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Superstring Field Theories with Manifest Spacetime Supersymmetry

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Dedico esta tese à minha família

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Resumo

Esta tese apresenta um estudo sobre teorias de campos de supercordas no formalismo híbrido. Duas aplicações da formulação híbrida da teoria de campos de supercordas abertas serão apresentadas: a construção de soluções instanton na teoria de campos de supercordas abertas, e o cálculo da ação efetiva para os estados sem massa da supercorda, que no limite $\alpha' \to 0$ reproduz super Yang-Mills. Além disso, será apresentada a primeira construção da teoria de campos de cordas heteróticas com supersimetria $N = 1$, $D = 4$ manifesta no espaço-tempo.

Palavras Chaves: Supercordas; Teoria de campos de cordas; Supersimetria; Corda heterótica;

Áreas do conhecimento: Física; Teoria de supercordas.

Abstract

This thesis presents a study of superstring field theories using the hybrid formalism. Two applications of the hybrid formulation of open superstring field theory will be presented: the construction of instanton solutions in open superstring field theory, and the computation of the effective action for massless states of the string, which in the limit $\alpha' \to 0$ reproduces super Yang-Mills. In addition, the first construction of heterotic string field theory with manifest $N = 1$, $D = 4$ spacetime supersymmetry will be presented.

Palavras Chaves: Superstrings; String field theory; Supersymmetry; Heterotic string;

Áreas do conhecimento: Physics; Superstring theory.

Índice

Capítulo 1

Introduction

The usual worldsheet formulation of string and superstring theories, despite it's successes, is not entirely satisfactory for a number of reasons. The most obvious are that it's only defined perturbatively, and only on-shell quantities are well defined. In fact, even perturbatively the first quantized formulation is not complete, as there is no systematic and unambiguous way to deal with infrared divergencies and mass renormalization. String Field Theory was developed with the purpose of addressing these issues, and providing a more complete description of string theory. In particular, there is hope that it may provide a non-perturbative and background independent formulation of string theory. While these hopes have yet to be realized, SFT has proven useful for addressing off-shell and non-perturbative questions, perhaps the most notable application to date being tachyon condensation[1], where tachyonic solutions to the SFT equations of motion have been shown to describe the decay of non-supersymmetric D-branes [2]. In addition, SFT has provided the first complete definition of string perturbation theory [3].

Bosonic string field theories, both open and closed, are well established and have been studied extensively. Their quantization is based on the Batalin-Vilkovisky (BV) formalism. The open bosonic string field theory is usually formulated using Witten's star product [4]. This formulation is cubic at the classical level which, together with associativity of the star product, makes it particularly simple. This has led to many interesting results in tachyon condensation and analytic SFT solutions [5]. The closed bosonic string field theory is more complicated: it is nonpolynomial and has no associative product, being formulated instead in terms of L_∞ algebras [6]. Open SFT can also be formulated using homotopy algebras - in this case A_∞ - making it more analogous to the closed string [7].

Field theory formulations of superstrings present further challenges, particularly with relation to picture numbers of the string fields. Nevertheless, several different superstring field theories have been developed. Early proposals for open superstring field theory [8, 9] suffered from divergences coming from collisions of picture-changing operators [10], or from difficulties associated with the non-trivial kernel of picture-lowering operators [11]. The first fully consistent field theory formulation for the NS sector of open superstrings was Berkovits' WZW-like open SFT [12]. There, the issue of picture changing is resolved by working in the large Hilbert space, i.e. the Hilbert space including the zero mode ξ_0 from the bosonisation of the superconformal ghosts. Then it is possible to write an action for the NS sector without the need for any picture changing operators. This formulation was later extended to include the R sector [13]. A different formulation of open superstring field theory, in the small Hilbert space, was introduced in [14], based on an A_∞ homotopy algebra as a guiding principle. This A_∞ structure plays an important role in quantization based on the Batalin-Vilkovisky formalism, which has proved difficult in the WZW-like approach. In [15] it was shown that the homotopy-based and WZW-like formulations can be related by field redefinition and partial gauge-fixing.

The formulations described above are mostly based on the RNS formalism of the superstring. There is an alternative description using the hybrid formalism [16, 17]. This formalism is convenient for describing any compactification of the superstring to four dimensions which preserves $N = 1$, $D = 4$ supersymmetry. Then the hybrid description is manifestly $SO(3,1)$ super-Poincaré invariant, and has $N = 2$ worldsheet superconformal invariance. All the hybrid variables (in the $GSO(+)$ sector) are integer-moded, and there is no need to sum over spin structures, which is another advantage over the RNS formalism. Also, while the RNS vertex operators look very different in the R and NS sectors, the hybrid vertex operators are written in terms of four-dimensional superfields, where the NS and R sectors appear as the bosonic and fermionic fields, respectively. Scattering amplitudes can be computed using the $N = 4$ topological method [18]. A classical field theory action for the open superstring in the hybrid formalism was constructed in [12]. It has manifest four dimensional super-Poincaré invariance and naturally includes both the NS and the R sectors. It is a generalization of the action for ten-dimensional super-Yang-Mills in terms of four-dimensional superfields [19].

Closed superstrings naturally display both the difficulties associated with picture prescriptions and the complexities of the closed string field vertices. There is an additional difficulty particular to the type IIB superstring in that the $R - R$ sector contains a four-form with self-dual field strength. The large Hilbert space approach was generalized to the NS sector of the heterotic string in [20, 21]. The basic ingredients are the WZW-like formulation of open superstring field theory and the closed string products of [6]. The heterotic field can be thought of as a product of an open superstring and an open bosonic string fields. The R sector was partially included in [22]. This construction was also generalized to the $NS - NS$ sector of type II superstrings in [23]. A different formulation of heterotic and type II superstring field theories, based on the small Hilbert space, was proposed by Sen [24, 25]. The no-go theorem for the self-dual form in type IIB superstrings is circumvented by including an additional ghost-like free string field which does not scatter. In this approach it is possible to construct both heterotic and type II string field theories not only at tree level but at the full quantum level. Closed superstring field theories using the hybrid formalism have not yet been developed, but I will present some results in that direction in this work.

In this thesis, I will present two applications of open SFT in the hybrid formalism, and construct a hybrid formulation of heterotic SFT. I first review, in chapter 2, the hybrid formalism and the open superstring field theory, as well as the construction of the WZW-lilke heterotic string field theory. Chapter 3 is dedicated to the construction of SFT instanton solutions, which was published in [26]. In chapter 4 I compute the low-energy effective action from the SFT action, confirming the expectation that it should be Super Yang-Mills theory, as published in [27]. In chapter 5 I present the construction of a linearized field theory for heterotic strings in the hybrid formalism, based on the paper [28].

Capítulo 2

Preliminaries

In this chapter we will give a short review of some background material that will be necessary for the next chapters. We start with some basic concepts of string field theory in sec. 2.1, and then review the construction of the WZW-like open superstring field theory in sec. 2.1.2. In secs. 2.2 and 2.2.2 we review closed SFT and the WZW-like heterotic SFT. Finally, in secs. 2.3 and 2.4 we introduce the hybrid formalism and construct the hybrid open superstring field theory. In all cases we will only deal with the classical (tree-level) theories, meaning we don't consider higher genus Riemann surfaces.

2.1 Open string field theory

In this section we will present some concepts which are essential to both open bosonic and open superstring field theory. String field theory is a field theoretical formulation of string theory, in which the infinite family of fields associated with string excitations are described by a space-time field theory action. In analogy with the familiar fields of QFT being functions of spacetime, we can intuitively think of the string field as a functional of the configurations of the string on spacetime, $\Phi\left[x^{\mu}(\sigma)\right]$ (although, more precisely, the string field will also depend on the ghost variables). This defines a Schrödinger picture of the string field. For most purposes, it will be more convenient to work with the states $|\Phi\rangle$ defined by $\Phi[x(\sigma)] = \langle x(\sigma)|\Phi\rangle$. Now we can simply treat the string field as a state in the worldsheet CFT. A basis of the space of states is given by states of the form $|A\rangle = A(0)|0\rangle$ where $A(0)$ is a vertex operator inserted at the origin of a unit half-disk and $|0\rangle$ is the $SL(2,\mathbb{R})$ vacuum. More generally, we can think of the string field as a punctured unit half-disk with a local coordinate and with insertions (not necessarily at the origin and not necessarily local).

The physical states of the string are usually defined by the BRST cohomology. In SFT, the condition of BRST invariance becomes the classical equation of motion of the free string field, while the identification by BRST exact states becomes a gauge invariance

$$Q\Phi = 0, \quad \delta\Phi = Q\Lambda \tag{2.1}$$

Here we have dropped the bracket and simply denoted the string field by $\Phi = |\Phi\rangle$. In addition, there should be a restriction on the ghost number of the classical string field.

Before we can construct SFT actions, it is also important to define a special type of conjugation (distinct from hermitian conjugation) called *BPZ conjugation*. Given a state $|A\rangle = A(0)|0\rangle$, it's BPZ conjugate is defined by

$$\langle A| \equiv \langle 0| I \circ A(0) \tag{2.2}$$

where I denotes the conformal mapping $I(z) = -1/z$. We also define the *BPZ product* of two states A and B as $\langle A|B\rangle$. In terms of modes, we have for a conformal primary ϕ that

$$\phi_n^T = (-1)^{n+h}\phi_{-n}, \tag{2.3}$$

where the superscript T denotes BPZ conjugation and h the conformal weight. This definition allows us to define a reality condition for the string field, namely

$$|\Phi\rangle^\dagger = \langle\Phi| \tag{2.4}$$

where † denotes hermitian conjugation.

These definitions are enough to write a free field action. Adding interactions involves defining the so-called string field vertices, which will be the higher order terms in the action. They should be defined in such a way that perturbative string scattering amplitudes can be computed though Feynman diagrams, reproducing the worldsheet method results when that external states are on-shell. The way to do this is not unique, reflecting the freedom of performing field redefinitions. In the case of the open bosonic string, however, there is a particular choice, discovered by Witten [4], which makes the theory simpler. It involves defining an associative product in the space of string fields, the so-called star product: first we arbitrarily choose a point in the string (not one of the endpoints) which we will call the midpoint; then the star product between two string fields is defined by gluing one half of one string with one half of the other. In the Schrödinger picture, the field is now a function of two halves $x_L^\mu(\sigma)$ and $x_R^\mu(\sigma)$ of the string, separated by the midpoint. The product of two string fields A and B is schematically (ignoring ghosts) given by

$$(A * B)\left[x_L^\mu(\sigma), x_R^\mu(\sigma)\right] = \int \mathcal{D}x^\mu(\sigma) A\left[x_L^\mu(\sigma), x^\mu(\sigma)\right] B\left[x^\mu(\sigma), x_R^\mu(\sigma)\right] \tag{2.5}$$

For practical computations, there is a more convenient definition of the product in terms of CFT correlators. For example, we can map three states A, B and C, viewed as three half-disks with local coordinates z_1, z_2 and z_3, to a unit disk with the following conformal maps:

$$g_1(z_1) = e^{-\frac{2\pi i}{3}}\left(\frac{1+iz_1}{1-iz_1}\right)^{\frac{2}{3}} \tag{2.6}$$

$$g_2(z_2) = \left(\frac{1+iz_2}{1-iz_2}\right)^{\frac{2}{3}} \tag{2.7}$$

$$g_3(z_3) = e^{\frac{2\pi i}{3}}\left(\frac{1+iz_3}{1-iz_3}\right)^{\frac{2}{3}} \tag{2.8}$$

Then the following correlator gives the overlap $\langle C|A * B\rangle$:

$$\langle C|A * B\rangle = \langle g_1 \circ A(0) g_2 \circ B(0) g_3 \circ C(0)\rangle_{\text{disk}} . \tag{2.9}$$

Other useful representations of the product can be found in [29].

Two crucial properties of this product are:

-The BRST operator acts as a derivative: $Q(A * B) = Q(A) * B + (-1)^A A * Q(B)$, where $(-1)^A$ denotes the grassmanality of A.

-Ciclicity under BPZ product: $\langle A|B * C\rangle = \langle C|A * B\rangle$ = $\langle B|C * A\rangle$.

2.1.1 Open bosonic SFT

Although we will be mostly concerned with superstrings in this thesis, it will be useful to present Witten's cubic, Chern-Simons-like SFT. The open bosonic string field has ghost number 1 and is grassmann odd. The action reads

$$S = -\frac{1}{2}\langle\Phi \mid Q\Phi\rangle + \frac{1}{3}g\langle\Phi \mid \Phi * \Phi\rangle, \tag{2.10}$$

where g is the open string coupling constant. Note that these are the only terms we can write with ghost number 3 using only Φ, Q and the star product. This action invariant under the gauge transformation

$$\delta\Phi = Q\Lambda + \Phi * \Lambda - \Lambda * \Phi \tag{2.11}$$

and it's variation gives the equation of motion

$$Q\Phi + \Phi * \Phi = 0 \tag{2.12}$$

It can be shown that this theory gives the correct open string amplitudes [30].

2.1.2 WZW-like open superstring field theory

The first attempt to define a field theory for open superstrings was a straightforward generalization of the open bosonic SFT [8], but there is an additional complication: we now need to be careful with the picture number. If we choose the string field to have picture number -1 in the NS sector, then we need a picture changing operator on the cubic term in order for it to have the correct (non-vanishing) picture number. However, it was shown in [10] that this leads to divergences in scattering amplitudes when these picture changing operators collide.

Another, more successful approach was proposed in [12]: we work in the large Hilbert space, taking the string field to have picture number 0. We will see that this allows us to write a SFT action with no picture changing insertions.

We start by noting that the critical $N = 1$ superstring can be embedded in an $N = 2$ string with $N = 2$ generators T, G^+, G^- and J, as described in [31]. For the RNS superstring, we can identify the fermionic generators with the BRST charge and the b ghost, $G^+ = j_{BRST}$ and $G^- = b$, while T is the energy-momentum tensor and J the ghost number $J = cb + \eta\xi$. However, the following method can be more generally applied to any critical $N = 2$ string, for example the open string describing $(2,2)$ self-dual Yang-Mills [32]. Thus, we consider superconformal generators T, G^+, G^- and J of a twisted c = 6 N = 2 algebra, with OPEs:

$$G^+(z_1)G^-(z_2) \to \frac{2}{(z_1 - z_2)^3} + \frac{J(z_2)}{(z_1 - z_2)^2} + \frac{T(z_2)}{z_1 - z_2} \tag{2.13}$$

$$T(z_1)T(z_2) \to \frac{2T(z_2)}{(z_1 - z_2)^2} + \frac{\partial T(z_2)}{z_1 - z_2} \tag{2.14}$$

This $N = 2$ algebra can be extended to a small $N = 4$ superconformal algebra by defining two new spin-one generators J^{++} and J^{--}, and two new spin-3/2 generators, $\tilde{G}^{+}$ and $\tilde{G}^{-}$, defined by

$$J^{++} = e^{iH}, \quad J^{--} = e^{-iH}, \tag{2.15}$$

$$\tilde{G}^{+} = \oint dz\, G^{-}(z) e^{iH}, \quad \tilde{G}^{-} = -\oint dz\, G^{+}(z) e^{-iH} \tag{2.16}$$

where $J_{N=2} = \partial H$, and in the second line the integrals are around e^{iH} or e^{-iH}. For the RNS string, $\tilde{G}^{+} = \eta$, $J^{++} = c\eta$ and $J^{--} = b\xi$.

We now construct a SFT action. The string field Φ is taken to be $U(1)$ neutral and grassmann even. The linearized equations of motion from the quadratic term should reproduce the physical spectrum, while the cubic term should give the correct on-shell three-point amplitude. For an RNS string field in the large Hilbert space, the linearized equation of motion is $\eta_0 Q\Phi = 0$, or

$$\tilde{G}^{+} G^{+} \Phi = 0, \tag{2.17}$$

with gauge transformations

$$\delta\Phi = G^{+}\Lambda + \tilde{G}^{+}\bar{\Lambda} \tag{2.18}$$

Also, the three-point tree amplitude at zero instanton number is given by:

$$\langle \Phi(z_1)\, G^{+}\Phi(z_2)\, \tilde{G}^{+}\Phi(z_3) \rangle \tag{2.19}$$

Thus, we conclude that the action to cubic order must be given by

$$\int \left((G^{+}\Phi)(\tilde{G}^{+}\Phi) + \Phi\{G^{+}\Phi, \tilde{G}^{+}\Phi\} \right) \tag{2.20}$$

where the string fields are multiplied using Witten's star product. However, this cannot be the full action, because it is not gauge invariant under any non-linear version of 2.18. The form of the action to all orders is fixed up to field redefinitions by requiring gauge invariance. It was found in [12] to be a generalization of the WZW model

$$\frac{1}{2g^2} \int \left(\left(e^{-\Phi} G^{+} e^{\Phi} \right) \left(e^{-\Phi} \tilde{G}^{+} e^{\Phi} \right) - \int_0^1 dt \left(e^{-t\Phi} \partial_t e^{t\Phi} \right) \left\{ e^{-t\Phi} G^{+} e^{t\Phi}, e^{-t\Phi} \tilde{G}^{+} e^{t\Phi} \right\} \right) \tag{2.21}$$

which is invariant under the gauge transformations

$$\delta e^{\Phi} = \left(G^{+}\Lambda\right) e^{\Phi} + e^{\Phi}\left(\tilde{G}^{+}\bar{\Lambda}\right) \tag{2.22}$$

which is the non-linear version of 2.18. Meanwhile the non-linear generalization of 2.17 is

$$\tilde{G}^{+}\left(e^{-\Phi}G^{+}e^{\Phi}\right) = 0. \tag{2.23}$$

For the *NS* sector of the RNS string, the action is

$$\frac{1}{2g^2}\int\left(\left(e^{-\Phi}Qe^{\Phi}\right)\left(e^{-\Phi}\eta_0 e^{\Phi}\right) - \int_0^1 dt\left(e^{-t\Phi}\partial_t e^{t\Phi}\right)\left\{e^{-t\Phi}Qe^{t\Phi}, e^{-t\Phi}\eta_0 e^{t\Phi}\right\}\right) \tag{2.24}$$

where Φ has zero ghost number and zero picture. Note that all terms in the action have picture number 1 (which is the correct picture in the large Hilbert space), with no need for picture changing operators. Obviously this action cannot include the R sector, since states in this sector must have half-integer picture. The R states must be included in a separate string field. A generalization of the above action including the R sector was achieved in [13], but we will not review it here, since it will not be useful for the next chapters.

2.2 Closed string field theory

Closed string fields can be defined in a similar way to the open case, simply taking instead a CFT without boundary. There are also a couple constraints we must impose on the space of closed string fields with no open analogue, to which we will get shortly. Unlike in the open case, there is no known choice of vertices which makes the theory particularly simple. In fact, it has been shown that covariant closed SFT cannot be cubic, and is suspected to necessarily be non-polynomial [33]. There is also no analogue of the string product - instead of an associative product, closed SFTs are constructed using the language of homotopy algebras.

We define the space of closed string fields to be constrained by the so-called subsidiary conditions [6]

$$b_0^{-}\Phi = 0, \quad L_0^{-}\Phi = 0 \tag{2.25}$$

where $b_0^{-} = b_0 - \bar{b}_0$ and $L_0^{-} = L_0 - \bar{L}_0$. It is well known that the physical states of closed string theories satisfy these conditions, and it is not known how to write a kinetic term for the closed SFT action without imposing them. These constraints should also br imposed on gauge parameters.

We also define an inner product in the space of closed string fields by

$$\langle A, B\rangle = \langle A|c_0^-|B\rangle \tag{2.26}$$

where $c_0^- = \frac{1}{2}(c_0 - \bar{c}_0)$. We will see later that the c_0^- insertion is necessary in order to have the correct ghost number inside the correlator. Now we can see the necessity of the b_0^- condition, since the inner product would be degenerate without it.

In order construct interacting closed SFT's, we define closed string products denote by brackets

$$[B_1, B_2, \cdots, B_n]_g \tag{2.27}$$

for $n \geq 0$ and $g \geq 0$, where g denotes the genus: each such product involves an integral over a region of the moduli space of Riemann surfaces of genus g with n punctures. This region is called a string vertex, and should avoid surfaces close to degeneration. In this thesis we will only work at genus 0, so from now on we drop the genus subscript. The B_i are n states in the Hilbert space $\mathcal{H}$ of closed string fields, satisfying the constraints 2.25. The products also satisfy the same constraints

$$\left(b_0 - \bar{b}_0\right)[B_1, B_2, \cdots, B_n] = \left(L_0 - \bar{L}_0\right)[B_1, B_2, \cdots, B_n] = 0. \tag{2.28}$$

and are linear in the inputs. Thus, the products define multilinear maps from $\mathcal{H}^{\otimes n}$ to $\mathcal{H}$. They are also graded-commutative.

For $n = 0$ or 1, this is not really a product. For $n = 0$ we simply define the bracket to give a special element in $\mathcal{H}$. In particular, for $g = 0$ we take it to be zero: $[\,\cdot\,]_0 \equiv 0$. For $n = 1$, the "product" is simply a linear map from $\mathcal{H}$ to itself. When $g = 0$ it is simply given by the BRST operator

$$[B] = QB \tag{2.29}$$

Each product is defined with a number of b and $\bar{b}$ ghost insertions, such that the ghost number is given by

$$gh\left([B_1, B_2, \ldots, B_i, B_{i+1}, \ldots, B_n]\right) = 3 - 2n + \sum_{i=1}^{n} gh\left(B_i\right). \tag{2.30}$$

where gh denotes the ghost number.

The last important property of the string products is the so-called *main identity*. At genus zero, it reads

$$\sum_{\{i_l, j_k\}; l,k \geqslant 0, l+k=n \geqslant 0} \sigma\left(i_l, j_k\right)\left[B_{i_1}, \ldots, B_{i_l}, \left[B_{j_1}, \ldots, B_{j_k}\right]\right] = 0. \tag{2.31}$$

where the sum runs over all different ways of partitioning $\{1, 2, \ldots, n\}$ into two subsets $\{i_1, \ldots, i_l\}$ and $\{j_1, \ldots, j_k\}$. This corresponds to an L_∞ algebra. For $n = 1$, using 2.29, this simply states the nilpotency of the BRST operator. For $n = 2$, it implies that Q acts as a derivation on the $n = 2$ product. For $n = 3$ we get the more unusual relation

$$\begin{gathered} Q\left[B_1, B_2, B_3\right] + \left[QB_1, B_2, B_3\right] + (-)^{B_1}\left[B_1, QB_2, B_3\right] + (-)^{B_1+B_2}\left[B_1, B_2, QB_3\right] \\ +(-)^{B_1}\left[B_1, \left[B_2, B_3\right]\right] + (-)^{B_2(1+B_1)}\left[B_2, \left[B_1, B_3\right]\right] + (-)^{B_3(1+B_1+B_2)}\left[B_3, \left[B_1, B_2\right]\right] = 0 \end{gathered} \tag{2.32}$$

which tells us that Q is not a derivation of the $n = 3$ product, and it's failure to be a derivative corresponds to the failure of the $n = 2$ product to satisfy a Jacobi identity. 2.31 implies that Q is not a derivative of any product with $n \geq 3$.

We also define a set of multilinear functions by simply combining the string products 2.27 and the inner product 2.26:

$$\{A, B_1, B_2, \cdots, B_n\}_g \equiv \left\langle A, [B_1, B_2, \cdots B_n]_g \right\rangle \tag{2.33}$$

They are also graded commutative. For the case with no inputs, $\{\cdot\}_g$, is simply a number, which we define to be zero for $g = 0$, $\{\cdot\} \equiv 0$.

2.2.1 Closed bosonic SFT

Let's now write the action for closed bosonic SFT. The closed bosonic string field has ghost number 2 and is grassmann even. We will first derive the kinetic term. Much like in the open case, we want to define a quadratic action which gives the linearized equation and gauge invariance

$$Q\Phi = 0, \quad \delta\Phi = Q\Lambda \tag{2.34}$$

where the gauge parameter also satisfies the subsidiary conditions:

$$b_0^- \Lambda = 0, \quad L_0^- \Lambda = 0 \tag{2.35}$$

One can show that such an action is given by

$$S_2 = \frac{1}{2}\langle \Phi, Q\Phi \rangle. \tag{2.36}$$

Here we see that the c_0^- insertion in the inner product is necessary in order to have ghost number 6.

