6] $Y^{[\mu\nu]\alpha} = -Y^{[\nu\mu]\alpha}$. We also have $A_{[\mu\nu]} = (A_{\mu\nu} - A_{\nu\mu})/2$ and $A_{(\mu\nu)} = (A_{\mu\nu} + A_{\nu\mu})/2$. Moreover, our signature will be $\eta_{\mu\nu} = (-, +, \cdots, +)$.

the parent action to derive the massless models of spin-1 dual to Maxwell. We will use dimensional reduction to derive massive parent actions from the massless ones and deduce massive spin-1 models dual to the Proca theory. We end the chapter with a brief use of alternative methods to derive dual models. In chapter 3 we will make an introduction to interactions using the new massless and massive models of spin-1 following the use of parent actions. Finally, in chapter 4 we study dual massive models with a new perspective, which allows us to make general statements about general interactions in perturbation theory and also some preliminary statements beyond perturbation theory.

2 FREE THEORY

We start with a definition of duality and introduce the parent action method as a tool to find dual models. We use massless parent actions to find dual models to Maxwell and introduce the Kaluza-Klein dimensional reduction to find massive parent actions, so that dual models to Proca can be found.

2.1 MASSLESS DUAL MODELS

In classical field theory, two actions will be dual if one can establish a map between the field configurations of those two theories. Usually, the equations of motion are used to show that such map exists, and the theories can be interacting or free. The parent action procedure (DESER; JACKIW, 1984; HJELMELAND; LINDSTROM, 1997) will be used as a tool to obtain dual theories. The general scheme in the massless case, for free theories, is to take $L = B^2 + B\hat{D}A$ in the Lagrangian path integral formalism,

$$Z = \int \mathcal{D}B\mathcal{D}A\, e^{i\int d^Dx\left[B^2+B\hat{D}A\right]} \tag{2.1.1}$$

where $\hat{D}$ is some differential operator. A Gaussian integral on B leads to

$$Z \propto \int \mathcal{D}A\, e^{-i\int d^Dx\frac{(\hat{D}A)^2}{4}}, \tag{2.1.2}$$

a theory that depends only on A. On the other hand, the integral on A enforces the functional constraint $\hat{D}B = 0$ and by replacing the solution $B = \hat{D}'C$, where $\hat{D}'$ is a new differential operator, we have

$$Z \propto \int \mathcal{D}C\, e^{i\int d^Dx(\hat{D}'C)^2}. \tag{2.1.3}$$

Since $L = -(\hat{D}A)^2/4$ and $L = (\hat{D}'C)^2$ were derived from the same master action we say that they are dual.

Our starting point is the first order action below where $\{a_1, a_2, a_3, b, c, d\}$ are real constants to be determined and $e_{\mu\nu}$ is an arbitrary rank-2 tensor[1]

$$S = \int d^Dx\left[(a_1e_{\mu\nu} + a_2e_{\nu\mu})\,\partial^\mu A^\nu + b\,e_{\mu\nu}e^{\mu\nu} + c\,e_{\mu\nu}e^{\nu\mu} + d\,e^2 + a_3\,e\,\partial^\mu A_\mu\right]. \tag{2.1.4}$$

If $a_1 = \pm a_2$ the requirement that the functional integral on $e_{\mu\nu}$ leads to Maxwell's theory partially fixes the remaining constants. After an adequate choice of the constants, it is possible to show that the

[1] where $e \equiv e^\mu_\mu$

first order lagrangian reduces to[2]

$$S_{AN} = \int d^D x \left[\frac{1}{4} V_{[\mu\nu]} V^{[\mu\nu]} - V^{[\mu\nu]} \partial_{[\mu} A_{\nu]} \right] \tag{2.1.5a}$$

$$S_S = \int d^D x \left[W^{\mu\nu} W_{\mu\nu} - \frac{W^2}{D-1} + 2W^{\mu\nu} \partial_{(\mu} A_{\nu)} \right], \tag{2.1.5b}$$

for $a_1 = -a_2$ and $a_1 = a_2$ respectively, here $W_{\mu\nu} = e_{(\mu\nu)}$ and $V_{[\mu\nu]} = e_{[\mu\nu]}$. The above actions are the spin-1 analogs of (1.0.3) and (1.0.2) respectively. From now on we use the notation: $\Big\langle F(x) \Big\rangle \equiv \int d^D x F(x)$.

2.1.1 Antisymmetric auxiliary field $(V_{[\mu\nu]} = -V_{[\nu\mu]})$

For S_{AN}, the gaussian integral on $V_{[\mu\nu]}$ leads to Maxwell's theory. If instead one integrates on A_μ the result is the constraint $\partial_\alpha V^{[\alpha\beta]} = 0$. Replacing the general solution[3] $V^{[\mu\nu]} = \partial_\alpha T^{[\alpha\mu\nu]}$ in S_{AN} the action becomes

$$S_{TT} = \frac{1}{4} \Big\langle \partial_\alpha T^{[\alpha\mu\nu]} \partial^\beta T_{[\beta\mu\nu]} \Big\rangle. \tag{2.1.6a}$$

Its equations of motion are just

$$\partial^{[\nu} \partial_\alpha T^{[\alpha\beta\mu]\,]} = 0. \tag{2.1.6b}$$

By the Poincaré lemma we infer that $\partial^\alpha T_{[\alpha\beta\mu]} = \partial_\beta A_\mu - \partial_\mu A_\beta$ with A_μ some vector field. Because the field is antisymmetric we identically have

$$\partial^\beta \partial^\alpha T_{[\alpha\beta\mu]} = 0 = \Box A_\mu - \partial_\mu \partial^\beta A_\beta = 0, \tag{2.1.7}$$

therefore the identity $\partial^\beta \partial^\alpha T_{[\alpha\beta\mu]} = 0$ implies Maxwell's equations.

The antissymmetric case already illustrates a general feature of dual field theories: there is an exchange between identities and dynamical equations. In a geometric formulation of Maxwell's theory the existence of the vector potential follows from the Bianchi identity

$$\partial_{[\alpha} F_{\mu\nu]} = 0, \tag{2.1.8}$$

which shows that $F_{\mu\nu}$ is the exterior derivative of A_μ, while the Maxwell's equations are dynamic. In the dual model S_{TT} the existence of the vector potential is dynamical, it comes from the equations of motion (2.1.6b), and Maxwell's equations for such vector potential originate from the identity $\partial^\alpha \partial^\beta T_{[\alpha\beta\mu]} = 0$. This exchange occurs in the Maxwell theory in the vacuum written in terms of electromagnetic fields when we replace $(\vec{E}, \vec{B}) \to (-\vec{B}, \vec{E})$, the Faraday-Lenz equations and

[2] Where the labels AN stands for antisymmetric and S stands for symmetric, and $W \equiv W^\mu_\mu$.

[3] The tensor $T^{[\alpha\mu\nu]}$ is totally antisymmetric in the three indices.

$\nabla \cdot \vec{B} = 0$ (identities) become Ampere's and Gauss law (dynamic) in the vacuum and vice versa.

It is worthy remarking that in $D = 4$ S_{TT} does not represent a new possibility for massless spin-1 because it is identical to Maxwell's theory. This happens because for $D = 4$, A_μ and $T_{[\mu\nu\alpha]}$ have 4 independent components[4], and in fact they can be related by $T_{[\mu\nu\alpha]} = \epsilon_{\mu\nu\alpha\beta}A^\beta$, where $\epsilon_{\mu\nu\alpha\beta}$ is the Levi-Civita symbol; replacing this in S_{TT} the result is Maxwell's action. For $D > 4$ this trivial equivalence does not hold, once that our interest is the dimensional reduction of theories from $D = 5$ to $D = 4$ the S_{TT} model is a non trivial dual theory to be considered in $D = 5$.

Another useful concept is that of a dual parent action. Given two parent actions they will be called dual if, by functional integration, the same theories can be obtained from both. Consider, for instance

$$S^*_{AN} = \left\langle -\frac{1}{4}V^{[\mu\nu]}V_{[\mu\nu]} + \frac{1}{2}V^{[\mu\nu]}\partial^\alpha T_{[\alpha\mu\nu]} \right\rangle, \tag{2.1.9}$$

by a Gaussian integral on $V_{[\mu\nu]}$ the theory S_{TT} is recovered. While the functional integral over $T_{[\alpha\mu\nu]}$ leads to Maxwell's theory. Therefore S_{AN} and S^*_{AN} are called dual parent actions. The idea of a dual parent action is that by using a new action which generates one of the dual actions one could establish new duality relations between the theories already found and possible new ones. A benefit of a dual parent action was noted in (DALMAZI; SANTOS, 2020) where a dual parent action was used to obtain a new massive action for spin-2 particles. In $D = 4$ because A_α and $T_{[\alpha\beta\mu]}$ are equivalent S_{AN} and S^*_{AN} are not really different, by writing $T_{[\mu\nu\alpha]} = \epsilon_{\mu\nu\alpha\beta}A^\beta$ in S^*_{AN}, and denoting $\tilde{V}_{[\mu\nu]} = \epsilon_{\mu\nu\alpha\beta}V^{[\alpha\beta]}$ is possible to see that S^*_{AN} and S_{AN} are the same action.

The Kaluza-Klein dimensional reduction of S_{AN} was already worked out in (KHOUDEIR, 1998) it leads to the massive models of Cremmer-Sherk (CREMMER; SCHERK, 1974), Kalb-Ramond (KALB; RAMOND, 1974) and Maxwell-Proca (Proca, Al., 1936). In (DALMAZI; SANTOS, 2020) the dimensional reduction of S^*_{AN} was made and the same three massive models of (KHOUDEIR, 1998) were recovered.

2.1.2 Symmetric auxiliary field ($W_{\mu\nu} = W_{\nu\mu}$)

The parent action S_S (2.1.5b) was first introduced in (KHOUDEIR; MONTEMAYOR; URRUTIA, 2008) in $D = 4$ and further investigated in (DALMAZI; SANTOS, 2011) in D dimensions. The Gaussian integral on $W_{\mu\nu}$ leads to Maxwell's theory, on the other hand the integration on A_μ leads to the constraint $\partial^\mu W_{\mu\nu} = 0$ whose solution $W_{\mu\nu} = \partial^\alpha\partial^\beta B_{[\alpha\mu][\beta\nu]}$ gives rise to a theory with higher derivatives

$$S_{HD} = \left\langle \left(\partial^\alpha\partial^\beta B_{[\mu\alpha][\nu\beta]}\right)^2 - \frac{\left(\partial^\alpha\partial^\beta B^\mu_{\alpha\mu\beta}\right)^2}{D-1} \right\rangle, \tag{2.1.10}$$

where $B_{[\mu\nu][\alpha\beta]}$ has the symmetry properties of the Riemann tensor. For $D = 3 + 1$ it has been shown (DESER; TOWNSEND; SIEGEL, 1981) that this higher order theory describes two degrees of freedom

[4] See (WEINBERG, 1972) section 4.11 for a general account of p-forms

without ghosts. The equations of motion are

$$\begin{aligned}&\partial_\alpha\partial_\theta\tilde{B}_{(\mu\nu)}-\partial_\mu\partial_\theta\tilde{B}_{(\alpha\nu)}+\partial_\mu\partial_\nu\tilde{B}_{(\alpha\theta)}-\partial_\alpha\partial_\nu\tilde{B}_{(\mu\theta)}\\&-\frac{1}{D-1}\Big(\partial_\alpha\partial_\theta\tilde{B}\eta_{\mu\nu}-\partial_\mu\partial_\theta\tilde{B}\eta_{\alpha\nu}+\partial_\mu\partial_\nu\tilde{B}\eta_{\alpha\theta}-\partial_\alpha\partial_\nu\tilde{B}\eta_{\mu\theta}\Big)=0,\end{aligned}\tag{2.1.11}$$

with $\tilde{B}_{(\mu\nu)}=\partial^\alpha\partial^\beta B_{[\mu\alpha][\nu\beta]}$ and $\tilde{B}=\tilde{B}^\mu{}_\mu$. The general solution of these equations[5] is $\tilde{B}_{(\mu\nu)}=\partial_{(\mu}A_{\nu)}-\eta_{\mu\nu}\partial_\beta A^\beta$, because (2.1.11) is equivalent to $R^L_{\mu\nu\alpha\beta}(h)=0$, where $R^L_{\mu\nu\alpha\beta}$ is the linearized Riemman tensor and $h_{\mu\nu}=\tilde{B}_{(\mu\nu)}-\eta_{\mu\nu}\tilde{B}/(D-1)$, whose general solution is $h_{\mu\nu}=\partial_\mu A_\nu+\partial_\nu A_\mu$. Maxwell's equations for this vector potential follow from the identity $\partial^\mu\tilde{B}_{(\mu\nu)}=0$. Again, Maxwell's equations come from an identity and the existence of the vector potential is a dynamical result, which is expected for an action dual to Maxwell. S_S is invariant under the $U(1)$ gauge transformation

$$\delta A_\mu=\partial_\mu\Lambda(x),\qquad \delta W_{\mu\nu}=(\eta_{\mu\nu}\Box-\partial_\mu\partial_\nu)\,\Lambda(x).\tag{2.1.12}$$

A parent action dual to S_S is given by

$$S_S^*=\Big\langle-W_{\mu\nu}W^{\mu\nu}+2W^{\mu\nu}\partial^\alpha\partial^\beta B_{[\mu\alpha][\nu\beta]}+W^2\Big\rangle,\tag{2.1.13}$$

the Gaussian integral on $W_{\mu\nu}$ gives S_{HD}, and the integral on $B_{[\mu\nu][\alpha\beta]}$ leads to Maxwell. S_S^* is invariant under

$$\delta W_{\mu\nu}=\partial_\mu\partial_\nu\Lambda\qquad \delta B_{[\mu\nu][\alpha\beta]}=(\eta_{\nu\alpha}\eta_{\mu\beta}-\eta_{\nu\beta}\eta_{\mu\alpha})\,\Lambda+\Omega^T_{[\mu\nu][\alpha\beta]},\tag{2.1.14}$$

with $\Omega^T_{[\mu\nu][\alpha\beta]}$ transverse in any of the boxes, $\partial^\mu\Omega^T_{[\mu\nu][\alpha\beta]}=0$. The dimensional reduction of S_S and S_S^* will be made latter.

2.1.3 Non-Symmetric auxiliary field

So far only the sub-cases $a_1=\pm a_2$ of S have been analysed, now we shall focus in the $|a_1|\neq|a_2|$ case. In this regime it is possible to make a change of variables in (2.1.4) to obtain an equivalent action in the form

$$S_I=\Big\langle e_{\mu\nu}\partial^\mu A^\nu+be_{\mu\nu}e^{\mu\nu}+ce_{\mu\nu}e^{\nu\mu}+de^2+a_3e\partial^\mu A_\mu\Big\rangle,\tag{2.1.15}$$

and for $a_3\neq-1/D$ one can make the new invertible change of variables $\tilde{e}_{\mu\nu}=e_{\mu\nu}+a_3\eta_{\mu\nu}e$ such that the action becomes[6]

$$\tilde{S}_I=\Big\langle e_{\mu\nu}\partial^\mu A^\nu+be_{\mu\nu}e^{\mu\nu}+ce_{\mu\nu}e^{\nu\mu}+de^2\Big\rangle\tag{2.1.16}$$

[5] where $\partial_{(\mu}A_{\nu)}=(\partial_\mu A_\nu+\partial_\nu A_\mu)/2$

[6] we come back to $e_{\mu\nu}$ via $\tilde{e}_{\mu\nu}\to e_{\mu\nu}$, and also perform the redefinition $\tilde{d}\to d$

which reproduces Maxwell, when $e_{\mu\nu}$ is integrated, if $d = -(b+c)/(D-1)$ and $b(b^2-c^2) \neq 0$, therefore we have hence forth

$$S_{NS} = \left\langle e_{\mu\nu}\partial^\mu A^\nu + b e_{\mu\nu}e^{\mu\nu} + c e_{\mu\nu}e^{\nu\mu} - \frac{(b+c)}{D-1}e^2 \right\rangle, \tag{2.1.17}$$

which is invariant under the $U(1)$ gauge transformation

$$\delta A_\mu = \partial_\mu \Lambda \qquad \delta e_{\mu\nu} = \frac{1}{2(b+c)}\left(\eta_{\mu\nu}\Box\Lambda - \partial_\mu\partial_\nu\Lambda\right). \tag{2.1.18}$$

In fact the functional integral on $e_{\mu\nu}$ in (2.1.17) gives

$$S_M = \left\langle -\frac{bF^{\mu\nu}F_{\mu\nu}}{8(b^2-c^2)} \right\rangle. \tag{2.1.19}$$

Notice that $b/(b^2-c^2) > 0$ requires $b > 0$ if $|b| > |c|$ or $b < 0$ if $|b| < |c|$.

The functional integral on A_μ leads to the constraint $\partial^\mu e_{\mu\nu} = 0$ whose solution $e_{\mu\nu} = \partial^\alpha B_{[\alpha\mu]\nu}$ furnishes a new model of massless spin-1 particles

$$S_{BB} = \left\langle b\partial^\beta B_{[\beta\mu]\nu}\partial_\alpha B^{[\alpha\mu]\nu} + c\partial^\beta B_{[\beta\mu]\nu}\partial_\alpha B^{[\alpha\nu]\mu} - \frac{(b+c)}{D-1}\left(\partial^\gamma B_\gamma\right)^2 \right\rangle, \tag{2.1.20}$$

where $B_{[\alpha\beta]\mu}$ is antisymmetric in the first two indices and $B_\gamma = B_{[\gamma\mu]}{}^{\mu}$. This action is invariant under the gauge transformations

$$\delta B_{[\beta\mu]\nu} = \left[\eta_{\beta\nu}\partial_\mu\Lambda - \eta_{\mu\nu}\partial_\beta\Lambda\right] + \Omega^T_{[\beta\mu]\nu}, \tag{2.1.21}$$

where $\partial^\beta\Omega^T_{[\beta\mu]\nu} = 0$. The equations of motion $\delta S_{BB} = 0$ are

$$b\left(\partial_\gamma e_{\mu\nu} - \partial_\mu e_{\gamma\nu}\right) + c\left(\partial_\gamma e_{\nu\mu} - \partial_\mu e_{\nu\gamma}\right) - \frac{(b+c)}{D-1}\left(\partial_\gamma e\eta_{\mu\nu} - \partial_\mu e\eta_{\gamma\nu}\right) = 0, \tag{2.1.22}$$

with $e_{\mu\nu} = \partial^\alpha B_{[\alpha\mu]\nu}$. These equations can be rearranged in a simple form

$$\partial_\gamma \bar{e}_{\mu\nu} - \partial_\mu \bar{e}_{\gamma\nu} = 0, \tag{2.1.23}$$

where

$$\bar{e}_{\mu\nu} \equiv b e_{\mu\nu} + c e_{\nu\mu} - \frac{(b+c)}{D-1}e\eta_{\mu\nu}. \tag{2.1.24}$$

The general solution of (2.1.23) is $\bar{e}_{\mu\nu} = \partial_\mu A_\nu$, when plugged into (2.1.24) gives

$$e_{\mu\nu} = \frac{1}{b^2-c^2}\left(b\partial_\mu A_\nu - c\partial_\nu A_\mu\right) - \frac{\eta_{\mu\nu}}{b+c}\partial^\mu A_\mu, \tag{2.1.25}$$

Maxwell's equations come from the divergence of $e_{\mu\nu}$ in the left index,

$$\partial^{\mu} e_{\mu\nu} = \frac{b}{b^2 - c^2} \left(\Box A_{\nu} - \partial_{\nu} \partial^{\mu} A_{\mu} \right) = 0. \tag{2.1.26}$$

Thus, Maxwell's equations here come from an identity, since $e_{\mu\nu} = \partial^{\alpha} B_{[\alpha\mu]\nu}$, while the existence of the vector potential is dynamic which reinforces the fact that S_{BB} is dual to Maxwell, just like S_{TT} and S_{HD}

A parent action dual to S_{NS} is given by

$$S^{*}_{NS} = \left\langle -\frac{1}{2} e^{\mu\nu} e_{\mu\nu} + \frac{1}{2} e^2 + e^{\mu\nu} \partial^{\alpha} B_{[\alpha\mu]\nu} \right\rangle, \tag{2.1.27}$$

where we have chosen $b = 1/2$ and $c = 0$ for simplicity. An integral on $B_{[\mu\nu]\alpha}$ leads to Maxwell, and the integral on $e_{\mu\nu}$ gives S_{BB}. The dimensional reduction of S_{NS} and S^{*}_{NS} will be investigated latter. Notice that S^{*}_{NS} is invariant under[7]

$$\delta B_{[\beta\mu]\nu} = \left[\eta_{\beta\nu} \partial_{\mu} \Lambda - \eta_{\mu\nu} \partial_{\beta} \Lambda \right] + \Omega^{T}_{[\beta\mu]\nu} \qquad \delta e_{\mu\nu} = \partial_{\nu} \partial_{\mu} \Lambda, \tag{2.1.28}$$

In summary, by using antisymmetric, symmetric and non-symmetric auxiliary fields we have obtained respectively the higher rank models S_{TT}, S_{HD} and the new one S_{BB}, dual to the Maxwell theory.

