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Massive vertex operators in string theory

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MASSIVE VERTEX OPERATORS IN STRING THEORY

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Resumo

Neste trabalho revisaremos a obtenção de operadores de vértice em teoria de cordas. Então construiremos o operador de vértice para os primeiros estados massivos da supercorda aberta em termos dos supercampos de super-Yang-Mills em d=10 a partir da expansão em produto de operadores dos vértices não massivos, utilizando o formalismo de espinores puros.

Palavras Chaves: Supercorda; Super-Yang-Mills; Espinor puro.

Áreas do conhecimento: Ciências exatas e da Terra; Física de Partículas e Campos.

Abstract

In this work we review the construction of string theory vertex operators. We then construct the vertex operator for the first massive states of the open superstring in terms of d=10 super-Yang-Mills superfields using the operator product expansions of massless vertex operators in the pure spinor formalism.

Key Words: Superstring; Super-Yang-Mills; Pure spinor.

Areas of Knowledge: Physical Sciences and Mathematics; Elementary Particle Physics and Fields.

Contents

Chapter 1

Introduction

String theory can be formulated as a two-dimensional conformal field theory where the spacetime amplitudes of physical particles are given by correlation functions of vertex operators on the worldsheet [1] [2]. These operators are local fields in one-to-one correspondence with Hilbert space states of the string conformal field theory [3].

The interest in string theory occurs because it is a framework that describes quantum gravitational interactions consistently. In particular, within the spectrum that emerges from closed strings, one can identify massless spin-2 particles whose characteristics are compatible with those required to be identified as mediators of gravity [4]. Furthermore, its spectrum contains a tower of massive states with higher spins, which are crucial for the theory's consistency. By incorporating the contributions of infinite massive states, one can show that non-renormalizable divergences in the loop amplitudes cancel each other.

Although bosonic string theory does not contain fermions, and the presence of excitations with a negative mass-squared implies a non-stable vacuum, it constitutes a laboratory to explore techniques to be applied in more realistic theories [3]. In turn, the RNS superstring presents fermionic modes [5] and a mechanism for eliminating tachyonic modes [6], which also makes the spectrum of the theory supersymmetric. However, its lack of manifest spacetime supersymmetry leads to technical difficulties in amplitude computations, such as the need for sum over spin structures [7].

The pure spinor formalism for the superstring has all spacetime symmetries manifest [8, 9]. This feature allows the construction of super-Poincaré covariant expressions for vertex operators through its quantization [10, 11]. These operators correspond to physical states in the cohomology [12] of the BRST charge $Q = \oint dz \lambda^\alpha d_\alpha$ expressed in terms of a ten-dimensional worldsheet spinor λ^α satisfying the pure spinor condition and the worldsheet variable d_α for the spacetime supersymmetric derivative. The knowledge of vertex operators makes it possible to establish the equivalence of superstring amplitudes in the pure spinor and RNS formalisms [13]. Nevertheless, a superfield description of superstring massive vertex operators remains an open problem.

In order to construct open superstring unintegrated vertex operators of mass $m^2 = \frac{n}{\alpha'}$ ($n \in \mathbb{N}$ and α' is the inverse string tension) using pure spinor formalism, one can write every possible combination of worldsheet fields with ghost number 1 and conformal weight n, and contract them with d=10 superfields. The onshell condition provides relations between these d=10 superfields [10]. For integrated operators, one needs to use the descent relation to constrain the d=10 superfields [14].

Although straightforward, this method becomes quite involved at higher mass levels, and it is convenient to resort to other ways of building the corresponding vertex operators. For this reason, in this dissertation, we address a method for obtaining massive operators, using the fact that these states can decay to massless ones and, therefore, can be interpreted as resonances in the massless S-matrix [5][15][16].

In chapter 2, we will review the construction of the vertex operators corresponding to bosonic open string states with mass $m^2 = -\frac{1}{\alpha'}, 0, \frac{1}{\alpha'}$, and RNS open string states with mass $m^2 = 0, \frac{1}{\alpha'}$. We will use the bosonic string as an example to illustrate the method discussed in chapter 3, and the RNS superstring will be employed to elucidate the predicted physical states for the first massive level of the open superstring.

In chapter 3, the open string unintegrated vertex operator at the first massive level will be constructed from the operator product expansion between a massless integrated and a massless unintegrated vertex operator using pure spinor formalism CFT [17]. This massive vertex will be BRST invariant by construction and expressed in terms of super-Yang-Mills d=10 superfields, which have well-known theta expansion [18]. This result can be generalized for any higher mass level and used to compute scattering amplitudes with massive vertex using all the machinery known for massless scattering amplitude computations [19].

Chapter 2

Vertex operators in Bosonic and RNS string

In section 2.1, we will use the bosonic string CFT to explain the procedure for the construction of massive vertex operators involving the OPEs between integrated and unintegrated ones. This method will be the main computation of the next chapter, where we will obtain the vertex operator corresponding to the first massive states of the open superstring. It is interesting to see how the physical information obtained by this vertex can be deduced using RNS formalism. This will be done in section 2.2. We do not want to be pedagogical or exhaustive, but fix the notation and present some conformal field theory techniques used in the next chapter. This review was written based on the following references: [3], [5], [20], and [21]. Further details can be found in these references.