The full nonlinear closed SFT classical action reads:

$$S = \frac{1}{\kappa^2} \sum_{n=2}^{\infty} \frac{\kappa^n}{n!} \{\Phi^n\}, \tag{2.37}$$

where κ is the coupling constant, and Φ^n denotes the n-product with n inputs of Φ. One can show that it is invariant under the gauge transformations

$$\delta\Phi = \sum_{n=0}^{\infty} \frac{\kappa^n}{n!} [\Phi^n, \Lambda] \tag{2.38}$$

and it's variation gives the equation of motion

$$Q\Phi + \sum_{n=2}^{\infty} \frac{\kappa^{n-1}}{n!} [\Phi^n] = 0 \tag{2.39}$$

2.2.2 WZW-like heterotic string field theory

The large Hilbert space approach of the WZW-like open superstring field theory was generalized to heterotic strings in [20, 21]. The basic idea is to take a left-right product of the open superstring field with the open bosonic string field, resulting in a heterotic string field. This field has total (holomorphic plus anti-holomorphic) ghost-number 1 and picture number 0, and is Grassmann odd. Like the closed bosonic string field, it is required to satisfy the subsidiary conditions 2.25. The linearized equation of motion takes the same form as for the open superstring:

$$Q\eta\Phi = 0 \tag{2.40}$$

except now the BRST operator includes an anti-holomorphic part. Defining the inner product with a c_0^-, similar to closed bosonic SFT, it is straightforward to show that the kinetic term should be

$$S_2 = \frac{1}{2}\langle \eta\Phi, Q\Phi \rangle \tag{2.41}$$

which has the linearized gauge invariance

$$\delta^{(0)}\Phi = Q\Lambda + \eta\Omega. \tag{2.42}$$

Like the open superstring field theory, the heterotic SFT action is non-polynomial. The string products are defined in essentially the same way as for the closed bosonic SFT. For the cubic order, there is only one term that can be written with the right picture and ghost number, using only Φ, Q and η, namely

$$S_3 = \frac{\kappa}{3}\langle \eta\Phi, [\Phi, Q\Phi] \rangle, \tag{2.43}$$

where κ is the gravitational constant, and the factor of $1/3$ is included for convenience. Note that $[\Phi, \Phi]$ vanishes, on account of the product being graded commutative and Φ being Grassmann-odd, so this is indeed the only possible term. It is easy to check that the gauge transformations generalize to

$$\delta_\Lambda\Phi = Q\Lambda + \frac{\kappa}{2}[\Phi, Q\Lambda] + \mathcal{O}\left(\kappa^2\right) \tag{2.44}$$

$$\delta_\Omega\Phi = \eta\Omega + \frac{\kappa}{2}[\eta\Omega, \Phi] + \mathcal{O}\left(\kappa^2\right) \tag{2.45}$$

We could proceed deriving the action order by order, but in [21] a method was developed to determine the form of the full classical action. It takes a modified WZW-like form. To understand the construction, we first rewrite the open SFT action 2.24 in the modified form which generalizes to the heterotic string. One can show that 2.24 is equivalent to

$$S = -\frac{1}{g^2}\int_0^1 dt \left\langle (\eta A_t)\, A_Q \right\rangle \tag{2.46}$$

where $A_Q = e^{-\Phi(t)}Qe^{\Phi(t)}$, $A_t = e^{-\Phi(t)}\partial_t e^{\Phi(t)}$. In 2.24, $\Phi(t) = t\Phi$, but more generally we can take any function such that $\Phi(0) = 0$ and $\Phi(1) \equiv \Phi$. Note that A_Q is a pure gauge field in the *bosonic* SFT, associated with the parameter $\Phi(t)$. This fact will be important for the generalization to heterotic strings. Note also that A_Q and A_t satisfy

$$\partial_t A_Q = Q' A_t \tag{2.47}$$

where Q' is the BRST operator of the bosonic open SFT action expanded about A_Q: $Q' = Q + [A_Q, .\}$, where $[., .\}$ denotes a graded commutator.

To construct the heterotic SFT, we define, in analogy with the above construction for the open superstring, closed string fields Ψ_Q and Ψ_t satisfying

$$\partial_t \Psi_Q = Q' \Psi_t, \tag{2.48}$$

where Q' is the BRST operator of the bosonic closed string field theory action expanded about Φ_Q,

$$Q' \equiv Q + \sum_{n=1}^{\infty} \frac{\kappa^n}{n!} \left[\Phi_Q^n, .\right], \tag{2.49}$$

where now the brackets denote the closed string field products. Unlike in the open superstring case, Ψ_Q and Ψ_t do not have simple forms in terms of the fundamental heterotic string field $\Phi(t)$. Instead, Ψ_Q is a pure gauge field in the *bosonic* SFT. Such a pure gauge field can be constructed by solving the differential equation

$$\partial_\tau G(\tau\Phi) = Q\Phi + \sum_{n=1}^{\infty} \frac{\kappa^n}{n!} \left[G(\tau\Phi)^n, \Phi\right] \equiv Q'_G \Phi \tag{2.50}$$

with the initial condition $G(0) = 0$. Then Ψ_Q is simply given by

$$\Psi_Q \equiv G(\Phi(t)). \tag{2.51}$$

where again $\Phi(t)$ connects $0 = \Phi(t = 0)$ and $\Phi = \Phi(t = 1)$. Now Ψ_t can be determined from 2.48. First, we consider a field $H(V, XV; \tau)$ dependent on an auxiliary parameter τ, such that $H(V, \partial_t\Phi(t); 0) = 0$ and $H(V, \partial_t\Phi(t); 1) = \Psi_t$. The one can show that 2.48 is implied by

$$\partial_\tau H(V, \partial_t V; \tau) = \partial_t V + \kappa[V, H(V, \partial_t V; \tau)]'_G, \tag{2.52}$$

$$Q'_G H(V, \partial_t V; \tau) = \partial_t G(\tau V) \quad \text{at} \quad \tau = 0 \tag{2.53}$$

where $[.,.]'_G$ denotes the string product around $G(\tau V)$,

$$[B_1, B_2, \ldots, B_m]' \equiv \sum_{n=0}^{\infty} \frac{\kappa^n}{n!} [G^n, B_1, B_2, \ldots, B_m] \tag{2.54}$$

It is straightforward to solve both 2.50 and 2.52 order by order in κ, and thus determine Ψ_Q and Ψ_t.

Now we can write the action, analogously to 2.46, as

$$S = \frac{2}{\alpha'} \int_0^1 dt \left\langle \eta \Psi_t, \Psi_Q \right\rangle \tag{2.55}$$

See [21] for a proof of gauge invariance.

2.3 The hybrid formalism

We now introduce the hybrid formalism, with which we will work through most of this thesis. This formalism is useful for describing superstrings compactified on Calabi-Yau manifolds, and has the advantage of making $SO(3,1)$ super-Poincaré invariance manifest. All the hybrid variables (in the $GSO(+)$ sector) are integer-moded, and there is no need to sum over spin structures, which is another advantage over the RNS formalism. Also, while the RNS vertex operators look very different in the R and NS sectors, the hybrid vertex operators are written in terms of four-dimensional superfields, where the NS and R sectors appear as the bosonic and fermionic fields, respectively. Scattering amplitudes can be computed using the $N = 4$ topological method [18].

The hybrid variables consist of eleven free bosons $(x^m, \rho, x^j, \bar{x}_j)$ and fourteen free fermions $(\theta^\alpha, \bar{\theta}^{\dot{\alpha}}, p^\alpha, \bar{p}^{\dot{\alpha}}, \psi^j, \bar{\psi}_j)$ with OPE's

$$\begin{gathered} x^m(z_1)x^n(z_2) \sim -\log|z_1 - z_2|\delta^{mn}, \quad \rho(z_1)\rho(z_2) \sim -\log(z_1 - z_2) \\ p_\alpha(z_1)\theta^\beta(z_2) \sim \delta_\alpha^\beta (z_1 - z_2)^{-1}, \quad \bar{p}_{\dot{\alpha}}(z_1)\theta^{\dot{\beta}}(z_2) \sim \delta_{\dot{\alpha}}^{\dot{\beta}} (z_1 - z_2)^{-1} \\ x^i(z_1)\bar{x}_j(z_2) \sim -\log|z_1 - z_2|\delta_j^i, \quad \psi^i(z_1)\bar{\psi}_j(z_2) \sim (z_1 - z_2)^{-1}\delta_j^i \end{gathered} \tag{2.56}$$

Here $m, \alpha, \dot{\alpha}$ are four dimensional vector and spinor indices. θ, $\bar{\theta}$ and x^m together are the coordinates of $N = 1$, $D = 4$ superspace, while p and $\bar{p}$ are conjugate to θ and $\bar{\theta}$. Meanwhile, $i, j = 1$to 3 are $SU(3)$ indices describing a six-dimensional compactified manifold. ρ is a chiral boson analogous to the RNS ϕ coming from the bosonization of the superconformal ghosts.

The $N = 1$, $D = 4$ supersymmetry generators take the form

$$q_\alpha = \oint dz \left(p_\alpha - \frac{i}{2}\bar{\theta}^{\dot{\alpha}}\partial x_{\alpha\dot{\alpha}} - \frac{1}{8}\theta_\alpha \partial(\bar{\theta})^2 \right) \tag{2.57}$$

$$\bar{q}_{\dot{\alpha}} = -\oint dz \left(\bar{p}_{\dot{\alpha}} - \frac{i}{2}\bar{\theta}^{\dot{\alpha}}\partial x_{\alpha\dot{\alpha}} - \frac{1}{8}\bar{\theta}_{\dot{\alpha}} \partial(\theta)^2 \right) \tag{2.58}$$

Note that this satisfies the four dimensional supersymmetry algebra, in contrast to the RNS formalism where the algebra only closes up to picture changing [17].

To formulate the $N = 4$ method, we define a set of twisted $c = 6$, N $= 4$ superconformal generators in terms of these variables as

$$T = \frac{1}{2}\partial x^m \partial x_m + p_\alpha \partial\theta^\alpha + \bar{p}_{\dot{\alpha}}\partial\bar{\theta}^{\dot{\alpha}} + \frac{1}{2}\partial\rho\partial\rho - \frac{i}{2}\partial^2\rho + \partial x^j \partial\bar{x}_j + i\psi^j\partial\bar{\psi}_j \tag{2.59}$$

$$G^+ = e^\rho d^2 + \psi^j \partial\bar{x}_j, \quad G^- = e^{-\rho}\bar{d}^2 + \bar{\psi}_j \partial x^j \tag{2.60}$$

$$\tilde{G}^+ = e^{-2\rho+iH_C}\bar{d}^2 + \frac{1}{2}e^{-\rho}\epsilon_{jkl}\psi^j\psi^k\partial x^l, \quad \tilde{G}^- = e^{2\rho-iH_C}d^2 + \frac{1}{2}e^{\rho}\epsilon^{jkl}\bar{\psi}_j\bar{\psi}_k\partial\bar{x}_l \tag{2.61}$$

$$J = -\partial\rho + iJ_C, \quad J^{++} = e^{-\rho+iH_C}, \quad J^{--} = e^{\rho-iH_C} \tag{2.62}$$

where

$$d_\alpha = p_\alpha + \frac{i}{2}\bar{\theta}^{\dot{\alpha}}\partial x_{\alpha\dot{\alpha}} - \frac{1}{4}(\bar{\theta})^2\partial\theta_\alpha + \frac{1}{8}\theta_\alpha\partial(\bar{\theta})^2 \tag{2.63}$$

$$\bar{d}_{\dot{\alpha}} = \bar{p}_{\dot{\alpha}} + \frac{i}{2}\theta^\alpha\partial x_{\alpha\dot{\alpha}} - \frac{1}{4}(\theta)^2\partial\bar{\theta}_{\dot{\alpha}} + \frac{1}{8}\bar{\theta}_{\dot{\alpha}}\partial(\theta)^2 \tag{2.64}$$

$J_C = \psi^j\bar{\psi}_j = \partial H_C$ and $x_{\alpha\dot{\alpha}} = \sigma^m_{\alpha\dot{\alpha}}x_m$. Note that d and $\bar{d}$ are defined so as to not have any singularities with the supersymmetry generators. The plus and minus superscripts indicate the $U(1)$ (i.e. J) charge. These generators satisfy the twisted small $N = 4$ algebra

$$T(z)T(0) \sim \frac{2}{z^2}T(0) + \frac{1}{z}\partial T(0) \tag{2.65}$$

$$G^+\tilde{G}^- \sim G^-\tilde{G}^+ \sim 0 \tag{2.66}$$

$$G^+(z)G^-(0) \sim \frac{2}{z^3} + \frac{J(0)}{z^2} + \frac{T(0)}{z} \tag{2.67}$$

$$\tilde{G}^+(z)\tilde{G}^-(0) \sim -\frac{2}{z^3} - \frac{J(0)}{z^2} - \frac{T(0)}{z} \tag{2.68}$$

with the twist given by $T(z) = T^{(untwisted)}(z) + \frac{1}{2}\partial J(z)$.

After the twist, the conformal weights of the worldsheet fields are as follows:

Field	$x^m, x^j, \bar{x}_j$	ψ^j	$\bar{\psi}_j$	$\theta, \bar{\theta}$	$e^{n\rho}$
Conformal weight	0	0	1	0	$-\frac{(n^2+n)}{2}$

which implies, for the $N = 4$ generators

Generator	T	$G^+, \tilde{G}^+$	$G^-, \tilde{G}^-$	J	J^{++}	J^{--}
Conformal weight	2	1	2	1	0	2

In Minkowski signature, hermitian conjugation is defined to exchange θ with $\bar{\theta}$, p with $\bar{p}$ and x^i with $\bar{x}_i$, while x^m is real. The definition for ρ, ψ^i and $\bar{\psi}_i$ is a bit more subtle. It seems natural to take ψ and $\bar{\psi}$ to be conjugates, and ρ anti-hermitian, but this would flip the sign of the $U(1)$ charge. So we complement this conjugation with an $SU(2)$ rotation taking $J \to -J$, $J^{++} \to J^{--}$, and $J^{--} \to J^{++}$. This means that

$$(e^\rho)^\dagger = e^{-2\rho+iH_C}, \quad (\psi^i)^\dagger = \frac{1}{2}e^{-\rho}\epsilon_{ijk}\psi^j\psi^k, \quad (\bar{\psi}_i)^\dagger = \frac{1}{2}e^{\rho}\epsilon^{ijk}\psi_j\psi_k \tag{2.69}$$

which also implies $(G^\pm)^\dagger = \tilde{G}^\pm$.

It will be useful to break the $N = 4$ generators into "four-dimensional" and "six-dimensional" parts as

$$G_4^+ = e^\rho d^2, \quad G_6^+ = \psi^j\partial\bar{x}_j, \quad \tilde{G}_4^+ = e^{-2\rho+iH_C}\bar{d}^2, \quad \tilde{G}_6^+ = \frac{1}{2}e^{-\rho}\epsilon_{jkl}\psi^j\psi^k\partial x^l \tag{2.70}$$

$$G_4^- = e^{-\rho}\bar{d}^2, \quad G_6^- = \bar{\psi}_j\partial x^j, \quad \tilde{G}_4^- = e^{2\rho-iH_C}d^2, \quad \tilde{G}_6^- = \frac{1}{2}e^{\rho}\epsilon^{jkl}\bar{\psi}_j\bar{\psi}_k\partial\bar{x}_l \tag{2.71}$$

Note however that the two sets are not decoupled. Specifically, $\tilde{G}_4^+$ has a pole with G_6^-, $\tilde{G}_6^+$ with G_4^-, $\tilde{G}_4^-$ with G_6^+ and $\tilde{G}_6^-$ with G_4^+.

Physical states are defined by the condition 2.17, up to the transformations 2.18. However, this still leaves an infinite family of vertex operators for each physical state, of which we should select a unique representative. In the RNS formalism, this is done by fixing the picture number. In terms of the hybrid variables, the picture number operator is

$$P = \frac{1}{2\pi}\oint dz\left(-\partial\rho - \frac{1}{2}p_\alpha\theta^\alpha + \frac{1}{2}\bar{p}_{\dot{\alpha}}\bar{\theta}^{\dot{\alpha}}\right), \tag{2.72}$$

which does not commute with the supersymemtry generators 2.57. Thus, fixing the picture of physical states would spoil manifest supersymmetry. So instead we simply fix the ρ charge, i.e. the charge under $-\partial\rho$.

The prescription for computing tree-level amplitudes is

$$\langle \Phi_1(z_1)\,\tilde{G}^+(\Phi_2(z_2))\,G^+(\Phi_3(z_3))\prod_{r=4}^{n}\int dz_r\,G^-\left(G^+(\Phi_r(z_r))\right)\rangle \tag{2.73}$$

where the Φ_i are vertex operators. The basic correlators are given by

$$\left\langle \theta^2\bar{\theta}^2\right\rangle_\theta = 1 \tag{2.74}$$

$$\left\langle e^{-\rho+iH_C}\right\rangle_{\rho,\psi} = 1 \tag{2.75}$$

$$\left\langle e^{ik_1x}(z_1)\,e^{ik_2x}(z_2)\right\rangle_x = i(2\pi)^4\delta^4(k_1+k_2)\,|z_1-z_2|^{-k_1\cdot k_2} \tag{2.76}$$

The hybrid variables can be related to RNS variables by the following field redefinitions:

$$\theta^\alpha = e^{\frac{i}{2}(-i\phi\pm(\sigma_0+\sigma_1)-H_C)},\quad \bar{\theta}^{\dot{\alpha}} = c\xi e^{\frac{i}{2}(3i\phi\pm(\sigma_0-\sigma_1)+H_C)} \tag{2.77}$$

$$p_\alpha = e^U e^{\frac{i}{2}(i\phi\pm(\sigma_0+\sigma_1)+H_C)}e^{-U},\quad \bar{p}_{\dot{\alpha}} = e^U\left(b\eta e^{\frac{i}{2}(-3i\phi\pm(\sigma_0-\sigma_1)-H_C)}\right)e^{-U} \tag{2.78}$$

$$x^m_{hybrid} = e^U x^m_{RNS}e^{-U},\quad x^j_{hybrid} = x^j_{RNS},\quad \bar{x}^{hybrid}_j = \bar{x}^{RNS}_j + ic\xi e^{-\phi}\bar{\psi}^{RNS}_j, \tag{2.79}$$

$$\psi^j_{hybrid} = i\eta e^\phi\psi^j_{RNS} + ic\partial x^j_{RNS},\quad \bar{\psi}^{hybrid}_j = -i\xi e^{-\phi}\bar{\psi}^{RNS}_j \tag{2.80}$$

$$\partial\rho = 3\partial\phi + icb + 2i\xi\eta - i\psi^j_{RNS}\bar{\psi}^{RNS}_j \tag{2.81}$$

where the σ's come from bosonizing the ψ^m, and

$$U = \frac{1}{2\pi}\oint dz\,c\xi e^{-\phi}\left(\frac{1}{2}\psi_m\partial x^m + \bar{\psi}^{RNS}_j\partial x^j_{RNS}\right) \tag{2.82}$$

2.4 Hybrid open superstring field theory

With the hybrid formalism, we can expect to be able to make open superstring field theory manifestly $D=4$ spacetime supersymmetric. This was done in [12]. Since picture number is not preserved by supersymmetry, instead of fixing the picture number we fix the ambiguity in the physical states by substituting 2.17 by the stronger condition

$$\left(G^+ + \tilde{G}^+\right)\Phi = 0 \tag{2.83}$$

Since $G^+ + \tilde{G}^+$ is nilpotent, this has the gauge invariance

$$\delta\Phi = \left(G^+ + \tilde{G}^+\right)\Lambda \tag{2.84}$$

We are now going to separate Φ into three string fields. To do this, first we expand Φ in eigenvalues of the ρ-charge, $\Phi = \sum_{n=-\infty}^{\infty}\Phi_n$. The linearized equation of motion and gauge invariance break up into

$$G_4^+\Phi_n + G_6^+\Phi_{n+1} + \tilde{G}_6^+\Phi_{n+2} + \tilde{G}_4^+\Phi_{n+3} = 0 \tag{2.85}$$

and

$$\delta\Phi_n = G_4^+\Lambda_{n-1} + G_6^+\Lambda_n + \tilde{G}_6^+\Lambda_{n+1} + \tilde{G}_4^+\Lambda_{n+2}. \tag{2.86}$$

Now note that both G_4^+ and $\tilde{G}_4^+$ have trivial cohomology. This means we can use 2.85 to write all Φ_n, up to gauge transformations, in terms of only three, which we choose to be Φ_{-1}, Φ_0 and Φ_1. The linearized equations of motion and gauge invariances for these three fields are

$$\tilde{G}_4^+G_4^+\Phi_{-1} + \tilde{G}_4^+G_6^+\Phi_0 + \tilde{G}_4^+\tilde{G}_6^+\Phi_1 = 0, \tag{2.87}$$
$$\left(\tilde{G}_6^+G_6^+ + \tilde{G}_4^+G_4^+\right)\Phi_0 + \tilde{G}_4^+G_6^+\Phi_1 + \tilde{G}_6^+G_4^+\Phi_{-1} = 0, \tag{2.88}$$
$$G_4^+G_6^+\Phi_{-1} + G_4^+\tilde{G}_6^+\Phi_0 + G_4^+\tilde{G}_4^+\Phi_1 = 0. \tag{2.89}$$

and

$$\delta\Phi_{-1} = G_4^+\Lambda_{-2} + G_6^+\Lambda_{-1} + \tilde{G}_6^+\Lambda_0 + \tilde{G}_4^+\Lambda_1, \tag{2.90}$$
$$\delta\Phi_0 = G_4^+\Lambda_{-1} + G_6^+\Lambda_0 + \tilde{G}_6^+\Lambda_1 + \tilde{G}_4^+\Lambda_2, \tag{2.91}$$
$$\delta\Phi_1 = G_4^+\Lambda_0 + G_6^+\Lambda_1 + \tilde{G}_6^+\Lambda_2 + \tilde{G}_4^+\Lambda_3. \tag{2.92}$$

The form of the full open SFT classical action can be guessed from the action for ten-dimensional Super Yang-Mills in terms of four-dimensional superfields

[19], which reads

$$S_{SYM} = \mathrm{Tr}\frac{1}{2g^2}\int d^{10}x \Big[-2\int d^2\theta W^\alpha W_\alpha$$
$$+\int d^4\theta \left(\left(e^{-v}\bar\partial_j e^v\right)\left(e^{-v}\partial^j e^v\right) - \int_0^1 dt \left(e^{-tv}\partial_t e^{tv}\right)\left\{ e^{-tv}\bar\partial_j e^{tv}, e^{-tv}\partial^j e^{tv} \right\} \right) \Big)$$
$$+2\int d^4\theta \left(\left(\partial^j e^{-v}\right)\bar\omega_j e^v + e^v\omega^j\left(\bar\partial_j e^{-v}\right) + e^{-v}\bar\omega_j e^v \omega^j \right)$$
$$+\int d^2\theta \epsilon_{jkl}\left(\omega^j\partial^k\omega^l + \frac{2}{3}\omega^j\omega^k\omega^l\right) + \int d^2\bar\theta\epsilon^{jkl}\left(\bar\omega_j\bar\partial_k\bar\omega_l - \frac{2}{3}\bar\omega_j\bar\omega_k\bar\omega_l\right)\Big]. \tag{2.93}$$

where $W_\alpha = \bar D^2(e^{-v}D_\alpha e^v)$. v is the real superfield containing the four-dimensional part of the gauge field, and ω_j and $\bar\omega^j$ are the chiral and anti-chiral superfields containing the six-dimensional part of the gauge field. Note that all fields are allowed to depend on the full ten dimensions.

The SFT generalization of this is

$$S_{Hybrid} =$$
$$\frac{1}{2g^2}\Big\langle \left(e^{-V}G_4^+e^V\right)\left(e^{-V}\tilde G_4^+e^V\right) - \int_0^1 dt\left(e^{-tV}\partial_t e^{tV}\right)\left\{e^{-tV}G_4^+e^{tV}, e^{-tV}\tilde G_4^+ e^{tV}\right\}$$
$$+\left(e^{-V}G_6^+e^V\right)\left(e^{-V}\tilde G_6^+e^V\right) - \int_0^1 dt\left(e^{-tV}\partial_t e^{tV}\right)\left\{e^{-tV}G_6^+e^{tV}, e^{-tV}\tilde G_6^+ e^{tV}\right\}$$
$$+2\left(\left(\tilde G_6^+e^{-V}\right)\bar\Omega e^V + e^V\Omega\left(G_6^+e^{-V}\right) + e^{-V}\bar\Omega e^V\Omega\right)$$
$$-\left(\Omega\tilde G_6^+\Phi_1 - \frac{2}{3}\Omega\Omega\Phi_1\right) + \left(\bar\Omega G_6^+\Phi_{-1} + \frac{2}{3}\bar\Omega\bar\Omega\Phi_{-1}\right)\Big\rangle. \tag{2.94}$$

where $\Omega = \tilde G_4^+\Phi_1$ and $\bar\Omega = G_4^+\Phi_{-1}$, and again the fields are multiplied using the star product. This is invariant under the gauge tranformations

$$\delta e^V = G_4^+\Lambda_{-1}e^V + e^V\tilde G_4^+\Lambda_2 + e^V\left(\mathcal{G}_6^+\Lambda_0 + \tilde{\mathcal{G}}_6^+\Lambda_1\right), \tag{2.95}$$
$$\delta\Omega = -\tilde G_6^+\tilde G_4^+\Lambda_2 + [\Omega, \tilde G_4^+\Lambda_2] + \tilde G_4^+\left(\mathcal{G}_4^+\Lambda_0 + \mathcal{G}_6^+\Lambda_1\right), \tag{2.96}$$
$$\delta\bar\Omega = -G_6^+G_4^+\Lambda_{-1} - [\bar\Omega, G_4^+\Lambda_{-1}] + G_4^+\left(e^V\left(\tilde{\mathcal{G}}_6^+\Lambda_0 + \tilde{\mathcal{G}}_4^+\Lambda_1\right)e^{-V}\right). \tag{2.97}$$

where we define convariantized versions of the $N=4$ generators as

$$\mathcal{G}_4^+ = e^{-V}G_4^+e^V, \quad \tilde{\mathcal{G}}_4^+ = \tilde G_4^+, \quad \mathcal{G}_6^+ = e^{-V}\left(G_6^+ + \bar\Omega\right)e^V \quad \tilde{\mathcal{G}}_6^+ = \tilde G_6^+ - \Omega. \tag{2.98}$$

The equations of motion implied by the action 2.94 can be neatly written in terms of these covariantized generators as

$$\left\{\mathcal{G}_4^+, \tilde{\mathcal{G}}_4^+\right\} = -\left\{\mathcal{G}_6^+, \tilde{\mathcal{G}}_6^+\right\}, \tag{2.99}$$
$$2\left\{\mathcal{G}_4^+, \tilde{\mathcal{G}}_6^+\right\} = -\left\{\mathcal{G}_6^+, \mathcal{G}_6^+\right\}, \quad 2\left\{\tilde{\mathcal{G}}_4^+, \mathcal{G}_6^+\right\} = -\left\{\tilde{\mathcal{G}}_6^+, \tilde{\mathcal{G}}_6^+\right\}.$$

Capítulo 3

Instanton Solutions in Open Superstring Field Theory

As explained in the introduction, open superstring field theory (SFT) can be a powerful method to obtain non-perturbative information about superstring theory, with the best example being tachyon condensation. In addition to the non-supersymmetric tachyonic solutions, the equations of motion of SFT also contains spacetime supersymmetric solutions which can be studied. In particular, it would be interesting to use SFT to search for stringy generalizations of instanton solutions of super-Yang-Mills which give rise to non-perturbative effects in gauge theories. Note that instanton solutions in SFT have been studied in [34, 35, 36, 37] in the RNS formalism starting from a system of $D3$- and $D(-1)$-branes. Here we will take a different approach, starting with the Yang-Mills instanton and calculating massive stringy corrections. It would be very interesting to compare these two approaches, although this is expected to be non-trivial since they correspond to perturbative expansions in opposite limits.