2.2 MASSIVE DUAL MODELS

The Kaluza-Klein dimensional reduction is introduced as a method to obtain massive theories from massless ones. We apply the dimensional reduction to the parent actions of the previous section, in order to obtain massive parent actions and explore what dual massive theories can be obtained from these parent actions. We also comment on alternative methods to obtain massive dual models.

2.2.1 KK Dimensional Reduction

The beginning of Kaluza-Klein (KK) types of dimensional reduction, of one extra cyclical dimension, goes back to Nordström in 1914 (NORDSTROM, 1914). Originally the aim was a unified theory of gravity and electromagnetism, electromagnetism was supposed to be a four-dimensional manifestation of gravity in five dimensions. Nordström tried to use a scalar theory of gravity that he was developing. The same idea was pursued later by Kaluza, and then Klein, this time using Einstein's General Relativity in five dimensions.

The KK dimensional reduction that will be performed here is to be regarded only as a tool for getting massive field theories out of massless ones. We can get a hint that massless particles in $D+1$ dimensions are related to massive particles in D dimensions by noticing that Wigner's little group (WIGNER, 1939) for massive particles in D dimensions is the $SO(D-1)$, and for massless particles

[7] Notice that the first term is the $U(1)$ symmetry of Maxwell via $\bar{e}_{\mu\nu} = \partial_{\mu} A_{\nu}$

in $D+1$ is $ISO(D-1)$ (BEKAERT; BOULANGER, 2021). The group of translations and rotations in a Euclidean space of $D-1$ dimensions. Since $SO(D-1)$ is a subgroup of $ISO(D-1)$ we can see that the little groups of massless particles in $D+1$ dimensions and those of massive particles in D dimensions are related. Moreover, in $D=4$, the massless particle states are usually defined as a trivial representation of the translation part of $ISO(2)$ (WEINBERG, 1995), so that their quantum numbers come from the representation of $SO(2)$[8]. If this is also the case in higher dimensions, then the same group would define the quantum numbers of massive particles in D and massless ones in $D+1$: the group $SO(D-1)$.

For instance, let's consider Maxwell's theory in $D+1$ dimensions, the signature is $(-,+,\cdots,+)$ and one of the directions, say y, has a circular topology with length L: y is the same as $y+L$. This condition requires the fields to be periodic in the y coordinate. If $\bar{A}^M$ is the vector potential in $D+1$, with capital letter indices for the $D+1$ coordinates $M=0,1,\cdots,D$, the Fourier decomposition is

$$\bar{A}^y(x,y)=\sum_{n=1}^{\infty}\sqrt{\frac{1}{r\pi}}\phi_n(x)\sin\left(\frac{ny}{r}\right), \qquad \bar{A}^\mu(x,y)=\sum_{n=0}^{\infty}\sqrt{\frac{1}{r\pi}}A_n^\mu(x)\cos\left(\frac{ny}{r}\right). \tag{2.2.1}$$

Where x^μ are the coordinates of the remaining D directions and r is the compactification radius defined by $2\pi r=L$. Under the transformation $y\to -y$ a tensor $T_{y_1,y_2,\cdots,y_n,\mu,\cdots\nu}$ transforms like $T_{y_1,y_2,\cdots,y_n,\mu,\cdots\nu}\to(-1)^nT_{y_1,y_2,\cdots,y_n,\mu,\cdots\nu}$. Therefore, the Fourier decomposition for tensors with an even number of components in the y direction depends only on the cosine and for an odd number of components the tensor depends only on the sine. The action in $D+1$ dimensions has the usual form

$$S_M^{D+1}=-\frac{1}{4}\int_0^L dy\int d^Dx\bar{F}_{MN}\bar{F}^{MN}, \tag{2.2.2}$$

with $\bar{F}^{MN}=\partial^M\bar{A}^N-\partial^N\bar{A}^M$. Using (2.2.1) and integrating on the y coordinate above one gets

$$\begin{aligned} S_M^{D+1}&=-\frac{1}{4}\int_0^L dy\int d^Dx\left(2\bar{F}^{y\mu}\bar{F}_{y\mu}+\bar{F}^{\mu\nu}\bar{F}_{\mu\nu}\right)\\ &=-\frac{1}{4}\int d^Dx\left[\sum_{n=1}\left(2\frac{n^2}{r^2}\left(A_n^\mu(x)+\frac{r}{n}\partial^\mu\phi_n(x)\right)^2+F_n^{\mu\nu}F_{\mu\nu}^n\right)+2F_0^{\mu\nu}F_{\mu\nu}^0\right]. \end{aligned} \tag{2.2.3}$$

By defining the mass $m_n=n/r$ it is clear that the action in D dimensions is a sum of massive actions plus a massless one. The possibility of getting massive field theories from massless ones is our reason to use KK dimensional reduction. From now on we shall make the dimensional reduction considering only the term $n=1$ in (2.2.1), once the higher terms give the same theory with a different mass, and $n=0$ furnishes a massless theory which is already known. Neglecting the massless theory and the massive theories with different mass, (2.2.3) becomes the Maxwell-Proca action with a Stueckelberg

[8] Although $SO(2)$ is a one dimensional Abelian group of continuous transformations the helicity quantum number is still quantized due to the topology of the group (WEINBERG, 1995)

field, where $m = 1/r$,

$$S_{MP} = \left\langle -\frac{1}{4}F_{\mu\nu}F^{\mu\nu} - \frac{m^2}{2}\left(A_\mu + \frac{\partial_\mu \Psi}{m}\right)^2 \right\rangle. \tag{2.2.4}$$

A key feature of the dimensional reduction method already appears in (2.2.4), namely, the final massive theory is still gauge invariant:

$$\delta A_\mu = \partial_\mu \Lambda \tag{2.2.5}$$

$$\delta \Psi = -m\Lambda. \tag{2.2.6}$$

2.2.2 Antissymmetric spin one

Despite the antisymmetric case being well known in the literature, we shall explore it in detail here as an introduction to the symmetric and non-symmetric cases. In $D+1$ the action S_{AN} is

$$S_{AN}^{D+1} = \int d^{D+1}x \left[\frac{1}{4}V_{[MN]}V^{[MN]} - V^{[MN]}\partial_{[M}A_{N]}\right], \tag{2.2.7}$$

with the Fourier decomposition

$$V_{y\mu}(x,y) = \sqrt{\frac{m}{\pi}}B_\mu(x)\sin(my) \quad V_{\mu\nu}(x,y) = \sqrt{\frac{m}{\pi}}V_{\mu\nu}(x)\cos(my),$$
$$A_y(x,y) = \sqrt{\frac{m}{\pi}}\Psi(x)\sin(my) \quad A_\mu(x,y) = \sqrt{\frac{m}{\pi}}A_\mu(x)\cos(my),$$

after integrating on the cyclic coordinate $x^{D+1} = y$, the massive action in D dimensions becomes

$$S_{AN}^m = \left\langle \frac{1}{4}V_{\mu\nu}V^{\mu\nu} - V^{\mu\nu}\partial_\mu A_\nu + \frac{B_\mu B^\mu}{2} + B^\mu\left(mA_\mu + \partial_\mu\Psi\right) \right\rangle. \tag{2.2.8}$$

From the gauge symmetry of S_{AN} by $\delta A_M = \partial_M \Lambda(x,y)$ in $D+1$, and with $\Lambda(x,y) = \sqrt{\frac{m}{\pi}}\Lambda(x)\cos(my)$, we see that S_{AN}^m is invariant by the $U(1)$ transformation:

$$\delta A_\mu = \partial_\mu \Lambda(x) \qquad \delta\Psi = -m\Lambda(x), \tag{2.2.9}$$

showing that Ψ is a Stueckelberg field. The functional generator is

$$Z = \int \mathcal{D}V_{\mu\nu}\mathcal{D}A_\mu\mathcal{D}B_\mu\mathcal{D}\Psi \exp\left[i\left\langle \frac{1}{4}V_{\mu\nu}V^{\mu\nu} - V^{\mu\nu}\partial_\mu A_\nu + \frac{B_\mu B^\mu}{2} + B^\mu\left(mA_\mu + \partial_\mu\Psi\right) \right\rangle\right]. \tag{2.2.10}$$

The integral of an exponential in the quadratic fields is proportional to the value of the exponential calculated with the fields that extremize the action. The value of $V_{[\mu\nu]}$ that extremizes the action is

$V_{[\mu\nu]} = F_{\mu\nu}(A)$, therefore the integral on $V_{[\mu\nu]}$ gives

$$Z = C \int \mathcal{D}A_\mu \mathcal{D}B_\mu \mathcal{D}\Psi \exp\left[i\left\langle -\frac{1}{4}F_{\mu\nu}F^{\mu\nu} + \frac{B_\mu B^\mu}{2} + B^\mu \left(mA_\mu + \partial_\mu \Psi\right) \right\rangle\right], \qquad (2.2.11)$$

where C is a divergent constant of integration. Doing the integral on B_μ we get

$$Z = C\bar{C} \int \mathcal{D}A_\mu \mathcal{D}\Psi \exp\left[i\left\langle -\frac{1}{4}F_{\mu\nu}F^{\mu\nu} - \frac{1}{2}\left(mA_\mu + \partial_\mu \Psi\right)^2 \right\rangle\right], \qquad (2.2.12)$$

the Maxwell-Proca action (2.2.4).

It's well known that

$$\int dx e^{ikx} \propto \delta(k). \qquad (2.2.13)$$

In the path integral formalism, the integral of a linear field in the action gives a functional version of the delta distribution. If, instead of the B_μ integral in (2.2.11), we integrate the Ψ field the functional generator becomes

$$Z = \tilde{C} \int \mathcal{D}A_\mu \mathcal{D}B_\mu \exp\left[i\left\langle -\frac{1}{4}F_{\mu\nu}F^{\mu\nu} + \frac{B_\mu B^\mu}{2} + B^\mu m A_\mu \right\rangle\right] \delta\left(\partial_\mu B^\mu\right). \qquad (2.2.14)$$

The B_μ integral selects the B^μ fields with null divergence or $B^\mu = \partial_\alpha B^{[\alpha\mu]}$, for any arbitrary antisymmetric tensor $B^{[\alpha\mu]}$, such that

$$Z = \tilde{C} \int \mathcal{D}A_\mu \mathcal{D}B_{[\alpha\mu]} \exp\left[i\left\langle -\frac{1}{4}F_{\mu\nu}F^{\mu\nu} + \frac{\partial^\alpha B_{[\alpha\mu]}\partial_\alpha B^{[\alpha\mu]}}{2} - \frac{m}{2}B^{[\alpha\mu]}F_{\alpha\mu} \right\rangle\right]. \qquad (2.2.15)$$

This is the Cremmer-Scherk action whose content is also massive spin-one particles[9].

Finally, the integral of (2.2.10) on A_μ gives the functional constraint

$$mB_\mu + \partial^\nu V_{[\nu\mu]} = 0, \qquad (2.2.16)$$

which solving for B_μ returns

$$Z = \int \mathcal{D}V_{\mu\nu} \exp\left[i\left\langle \frac{1}{4}V_{\mu\nu}V^{\mu\nu} + \frac{\partial^\alpha V_{[\alpha\mu]}\partial_\beta V^{[\beta\mu]}}{2m^2} \right\rangle\right], \qquad (2.2.17)$$

the Kalb-Ramond model.

From the path integral perspective, observables always come from the functional generator. Two theories whose functional generators differ by a constant will give the same observable quantities and in this sense are called dual. From this point of view we have just proved the duality between the

[9] In $D = 4$ we can make the replacement $B_{\mu\nu} \to \epsilon_{\mu\nu\alpha\beta}\tilde{B}^{\alpha\beta}$, the last term of (2.2.15) becomes topological and the Cremmer-Scherk model is also called topologically massive BF model.

Maxwell-Proca theory with Stueckelberg (2.2.12), the Cremmer-Scherck model (2.2.15) and the Kalb-Ramond model (2.2.17), also called massive two-form model, starting from (2.2.10). The reduction presented here appears in (KHOUDEIR, 1998) from $D=5$ to $D=4$. Another perspective on duality among certain classes of theories and an issue regarding the path integral derivation of dual models, will be given in 4.

We can summarize the results of this section with the following scheme

$$S^m_{AN} \to \begin{cases} S_{MP} \\ S_{CS} \\ S_{KR} \end{cases}, \tag{2.2.18}$$

where S_{MP}, S_{CS} and S_{KR} are respectively (2.2.12), (2.2.15), (2.2.17).

2.2.3 Symmetric spin one

Now we shall proceed with the dimensional reduction of S_S and S^*_S. The parent action S_S in $D+1$ is

$$S_{(D+1),S} = \int d^{D+1}x \left(W^{MN}W_{MN} - \frac{W^2}{D} + 2W^{MN}\partial_{(M}A_{N)} \right), \tag{2.2.19}$$

the fields in $D+1$ can be decomposed as

$$\begin{aligned} W_{yy}(x,y) &= \sqrt{\frac{m}{\pi}}\phi(x)\cos(my), \quad W_{y\mu}(x,y) = \sqrt{\frac{m}{\pi}}B_\mu(x)\sin(my) \\ W_{\mu\nu}(x,y) &= \sqrt{\frac{m}{\pi}}W_{\mu\nu}(x)\cos(my), \quad A_y(x,y) = \sqrt{\frac{m}{\pi}}\Psi(x)\sin(my) \\ A_\mu(x,y) &= \sqrt{\frac{m}{\pi}}A_\mu(x)\cos(my), \end{aligned}$$

with the following expansion for the gauge parameter

$$\Lambda(x,y) = \sqrt{\frac{m}{\pi}}\Lambda(x)\cos(my). \tag{2.2.20}$$

After integrating the y coordinate one gets the reduced action in D dimensions

$$S^m_S = \left\langle W_{\mu\nu}W^{\mu\nu} - \frac{1}{D}(W+\phi)^2 + 2W^{\mu\nu}\partial_{(\mu}A_{\nu)} + \phi^2 + 2B_\mu B^\mu + 2m\phi\Psi + 2B_\mu\left(-mA^\mu + \partial^\mu\Psi\right) \right\rangle. \tag{2.2.21}$$

The above action is invariant by the gauge transformations

$$\delta\Psi = -m\Lambda, \qquad \delta\phi = \Box\Lambda, \tag{2.2.22}$$

$$\delta A_\mu = \partial_\mu\Lambda, \qquad \delta B_\mu = m\partial_\mu\Lambda, \tag{2.2.23}$$

$$\delta W_{\mu\nu} = \left[\left(\Box - m^2\right)\eta_{\mu\nu} - \partial_\mu\partial_\nu\right]\Lambda, \tag{2.2.24}$$

which comes from the invariance of the action (2.2.19) in $D+1$ dimensions by

$$\delta A_M = \partial_M\Lambda(x,y) \qquad \delta W_{MN} = \left(\eta_{MN}\Box - \partial_M\partial_N\right)\Lambda(x,y). \tag{2.2.25}$$

The functional integration on ϕ, W and then Ψ leads to the well known Cremmer-Scherk model once again

$$S_{CS} = \left\langle -\frac{1}{4}F_{\mu\nu}(A)F^{\mu\nu}(A) - \frac{m}{2}F^{\mu\nu}(A)B_{[\mu\nu]} + \frac{1}{2}\partial^\alpha B_{[\alpha\mu]}\partial_\beta B^{[\beta\mu]}\right\rangle, \tag{2.2.26}$$

with $B_{[\mu\nu]}$ an antisymmetric tensor which comes from the solution $B_\mu = mA_\mu + \partial^\alpha B_{[\alpha\mu]}$ of the functional constraint $\partial^\mu\left(B_\mu - mA_\mu\right) = 0$, arising from the Ψ integral. If the integral on Ψ is exchanged with the B_μ integral, the result is Maxwell-Proca with a Stuckelberg field

$$S_{MP} = \left\langle -\frac{1}{4}F_{\mu\nu}F^{\mu\nu} - \frac{m^2}{2}\left(A_\mu + \frac{\partial_\mu\Psi}{m}\right)^2\right\rangle. \tag{2.2.27}$$

Integrating (2.2.21) on ϕ, B and A we get

$$S^I = \left\langle W^{\mu\nu}W_{\mu\nu} - \frac{W^2}{D-1} + \frac{2}{m^2}\partial^\alpha W_{\alpha\mu}\partial_\beta W^{\beta\mu} + 2m\frac{W\Psi}{D-1} - \frac{m^2\Psi^2 D}{D-1} + \frac{2}{m}\Psi\partial^2 W\right\rangle \tag{2.2.28}$$

fixing the gauge $\Psi = 0$ in the action, this is allowed since Ψ is a Stueckelberg field (MOTOHASHI; SUYAMA; TAKAHASHI, 2016), we obtain

$$S^m = \left\langle W^{\mu\nu}W_{\mu\nu} - \frac{W^2}{D-1} + \frac{2}{m^2}\partial^\alpha W_{\alpha\mu}\partial_\beta W^{\beta\mu}\right\rangle. \tag{2.2.29}$$

which was already deduced in (DALMAZI; SANTOS, 2011). If instead of fixing the gauge $\Psi = 0$ we integrate over Ψ we get a fourth order model physically equivalent to (2.2.29).

Now we shift to the dual master action S_S^*, in $D+1$ dimensions

$$S^*_{(D+1),S} = \int d^{D+1}x\left[-W_{MN}W^{MN} + 2W^{MN}\partial^O\partial^P B_{[MO][NP]} + W^2\right]. \tag{2.2.30}$$

Taking the Fourier decomposition

$$W_{yy}(x,y)=\sqrt{\frac{m}{\pi}}\phi(x)\cos(my)\qquad W_{y\mu}(x,y)=\sqrt{\frac{m}{\pi}}B_\mu(x)\sin(my)$$
$$W_{\mu\nu}(x,y)=\sqrt{\frac{m}{\pi}}W_{\mu\nu}(x)\cos(my)\qquad B_{[y\mu][\alpha\beta]}(x,y)=\sqrt{\frac{m}{\pi}}Z_{\mu[\alpha\beta]}(x)\sin(my)$$
$$B_{[y\mu][y\alpha]}(x,y)=\sqrt{\frac{m}{\pi}}N_{\mu\alpha}(x)\cos(my)\qquad B_{[\nu\mu][\beta\alpha]}(x,y)=\sqrt{\frac{m}{\pi}}B_{[\nu\mu][\beta\alpha]}(x)\cos(my),$$

where $N_{\mu\nu}=N_{(\mu\nu)}$, the integral on y leads to the massive parent action

$$S_S^{*m}=\Big\langle -W_{\mu\nu}W^{\mu\nu}-2B^\mu B_\mu+2W\phi+W^2+2\phi\partial^2N+4mB^\mu\partial^\alpha N_{\alpha\mu}+4B^\mu\partial^\alpha\partial^\beta Z_{\alpha[\mu\beta]}$$
$$-2m^2W^{\mu\nu}N_{\mu\nu}-4mW^{\mu\nu}\partial^\alpha Z_{\mu[\nu\alpha]}+2W^{\mu\nu}\partial^\alpha\partial^\beta B_{[\mu\alpha][\nu\beta]}\Big\rangle. \tag{2.2.31}$$

With the following decomposition for the gauge parameters

$$\Lambda(x,y)=\sqrt{\frac{m}{\pi}}=\Lambda(x)\cos(my)\qquad \Omega^T_{[y\mu][\alpha\beta]}(x,y)=\sqrt{\frac{m}{\pi}}\Gamma_{\mu[\alpha\beta]}(x)\sin(my) \tag{2.2.32}$$
$$\Omega^T_{[y\mu][y\alpha]}(x,y)=\sqrt{\frac{m}{\pi}}\Theta_{(\mu\alpha)}(x)\cos(my)\qquad \Omega^T_{[\nu\mu][\beta\alpha]}(x,y)=\sqrt{\frac{m}{\pi}}\omega_{[\nu\mu][\beta\alpha]}(x)\cos(my), \tag{2.2.33}$$

the action S_S^{*m} is invariant by

$$\delta\phi=-m^2\Lambda,\qquad \delta N_{\mu\alpha}=-\eta_{\mu\alpha}\Lambda+\Theta_{(\mu\alpha)}, \tag{2.2.34}$$
$$\delta W_{\mu\nu}=\partial_\mu\partial_\nu\Lambda\qquad \delta B_\mu=-m\partial_\mu\Lambda \tag{2.2.35}$$
$$\delta B_{[\mu\nu][\alpha\beta]}=(\eta_{\mu\beta}\eta_{\nu\alpha}-\eta_{\mu\alpha}\eta_{\nu\beta})\Lambda+\omega_{[\mu\nu][\alpha\beta]} \tag{2.2.36}$$
$$\delta Z_{\mu[\alpha\beta]}=\Gamma_{\mu[\alpha\beta]} \tag{2.2.37}$$

which is the version in D dimensions of

$$\delta W_{MN}=\partial_M\partial_N\Lambda(x,y)\qquad \delta B_{[MN][OP]}=(\eta_{NO}\eta_{MP}-\eta_{NP}\eta_{OM})\Lambda(x,y)+\Omega^T_{[MN][OP]}. \tag{2.2.38}$$

Due to the $D+1$ condition $\partial^M\Omega^T_{[MN][OP]}=0$ we have the following relations among the gauge parameters in D dimensions:

$$\partial^\mu\Gamma_{\mu[\alpha\beta]}=0,\qquad -m\Theta_{(\alpha\beta)}+\partial^\mu\Gamma_{\beta[\mu\alpha]}=0, \tag{2.2.39}$$
$$m\Gamma_{\alpha[\gamma\lambda]}+\partial^\mu\omega_{[\mu\alpha][\gamma\lambda]}=0. \tag{2.2.40}$$

The integral on $B_{[\mu\alpha][\nu\beta]}$, $Z_{\alpha[\mu\beta]}$ and $N_{\mu\nu}$ leads to Maxwell-Proca with a Stueckelberg field

$$S_{MP} = \left\langle -\frac{1}{4}F_{\mu\nu}F^{\mu\nu} - \frac{m^2}{2}\left(A_\mu + \frac{\partial_\mu \Psi}{m}\right)^2 \right\rangle. \tag{2.2.41}$$

On the other hand, the integral on $W_{\mu\nu}$, B_μ and ϕ recovers (2.2.29) with Stueckelberg fields, when the Stuckelberg fields are set to zero (2.2.29) is recovered with $W_{\mu\nu}$ in (2.2.29) replaced by $N_{\mu\nu}$. The Cremmer-Schercker model will result from integrations in $B_{[\mu\alpha][\nu\beta]}$, B_μ and ϕ.