2.1 Bosonic string

The bosonic string fixed action can be written in complex coordinates as

$$S = \int d^2z (\frac{1}{\alpha'}\partial x^m \bar{\partial} x^m + b\bar{\partial}c + \bar{b}\partial\bar{c}), \tag{2.1}$$

where $x^m(z, \bar{z})$ is the spacetime embedding of the string; $b(z)$ and $c(z)$ are fermionic ghosts degrees of freedom resulting from gauge fixing the metric, and $\partial = \frac{\partial}{\partial z}$. They account for unphysical gauge degrees of freedom, maintaining Lorentz covariance [3]. This action has a residual conformal symmetry that must be preserved in quantization to ensure the vanishing of Weyl anomaly [22], and the Noether charge related to conformal transformations is given by the energy-momentum tensor by varying 2.1 with respect to the metric. The holomorphic conformal charge is

$$Q_v = \oint_{C_0} dz v(z) T(z), \tag{2.2}$$

$$\begin{aligned} T(z) &= T_{matter} + T_{ghost} \\ &= -\frac{1}{\alpha'}\partial x^m \partial x^m + 2\partial c b + c\partial b, \end{aligned} \tag{2.3}$$

where $v(z)$ is holomorphic, and C_z is a contour around z. We are working on a Fock space given by the ghost (c^z), anti-ghost (b_z), and coordinate (x^m) excitations. Furthermore, one can show that the conformal anomaly on a curved world-sheet vanishes according to the central charge of 2.1 [22], defined below in 2.19, which will give us the constraint on spacetime dimension ($d = 26$).

Propagators are solutions of Schwinger-Dyson equation,

$$\langle \phi_i(z_1, \bar{z}_1) \frac{\delta S}{\delta \phi_j}(z_2, \bar{z}_2) \rangle = \delta_i^j \delta^{(2)}(z_{12}, \bar{z}_{12}), \tag{2.4}$$

thus, for the action 2.1, one has the following correlations on the complex plane,

$$\langle x^m(z_1, \bar{z}_1) x^n(z_2, \bar{z}_2) \rangle = -\frac{\alpha'}{2}\delta^{mn} ln|z_{12}|^2, \tag{2.5}$$

$$\langle b(z_1)c(z_2) \rangle = \frac{1}{z_{12}}, \tag{2.6}$$

where $z_{12} \equiv z_1 - z_2$.

Open string states are defined in the world-sheet boundary (the real axis $z \in \mathbb{R}$) [3]. The relevant open bosonic string OPEs are therefore the same as 2.5 and 2.6, but with the fields inserted at the world-sheet boundary, and the shift $\alpha' \to 4\alpha'$, which correspond to the doubling trick procedure [23]. Instead of working with the energy-momentum tensor holomorphic (T) and anti-holomorphic ($\tilde{T}$) components in the upper-half complex plane ($\mathrm{Im}(z) \geq 0$) and implement the boundary condition $T = \tilde{T}$, if $\mathrm{Im}(z) = 0$; one can perform the doubling trick:

$$T(z) := \tilde{T}(\bar{z}), \qquad \mathrm{Im}(z) < 0, \tag{2.7}$$

and work with $T(z)$, which is holomorphic in the whole complex plane [3]. In the following equations, we will adopt this doubling trick procedure, and the convention $2\alpha' = 1$ will be used to simplify the notation in some equations. α' has spacetime dimension $[\alpha'] = 2$, so it can be inserted from dimensional analysis, and the open string correlation function becomes

$$\langle x^m(z_1)x^n(z_2)\rangle = -\delta^{mn} ln z_{12}, \tag{2.8}$$

$$\langle b(z_1)c(z_2)\rangle = \frac{1}{z_{12}}. \tag{2.9}$$

In order to construct the vertex operators, it is useful to note the existence of a residual symmetry group after the gauge fixing, with generators $\{G_i\}$ satisfying an algebra

$$[G_i, G_j] = i f^k_{ij} G_k, \tag{2.10}$$

and associated with ghosts b_i, c^i, such that $\{c^i, b_j\} = \delta^i_j$, and the other commutators being zero [3]. The set of physical states is the cohomology of the nilpotent BRST operator

$$Q_{BRST} = c^i(G^x_i + \frac{1}{2}G^g_i), \tag{2.11}$$

where $G^g_i \equiv -i f^k_{ij} c^j b_k$.