After compactifying to four dimensions, the massless states of the open superstring include $D = 4$ super-Yang-Mills fields and the instanton solution is obtained by requiring the four-dimensional Yang-Mills field-strength to be self-dual. However, it is difficult to generalize the concept of self-dual field strength to SFT since the only gauge-invariant quantity that can be constructed from the string field Φ is $Q\Phi$, which vanishes on-shell. Nevertheless, one can instead define the four-dimensional instanton solution as a localized half-BPS solution to the equations of motion, i.e. a solution which is annihilated by half of the $N = 1$ $D = 4$ spacetime supersymmetries. In this chapter, we will generalize this half-BPS definition of the instanton solution in SFT and will find the first stringy correction using the hybrid formalism. Although we will be unable to find an exact solution to the SFT equations of motion which generalizes the self-dual instanton, we will find the first stringy correction to the instanton solution which corresponds to turning on certain massive spin-2 and spin-0 fields.

In section 3.1 we review self-dual instanton solutions in super-Yang-Mills theory. In section 3.2 we introduce the central problem of this chapter - solving the half-BPS condition in open superstring field theory. Our strategy for tackling this problem is to write a series expansion for the string field and solve order by order in the expansion parameter, as explained in section 3.3. In section 3.4 we review the BPST instanton in super-Yang-Mills and generalize it to SFT using the star product. In section 3.5 we calculate the first stringy correction to the BPST instanton and show it corresponds to turning on certain off-shell massive fields by comparing with the vertex operators for the first massive level of the string.

This chapter is based on the paper [26].

3.1 Super-Yang-Mills Instantons

$N = 1$ $D = 4$ super-Yang-Mills can be described by a vector superfield $V^i(x, \theta, \bar{\theta})$ where i is a gauge index and $(x^m, \theta^\alpha, \bar{\theta}^{\dot{\alpha}})$ are the usual $N = 1$ $D = 4$ superspace variables. We also define $V = V^i T^i$ where T^i are the generators of the gauge group. Supersymmetry transformations are generated by acting on V with the supersymmetry generators q_α and $\bar{q}_{\dot{\alpha}}$ (see appendix A).

We want to find field configurations that are preserved by half of this supersymmetry. Naively, this condition would be formulated as

$$(\epsilon q)V = 0, \tag{3.1}$$

where ϵ^α is a supersymmetry parameter.

However, that is not quite correct. We must keep in mind that the theory has a gauge symmetry, given by

$$\delta_{\text{gauge}} V = \mathscr{L}_{V/2}[\Lambda - \bar{\Lambda} - i \coth(\mathscr{L}_{V/2})(\Lambda + \bar{\Lambda})], \tag{3.2}$$

where Λ and $\bar{\Lambda}$ are chiral and antichiral, respectively, and $\mathscr{L}$ is the Lie derivative:

$$\mathscr{L}_V(A) = [V, A] \tag{3.3}$$

So the correct condition is not that $(\epsilon q)V$ vanishes, but that it is pure gauge. In other words, V is preserved by half of the supersymmetry if there are Λ and $\bar{\Lambda}$

such that

$$(\epsilon q)V = -\delta_{\text{gauge}}V. \tag{3.4}$$

Of course, one could also consider states preserved by the antichiral supersymmetry, for which the analysis is similar.

To see what eq. 3.4 implies, we first write V in Wess-Zumino gauge and assume that the fermions vanish (since we are concerned with classical field configurations):

$$V = -\theta\sigma^m\bar{\theta}v_m + \frac{1}{2}\theta^2\bar{\theta}^2 D. \tag{3.5}$$

Since $V^3 = 0$ in Wess-Zumino gauge, the gauge transformation of eq. 3.2 becomes

$$\delta_{\text{gauge}}V = \Lambda + \bar{\Lambda} + \frac{1}{2}[V, \Lambda - \bar{\Lambda}] + \frac{1}{12}[V, [V, \Lambda + \bar{\Lambda}]]. \tag{3.6}$$

The most general Λ can be written in components as

$$\Lambda = A(x) + \sqrt{2}\theta\psi(x) + \theta^2 F(x) + i\theta\sigma^m\bar{\theta}\partial_m A(x) + \frac{i}{\sqrt{2}}\theta^2\bar{\theta}\bar{\sigma}^m\partial_m\psi(x) + \frac{1}{2}\theta^2\bar{\theta}^2\Box A(x). \tag{3.7}$$

Similarly, the most general $\bar{\Lambda}$ is

$$\bar{\Lambda} = A^* + \sqrt{2}\bar{\theta}\bar{\psi} + \theta^2 F^* - i\theta\sigma^m\bar{\theta}\partial_m A^* + \frac{i}{\sqrt{2}}\bar{\theta}^2\theta\sigma^m\partial_m\bar{\psi} + \frac{1}{2}\theta^2\bar{\theta}^2\Box A^*. \tag{3.8}$$

The gauge transformation is then

$$\begin{aligned}\delta_{\text{gauge}}V = A + A^* + \sqrt{2}\theta\psi + \sqrt{2}\bar{\theta}\bar{\psi} + \theta^2 F + \bar{\theta}^2 F^* +\\ \theta\sigma^m\bar{\theta}\left(i\partial_m\left(A - A^*\right) - \frac{1}{2}\left[v_m, A + A^*\right]\right)\\ +\frac{i}{\sqrt{2}}\theta^2\left(\bar{\theta}\bar{\sigma}^m\partial_m\psi + \frac{1}{2}\left[v_m, \psi\sigma^m\bar{\theta}\right]\right) + \frac{i}{\sqrt{2}}\bar{\theta}^2\left(\theta\sigma^m\partial_m\bar{\psi} + \frac{1}{2}\left[v_m, \bar{\psi}\bar{\sigma}^m\theta\right]\right) +\\ \frac{1}{2}\theta^2\bar{\theta}^2\left(\frac{1}{2}\Box\left(A + A^*\right) - \frac{i}{2}\left[v_m, \partial^m\left(A + A^*\right)\right] +\right.\\ \left. v^2\left(A + A^*\right) - 2v^m\left(A + A^*\right)v_m + \left(A + A^*\right)v^2\right).\end{aligned} \tag{3.9}$$

On the other hand, the supersymmetry transformation is

$$(\epsilon q)V = -\epsilon\sigma^m\bar{\theta}v_m + \epsilon\theta\bar{\theta}^2 D + \frac{i}{2}\epsilon\sigma^m\bar{\sigma}^n\theta\bar{\theta}^2\partial_m v_n \tag{3.10}$$

Plugging eqs. 3.9 and 3.10 into eq. 3.4 and splitting it in components, we find the conditions:

$$A + A^* = \psi^\alpha = F = F^* = \partial_m(A - A^*) = 0, \tag{3.11}$$

$$\bar{\psi}_{\dot{\alpha}} = \frac{1}{\sqrt{2}} (\epsilon\sigma^m)_{\dot{\alpha}} v_m, \tag{3.12}$$

$$i\,(\sigma^{mn})_\beta^{\ \alpha} F_{mn} = \delta_\beta^\alpha D \tag{3.13}$$

where σ^{mn} is defined in appendix A. Since σ^{mn} is traceless, we conclude that

$$D = 0 \tag{3.14}$$

and

$$(\sigma^{mn})_\beta^{\ \alpha} F_{mn} = 0, \tag{3.15}$$

which tells us that the gauge field must be anti-self-dual.

3.2 String Field Theory Instantons

We now want to generalize the Yang-Mills instanton solutions to SFT. One approach to this problem was developed in [34, 35, 36, 37] by studying marginal deformations corresponding to the blow-up of a zero-size D(-1) brane on a background of D3-branes, and calculating the massless stringy corrections to the instanton profile. Here we will start with the super-Yang-Mills instanton profile and calculate the first stringy corrections using supersymmetry and the equations of motion. We will see that these two methods of computing stringy corrections correspond to opposite limits of small and large instantons.

In principle, instantons in SFT can be found using the methods of the previous section and are defined to be field configurations whose superymmetry variation is pure gauge:

$$(\epsilon q)V = -\delta_{\text{gauge}} V. \tag{3.16}$$

However, unlike the super-Yang-Mills case, there is no gauge choice where $V^3 = 0$. So the gauge transformation is now an infinite series in V:

$$\delta_{\text{gauge}} V = \Lambda + \bar{\Lambda} + \frac{1}{2}[V, \Lambda - \bar{\Lambda}] + \frac{1}{12}[V, [V, \Lambda + \bar{\Lambda}]] + \mathcal{O}(V^3), \tag{3.17}$$

where the products are to be interpreted as star products. The gauge parameters

Λ and $\bar{\Lambda}$ in eq. 3.17 have to satisfy

$$\tilde{G}^{+}\Lambda = 0 \tag{3.18}$$

and

$$G^{+}\bar{\Lambda} = 0 \tag{3.19}$$

Although we will be unable to analytically solve eq. 3.16 because of the infinite number of terms, one can solve eq. 3.16 perturbatively by expanding around the linearized massless solutions to V, Λ and $\bar{\Lambda}$. In the rest of this chapter, we will use this perturbative method to find the first stringy corrections to the super-Yang-Mills instanton solution.

3.3 Series Expansion of the Half-BPS Condition

To perform a perturbative expansion, first write the string field V as a power series in some parameter λ:

$$V = \lambda V_0 + \lambda^2 V_1 + \lambda^3 V_2 + \ldots. \tag{3.20}$$

Similarly, write

$$\Lambda = \lambda\Lambda_0 + \lambda^2\Lambda_1 + \lambda^3\Lambda_2 + \ldots \tag{3.21}$$

and

$$\bar{\Lambda} = \lambda\bar{\Lambda}_0 + \lambda^2\bar{\Lambda}_1 + \lambda^3\bar{\Lambda}_2 + \ldots. \tag{3.22}$$

We can then break up eq. 3.16 in powers of λ:

$$(\epsilon q)V_0 = \Lambda_0 + \bar{\Lambda}_0, \tag{3.23}$$

$$(\epsilon q)V_1 = \Lambda_1 + \bar{\Lambda}_1 + \frac{1}{2}[V_0, \Lambda_0 - \bar{\Lambda}_0], \tag{3.24}$$

and so on. The goal is to first determine V_0 and then recursively determine V_{i+1} from V_i.

Finding V_0 is relatively easy because eq. 3.23 is linear and thus the massless and massive states decouple. Actually, since we are working with off-shell string theory, the expressions "massless" and "massive" need some clarification. By "massless" we mean any state that depends only on the worldsheet zero modes of x^m, θ^α and $\bar{\theta}_{\dot{\alpha}}$ and not on their derivatives nor on p^α. That is, they are the states that would be massless if they were on-shell. All remaining states will be called "massive".

Let us choose a gauge such that

$$V_0 = -\theta\sigma^m\bar{\theta}v_{0m} + \frac{1}{2}\theta^2\bar{\theta}^2 D_0 + V_{0,\text{massive}}, \tag{3.25}$$

where $V_{0,\text{massive}}$ contains the massive components.

It is trivial to adapt the computations of section 3.1 to show that

$$D_0 = 0 \tag{3.26}$$

and

$$\sigma^{mn}(\partial_m v_{0n} - \partial_n v_{0m}) = 0. \tag{3.27}$$

Later we will describe an explicit solution of this condition for the case of a $U(2)$ gauge group.

What about the massive part? We will argue here that massive on-shell states cannot contribute to the instanton solution since they are not localized in spacetime. Note that in Euclidean space, the solution to the linearized equation of motion $(\delta^{mn}\partial_m\partial_n - M^2)\Phi = 0$ is

$$\Phi = \int d^4k f(k) e^{k\cdot x}\delta(|k| - M). \tag{3.28}$$

Suppose $f(\hat{k})$ is non-zero for some $\hat{k}^m$ satisifying $|\hat{k}| = M$. Then it is easy to see that $\Phi(x)$ diverges exponentially as e^{cM^2} in the direction $x^m = c\hat{k}^m$ for c large. So there are no localized solutions to $(\delta^{mn}\partial_m\partial_n - M^2)\Phi = 0$ in Euclidean space which can contribute to the instanton solution.

So the linearized condition of eq. 3.23 together with the requirement that the solution is localized in spacetime implies that

$$V_0 = -\theta\sigma^m\bar{\theta}v_{0m}, \tag{3.29}$$

where v_{0m} satisfies

$$\sigma^{mn}(\partial_m v_{0n} - \partial_n v_{0m}) = 0. \tag{3.30}$$

A specific choice for the compensating gauge transformation is

$$\Lambda_0 = 0, \tag{3.31}$$

and

$$\bar{\Lambda}_0 = -(\epsilon\sigma^m\bar{\theta})v_{0m} \tag{3.32}$$

Equipped with this, we can now try to tackle eq. 3.24. We will find that due to the complexity of the star product, $V_0 * \bar{\Lambda}$ contains massive components even though V_0 and $\bar{\Lambda}$ contain only massless components. So unlike V_0, V_1 will contain massive components.

3.4 BPST Instanton

To gain intuition, it will be useful to consider an explicit super-Yang-Mills instanton solution and see how it can be extended to string field theory. For an $SU(2)$ gauge group, the $k = 1$ instanton can be written in a simple form which is the BPST instanton [38]:

$$v^i_m = 2\frac{\bar{\eta}^i_{mn}(x - x_0)^n}{(x - x_0)^2 + \rho^2}. \tag{3.33}$$

x^m_0 and ρ are two fixed parameters representing the center and the size of the instanton, and $\bar{\eta}^i_{mn}$ are the 't Hooft symbols [39]:

$$\bar{\eta}_{imn} = \epsilon_{imn4} - \delta_{im}\delta_{n4} + \delta_{in}\delta_{m4} \tag{3.34}$$

where $i = 1, 2, 3$ is an $SU(2)$ index and $m, n = 1, 2, 3, 4$ are Euclidean spacetime indices. Note that the t'Hooft symbols are anti-self-dual in their Lorentz indices:

$$\bar{\eta}_{imn} = -\frac{1}{2}\epsilon_{mnpq}\bar{\eta}_{ipq}. \tag{3.35}$$

To show that eq. 3.33 does indeed represent an instanton, we have to show that the corresponding field strength,

$$F^i_{mn} = \partial_m v^i_n - \partial_n v^i_m + \epsilon^{ijk}v^j_m v^k_n, \tag{3.36}$$

is anti-self-dual. The derivative of v^i_m is

$$\partial_m v^i_n = 2\frac{\bar{\eta}_{inm}[(x - x_0)^2 + \rho^2] - 2\bar{\eta}_{inp}(x - x_0)^p(x - x_0)_m}{[(x - x_0)^2 + \rho^2]^2}, \tag{3.37}$$

whereas the quadratic part of the field strength is

$$\epsilon_{ijk} v^j_m v^i_n = 4 \frac{\epsilon_{ijk} \bar{\eta}_{jmp} \bar{\eta}_{knq} (x - x_0)^p (x - x_0)^q}{[(x - x_0)^2 + \rho^2]^2}. \tag{3.38}$$

Using the identity

$$\epsilon_{ijk} \bar{\eta}_{jmp} \bar{\eta}_{knq} = \delta_{mn} \bar{\eta}_{ipq} + \delta_{pq} \bar{\eta}_{imn} - \delta_{mq} \bar{\eta}_{ipn} - \delta_{pn} \bar{\eta}_{imq}, \tag{3.39}$$

we obtain

$$\epsilon_{ijk} v^j_m v^i_n = 4 \frac{\bar{\eta}_{imn} (x - x_0)^2 - \bar{\eta}_{ipn} (x - x_0)^p (x - x_0)_m - \bar{\eta}_{imp} (x - x_0)^p (x - x_0)_n}{[(x - x_0)^2 + \rho^2]^2}. \tag{3.40}$$

Then the field strength is

$$F^i_{mn} = -4 \frac{\bar{\eta}_{imn} \rho^2}{[(x - x_0)^2 + \rho^2]^2} \tag{3.41}$$

and the anti-self-duality of the field strength follows from the anti-self-duality of $\bar{\eta}_{imn}$.

To connect this with what we were doing in the previous section, let us expand v^i_m in powers of $1/\rho^2$:

$$v^i_m = \sum_{r=0}^{\infty} \frac{v^i_{rm}}{\rho^{2(r+1)}}, \tag{3.42}$$

where

$$v^i_{rm} = 2(-1)^r \bar{\eta}^i_{mn} (x - x_0)^n (x - x_0)^{2r}. \tag{3.43}$$

That is, we are working with a large instanton expansion which has the form of 3.20 where the expansion parameter is $\lambda = \frac{1}{\rho^2}$. This is different from [34][35][36][37] where they expand the solution in ρ corresponding to a small instanton expansion.

Now we would like to find the corrections to this instanton solution in superstring field theory - that is, a solution to eq. 3.4 where we take the massless part of the string field to have the form of the BPST instanton. Note that the gauge group should now be $U(2)$ rather than $SU(2)$. Since the star product in non-commutative, the massive levels of the string field will have singlet contributions, even if the massless level doesn't. We formally define the object v^i_m as

$$v^i_m = 2 \frac{\bar{\eta}^i_{mn} (x - x_0)^n}{(x - x_0)^2 + \rho^2}, \tag{3.44}$$

where the products are to be interpreted as $\otimes$ products, and $\otimes$ denotes the truncation of the star product to the massless level (see appendix B). To give meaning to division by x^m, we invoke the expansion in $1/\rho^2$ of eq. 3.42 where

$$v^i_{rm} = 2(-1)^r \bar{\eta}^i_{mn} (x - x_0)^n \otimes \left((x - x_0)^2\right)^{r_\otimes} \tag{3.45}$$

and $r_\otimes$ in the exponent means we multiply r times with the $\otimes$ product. Also, x^2 means the normal ordered product - that is, no contractions between the x's.

As we showed in the last section, the massive part of V_0 is trivial. Thus we can take V_0 to be the linear part of the BPST instanton (we will fix the position to be at the origin):

$$V^0_0 = 0, \quad V^i_0 = -2\theta\sigma^m\bar{\theta}\bar{\eta}^i_{mn}x^n \tag{3.46}$$

where V^0_0 denotes the $U(1)$ component of the $U(2)$ gauge group. We can also take the massless part of V_1 to be

$$V^i_{1,massless} = -\theta\sigma^m\bar{\theta}\bar{v}^i_{1m} = 2\theta\sigma^m\bar{\theta}\bar{\eta}^i_{mn}x^n \otimes x^2. \tag{3.47}$$

but this will have massive corrections.[1] Both the equations of motion and the half-BPS condition for the massless fields reduce, up to this order, to their form in super-Yang-Mills. Thus, it is immediate that 3.46, 3.47 are solutions. At higher orders, since the star product of massive fields can have massless components, we expect there will be both massive and massless corrections. That is, $V^i_{r,massless} = -\theta\sigma^m\bar{\theta}\bar{v}^i_{rm}$ is not a solution to the equations for $r \geq 2$.

In the next section we'll compute the first massive level of V_1.

3.5 The First Stringy Correction

We now want to compute the first stringy correction to the BPST instanton, which is the first massive level of V_1. To that end, we must solve the half-BPS equation at the quadratic level (i.e. $O(1/\rho^4)$ terms) which corresponds to eq. 3.24. Although V_0, Λ_0 and $\bar{\Lambda}_0$ are massless, the star products in the commutator will contain massive terms which will imply that V_1 has a nontrivial massive part.

In addition to the expansion 3.20, we will also use level truncation. We define the $n-$th mass level of the string field as the component with conformal weight

[1] Note that the massless contribution to V_1 is ambiguous, since one can always shift V_1 by any linearized massless solution. One could try to remove this ambiguity by fixing the properties of the half-BPS solution, but we will not attempt this here.

n at zero momentum. Note that each term in the perturbative expansion has, in principle, components at all mass levels. Here we truncate to the first level.

The first massive contribution to the half-BPS solution is not unique - if V_1 is a solution to eq. 3.24, then $V_1' = V_1 + q^2\Omega$ is also a solution for any field Ω since $(\epsilon q)q^2\Omega = 0$. We can ask if there is a half-BPS solution which also satisfies the SFT equations of motion.[2] The answer is yes, and we will show that the first massive contribution to this solution is unique if we also require that it vanishes at infinity, i.e. it is localized in spacetime like the Yang-Mills instanton. In other words, $q^2\Omega$ will be uniquely determined if we require that V_1' satisfies the SFT equations of motion and is localized in spacetime. This first stringy correction to the BPST instanton will be interpreted as turning on some off-shell massive fields. To see what kinds of fields these are, we will compare our result with the first massive states of the superstring which on-shell consists of a massive spin-two multiplet and two massive scalar multiplets [40, 41]. We will find that our solution corresponds to turning on a spin-two and a scalar field.

In subsection 3.5.1 we find the general solution to eq. 3.24 when V_0 is given by the linearized BPST solution. In subsection 3.5.2 we find the unique first massive contribution to the half-BPS solution that also satisfies the equations of motion and vanishes at infinity. And in subsection 3.5.3 we compare our solution with the vertex operator for the first massive level.

3.5.1 Half-BPS solutions

In this section we will solve the equation

$$(\epsilon q)V_1 = \Lambda_1 + \bar{\Lambda}_1 + \frac{1}{2}\left[V_0, \Lambda_0 - \bar{\Lambda}_0\right] \tag{3.48}$$

given the linearized BPST solution for V_0

$$V_0^i = -\theta\sigma^m\bar{\theta}v_{0m}^i(x) = -2\theta\sigma^m\bar{\theta}\bar{\eta}^i_{mn}x^n \tag{3.49}$$

$$\bar{\Lambda}_0^i = -2\epsilon\sigma^m\bar{\theta}\bar{\eta}^i_{mn}x^n \tag{3.50}$$

$$\Lambda_0^i = 0 \tag{3.51}$$

[2] Note that the half-BPS solution is required to be annihilated by half of the N=1 d=4 supersymmetries, i.e. two supersymmetries. If one required that the half-BPS solution was annihilated by half of the N=4 d=4 supersymmetries, i.e. 8 supersymmetries, one expects that it would automatically satisfy the SFT equations of motion. We would like to thank discussions with Carlo Maccaferri and Jakub Vosmera on this point.

As noted before, we can add any q^2-exact term to a solution V_1 and still have a solution. For the massive part $V_{1,massive}$, we'll show that the solution for the first massive level is unique up to a q^2-exact term, and up to gauge transformations. If, in addition to eq. 3.48, we require that the solution satisfies the equations of motion and vanishes at infinity, we will find that the q^2-exact term is fixed uniquely. From now on, we'll use V_1 instead of $V_{1,massive}$ to denote the first massive level since we'll only deal with this level in this section. Similarly, Λ_1 and $\bar{\Lambda}_1$ will denote the first massive level of the gauge parameters.

First we have to calculate the commutator on the right-hand side of eq. 3.48, which involves evaluating a star product. Then we substitute the general form of the string field and gauge parameters at the first massive level. This is given by the fields that have conformal weight 1 at zero momentum, and which have mass squared $M^2 = 2$ when on-shell.