Schematically the symmetric master action S_S^m provides the dual models

$$S_S^m \to \begin{cases} S_{CS} \\ S_{MP} \ , \\ S^m \end{cases} \tag{2.2.42}$$

whereas the dual master action S_S^{*m} gives the same results.

2.2.4 Non-Symmetric Spin One

For the dimensional reduction of S_{NS} from $D+1$ to D (see (2.1.17)), we take the Fourier decomposition

$$\begin{aligned} e_{yy}(x,y) &= \sqrt{\frac{m}{\pi}}\phi(x)\cos(my) \qquad & e_{y\mu}(x,y) &= \sqrt{\frac{m}{\pi}}S_\mu(x)\sin(my) \\ e_{\mu y}(x,y) &= \sqrt{\frac{m}{\pi}}B_\mu(x)\sin(my) \qquad & e_{\mu\nu}(x,y) &= \sqrt{\frac{m}{\pi}}e_{\mu\nu}(x)\cos(my) \\ A_y(x,y) &= \sqrt{\frac{m}{\pi}}\Psi(x)\sin(my) \qquad & A_\mu(x,y) &= \sqrt{\frac{m}{\pi}}A_\mu(x)\cos(my), \end{aligned}$$

The massive parent action, after the integral on y,

$$\begin{aligned} S_{NS}^m = \Big\langle & m\Psi\phi - mS_\mu A^\mu + B_\mu\partial^\mu\Psi + e_{\mu\nu}\partial^\mu A^\nu + b\left(\phi^2 + S_\mu S^\mu + B_\mu B^\mu + e_{\mu\nu}e^{\mu\nu}\right) \\ & + c\left(\phi^2 + 2S_\mu B^\mu + e_{\mu\nu}e^{\nu\mu}\right) - \frac{(b+c)}{D}(\phi+e)^2 \Big\rangle, \end{aligned} \tag{2.2.43}$$

is invariant by the gauge transformations

$$\delta\phi = \frac{\Box\Lambda}{2(b+c)} \qquad \delta S_\mu = \frac{m\partial_\mu\Lambda}{2(b+c)} \tag{2.2.44}$$

$$\delta B_\mu = \frac{m\partial_\mu\Lambda}{2(b+c)} \qquad \delta e_{\mu\nu} = \frac{1}{2(b+c)}\left(\eta_{\mu\nu}(\Box - m^2) - \partial_\mu\partial_\nu\right)\Lambda \tag{2.2.45}$$

$$\delta\Psi = -m\Lambda \qquad \delta A_\mu = \partial_\mu\Lambda. \tag{2.2.46}$$

As usual these come from the higher dimensional version of:

$$\delta A_M = \partial_M \Lambda(x,y) \qquad \delta e_{MN} = \frac{1}{2(b+c)} \left(\eta_{MN}\Box - \partial_{MN}\right) \Lambda(x,y). \tag{2.2.47}$$

The integral of (2.2.43) on $e_{\mu\nu}$ leads to the intermediate action[10]

$$S_I = \left\langle bB_\mu B^\mu - mA^\mu S_\mu + 2cB^\mu S_\mu + bS_\mu S^\mu - \frac{m}{2(b+c)} A^\mu \partial_\mu \phi + B^\mu \partial_\mu \Psi - \frac{b}{8(b^2-c^2)} F_{\mu\nu}F^{\mu\nu} \right\rangle. \tag{2.2.48}$$

The integral on Ψ leads to the Cremmer-Schercker model if instead, one integrates on B_μ and S_μ the dual model is Maxwell-Proca with a Stueckelberg field. Starting from (2.2.43) the integral on S, B, A and ϕ results in a non-symmetric model

$$\begin{aligned} S[e,\Psi] = \Bigg\langle be_{\mu\nu}e^{\mu\nu} + ce_{\mu\nu}e^{\nu\mu} - \frac{(b+c)}{D-1}e^2 + \frac{me\Psi}{D-1} - \frac{Dm^2\Psi^2}{4(b+c)(D-1)} - \frac{\partial^\alpha\Psi\partial_\alpha\Psi}{4b} \\ - \frac{c\Psi\partial^2 e}{bm} + \frac{(b^2-c^2)}{bm^2}\partial^\alpha e_{\alpha\beta}\partial_\mu e^{\mu\beta} \Bigg\rangle, \end{aligned} \tag{2.2.49}$$

where Ψ is a Stueckelberg field. It is more convenient to express the equations of motion coming from $\delta_e S[e,\Psi] = 0$ in terms of $\tilde{e}_{\mu\nu} \equiv be_{\mu\nu} + ce_{\nu\mu}$

$$\tilde{e}_{\mu\nu} - \frac{\tilde{e}\eta_{\mu\nu}}{D-1} + \frac{m\eta_{\mu\nu}\Psi}{2(D-1)} - \frac{c}{2bm}\partial_\mu\partial_\nu\Psi - \frac{1}{bm^2}\partial_\mu\left(b\partial^\alpha\tilde{e}_{\alpha\nu} - c\partial^\alpha\tilde{e}_{\nu\alpha}\right) = 0, \tag{2.2.50}$$

whose trace is:

$$-\frac{\tilde{e}}{D-1} + \frac{mD\Psi}{2(D-1)} - \left[\frac{c}{2bm}\Box\Psi + \frac{(b-c)}{bm^2}\partial^2\tilde{e}\right] = 0. \tag{2.2.51}$$

Using the Ψ equation of motion from (2.2.49):

$$\frac{1}{2(b+c)}\Box\Psi - \frac{1}{m}\partial^2\tilde{e} = 0, \tag{2.2.52}$$

we can rewrite (2.2.50) as

$$\tilde{e}_{\mu\nu} - \frac{m}{2}\eta_{\mu\nu}\Psi + \eta_{\mu\nu}\frac{\partial^2\tilde{e}}{m^2} - \frac{\partial_\mu\partial^\alpha\tilde{e}_{\alpha\nu}}{m^2} + \frac{c}{bm}\partial_\mu\left(\frac{\partial^\alpha\tilde{e}_{\nu\alpha}}{m} - \frac{\partial_\nu\psi}{2}\right) = 0. \tag{2.2.53}$$

Taking the ∂^ν divergence we have

$$\partial^\nu\tilde{e}_{\mu\nu} - \partial_\mu\frac{m\Psi}{2} = 0, \tag{2.2.54}$$

[10] The field ϕ is also eliminated by the integral on $e_{\mu\nu}$

which allows us to write

$$\tilde{e}_{\mu\nu} - \eta_{\mu\nu} m \frac{\Psi}{2} + \eta_{\mu\nu} \frac{\partial^2 \tilde{e}}{m^2} - \frac{\partial_\mu \partial^\alpha \tilde{e}_{\alpha\nu}}{m^2} = 0. \tag{2.2.55}$$

Taking the ∂^μ divergence

$$\partial^\mu \tilde{e}_{\mu\nu} - \frac{\Box \partial^\mu \tilde{e}_{\mu\nu}}{m^2} - \frac{m \partial_\nu \Psi}{2} + \frac{\partial_\nu \partial^2 \tilde{e}}{m^2} = 0 \tag{2.2.56}$$

The above equation, and the Ψ equation (2.2.52) will be mapped in Maxwell-Proca equations with Stueckelberg field, if one defines $-\partial^\alpha \tilde{e}_{\alpha\nu}/m^2 \equiv A_\nu$ and $\Psi_{STCK} \equiv \Psi / [2(b+c)]$ showing that (2.2.49) is dual to massive spin-1. Forgetting for a moment that $A_\nu = -\partial^\alpha \tilde{e}_{\alpha\nu}/m^2$ the role of the second order equations (2.2.55) is to define the rank two tensor as a function of A_ν and Ψ_{STCK} which in turn are fixed by (2.2.52) and (2.2.56). The $e_{\mu\nu}$ obtained in this way satisfies all the equations of motion and constraint equations, being therefore a solution. In fact, any rank two tensor can be decomposed as $\tilde{e}_{\mu\nu} = \eta_{\mu\nu}\phi + \partial_\mu f_\nu + e^{LT}_{\mu\nu}$, where $e^{LT} = 0 = \partial^\mu e^{LT}_{\mu\nu}$, (2.2.55) fix $e^{LT}_{\mu\nu} = 0$. We can see more clearly that Ψ is a Stueckelberg compensating field by defining the gauge invariant rank-two tensor

$$\bar{e}_{\mu\nu} = e_{\mu\nu} + \frac{1}{2(b+c)} \left(\eta_{\mu\nu}(\Box - m^2) - \partial_\mu \partial_\nu \right) \frac{\Psi}{m}. \tag{2.2.57}$$

In terms of this new gauge invariant field (2.2.49) is

$$S_{NS}[\bar{e}] = \left\langle b \bar{e}_{\mu\nu} \bar{e}^{\mu\nu} + c \bar{e}_{\mu\nu} \bar{e}^{\nu\mu} - \frac{(b+c)}{D-1} \bar{e}^2 + \frac{(b^2 - c^2)}{b m^2} \partial^\alpha \bar{e}_{\alpha\beta} \partial_\mu \bar{e}^{\mu\beta} \right\rangle, \tag{2.2.58}$$

a massive spin one Lagrangian without gauge symmetry if we set $\bar{\Psi} = 0$. The model S_{NS} is an original result of this dissertation and can not be reduced to its predecessors (2.2.17) and (2.2.29) via any local field redefinition.

On the other hand, if one integrates (2.2.43) on S, ϕ and A we get a new model

$$\begin{aligned} S[e,B] = \Bigg\langle & b e_{\mu\nu} e^{\mu\nu} + c e_{\mu\nu} e^{\nu\mu} - \frac{(b+c)}{D} e^2 + \frac{b}{m^2} \partial^\alpha e_{\alpha\beta} \partial_\mu e^{\mu\beta} + b B^\mu B_\mu + (b+c) \frac{(D-1)}{D m^2} (\partial^\mu B_\mu)^2 \\ & -2 \frac{(b+c)}{D m} e \partial^\mu B_\mu - \frac{2c}{m} B^\mu \partial^\alpha e_{\alpha\mu} \Bigg\rangle \end{aligned} \tag{2.2.59}$$

This dual theory mixes the rank-2 tensor with a vector field.

For S^*_{NS}, see (2.1.27), we take the Fourier decomposition in $D+1$

$$
\begin{aligned}
e_{yy}(x,y) &= \sqrt{\frac{m}{\pi}}\Psi(x)\cos(my) & e_{y\mu}(x,y) &= \sqrt{\frac{m}{\pi}}S_\mu(x)\sin(my)\\
e_{\mu y}(x,y) &= \sqrt{\frac{m}{\pi}}\tilde{B}_\mu(x)\sin(my) & e_{\mu\nu}(x,y) &= \sqrt{\frac{m}{\pi}}e_{\mu\nu}(x)\cos(my)\\
B_{[y\mu]y}(x,y) &= \sqrt{\frac{m}{\pi}}A_\mu(x)\cos(my) & B_{[y\mu]\nu}(x,y) &= \sqrt{\frac{m}{\pi}}K_{\mu\nu}(x)\sin(my),\\
B_{[\mu\nu]y}(x,y) &= \sqrt{\frac{m}{\pi}}H_{[\mu\nu]}(x)\sin(my) & B_{[\mu\nu]\alpha}(x,y) &= \sqrt{\frac{m}{\pi}}B_{[\mu\nu]\alpha}(x)\cos(my),
\end{aligned}
$$

the massive parent action becomes

$$
\begin{aligned}
S^{*m}_{NS} = \Bigg\langle &-\frac{1}{2}\left(\Psi^2 + S_\mu S^\mu + \tilde{B}^\mu\tilde{B}_\mu + e_{\mu\nu}e^{\mu\nu}\right) + \frac{1}{2}\left(\Psi + e\right)^2 - mA_\mu\tilde{B}^\mu + me^{\mu\nu}K_{\mu\nu}\\
&+\partial_\mu\Psi A^\mu - S^\nu\partial^\mu K_{\mu\nu} + \tilde{B}^\nu\partial^\mu H_{\mu\nu} + e^{\alpha\beta}\partial^\mu B_{[\mu\alpha]\beta}\Bigg\rangle
\end{aligned}
\tag{2.2.60}
$$

If we take the following decomposition for the gauge parameters

$$
\Lambda(x,y) = \sqrt{\frac{m}{\pi}}\Lambda(x)\cos(my) \qquad \Omega^T_{[y\mu]y}(x,y) = \sqrt{\frac{m}{\pi}}\lambda_\mu(x)\cos(my) \tag{2.2.61}
$$

$$
\Omega^T_{[y\mu]\nu}(x,y) = \sqrt{\frac{m}{\pi}}G_{\mu\nu}(x)\sin(my) \qquad \Omega^T_{[\alpha\mu]y}(x,y) = \sqrt{\frac{m}{\pi}}\Gamma_{[\alpha\mu]}(x)\sin(my) \tag{2.2.62}
$$

$$
\Omega^T_{[\mu\nu]\alpha}(x,y) = \sqrt{\frac{m}{\pi}}\Omega_{[\mu\nu]\alpha}(x)\cos(my), \tag{2.2.63}
$$

the action will be invariant by the gauge transformations

$$
\delta\Psi = -m^2\Lambda \qquad \delta S_\mu = -m\partial_\mu\Lambda \tag{2.2.64}
$$

$$
\delta\tilde{B}_\mu = -m\partial_\mu\Lambda \qquad \delta e_{\mu\nu} = \partial_\mu\partial_\nu\Lambda \tag{2.2.65}
$$

$$
\delta A_\mu = \partial_\mu\Lambda + \lambda_\mu \qquad \delta K_{\mu\nu} = m\eta_{\mu\nu}\Lambda + G_{\mu\nu} \tag{2.2.66}
$$

$$
\delta H_{[\mu\nu]} = \Gamma_{[\mu\nu]} \qquad \delta B_{[\mu\nu]\alpha} = \left(\eta_{\mu\alpha}\partial_\nu - \eta_{\nu\alpha}\partial_\mu\right)\Lambda + \Omega_{[\mu\nu]\alpha}, \tag{2.2.67}
$$

coming from

$$
\delta B_{[MN]O} = \left(\eta_{MO}\partial_N - \eta_{ON}\partial_M\right)\Lambda(x,y) + \Omega^T_{[MN]O} \qquad \delta e_{MN} = \partial_M\partial_N\Lambda(x,y). \tag{2.2.68}
$$

Due to the constraint $\partial^M\Omega^T_{[MN]O} = 0$ in $D+1$ we have

$$
-m\lambda_\alpha + \partial^\mu\Gamma_{[\mu\alpha]} = 0, \qquad mG_{\alpha\beta} + \partial^\mu\Omega_{[\mu\alpha]\beta} = 0. \tag{2.2.69}
$$

The integral on $e_{\mu\nu}$, A_μ and $K_{\mu\nu}$ results in Maxwell-Proca with Stueckelberg field, on the other hand integrating Ψ, $K_{\mu\nu}$ and $\tilde{B}_\mu$ one gets Cremmer-Scherck. Integration on $e_{\mu\nu}$, S_μ and A_μ leads to an apparently massive version of (2.1.20)

$$S^m_{BB} = \Big\langle \frac{1}{2}\partial_\alpha B^{[\alpha\mu]\nu}\partial^\beta B_{[\beta\mu]\nu} - \frac{(\partial_\alpha B^\alpha)^2}{2(D-1)} - \frac{m}{D-1}\tilde{\Psi}\partial_\beta B^\beta - \frac{Dm^2\tilde{\Psi}^2}{2(D-1)} - \frac{\partial_\alpha\tilde{\Psi}\partial^\alpha\tilde{\Psi}}{2} - \frac{m^2K^2}{2(D-1)}$$
$$- \frac{m^2K\tilde{\Psi}}{D-1} - \frac{mK\partial_\beta B^\beta}{D-1} + mK^{\alpha\gamma}\partial^\beta B_{[\beta\alpha]\gamma} + \frac{1}{2}\partial^\mu K_{\mu\nu}\partial_\alpha K^{\alpha\nu} + \frac{m^2}{2}K_{\alpha\gamma}K^{\alpha\gamma}\Big\rangle, \tag{2.2.70}$$

with $\tilde{\Psi} = \Psi/m$. However, the fields $\tilde{\Psi}$ and $B_{[\alpha\mu]\nu}$ are Stueckelberg fields in (2.2.70), fixing $\tilde{\Psi} = B_{[\alpha\mu]\nu} = 0$ we are left with (2.2.58) with $b = 1/2$ and $c = 0$ and $\bar{e}_{\mu\nu} \to K_{\mu\nu}$.

The non-symmetric master action S^m_{NS} provides the dual models

$$S^m_{NS} \to \begin{cases} S_{CS} \\ S_{MP} \\ S_{NS}[\bar{e}] \\ S[e,B] \end{cases}. \tag{2.2.71}$$

The other hand the dual master action S^{*m}_{NS} leads to the same models except for $S[e,B]$.

2.2.5 Alternative Methods

There are several methods to obtaining massive dual theories. The dimensional reduction performed so far is just one among many options. We will use here two alternative methods, which will give dual theories not obtained in the dimensional reduction.

The first method is due to (SMAILAGIC; SPALLUCCI, 2001) and consists of three steps. We start by promoting the symmetry of the kinetic term of the massive Lagrangian to a symmetry of the whole massive theory through a Stueckelberg field. For example, in the Proca theory, the symmetry in the kinetic term is by $\delta A_\mu = \partial_\mu\Lambda$, we modify the mass term via $A_\mu \to A_\mu + \partial_\mu\Psi/m$ in order to get

$$S_{MP} = \Big\langle -\frac{1}{4}F_{\mu\nu}F^{\mu\nu} - \frac{m^2}{2}\left(A_\mu + \frac{\partial_\mu\Psi}{m}\right)^2\Big\rangle. \tag{2.2.72}$$

Then we reduce the order of the mass term with a vector field B_μ such that

$$S_I = \Big\langle -\frac{1}{4}F_{\mu\nu}F^{\mu\nu} + mB^\mu\left(A_\mu + \frac{\partial_\mu\Psi}{m}\right) + \frac{B_\mu B^\mu}{2}\Big\rangle, \tag{2.2.73}$$

Now we integrate on the Stueckelberg Ψ field in order to get a dual theory. In fact, the integral of (2.2.73) on Ψ gives the Cremmer-Sherck action which is in fact a dual massive spin one model.

If we take the non-symmetric dual model (2.2.58)[11] the symmetry transformation of the kinetic

[11] We make $\bar{e}_{\mu\nu} \to e_{\mu\nu}$.

term is $\delta e_{\mu\nu} = \partial^{\alpha}\Lambda_{[\alpha\mu]\nu}$. The action invariant by this symmetry transformation is

$$S_I = \left\langle \frac{(b^2-c^2)}{b}\partial_\alpha e^{\alpha\mu}\partial^\beta e_{\beta\mu} + m^2\left[b\tilde{e}_{\mu\nu}\tilde{e}^{\mu\nu} + c\tilde{e}_{\mu\nu}\tilde{e}^{\nu\mu} - \frac{(b+C)}{D-1}\tilde{e}^2\right]\right\rangle \tag{2.2.74}$$

$$\tilde{e}_{\mu\nu} \equiv e_{\mu\nu} - \frac{\partial^\alpha B_{[\alpha\mu]\nu}}{m}, \tag{2.2.75}$$

where $B_{[\alpha\mu]\nu}$ is a Stueckelberg field, $\delta B_{[\alpha\mu]\nu} = m\Lambda_{[\alpha\mu]\nu}$. We reduce the order of the mass term with the tensor $f_{\mu\nu}$ such that

$$S_I[f] = \left\langle \frac{(b^2-c^2)}{b}\partial_\alpha e^{\alpha\mu}\partial^\beta e_{\beta\mu} - \frac{b}{4(b^2-c^2)}f_{\mu\nu}f^{\mu\nu} + \frac{c}{4(b^2-c^2)}f_{\mu\nu}f^{\nu\mu} + \frac{f^2}{4(b+c)} + f^{\mu\nu}\left(e_{\mu\nu} - \frac{\partial^\alpha B_{[\alpha\mu]\nu}}{m}\right)\right\rangle. \tag{2.2.76}$$

The integral on the Stueckelberg $B_{[\alpha\mu]\nu}$ leads to the constraint $\partial^\alpha f^{\mu\nu} - \partial^\mu f^{\alpha\nu} = 0$, whose solution is $f^{\mu\nu} = \partial^\mu A^\nu$. This leads to the new dual model

$$S_I^* = \left\langle \frac{(b^2-c^2)}{b}\partial_\alpha e^{\alpha\mu}\partial^\beta e_{\beta\mu} + m^2\left[\frac{-b}{8(b^2-c^2)}F_{\mu\nu}F^{\mu\nu} + e^{\mu\nu}\partial_\mu A_\nu\right]\right\rangle. \tag{2.2.77}$$

The equations of motion are

$$\frac{b}{2(b^2-c^2)}\partial^\mu F_{\mu\nu} - \partial^\mu e_{\mu\nu} = 0 \tag{2.2.78}$$

$$-\frac{2(b^2-c^2)}{b}\partial_\mu\partial^\alpha e_{\alpha\nu} + m^2\partial_\mu A_\nu = 0, \tag{2.2.79}$$

using the first equation in the second one we have

$$\partial_\mu\left(\partial^\alpha F_{\alpha\nu} - m^2 A_\nu\right) = 0 \tag{2.2.80}$$

whose solution is the Proca equation for A_ν showing that (2.2.77) is in fact dual to Proca.

The second method relies on the parent action. The idea now is to construct a massive parent action by adding a mass term to the massless parent action (2.1.17). Take the action

$$S = \left\langle be_{\mu\nu}e^{\mu\nu} + ce_{\mu\nu}e^{\nu\mu} - \frac{(b+c)e^2}{D-1} + e_{\mu\nu}\partial^\mu A^\nu - \frac{bm^2A_\mu A^\mu}{4(b^2-c^2)}\right\rangle, \tag{2.2.81}$$

we define a parent action S_P by adding a second term without content

$$S_P^m = \left\langle be_{\mu\nu}e^{\mu\nu} + ce_{\mu\nu}e^{\nu\mu} - \frac{(b+c)e^2}{D-1} + e_{\mu\nu}\partial^\mu A^\nu - \frac{1}{2}\left(e_{[\mu\nu]} - \tilde{e}_{[\mu\nu]}\right)\left(F^{\mu\nu}(A) - F^{\mu\nu}(\tilde{A})\right) - \frac{bm^2A_\mu A^\mu}{4(b^2-c^2)}\right\rangle, \tag{2.2.82}$$

notice that the integral on $\tilde{e}_{\mu\nu}$ or $\tilde{A}_\mu$ recovers (2.2.81). On the other hand, if we integrate on $e_{[\mu\nu]}$ and A_μ we get the new dual model, $U(1)$ invariant,

$$S_P^m(\bar{e}_{\mu\nu},\tilde{A}_\mu) = \Big\langle (b+c)\bar{e}_{(\mu\nu)}\bar{e}^{(\mu\nu)} - \frac{(b+c)\bar{e}^2}{(D-1)} + \frac{(b^2-c^2)}{bm^2}\partial^\mu\bar{e}_{\mu\alpha}\partial_\nu\bar{e}^{\nu\alpha} - \frac{1}{2}\bar{e}_{[\mu\nu]}F^{\mu\nu}(\tilde{A}) - \frac{F^{\mu\nu}(\tilde{A})F_{\mu\nu}(\tilde{A})}{16(b-c)}\Big\rangle, \quad (2.2.83)$$

where $\bar{e}_{\mu\nu} \equiv e_{(\mu\nu)} + \tilde{e}_{[\mu\nu]}$.

Its equations of motion are

$$(b+c)\bar{e}_{(\mu\nu)} - \frac{(b+c)\eta_{\mu\nu}\bar{e}}{(D-1)} - \frac{(b^2-c^2}{bm^2}\partial_\mu\partial^\gamma\bar{e}_{\gamma\nu} - \frac{F_{\mu\nu}}{4} = 0 \quad (2.2.84)$$

$$\frac{\partial^\mu F_{\mu\nu}}{4(b-c)} + \partial^\mu\bar{e}_{[\mu\nu]} = 0, \quad (2.2.85)$$

the trace and ∂^ν divergence of (2.2.84) lead to

$$-\frac{(b+c)\bar{e}}{(D-1)} - \frac{(b^2-c^2)\partial^2\bar{e}}{bm^2} = 0 \quad (2.2.86)$$

$$(b+c)\partial^\nu\bar{e}_{(\mu\nu)} - \frac{\partial^\nu F_{\mu\nu}}{4} = 0. \quad (2.2.87)$$

From the second equation we have $\partial^2\bar{e} = 0$, and from (2.2.84) in (2.2.87) it comes

$$(b+c)\partial^\nu\bar{e}_{(\mu\nu)} - (b-c)\partial^\nu\bar{e}_{[\nu\mu]} = 0. \quad (2.2.88)$$

Finally, if we apply ∂^μ on (2.2.84) we get

$$-\frac{2(b^2-c^2)}{bm^2}\left(\Box - m^2\right)\partial^\alpha\bar{e}_{\alpha\nu} = 0. \quad (2.2.89)$$

3 INTERACTIONS

Up to now our investigation of duality was restricted to free theories at the classical level. Here we approach the study of duality at the quantum level with the usual method found in the literature: examining the correlation functions with path integrals. We investigate the most interesting models of the last chapter: S_{BB} and $S[\bar{e}]$ given in (2.1.20) and (2.2.58) respectively.

3.1 MASSLESS CASE

We shall introduce an external source in the parent action (2.1.16) to search for a map between correlation functions

$$S_{NS}^{EF} = \left\langle be_{\mu\nu}e^{\mu\nu} + ce_{\mu\nu}e^{\nu\mu} - \frac{(b+c)e^2}{D-1} + \partial^\mu A^\nu e_{\mu\nu} + e^{\mu\nu}J_{[\mu\nu]}\right\rangle. \tag{3.1.1}$$

From the gauge symmetry (2.1.18) we see that $e_{[\mu\nu]}$ is gauge invariant as well as the source term above. The integral on $e_{\mu\nu}$ leads to

$$S_M^{EF} = \left\langle -\frac{bF_{\mu\nu}F^{\mu\nu}}{8(b^2-c^2)} - \frac{J^{[\mu\nu]}F_{\mu\nu}}{4(b-c)} - \frac{J_{[\mu\nu]}J^{[\mu\nu]}}{4(b-c)}\right\rangle, \tag{3.1.2}$$

and an integral of (3.1.1) on A_μ recovers the S_{BB} model with sources

$$S_{BB}^{EF} = \left\langle b\partial^\beta B_{[\beta\mu]\nu}\partial_\alpha B^{[\alpha\mu]\nu} + c\partial^\beta B_{[\beta\mu]\nu}\partial_\alpha B^{[\alpha\nu]\mu} - \frac{(b+c)}{D-1}\left(\partial^\gamma B_\gamma\right)^2 + \partial^\beta B_{[\beta\mu]\nu}J^{[\mu\nu]}\right\rangle. \tag{3.1.3}$$

The correlation functions come from the derivative of the functional generator with respect to $J_{[\mu\nu]}$ on the "point"$J_{[\mu\nu]} = 0$. Therefore one can read off the dual map between correlation functions, up to contact terms, from functional derivatives of (3.1.2) and (3.1.3):

$$-\frac{F_{\mu\nu}}{4(b-c)} \longleftrightarrow \frac{1}{2}\left(\partial^\alpha B_{[\alpha\mu]\nu} - \partial^\alpha B_{[\alpha\nu]\mu}\right). \tag{3.1.4}$$

We note that this map between the usual Maxwell theory and the S_{BB} model is gauge invariant and compatible with (2.1.25), which was inspired only by the analysis of the classical equations of motion.

Not all gauge invariants of the S_{BB} theory are given in terms of $e_{[\mu\nu]}[B]$, where $e_{\mu\nu}[B] \equiv \partial^\alpha B_{[\alpha\mu]\nu}$. We also have gauge invariants in terms of $e_{(\mu\nu)}[B]$. We can show that these gauge invariants are also mapped in $F_{\mu\nu}$ through the following parent action

$$\tilde{S}_{NS}^{EF} = \left\langle be_{\mu\nu}e^{\mu\nu} + ce_{\mu\nu}e^{\nu\mu} - \frac{(b+c)e^2}{D-1} + \partial^\mu A^\nu e_{\mu\nu} + e^{\mu\nu}T_{(\mu\nu)}\right\rangle \tag{3.1.5}$$

$$\Box\theta_{\mu\nu}T^{(\mu\nu)} = 0, \tag{3.1.6}$$

with $\theta_{\mu\nu} = \eta_{\mu\nu} - \partial_\mu \partial_\nu / \Box$. The condition (3.1.6) on the source $T_{(\mu\nu)}$ insures the gauge invariance of the coupling. If we integrate on $e_{\mu\nu}$ or A_μ we obtain, respectively

$$S_M = \left\langle -\frac{bF_{\mu\nu}F^{\mu\nu}}{8(b^2-c^2)} + \frac{T^{(\mu\nu)}}{2(b+C)}\left(\eta_{\mu\nu}\partial^\alpha A_\alpha - \partial_{(\mu}A_{\nu)}\right) + \frac{T^{(\mu\nu)}}{4(b+c)}\left(\eta_{\mu\nu}T - T_{(\mu\nu)}\right)\right\rangle \tag{3.1.7}$$

$$S_{BB} = \left\langle b\partial^\beta B_{[\beta\mu]\nu}\partial_\alpha B^{[\alpha\mu]\nu} + c\partial^\beta B_{[\beta\mu]\nu}\partial_\alpha B^{[\alpha\nu]\mu} - \frac{(b+c)}{D-1}\left(\partial^\gamma B_\gamma\right)^2 + \partial^\beta B_{[\beta\mu]\nu}T^{(\mu\nu)}\right\rangle. \tag{3.1.8}$$

We can decompose $T_{(\mu\nu)}$ in helicity variables:

$$T_{(00)} = s, \qquad T_{(0j)} = s_j^t + \partial_j\sigma, \qquad T_{(ij)} = s_{ij}^{tt} + \partial_i\sigma_j + \partial_j\sigma_i + \nabla^2\bar{\theta}_{ij}S + \partial_i\partial_j\bar{S}, \tag{3.1.9}$$

$$\partial^j s_j = \partial^j\sigma_j = \partial^i s_{ij}^{tt} = \delta_{ij}s_{ij}^{tt} = 0, \qquad \bar{\theta}_{ij} = \delta_{ij} - \frac{\partial_i\partial_j}{\nabla^2}, \tag{3.1.10}$$

and use (3.1.6) to eliminate one of them

$$\Box\theta_{\mu\nu}T^{(\mu\nu)} = 0 \to s = (D-2)\Box S - \ddot{S} - 2\dot{\sigma}. \tag{3.1.11}$$

Using this helicity decomposition in the coupling term $e_{\mu\nu}T^{(\mu\nu)}$ in (3.1.8), the independent components of $T_{(\mu\nu)}$ are coupled to gauge invariants

$$\begin{aligned} T^{(\mu\nu)}e_{(\mu\nu)} = &-\frac{2\sigma\partial_j I_j}{\nabla^2} + \frac{2s_j\bar{\theta}_{jk}I_k}{\nabla^2} + \frac{S\nabla^2\bar{\theta}_{ij}I_{ij}}{\nabla^2} + \frac{\bar{S}\partial_i\partial_j I_{ij}}{\nabla^2} - \frac{2\sigma_j\partial_i\bar{\theta}_{jk}I_{ik}}{\nabla^2} \\ &+ s_{ij}^{tt}\left(\frac{\bar{\theta}_{ik}\bar{\theta}_{jl}I_{kl}}{\nabla^2} - \frac{\bar{\theta}_{ij}\bar{\theta}_{kl}}{D-2}\frac{I_{kl}}{\nabla^2}\right), \end{aligned} \tag{3.1.12}$$

where we have the gauge invariants

$$I_j \equiv \nabla^2 e_{(0j)}[B] - \partial_0\partial_j e_{00}[B], \qquad I_{ij} \equiv \nabla^2 e_{(ij)}[B] + \left(\Box\delta_{ij} - \partial_i\partial_j\right)e_{00}[B]. \tag{3.1.13}$$

From functional derivatives of (3.1.7) and (3.1.8) with respect to the independent components of $T_{(\mu\nu)}$ and fixing $T_{(\mu\nu)} = 0$ at the end we have the following gauge invariant map between the Maxwell and S_{BB} model up to contact terms

$$\frac{\partial_i\left(\partial_j F_{0i} - \partial_0 F_{ij}\right)}{2(b+c)} \longleftrightarrow I_j \tag{3.1.14}$$

$$\frac{\partial_k\left(2\delta_{ij}\partial_0 F_{0k} - \partial_i F_{kj} - \partial_j F_{ki}\right)}{2(b+c)} \longleftrightarrow I_{ij}. \tag{3.1.15}$$

Thus, we end up mapping the symmetric and antisymmetric parts of $e_{\mu\nu}$ in the new massless spin-1 model and the usual Maxwell's theory.

3.2 MASSIVE CASE

First, we note that from the parent action

$$S_P = \left\langle be_{\mu\nu}e^{\mu\nu} + ce_{\mu\nu}e^{\nu\mu} - \frac{(b+c)e^2}{D-1} + \partial_\mu A_\nu e^{\mu\nu} - \frac{bm^2 A_\mu A^\mu}{4(b^2-c^2)} \right\rangle \tag{3.2.1}$$

it is possible to get Maxwell-Proca after a functional integral on $e_{\mu\nu}$. The non-symmetric model (2.2.49), without the Stueckelberg field Ψ, is obtained after an integral on A_μ. In order to establish the quantum map we shall use (bellow $T_{\mu\nu} \neq T_{\nu\mu}$)

$$S_P[J,T] = \left\langle be_{\mu\nu}e^{\mu\nu} + ce_{\mu\nu}e^{\nu\mu} - \frac{(b+c)e^2}{D-1} + \partial_\mu A_\nu e^{\mu\nu} - \frac{bm^2 A_\mu A^\mu}{4(b^2-c^2)} + A_\mu J_\mu + e_{\mu\nu}T^{\mu\nu} \right\rangle. \tag{3.2.2}$$

Integrating on A_μ gives

$$S^{EF}[e] = \left\langle be_{\mu\nu}e^{\mu\nu} + ce_{\mu\nu}e^{\nu\mu} - \frac{(b+c)e^2}{D-1} + \frac{(b^2-c^2)}{bm^2}\partial_\mu e^{\mu\nu}\partial^\alpha e_{\alpha\nu} + e_{\mu\nu}T^{\mu\nu} \right.$$
$$\left. + \frac{(b^2-c^2)}{bm^2}\left(J^\mu J_\mu + 2e^{\mu\nu}\partial_\mu J_\nu\right) \right\rangle, \tag{3.2.3}$$

on the other hand the integral of (3.2.2) on $e_{\mu\nu}$ leads to

$$S^{EF}[A] = \left\langle \frac{-bF_{\mu\nu}F^{\mu\nu}}{8(b^2-c^2)} - \frac{bm^2 A_\mu A^\mu}{4(b^2-c^2)} + A_\mu J^\mu - \frac{T^{\mu\nu}}{2(b^2-c^2)}\left(b\partial_\mu A_\nu - c\partial_\nu A_\mu - (b-c)\eta_{\mu\nu}\partial_\mu A^\mu\right) \right.$$
$$\left. - \frac{T^{\mu\nu}}{4(b^2-c^2)}\left(bT^{\mu\nu} - cT^{\nu\mu} - (b-c)\eta_{\mu\nu}T\right) \right\rangle \tag{3.2.4}$$

Therefore the dual maps are, up to contact terms,

$$A_\mu \longleftrightarrow \frac{-2(b^2-c^2)}{bm^2}\partial^\alpha e_{\alpha\mu} \tag{3.2.5}$$

$$-\frac{1}{2(b^2-c^2)}\left[b\partial_\mu A_\nu - c\partial_\nu A_\mu - (b-c)\eta_{\mu\nu}\partial_\alpha A^\alpha\right] \longleftrightarrow e_{\mu\nu} \tag{3.2.6}$$

between the correlation functions of Maxwell-Proca and the $S[e]$ model.

In order to investigate the duality in the presence of interactions we propose the parent action

$$S_P = \left\langle be_{\mu\nu}e^{\mu\nu} + ce_{\mu\nu}e^{\nu\mu} - \frac{(b+c)e^2}{D-1} + \partial_\mu A_\nu e^{\mu\nu} - \frac{bm^2 A_\mu A^\mu}{4(b^2-c^2)} + j_\mu A^\mu + e_{\mu\nu}t^{\mu\nu} + \mathcal{L}_M \right\rangle, \tag{3.2.7}$$

where j_μ and $t_{\mu\nu}$ are functions of matter fields ψ_l, while $\mathcal{L}_M$ is the free matter lagrangian. The integral on A_μ leads to

$$S^I[e] = \left\langle be_{\mu\nu}e^{\mu\nu} + ce_{\mu\nu}e^{\nu\mu} - \frac{(b+c)e^2}{D-1} + \frac{(b^2-c^2)}{bm^2}\partial_\mu e^{\mu\nu}\partial^\alpha e_{\alpha\nu} + e_{\mu\nu}t^{\mu\nu} + \mathcal{L}_M \right.$$
$$\left. + \frac{(b^2-c^2)}{bm^2}\left(j^\mu j_\mu + 2e^{\mu\nu}\partial_\mu j_\nu\right)\right\rangle, \qquad (3.2.8)$$

whose equations of motion $\delta_e S^I[e] = 0$ are

$$be_{\mu\nu} + ce_{\nu\mu} - \frac{(b+c)e}{D-1}\eta_{\mu\nu} - \frac{(b^2-c^2)}{bm^2}\partial_\mu\partial^\alpha e_{\alpha\nu} + \frac{t_{\mu\nu}}{2} + \frac{(b^2-c^2)}{bm^2}\partial_\mu j_\nu = 0, \qquad (3.2.9)$$

and the trace is

$$-\frac{(b+c)e}{D-1} - \frac{(b^2-c^2)}{bm^2}\partial^2 e + \frac{t}{2} + \frac{(b^2-c^2)}{bm^2}\partial^\alpha j_\alpha = 0. \qquad (3.2.10)$$

Replacing the trace back in (3.2.9) we get

$$E_{\mu\nu} \equiv be_{\mu\nu} + ce_{\nu\mu} - \left[-\frac{(b^2-c^2)}{bm^2}\partial^2 e + \frac{t}{2} + \frac{(b^2-c^2)}{bm^2}\partial^\alpha j_\alpha\right]\eta_{\mu\nu} - \frac{(b^2-c^2)}{bm^2}\partial_\mu\partial^\alpha e_{\alpha\nu}$$
$$+\frac{t_{\mu\nu}}{2} + \frac{(b^2-c^2)}{bm^2}\partial_\mu j_\nu = 0. \qquad (3.2.11)$$

From $bE_{\mu\nu} - cE_{\nu\mu} = 0$ we have

$$e_{\mu\nu} = -\frac{1}{2(b^2-c^2)}\left[bt_{\mu\nu} - ct_{\nu\mu} - (b-c)\eta_{\mu\nu}t\right]$$
$$+ \frac{1}{bm^2}\left[b\partial_\mu(\partial^\alpha e_{\alpha\nu} - j_\nu) - c\partial_\nu(\partial^\alpha e_{\alpha\mu} - j_\mu) - \eta_{\mu\nu}(b-c)(\partial^2 e - \partial_\mu j^\mu)\right], \qquad (3.2.12)$$

the ∂^μ divergence gives

$$\frac{(\Box - m^2)}{m^2}\left(\partial^\alpha e_{\alpha\nu} - j_\nu\right) - \frac{\partial_\nu}{m^2}\left(\partial^2 e - \partial^\mu j_\mu\right) - \frac{1}{2(b^2-c^2)}\left[b\partial^\alpha t_{\alpha\nu} - c\partial^\alpha t_{\nu\alpha} - (b-c)\partial_\nu t\right]$$
$$-j_\nu = 0, \qquad (3.2.13)$$

The integral of (3.2.7) on $e_{\mu\nu}$ gives

$$S^I_{MP} = \left\langle -\frac{bF_{\mu\nu}F^{\mu\nu}}{8(b^2-c^2)} - \frac{m^2 bA^\mu A_\mu}{4(b^2-c^2)} + A_\mu j^\mu - \frac{t^{\mu\nu}}{4(b^2-c^2)}\left[bt_{\mu\nu} - ct_{\nu\mu} - (b-c)\eta_{\mu\nu}t\right] \right.$$
$$\left. -\frac{t^{\mu\nu}}{2(b^2-c^2)}\left[b\partial_\mu A_\nu - c\partial_\nu A_\mu - (b-c)\eta_{\mu\nu}\partial^\alpha A_\alpha\right] + \mathcal{L}_M\right\rangle, \qquad (3.2.14)$$

the equations of motion $\delta_A S^I_{MP} = 0$ are

$$\frac{b}{2(b^2-c^2)}\left(\Box A_\nu - \partial_\nu \partial^\mu A_\mu\right) - \frac{m^2 b A_\nu}{2(b^2-c^2)} + j_\nu + \frac{1}{2(b^2-c^2)}\left[b\partial^\alpha t_{\alpha\nu} - c\partial^\alpha t_{\nu\alpha} - (b-c)\partial_\nu t\right] = 0. \tag{3.2.15}$$

We see from (3.2.13) and (3.2.15) that there is a map between the vector field

$$\tilde{A}_\nu \equiv -\frac{2(b^2-c^2)}{bm^2}\left(\partial^\alpha e_{\alpha\nu} - j_\nu\right), \tag{3.2.16}$$

in (3.2.13) and A_μ in the equations (3.2.15). This map is preserved when one takes into account the equations for matter fields. In fact, using (3.2.12), $\delta_\psi S^I[e] = 0$ gives

$$\frac{\delta \mathcal{L}_M}{\delta \psi_l} - \frac{2(b^2-c^2)}{bm^2}\left(\partial^\alpha e_{\alpha\nu} - j_\nu\right)\frac{\delta j^\nu}{\delta \psi_l} + \left\{ -\frac{1}{2(b^2-c^2)}\left[b t_{\mu\nu} - c t_{\mu\nu} - (b-c)\eta_{\mu\nu} t\right]\right.$$
$$\left. + \frac{1}{bm^2}\left[b\partial_\mu(\partial^\alpha e_{\alpha\nu} - j_\nu) - c\partial_\nu(\partial^\alpha e_{\alpha\mu} - j_\mu) - \eta_{\mu\nu}(b-c)(\partial^2 e - \partial_\mu j^\mu)\right]\right\}\frac{\delta t^{\mu\nu}}{\delta \psi_l} = 0. \tag{3.2.17}$$

The matter equations $\delta_\psi S^I_{MP}$ of (3.2.14) are

$$\frac{\delta \mathcal{L}_M}{\delta \psi_l} - \left\{\frac{\left[b t_{\mu\nu} - c t_{\nu\mu} - (b-c) t \eta_{\mu\nu}\right]}{2(b^2-c^2)} + \frac{\left[b\partial_\mu A_\nu - c\partial_\nu A_\mu - (b-c)\eta_{\mu\nu}\partial^\mu A_\mu\right]}{2(b^2-c^2)}\right\}\frac{\delta t^{\mu\nu}}{\delta \psi_l}$$
$$+ A_\mu \frac{\delta j^\mu}{\delta \psi_l} = 0. \tag{3.2.18}$$

Therefore the vector field (3.2.16) appears on (3.2.17) in the same way as A_μ in (3.2.18), leading to a map between the interacting theories $S^I[e]$ and S^I_{MP}. The mixing (3.2.16) between matter fields and spin-1 fields and the presence of quadratic terms in the matter sources has appeared before in (MALUF et al., 2020).