The commutator will contain terms that are not ϵq-exact, which must therefore be cancelled by the gauge terms. This will fix Λ_1 and $\bar{\Lambda}_1$, up to ϵq-exact gauge parameters (that is, up to $\epsilon q(\Lambda_1' + \bar{\Lambda}_1')$ where Λ_1' and $\bar{\Lambda}_1'$ are gauge parameters). Once the gauge parameters are fixed, we have determined $\epsilon q V_1$, which means we have determined V_1 up to a q^2-exact term.

At the first massive level, the commutator $[V_0, \bar{\Lambda}_0]$ satisfies

$$\frac{1}{2}[V_0, \bar{\Lambda}_0] = \frac{1}{2} V_0^i * \bar{\Lambda}_0^j \{\sigma^i, \sigma^j\} = V_0^i * \bar{\Lambda}_0^i \sigma^0 \tag{3.52}$$

where σ^0 is i times the 2×2 identity. This is because $(V_0^i * \bar{\Lambda}_0^j) = -(\bar{\Lambda}_0^j * V_0^i)$ at this level. Details on the calculation of the star product can be found in appendix B, and one finds at the first massive level that

$$V_0^i * \bar{\Lambda}_0^i = \partial\theta^\alpha V_\alpha + \partial\bar{\theta}^{\dot{\alpha}} \bar{V}_{\dot{\alpha}} + \Pi^m U_m \tag{3.53}$$

where

$$\Pi^m = \partial x^m - \frac{i}{2}\theta\sigma^m\partial\bar{\theta} - \frac{i}{2}\bar{\theta}\bar{\sigma}^m\partial\theta \tag{3.54}$$

and

$$V_\alpha = -\frac{2}{3\sqrt{3}}(\sigma^m\bar{\theta})_\alpha \epsilon\sigma^n\bar{\theta} v^i_{0m} \otimes v^i_{0n} \tag{3.55}$$

$$\bar{V}_{\dot{\alpha}} = \frac{2}{3\sqrt{3}}\left((\theta\sigma^m)_{\dot{\alpha}}\epsilon\sigma^n\bar{\theta} - \theta\sigma^m\bar{\theta}(\epsilon\sigma^n)_{\dot{\alpha}}\right) v^i_{0m} \otimes v^i_{0n}(y) \tag{3.56}$$

$$U_m = \frac{4}{3\sqrt{3}}\theta\sigma^p\bar{\theta}\epsilon\sigma^q\bar{\theta}\left(\bar{\eta}^i_{pm} v^i_{0q} - v^i_{0p}\bar{\eta}^i_{qm}\right) \tag{3.57}$$

where $y^m = x^m + \frac{i}{2}\theta\sigma^m\bar{\theta}$.

The general form of the string field at the first massive level is [40]

$$\begin{aligned} V_1 =& d^\alpha W_\alpha(x,\theta,\bar{\theta}) + \bar{d}^{\dot{\alpha}}\bar{W}_{\dot{\alpha}}(x,\theta,\bar{\theta}) + \Pi^m W_m(x,\theta,\bar{\theta}) + \partial\theta^\alpha V_\alpha(x,\theta,\bar{\theta}) \\ &+ \partial\bar{\theta}^{\dot{\alpha}}\bar{V}_{\dot{\alpha}}(x,\theta,\bar{\theta}) + i\left(\partial\rho - \partial H_C\right) B(x,\theta,\bar{\theta}) + \left(\partial H_C - 3\partial\rho\right) C(x,\theta,\bar{\theta}) \end{aligned} \tag{3.58}$$

and the gauge transformations are

$$\begin{aligned} \Lambda_1 =& G^+\left(e^{-i\rho}\left(\bar{d}^{\dot{\alpha}}\bar{E}_{\dot{\alpha}}(x,\theta,\bar{\theta}) + \partial\theta^\alpha B_\alpha(x,\theta,\bar{\theta})\right)\right) \\ =& -2i\Pi_{\alpha\dot{\alpha}}D^\alpha\bar{E}^{\dot{\alpha}} + \bar{d}^{\dot{\alpha}}D^2\bar{E}_{\dot{\alpha}} + d^\alpha\left(2B_\alpha - i\partial_{\alpha\dot{\alpha}}\bar{E}^{\dot{\alpha}}\right) + \partial\theta^\alpha D^2 B_\alpha \\ &+ \partial\bar{\theta}^{\dot{\alpha}}\left(-i\partial_{\alpha\dot{\alpha}}B^\alpha + 4\bar{E}_{\dot{\alpha}} - \frac{1}{2}\partial^2\bar{E}_{\dot{\alpha}}\right) + i\partial\rho\left(2D^\alpha B_\alpha - i\partial_{\alpha\dot{\alpha}}D^\alpha\bar{E}_{\dot{\alpha}}\right) \\ \bar{\Lambda}_1 =& \tilde{G}^+\left(e^{2i\rho - iH_C}\left(d^\alpha E_\alpha(x,\theta,\bar{\theta}) + \partial\bar{\theta}^{\bar{\alpha}}\bar{B}_{\dot{\alpha}}(x,\theta,\bar{\theta})\right)\right) \\ =& -2i\Pi_{\alpha\dot{\alpha}}\bar{D}^{\dot{\alpha}}E^\alpha + d^\alpha\bar{D}^2E_\alpha + \bar{d}^{\dot{\alpha}}\left(2\bar{B}_{\dot{\alpha}} - i\partial_{\alpha\dot{\alpha}}E^\alpha\right) + \partial\bar{\theta}^{\bar{\alpha}}\bar{D}^2\bar{B}_{\dot{\alpha}} \\ &+ \partial\theta^\alpha\left(-i\partial_{\alpha\dot{\alpha}}\bar{B}^{\dot{\alpha}} + 4E_\alpha - \frac{1}{2}\partial^2E_\alpha\right) + i\partial\left(H_C - 2\rho\right)\left(2\bar{D}^{\dot{\alpha}}\bar{B}_{\dot{\alpha}} - i\partial_{\alpha\dot{\alpha}}\bar{D}^{\dot{\alpha}}E_\alpha\right) \end{aligned} \tag{3.59}$$

where D and $\bar{D}$ are defined in appendix A. We can use the gauge freedom to fix V_1 in the form

$$V_1 = \Pi^m W_m(x,\theta,\bar{\theta}) + i\left(\partial\rho - \partial H_C\right) B(x,\theta,\bar{\theta}) + \left(\partial H_C - 3\partial\rho\right) C(x,\theta,\bar{\theta}) \tag{3.60}$$

Note that there are no terms proportional to d or $\bar{d}$ in either V_1 or in the commutator. Fixing the d and $\bar{d}$ terms to zero in 3.59, the gauge parameters take the form

$$\Lambda_1 = -\frac{1}{2} i\Pi_{\alpha\dot{\alpha}} D^{\alpha} \bar{E}^{\dot{\alpha}} + \frac{1}{4} \partial\theta^{\alpha} i\partial_{\alpha\dot{\alpha}} D^2 \bar{E}^{\dot{\alpha}} + \partial\bar{\theta}^{\dot{\alpha}} \left(\bar{E}_{\dot{\alpha}} - \frac{1}{8} \bar{D}^2 D^2 \bar{E}_{\dot{\alpha}} \right) \tag{3.61}$$

$$-\frac{i}{4} \partial\rho D^{\alpha} \bar{D}^2 E_{\alpha} \tag{3.62}$$

$$\bar{\Lambda}_1 = -\frac{1}{2} i\Pi_{\alpha\dot{\alpha}} \bar{D}^{\dot{\alpha}} E^{\alpha} + \frac{1}{4} \partial\bar{\theta}^{\dot{\alpha}} i\partial_{\alpha\dot{\alpha}} \bar{D}^2 E^{\alpha} + \partial\theta^{\alpha} \left(E_{\alpha} - \frac{1}{8} D^2 \bar{D}^2 E_{\alpha} \right) \tag{3.63}$$

$$-\frac{i}{4} \partial(H_C - 2\rho) \bar{D}^{\dot{\alpha}} D^2 \bar{E}_{\dot{\alpha}}$$

where we have rescaled E_{α} and $\bar{E}_{\dot{\alpha}}$ by a factor of four for later convenience.

Equation 3.48 can be decomposed into five equations by separating the terms proportional to $\partial\theta^{\alpha}$, $\partial\bar{\theta}^{\dot{\alpha}}$, Π^m, $\partial\rho$ and ∂H_C. The first two will fix E_{α} and $\bar{E}_{\dot{\alpha}}$, up to ϵq-exact terms. The other equations, in turn, will determine V_1.

The terms proportional to $\partial\theta^{\alpha}$ and $\partial\bar{\theta}^{\dot{\alpha}}$ give

$$V_{\alpha} = E_{\alpha} - \frac{1}{8} D^2 \bar{D}^2 E_{\alpha} + \frac{i}{4} \partial_{\alpha\dot{\alpha}} D^2 \bar{E}^{\dot{\alpha}} \tag{3.64}$$

$$\bar{V}_{\dot{\alpha}} = \bar{E}_{\dot{\alpha}} - \frac{1}{8} \bar{D}^2 D^2 \bar{E}_{\dot{\alpha}} + \frac{i}{4} \partial_{\alpha\dot{\alpha}} \bar{D}^2 E^{\alpha} \tag{3.65}$$

which are solved by

$$E_{\alpha} = V_{\alpha} + \frac{1}{8} D^2 \bar{D}^2 V_{\alpha} - \frac{i}{4} \partial_{\alpha\dot{\alpha}} D^2 \bar{V}^{\dot{\alpha}} + E'_{\alpha} \tag{3.66}$$

$$\bar{E}_{\dot{\alpha}} = \bar{V}_{\dot{\alpha}} + \frac{1}{8} \bar{D}^2 D^2 \bar{V}_{\dot{\alpha}} - \frac{i}{4} \partial_{\alpha\dot{\alpha}} \bar{D}^2 V^{\alpha} + \bar{E}'_{\dot{\alpha}} \tag{3.67}$$

where E'_{α} and $\bar{E}'{}_{\dot{\alpha}}$ are solutions to the homogeneous equations, i.e. they satisfy

$$E'_{\alpha} = -\frac{i}{2} \partial_{\alpha\dot{\alpha}} D^2 \bar{E}'^{\dot{\alpha}}, \quad \bar{E}'_{\dot{\alpha}} = -\frac{i}{2} \partial_{\alpha\dot{\alpha}} \bar{D}^2 E'^{\alpha} \tag{3.68}$$

$$\left(\partial^n \partial_n + 1\right) E'_{\alpha} = \left(\partial^n \partial_n + 1\right) \bar{E}'_{\dot{\alpha}} = 0. \tag{3.69}$$

As we'll see shortly, E'_{α} and $\bar{E}'{}_{\dot{\alpha}}$ are also ϵq-exact.

The terms proportional to Π^m give the equation

$$\epsilon q W_m + U_m = -\frac{i}{2} \left(\bar{\sigma}_m\right)^{\alpha\dot{\alpha}} D_{\alpha} \bar{E}_{\dot{\alpha}} - \frac{i}{2} \left(\bar{\sigma}_m\right)^{\alpha\dot{\alpha}} \bar{D}_{\dot{\alpha}} E_{\alpha} \tag{3.70}$$

Note that

$$\begin{aligned}
\bar{\sigma}_m^{\dot{\alpha}\alpha} D_\alpha \bar{V}_{\dot{\alpha}} =& \frac{2}{3\sqrt{3}} \left[\left(-2\epsilon\sigma^q\bar{\theta}\delta_m^p + \epsilon\sigma^q\bar{\sigma}_m\sigma^p\bar{\theta}\right) v_{0p}^i \otimes v_{0q}^i(y) \right. \\
& \left. -2i\theta\sigma^p\bar{\sigma}_m\sigma^n\bar{\theta}\epsilon\sigma^q\bar{\theta}\bar{\eta}_{pn}^i v_{0q}^i + 2i\theta\sigma^p\bar{\theta}\epsilon\sigma^q\bar{\sigma}_m\sigma^n v_{0p}^i\bar{\eta}_{qn}^i \right] \\
=& \frac{2}{3\sqrt{3}} \left[(-4\epsilon\sigma^q\bar{\theta}\delta^{mp} + \epsilon\sigma_m\bar{\theta}\delta^{pq}) v_{0p}^i \otimes v_{0q}^i(y) \right. \\
& \left. +4i\theta\sigma^p\bar{\theta}\epsilon\sigma^q\bar{\theta}\left(\bar{\eta}_{pm}^i v_{0q}^i - v_{0p}^i\bar{\eta}_{qm}^i\right)\right] \\
=& \frac{2}{3\sqrt{3}} (-4\epsilon\sigma^q\bar{\theta}\delta^{mp} + \epsilon\sigma_m\bar{\theta}\delta^{pq}) v_{0p}^i \otimes v_{0q}^i(y) + 2iU_m
\end{aligned} \tag{3.71}$$

In the second equality we used the fact that $\sigma^{mn}\bar{\eta}_{mn}^i = 0$ and $\left(\bar{\sigma}^{(m}\sigma^{n)}\right)^{\dot{\alpha}}_{\dot{\beta}} = -2\delta^{mn}\delta_{\dot{\alpha}}^{\dot{\beta}}$. So eq. 3.70 becomes

$$\begin{aligned}
\epsilon q W_m =& \frac{i}{3\sqrt{3}} (4\epsilon\sigma^q\bar{\theta}\delta^{mp} - \epsilon\sigma_m\bar{\theta}\delta^{pq}) v_{0p}^i \otimes v_{0q}^i(y) \\
& - \frac{i}{2}\bar{\sigma}_m^{\alpha\dot{\alpha}} D_\alpha \left(\frac{1}{8}\bar{D}^2 D^2 \bar{V}_{\dot{\alpha}} - \frac{i}{4}\partial_{\beta\dot{\alpha}}\bar{D}^2 V^\beta + \bar{E}'_{\dot{\alpha}}\right) - \frac{i}{2}\left(\bar{\sigma}_m\right)^{\alpha\dot{\alpha}} \bar{D}_{\dot{\alpha}} E_\alpha
\end{aligned} \tag{3.72}$$

Finally, the equations for C and B are

$$\epsilon q(C + iB) = \frac{i}{4} D^\alpha \bar{D}^2 E_\alpha \tag{3.73}$$

$$\epsilon q(C - iB) = -\frac{i}{4} \bar{D}^{\dot{\alpha}} D^2 \bar{E}_{\dot{\alpha}} \tag{3.74}$$

In order for equations 3.72-3.74 to be consistent, we must verify that the right-hand sides are ϵq-exact. Note that any field of the form $\epsilon\psi(y,\bar{\theta})$ is ϵq-exact: $\epsilon\psi(y,\bar{\theta}) = \epsilon q(\theta\psi(y,\bar{\theta}))$. It's then easy to verify that V_α and $D^2\bar{V}_{\dot{\alpha}}$ are ϵq-exact, as are any derivatives of these quantities, since q anti-commutes with D and $\bar{D}$. Thus, the equations are consistent as long as we require that $D_\alpha\bar{E}'_{\dot{\alpha}} + \bar{D}_{\dot{\alpha}}E'_\alpha$, $D^\alpha\bar{D}^2E'_\alpha$ and $\bar{D}^{\dot{\alpha}}D^2\bar{E}'_{\dot{\alpha}}$ are ϵq-exact.

One might suspect that the E'_α and $\bar{E}'_{\dot{\alpha}}$ terms simply give a gauge transformation of V_1. This is indeed the case, as we'll now show. First, note that requiring $D^\alpha\bar{D}^2E'_\alpha = \epsilon q(F)$ implies, by the first eq. in 3.68, that $\bar{E}'_{\dot{\alpha}} = -\frac{1}{2}\epsilon q(\bar{D}_{\dot{\alpha}}F) = \epsilon q(\bar{E}''_{\dot{\alpha}})$. Similarly, $\bar{D}^{\dot{\alpha}}D^2\bar{E}'_{\dot{\alpha}} = \epsilon q(\bar{F})$ implies $E'_\alpha = \epsilon q(E''_\alpha)$. Note that E''_α and $\bar{E}''_{\dot{\alpha}}$ are only defined up to $q^2(something)$. Also, they satisfy eqs. 3.68 and 3.69 up to $q^2(something)$ - in other words, we can choose E''_α and $\bar{E}''_{\dot{\alpha}}$ such that they satisfy those conditions. That means they only change Λ_1 and $\bar{\Lambda}_1$ by ϵq-exact gauge parameters, which in turn means that they only contribute pure gauge terms to V_1.

Thus, from eqs. 3.72-3.74, we have uniquely determined $\epsilon q(V_1)$ up to a gauge transformation. This means we also have determined V_1, up to a gauge transformation and up to $q^2(something)$. The particular solution found by simply eliminating ϵq in eqs. 3.72-3.74 (after writing the right-hand sides as $\epsilon q(something)$ in the way described in the previous paragraphs) is

$$\begin{aligned}
\tilde{V}_1^0 =& \frac{2i}{3\sqrt{3}} \Pi^m \left[(2\theta\sigma^q\bar{\theta}\delta_m^p - \theta\sigma_m\bar{\theta}\delta^{pq}) v_{0p}^i \otimes v_{0q}^i(y) - 72\theta\sigma_m\bar{\theta} \right] \\
&- \frac{8}{\sqrt{3}} \partial(5\rho - 3H_C)\theta\sigma^m\bar{\theta}y_m \\
V_1^i =& 0
\end{aligned} \tag{3.75}$$

where the x products, again, are normal ordered, i.e. no contractions.

3.5.2 Equations of motion

We now want to verify that we can find a half-BPS solution which satisfies the equation of motion, given, to quadratic order, by

$$\tilde{G}^+ G^+ V_1 = \frac{1}{2} \{ G^+ V_0, \tilde{G}^+ V_0 \} \tag{3.76}$$

Although we will only solve for the first massive level, we can show that the equation of motion can be solved for all massive levels. There is a solution if the right-hand side is $\tilde{G}^+ G^+$-exact. Note that it is annihilated by both $\tilde{G}^+$ and G^+. In other words, there is an obstruction to solving the equation if and only if the right hand side is a non-trivial state in the cohomology of $\tilde{G}^+ G^+$. This is equivalent to asking whether there is an on-shell massive state that can decay into two self-dual massless states. But V_0 is not an eigenstate of momentum and is a polynomial in x. In momentum space, it is therefore a linear combination of delta functions and derivatives of delta functions which vanishes at non-zero momentum. So there can be no such on-shell massive state and there is no obstruction to solving the equation of motion for V_1 at any mass level.

The solution $\tilde{V}_1$ written above does not satisfy the equation of motion, but we can find a field V_1 that does, and which differs from $\tilde{V}_1$ by a q^2-exact term. This solution V_1 is unique if we also require that it vanishes at infinity.

Calculating the anticommutator between $G^+ V_0$ and $\tilde{G}^+ V_0$ again involves evaluating a star product:

$$\frac{1}{2}\left\{G^+V_0,\tilde{G}^+V_0\right\} = \frac{1}{2}(G^+V_0^i * \tilde{G}^+V_0^j)\{\sigma^i,\sigma^j\} = (G^+V_0^i * \tilde{G}^+V_0^i)\sigma^0 \tag{3.77}$$

where

$$\begin{aligned} G^+V_0 &= e^{i\rho}(2d^\alpha D_\alpha - i\partial\bar{\theta}^{\dot{\alpha}}\partial_{\alpha\dot{\alpha}}D^\alpha)V_0, \\ \tilde{G}^+V_0 &= e^{-2i\rho+iH_C}2\bar{d}^{\dot{\alpha}}\bar{D}_{\dot{\alpha}}V_0, \end{aligned} \tag{3.78}$$

and we have used that $\partial_{\alpha\dot{\alpha}}\bar{D}^{\dot{\alpha}}V_0 = 0$.

As before, let's separate the terms proportional to Π^m, d_α, $\bar{d}_{\dot{\alpha}}$, $\partial\rho$ and ∂H_C. Note that acting with G^+ or $\tilde{G}^+$ on the equation gives zero, so not all terms are independent. For example, we don't need to check the terms proportional to $\partial\theta^\alpha$ and $\partial\bar{\theta}^{\dot{\alpha}}$. Details on the star product can again be found in appendix B, and on the right-hand side of 3.76 we find (times an overall factor of $e^{-i\rho+iH_C}$):
Terms proportional to Π^m:

$$\begin{aligned} &-\frac{16i}{3\sqrt{3}}\bar{\sigma}_m^{\dot{\alpha}\alpha}D_\alpha V_0^i\otimes\bar{D}_{\dot{\alpha}}V_0^i - \frac{8}{3\sqrt{3}}\Big(\partial_m\bar{D}^{\dot{\alpha}}D_\alpha V_0^i\otimes D^\alpha\bar{D}_{\dot{\alpha}}V_0^i \\ &-\bar{D}^{\dot{\alpha}}D_\alpha V_0^i\otimes\partial_m D^\alpha\bar{D}_{\dot{\alpha}}V_0^i + i\partial_{\alpha\dot{\alpha}}D^\alpha V_0^i\otimes\partial_m\bar{D}^{\dot{\alpha}}V_0^i\Big) \\ &= \frac{16i}{3\sqrt{3}}\left(2\theta\sigma^q\bar{\theta}\delta_m^p - \theta\sigma_m\bar{\theta}\delta^{pq}\right)v_{0p}^i\otimes v_{0q}^i(y) + \frac{192i}{\sqrt{3}}\theta\sigma_m\bar{\theta} \end{aligned} \tag{3.79}$$

Terms proportional to d^α:

$$-\frac{16}{3\sqrt{3}}\bar{D}_{\dot{\alpha}}D_\alpha V_0^i\otimes\bar{D}^{\dot{\alpha}}V_0^i = -\frac{16}{3\sqrt{3}}\theta_\alpha v_{0p}^i\otimes v_0^{ip}(y) \tag{3.80}$$

Terms proportional to $\bar{d}^{\dot{\alpha}}$:

$$-\frac{16}{3\sqrt{3}}D_\alpha\bar{D}_{\dot{\alpha}}V_0^i\otimes D^\alpha V_0^i = -\frac{16}{3\sqrt{3}}\left(-\bar{\theta}_{\dot{\alpha}}v_{0p}^i\otimes v_0^{ip}(y) + 2i\bar{\theta}^2(\theta\sigma^m)_{\dot{\alpha}}\bar{\eta}_{mn}^i v^{in}(y)\right) \tag{3.81}$$

Terms proportional to $\partial(H_C - 3\rho)$:

$$\begin{aligned} &\frac{8i}{3\sqrt{3}}\left(\bar{D}^{\dot{\alpha}}D_\alpha V_0^i\otimes D^\alpha\bar{D}_{\dot{\alpha}}V_0^i + i\partial_{\alpha\dot{\alpha}}D^\alpha V_0^i\otimes\bar{D}^{\dot{\alpha}}V_0^i\right) \\ &= -\frac{16i}{3\sqrt{3}}v_{0p}^i\otimes v_0^{ip}(y) - \frac{64}{\sqrt{3}}\theta\sigma^m\bar{\theta}y_m \end{aligned} \tag{3.82}$$

And on the left-hand side of 3.76 we find:
Terms proportional to Π^m:

$$-\left\{\bar{D}^2, D^2\right\} W_m + 8W_m - 4\partial^n\partial_n W_m - 8\partial_m B - 12\,(\bar{\sigma}_m)^{\dot{\alpha}\alpha}\,[\bar{D}_{\dot{\alpha}}, D_\alpha]\,C$$
$$= \frac{16i}{3\sqrt{3}}\left(2\theta\sigma^q\bar{\theta}\delta^p_m - \theta\sigma_m\bar{\theta}\delta^{pq}\right) v^i_{0p}\otimes v^i_{0q}(y) + \frac{192i}{\sqrt{3}}\theta\sigma_m\bar{\theta} + \frac{128}{\sqrt{3}}y_m \tag{3.83}$$

Terms proportional to d^α:

$$4i\,(\sigma^m)_{\alpha\dot{\alpha}}\,\bar{D}^{\dot{\alpha}}W_m - 24\partial_{\alpha\dot{\alpha}}\bar{D}^{\dot{\alpha}}C + D_\alpha\bar{D}^2(18iC + 2B)$$
$$= -\frac{16}{3\sqrt{3}}\theta_\alpha v^i_{0p}\otimes v^{ip}_0(y) \tag{3.84}$$

Terms proportional to $\bar{d}^{\dot{\alpha}}$:

$$4i\,(\sigma^m)_{\alpha\dot{\alpha}}\,D^\alpha W_m + 24\partial_{\alpha\dot{\alpha}}D^\alpha C - \bar{D}_{\dot{\alpha}}D^2(18iC - 2B)$$
$$= -\frac{16}{3\sqrt{3}}\left(-\bar{\theta}_{\dot{\alpha}}v^i_{0p}\otimes v^{ip}_0(y) + 2\bar{\theta}^2(\theta\sigma^m)_{\dot{\alpha}}\bar{\eta}^i_{mn}v^{in}(y)\right) - \frac{256}{\sqrt{3}}\bar{\theta}_{\dot{\alpha}} \tag{3.85}$$

Terms proportional to $i\partial(\rho - H_C)$:

$$-4\partial^m W_m - 8B - \frac{1}{2}\left\{\bar{D}^2, D^2\right\}B + 3\partial_{\alpha\dot{\alpha}}\left[\bar{D}^{\dot{\alpha}}, D^\alpha\right]C$$
$$= -\frac{128}{\sqrt{3}} \tag{3.86}$$

Terms proportional to $\partial(H_C - 3\rho)$:

$$-2\bar{\sigma}^{\dot{\alpha}\alpha}_m\,[\bar{D}_{\dot{\alpha}}, D_\alpha]\,W^m - 8C + \frac{11}{2}\left\{\bar{D}^2, D^2\right\}C + 16\partial_m\partial^m C + \partial_{\alpha\dot{\alpha}}\left[\bar{D}^{\dot{\alpha}}, D^\alpha\right]B$$
$$= -\frac{32i}{3\sqrt{3}}v^i_{0p}\otimes v^{ip}_0(y) - \frac{64}{\sqrt{3}}\theta\sigma^m\bar{\theta}y_m + \frac{128i}{\sqrt{3}} \tag{3.87}$$

We conclude that $\tilde{V}_1$ does not satisfy the equations of motion. However, the following V_1 can be defined which differs from $\tilde{V}_1$ by a q^2-exact term and which satisfies the equations of motion:

$$\begin{aligned} V_1^0 =&\tilde{V}_1^0 - \frac{64}{\sqrt{3}}\Pi^m y_m - \frac{8i}{\sqrt{3}}(\partial H_C - 3\partial\rho)y^2 + \frac{64}{\sqrt{3}}i(\partial\rho - \partial H_C) \\ =&\frac{2i}{3\sqrt{3}}\Pi^m \left[(2\theta\sigma^q\bar{\theta}\delta_m^p - \theta\sigma_m\bar{\theta}\delta^{pq})v_{0p}^i \otimes v_{0q}^i(y) - 72\theta\sigma_m\bar{\theta} + 96iy_m\right] \\ &+ \frac{8}{\sqrt{3}}(\theta\sigma^m\bar{\theta}y_m - iy^2)(\partial H_C - 3\partial\rho) + \frac{16}{\sqrt{3}}(i\theta\sigma^m\bar{\theta}y_m + 4)i(\partial\rho - \partial H_C) \\ V_1^i =&0 \end{aligned} \tag{3.88}$$

We can still add to V_1 a term $q^2\Phi$ where Φ satisfies $\tilde{G}^+\left(G^+(\Phi)\right) = 0$, i.e. Φ describes an on-shell massive field. But as explained below eqn. 3.28, any non-zero on-shell massive field will diverge exponentially when $x \to \infty$. So in order to have an instanton solution that vanishes at infinity, we should set Φ to zero. Note that the stringy contribution V_1 of eqn. 3.88 naively diverges quadratically in x^m when $x \to \infty$, but this is expected since it was found using an expansion in $\frac{1}{\rho^2}$. After summing over all orders in $\frac{1}{\rho^2}$, one expects that the stringy contribution will vanish at infinity like the super-Yang-Mills instanton. However, adding an on-shell massive contribution to V_1 would diverge exponentially when $x \to \infty$ and could never cancel after summing over all orders in $\frac{1}{\rho^2}$. We therefore have a unique solution.