4 CANONICAL ANALYSIS

We adopt the point of view introduced in (BARBOSA, 2022) for a class of massive dual models, which will be defined as Kalb-Ramond type duals. The advantage of this approach is that it provides a systematical framework to understand the free theories and make general statements about the interacting dual models, in the perturbative sense. We shall see that some of the so called "dual models"have their origin in a change of coordinates.

4.1 FREE KALB-RAMOND

Consider again the Kalb-Ramond theory (2.2.17)

$$S_{KR} = \int d^4x \left[\frac{(\partial^\mu B_{\mu\nu})^2}{2} + \frac{m^2}{4} B_{\mu\nu} B^{\mu\nu} \right], \tag{4.1.1}$$

with the usual notation for the antisymmetric field: $B_{\mu\nu} = -B_{\nu\mu}$. In order to see the spin-1 content of this theory we will canonically quantize it. The conjugate momenta are

$$\Pi^{0i} = -\partial_\mu B^{\mu i}, \qquad \Pi^{ij} = 0. \tag{4.1.2}$$

The spatial components of the momenta are zero and become therefore constraints. We will be using the Dirac-Bergmann algorithm to deal with constrained systems. The Hamiltonian is

$$H = \int d^3x \left[\frac{1}{2} \Pi^{0i} \Pi^{0i} - B_{ji} \partial^j \Pi^{0i} + \frac{1}{2} \partial^j B_{0j} \partial^k B_{0k} - \frac{m^2}{4} B_{\mu\nu} B^{\mu\nu} + \lambda_{ij} \Pi^{ij} \right], \tag{4.1.3}$$

where λ_{ij} are some Lagrange multipliers. Due to the antisymmetry of $B_{\mu\nu}$ the Poisson brackets are

$$\{B_{\mu\nu}(\boldsymbol{x}, t)\, ,\, \Pi^{\alpha\beta}(\boldsymbol{y}, t)\} = \left(\delta^\alpha_\mu \delta^\beta_\nu - \delta^\alpha_\nu \delta^\beta_\mu \right) \delta^3(\boldsymbol{x} - \boldsymbol{y}). \tag{4.1.4}$$

The consistency condition of the primary constraint $\Pi^{ij} = 0$ generates a new constraint

$$\xi^{ij} \equiv \{\Pi^{ij}\, ,\, H\} = \partial^i \Pi^{0j} - \partial^j \Pi^{0i} + m^2 B^{ij} = 0, \tag{4.1.5}$$

whose consistency condition fixes λ_{ij} ending the algorithm.

The reduced Hamiltonian can be easily expressed as a function of the independent variables (B_{0i}, Π^{0i}), it is definite positive and reads

$$H^R_{KR} = \int d^3x \left[\frac{1}{2} \Pi^{0i} \Pi^{0i} + \frac{1}{2} \partial^j B_{0j} \partial^l B_{0l} + \frac{m^2}{2} B_{0j} B_{0j} + \frac{m^2}{4} \left(\partial^i \Pi^{0j} - \partial^j \Pi^{0i} \right)^2 \right]. \tag{4.1.6}$$

The non-zero Dirac-brackets among the independent variables B_{0i} and Π^{0i} are

$$\{B_{0i}(\boldsymbol{x},t)\,,\,\Pi^{0j}(\boldsymbol{y},t)\}_D = \delta_i^j\delta^3(\boldsymbol{x}-\boldsymbol{y}). \tag{4.1.7}$$

In order to quantize the theory we should promote fields to operators, the reduced Hamiltonian (4.1.6) to the Hamiltonian operator and the commutation relations between fields will be given by

$$\left[B_{0i}(\boldsymbol{x},t)\,,\,\Pi^{0j}(\boldsymbol{y},t)\right] = i\,\{B_{0i}(\boldsymbol{x},t)\,,\,\Pi^{0j}(\boldsymbol{y},t)\}_D = i\,\delta_i^j\delta^3(\boldsymbol{x}-\boldsymbol{y}). \tag{4.1.8}$$

The Heisenberg equations of motion are

$$\dot{B}_{0i} = i\,\left[H_{KR}^R\,,\,B_{0i}\right] = \Pi^{0i} + \partial^j B_{ji}\,, \qquad \dot{\Pi}^{0i} = i\,\left[H_{KR}^R\,,\,\Pi^{0i}\right] = \partial^i\partial^j B_{0j} - m^2 B_{0i}, \tag{4.1.9}$$

with these equations we can recover the covariant equations of motion from (4.1.1):

$$m^2 B_{\mu\nu} - \partial_\mu\partial^\alpha B_{\alpha\nu} + \partial_\nu\partial^\alpha B_{\alpha\mu} = 0. \tag{4.1.10}$$

The divergence of (4.1.10) is the Klein-Gordon equation:

$$(\Box - m^2)\partial^\alpha B_{\alpha\nu} = 0. \tag{4.1.11}$$

The general solution for $B_{\mu\nu}$ in (4.1.10) is

$$B_{\mu\nu} = \frac{F_{\mu\nu}(A)}{m}, \qquad F_{\mu\nu}(A) \equiv \partial_\mu A_\nu - \partial_\nu A_\mu, \tag{4.1.12}$$

with A_μ being the general solution for (4.1.11)

$$\partial^\alpha B_{\alpha\nu} \equiv m A_\nu, \tag{4.1.13}$$

$$(\Box - m^2)A_\nu = 0, \qquad \partial^\nu A_\nu = 0. \tag{4.1.14}$$

Given (4.1.9), (4.1.12) and (4.1.13) we can show that

$$B_{0i} = \frac{1}{m}\left(\dot{A}_i - \partial_i A_0\right), \qquad \Pi^{0i} = -m A_i, \tag{4.1.15}$$

using the commutation relations (4.1.8) we have

$$[A_i(\boldsymbol{y},t)\,,(\dot{A}_j - \partial_j A_0)(\boldsymbol{x},t)] = i\,\delta_i^j\delta^3(\boldsymbol{x}-\boldsymbol{y}) \tag{4.1.16a}$$

$$[A_i(\boldsymbol{y},t)\,,\,A_j(\boldsymbol{x},t)] = [(\dot{A}_i - \partial_i A_0)(\boldsymbol{y},t)\,,\,(\dot{A}_j - \partial_j A_0)(\boldsymbol{x},t)] = 0. \tag{4.1.16b}$$

The variables A_i and $\pi_i \equiv \dot{A}_i - \partial_i A_0$ are also canonical, in particular we can use (4.1.13) to write A_0 as an auxiliary variable

$$A_0 = -\frac{\boldsymbol{\nabla}\cdot\boldsymbol{\pi}}{m}. \tag{4.1.17}$$

Therefore, (4.1.15) is to be understood as a canonical transformation

$$B_{0i} = \frac{\pi^i}{m}, \qquad \Pi^{0i} = -mA_i \tag{4.1.18}$$

and we can use these new coordinates to rewrite (4.1.6), in which case the Hamiltonian reads

$$H^R_{KR} = \int \frac{d^3x}{2}\left[m^2\boldsymbol{A}^2 + \frac{(\boldsymbol{\nabla}\cdot\boldsymbol{\pi})^2}{m^2} + \boldsymbol{\pi}^2 + (\boldsymbol{\nabla}\times\boldsymbol{A})^2\right]. \tag{4.1.19}$$

The set of equations (4.1.16), (4.1.17) and (4.1.19) are what define the canonical structure derived from the Proca Lagrangian (2.2.4) without Stueckelberg fields:

$$L_P = -\frac{1}{4}F_{\mu\nu}F^{\mu\nu} - \frac{m^2}{2}A_\mu A^\mu. \tag{4.1.20}$$

The Lagrangians (4.1.1) and (4.1.20) when canonically quantized give the same Hamiltonian, but in different canonical coordinates. This is the reason for Kalb-Ramond and Proca being associated to the same content.

Starting with the basic postulates of quantum mechanics [1] the first consequence of adding special relativity is the existence of a conserved Hamiltonian (WEINBERG, 1995). So, the Hamiltonian is the natural starting point in a relativistic quantum mechanics. In particular, the Hamiltonian of a free theory is

$$H_0 = \frac{1}{2}\sum_\sigma \int d^3p\, p^0 a^\dagger(\boldsymbol{p},\sigma)a(\boldsymbol{p},\sigma), \tag{4.1.21}$$

where σ are the spin projections of the particle and $a(\boldsymbol{p},\sigma)$, $a^\dagger(\boldsymbol{p}',\sigma')$ are creation and annihilation operators:[2]

$$\left[a(\boldsymbol{p},\sigma)\,,\,a^\dagger(\boldsymbol{p}',\sigma')\right] = \delta_{\sigma\sigma'}\delta^3(\boldsymbol{p}-\boldsymbol{p}'), \qquad \left[a(\boldsymbol{p},\sigma)\,,\,a(\boldsymbol{p}',\sigma')\right] = 0. \tag{4.1.22}$$

We can use the creation and annihilation operators to construct field canonical variables $(q^i(\boldsymbol{x},t), p^i(\boldsymbol{x},t))$ so that (4.1.21) is a functional of these variables and

$$\left[q^i(\boldsymbol{x},t)\,,\,p^j\boldsymbol{y},t)\right] = \delta^{ij}\delta^3(\boldsymbol{x}-\boldsymbol{y}), \tag{4.1.23a}$$

$$\left[q^i(\boldsymbol{x},t)\,,\,q^j(\boldsymbol{x},t)\right] = \left[p^i(\boldsymbol{x},t)\,,\,p^j(\boldsymbol{x},t)\right] = 0, \tag{4.1.23b}$$

$$\dot{q}^i(\boldsymbol{x},t) = i\left[q^i\,,\,H\right], \qquad \dot{p}^i(\boldsymbol{x},t) = i\left[p^i\,,\,H\right]. \tag{4.1.23c}$$

Finally, the Lagrangian is introduced as the Legendre transform of the Hamiltonian

$$L = \int d^3x \sum_i p^i\dot{q}^i - H[q,p]. \tag{4.1.24}$$

[1] Physical states are described by rays in a Hilbert space, observables are represented by Hermitian operators and Born's probabilistic interpretation

[2] These are creation and annihilation operators for bosons, for fermions one should use the anti commutation relations

The fact that a Lagrangian has spin s means that when canonically quantized it gives the right answer, that is (4.1.21).

The advantage of this point of view is that it explains some features of Kalb-Ramond-like dualities. The construction of canonical variables from the creation and annihilation operators is not unique, so we can rewrite (4.1.21) as a function of different canonical coordinates and they are all equivalent. Naturally, the Legendre transformation of the same Hamiltonian in different canonical coordinates will give different Lagrangians. However, all these Lagrangians are equivalent since they furnish the same Hamiltonian. It was shown in (BARBOSA, 2022) how one can start from (4.1.21) and by different choices of canonical variables recover either the Proca or Kalb-Ramond Lagrangians. One may wonder if there are additional choices of coordinates that lead to Lagrangians that are neither (4.1.20) or (4.1.1), but also have spin-1 content. It is shown in the next section that this is the case for the non-symmetric (2.2.49) spin-1 model. The ambiguity in the choice of canonical coordinates is not exclusive to massive spin-1. It is possible to show that the Curtright-Freund model for spin-0 (CURTRIGHT; FREUND, 1980) is also related to massive Klein-Gordon by canonical coordinate changes. The same relations should also exist for higher spins. It has been emphasized in the literature that dual Lagrangians like (4.1.1) and (4.1.20) describe the same physics for non-zero mass, but have different content in the massless limit. This can be easily understood from our point of view. The coordinate change (4.1.15) is not defined for zero mass, so in the massless limit we can not relate these Lagrangians by coordinate changes.

Before closing this section we shall make a comment on the correlation functions. Similarly to what was done in 3.2, using the parent action method is possible to show that the Kalb-Ramond and Proca field strength propagators differ by contact terms, that is

$$\langle B_{\mu\nu}(x)B_{\alpha\beta}(y)\rangle = \langle F_{\mu\nu}(x)F_{\alpha\beta}(y)\rangle + t_{[\mu\nu][\alpha\beta]}\,\delta^4(x-y), \tag{4.1.25}$$

where $t_{[\mu\nu][\alpha\beta]}$ is some tensor. However, from our canonical analysis these propagators should always be equal, since $B_{\mu\nu} \propto F_{\mu\nu}(A)$ is an identity between quantum fields. This is not a contradiction, or even a new problem. For instance, the Proca field propagators calculated from the canonical and path integral formalism are respectively

$$\langle A_\mu(x)A_\nu(y)\rangle_C = \Delta_{\mu\nu}(x,y) + m^{-2}\delta^4(x-y)\delta^0_\mu\delta^0_\nu, \tag{4.1.26}$$

$$\langle A_\mu(x)A_\nu(y)\rangle_{PI} = \Delta_{\mu\nu}(x,y), \tag{4.1.27}$$

where $\Delta_{\mu\nu}(x,y)$ is the usual covariant propagator

$$\Delta_{\mu\nu}(x,y) \equiv (2\pi)^{-4}\int d^4q \frac{(\eta_{\mu\nu}+m^{-2}q_\mu q_\nu)\,e^{iq\cdot(x-y)}}{q^2+m^2-i\epsilon} \tag{4.1.28}$$

This does not pose a problem to the equivalence of path integral and canonical formulations because in the context of free theories we can not compare the points $x = y$. The comparison has to be done in the context that contact terms matter, that is the Feynman rules for interacting theories. In this example the vertex derived from the canonical and the path integral formalism are not equal at first. However, by summing the effect of contact terms in the canonical formalism is possible to redefine the perturbation theory, so that the effective Feynman rules have the same vertex coming from the Path integral and the propagators to be used are just $\Delta_{\mu\nu}(x,y)$. A similar mechanism applies to the relation of Feynman rules derived from Kalb-Ramond and Proca theories.

4.2 NON-SYMMETRIC MODEL

Instead of working with (2.2.49) it will be more convenient to express the action in terms of

$$\tilde{e}_{\mu\nu} \equiv be_{\mu\nu} + ce_{\nu\mu}, \tag{4.2.1}$$

with the gauge fixing $\Psi = 0$. Taking the redefinition $\tilde{e}_{\mu\nu} \to e_{\mu\nu}$ the action reads

$$S_{NS} = \left\langle \frac{(be_{\mu\nu}e^{\mu\nu} - ce_{\mu\nu}e^{\nu\mu})}{(b^2-c^2)} + \frac{d\,e^2}{(b+c)^2} + \frac{(b\partial^\alpha e_{\alpha\nu} - c\partial^\alpha e_{\nu\alpha})^2}{bm^2(b^2-c^2)} \right\rangle. \tag{4.2.2}$$

The conjugate momenta are

$$\Pi^{00} = -2\frac{\left(b\partial_\beta e^{\beta 0} - c\partial_\beta e^{0\beta}\right)}{bm^2(b+c)} \qquad \Pi^{0l} = -2\frac{\left(b\partial_\beta e^{\beta l} - c\partial_\beta e^{l\beta}\right)}{m^2(b^2-c^2)} \tag{4.2.3}$$

$$\Pi^{l0} = \frac{2c}{b}\frac{\left(b\partial_\beta e^{\beta l} - c\partial_\beta e^{l\beta}\right)}{m^2(b^2-c^2)} \qquad \Pi^{ij} = 0, \tag{4.2.4}$$

in addition to $\Pi^{ij} = 0$ we also have the primary constraint

$$c\Pi^{0l} + b\Pi^{l0} = 0. \tag{4.2.5}$$

The Hamiltonian is given by

$$H = \int d^3x \Bigg[-\frac{bm^2(b+c)}{4(b-c)}(\Pi^{00})^2 + \frac{m^2(b^2-c^2)}{4b}\Pi^{0l}\Pi^{0l} - \frac{\left(b\partial_l e^{l0} - c\partial_l e^{0l}\right)}{b-c}\Pi^{00} - \frac{de^2}{(b+c)} \\ \frac{\left(b\partial_k e^{kl} - c\partial_k e^{lk}\right)}{b}\Pi^{0l} - \frac{(be_{\mu\nu}e^{\mu\nu} - ce_{\mu\nu}e^{\nu\mu})}{b^2-c^2} + \lambda_{ij}\Pi^{ij} + \lambda_l\left(b\Pi^{l0} + c\Pi^{0l}\right) \Bigg]. \tag{4.2.6}$$

The consistency condition of $b\Pi^{l0} + c\Pi^{0l} = 0$ generates the new constraint

$$\Gamma^l \equiv (b+c)\partial^l\Pi^{00} + 2e^{l0} = 0, \tag{4.2.7}$$

whose consistency condition fixes λ_l.