3.5.3 Meaning of the massive fields

The first massive level of the on-shell string field describes two massive scalar multiplets and a massive spin-two multiplet. The on-shell field can be gauge-fixed to the form

$$\Pi^m\left(\hat{W}_m + 2i\partial_m(F - \bar{F})\right) - \frac{1}{8}\left(\partial\rho - \partial H_C\right)\left[D^2, \bar{D}^2\right]\left(F + \bar{F}\right) + \left(\partial H_C - 3\partial\rho\right)\left(F + \bar{F}\right) \tag{3.89}$$

where F and $\bar{F}$ are the chiral and antichiral superfields for the scalar multiplets, and $\hat{W}_m$ is the superfield for the spin-two multiplet. In components,

$$F = X(y) + \theta\xi(y) + \theta^2 Y(y), \quad \bar{F} = \bar{X}(\bar{y}) + \bar{\theta}\bar{\xi}(\bar{y}) + \bar{\theta}^2\bar{Y}(\bar{y}) \tag{3.90}$$

and

$$\hat{W}_m = C_m + i\theta\chi_m - i\bar{\theta}\bar{\chi}_m + \theta\sigma^n\bar{\theta}\left(v_{mn} + \frac{1}{4}\epsilon_{mnpq}\partial^p C^q\right) + i\theta\theta\bar{\theta}\bar{\lambda}_m - i\bar{\theta}\bar{\theta}\theta\lambda_m - \theta\theta\bar{\theta}\bar{\theta}\frac{1}{16}\partial^n\partial_n C_m \tag{3.91}$$

where X, $\bar{X}$, Y and $\bar{Y}$ are massive real bosons, ξ_α and $\bar{\xi}_{\dot{\alpha}}$ are massive spinors, v_{mn} is a massive symmetric traceless tensor, C_m is a massive vector, and χ and λ are massive spin-3/2 fermions [40, 41].

This is to be compared to our result of eq. 3.88. Since V_1 is not on-shell, it doesn't take the form 3.89. Nevertheless, we can still identify the components of V_1 with certain components of F, $\bar{F}$ and $\hat{W}_m$. Comparing eqn. 3.90 and 3.91 with eqn. 3.88, one finds that the non-zero component fields are

$$X^0(y) = -\frac{8i}{\sqrt{3}}y^2 + \frac{16i}{\sqrt{3}}, \quad v^0_{mn} = v^i_{0(m} \otimes v^i_{0n)} - 2v^i_{0p} \otimes v^{ip}_0 \delta_{mn} \tag{3.92}$$

where $v^i_{0(m} \otimes v^i_{0n)} = v^i_{0m} \otimes v^i_{0n} + v^i_{0n} \otimes v^i_{0m}$. We can therefore interpret the correction to the BPST instanton as turning on the $U(1)$ component of a massive scalar field and a massive spin-two field.

Capítulo 4

Super Yang-Mills Action from Hybrid Superstring Field Theory

String Field Theory provides an off-shell description of string theory. As such, it's expected that, upon integrating out the massive fields, we should obtain an effective field theory describing the low-energy limit of string theory. Effective actions were computed from bosonic open string field theory in [42, 43] using level truncation. This was generalized to Open Superstring Field Theory (OSFT) in [44], where the authors found a method that allows one to integrate out all massive states at once, no truncation required. Starting from the WZW-like formulation for the NS sector of OSFT [12], they computed the effective action to quartic order, and showed that it reproduces the Yang-Mills action, up to α' corrections (not super Yang-Mills because the R sector was not included). One can argue by dimensional analysis that there cannot be any higher order terms at $\alpha' \to 0$, so that the effective theory coincides exactly with Yang-Mills in this limit.

If we follow a similar method using the hybrid open superstring field theory, the resulting effective theory is expected to reproduce ten-dimensional super Yang-Mills (SYM) in terms of four-dimensional superfields [19] (plus α' corrections). In this chapter we will check that this is true at the quartic level. Although we do not compute the full action explicitly, we will argue from gauge invariance that it must reproduce SYM.

In section 4.1 we compute the massless contribution to the effective action, and find that it coincides with the SYM action to cubic order, but not quartic. In section 4.2 we integrate out the massive levels exactly, and find that, upon adding this contribution, the effective action coincides with SYM up to quartic order. In section 4.3 we obtain the full gauge transformations at $\alpha' \to 0$ and argue that it implies that the full effective action is the same as SYM in this limit.

This chapter is based on the paper [27].

4.1 Massless contribution to the effective action

We start by computing the massless contribution to the low-energy effective action. We will suppress α' corrections. This means we only consider terms with two derivatives (where D and $\bar{D}$ count as "half a derivative"). In this section we compute the massless contribution up to quartic order - i.e. we simply truncate the string fields and their products to massless order in the hybrid SFT action 2.94. This should result in the SYM action up to cubic order, but there should be a mismatch at quartic order, since at this order we will also have propagation of massive fields. We will now show that this is indeed the case.

At the massless level, the three string fields take the forms

$$V = V(x,\theta,\bar{\theta}), \quad \Phi_{-1} = e^{-\rho}\psi^j\bar{\sigma}_j(x,\theta,\bar{\theta}), \quad \Phi_1 = e^{\rho}\bar{\psi}_j\sigma^j(x,\theta,\bar{\theta}) \tag{4.1}$$

where x stands for all the ten directions. We expect that V should be the same as the v in 2.4, while Φ_1 and Φ_{-1} should be related to the chiral and anti-chiral superfields, in a manner to be made precise shortly. Acting with the G's, we get

$$G_4^+V = 2e^{\rho}dDV + \partial(e^{\rho})D^2V - ie^{\rho}\partial\bar{\theta}_{\dot{\alpha}}\partial^{\dot{\alpha}\alpha}D_\alpha V, \tag{4.2}$$

$$\tilde{G}_4^+V = 2e^{-2\rho+iH}\bar{d}\bar{D}V + \partial(e^{-2\rho+iH})\bar{D}^2V - ie^{-2\rho+iH}\partial\theta_\alpha\partial^{\dot{\alpha}\alpha}\bar{D}_{\dot{\alpha}}V \tag{4.3}$$

$$G_6^+V = \psi^j\partial_jV, \quad \tilde{G}_6^+V = e^{-\rho}\frac{1}{2}\epsilon_{ijk}\psi^i\psi^j\bar{\partial}^kV \tag{4.4}$$

$$\Omega = \frac{1}{2}e^{-\rho}\epsilon_{ijk}\psi^i\psi^j\bar{D}^2\sigma^k, \quad \bar{\Omega} = \psi^jD^2\bar{\sigma}_j \tag{4.5}$$

$$\tilde{G}_6^+\Phi_1 = \epsilon_{ijk}\psi^i\bar{\partial}^j\sigma^k, \quad G_6^+\Phi_{-1} = e^{-\rho}\psi^i\psi^j\partial_i\bar{\sigma}_j \tag{4.6}$$

Now we see that Ω and $\bar{\Omega}$ contain a chiral and an anti-chiral superfield, respectively. Presumably $\bar{D}^2\sigma^i = \omega^i$ and $D^2\bar{\sigma}_i = \bar{\omega}_i$.

We start by computing the third term in 2.94 for massless V. From the relations above, we have

$$\left(e^{-V}G_6^+e^V\right)\left(e^{-V}\tilde{G}_6^+e^V\right) = \left(e^{-V}\bar{\partial}^ie^V\right)\left(e^{-V}\partial_le^V\right)\frac{1}{2}e^{-\rho}\epsilon_{ijk}\psi^j\psi^k\psi^l \tag{4.7}$$

The $e^{-\rho}\psi^3$ factor gives precisely the basic correlator 2.75, and all the OPE's are regular. Therefore, we obtain precisely the corresponding term in the super-Yang-Mills action. The same logic follows for all terms in 2.94 but the first two. These are a bit more involved, since we have to take the OPE's of the d's and $\bar{d}$'s coming from

G_4^+ and $\tilde{G}_4^+$, as well as the ρ exponentials. Thus we see that all the terms in 2.4 involving the chiral or anti-chiral superfields or the six-dimensional derivatives of v are matched exactly by the truncation of 2.94, to all orders. Next we will investigate the first two terms in 2.94.

4.1.1 Cubic term

The cubic term in the four dimensional part of the SFT action is

$$S^{(3)} = -\frac{1}{6}\left\langle V\{G_4^+V, \tilde{G}_4^+V\}\right\rangle \tag{4.8}$$

which is to be compared with the SYM cubic term

$$S^{(3)}_{SYM} = -\mathrm{Tr}\int d^{10}x d^2\theta\, \bar{D}^2[D^\alpha V, V]\bar{D}^2 D_\alpha V = -\mathrm{Tr}\int d^{10}x d^4\theta\, [D^\alpha V, V]\bar{D}^2 D_\alpha V \tag{4.9}$$

where we have transformed a total $\bar{D}^2$ derivative into a $d^2\bar{\theta}$ integral. This integral is antysimmetric with respect to flipping the chiralities (i.e., $D \leftrightarrow \bar{D}$), and it will be convenient to write it in a way that makes this manifest. Using integration by parts and cyclicity, one can show that

$$S^{(3)}_{SYM} = \mathrm{Tr}\int d^{10}x d^4\theta\, V[\bar{D}^2 V, D^2 V] + 2V[D^\alpha \bar{D}_{\dot{\alpha}} V, \bar{D}^{\dot{\alpha}} D_\alpha V] \tag{4.10}$$

Now we compute 4.8 for massless V. We will compute the correlators in the upper half plane (UHP), where the prescription for the star product is that the operators be inserted at $-\sqrt{3}$, 0 and $\sqrt{3}$. Using 4.2, 4.3 we get

$$\begin{aligned}\left\langle G_4^+ V V \tilde{G}_4^+ V\right\rangle &= 48\left\langle dDVVd\bar{D}V\right\rangle - 8\sqrt{3}\left(\left\langle D^2VVd\bar{D}V\right\rangle - \left\langle dDVV\bar{D}^2V\right\rangle\right) \\ &-2\left\langle D^2VV\bar{D}^2V\right\rangle - 24i\left(\left\langle dDVV\partial\theta_\alpha\partial^{\dot{\alpha}\alpha}\bar{D}_{\dot{\alpha}}V\right\rangle + \left\langle \partial\bar{\theta}_{\dot{\alpha}}\partial^{\dot{\alpha}\alpha}D_\alpha VVd\bar{D}V\right\rangle\right)\end{aligned} \tag{4.11}$$

Here we hid the positions of the operators for ease of notation. On the r.h.s., we already computed the ρ and ψ correlators. Wherever there is no ρ or ψ dependence, the brackets are to be understood as just the x and θ correlators. The other term in 4.8 is given by hermitian conjugation, which translates into $D \leftrightarrow \bar{D}$ with a minus sign.

The correlators give

$$\left\langle dDV(-\sqrt{3})V(0)d\bar{D}V(\sqrt{3})\right\rangle = \Big\langle \frac{1}{6}\bar{D}^{\dot{\alpha}}D_\alpha VD^\alpha V\bar{D}_{\dot{\alpha}}V - \frac{1}{6}D_\alpha V\bar{D}^{\dot{\alpha}}VD^\alpha\bar{D}_{\dot{\alpha}}V$$
$$+\frac{1}{12}\bar{D}^{\dot{\alpha}}D_\alpha VVD^\alpha\bar{D}_{\dot{\alpha}}V + \frac{1}{6}D_\alpha V[D^\alpha,\bar{D}^{\dot{\alpha}}]V\bar{D}_{\dot{\alpha}}V + \frac{i}{24}\partial_{\alpha\dot{\alpha}}D^\alpha VV\bar{D}^{\dot{\alpha}}V \tag{4.12}$$
$$-\frac{i}{24}D^\alpha VV\partial_{\alpha\dot{\alpha}}\bar{D}^{\dot{\alpha}}V\Big\rangle$$

$$\left\langle dDV(-\sqrt{3})V(0)\bar{D}^2V(\sqrt{3})\right\rangle = \frac{1}{\sqrt{3}}\left\langle D_\alpha VD^\alpha V\bar{D}^2V + \frac{1}{2}D_\alpha VVD^\alpha\bar{D}^2V\right\rangle \tag{4.13}$$

$$\left\langle dDV(-\sqrt{3})V(0)\partial\theta_\alpha\partial^{\dot{\alpha}\alpha}\bar{D}_{\dot{\alpha}}V(\sqrt{3})\right\rangle = -\frac{i}{12}\left\langle D_\alpha VV\partial^{\dot{\alpha}\alpha}\bar{D}_{\dot{\alpha}}V\right\rangle \tag{4.14}$$

Now, we again use integration by parts and cyclicity to write this in a form in which there is either two or zero D's or $\bar{D}$'s on each V, as in 4.10. Substituting in 4.1.1, we obtain

$$\left\langle G_4^+VV\tilde{G}_4^+V\right\rangle = 12\left\langle\bar{D}^{\dot{\alpha}}D_\alpha VVD^\alpha\bar{D}_{\dot{\alpha}}V\right\rangle + 6\left\langle D^2VV\bar{D}^2V\right\rangle \tag{4.15}$$

Adding the conjugate and the factor of $1/6$, this gives precisely 4.10.

4.1.2 Quartic term

The SYM quartic term is symmetric with respect to $D \leftrightarrow \bar{D}$. Following the method used for the cubic term, one can show that it can be written as

$$S^{(4)}_{SYM} = -\mathrm{Tr}\frac{1}{12}\int d^{10}xd^4\theta\,[D^\alpha,\bar{D}^{\dot{\alpha}}]V(\{D_\alpha V,[V,\bar{D}_{\dot{\alpha}}V]\} - \{\bar{D}_{\dot{\alpha}}V,[V,D_\alpha V]\})$$
$$-2[D_\alpha V,\{\bar{D}^{\dot{\alpha}}V,D^\alpha V\}]\bar{D}_{\dot{\alpha}}V - [V,[V,D^2V]]\bar{D}^2V \tag{4.16}$$
$$+i([V,[V,D^\alpha V]]\partial_{\alpha\dot{\alpha}}\bar{D}^{\dot{\alpha}}V + [V,[V,\bar{D}^{\dot{\alpha}}V]]\partial_{\alpha\dot{\alpha}}D^\alpha V)$$

On the OSFT side, the quartic term for the four-dimensional part of the action is

$$S_4^{(4)} = \frac{1}{4!}\left\langle G_4V\tilde{G}_4VV^2 + G_4VV^2\tilde{G}_4V - 2G_4VV\tilde{G}_4VV\right\rangle \tag{4.17}$$

One might think that computing this term would require a lot more work than the cubic one, but we will now argue that all the relevant quartic correlators can be immediately read off from the cubic ones, up to simple factors. First, note that now the operators should be inserted at -1, 0, 1 and ∞. By cyclicity, we can always choose to put a V with no G's at infinity, and then all contractions with this V vanish - it becomes just a spectator. Thus the correlators take the same form as the cubic ones, with an extra V inserted at infinity, and with a rescaling by $1/\sqrt{3}$. The result, after organizing the terms to look as close to 4.1.2 as possible, is

$$S^{(4)}_{Massless} = S^{(4)}_{SYM} + \frac{1}{4}\mathrm{Tr}\int d^{10}x d^4\theta\, D_\alpha V \bar{D}_{\dot\alpha} V D^\alpha V \bar{D}^{\dot\alpha} V - 2 D_\alpha V \bar{D}_{\dot\alpha} V \bar{D}^{\dot\alpha} V D^\alpha V \tag{4.18}$$

As expected, there is a mismatch. We will see in the next section that the extra terms are precisely cancelled by the contribution to the action from integrating out the higher mass levels.

4.2 Massive contribution

To integrate out the massive fields, we will use the methods of [44], which allow us to obtain the exact answer, without any level truncation. We write the string fields as

$$\Phi_i = \Phi_i^0 + R_i \tag{4.19}$$

where Φ_i^0 contain the massless states, and R_i contain the fields we want to integrate out. We now solve the equations of motion for R_i in terms of Φ_i^0. Since we are only computing the quartic term in the action, we only need the solution to quadratic order. Expanding the equations 2.99 we get:

$$(\tilde{G}_4^+ G_4^+ + \tilde{G}_6^+ G_6^+)V + \tilde{G}_6^+\bar{\Omega} - G_6^+\Omega - \frac{1}{2}\{\tilde{G}_4^+ V, G_4^+ V\} - \frac{1}{2}\{\tilde{G}_6^+ V, G_6^+ V\} \tag{4.20}$$
$$+\frac{1}{2}[\tilde{G}_6^+\bar{\Omega}, V] + \frac{1}{2}[G_6^+\Omega, V] - \{\bar{\Omega}, \tilde{G}_6^+ V\} - \{\Omega, G_6^+ V\} = O(\Phi_i^3)$$

$$\tilde{G}_6^+ G_4^+ V + G_6^+\bar{\Omega} - G_4^+\Omega - \frac{1}{2}\{\tilde{G}_6^+ V, G_4^+ V\} + \frac{1}{2}[G_6^+\bar{\Omega}, V] + \frac{1}{2}[G_4^+\Omega, V] \tag{4.21}$$
$$-\{\Omega, G_4^+ V\} + \bar{\Omega}^2 = O(\Phi_i^3)$$

$$\tilde{G}_4^+ G_6^+ V + \tilde{G}_4^+\bar{\Omega} - \tilde{G}_6^+\Omega - \frac{1}{2}\{\tilde{G}_4^+ V, G_6^+ V\} + \frac{1}{2}[\tilde{G}_4^+\bar{\Omega}, V] + \frac{1}{2}[\tilde{G}_6^+\Omega, V] \tag{4.22}$$
$$-\{\bar{\Omega}, \tilde{G}_4^+ V\} + \Omega^2 = O(\Phi_i^3)$$

which are solved to quadratic order by

$$R_{-1} = -\frac{1}{2}\left(\frac{G_4^-}{L_0}[V^0, G_6^+ V^0] + \frac{G_4^-}{L_0}[V^0, G_4^+ \Phi_{-1}^0] + \frac{G_6^-}{L_0}[\Phi_{-1}^0, G_4^+ \Phi_{-1}^0]\right) \tag{4.23}$$

$$R_0 = -\frac{1}{2}\left(\frac{G_4^-}{L_0}[V^0, G_4^+ V^0] + \frac{G_6^-}{L_0}[V^0, G_6^+ V^0] + \frac{G_4^-}{L_0}[V^0, G_6^+ \Phi_1^0] + \frac{G_6^-}{L_0}[V^0, G_4^+ \Phi_{-1}^0]\right) \tag{4.24}$$

$$R_1 = -\frac{1}{2}\left(\frac{G_6^-}{L_0}[V^0, G_4^+ V^0] + \frac{G_4^-}{L_0}[V^0, G_4^+ \Phi_1^0] + \frac{G_4^-}{L_0}[\Phi_1^0, G_6^+ \Phi_1^0]\right) \tag{4.25}$$

Now we obtain the contribution from R_i to the quartic term by simply substituting these solutions into the R_i kinetic terms. Note that we have now four G operators, which will become simple derivatives on the massless fields, as in 4.2-4.6. Since we do not want the higher derivative corrections, we consider only the terms with two derivatives (again, D and $\bar{D}$ count as half). That means that each G operator can only contribute one D or $\bar{D}$, which implies that the only nonzero contribution is from the four-dimensional term, namely

$$\left\langle [\tilde{G}_4^+ V^0, G_4^+ V^0], \frac{G_4^-}{L_0}[V^0, G_4^+ V^0] \right\rangle \tag{4.26}$$

where $<,>$ denotes the BPZ product. This is as should be, since, as we have seen, the four-dimensional term is the only one for which the massless contribution does not match exactly the SYM action. Since the R_i's will no longer appear, we will drop the 0 superscript on V.

Now we use the crucial trick from [44], which allows us to rewrite the BPZ product of two star products, such as 4.26, as a correlator with insertion points $-1/a$, $1/a$, a, $-a$, with $a = \sqrt{2} - 1$. We will not review the proof here. Then we evaluate the action of the G operators as in 4.2-4.6. It might seem like we will get a multitude of terms, but in fact most of them will have more than two derivatives. The only terms that interest us are

$$\left\langle [\tilde{G}_4^+ V(-\frac{1}{a}), G_4^+ V(\frac{1}{a})] \frac{G_4^-}{L_0}[V(a), G_4^+ V(-a)] \right\rangle = \tag{4.27}$$

$$2\left\langle [e^{-2\rho+iH} \bar{d}\bar{D}V, e^{\rho} dDV] \frac{a}{L_0}([e^{-\rho}\bar{d}\bar{D}V, e^{\rho}dDV] - i[V, \Pi^{\dot{\alpha}\alpha}\bar{D}_{\dot{\alpha}}D_{\alpha}V] \right.$$
$$\left. +2[V, \partial\theta^{\alpha}D_{\alpha}V]) \right\rangle + 2\left\langle [\partial e^{-2\rho+iH}\bar{D}^2 V, e^{\rho}dDV] \frac{a}{L_0}[V, \partial\theta^{\alpha}D_{\alpha}V] \right\rangle + ...$$

where $\Pi^{\dot{\alpha}\alpha} = \partial x^{\dot{\alpha}\alpha} - \frac{i}{2}\left(\theta^{\alpha}\partial\bar{\theta}^{\dot{\alpha}} + \bar{\theta}^{\dot{\alpha}}\partial\theta^{\alpha}\right)$, and the ellipsis indicate higher derivative terms.

We now use the Schwinger parametrization

$$\frac{1}{L_0} = \int_0^\infty dt\, e^{-tL_0} \tag{4.28}$$

This will have the effect of shifting the positions of the insertions in the second commutator from a and $-a$ to $b \equiv e^{-t}a$ and $-b$, plus adding an overall factor of e^{-t}

$$\left\langle [A(-\frac{1}{a}), B(\frac{1}{a})]\frac{a}{L_0}[C(a), D(-a)] \right\rangle = \int_0^\infty dt\, b \left\langle [A(-\frac{1}{a}), B(\frac{1}{a})][C(b), D(-b)] \right\rangle \tag{4.29}$$

We must now evaluate the correlators. Let us start with the first term on the r.h.s. of 4.27. Note that, since we do not want any more derivatives on the V's, we should only contract the d's and $\bar{d}$'s among themselves. This results in, for example

$$\begin{aligned}&\left\langle \bar{d}_{\dot{\alpha}}(-\frac{1}{a})d_{\alpha}(\frac{1}{a})\bar{d}_{\dot{\beta}}(b)d_{\beta}(-b) \right\rangle = \\ &\epsilon_{\alpha\beta}\epsilon_{\dot{\alpha}\dot{\beta}}\frac{1}{(b-\frac{1}{a})^2(b+\frac{1}{a})^2}\left(\frac{1}{2b}(-\frac{2}{a}+a(b^2+\frac{1}{a^2}))-1-\frac{ab}{2}\right)\end{aligned} \tag{4.30}$$

while the corresponding ρ and ψ correlators give

$$\left\langle e^{-2\rho+iH}(-\frac{1}{a})e^{\rho}(\frac{1}{a})e^{-\rho}(b)e^{\rho}(-b) \right\rangle_{\rho,\psi} = -\frac{8b}{a^2}\left(\frac{b-\frac{1}{a}}{b+\frac{1}{a}}\right)^3 \tag{4.31}$$

We can obtain the other orderings of the insertions by simply changing the sign of a and/or b.

For the second term, we have

$$\left\langle \bar{d}_{\dot{\alpha}}(-\frac{1}{a})d_{\alpha}(\frac{1}{a})\Pi^{\dot{\beta}\beta}(b) \right\rangle = -i\delta_{\alpha}^{\beta}\delta_{\dot{\alpha}}^{\dot{\beta}}\left(\frac{a}{2(b-\frac{1}{a})^2} + \frac{a}{2(b+\frac{1}{a})^2} + \frac{2}{a}\frac{1}{(b^2-\frac{1}{a^2})^2}\right)$$

On the third term, the d must contract with the $\partial\theta$, and the $\bar{d}$ will contract with the V's to give $\bar{D}V$. And on the last term, there is only one d, which again must contract with $\partial\theta$.

Putting everything together and integrating on t, we get

$$\frac{1}{4}\left\langle [\tilde{G}_4^+ V^0, G_4^+ V^0], \frac{G_4^-}{L_0}[V^0, G_4^+ V^0] \right\rangle = \tag{4.32}$$
$$-\frac{1}{4}\mathrm{Tr}\int d^{10}x d^4\theta\, D_\alpha V \bar{D}_{\dot{\alpha}} V D^\alpha V \bar{D}^{\dot{\alpha}} V - 2D_\alpha V \bar{D}_{\dot{\alpha}} V \bar{D}^{\dot{\alpha}} V D^\alpha V$$

which precisely cancels the extra terms in 4.18. Thus the effective action from OSFT coincides with SYM to quartic order. In the next section we will argue, based on gauge invariance, that they should coincide to all orders, even though we do not explicitly compute the effective action.

4.3 Gauge invariance and the full action

We have shown that the effective action computed from OSFT in the hybrid formalism reproduces, up to quartic order, super Yang-Mills theory plus higher derivative corrections. Differently than the Yang-Mills action, which is quartic, the SYM action 2.4 is non-polynomial in the prepotential. We would like then to generalize the result to all orders. Explicit computation of the action using the same methods as above seems impractical, as the calculations become more involved at each order. However, the SYM is actually quartic in Wess-Zumino gauge. If the gauge transformations are the same in the two theories to all orders, we can conclude that the actions must be as well.[1]

Consider the gauge transformations with parameters Σ and $\bar{\Sigma}$ in 2.95-2.97. The other gauge transformations do not appear at the massless level. Restricting the gauge parameters Λ_{-1} and Λ_2 to be massless, i.e. $\Lambda_{-1} = e^{-\rho}\lambda(x,\theta,\bar{\theta})$ and $\Lambda_2 = e^{2\rho - iH_C}\bar{\lambda}(x,\theta,\bar{\theta})$, we get $\bar{\Sigma} = D^2\lambda$ and $\Sigma = \bar{D}^2\bar{\lambda}$. Now, if we also truncate the string fields and their products to the massless levels (the star product simply reduces to the usual field product), these transformations reduce exactly to the SYM gauge transformations.

Now we must ask whether there are any massive contributions to the massless gauge transformations. Remember we are ignoring α' corrections - here this means that the string fields should appear with no derivatives in the gauge transformations, and the gauge parameters should only appear with the D^2 or $\bar{D}^2$. Note that in 4.23 the massless fields always appear with derivatives in the massive terms. Although we only have the explicit solution to quadratic order,

[1] I thank Nathan Berkovits for suggesting this idea.

we can argue that the higher orders will always include derivatives, since there is no expression we can write with only products of the string fields (with no G operators, i.e. no derivatives) which is purely massive. Thus we see that the R_i in 4.19 can be ignored as far as the massless gauge transformations are concerned. Massive states will still appear in the star products, but it's easy to see that also those can only contribute higher derivative terms to the gauge transformations. We therefore conclude that the effective action from OSFT has the same gauge invariance as SYM, and thus the action should be the same to all orders.

Capítulo 5

Heterotic String Field Theory in the Hybrid Formalism

A classical action for the Neveu-Schwarz sector of heterotic superstring field theory (SFT) was first constructed in [21, 20]. The basic ingredients are the WZW-like large Hilbert space formulation of open superstring field theory and the closed string field products. The heterotic field can be thought of as a left-right product of an open superstring field and an open bosonic string field. This construction was extended to include the Ramond sector up to quartic order in fermions in [22].

In the previous chapters we have explored some applications of the hybrid formalism for open superstring field theory. It would be useful to have a similar formulation of heterotic closed SFT, which at the massless level would correspond to a superspace formulation of $N = 1$ $d = 10$ supergravity in terms of four-dimensional superfields. In this chapter we take a first step towards this goal and construct the quadratic term in the heterotic string field theory action with manifest $N = 1$ $d = 4$ super-Poincaré invariance in terms of three string fields, similar to what was done for the open superstring. We then analyze explicitly the massless contribution to the action and show that, upon restricting to the Calabi-Yau (CY) independent sector, it provides a description of four dimensional supergravity plus a tensor multiplet in $N = 1$ $d = 4$ superspace. Work is in progress on extending this quadratic action to the full non-linear closed string field theory action of the heterotic superstring.

In section 5.1 we obtain the linearized equations of motion and gauge invariances describing the heterotic spectrum, and construct the corresponding quadratic action. In section 5.2 we explicitly evaluate the action for the massless sector and show that the CY independent sector correctly describes supergravity in $N = 1$ $d = 4$ superspace. We also show how to modify the action for the massless sector of heterotic SFT in order to obtain a real action for ten-dimensional supergravity. After that we compare with the RNS formulation of heterotic string field theory.

Finally, we discuss our results in section 5.4.

This chapter is based on the paper [28].

5.1 Linearized action

In this section we will find that all the physical states of the heterotic string can be described using three string fields, and will construct a quadratic action for heterotic string field theory with manifest four dimensional spacetime supersymmetry.

We take the string field Φ to be a direct product of a bosonic open string in the antiholomorphic sector and a hybrid open superstring in the holomorphic sector. Φ is grassmann odd with total ghost number 1, i.e. $\oint(-\partial\rho + J_C - \bar{b}\bar{c})\Phi = \Phi$. On-shell states should satisfy the linearized equation of motion

$$\tilde{G}^+(G^+ + \bar{Q})\Phi = 0 \tag{5.1}$$

where $\bar{Q} = \int d\bar{z}(\bar{c}\bar{T} + \bar{c}\bar{\partial}\bar{c}\bar{b})$ is the antiholomorphic BRST operator.

In addition, the string field should satisfy the hybrid version of the b_0^- and L_0^- conditions on closed string fields which are

$$(G^- - \bar{b})_0\Phi = 0, \quad L_0^-\Phi = 0. \tag{5.2}$$

Here we run into an issue with the reality condition: in the hybrid formalism, hermitian conjugation (in Minkowski signature) is defined in such a way that $G^{\pm\dagger} = \tilde{G}^{\pm}$, and then the first condition of 5.2 is evidently not real. For now we will simply work in signature $d = (2,2)$ or $d = (5,5)$ so that all the operators and fields are real. It will later be shown that the physical spectrum can be analytically continued to Minkowski signature.

To fix the ambiguity that comes from picture changing, rather than fixing the picture number, we consider the more restrictive equation

$$(G^+ + \bar{Q} + \tilde{G}^+)\Phi = 0 \tag{5.3}$$

with the gauge invariance

$$\delta\Phi = (G^+ + \bar{Q} + \tilde{G}^+)\Lambda \tag{5.4}$$

just as was done for the open superstring. The gauge parameter Λ should satisfy the same conditions 5.2.

Now the idea would be to expand Φ in eigenvalues of the ρ charge:

$$\Phi = \sum_{n=-\infty}^{\infty} \Phi_n \tag{5.5}$$

where $-\partial\rho\Phi_n = n\Phi_n$. Our hope is that we can describe Φ in terms of only a finite number of the Φ_n's. But here we encounter a problem in the fact that $G^- - \bar{b}$ does not have a well defined ρ-charge, which means that the subsidiary condition will mix different Φ_n. Explicitly, we have $(G_4^-)_0\Phi_n = -(G_6^- - \bar{b})_0\Phi_{n-1}$.

To avoid this, we will solve the $(G^- - \bar{b})_0$ constraint as

$$\Phi = (G^- - \bar{b})_0\Sigma \tag{5.6}$$

and work with Σ instead of Φ. Note that we can choose Σ to be annihilated by $\bar{c}_0$, and then it is expressed in terms of Φ as $\Sigma = \bar{c}_0\Phi$. Now we have a constraint $\bar{c}_0$ with a well defined ρ-charge. The price we pay is that the equations for Σ take a more complicated form, namely 5.3 implies that

$$(G^+ + \bar{Q} + \tilde{G}^+ + 2\sum_{n=0}^{\infty} n\bar{c}_{-n}\bar{c}_n(G^- - \bar{b})_0)\Sigma = 0 \tag{5.7}$$

with gauge invariance

$$\delta\Sigma = (G^+ + \bar{Q} + \tilde{G}^+ + 2\sum_{n=0}^{\infty} n\bar{c}_{-n}\bar{c}_n(G^- - \bar{b})_0)\Lambda \tag{5.8}$$

where Λ is also annihilated by $\bar{c}_0$. Note that $(G^+ + \bar{Q} + \tilde{G}^+ + 2\sum_{n=0}^{\infty} n\bar{c}_{-n}\bar{c}_n(G^- - \bar{b})_0)$ is nilpotent (in the space of string fields annihilated by L_0^-) and anticommutes with $\bar{c}_0$. Now we can expand Σ in eigenvectors of the ρ-charge, $\Sigma = \sum_{-\infty}^{\infty}\Sigma_n$, with the simple constraints

$$\bar{c}_0\Sigma_n = 0, \quad L_0^-\Sigma_n = 0 \tag{5.9}$$

The equations of motion and gauge invariances become

$$G_4'^+\Sigma_n + G_6'^+\Sigma_{n+1} + \tilde{G}_6'^+\Sigma_{n+2} + \tilde{G}_4'^+\Sigma_{n+3} = 0 \tag{5.10}$$

$$\delta\Sigma_n = G_4'^+\Lambda_{n-1} + G_6'^+\Lambda_n + \tilde{G}_6'^+\Lambda_{n+1} + \tilde{G}_4'^+\Lambda_{n+2} \tag{5.11}$$

where

$$G_4'^+ = G_4^+, \quad \tilde{G}_4'^+ = \tilde{G}_4^+, \tag{5.12}$$

$$G_6'^+ = G_6^+ + \bar{Q} + 2\sum_{n=0}^{\infty} n\bar{c}_{-n}\bar{c}_n(G_6^- - \bar{b})_0, \quad \tilde{G}_6'^+ = \tilde{G}_6^+ + 2\sum_{n=0}^{\infty} n\bar{c}_{-n}\bar{c}_n(G_4^-)_0 \tag{5.13}$$

Note that each of $G_4'^+$, $G_6'^+$, $\tilde{G}_6'^+$ and $\tilde{G}_4'^+$ anti-commutes with $\bar{c}_0$. Note also that $G_4'^+$ and $\tilde{G}_4'^+$ are nilpotent and have trivial cohomology, which means we can use 5.10 to write all Σ_n in terms of only three, which we choose to be Σ_{-1}, Σ_0 and Σ_1.

The equations of motion for these three string fields are

$$\tilde{G}_4'^+(G_4'^+\Sigma_{-1} + G_6'^+\Sigma_0 + \tilde{G}_6'^+\Sigma_1) = 0 \tag{5.14}$$

$$(\tilde{G}_6'^+G_6'^+ + \tilde{G}_4'^+G_4'^+)\Sigma_0 - G_6'^+\tilde{G}_4'^+\Sigma_1 + \tilde{G}_6'^+G_4'^+\Sigma_{-1} = 0 \tag{5.15}$$

$$G_4'^+(G_6'^+\Sigma_{-1} + \tilde{G}_6'^+\Sigma_0 + \tilde{G}_4^+\Sigma_1) = 0 \tag{5.16}$$

with linear gauge transformations

$$\delta\Sigma_{-1} = G_4'^+\Lambda_{-2} + G_6'^+\Lambda_{-1} + \tilde{G}_6'^+\Lambda_0 + \tilde{G}_4'^+\Lambda_1 \tag{5.17}$$

$$\delta\Sigma_0 = G_4'^+\Lambda_{-1} + G_6'^+\Lambda_0 + \tilde{G}_6'^+\Lambda_1 + \tilde{G}_4'^+\Lambda_2 \tag{5.18}$$

$$\delta\Sigma_1 = G_4'^+\Lambda_0 + G_6'^+\Lambda_1 + \tilde{G}_6'^+\Lambda_2 + \tilde{G}_4'^+\Lambda_3 \tag{5.19}$$

with

$$\bar{c}_0\Lambda_n = 0, \quad L_0^-\Lambda_n = 0 \tag{5.20}$$

We should note that $G_6'^+$ and $\tilde{G}_6'^+$ are not nilpotent like G_6^+ and $\tilde{G}_6^+$, and also that $G_4'^+$ does not anti-commute with $\tilde{G}_6'^+$ and $G_6'^+$ with $\tilde{G}_4'^+$. Explicitly, we have

$$(G_6'^+)^2 = 2\sum_{n=0}^{\infty} n\bar{c}_{-n}\bar{c}_n(L_6 - \bar{L})_0, \tag{5.21}$$

$$(\tilde{G}_6'^+)^2 = -\{\tilde{G}_4'^+, G_6'^+\} = -2\sum_{n=0}^{\infty} n\bar{c}_{-n}\bar{c}_n e^{-2\rho}\epsilon_{ijk}\psi^i\psi^j\partial x^k d^2, \tag{5.22}$$

$$\{G_4'^+, \tilde{G}_6'^+\} = 2\sum_{n=0}^{\infty} n\bar{c}_{-n}\bar{c}_n(L_4)_0, \tag{5.23}$$

$$\{G_4'^+, \tilde{G}_4'^+\} = -\{G_6'^+, \tilde{G}_6'^+\} = \partial\rho e^{-\rho+iH_C} \tag{5.24}$$

where L_4 and L_6 are the four-dimensional and six-dimensional parts of the holomorphic energy-momentum tensor, respectively, and $\bar{L}$ is the antiholomorphic energy-momentum tensor. All other anticommutators vanish. Still, the fact that $G_4'^+ + G_6'^+ + \tilde{G}_6'^+ + \tilde{G}_4'^+$ is nilpotent ensures gauge invariance.

Now we would like to define a quadratic action whose variation gives the above equations. In order to do this, we define an inner product with a $\bar{b}_0$ insertion, $\langle A, B\rangle \equiv \langle A|\bar{b}_0|B\rangle$. Although $G_6'^+$ does not anti-commute with $\bar{b}_0$, the $\bar{c}_0$ constraint ensures that the anti-commutator vanishes inside the brackets, such that

$$\langle A, G'B\rangle = (-1)^A \langle G'A, B\rangle \tag{5.25}$$

where G' stands for any of $G_4'^+$, $G_6'^+$, $\tilde{G}_6'^+$ and $\tilde{G}_4'^+$, and $(-1)^A$ denotes the grassmanality of A.

With this definition, the action which reproduces the equations 5.14-5.16 and gauge transformations of 5.17-5.19 is

$$\begin{aligned} S_2 = \frac{1}{2}\left(\left\langle \tilde{G}_4'^+\Sigma_0, G_4'^+\Sigma_0\right\rangle + \left\langle \tilde{G}_6'^+\Sigma_0, G_6'^+\Sigma_0\right\rangle + \left\langle \Omega, \tilde{G}_6'^+\Sigma_1\right\rangle + \left\langle G_6'^+\Sigma_{-1}, \bar{\Omega}\right\rangle\right) \\ + \left\langle \tilde{G}_6'^+\Sigma_0, \bar{\Omega}\right\rangle + \left\langle \Omega, G_6'^+\Sigma_0\right\rangle + \left\langle \Omega, \bar{\Omega}\right\rangle \end{aligned} \tag{5.26}$$

where $\Omega = \tilde{G}_4'^+\Sigma_1$, $\bar{\Omega} = G_4'^+\Sigma_{-1}$. This is formally the same as the open superstring field action [12].

Note that we can also define string fields of ghost number one which are annihilated by $\bar{b}_0$ as $\Phi_n' = \bar{b}_0\Sigma_n$. Then, if we redefine the inner product as $\langle A, B\rangle = \langle A|\bar{c}_0|B\rangle$, we can write the action 5.1 in the same form in terms of Φ_n'. This form is more similar to the usual formulations of closed string field theories, and in particular the RNS formulation of heterotic SFT, so it might be more suitable for constructing the nonlinear orders. Note however that $\bar{b}_0$ does not anti-commute with $G_6'^+$, so the gauge transformations do not take the same form - instead there will be a $\bar{b}_0$ in front of every term in 5.17, and the parameters Λ_n are still annihilated by $\bar{c}_0$.

5.2 Massless level

Having constructed a quadratic action, it will be interesting to analyze it explicitly for the massless level. We expect this should give a description of ten-dimensional supergravity in terms of four-dimensional superfields. We will ignore the right-moving fermionic worldsheet variables, that is the states transforming under $E8 \times E8$ or $SO(32)$. Then the most general form of the string fields at the massless level is

$$\bar{b}_0\Sigma_0 = \bar{c}\bar{\partial}x_M H^M(x,\theta,\bar{\theta}) + \psi^j C_j(x,\theta,\bar{\theta}) \tag{5.27}$$

$$\bar{b}_0\Sigma_1 = e^{\rho}F + e^{\rho}\bar{\psi}_j\bar{c}\bar{\partial}x^M \bar{A}^j_M(x,\theta,\bar{\theta}) \tag{5.28}$$

$$\bar{b}_0\Sigma_{-1} = e^{-\rho}\bar{c}\bar{\partial}^2\bar{c}B(x,\theta,\bar{\theta}) + e^{-\rho}\psi^j\bar{c}\bar{\partial}x_M A^M_j(x,\theta,\bar{\theta}) + \frac{1}{2}e^{-\rho}\psi^i\psi^j\epsilon_{ijk}\bar{C}^k(x,\theta,\bar{\theta}) \tag{5.29}$$

where M runs over all the ten dimensions, while i, j, k are $SU(3)$ indices describing the six compactified directions. All the fields in the above expansions have conformal weight zero, except for F which has conformal weight 1. The graviton is contained in the fields H^M, A^M_j and $\bar{A}^j_M$, while B is related to the ghost dilaton. For the gauge parameters, we have:

$$\bar{b}_0\Lambda_{-2} = e^{-2\rho}\partial\theta^{\alpha}\left(\bar{c}\bar{\partial}^2\bar{c}\omega_{\alpha}(x,\theta,\bar{\theta}) + \psi^j\bar{c}\bar{\partial}x_M\omega^M_{j\alpha}(x,\theta,\bar{\theta}) + \frac{1}{2}\epsilon_{ijk}\psi^i\psi^j\bar{\omega}^k_{\alpha}(x,\theta,\bar{\theta})\right) \tag{5.30}$$

$$\bar{b}_0\Lambda_{-1} = e^{-\rho}\left(\bar{c}\bar{\partial}x_M\bar{N}^M(x,\theta,\bar{\theta}) + \psi^j\lambda_j(x,\theta,\bar{\theta})\right) \tag{5.31}$$

$$\bar{b}_0\Lambda_0 = P(x,\theta,\bar{\theta}) \tag{5.32}$$

$$\bar{b}_0\Lambda_1 = e^{\rho}\bar{\psi}_j\bar{\lambda}^j(x,\theta,\bar{\theta}) \tag{5.33}$$

$$\bar{b}_0\Lambda_2 = \frac{1}{2}e^{2\rho}\epsilon^{ijk}\bar{\psi}_i\bar{\psi}_j\partial\bar{\theta}^{\dot{\alpha}}\omega_{k\dot{\alpha}}(x,\theta,\bar{\theta}) + e^{2\rho-iH_C}\bar{c}\bar{\partial}x_M N^M(x,\theta,\bar{\theta}) \tag{5.34}$$

$$\bar{b}_0\Lambda_3 = e^{3\rho-iH_C}\left(\Psi + \partial\bar{\psi}_j\partial\bar{\theta}^{\dot{\alpha}}\bar{c}\bar{\partial}x^M\omega^j_{M\dot{\alpha}}(x,\theta,\bar{\theta})\right) \tag{5.35}$$

Here we have used some of the gauge for gauge freedom to fix this form of the gauge parameters. All the parameters in the above expansions have conformal weight zero, except for Ψ which has conformal weight 3. This parameter can be used to fix the form of F to

$$F = \partial\bar{\theta}^{\dot{\alpha}}\bar{\beta}_{\dot{\alpha}}(x,\theta,\bar{\theta}) \tag{5.36}$$

leaving the residual gauge symmetry

$$\delta\bar{\beta}_{\dot{\alpha}} = \bar{D}^2\alpha_{\dot{\alpha}} \tag{5.37}$$

Evaluating the action 5.1 at the massless level, we get

$$S_2 = \frac{1}{8} \int d^{10}x d^4\theta \tag{5.38}$$

$$\Big[H^M \left(D\bar{D}^2 D H_M + 4\partial^j \partial_j H_M - 2\partial_M \bar{D}\bar{\beta} - 8\bar{D}^2 \partial_k \bar{A}^k_M - 8D^2 \partial^j A_{jM} - 8\partial_M \partial^j C_j \right) +$$
$$C_i \left(8\partial^i \partial^j C_j + 2\sqrt{2}\epsilon^{ijk} \bar{D}^2 \partial_j C_k + 8\partial^M \bar{D}^2 \bar{A}^i_M + 4\partial^i D^2 B - 2\bar{D}^2 D^2 \bar{C}^i + 4\partial^i \bar{D}\bar{\beta} \right) +$$
$$D^2 \bar{C}^i \left(-4\partial_i B + 8\partial_M A^M_i - 2\sqrt{2}\epsilon_{ijk} \partial^j \bar{C}^k \right) - 8\sqrt{2} D^2 A^M_i \epsilon^{ijk} \partial_j A_{kM} +$$
$$\bar{D}^2 \bar{A}^i_M \left(-8D^2 A^M_i + 8\sqrt{2}\epsilon_{ijk} \partial^j \bar{A}^{kM} \right) - \bar{D}\bar{\beta} \left(D^2 B + \frac{1}{2}\bar{D}\bar{\beta} \right) \Big]$$

which is invariant under the gauge transformations

$$\delta H^M = D^2 \bar{N}^M + \bar{D}^2 N^M + \partial^M P \tag{5.39}$$
$$\delta A^M_i = \partial_i \bar{N}^M + \partial^M \lambda_i + D\omega^M_i \tag{5.40}$$
$$\delta \bar{A}^i_M = \partial^i N_M + \partial_M \bar{\lambda}^i + \bar{D}\bar{\omega}^i_M \tag{5.41}$$
$$\delta C_i = \partial_i P - D^2 \lambda_i + 2\sqrt{2}\epsilon_{ijk} \partial^j \bar{\lambda}^k + \bar{D}\omega_i \tag{5.42}$$
$$\delta \bar{C}^i = \partial^i P - 2\sqrt{2}\epsilon^{ijk} \partial_j \lambda_k - \bar{D}^2 \bar{\lambda}^i + D\bar{\omega}^i \tag{5.43}$$
$$\delta B = 2\partial_M \bar{N}^M - \frac{1}{2}\bar{D}^2 P + 4\partial^j \lambda_j + D\omega \tag{5.44}$$
$$\delta \bar{\beta}_{\dot{\alpha}} = D^2 \bar{D}_{\dot{\alpha}} P - 4\bar{D}_{\dot{\alpha}} \partial_j \bar{\lambda}^j - 4\partial^j \omega_{j\dot{\alpha}} + \bar{D}^2 \alpha_{\dot{\alpha}} \tag{5.45}$$

As mentioned above, we expect this to describe ten-dimensional supergravity. This is not immediately clear, especially given that the action 5.38 is not even real in Minkowski signature. To simplify the problem, we look in the next subsection at the compactification independent part of the action. Then in subsection 5.2.3 we will modify the action 5.38 in order to make it real in Minkowski space. After that, we will compare these results to the RNS formalism.