The consistency condition $\dot{\Pi}^{ij} = 0$ generates the new constraint

$$\chi^{ij} \equiv \{\Pi^{ij}\,,\,H\} = \frac{b\partial^i\Pi^{0j} - c\partial^j\Pi^{0i}}{b} + 2\frac{be^{ij} - ce^{ji}}{b^2-c^2} + 2\frac{d\delta^{ij}e}{(b+c)^2} = 0. \tag{4.2.8}$$

The consistency condition $\dot{\chi}^{ij} = 0$ gives[3]

$$\phi^{ij} \equiv \{\chi^{ij}\,,\,H\} = -\frac{c\partial^j\partial^i\Pi^{00}}{b} + 2\frac{b\partial^i e^{0j} - c\partial^j e^{0i}}{b^2-c^2} - 2\frac{c\left(b\partial^i e^{j0} - c\partial^j e^{i0}\right)}{b(b^2-c^2)}$$
$$2\frac{(b\lambda^{ij} - c\lambda^{ji})}{b^2-c^2} + \frac{dbm^2\delta^{ij}\Pi^{00}}{b^2-c^2} + 2\frac{d\delta^{ij}\left(b\partial_l e^{l0} - c\partial_l e^{0l}\right)}{(b+c)((b^2-c^2)} + 2\frac{d\delta^{ij}\delta^{lm}\lambda_{lm}}{(b+c)^2} = 0, \tag{4.2.9}$$

whose trace is a new constraint

$$\phi \equiv \delta^{ij}\phi^{ij} = -\frac{c\nabla^2\Pi^{00}}{b} - \frac{bm^2\Pi^{00}}{b-c} + 2\frac{b\partial_l e^{0l}}{b^2-c^2} - 2\frac{\left(b^2-c^2+bc\right)\partial_l e^{l0}}{b(b^2-c^2)} = 0. \tag{4.2.10}$$

The time derivative $\dot{\phi}$ leads to a new constraint

$$\{\phi\,,\,H\} = -\frac{2bm^2 e}{(b^2-c^2)(D-1)} = 0 \tag{4.2.11}$$

and the time derivative $\dot{e}$ fixes $\delta^{ij}\lambda_{ij}$ ending the algorithm. The set of the constraints which are all second class, is

$$\Pi^{ij} = 0, \qquad b\Pi^{l0} + c\Pi^{0l} = 0, \qquad e = 0, \tag{4.2.12}$$
$$2e^{l0} + (b+c)\partial^l\Pi^{00} = 0, \qquad \frac{b\partial^i\Pi^{0j} - c\partial^j\Pi^{0i}}{b} + 2\frac{be^{ij} - ce^{ji}}{b^2-c^2} = 0, \tag{4.2.13}$$
$$\frac{c\nabla^2\Pi^{00}}{b} - \frac{bm^2\Pi^{00}}{b-c} + 2\frac{b\partial_l e^{0l}}{b^2-c^2} - 2\frac{\left(b^2-c^2+bc\right)\partial_l e^{l0}}{b(b^2-c^2)} = 0. \tag{4.2.14}$$

It will be convenient to work with the equivalent set of constraints

$$\chi_1^{ij} \equiv e^{ij} + \frac{(b^2-c^2)\partial^i\Pi^{0j}}{2b} = 0, \qquad \chi_2^{ij} \equiv \Pi^{ij} = 0, \qquad \chi_3^l \equiv b\Pi^{l0} + c\Pi^{0l} = 0, \tag{4.2.15}$$
$$\chi_4^l \equiv e^{l0} + \frac{(b+c)\partial^l\Pi^{00}}{2} = 0, \qquad \chi_5 \equiv \left(\nabla^2 - m^2\right)\Pi^{00} + 2\frac{\partial_l e^{0l}}{(b+c)} = 0, \tag{4.2.16}$$
$$\chi_6 \equiv e^{00} + \frac{(b^2-c^2)\partial_l\Pi^{0l}}{2b} = 0. \tag{4.2.17}$$

The reduced Hamiltonian reads[4]

$$H^R = \int d^3x\left[-\frac{b\Pi^{00}\partial_l e^{0l}}{2(b-c)} + \frac{(b^2-c^2)m^2\mathbf{\Pi}^2}{4b} + \frac{(b^2-c^2)(\mathbf{\nabla}\times\mathbf{\Pi})^2}{4b} + \frac{be_{0l}e_{0l}}{b^2-c^2}\right]. \tag{4.2.18}$$

Is not obvious that the above Hamiltonian is positive defined, in order to see this consider the decomposition

$$e_{0l} = \partial_l\phi + v_l^t, \tag{4.2.19}$$

[3] Recall that $d = -(b+c)/D-1$

[4] Note that $\mathbf{\Pi}$ is the vector Π^{0l}

where $\partial_l v_l^t = 0$. Using the constraint $\chi_5 = 0$ we can show that

$$H^R = \int d^3x \left[\frac{b}{b^2 - c^2} \left(v_l^t v_l^t + \frac{m^2 \phi \, (-\nabla^2) \, \phi}{m^2 - \nabla^2} \right) + \frac{(b^2 - c^2) m^2 \boldsymbol{\Pi}^2}{4b} + \frac{(b^2 - c^2)(\boldsymbol{\nabla} \times \boldsymbol{\Pi})^2}{4b} \right],$$

which is positive defined.

Given the constraint matrix

$$C_{NM} \equiv \{\chi_N \, , \, \chi_M\}, \tag{4.2.20}$$

one can show that the matrix elements of the inverse are

$$\begin{aligned}
&\left(C^{-1}\right)_{\chi_2^{lk}(\boldsymbol{x}) \, , \, \chi_1^{ij}(\boldsymbol{y})} = \delta_i^l \delta_j^k \delta^3(\boldsymbol{x} - \boldsymbol{y}), \qquad \left(C^{-1}\right)_{\chi_3^{m}(\boldsymbol{x}) \, , \, \chi_2^{ij}(\boldsymbol{y})} = -\frac{(b^2 - c^2)}{2bm^2} \partial_{\boldsymbol{x}}^m \partial_{\boldsymbol{x}}^i \partial_{\boldsymbol{x}}^j \delta^3(\boldsymbol{x} - \boldsymbol{y}), \\
&\left(C^{-1}\right)_{\chi_6(\boldsymbol{x}) \, , \, \chi_2^{ij}(\boldsymbol{y})} = \frac{(b - c)}{bm^2} \partial_{\boldsymbol{x}}^i \partial_{\boldsymbol{x}}^j \delta^3(\boldsymbol{x} - \boldsymbol{y}), \qquad \left(C^{-1}\right)_{\chi_4^{l}(\boldsymbol{x}) \, , \, \chi_3^{p}(\boldsymbol{y})} = \frac{\delta_p^l \delta^3(\boldsymbol{x} - \boldsymbol{y})}{b} - \frac{c \partial_{\boldsymbol{x}}^l \partial_{\boldsymbol{x}}^p \delta^3(\boldsymbol{x} - \boldsymbol{y})}{b^2 m^2}, \\
&\left(C^{-1}\right)_{\chi_5(\boldsymbol{x}) \, , \, \chi_3^{p}(\boldsymbol{y})} = -\frac{(b + c) \partial_{\boldsymbol{x}}^p \delta^3(\boldsymbol{x} - \boldsymbol{y})}{2m^2 b}, \qquad \left(C^{-1}\right)_{\chi_6(\boldsymbol{x}) \, , \, \chi_4^{p}(\boldsymbol{y})} = \frac{2c \partial_{\boldsymbol{y}}^p \delta^3(\boldsymbol{x} - \boldsymbol{y})}{b(b + c) m^2}, \\
&\left(C^{-1}\right)_{\chi_6(\boldsymbol{x}) \, , \, \chi_5(\boldsymbol{y})} = \frac{\delta^3(\boldsymbol{x} - \boldsymbol{y})}{m^2},
\end{aligned}$$

the other matrix elements are either zero or fixed by antisymmetry of the inverse matrix.

With the inverse of the constraint matrix we can show that

$$\{e_{0l}(\boldsymbol{x}, t) \, , \, \Pi^{0k}(\boldsymbol{y}, t)\}_D = \left(\delta_l^k + \frac{\partial_{\boldsymbol{x}}^l \partial_{\boldsymbol{y}}^k}{m^2} \right) \delta^3(\boldsymbol{x} - \boldsymbol{y}), \qquad \{e_{0l}(\boldsymbol{x}, t) \, , \, \Pi^{00}(\boldsymbol{y}, t)\}_D = 0, \tag{4.2.21}$$

$$\{\Pi^{0k}(\boldsymbol{x}, t) \, , \, \Pi^{00}(\boldsymbol{y}, t)\}_D = 2 \frac{\partial_{\boldsymbol{y}}^k \delta^3(\boldsymbol{x} - \boldsymbol{y})}{m^2 (b + c)}, \qquad \{e_{0l}(\boldsymbol{x}, t) \, , \, e_{0k}(\boldsymbol{y}, t)\}_D = 0, \tag{4.2.22}$$

$$\{\Pi^{0l}(\boldsymbol{x}, t) \, , \, \Pi^{0k}(\boldsymbol{y}, t)\}_D = 0. \tag{4.2.23}$$

In order to apply canonical quantization to this field theory, we promote the fields to operators, the reduced Hamiltonian to the Hamiltonian operator and postulate the commutators

$$\left[e_{0l}(\boldsymbol{x}, t) \, , \, \Pi^{0k}(\boldsymbol{y}, t) \right] = i \, \{e_{0l}(\boldsymbol{x}, t) \, , \, \Pi^{0k}(\boldsymbol{y}, t)\}_D, \tag{4.2.24a}$$

$$\left[e_{0l}(\boldsymbol{x}, t) \, , \, \Pi^{00}(\boldsymbol{y}, t) \right] = 0, \qquad \left[\Pi^{0k}(\boldsymbol{x}, t) \, , \, \Pi^{00}(\boldsymbol{y}, t) \right] = i \, \{\Pi^{0k}(\boldsymbol{x}, t) \, , \, \Pi^{00}(\boldsymbol{y}, t)\}_D, \tag{4.2.24b}$$

$$\left[e_{0l}(\boldsymbol{x}, t) \, , \, e_{0k}(\boldsymbol{y}, t) \right] = \left[\Pi^{0l}(\boldsymbol{x}, t) \, , \, \Pi^{0k}(\boldsymbol{y}, t) \right] = 0. \tag{4.2.24c}$$

We can use the Hamiltonian equations of motion and constraints to recover

$$m^2 e_{\mu\nu} = \partial_\mu \partial^\alpha e_{\alpha\nu}, \qquad \partial^\nu e_{\alpha\nu} = 0. \tag{4.2.25}$$

Using the same arguments of the Kalb-Ramond case we can show that the general solution for (4.2.25) is

$$e_{\mu\nu} = \sqrt{\frac{b^2 - c^2}{2b}} \partial_\mu A_\nu, \tag{4.2.26}$$

$$(\Box - m^2) A_\nu = 0, \qquad \partial^\nu A_\nu = 0, \tag{4.2.27}$$

where the square root factor is chosen for normalization reasons. Using the constraints of the theory we can show that

$$\Pi^{0l} = -\frac{2b}{m^2(b^2 - c^2)} \partial_\alpha e^{\alpha l} = -\sqrt{\frac{2b}{b^2 - c^2}} A^l, \tag{4.2.28}$$

$$\sqrt{\frac{2b}{b^2 - c^2}} \left(e_{0l} - e_{l0}\right) = \sqrt{\frac{2b}{b^2 - c^2}} \left(e_{0l} - \frac{(b+c)\partial^l \Pi^{00}}{2}\right) = \dot{A}_l - \partial_l A_0 \equiv \pi^l. \tag{4.2.29}$$

Using (4.2.24) we can show that

$$\left[A_l(\boldsymbol{x}, t)\,,\, \pi^k(\boldsymbol{y}, t)\right] = i\,\delta^3(\boldsymbol{x} - \boldsymbol{y}), \tag{4.2.30}$$

$$\left[A_l(\boldsymbol{x}, t)\,,\, A_k(\boldsymbol{y}, t)\right] = \left[\pi^l(\boldsymbol{x}, t)\,,\, \pi^k(\boldsymbol{y}, t)\right] = 0. \tag{4.2.31}$$

We can use the constraints $\chi_4^l = 0$ and $\chi_5 = 0$ to show that

$$e_{0l} = \sqrt{\frac{b^2 - c^2}{2b}} \left(\pi^l - \frac{\partial^l \boldsymbol{\nabla} \cdot \boldsymbol{\pi}}{m^2}\right), \qquad A_0 = -\frac{\boldsymbol{\nabla} \cdot \boldsymbol{\pi}}{m^2} \tag{4.2.32}$$

In terms of the variables, A_l and π^l (4.2.18) is just the Proca Hamiltonian.

4.3 PERTURBATION THEORY

Here we follow the textbook approach of (WEINBERG, 1995). The canonical duality between free theories of Kalb-Ramond and Proca allows us to immediately say something about the relation between interacting theories. To understand why this is possible we do a quick review on the interaction picture. Consider the Hamiltonian H as a functional of canonical fields $\Psi_l(\boldsymbol{x}, t), \Pi_l(\boldsymbol{x}, t)$, in the Heisenberg picture. The interaction picture fields are related to the Heisenberg picture ones by a similarity transformation

$$\Psi_l^I(\boldsymbol{x}, t) = e^{iH_0 t} \Psi_l(\boldsymbol{x}, 0) e^{-iH_0 t} \tag{4.3.1}$$

$$\Pi_l^I(\boldsymbol{x}, t) = e^{iH_0 t} \Pi_l(\boldsymbol{x}, 0) e^{-iH_0 t} \tag{4.3.2}$$

where H_0 is the free part of the full Hamiltonian and the upper index I stands for interaction picture. Applying this similarity transformation to H gives

$$e^{iH_0 t} H e^{-iH_0 t} = e^{iH_0 t}(H_0 + V) e^{-iH_0 t} = H_0 + V(t), \tag{4.3.3}$$

where V is the interacting part of the full Hamiltonian. The term $V(t)$ is an integral of the Hamiltonian density $\mathcal{H}(x)$:

$$V(t) = \int d^3x\, \mathcal{H}(x), \tag{4.3.4}$$

so that the S-matrix (WHEELER, 1937; HEISENBERG, 1943a; HEISENBERG, 1943b) is given by (DYSON, 1949)

$$S = \mathbf{1} + \sum_{n=1}^{\infty} \frac{(-i)^n}{n!} \int d^4x_1 \cdots d^4x_n T\{\mathcal{H}(x_1) \cdots \mathcal{H}(x_n)\}, \tag{4.3.5}$$

where T is the time ordering product. The interaction picture Hamiltonian $\mathcal{H}(x)$ is a function of the free fields Ψ_l^I and Π_l^I. Expressing Π_l^I as a function of Ψ_l^I we can rewrite $\mathcal{H}(x)$ in terms of Ψ_l^I and its derivatives. The propagator of the fields Ψ_l^I in $\mathcal{H}(x)$ is defined by:

$$-i\Delta_{lm}(x, y) \equiv \langle T\{\Psi_l^I(x)\Psi_m^I(y)\}\rangle_0. \tag{4.3.6}$$

We already noted that the free canonical variables of Kalb-Ramond and Proca are related by (4.1.15) so any function of the free variables of Kalb-Ramond can be rewritten in terms of Proca variables. Therefore, any $\mathcal{H}(x)$ derived from a Kalb-Ramond Lagrangian will be a function of Proca fields. Given a Hamiltonian density $\mathcal{H}(x)$ involving Proca fields we generally expect that it can be derived from the canonical quantization of a Proca Lagrangian. This leads to our definition of perturbative equivalence between Kalb-Ramond and Proca: Given an interacting Kalb-Ramond Lagrangian from which we derive a set of Feynman rules, we expect that there is an interacting Proca Lagrangian such that the same set of Feynman rules can be derived.

In practice it may be difficult to explicitly write $\mathcal{H}(x)$ as a function of the independent canonical variables for which (4.1.15) is valid, but there should be a solution in principle. We shall see that the equivalence between Feynman rules can also be supported in the context of path integral.

4.4 SCALAR INTERACTION

In order to illustrate our previous argument consider the interacting theory

$$S = \int d^4x \left[-\frac{1}{2}\partial_\mu\phi\partial^\mu\phi + \frac{1}{2}\partial_\alpha B^{\alpha\mu}\partial^\nu B_{\nu\mu} + \frac{m^2}{4} B_{\mu\nu}B^{\mu\nu} + \frac{\lambda}{4}\phi^2 B_{\mu\nu}B^{\mu\nu} \right], \tag{4.4.1}$$

where $\lambda > 0$. The conjugate momenta are now

$$\pi = \dot{\phi}, \qquad \Pi^{0i} = -\partial_\alpha B^{\alpha i}, \qquad \Pi^{ij} = 0, \tag{4.4.2}$$

and the Hamiltonian is

$$H = \int d^3x \left[\frac{1}{2}\pi^2 + \frac{1}{2}\Pi^{0i}\Pi^{0i} + \Pi^{0i}\partial^j B_{ji} + \frac{1}{2}(\boldsymbol{\nabla}\phi)^2 + \frac{1}{2}(\partial^i B_{0i})^2 + \Lambda_{ij}\Pi^{ij} \right.$$
$$\left. -\frac{1}{4}(m^2 + \lambda\phi^2) B_{\mu\nu}B^{\mu\nu} \right]. \tag{4.4.3}$$

The consistency condition of the primary constraint now gives

$$\partial^i\Pi^{0j} - \partial^j\Pi^{0i} + m^2\left(1 + \frac{\lambda\phi^2}{m^2}\right)B_{ij} = 0, \tag{4.4.4}$$

whose time derivative once again fixes Λ_{ij}. The reduced Hamiltonian is

$$H^R_{KR} = \int d^3x \left[\frac{1}{2}\pi^2 + \frac{1}{2}\Pi^{0i}\Pi^{0i} + \frac{1}{2}(\boldsymbol{\nabla}\phi)^2 + \frac{1}{2}(\partial^i B_{0i})^2 + \frac{m^2}{2}\left(1 + \frac{\lambda\phi^2}{m^2}\right) B_{0i}B_{0i} \right.$$
$$\left. + \frac{1}{2m^2}\left(1 + \frac{\lambda\phi^2}{m^2}\right)^{-1} (\boldsymbol{\nabla}\times\boldsymbol{\Pi})^2 \right] \tag{4.4.5}$$

In order to quantize the theory we note that ϕ and Π^{0j} have zero Poisson-brackets with all constraints of the theory. Therefore, we postulate the only non-zero commutators among independent canonical variables

$$[\phi\,,\,\pi] = i\,\delta^3(\boldsymbol{x}-\boldsymbol{y}), \qquad [B_{0i}\,,\,\Pi^{0j}] = i\,\delta^i_j\delta^3(\boldsymbol{x}-\boldsymbol{y}). \tag{4.4.6}$$

As usual, for the quantized theory the function

$$\left(1 + \frac{\lambda\phi^2}{m^2}\right)^{-1}, \tag{4.4.7}$$

is to be understood in terms of its Taylor series. The interacting Hamiltonian of the interaction picture is

$$\mathcal{H}_{NC}(x) = \frac{\lambda\phi^2}{2}B_{0i}B_{0i} + \frac{1}{2m^2}(\boldsymbol{\nabla}\times\boldsymbol{\Pi})^2\left(\sum_{n=1}^{\infty}(-1)^n\left(\frac{\lambda\phi^2}{m^2}\right)^n\right), \tag{4.4.8}$$

where the label NC stands for non-covariant Hamiltonian density. Since we are in the interaction picture we can use the free field relation (4.1.12)

$$\mathcal{H}_{NC}(x) = \frac{\lambda\phi^2}{2m^2}F_{0i}F_{0i} + \frac{1}{4}\left(\sum_{n=1}^{\infty}(-1)^n\left(\frac{\lambda\phi^2}{m^2}\right)^n\right)F_{ij}F_{ij}, \tag{4.4.9}$$

Therefore, given (4.4.1) we can rewrite its interaction Hamiltonian as function of Proca fields, as argued in the previous section.

Now, we consider the interacting Proca Lagrangian

$$S = \int d^4x \left[-\frac{1}{4}\left(1+\frac{\lambda\phi^2}{m^2}\right)^{-1} F_{\mu\nu}F^{\mu\nu} - \frac{m^2}{2}A_\mu A^\mu - \frac{1}{2}\partial_\mu\phi\partial^\mu\phi \right], \tag{4.4.10}$$

the conjugate momenta are

$$\bar{\Pi}^0 = 0, \qquad \bar{\Pi}^i = -\left(1+\frac{\lambda\phi^2}{m^2}\right)^{-1} F^{0i}, \qquad \pi = \dot{\phi}, \tag{4.4.11}$$

so that the Hamiltonian is

$$H = \int d^3x \Bigg[\left(1+\frac{\lambda\phi^2}{m^2}\right)\frac{\bar{\boldsymbol{\Pi}}^2}{2} + \frac{\pi^2}{2} - A_0 \boldsymbol{\nabla}\cdot\bar{\boldsymbol{\Pi}} + \left(1+\frac{\lambda\phi^2}{m^2}\right)^{-1}\frac{(\boldsymbol{\nabla}\times\boldsymbol{A})^2}{2} + \frac{(\nabla\phi)^2}{2} + \frac{m^2}{2}A_\mu A^\mu + \bar{\lambda}\bar{\Pi}^0 \Bigg]. \tag{4.4.12}$$

The consistency condition of the primary constraint $\bar{\Pi}^0 = 0$ gives

$$m^2 A_0 + \boldsymbol{\nabla}\cdot\bar{\boldsymbol{\Pi}} = 0, \tag{4.4.13}$$

whose consistency condition fixes $\bar{\lambda}$. The reduced Hamiltonian is

$$H_P^R = \int \frac{d^3x}{2} \Bigg[\left(1+\frac{\lambda\phi^2}{m^2}\right)\bar{\boldsymbol{\Pi}}^2 + \pi^2 + \left(1+\frac{\lambda\phi^2}{m^2}\right)^{-1}(\boldsymbol{\nabla}\times\boldsymbol{A})^2 + (\boldsymbol{\nabla}\phi)^2 + m^2\boldsymbol{A}^2 + \frac{(\boldsymbol{\nabla}\cdot\bar{\boldsymbol{\Pi}})^2}{m^2} \Bigg]. \tag{4.4.14}$$

The non zero Dirac-brackets among canonical variables motivate the commutation relations:

$$\left[A_i(\boldsymbol{x},t)\,,\,\bar{\Pi}^j(\boldsymbol{y},t)\right] = \delta_i^j\,\delta^3(\boldsymbol{x}-\boldsymbol{y})\,, \qquad [\phi(\boldsymbol{x},t)\,,\,\pi(\boldsymbol{y},t)] = \delta^3(\boldsymbol{x}-\boldsymbol{y}). \tag{4.4.15}$$

The interaction picture Hamiltonian is

$$\mathcal{H}_{NC}(x) = \frac{\lambda\phi^2\bar{\boldsymbol{\Pi}}^2}{2m^2} + \left(\sum_{n=1}^{\infty}(-1)^n\left(\frac{\lambda\phi^2}{m^2}\right)^n\right)\frac{(\boldsymbol{\nabla}\times\boldsymbol{A})^2}{2}, \tag{4.4.16}$$

using the interaction picture relation $\bar{\Pi}^i = -F^{0i}$ we can rewrite the Hamiltonian density

$$\mathcal{H}_{NC}(x) = \frac{\lambda\phi^2}{2m^2}F_{0i}F_{0i} + \frac{1}{4}\left(\sum_{n=1}^{\infty}(-1)^n\left(\frac{\lambda\phi^2}{m^2}\right)^n\right)F_{ij}F_{ij}. \tag{4.4.17}$$

This is the same interacting Hamiltonian (4.4.9) derived from (4.4.1), as expected from our discussion of the interaction picture.

In order to derive the covariant Feynman rules associated with (4.4.17) we first rewrite the Hamiltonian density as

$$\mathcal{H}_{NC}(x) = -\frac{1}{2}F^{\mu\nu}O_{\mu\nu\alpha\beta}F^{\alpha\beta}, \tag{4.4.18a}$$

$$O_{\mu\nu\alpha\beta} \equiv -\frac{(\lambda\phi^2/m^2)^2}{4\,(1+\lambda\phi^2/m^2)}\left(n_\mu n_\alpha \eta_{\nu\beta} - n_\nu n_\alpha \eta_{\mu\beta} - n_\mu n_\beta \eta_{\nu\alpha} + n_\nu n_\beta \eta_{\mu\alpha}\right)$$
$$+\frac{\lambda\phi^2}{4m^2}\left(1-\frac{\lambda\phi^2/m^2}{1+\lambda\phi^2/m^2}\right)\left(\eta_{\alpha\mu}\eta_{\beta\nu} - \eta_{\beta\mu}\eta_{\alpha\nu}\right), \tag{4.4.18b}$$

where $n_\mu \equiv (1,0,0,0)$. The propagators associated to this perturbation theory are

$$\langle T\{\phi(x)\phi(y)\}\rangle_0 = -i\,\Delta_F^{m=0}(x,y), \tag{4.4.19}$$

$$\langle T\{F^{\mu\nu}(x)F^{\alpha\beta}(y)\}\rangle_0 = -i\Big[\delta^4(x-y)\left(\delta_0^\mu\delta_0^\alpha\eta^{\nu\beta} - \delta_0^\mu\delta_0^\beta\eta^{\nu\alpha} - \delta_0^\nu\delta_0^\alpha\eta^{\mu\beta} + \delta_0^\nu\delta_0^\beta\eta^{\mu\alpha}\right)$$
$$-\partial^\mu\partial^\alpha\Delta^{\nu\beta}(x,y) + \partial^\mu\partial^\beta\Delta^{\nu\alpha}(x,y) + \partial^\nu\partial^\alpha\Delta^{\mu\beta}(x,y) - \partial^\nu\partial^\beta\Delta^{\mu\alpha}(x,y)\Big], \tag{4.4.20}$$

where $\Delta^{\mu\nu}(x,y)$ is given in (4.1.28) and

$$\Delta_F^{m=0}(x,y) \equiv (2\pi)^{-4}\int d^4q\frac{e^{iq\cdot(x-y)}}{q^2-i\epsilon}. \tag{4.4.21}$$

For an interaction of the form of (4.4.18a) we can use the methods of (LEE; YANG, 1962) to sum the effect of contact terms to the perturbation theory. We represent the full propagator (4.4.20) as a sum of straight and spring lines, with the spring lines representing the contact terms. The effect of inserting $n-1$ spring lines between two straight lines is[5]

$$I_n = i\,\frac{4^{n-1}}{2}\int d^4x\, F^{\alpha\beta}F^{\mu\nu}O_{\alpha\beta 0i}O_{0i0j}\cdots O_{0p\kappa\xi}$$
$$= \frac{i}{2}\int d^4x F^{0i}F^{0i}\left(-\frac{\lambda\phi^2}{m^2}\right)^n, \qquad n\geq 2 \tag{4.4.22}$$

and the sum over all n, including n=1 which is (4.4.17) is the new vertex

$$(-i)\int d^4x\left(-\frac{\lambda\phi^2}{4m^2}\frac{F_{\mu\nu}F^{\mu\nu}}{(1+\lambda\phi^2/m^2)}\right). \tag{4.4.23}$$

We should also consider the contribution to perturbation theory coming from loops of spring lines. The contribution of a loop with n springs is

$$\bar{I}_n = -\frac{3}{2}\delta^4(0)\int d^4x\frac{(-1)^{n-1}}{n}\left(\frac{\lambda\phi^2}{n}\right), \qquad n\geq 1, \tag{4.4.24}$$

[5] Notice that we have n factors of $O_{\mu\alpha\beta\nu}$

the sum over all loops gives the contribution

$$-\frac{3}{2}\delta^4(0)\int d^4x\,\ln\left(1+\lambda\phi^2/m^2\right). \tag{4.4.25}$$

The covariant Hamiltonian density to be used in the power series (4.3.5) with the covariant part of (4.4.20) is

$$\mathcal{H}_C(x) = -\frac{\lambda\phi^2}{4m^2}\frac{F_{\mu\nu}F^{\mu\nu}}{(1+\lambda\phi^2/m^2)} - \frac{3\,i}{2}\delta^4(0)\ln\left(1+\lambda\phi^2/m^2\right). \tag{4.4.26}$$

Starting with either (4.4.1) or (4.4.10) we will obtain the same covariant Feynman rules given by (4.4.26). This is straightforward in the canonical formalism because both Lagrangians gave the same $\mathcal{H}_{NC}(x)$ from which the Feynman rules will be derived. However, there are two drawbacks. First we have to solve for the auxiliary variables in order to compare the interacting Hamiltonian densities, which can be very difficult depending on the interaction. Second, the Feynman rules are not covariant from the start, we have to sum the effect of contact terms to obtain the covariant perturbation theory. We will explore now the path integral formulation of this theory, where the Feynman rules are covariant from the start.

4.5 PATH INTEGRAL

The path integral associated to (4.4.1) and (4.4.10) are

$$\langle Vac,in|Vac,out\rangle = \mathcal{C}\int \mathcal{D}\phi\mathcal{D}B_{[\mu\nu]}\, e^{i\int d^4x\left[\mathcal{L}^0_{KR}+\frac{\lambda}{4}\phi^2 B_{\mu\nu}B^{\mu\nu}-\frac{3\,i}{2}\delta^4(0)\ln\left(1+\lambda\phi^2/m^2\right)\right]}, \tag{4.5.1a}$$

$$\langle Vac,in|Vac,out\rangle = \mathcal{C}'\int \mathcal{D}\phi\mathcal{D}A_\mu\, e^{i\int d^4x\left[\mathcal{L}^0_{P}+\frac{\lambda\phi^2}{4m^2}\frac{F_{\mu\nu}F^{\mu\nu}}{(1+\lambda\phi^2/m^2)}+\frac{3\,i}{2}\delta^4(0)\ln\left(1+\lambda\phi^2/m^2\right)\right]}, \tag{4.5.1b}$$

where

$$\mathcal{L}^0_{KR} \equiv -\frac{1}{2}\partial_\mu\phi\partial^\mu\phi + \frac{1}{2}\partial_\alpha B^{\alpha\mu}\partial^\nu B_{\nu\mu} + \frac{m^2}{4}B_{\mu\nu}B^{\mu\nu} \tag{4.5.1c}$$

$$\mathcal{L}^0_{P} \equiv -\frac{1}{2}\partial_\mu\phi\partial^\mu\phi - \frac{1}{4}F_{\mu\nu}F^{\mu\nu} - \frac{m^2}{2}A_\mu A^\mu. \tag{4.5.1d}$$

The covariant perturbation theory derived from (4.5.1b) clearly agrees with the one derived from (4.4.26). However, from (4.5.1a) is not obvious that we will get the same perturbation theory. In order to see the equivalence we recast the problem in the language of the canonical formalism. The set of Feynman rules coming from (4.5.1a) are the same that one would obtain from

$$\mathcal{H}_{PI}(x) = -\frac{\lambda\phi^2}{4}B_{\mu\nu}B^{\mu\nu} + \frac{3\,i}{2}\delta^4(0)\ln\left(1+\lambda\phi^2/m^2\right), \tag{4.5.2}$$

where the $B_{\mu\nu}$ propagator is

$$\langle T\{B^{\mu\nu}(x)B^{\alpha\beta}(y)\}\rangle_0 = -i\Bigg[-\frac{\delta^4(x-y)}{m^2}\left(\eta^{\mu\alpha}\eta^{\nu\beta}-\eta^{\mu\beta}\eta^{\nu\alpha}\right) + \frac{1}{m^2}\left[-\partial^\mu\partial^\alpha\Delta^{\nu\beta}(x,y)+\partial^\mu\partial^\beta\Delta^{\nu\alpha}(x,y)+\partial^\nu\partial^\alpha\Delta^{\mu\beta}(x,y)-\partial^\nu\partial^\beta\Delta^{\mu\alpha}(x,y)\right]\Bigg]. \tag{4.5.3}$$

The interacting Hamiltonian density (4.5.2) and the propagator (4.5.3) have the same structure of (4.4.17) and (4.4.20), so that we can use the same methods of the previous section to sum the effect of contact terms. The sum of spring lines between two straight lines gives the corrected vertex:

$$-i\int d^4x\left[-\frac{\lambda\phi^2 B_{\mu\nu}B^{\mu\nu}}{4\left(1+\lambda\phi^2/m^2\right)}\right], \tag{4.5.4}$$

and the contribution of loops is

$$-3\delta^4(0)\int d^4x \ln\left(1+\lambda\phi^2/m^2\right). \tag{4.5.5}$$

When we add this loop contribution to (4.5.2) we revert the sign of the delta term and end up with the effective Hamiltonian

$$\mathcal{H}_{eff}(x) = -\frac{\lambda\phi^2 B_{\mu\nu}B^{\mu\nu}}{4\left(1+\lambda\phi^2/m^2\right)} - \frac{3\,i}{2}\delta^4(0)\ln\left(1+\lambda\phi^2/m^2\right). \tag{4.5.6}$$

This effective Hamiltonian is to be used in perturbation theory calculations with the second line of (4.5.3), that is the covariant part of (4.4.20) multiplied by m^{-2}. This is exactly the perturbation theory that we would obtain from a path integral where the free Lagrangian is Proca and the interaction Lagrangian is $\mathcal{L}_I = -\mathcal{H}_{eff}(x)$ with $B_{\mu\nu}$ replaced by $F_{\mu\nu}/m$, which is just (4.5.1b).

Our specific choice of interaction should not divert our attention from the generality of this result. For any interaction Lagrangian in the path integral of Kalb-Ramond, the propagator will always be (4.5.3) since it is derived from the free Lagrangian. Therefore, the Feynman rules will always contain contact terms. We did use one specific technique to sum the effect of this contact terms, which works for interaction we had, but the sum itself is not interaction dependent. The redefinition of the vertex involves a sum of a subset of all diagrams contributing to the S matrix, which we can always do in principle. This is the path integral version of our argument for perturbative equivalence between Kalb-Ramond and Proca, defended in canonical terms at the end of 4.3. Given a general Kalb-Ramond Lagrangian in the path integral, the corresponding Feynman rules are always equivalent to the Feynman rules derived from a Proca Lagrangian once we sum the effect of contact terms.

4.6 PARENT ACTION

Despite of having formulated the notion of perturbative duality with general arguments, in the path integral and canonical formalism, we are left with a practical problem: given a Kalb-Ramond Lagrangian how do we know what is the dual Proca Lagrangian? For the simple theories explored here this can be guessed from the parent action method (DESER; JACKIW, 1984). Consider

$$S_P = \int d^4x \bigg[-\frac{m^2}{2}A_\mu A^\mu + \frac{m^2}{4}B_{\mu\nu}B^{\mu\nu} - mB^{\mu\nu}\partial_\mu A_\nu + \frac{\lambda\phi^2}{4}B_{\mu\nu}B^{\mu\nu} - \frac{\partial_\mu\phi\partial^\mu\phi}{2}\bigg], \tag{4.6.1}$$

if we do the coordinate change

$$B_{\mu\nu} \to \bar{B}_{\mu\nu} = B_{\mu\nu} - \left(1+\frac{\lambda\phi^2}{m^2}\right)^{-1}\frac{F_{\mu\nu}(A)}{m} \tag{4.6.2a}$$

$$\phi \to \phi, \tag{4.6.2b}$$

then (4.6.1) reads

$$S_P = \int d^4x \bigg[-\frac{1}{4}\left(1+\frac{\lambda\phi^2}{m^2}\right)^{-1} F_{\mu\nu}F^{\mu\nu} - \frac{m^2}{2}A_\mu A^\mu - \frac{1}{2}\partial_\mu\phi\partial^\mu\phi + \frac{m^2}{4}\left(1+\frac{\lambda\phi^2}{m^2}\right)\bar{B}_{\mu\nu}\bar{B}^{\mu\nu}\bigg]. \tag{4.6.3}$$

This is just the Proca action (4.4.10) with an extra term, quadratic in $\bar{B}_{\mu\nu}$ that can be droped since its equations of motion are $\bar{B}_{\mu\nu} = 0$. On the oder hand, we could integrate by parts in (4.6.1) so that

$$S_P = \int d^4x \bigg[-\frac{m^2}{2}A_\mu A^\mu + \frac{m^2}{4}B_{\mu\nu}B^{\mu\nu} + m\partial_\mu B^{\mu\nu} A_\nu + \frac{\lambda\phi^2}{4}B_{\mu\nu}B^{\mu\nu} - \frac{\partial_\mu\phi\partial^\mu\phi}{2}\bigg], \tag{4.6.4}$$

and then we can do the coordinate change

$$B_{\mu\nu} \to B_{\mu\nu} \tag{4.6.5a}$$

$$A_\mu \to \bar{A}_\mu = A_\mu - \frac{\partial^\nu B_{\mu\nu}}{m}, \tag{4.6.5b}$$

and the new action is

$$S_P = \int d^4x \bigg[-\frac{1}{2}\partial_\mu\phi\partial^\mu\phi + \frac{1}{2}\partial_\alpha B^{\alpha\mu}\partial^\nu B_{\nu\mu} + \frac{m^2}{4}B_{\mu\nu}B^{\mu\nu} + \frac{\lambda}{4}\phi^2 B_{\mu\nu}B^{\mu\nu} - \frac{m^2}{2}\bar{A}_\mu\bar{A}^\mu\bigg]. \tag{4.6.6}$$

This is the interacting Kalb-Ramond model (4.4.1) with an extra term $\bar{A}_\mu\bar{A}^\mu$ that can also be dropped. Summarazing we can recover (4.4.1) or (4.4.10) from the same action by coordinate changes and integration by parts.The coordinate changes we did are invertible so in principle they do not affect the content of the theory. Moreover, for fields that vanish at infinity it is known that integration by parts in the Lagrangian also does not affect the quantum structure (WEINBERG, 1995). Therefore, the parent action (4.6.1) suggests that (4.4.1) and (4.4.10) have the same content, which we already checked at the level of the Feynman rules in the previous sections.

The coordinate changes we have just defined are non perturbative, this suggests that the relation between (4.4.1) and (4.4.10) does not hold only in perturbation theory, but goes beyond that. Adopting the definition

$$\boldsymbol{q}^P(\boldsymbol{x},t) \equiv \boldsymbol{A}(\boldsymbol{x},t), \qquad \boldsymbol{\pi}_P(\boldsymbol{x},t) \equiv \bar{\boldsymbol{\Pi}}(\boldsymbol{x},t); \tag{4.6.7}$$

$$q_i(\boldsymbol{x},t) \equiv B_{0i}(\boldsymbol{x},t), \qquad \pi^i(\boldsymbol{x},t) \equiv \Pi^{0i}(\boldsymbol{x},t), \tag{4.6.8}$$

we can see that the coordinate change

$$q_i(\boldsymbol{x},t) = m^{-1}\pi^i_P(\boldsymbol{x},t), \qquad \pi^i(\boldsymbol{x},t) = -m\, q^P_i(\boldsymbol{x},t), \tag{4.6.9}$$

relates the Hamiltonians (4.4.5) and (4.4.14). These are coordinate changes between Heisenberg picture operators. Therefore, as in the free case the relation between the Kalb-Ramond and Proca Lagrangians is that they furnish the same Hamiltonian but in different canonical coordinates. The theories are not equivalent only at the level of Feynman rules, but even non perturbatively. Naturally, we are led to wonder if any two Kalb-Ramond and Proca Lagrangians related by coordinate transformations of the parent action will define the same Hamiltonian in different coordinates. In our simple examples, free and interacting, this is true, but this does not constitute a proof for general cases. That is, the fact that for simple interactions changes of coordinates at the Lagrangian level correspond to canonical coordinate changes at the Hamiltonian level does not guarantee that for any coordinate change in the Lagrangian there is a corresponding coordinate change for the Hamiltonians. However, it is a reasonable expectation that coordinate changes in the Lagrangian correspond to coordinate changes in the canonical formalism. Another relevant question is whether this mechanism will always work, that is, given a Kalb-Ramond Lagrangian we can always find a parent action so that the dual Proca theory can be defined. We already know that perturbatively Kalb-Ramond is always equivalent to Proca. If it was possible to connect any Kalb-Ramond Lagrangian to a Proca Lagrangian through the parent action the equivalence would always be true even non-perturbatively.

5 FINAL-REMARKS

It has been shown that the duality of massless field theories is a useful approach to investigate the duality of massive theories. Moreover, it is remarkable that the construction of a parent action with a non-symmetric rank two tensor allows us to establish dualities that are not possible either with a symmetric or antisymmetric field, in the massless and massive case, see the new spin-1 models (2.1.20) and (2.2.58) respectively. Those models are original results of the present work. It has also been shown how alternative methods can be used to derive new dual models. We have provided a preliminary approach to the presence of interactions and a different analysis from what is found in the literature.

Our search for dual models, from the beginning, was motivated by the possible new physics that they can offer when interactions are added, since we are searching applications for interacting theories in cosmology and astrophysics. Therefore, the search for new dual models at the free level is important, but the status of the duality when interactions are added is equally relevant.

Regarding the dual massive models without gauge symmetry of spin-1 and spin-0 (BARBOSA, 2022), our preliminary calculations suggests that, in the context of perturbation theory, no new interactions can be obtained from dual models, neither a strong-weak coupling relation. In the context of Feynman rules, there is nothing new to be gained with dual models of the Kalb-Ramond type, since they are always equivalent to the Feynman rules coming from a Proca Lagrangian once we sum the contact terms. In fact, as far as the Feynman rules are concerned it is not convenient to work with these dual models since the vertex can not be easily read from the starting Lagrangian. The effective Feynman rules to be used after the summation of the contact terms can give vertex that are not necessarily similar to the original interacting Lagrangian. About the non-perturbative content, in simple cases we have shown that dual massive models lead to the same Hamiltonian that could be derived from a textbook Proca Lagrangian for spin-1 and Klein-Gordon Lagrangian for spin-0, in different canonical coordinates. Of course, the relevant question is whether this will always be the case, that is if any Hamiltonian derived from a dual massive model can be associated with a Hamiltonian derived from the standard massive Lagrangians by coordinate changes. Even without such a theorem, it would be interesting to see if that is the case for physically motivated theories involving massive dual models like the one in (HEISENBERG; TRENKLER, 2020). Since our conclusions were obtained in the absence of gravity, further investigation should be performed in the presence of curved space. However, we would like to point out that a preliminary investigation (BUCHBINDER; KIRILLOVA; PLETNEV, 2008), for spin-0 and spin-1, shows that there is still an equivalence between dual massive models in curved space.

Our search for dual models of spin-1 was also motivated by the search of new dual models of spin-2. Due to the success of the parent action method in obtaining dual models for spin-1 we expect that the same method can be used in order to obtain dual models for spin-2. In fact there are already positive results in this direction (DALMAZI; SANTOS, 2020). Regarding our investigation of interactions in dual models without gauge symmetry we expect that the same techniques can be applied to higher spin models leading essentially to the same conclusions. In particular, the Curtright-Freund model of spin-2 (CURTRIGHT; FREUND, 1980) is a massive dual model without gauge symmetry and has been suggested in the context of modified gravity (ALSHAL; CURTRIGHT, 2019).

REFERÊNCIAS

ALSHAL, H.; CURTRIGHT, T. L. Massive Dual Gravity in N Spacetime Dimensions. **JHEP**, v. 09, p. 063, 2019.

BANDOS, I. et al. A non-linear duality-invariant conformal extension of Maxwell's equations. **Phys. Rev. D**, v. 102, p. 121703, 2020.

BARBOSA, F. A. d. S. Canonical analysis of Kalb-Ramond-Proca duality. **arXiv:2203.08867 [hep-th]**, 2022.

BEKAERT, X.; BOULANGER, N. The unitary representations of the Poincaré group in any spacetime dimension. **SciPost Phys. Lect. Notes**, v. 30, p. 1, 2021.

BOKULIĆ, A.; JURIĆ, T.; SMOLIĆ, I. Black hole thermodynamics in the presence of nonlinear electromagnetic fields. **Phys. Rev. D**, v. 103, n. 12, p. 124059, 2021.