5.2.1 Calabi-Yau independent states

We now consider only the CY independent states, by which we mean the massless string fields that do not depend on ψ^j, $\bar{\psi}_j$, x^j or $\bar{x}_j$. That leaves us with

$$\bar{b}_0 \Sigma_0 = H^m(x,\theta,\bar{\theta}) \bar{c}\bar{\partial} x_m, \quad \bar{b}_0 \Sigma_{-1} = e^{\rho} F, \quad \bar{b}_0 \Sigma_1 = e^{-\rho} B(x,\theta,\bar{\theta}) \bar{c}\bar{\partial}^2 \bar{c} \tag{5.46}$$

where m is a four-dimensional index. H^m should correspond to the supergravity superfield [45, 46]. The action 5.38 reduces to

$$\frac{1}{8}\int d^4x d^4\theta[H^m D\bar{D}^2 DH_m + \bar{D}\bar{\beta}(2\partial_m H^m - \frac{1}{2}\bar{D}\bar{\beta} - D^2 B)] \tag{5.47}$$

which has the gauge invariances

$$\delta H^m = \bar{D}^2\bar{N}^m + D^2 N^m + \partial^m P \tag{5.48}$$

$$\delta\bar{\beta} = D^2\bar{D}P \tag{5.49}$$

$$\delta B = 2\partial_m N^m - \frac{1}{2}\bar{D}^2 P + D\lambda \tag{5.50}$$

and equations of motion

$$D\bar{D}^2 DH^m - \partial^m\bar{D}\bar{\beta} = 0 \tag{5.51}$$

$$\bar{D}_{\dot{\alpha}}(2\partial_m H^m - \bar{D}\bar{\beta} - D^2 B) = 0 \tag{5.52}$$

$$D^2\bar{D}\bar{\beta} = 0 \tag{5.53}$$

Although, as discussed above, the action is not real in Minkowski signature, the solutions to the equations of motion can be chosen to be. To see this, first note we can define $E = \bar{D}\bar{\beta}$ and work with the constrained field E, satisfying $\bar{D}^2 E = 0$, instead of $\bar{\beta}$:

$$D\bar{D}^2 DH^m - \partial^m E = 0 \tag{5.54}$$

$$\bar{D}_{\dot{\alpha}}(2\partial_m H^m - E - D^2 B) = 0 \tag{5.55}$$

$$D^2 E = 0 \tag{5.56}$$

The third equation is the complex conjugate in Minkowski signature of the constraint $\bar{D}^2 E = 0$, and the first equation above is manifestly real if H^m and E are. The second equation can be written as $2\partial_m H^m - E - D^2 B - \bar{D}^2\bar{B} = 0$ for some $\bar{B}$. So if we choose B to be the complex conjugate of $\bar{B}$, this equation is also real. In the next subsection we will see how this gives a superspace description of four dimensional supergravity.

5.2.2 Supergravity

We expect the massless level of dimensionally reduced heterotic SFT to describe $N = 1$ $d = 4$ supergravity plus a tensor multiplet. One way to formulate this theory in four dimensional superspace is with the supergravity superfield H_m plus a linear superfield [47], i.e. a superfield satisfying $D^2 E = 0$ and $\bar{D}^2 E = 0$. The action is

$$S = \frac{1}{8} \int d^4x d^4\theta [H^m D\bar{D}^2 D H_m + 2E\partial^m H_m - \frac{1}{2} E^2] \tag{5.57}$$

which has the gauge invariances

$$\delta H^m = \bar{D}^2 \bar{N}^m + D^2 N^m + \partial^m P \tag{5.58}$$

$$\delta E = D\bar{D}^2 D P \tag{5.59}$$

and equations of motion

$$D\bar{D}^2 D H^m - \partial^m E = 0 \tag{5.60}$$

$$D^2 \bar{D}_{\dot{\alpha}} (2\partial_m H^m - E) = \bar{D}^2 D_\alpha (2\partial_m H^m - E) = 0 \tag{5.61}$$

This is equivalent to the equations obtained in the hybrid SFT since 5.61 implies that $2\partial_m H^m - E = D^2 B + \bar{D}^2 \bar{B}$ for some B and $\bar{B}$.

5.2.3 Real ten-dimensional supergravity action

Similarly to what we did in the previous subsection for the four-dimensional supergravity action, we can make the action 5.38 real by the following procedure: first we define $E = \bar{D}\bar{\beta}$. Obviously E is constrained, $\bar{D}^2 E = 0$. However, we can make E unconstrained by adding an extra superfield $\bar{B}$ such that the equation of motion from varying $\bar{B}$ is the conjugate of the last equation above, $\bar{D}^2 \left(\partial^i C_i + \partial_i \bar{C}^i - \frac{1}{4} E \right) = 0$, with E real. We then impose $B^\dagger = \bar{B}$. Thus, we propose the following real action

$$
\begin{aligned}
S_2 = \frac{1}{8}\int d^{10}x d^4\theta \\
H^M\left(D\bar{D}^2 D H_M + 4\partial^j\partial_j H_M - 2\partial_M E + 8\bar{D}^2\partial_k \bar{A}^k_M - 8D^2\partial^j A_{jM}\right. \\
\left.-4\partial_M\partial^j C_j - 4\partial_M\partial_j\bar{C}^j\right) + C_i\left(2\partial^i\partial^j C_j + 2\sqrt{2}\epsilon^{ijk}\bar{D}^2\partial_j C_k - 8\partial^M\bar{D}^2\bar{A}^i_M\right. \\
\left.+2\partial^i(D^2 B + \bar{D}^2\bar{B}) - 2\bar{D}^2 D^2\bar{C}^i + 2\partial^i E + 4\partial^i\partial_j\bar{C}^j\right) + \\
\bar{C}^i\left(2\partial_i\partial_j\bar{C}^j - 2\partial_i(\bar{D}^2\bar{B} + D^2 B) + 8D^2\partial_M A^M_i - 2\sqrt{2}\epsilon_{ijk}D^2\partial^j\bar{C}^k + 2\partial_i E\right) \\
-8\sqrt{2}D^2 A^M_i\epsilon^{ijk}\partial_j A_{kM} + \bar{D}^2\bar{A}^i_M\left(8D^2 A^M_i + 8\sqrt{2}\epsilon_{ijk}\partial^j\bar{A}^{kM}\right) \\
-E\left(D^2 B - \bar{D}^2\bar{B} + \frac{1}{2}E\right)
\end{aligned}
\tag{5.62}
$$

with gauge transformations

$$\delta H^M = D^2\bar{N}^M - \bar{D}^2 N^M + \partial^M P \tag{5.63}$$

$$\delta A^M_i = \partial_i\bar{N}^M + \partial^M\lambda_i + D\omega^M_i \tag{5.64}$$

$$\delta\bar{A}^i_M = \partial^i N_M + \partial_M\bar{\lambda}^i + \bar{D}\bar{\omega}^i_M \tag{5.65}$$

$$\delta C_i = \partial_i P - D^2\lambda_i - 2\sqrt{2}\epsilon_{ijk}\partial^j\bar{\lambda}^k + \bar{D}\omega_i \tag{5.66}$$

$$\delta\bar{C}^i = \partial^i P - 2\sqrt{2}\epsilon^{ijk}\partial_j\lambda_k + \bar{D}^2\bar{\lambda}^i + D\bar{\omega}^i \tag{5.67}$$

$$\delta B = 2\partial_M\bar{N}^M - \frac{1}{2}\bar{D}^2 P + 4\partial^j\lambda_j + D\omega \tag{5.68}$$

$$\delta E = D\bar{D}^2 DP - 2D^2\partial^j\lambda_j + 2\bar{D}^2\partial_j\bar{\lambda}^j - 2\partial_j D\bar{\omega}^j - 2\partial^j\bar{D}\omega_j \tag{5.69}$$

This action describes exactly the same physical states as 5.38.

5.2.4 Massless level in RNS

Let us do a similar derivation using the RNS formulation of [21], in order to compare. Here we work only in the NS sector. The string field has ghost number zero and picture zero. In the gauge $\xi_0\phi = 0$, the massless string field reads

$$
\begin{aligned}
\Phi = c\xi e^{-\phi}\psi^M\bar{c}\bar{\partial}x^N A_{MN} + 2ic\partial c^+\xi\partial\xi e^{-2\phi}\bar{c}\bar{\partial}x^M\bar{E}_M + c\partial c^+\xi e^{-\phi}\psi^M E_M \\
+ic\xi\partial\xi e^{-2\phi}\bar{c}\bar{\partial}^2\bar{c}B + ic\xi\eta\bar{B}
\end{aligned}
\tag{5.70}
$$

where $c^+ = \frac{1}{2}(c+\bar{c})$ and the barred and unbarred fields are independent.

The linearized action is

$$S = \int d^{10}x [A^{MN}(\frac{1}{4}\Box A_{MN} - \partial_M \bar{E}_N - \partial_N E_M)$$
$$-\bar{E}^M(\bar{E}_M + 2\partial_M \bar{B}) - E^M(E_M - 2\partial_M B) + 2\bar{B}\Box B] \tag{5.71}$$

with gauge invariances

$$\delta A_{MN} = \partial_N \xi_M + \partial_M \bar{\xi}_N \tag{5.72}$$
$$\delta E_M = \frac{1}{2}\Box \xi_M + \partial_M f \tag{5.73}$$
$$\delta \bar{E}_M = \frac{1}{2}\Box \bar{\xi}_M - \partial_M f \tag{5.74}$$
$$\delta B = -\frac{1}{2}\partial^M \bar{\xi}_M + f \tag{5.75}$$
$$\delta \bar{B} = \frac{1}{2}\partial^M \xi_M + f \tag{5.76}$$

The RNS and hybrid formalisms can be related by the field redefinitions 2.77-2.81. This leads to the following identifications of the massless fields:

$$A_{mN} \propto D\sigma_m \bar{D} H_N|, \tag{5.77}$$
$$A^j_N \propto \bar{D}^2 \bar{A}^j_N|, \quad A^N_j \propto D^2 A^N_j|, \tag{5.78}$$
$$E_m \propto D\sigma_m \bar{D}\bar{D}\bar{\beta}|, \quad \bar{E}_M \propto D^2\bar{D}^2 H_M|, \tag{5.79}$$
$$E_j \propto D^2\bar{D}^2 C_j|, \quad E^j \propto D^2\bar{D}^2\bar{C}^j| \tag{5.80}$$
$$B \propto D^2 B^{(hybrid)}|, \quad \bar{B} \propto \bar{D}\bar{\beta}| \tag{5.81}$$

where the vertical bar denotes the $\theta = \bar{\theta} = 0$ component. However, the hybrid formalism includes extra states with no RNS equivalent which will not be physical. To see this, first note that we can use N^M and $\bar{N}^M$ in 5.39 to gauge H^M to Wess-Zumino gauge, where only the $\theta\bar{\theta}$ and $\theta^2\bar{\theta}^2$ terms survive for bosonic states. $\omega^M_{i\alpha}$ can be used to gauge away all of A^M_i except the θ^2 and $\theta^2\bar{\theta}^2$ terms. The latter has no RNS equivalent, but it can be easily checked that it is auxiliary by deriving the equation of motion from varying 5.38 with respect to $\bar{A}^j_N$. The components of $\bar{A}^j_N$ work in an entirely analogous way. For C_i, we can use λ_i and $\omega_{i\alpha}$ to gauge away everything except the $\theta^2\bar{\theta}^2$ term (this works similarly to a Wess-Zumino gauge, but we have an extra parameter to gauge away the $\theta\bar{\theta}$ term). $\bar{C}^i$ is analogous. $\bar{\beta}_{\dot{\alpha}}$ can be gauged, using $\alpha_{\dot{\alpha}}$, to a form where no terms with two $\bar{\theta}$'s appear. Among the remaining bosonic components, the $\theta\bar{\theta}$ term can be shown to be auxiliary by the B equation of motion. Finally, B can be gauged, by ω_α, to the form $\theta^2 G(\bar{y}, \bar{\theta})$, where G is anti-chiral. The $\theta^2\bar{\theta}^2$ component can be shown to be auxiliary by the $\bar{\beta}$ equation of motion. All the remaining bosonic components of all fields are accounted for in 5.77-5.81. Thus, there are no extra physical states.

If we restrict to the CY independent states, it is easy to check that the RNS action

$$S = \int d^4x [A^{mn}(\frac{1}{4}\Box A_{mn} - \partial_m \bar{E}_n - \partial_n E_m)$$
$$-\bar{E}^m(\bar{E}_m + 2\partial_m \bar{B}) - E^m(E_m - 2\partial_m B) + 2\bar{B}\Box B] \tag{5.82}$$

and the hybrid action 5.47 are compatible (in the NS sector) upon identifying the fields as in 5.77-5.81. Note that the hybrid field, even in the NS CY independent sector, still includes some extra states that do not correspond to any RNS state, but these are all pure gauge. Of course, in addition, the hybrid field includes the R sector.

5.3 Difficulties with adding interactions

The obvious next step would be to add interactions to the theory. However, this is not straightforward, mainly because the linearized equations of motion involve zero modes of G^- and $\bar{b}$. To see this more explicitly, we can try to define the cubic terms in the action. The strategy will be to make simplifying assumptions which allow us to determine the cubic interactions in some sector, and then try to determine the complete cubic term by gauge invariance. This second step will fail. Our simplifying assumptions will be $\Sigma_{-1} = \Sigma_1 = 0$ and $G_4^-\Sigma_0 = 0$. Then it is easy to check that eqs.5.14-5.16 imply that Σ_0 satisfies the on-shell condition

$$(G^+ + \bar{Q})\tilde{G}^+(G^-)_0\Sigma_0 = 0 \tag{5.83}$$

or, since $\bar{c}_0\Sigma_0 = 0$,

$$(\{\bar{c}_0, \bar{Q}\}\tilde{G}^+(G^-)_0 + (G^+ + \bar{Q})\tilde{G}^+)\Sigma_0 = 0 \tag{5.84}$$

This means that the cubic vertex must reproduce the on-shell three-point amplitude for Σ_0,

$$\begin{aligned}\langle (G^- - \bar{b})_0 \Sigma_0 (G^+ + \bar{Q})(G^- - \bar{b})_0 \Sigma_0 \tilde{G}^+ (G^- - \bar{b})_0 \Sigma_0 \rangle = \\ \langle \Phi_0 G_4^+ \Phi_0 \tilde{G}_4^+ \Phi_0 \rangle + \langle \Phi_0 (G_6^+ + \bar{Q}) \Phi_0 \tilde{G}_6^+ \Phi_0 \rangle \end{aligned} \tag{5.85}$$

where $\Phi_0 = (G_6^- - \bar{b})_0 \Sigma_0$. When Σ_0 goes off-shell, we must use the closed string field products and multilinear functions, as defined in [6]. Thus, the cubic terms when $\Sigma_{-1} = \Sigma_1 = G_4^- \Sigma_0 = 0$ is

$$\{\Phi_0, G_4^+ \Phi_0, \tilde{G}_4^+ \Phi_0\} + \{\Phi_0, G_6^+ \Phi_0, \tilde{G}_6^+ \Phi_0\} \tag{5.86}$$

We should be able to deduce the other terms (involving Σ_{-1}, Σ_1 and $G_4^- \Sigma_0$) from gauge invariance. Let us try to lift the $G_4^- \Sigma_0 = 0$ assumption while keeping $\Sigma_1 = \Sigma_{-1} = 0$. In order to preserve gauge invariance, the linearized variation of the cubic term must be proportional to the linearized equation of motion, i.e.

$$\delta^{(1)} S_3 = -\langle (\tilde{G}_6'^+ G_6'^+ + \tilde{G}_4'^+ G_4'^+) \Sigma_0, \delta^{(2)} \Sigma_0 \rangle \tag{5.87}$$

But the linearized e.o.m. involves the zero modes of G^- and $\bar{b}$. Since these have conformal weights $(2,0)$ and $(0,2)$, they do not act as derivatives on the string field products, which prevents us writing a cubic term whose variation is 5.87 using only the G, $\tilde{G}$, G' and $\tilde{G}'$ operators, the string field products and the inner product.

5.4 Discussion

In this chapter we have constructed a linearized theory for heterotic string fields using the hybrid formalism. We found that the heterotic spectrum can be described in terms of three string fields, similarly to the open superstring. We can formulate the theory either in terms of ghost number 2 fields annihilated by $\bar{c}_0$ or ghost number 1 fields annihilated by $\bar{b}_0$. Reflecting these somewhat unusual constraints, the linearized equations of motion also take an unusual form involving the zero modes of G^- and $\bar{b}$. We also analyzed explicitly the massless level. Although the theory is not real in Minkowski signature, we found that the equations of motion in the massless compactification independent sector can be written in a real form, and describe four-dimensional supergravity plus a tensor multiplet in superspace. If we include the CY states, since all the NS physical states coincide with the RNS formulation, and the action has manifest spacetime supersymmetry, the full linearized action must describe ten-dimensional supergravity in terms of four-dimensional superfields.

Having defined a linearized theory, we might expect to be able to construct an interacting theory following the ideas of [21] and [12]. However, given that the equations of motion now involve G^- and $\bar{b}$ zero modes, the algebraic structures of the RNS formulation do not fully carry over. In particular, it is not clear how to construct string field products on which $G_6'^+$ and $\tilde{G}_6'^+$ would act in a simple way. We leave these questions for future work.

Capítulo 6

Conclusion

In this thesis, we explored some results in superstring field theories, both open and heterotic, using the hybrid formalism. We constructed an explicit open SFT solution corresponding to the leading stringy correction to the BPST instanton. After compactifying to four dimensions, the massless states of the open superstring include $D = 4$ super-Yang-Mills fields and the instanton solution is obtained by requiring the Yang-Mills field-strength to be self-dual. However, it is difficult to generalize the concept of self-dual field strength to SFT. One can instead define the four-dimensional instanton solution as a localized half-BPS solution to the equations of motion, i.e. a solution which is annihilated by half of the $N = 1$ $D = 4$ spacetime supersymmetries. An SFT solution was constructed perturbatively in the inverse of the instanton size - that is, a large instanton expansion. We found the first stringy correction to the instanton solution, corresponding to turning on certain massive spin-2 and spin-0 fields. The hybrid formalism might also be useful in other approaches to SFT instantons, such as the one developed in [34, 35, 36, 37]. Given the prominent role played by supersymmetry in instanton and BPS states, we can expect that a manifestly supersymmetric formulation will lead to many insights. In particular, it would be interesting to understand how to describe a $D3/D(-1)$ brane system, following [34], in the hybrid formulation.

We also constructed an effective action from open superstring field theory. Following the methods of [44], we showed that it reproduces super Yang-Mills in superspace, plus higher derivative corrections. It would also be interesting to relate this to [34], where it was shown that the contribution to the potential in the effective action is localized at the boundary of moduli space. This was shown generally for WZW-like theories with $N = 2$ worldsheet supersymmetry where the string field can be split in two by the $U(1)$ charge. In the hybrid formulation we already have such a splitting (into three) by a $U(1)$ charge before we can even write the action. Clearly the general argument from $N = 2$ would need some adaptation for the hybrid formalism, but at least the simple argument for the boundary terms being proportional to the full potential seems to translate easily.

For the heterotic string, we constructed a linearized SFT action with manifest $N = 1$ $D = 4$ supersymmetry. The massless level gives a formulation of ten-dimensional supergravity in terms of four-dimensional superfields. However, generalizing this to an interacting theory is not straightforward. The main difficulty in realizing a complete hybrid formulation of heterotic SFT lies in the $G^- - \bar{b}$ condition, and the fact that it does not have a well defined ρ charge. A formulation of SFT without such constraints was recently constructed [48, 49], and it would be interesting to see whether a supersymmetric formulation can be constructed using a similar approach. For bosonic closed SFT, this is achieved by adding an extra spurious free string field (inspired by Sen's approach to the R sector of the superstring in [25]). The b ghosts appear in the action, but only the quadratic term for the spurious string field. Although this has only been done for closed bosonic SFT, there is no obvious obstruction to applying similar methods to the heterotic string. This would make the splitting of the states into three string fields a straightforward matter (presumably we would also need several spurious fields).

The formulation of linearized ten-dimensional supergravity in terms of four-dimensional superfields is also novel, and we can ask whether it can be generalized to the full interacting theory. If this succeeds, it might also give some hints on how to construct the interacting heterotic SFT. I expect that these questions can be answered in future work.

Apêndice A

Notation and Conventions

We always use the mostly + convention for the spacetime metric. For the most part, we will not write Regge slope α' factors explicitly, instead setting $\alpha' = 2$.

A.1 RNS superstring

The superconformal ghosts β and γ are bosonized as follows

$$\beta(z) \cong e^{-\phi}\partial\xi, \quad \gamma(z) \cong \eta e^{\phi}, \quad \delta(\gamma) \cong e^{-\phi}, \quad \delta(\beta) \cong e^{\phi} \tag{A.1}$$

$$T^{\phi} = -\frac{1}{2}\partial\phi\partial\phi - \partial^2\phi, \quad T^{\eta\xi} = -\eta\partial\xi \tag{A.2}$$

$$\xi(z)\eta(w) \sim \frac{1}{z-w}, \quad \phi(z)\phi(w) \sim -\log(z-w). \tag{A.3}$$

The picture number operator is defined as

$$P = \oint dz\, \xi\eta - \partial\phi \tag{A.4}$$

and the picture raising and lowering operatos as

$$X(z) = \{Q_B, \xi(z)\} = G(z)\delta(\beta(z)) - \partial b(z)\delta'(\beta(z)) \tag{A.5}$$

$$Y(z) = c\partial\xi e^{-2\phi}(z) \tag{A.6}$$

A.2 $4D$ Supersymmetry

The Pauli matrices $\sigma^m_{\alpha\dot\beta}$, where $m = 0$ to 3 and $\alpha, \dot\alpha$ = 1 to 2 are four dimensional vector and spinor indices, respectively, are

$$\begin{aligned} \sigma^0 &= \begin{pmatrix} -1 & 0 \\ 0 & -1 \end{pmatrix}, \quad \sigma^1 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \\ \sigma^2 &= \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}, \quad \sigma^3 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} \end{aligned} \tag{A.7}$$

We also define barred matrices as

$$\bar\sigma^{m\dot\beta\alpha} = \epsilon^{\alpha\gamma}\epsilon^{\dot\beta\dot\rho}\sigma^m_{\gamma\dot\rho} \tag{A.8}$$

where the antisymmetric ϵ tensors are used to raise and lower spinor indices, and are defined as

$$\epsilon_{12} = -\epsilon_{21} = -1, \quad \epsilon^{12} = -\epsilon^{21} = 1, \quad \epsilon_{11} = \epsilon_{22} = \epsilon^{11} = \epsilon^{22} = 0 \tag{A.9}$$

for both $\epsilon_{\alpha\beta}$ and $\epsilon_{\dot\alpha\dot\beta}$.

The generators of Lorentz transformations for spinors are defined as

$$(\sigma^{mn})_\alpha^{\ \beta} = \frac{1}{4}\left(\sigma^m_{\alpha\dot\alpha}\bar\sigma^{n\dot\alpha\beta} - \sigma^n_{\alpha\dot\alpha}\bar\sigma^{m\dot\alpha\beta}\right) \tag{A.10}$$

$$(\bar\sigma^{mn})^{\dot\alpha}_{\ \dot\beta} = \frac{1}{4}\left(\bar\sigma^{m\dot\alpha\alpha}\sigma^n_{\alpha\dot\beta} - \bar\sigma^{n\dot\alpha\alpha}\sigma^m_{\alpha\dot\beta}\right) \tag{A.11}$$

We also use the following notation for contracting spinor indices:

$$\psi\chi \equiv \psi^\alpha\chi_\alpha, \quad \bar\psi\bar\chi \equiv \bar\psi^{\dot\alpha}\bar\chi_{\dot\alpha} = -(\psi\chi)^\dagger \tag{A.12}$$

We work in $N = 1$, $D = 4$ superspace with the usual variables $(x^m, \theta^\alpha, \bar\theta^{\dot\alpha})$, where θ is anticommuting and $\bar\theta^{\dot\alpha} = (\theta^\alpha)^\dagger$. The supersymmetry generators in superspace are

$$q_\alpha = \frac{\partial}{\partial\theta^\alpha} - \frac{i}{2}(\sigma^m\bar\theta)_\alpha\partial_m \tag{A.13}$$

and

$$\bar q_{\dot\alpha} = -\frac{\partial}{\partial\bar\theta^{\dot\alpha}} + \frac{i}{2}(\theta\sigma^m)_{\dot\alpha}\partial_m \tag{A.14}$$

which are easily shown to satisfy the supersymmetry algebra

$$\{q_\alpha, \bar{q}_{\dot{\alpha}}\} = i\sigma^m_{\alpha\dot{\alpha}}\partial_m \tag{A.15}$$

We will also often use the supersymmetric fermionic derivatives

$$D_\alpha = \frac{\partial}{\partial\theta^\alpha} + \frac{i}{2}(\sigma^m\bar{\theta})_\alpha\partial_m, \tag{A.16}$$

$$\bar{D}_{\dot{\alpha}} = \frac{\partial}{\partial\bar{\theta}^{\dot{\alpha}}} + \frac{i}{2}(\theta\sigma^m)_{\dot{\alpha}}\partial_m \tag{A.17}$$

which are easily shown to anti-commute with the supersymmetry generators.