BOULWARE, D. G.; DESER, S. Can gravitation have a finite range? **Phys. Rev. D**, American Physical Society, v. 6, p. 3368–3382, Dec 1972. Disponível em: <https://link.aps.org/doi/10.1103/PhysRevD.6.3368>.

BUCHBINDER, I. L.; KIRILLOVA, E. N.; PLETNEV, N. G. Quantum equivalence of massive antisymmetric tensor field models in curved space. **Phys. Rev. D**, American Physical Society, v. 78, p. 084024, Oct 2008. Disponível em: <https://link.aps.org/doi/10.1103/PhysRevD.78.084024>.

BURGESS, C. P. Quantum gravity in everyday life: General relativity as an effective field theory. **Living Rev. Rel.**, v. 7, p. 5–56, 2004.

CREMMER, E.; SCHERK, J. Spontaneous dynamical breaking of gauge symmetry in dual models. **Nuclear Physics B**, v. 72, n. 1, p. 117–124, 1974. ISSN 0550-3213. Disponível em: <https://www.sciencedirect.com/science/article/pii/0550321374902247>.

CURTRIGHT, T. L.; FREUND, P. G. O. MASSIVE DUAL FIELDS. **Nucl. Phys. B**, v. 172, p. 413–424, 1980.

DALMAZI, D.; SANTOS, A. L. R. dos. More on dual actions for massive spin-2 particles. **Classical and Quantum Gravity**, IOP Publishing, v. 37, n. 14, p. 145012, jul 2020. Disponível em: <https://doi.org/10.1088/1361-6382/ab9144>.

DALMAZI, D.; SANTOS, R. C. Spin-1 duality in d dimensions. **Phys. Rev. D**, American Physical Society, v. 84, p. 045027, Aug 2011. Disponível em: <https://link.aps.org/doi/10.1103/PhysRevD.84.045027>.

DESER, S. Selfinteraction and gauge invariance. **Gen. Rel. Grav.**, v. 1, p. 9–18, 1970.

DESER, S.; JACKIW, R. "self-duality" of topologically massive gauge theories. **Physics Letters B**, v. 139, n. 5, p. 371–373, 1984. ISSN 0370-2693. Disponível em: <https://www.sciencedirect.com/science/article/pii/0370269384918331>.

DESER, S.; TOWNSEND, P. K.; SIEGEL, W. Higher Rank Representations of Lower Spin. **Nucl. Phys. B**, v. 184, p. 333–350, 1981.

DONOGHUE, J. F. General relativity as an effective field theory: The leading quantum corrections. **Phys. Rev. D**, v. 50, p. 3874–3888, 1994.

DYSON, F. J. The S Matrix in Quantum Electrodynamics. **Phys. Rev.**, v. 75, p. 1736, 1949.

FERRARIS, M.; FRANCAVIGLIA, M.; REINA, C. Variational formulation of general relativity from 1915 to 1925 “palatini’s method” discovered by einstein in 1925. **Gen Relat Gravit**, v. 14, p. 243–254, 1982.

GIVEON, A.; PORRATI, M.; RABINOVICI, E. Target space duality in string theory. **Phys. Rept.**, v. 244, p. 77–202, 1994.

HARTNOLL, S. A. Lectures on holographic methods for condensed matter physics. **Class. Quant. Grav.**, v. 26, p. 224002, 2009.

HEISENBERG, L. Generalization of the Proca Action. **JCAP**, v. 05, p. 015, 2014.

HEISENBERG, L.; TRENKLER, G. Generalization of the 2-form interactions. **JCAP**, v. 05, p. 019, 2020.

HEISENBERG, W. Die ’beobachtbaren’ Grössen in der Theorie der Elementarteilchen. **Z. Phys.**, v. 120, p. 513, 1943.

HEISENBERG, W. Die ’beobachtbaren’ Grössen in der Theorie der Elementarteilchen. II. **Z. Physik.**, v. 120, p. 673, 1943.

HJELMELAND, S. E.; LINDSTROM, U. Duality for the non-specialist. **arXiv:hep-th/9705122**, 1997.

JACCARD, M.; MAGGIORE, M.; MITSOU, E. Bardeen variables and hidden gauge symmetries in linearized massive gravity. **Physical Review D**, v. 87, p. 044017, 2013.

JIMENEZ, J. B. et al. Born–Infeld inspired modifications of gravity. **Phys. Rept.**, v. 727, p. 1–129, 2018.

KALB, M.; RAMOND, P. Classical direct interstring action. **Phys. Rev. D**, American Physical Society, v. 9, p. 2273–2284, Apr 1974. Disponível em: <https://link.aps.org/doi/10.1103/PhysRevD.9.2273>.

KENNA-ALLISON, M. P. R. **Cosmological implications of extended massive gravity theories**. Tese (Doutorado) — University of Portsmouth, 2021.

KHOUDEIR, A. Topological mass mechanism from dimensional reduction. **Phys. Rev. D**, American Physical Society, v. 59, p. 027702, Dec 1998. Disponível em: <https://link.aps.org/doi/10.1103/PhysRevD.59.027702>.

KHOUDEIR, A.; MONTEMAYOR, R.; URRUTIA, L. F. Dimensional reduction as a method to obtain dual theories for massive spin two in arbitrary dimensions. **Phys. Rev. D**, American Physical Society, v. 78, p. 065041, Sep 2008. Disponível em: <https://link.aps.org/doi/10.1103/PhysRevD.78.065041>.

LEE, T. D.; YANG, C. N. Theory of charged vector mesons interacting with the electromagnetic field. **Phys. Rev.**, American Physical Society, v. 128, p. 885–898, Oct 1962. Disponível em: <https://link.aps.org/doi/10.1103/PhysRev.128.885>.

LIN, C. The Self-consistent Matter Coupling of a Class of Minimally Modified Gravity Theories. **JCAP**, v. 05, p. 037, 2019.

MALUF, R. V. et al. Dual equivalence between self-dual and topologically massive $B \wedge F$ models coupled to matter in $3 + 1$ dimensions. **Phys. Rev. D**, v. 102, n. 2, p. 025006, 2020.

MOFFAT, J. W. Black Holes in Modified Gravity (MOG). **Eur. Phys. J. C**, v. 75, n. 4, p. 175, 2015.

MOTOHASHI, H.; SUYAMA, T.; TAKAHASHI, K. Fundamental theorem on gauge fixing at the action level. **Phys. Rev. D**, American Physical Society, v. 94, p. 124021, Dec 2016. Disponível em: <https://link.aps.org/doi/10.1103/PhysRevD.94.124021>.

NOJIRI, S.; ODINTSOV, S.; OIKONOMOU, V. Modified gravity theories on a nutshell: Inflation, bounce and late-time evolution. **Physics Reports**, v. 692, p. 1–104, 2017. ISSN 0370-1573. Disponível em: <https://www.sciencedirect.com/science/article/pii/S0370157317301527>.

NORDSTROM, G. On the possibility of unifying the electromagnetic and the gravitational fields. **Phys. Z.**, v. 15, p. 504–506, 1914.

OLMO, G. J. Palatini approach to modified gravity: f(r) theories and beyond. **International Journal of Modern Physics D**, v. 20, n. 04, p. 413–462, 2011. Disponível em: <https://doi.org/10.1142/S0218271811018925>.

PELDÁN, P. Actions for gravity, with generalizations: a review. **Classical and Quantum Gravity**, IOP Publishing, v. 11, n. 5, p. 1087–1132, may 1994. Disponível em: <https://doi.org/10.1088/0264-9381/11/5/003>.

PERLMUTTER, S. et al. Measurements of ω and λ from 42 high-redshift supernovae. **The Astrophysical Journal**, American Astronomical Society, v. 517, n. 2, p. 565–586, jun 1999. Disponível em: <https://doi.org/10.1086/307221>.

Proca, Al. Sur la théorie ondulatoire des électrons positifs et négatifs. **J. Phys. Radium**, v. 7, n. 8, p. 347–353, 1936. Disponível em: <https://doi.org/10.1051/jphysrad:0193600708034700>.

QUINTIN, J.; YOSHIDA, D. Cuscuton gravity as a classically stable limiting curvature theory. **JCAP**, v. 02, p. 016, 2020.

RHAM, C. de. Massive Gravity. **Living Rev. Rel.**, v. 17, p. 7, 2014.

RHAM, C. de; GABADADZE, G.; TOLLEY, A. J. Resummation of Massive Gravity. **Phys. Rev. Lett.**, v. 106, p. 231101, 2011.

RHAM, C. de et al. Cosmology of Extended Proca-Nuevo. 10 2021.

RHAM, C. de; POZSGAY, V. New class of Proca interactions. **Phys. Rev. D**, v. 102, n. 8, p. 083508, 2020.

RIESS, A. G. et al. Observational evidence from supernovae for an accelerating universe and a cosmological constant. **Astron. J.**, v. 116, p. 1009–1038, 1998.

SARIDAKIS, E. N. et al. Modified Gravity and Cosmology: An Update by the CANTATA Network. 5 2021.

SCHMIDT-MAY, A.; STRAUSS, M. von. Recent developments in bimetric theory. **Journal of Physics A: Mathematical and Theoretical**, IOP Publishing, v. 49, n. 18, p. 183001, mar 2016. Disponível em: <https://doi.org/10.1088/1751-8113/49/18/183001>.

SEN, A. Strong - weak coupling duality in four-dimensional string theory. **Int. J. Mod. Phys. A**, v. 9, p. 3707–3750, 1994.

SMAILAGIC, A.; SPALLUCCI, E. The dual phases of massless/massive kalb-ramond fields. **Journal of Physics A: Mathematical and General**, IOP Publishing, v. 34, n. 32, p. L435–L440, aug 2001. Disponível em: <https://doi.org/10.1088/0305-4470/34/32/101>.

SOROKIN, D. P. Introductory Notes on Non-linear Electrodynamics and its Applications. In: . [S.l.: s.n.], 2021.

TASINATO, G. Cosmic Acceleration from Abelian Symmetry Breaking. **JHEP**, v. 04, p. 067, 2014.

VALENTINO, E. D. et al. In the realm of the hubble tensiona review of solutions. **Classical and Quantum Gravity**, IOP Publishing, v. 38, n. 15, p. 153001, jul 2021. Disponível em: <https://doi.org/10.1088/1361-6382/ac086d>.

WEINBERG, S. Photons and gravitons in s-matrix theory: Derivation of charge conservation and equality of gravitational and inertial mass. **Phys. Rev.**, American Physical Society, v. 135, p. B1049–B1056, Aug 1964. Disponível em: <https://link.aps.org/doi/10.1103/PhysRev.135.B1049>.

WEINBERG, S. **Gravitation and Cosmology: Principles and Applications of the General Theory of Relativity**. New York: John Wiley and Sons, 1972. ISBN 978-0-471-92567-5, 978-0-471-92567-5.

WEINBERG, S. Phenomenological Lagrangians. **Physica A**, v. 96, n. 1-2, p. 327–340, 1979.

WEINBERG, S. **The Quantum Theory of Fields**. [S.l.]: Cambridge University Press, 1995. v. 1.

WEINBERG, S. What is quantum field theory, and what did we think it is? In: **Conference on Historical Examination and Philosophical Reflections on the Foundations of Quantum Field Theory**. [S.l.: s.n.], 1996. p. 241–251.

WHEELER, J. A. On the mathematical description of light nuclei by the method of resonating group structure. **Phys. Rev.**, American Physical Society, v. 52, p. 1107–1122, Dec 1937. Disponível em: <https://link.aps.org/doi/10.1103/PhysRev.52.1107>.

WIGNER, E. P. On Unitary Representations of the Inhomogeneous Lorentz Group. **Annals Math.**, v. 40, p. 149–204, 1939.

APÊNDICE A – HELICITY DECOMPOSITION

Since the dual Maxwell theory (2.1.20) is a new model not known in the literature it is important to check its particle content, however the use of Hamiltonian methods would be cumbersome, so here we use helicity decomposition.

A.1 MAXWELL

Helicity decomposition analysis is an approach for investigating the content of a theory by a change of variables which does not involve time derivatives, so it does not change the canonical structure of the phase space. For an introduction see (JACCARD; MAGGIORE; MITSOU, 2013). As a first example consider

$$A_0 = \gamma \qquad\qquad A_j = v_j^t + \partial_j \theta, \tag{A.1.1}$$

in Maxwell's theory, despite the existence of derivatives this change of variables has an inverse because the boundary conditions require that the fields go to zero at infinity; the index t denotes transversality with respect to spatial divergence, $\partial^j v_j^t = 0$. Solving the gauge transformations of the Maxwell theory for the gauge parameter Λ

$$\delta A_j = \partial_j \Lambda \to \Lambda = \frac{\delta\left(\partial^j A_j\right)}{\nabla^2}, \tag{A.1.2}$$

it is possible to find a set of $(D-1)$ gauge invariants[1]

$$\delta\sigma = 0 \qquad\qquad \delta v_j^t = 0, \tag{A.1.3}$$

with $\sigma \equiv (\gamma - \dot{\theta})$. Because the action is gauge invariant it can be written in terms of the gauge invariants above

$$S_M = \left\langle -\frac{1}{4} F_{\mu\nu} F^{\mu\nu} \right\rangle = \left\langle \frac{v_j^t \Box v_j^t}{2} - \frac{\sigma \nabla^2 \sigma}{2} \right\rangle. \tag{A.1.4}$$

Maxwell's theory propagates a spatially transverse vector (v_j^t), which in D space-time dimensions has $D-2$ independent components, in agreement with the standard result of canonical analysis. Furthermore we see that there is also a non propagating degree of freedom, it corresponds to the longitudinal component of the electric field ($\nabla^2\sigma = -\boldsymbol{\nabla} \cdot \boldsymbol{E}$).

[1] Originally we have $\delta(\nabla^2\sigma) = 0$ but due to the boundary conditions σ is also gauge invariant

A.2 NEW SPIN-1 MODEL

Consider the helicty decomposition of the gauge parameter $\Omega^T_{[\mu\nu]\alpha}$ in (2.1.21)

$$\Omega^T_{[0j]0} = \lambda^t_j + \partial_j \phi \qquad \Omega^T_{[0i]j} = e_{ij} \tag{A.2.1}$$

$$\Omega^T_{[ij]0} = \partial_i u^t_j - \partial_j u^t_i + o^t_{[ij]} \qquad \Omega^T_{[ij]k} = \partial_i \omega^{Lt}_{jk} - \partial_j \omega^{Lt}_{ik} + \Gamma^{Lt}_{[ij]k}, \tag{A.2.2}$$

where the index "t"means spatial transversality, "T"is space-time transversality and Lt is transverse in the left index. The space-time transversality $\partial^\mu \Omega^T_{[\mu\nu]\alpha} = 0$ gives

$$\phi = 0 \qquad \partial^j e_{ji} = 0 \tag{A.2.3}$$

$$u^t_j = \frac{\dot{\lambda}^t_j}{\nabla^2} \qquad \omega^{Lt}_{jk} = \frac{\dot{e}^{Lt}_{jk}}{\nabla^2}. \tag{A.2.4}$$

For $b = 1/2,\ c = 0$ in (2.1.21) the gauge transformation for $B_{[0j]0}$ is

$$\delta B_{[0j]0} = \lambda^t_j + \partial_j \Lambda, \tag{A.2.5}$$

taking the divergence is possible to solve for Λ and then one can solve for λ^t_j as

$$\delta \left(B_{[0j]0} - \partial_j \frac{\partial^m B_{[0m]0}}{\nabla^2} \right) = \lambda^t_j. \tag{A.2.6}$$

From $\partial^j [\delta B_{[0j]k} = e^{Lt}_{jk} + \delta_{jk} \dot{\Lambda}]$ the first gauge invariant is given by

$$I_j = \partial^m B_{[0m]j} - \partial_j \frac{\partial^m \dot{B}_{[0m]0}}{\nabla^2}, \tag{A.2.7}$$

solving (2.1.21) for $o^t_{[jk]}$ and $\Gamma^t_{[ij]k}$, additional gauge invariants are found

$$\bar{I}^t_j = \partial^m B_{[mj]0} - \dot{B}_{[0j]0} + \partial_j \frac{\partial^m \dot{B}_{[om]0}}{\nabla^2}, \tag{A.2.8}$$

$$\tilde{I}^{Lt}_{jk} = \partial^m B_{[mj]k} + \partial_j \left(\frac{\partial^m \dot{B}_{[0m]0}}{\nabla^2} - \frac{\partial_k \partial^m \ddot{B}_{[0m]0}}{\nabla^4} \right) - \dot{B}_{[0j]k} + \partial_k \partial_j \frac{\partial^m B_{[0m]0}}{\nabla^2}$$
$$+ \delta_{jk} \left(\frac{\partial^m \ddot{B}_{[0m]0}}{\nabla^2} - \partial^m B_{[0m]0} \right). \tag{A.2.9}$$

The number of gauge invariants can be checked by a simple counting of gauge parameters. The gauge parameter Ω^T is subjected to D^2 conditions $\epsilon_{\nu\alpha} = \partial^\mu \Omega^T_{[\mu\nu]\alpha} = 0$, but these conditions are not independent, they are related by $\partial^\nu \epsilon_{\nu\alpha} = 0$. So, there are $D^2 - D$ independent relations for Ω^T, the amount of gauge invariants is the number of field components minus the number of independent gauge parameters, recalling also the $U(1)$ gauge parameter Λ, we have

$$\frac{D^2(D-1)}{2}-\left[\frac{D^2(D-1)}{2}-(D^2-D)+1\right]=D^2-D-1, \tag{A.2.10}$$

which is the same number of independent components encoded in $\{I_j,\bar{I}^t_j,\tilde{I}^{Lt}_{jk}\}$.

The helicity decomposition for $B_{[\mu\nu]\alpha}$ can be written as

$$B_{[0j]0}=v^t_j+\partial_j\Psi \qquad B_{[0j]k}=f_{jk} \tag{A.2.11}$$

$$B_{[jk]0}=\partial_j\mu^t_k-\partial_k\mu^t_j+K^t_{[jk]} \qquad B_{[jk]m}=\partial_j\phi^{Lt}_{km}-\partial_k\phi^{Lt}_{km}+\gamma^t_{[jk]m}, \tag{A.2.12}$$

in these variables the gauge invariants are

$$I_j=\partial^m f_{mj}-\partial_j\dot{\Psi} \qquad \bar{I}^t_j=\nabla^2\mu^t_j-\dot{v}^t_j \tag{A.2.13}$$

$$\tilde{I}^{Lt}_{jk}=\nabla^2\phi^{Lt}_{jk}-\dot{f}_{jk}-\delta_{jk}\Box\Psi+\partial_j\partial_k\Psi+\partial_j\left(\frac{\partial^m\dot{f}_{mk}}{\nabla^2}-\frac{\partial_k\ddot{\Psi}}{\nabla^2}\right), \tag{A.2.14}$$

and (2.1.20) in terms of these gauge invariants is

$$S_{BB}[I]=\left\langle\frac{1}{2}\left[-I_j^2-(\bar{I}^t_j)^2+(\tilde{I}^{Lt}_{jk})^2+\left(\partial_m\frac{\dot{I}_j}{\nabla^2}\right)^2\right]-\frac{1}{2(D-1)}\left(\tilde{I}+\frac{\partial^m\dot{I}_m}{\nabla^2}\right)^2\right\rangle, \tag{A.2.15}$$

where $\tilde{I}=\delta^{ij}\tilde{I}^{Lt}_{ij}$. This does not look like (A.1.4), but it does propagate the correct degrees of freedom. In order to see this, first notice that to derive the equations of motion one cannot extremize the action in terms of gauge invariants. The correct variables are $\{v^t_j,\Psi,f_{jk},\mu^t_j,K^t_{[jk]},\phi^{Lt}_{km},\gamma^t_{[jk]m}\}$, but it is possible to make a change to $\{v^t_j,\Psi,f_{jk},\bar{I}^t_j,K^t_{[jk]},\tilde{I}^{Lt}_{km},\gamma^t_{[jk]m}\}$ [2] . The equations of motion for $\bar{I}^t_j$ are $\bar{I}^t_j=0$ and the equations for $\tilde{I}^{LT}_{jk}$ are

$$\tilde{I}^{Lt}_{jk}-\frac{\delta_{jk}}{D-1}\left(\tilde{I}^{Lt}+\frac{\partial^j\partial^k\dot{f}_{jk}}{\nabla^2}-\ddot{\Psi}\right)=0. \tag{A.2.16}$$

The double divergence and trace of this equation leads to

$$\tilde{I}^{Lt}_{jk}=0 \qquad \partial^j\dot{I}_j=0, \tag{A.2.17}$$

the Ψ equation leads again to $\partial^j\dot{I}_j=0$. On the other hand the f_{jk} equations lead to

$$\partial_k\left(I_j-\frac{\ddot{I}_J}{\nabla^2}\right)=0, \tag{A.2.18}$$

[2] In the ordinary Maxwell theory the gauge invariant σ is linear in one helicity variable and the transverse vector v^t_j is another independent variable. So it is possible to treat all the gauge invariants as independent variables over which one extremizes the action

taking the ∂^k together with (A.2.17) leads to the propagating gauge invariant

$$\Box I_j = 0 \qquad \partial^j I_j = 0. \tag{A.2.19}$$

So the transverse invariant I_j plays the role of the transverse modes v_j^t of the Maxwell theory.