Apêndice B

Star Products

In this appendix we detail the calculation of the relevant star products. We'll use conformal mappings to the upper half complex plane (see [29]). The prescription we use is as follows: given a basis e^r of the string field space, the star product of two arbitrary string fields A and B is

$$A * B = \sum_r \left(e_r^C, A, B \right) e_r \tag{B.1}$$

where C denotes conjugate states. These are defined by

$$\left(e_r^C, e_s \right) = \delta_{rs} \tag{B.2}$$

where

$$(A, B) = \langle (I(z) \circ A(0))\, B(0) \rangle_{\text{UHP}} \tag{B.3}$$

The right-hand side is a correlator in the upper half plane, and $I(z)$ is the conformal transformation

$$I(z) = -\frac{1}{z} \tag{B.4}$$

The coefficient is given by the correlator

$$(A, B, C) = \langle f_{-1} \circ A(0) f_0 \circ B(0) f_1 \circ C(0) \rangle_{\text{UHP}} \tag{B.5}$$

with the conformal transformations

$$f_n(z) = \tan\left[\frac{n\pi}{3} + \frac{2}{3} \arctan z \right] \tag{B.6}$$

The correlators are given by

$$\left\langle e^{ik_1x}\left(z_1\right)e^{ik_2x}\left(z_2\right)\right\rangle = i(2\pi)^4\delta^4\left(k_1+k_2\right)\left|z_1-z_2\right|^{-k_1\cdot k_2} \tag{B.7}$$

$$\left\langle \theta^2\bar{\theta}^2 e^{-i\rho+iH_C}\right\rangle = 1 \tag{B.8}$$

B.1 Massless level

We will denote the massless level of the star product by $\otimes$. In the $p\theta$ sector, there are no non-trivial massless contributions. For example, it is immediate that

$$\theta^\alpha \otimes \theta^\beta = \theta^\alpha\theta^\beta \tag{B.9}$$

For the x^m, however, there are non-trivial (higher derivative/higher order in α') terms, even at the massless level. Computing the product of two functions of x^m is slightly tricky because x^m itself is not a well-behaved operator in the worldsheet theory. Rather, the actual operators are worldsheet derivatives of x^m and plane waves $e^{ik\cdot x}$. The solution is to assume that the functions of x^m we are interested in admit Fourier transform representations and to compute the star product of functions x^m in terms of the products of exponentials, that can be obtained by the usual methods. That is,

$$F(x)*G(x) = \int \frac{d^4k_1}{(2\pi)^4}\frac{d^4k_2}{(2\pi)^4}\tilde{F}\left(k_1\right)\tilde{G}\left(k_2\right)e^{ik_1x}*e^{ik_2x} \tag{B.10}$$

We need then to calculate the product $e^{ik_1x}*e^{ik_2x}$, which amounts to calculating the coefficient $\left(e^{-ikx},e^{ik_1x},e^{ik_2x}\right)$:

$$\begin{aligned}\left(e^{-ikx},e^{ik_1x},e^{ik_2x}\right) &= \left(\frac{8}{3}\right)^{-\left(k^2+k_2^2\right)/2}\left(\frac{2}{3}\right)^{-k_1^2/2}\left\langle e^{-ikx}(-\sqrt{3})e^{ik_1x}(0)e^{ik_2x}(\sqrt{3})\right\rangle \\ &= i(2\pi)^4 2^{-2k^2+2k_1\cdot k_2}\sqrt{3}^{k^2+k_1\cdot k_2}\delta^4\left(k_1+k_2-k\right)\end{aligned} \tag{B.11}$$

So the $\otimes$ product is

$$e^{ik_1x}\otimes e^{ik_2x} = e^{\ln 2\left[-2(k_1+k_2)^2+2k_1\cdot k_2^2\right]+\ln\sqrt{3}\left[(k_1+k_2)^2+k_1\cdot k_2^2\right]}e^{i(k_1+k_2)x} \tag{B.12}$$

We conclude that

$$F(x) \otimes G(x) = \tag{B.13}$$
$$\exp\left[-2\ln 2\left(\frac{\partial_F^2}{\partial x^2}+\frac{\partial_G^2}{\partial x^2}+\frac{\partial_F\partial_G}{\partial x^2}\right)-\ln\sqrt{3}\left(\frac{\partial_F^2}{\partial x^2}+\frac{\partial_G^2}{\partial x^2}+3\frac{\partial_F\partial_G}{\partial x^2}\right)\right]F(x)G(x)$$

B.2 First massive level

The star product between two fields $F(x,\theta,\bar{\theta})$, $G(x,\theta,\bar{\theta})$ will contain terms proportional to $\partial\theta^\alpha$, $\partial\bar{\theta}^{\dot{\alpha}}$ and Π^m. The conjugate states are proportional to d_α, $\bar{d}_{\dot{\alpha}}$ and Π_m, respectively. Thus, we need to calculate the coefficients $(d_\alpha V(x,\theta,\bar{\theta}),F,G)$, $(\bar{d}_{\dot{\alpha}}V(x,\theta,\bar{\theta}),F,G)$ and $(\Pi^m V(x,\theta,\bar{\theta}),F,G)$. The relevant conformal transformations are

$$f\circ\left(d_\alpha V(x,\theta,\bar{\theta})\right)(z) = f'(z)d_\alpha V(f(z))+\frac{f''(z)}{2f'(z)}D_\alpha V \tag{B.14}$$

$$f\circ\left(\bar{d}_{\dot{\alpha}} V(x,\theta,\bar{\theta})\right)(z) = f'(z)\bar{d}_{\dot{\alpha}} V(f(z))+\frac{f''(z)}{2f'(z)}\bar{D}_{\dot{\alpha}} V \tag{B.15}$$

$$f\circ\left(\Pi^m V(x,\theta,\bar{\theta})\right)(z) = f'(z)d_\alpha V(f(z))+\frac{f''(z)}{2f'(z)}\partial^m V \tag{B.16}$$

For the first coefficient we obtain

$$(d_\alpha V(x,\theta,\bar{\theta}),F,G) = \frac{8}{3}\left\langle d_\alpha V(-\sqrt{3})F(0)G(\sqrt{3})\right\rangle - \frac{2}{\sqrt{3}}\left\langle D_\alpha V(-\sqrt{3})F(0)G(\sqrt{3})\right\rangle \tag{B.17}$$
$$= -\frac{2}{3\sqrt{3}}(-1)^V\left(\left\langle V(-\sqrt{3})D_\alpha F(0)G(\sqrt{3})\right\rangle-(-1)^F\left\langle V(-\sqrt{3})F(0)D_\alpha G(\sqrt{3})\right\rangle\right)$$

where $(-1)^V$ is 1 if V is commuting and -1 if anticommuting. Note that $(-1)^V\partial\theta^\alpha W$ is the conjugate of $d_\alpha V$ if W is the conjugate of V. We conclude that

$$F * G\big|_{\partial\theta} = \frac{C}{2}\partial\theta^\alpha\left(-D_\alpha F\otimes G+F\otimes D_\alpha G\right) \tag{B.18}$$

where $\big|_{\partial\theta}$ denotes the terms in the star product proportional to $\partial\theta^{\alpha}$, and $C = \frac{4}{3\sqrt{3}}$. A similar calculation yields the terms proportional to $\partial\bar{\theta}^{\dot{\alpha}}$ and Π^m:

$$\begin{aligned}(F * G)_{M^2=2} =& \frac{C}{2}\left[\partial\theta^{\alpha}\left(-D_{\alpha}F \otimes G + F \otimes D_{\alpha}G\right) + \partial\bar{\theta}^{\dot{\alpha}}\left(-\bar{D}_{\dot{\alpha}}F \otimes G + F \otimes \bar{D}_{\dot{\alpha}}G\right)\right. \\ & \left. + \Pi^m\left(-\partial_m F \otimes G + F \otimes \partial_m G\right)\right]\end{aligned} \tag{B.19}$$

B.3 Products involving d and $\partial\theta$

To calculate the equations of motion, we also need to evaluate products of the form $d_{\alpha}F * \bar{d}_{\dot{\alpha}}G$ and $\bar{d}_{\dot{\alpha}}F * \partial\bar{\theta}^{\dot{\beta}}G$. At the massless level, we have

$$\begin{aligned}(V(x,\theta,\bar{\theta}), d_{\alpha}F, \bar{d}_{\dot{\alpha}}G) =& \frac{16}{9}\left\langle V(-\sqrt{3})d_{\alpha}F(0)\bar{d}_{\dot{\alpha}}G(\sqrt{3})\right\rangle \\ & + \frac{4}{3\sqrt{3}}\left\langle V(-\sqrt{3})d_{\alpha}F(0)\bar{D}_{\dot{\alpha}}G(\sqrt{3})\right\rangle\end{aligned} \tag{B.20}$$

$$(V(x,\theta,\bar{\theta}), \bar{d}_{\dot{\alpha}}F, \partial\bar{\theta}^{\dot{\beta}}G) = \frac{16}{27}\delta^{\dot{\beta}}_{\dot{\alpha}}\left\langle V(-\sqrt{3})F(0)G(\sqrt{3})\right\rangle \tag{B.21}$$

Therefore

$$\begin{aligned}d_{\alpha}F \otimes \bar{d}_{\dot{\alpha}}G =& C^2\left(\bar{D}_{\dot{\alpha}}F \otimes D_{\alpha}G - \frac{1}{2}(-1)^F D_{\alpha}\bar{D}_{\dot{\alpha}}F \otimes G\right. \\ & \left. + \frac{1}{2}(-1)^F F \otimes \bar{D}_{\dot{\alpha}}D_{\alpha}G - \frac{1}{4}D_{\alpha}F \otimes \bar{D}_{\dot{\alpha}}G\right)\end{aligned} \tag{B.22}$$

$$\bar{d}_{\dot{\alpha}}F \otimes \partial\bar{\theta}^{\dot{\beta}}G = C^2\delta^{\dot{\beta}}_{\dot{\alpha}}F \otimes G \tag{B.23}$$

Now we move on to the first massive level. To get the terms proportional to Π^m, we need the coefficient

$$\begin{aligned}&\left(\Pi^m V(x,\theta,\bar{\theta}), d_{\alpha}F, \bar{d}_{\dot{\alpha}}G\right) = C^2\left((\Pi^m V, \bar{D}_{\dot{\alpha}}F, D_{\alpha}G)\right. \\ & - \frac{1}{2}(-1)^F(\Pi^m V, D_{\alpha}\bar{D}_{\dot{\alpha}}F, G) + \frac{1}{2}(-1)^F(\Pi^m V, F, \bar{D}_{\dot{\alpha}}D_{\alpha}G) \\ & \left. - \frac{1}{4}(\Pi^m V, D_{\alpha}F, \bar{D}_{\dot{\alpha}}G)\right) + (-1)^F C^3\sigma^m_{\alpha\dot{\alpha}}(V, F, G)\end{aligned} \tag{B.24}$$

where the last term comes from the $O(1/(z_{12}z_{13}z_{23}))$ term in the OPE $\bar{d}_{\dot{\alpha}}(z_1)d_{\alpha}(z_2)\Pi^m(z_3)$.

For $d_{\dot\alpha}F * \partial\bar\theta^{\dot\beta}G$ we have simply

$$\left(\Pi^m V, \bar d_{\dot\alpha}F, \partial\bar\theta^{\dot\beta}G\right) = C^2\delta_{\dot\alpha}^{\dot\beta}\left(\Pi^m V, F, G\right) \tag{B.25}$$

We conclude that

$$d_\alpha F * \bar d_{\dot\alpha}G\big|_\Pi = C^2\left(\bar D_{\dot\alpha}F * D_\alpha G - \frac{1}{2}(-1)^F D_\alpha \bar D_{\dot\alpha}F * G\right. \tag{B.26}$$

$$\left.+\frac{1}{2}(-1)^F F * \bar D_{\dot\alpha}D_\alpha G - \frac{1}{4}D_\alpha F * \bar D_{\dot\alpha}G\right)\Bigg|_\Pi + (-1)^F iC^3\Pi^m_{\alpha\dot\alpha}F\otimes G$$

$$\bar d_{\dot\alpha}F * \partial\bar\theta^{\dot\beta}G\big|_\Pi = C^2\delta^{\dot\beta}_{\dot\alpha}F * G\big|_\Pi \tag{B.27}$$

To obtain the terms proportional to d and $\bar d$, we need the following coefficients:

$$\left(\partial\theta^\beta V(x,\theta,\bar\theta), d_\alpha F, \bar d_{\dot\alpha}G\right) = \tag{B.28}$$

$$-(-1)^V C^3\delta_\alpha^\beta\left((-1)^F\left\langle V(-\sqrt{3})\bar D_{\dot\alpha}F(0)G(\sqrt{3})\right\rangle + \frac{1}{2}\left\langle V(-\sqrt{3})F(0)\bar D_{\dot\alpha}G(\sqrt{3})\right\rangle\right)$$

$$\left(\partial\bar\theta^{\dot\beta} V(x,\theta,\bar\theta), d_\alpha F, \bar d_{\dot\alpha}G\right) = \tag{B.29}$$

$$-(-1)^V C^3\delta_{\dot\alpha}^{\dot\beta}\left(\left\langle V(-\sqrt{3})F(0)D_\alpha G(\sqrt{3})\right\rangle + \frac{1}{2}(-1)^F\left\langle V(-\sqrt{3})D_\alpha F(0)G(\sqrt{3})\right\rangle\right)$$

$$\left(\partial\theta^\beta V, \bar d_{\dot\alpha}F, \partial\bar\theta^{\dot\beta}G\right) = \left(\partial\bar\theta^{\dot\gamma}V, \bar d_{\dot\alpha}F, \partial\bar\theta^{\dot\beta}G\right) = 0 \tag{B.30}$$

Therefore

$$d_\alpha F * \bar d_{\dot\alpha}G\big|_d = -C^3 d_\alpha\left((-1)^F\bar D_{\dot\alpha}F\otimes G + \frac{1}{2}F\otimes\bar D_{\dot\alpha}G\right) \tag{B.31}$$

$$d_\alpha F * \bar d_{\dot\alpha}G\big|_{\bar d} = -C^3\bar d_{\dot\alpha}\left(F\otimes D_\alpha G + \frac{1}{2}(-1)^F D_\alpha F\otimes G\right) \tag{B.32}$$

$$\bar d_{\dot\alpha}F * \partial\bar\theta^{\dot\beta}G\big|_d = \bar d_{\dot\alpha}F * \partial\bar\theta^{\dot\beta}G\big|_{\bar d} = 0 \tag{B.33}$$

B.4 Products involving ρ and H_C

The last product we need for the equations of motion is $e^{i\rho} * e^{-2i\rho+iH_C}$. We will need the following conformal transformations

$$f \circ e^{ni\rho}(z) = \left(f'(z)\right)^{-(n^2+n)/2} e^{ni\rho}(f(z)) \tag{B.34}$$

$$f \circ e^{niH_C}(z) = \left(f'(z)\right)^{3(n^2-n)/2} e^{niH_C}(f(z)) \tag{B.35}$$

$$f \circ \partial\rho(z) = f'(z)\partial\rho(f(z)) + \frac{if''(z)}{2f'(z)} \tag{B.36}$$

$$f \circ \partial H_C(z) = f'(z)\partial H_C(f(z)) + \frac{3if''(z)}{2f'(z)} \tag{B.37}$$

The relevant coefficients are

$$\left(1, e^{i\rho}, e^{-2i\rho+iH_C}\right) = C^{-2} \tag{B.38}$$

$$\left(\partial\rho, e^{i\rho}, e^{-2i\rho+iH_C}\right) = -\frac{3i}{2}C^{-1} \tag{B.39}$$

$$\left(\partial H_C, e^{i\rho}, e^{-2i\rho+iH_C}\right) = -\frac{3i}{2}C^{-1} \tag{B.40}$$

where to conjugates of 1, ∂H_C and $\partial\rho$ are, respectively, $e^{-i\rho+iH_C}$, $-\partial H_C e^{-i\rho+iH_C}/3$ and
$\partial\rho e^{-i\rho+iH_C}$. Thus, the product we are looking for is (truncated to the first massive level)

$$e^{i\rho} * e^{-2i\rho+iH_C} = C^{-2}e^{-i\rho+iH_C} + \frac{i}{2}C^{-1}(\partial H_C - 3\partial\rho)e^{-i\rho+iH_C} \tag{B.41}$$

Referências

[1] K. Ohmori, "A Review on tachyon condensation in open string field theories," other thesis, 2, 2001.

[2] A. Sen, "Universality of the tachyon potential," *JHEP* **12** (1999) 027, arXiv:hep-th/9911116.

[3] A. Sen and B. Zwiebach, "String Field Theory: A Review," arXiv:2405.19421 [hep-th].

[4] E. Witten, "Noncommutative Geometry and String Field Theory," *Nucl. Phys. B* **268** (1986) 253–294.

[5] M. Schnabl, "Analytic solution for tachyon condensation in open string field theory," *Adv. Theor. Math. Phys.* **10** (2006) no. 4, 433–501, arXiv:hep-th/0511286.

[6] B. Zwiebach, "Closed string field theory: Quantum action and the B-V master equation," *Nucl. Phys. B* **390** (1993) 33–152, arXiv:hep-th/9206084.

[7] M. R. Gaberdiel and B. Zwiebach, "Tensor constructions of open string theories. 1: Foundations," *Nucl. Phys. B* **505** (1997) 569–624, arXiv:hep-th/9705038.

[8] E. Witten, "Interacting Field Theory of Open Superstrings," *Nucl. Phys. B* **276** (1986) 291–324.

[9] C. R. Preitschopf, C. B. Thorn, and S. A. Yost, "SUPERSTRING FIELD THEORY," *Nucl. Phys. B* **337** (1990) 363–433.

[10] C. Wendt, "Scattering Amplitudes and Contact Interactions in Witten's Superstring Field Theory," *Nucl. Phys. B* **314** (1989) 209–237.

[11] N. Berkovits, "Review of open superstring field theory," arXiv:hep-th/0105230.

[12] N. Berkovits, "SuperPoincare invariant superstring field theory," *Nucl. Phys. B* **450** (1995) 90–102, arXiv:hep-th/9503099. [Erratum: Nucl.Phys.B 459, 439–451 (1996)].

[13] H. Kunitomo and Y. Okawa, "Complete action for open superstring field theory," *PTEP* **2016** (2016) no. 2, 023B01, `arXiv:1508.00366 [hep-th]`.

[14] T. Erler, S. Konopka, and I. Sachs, "Resolving Witten's superstring field theory," *JHEP* **04** (2014) 150, `arXiv:1312.2948 [hep-th]`.

[15] T. Erler, Y. Okawa, and T. Takezaki, "A_∞ structure from the Berkovits formulation of open superstring field theory," `arXiv:1505.01659 [hep-th]`.

[16] N. Berkovits, "Covariant quantization of the Green-Schwarz superstring in a Calabi-Yau background," *Nucl. Phys. B* **431** (1994) 258–272, `arXiv:hep-th/9404162`.

[17] N. Berkovits, "A New description of the superstring," in *8th Jorge Andre Swieca Summer School: Particles and Fields*, pp. 390–418. 4, 1996. `arXiv:hep-th/9604123`.

[18] N. Berkovits and C. Vafa, "N=4 topological strings," *Nucl. Phys. B* **433** (1995) 123–180, `arXiv:hep-th/9407190`.

[19] N. Marcus, A. Sagnotti, and W. Siegel, "Ten-dimensional Supersymmetric Yang-Mills Theory in Terms of Four-dimensional Superfields," *Nucl. Phys. B* **224** (1983) 159.

[20] Y. Okawa and B. Zwiebach, "Heterotic string field theory," *JHEP* **07** (2004) 042, `arXiv:hep-th/0406212`.

[21] N. Berkovits, Y. Okawa, and B. Zwiebach, "WZW-like action for heterotic string field theory," *JHEP* **11** (2004) 038, `arXiv:hep-th/0409018`.

[22] K. Goto and H. Kunitomo, "Construction of action for heterotic string field theory including the Ramond sector," *JHEP* **12** (2016) 157, `arXiv:1606.07194 [hep-th]`.

[23] H. Matsunaga, "Nonlinear gauge invariance and WZW-like action for NS-NS superstring field theory," *JHEP* **09** (2015) 011, `arXiv:1407.8485 [hep-th]`.

[24] A. Sen, "Gauge Invariant 1PI Effective Superstring Field Theory: Inclusion of the Ramond Sector," *JHEP* **08** (2015) 025, `arXiv:1501.00988 [hep-th]`.

[25] A. Sen, "BV Master Action for Heterotic and Type II String Field Theories," *JHEP* **02** (2016) 087, arXiv:1508.05387 [hep-th].

[26] N. Berkovits, V. F. Juliatto, and U. M. Portugal, "Instanton solutions in open superstring field theory," *JHEP* **09** (2022) 005, arXiv:2110.07645 [hep-th].

[27] U. M. Portugal, "Super Yang-Mills action from Hybrid Superstring Field Theory," *JHEP* **01** (2025) 176, arXiv:2411.04208 [hep-th].

[28] N. Berkovits and U. M. Portugal, "Heterotic string field theory with manifest spacetime supersymmetry," *JHEP* **05** (2025) 050, arXiv:2412.15343 [hep-th].

[29] L. Rastelli and B. Zwiebach, "Tachyon potentials, star products and universality," *JHEP* **09** (2001) 038, arXiv:hep-th/0006240.

[30] B. Zwiebach, "A Proof that Witten's open string theory gives a single cover of moduli space," *Commun. Math. Phys.* **142** (1991) 193–216.

[31] N. Berkovits and C. Vafa, "On the Uniqueness of string theory," *Mod. Phys. Lett. A* **9** (1994) 653–664, arXiv:hep-th/9310170.

[32] H. Ooguri and C. Vafa, "N=2 heterotic strings," *Nucl. Phys. B* **367** (1991) 83–104.

[33] H. Sonoda and B. Zwiebach, "COVARIANT CLOSED STRING THEORY CANNOT BE CUBIC," *Nucl. Phys. B* **336** (1990) 185–221.

[34] C. Maccaferri and A. Merlano, "Localization of effective actions in open superstring field theory," *JHEP* **03** (2018) 112, arXiv:1801.07607 [hep-th].

[35] C. Maccaferri and A. Merlano, "Localization of effective actions in open superstring field theory: small Hilbert space," *JHEP* **06** (2019) 101, arXiv:1905.04958 [hep-th].

[36] L. Mattiello and I. Sachs, "On Finite-Size D-Branes in Superstring Theory," *JHEP* **11** (2019) 118, arXiv:1902.10955 [hep-th].

[37] J. Vošmera, "Generalized ADHM equations from marginal deformations in open superstring field theory," *JHEP* **12** (2019) 118, arXiv:1910.00538 [hep-th].

[38] A. A. Belavin, A. M. Polyakov, A. S. Schwartz, and Y. S. Tyupkin, "Pseudoparticle Solutions of the Yang-Mills Equations," *Phys. Lett. B* **59** (1975) 85–87.

[39] G. 't Hooft, "Computation of the Quantum Effects Due to a Four-Dimensional Pseudoparticle," *Phys. Rev. D* **14** (1976) 3432–3450. [Erratum: Phys.Rev.D 18, 2199 (1978)].

[40] N. Berkovits and M. M. Leite, "Superspace action for the first massive states of the superstring," *Phys. Lett. B* **454** (1999) 38–42, arXiv:hep-th/9812153.

[41] N. Berkovits and M. M. Leite, "First massive state of the superstring in superspace," *Phys. Lett. B* **415** (1997) 144–148, arXiv:hep-th/9709148.

[42] W. Taylor, "D-brane effective field theory from string field theory," *Nucl. Phys. B* **585** (2000) 171–192, arXiv:hep-th/0001201.

[43] E. Coletti, I. Sigalov, and W. Taylor, "Abelian and nonAbelian vector field effective actions from string field theory," *JHEP* **09** (2003) 050, arXiv:hep-th/0306041.

[44] N. Berkovits and M. Schnabl, "Yang-Mills action from open superstring field theory," *JHEP* **09** (2003) 022, arXiv:hep-th/0307019.

[45] W. Siegel, "Supergravity Superfields Without a Supermetric," HUTP-77/A068.

[46] W. Siegel and S. J. Gates, Jr., "Superfield Supergravity," *Nucl. Phys. B* **147** (1979) 77–104 HUTP-78/A019.

[47] J.-P. Derendinger, F. Quevedo, and M. Quiros, "The Linear multiplet and quantum four-dimensional string effective actions," *Nucl. Phys. B* **428** (1994) 282–330, arXiv:hep-th/9402007.

[48] Y. Okawa and R. Sakaguchi, "Closed string field theory without the level-matching condition," arXiv:2209.06173 [hep-th].

[49] H. Erbin and M. Médevielle, "Closed string theory without level-matching at the free level," *JHEP* **03** (2023) 091, arXiv:2209.05585 [hep-th].