In the bosonic string, one can represent the BRST charge as an integral of a conserved current density. So, the gauge fixed action has a symmetry whose current is

$$J_{BRST} = c\big(T_{matter} + \frac{1}{2}T_{ghost}\big), \tag{2.12}$$

and the BSRT charge can be written as

$$Q_{BRST} = \oint dz c\big(-\frac{1}{2}\partial x^n \partial x^n + \partial c b\big). \tag{2.13}$$

Physical states will correspond to conformal primary fields $\phi_{(h)}$ by

$$|h\rangle = \lim_{z\to 0} \phi_{(h)}(z)|0\rangle, \tag{2.14}$$

where $|0\rangle$ is the SL(2, $\mathbb{R}$) vacuum state, the image of identity under the state-operator correspondence. From holomorphicity at the origin, one has

$$L_n|0\rangle = 0,\ n \geq -1 \tag{2.15}$$

$$L_n = \oint z^{n+1} T(z), \tag{2.16}$$

and $\phi_{(h)}(z)$ are defined to satisfy the OPE

$$T(z_1)\phi_h(z_2) \sim \frac{h}{z_{12}^2}\phi_h(z_2) + \frac{1}{z_{12}}\partial\phi_h(z_2), \tag{2.17}$$

and have the expansion in Laurent modes

$$\phi_{(h)}(z) = \sum_{n=-\infty}^{\infty} \frac{(\phi_{(h)})_n}{z^{n+h}}, \qquad (\phi_{(h)})_n = \oint_{C_{z_2}} \frac{dz_1}{z_{12}} z_{12}^{m+h}\phi(z_2) \tag{2.18}$$

So the Hilbert space is constructed as a representation of the conformal algebra

$$T(z_1)T(z_2) \sim \frac{c}{2z_{12}^4} + \frac{2T(z_2)}{z_{12}^2} + \frac{\partial T(z_2)}{z_{12}}, \tag{2.19}$$

$$c = D - 26, \tag{2.20}$$

by acting combinations of raising operators ($L_{-n},\ n \geq 0$) on highest weight states $|h\rangle$. The highest weight states satisfy $L_n|h\rangle = 0$ and $L_0|h\rangle = h|h\rangle$. And the condition $D = 26$ implies the vanishing of Weyl anomaly [7], ensuring that Weyl symmetry holds after quantization.

We also have U(1) ghost current

$$J = - : bc :, \tag{2.21}$$

thus physical states, in addition to being in the cohomology of Q_{BRST}, are excitations of a vacuum with ghost number $-\frac{1}{2}$ [1].

In particular, the conformal weight $h = 1$ primary fields satisfies

$$[Q_{BRST}, \phi_{(1)}] = \partial(c\phi_{(1)}), \tag{2.22}$$

and when integrated over the boundary, they correspond to BRST invariant states. It will also be imposed that physical states are annihilated by $b_0 = \oint dz(zb)$ mode, which correspond to Lorenz gauge [3]. One, therefore, can write an integrated vertex operator using the conformal primary fields ($\partial^n x^m$) of weight n, that correspond to bosonic excited states, and the momentum ($P_m = \oint i\partial x_m$)

[1] This occurs because the ghost vacuum degenerates in a pair ($|\downarrow\rangle, |\uparrow\rangle$) with ghost number $(-1/2, +1/2)$ on the cylinder [3]. To translate the ghost number to radial frame, one has to take into account the non-tensor transformation of ghost current, which changes $Q_{radial} = N_{cylinder} + \frac{3}{2}$.

eigenstate $e^{ik\cdot x}$ [5],

$$\mathcal{V} = \int dz U(P,k,z), \tag{2.23}$$

$$U(P,k,z) = :Pe^{ik\cdot x}:, \tag{2.24}$$

where $P = P\big(\partial x^{\mu}, \partial^2 x^{\mu}, ..., \partial^n x^{\mu}\big)$ is a polynomial with weight $h = n$, and $\alpha' k^2 = 1 - n$. The normal ordering $: \cdot :$ is defined as [24]

$$: A(z_1)B(z_2) := \oint_{C_{z_2}} \frac{dz_1}{z_{12}} A(z_1)B(z_2). \tag{2.25}$$

One can also write the unintegrated version of 2.24,

$$V(P,k,z) = \ c(z)U(P,k,z), \tag{2.26}$$

which is BRST invariant as a consequence of 2.24 definition,

$$\begin{aligned}
[Q_{BRST}, V](z_2) &= \oint_{C_{z_2}} dz_1 \left(cT_{matter} + c(\partial c)b\right)(z_1)(cU)(z_2), \\
&= \oint_{C_{z_2}} dz_1\, c(z_1)\left(\frac{1}{z_{12}^2} cU + \frac{1}{z_{12}} \partial cU\right)(z_2) \\
&= (\partial c c + c\partial c)U(z_2) \\
&= 0,
\end{aligned} \tag{2.27}$$

and has the gauge freedom

$$V \longrightarrow V + [Q_{BRST}, \Lambda]. \tag{2.28}$$

If Λ has conformal weight $h = 0$ and is constructed out from matter fields, one has

$$\begin{aligned}
[Q_{BRST}, \Lambda](z_2) &= \oint_{C_{z_2}} dz_1 \big(cT_{matter}\big)(z_1)\Lambda(z_2) \\
&= \oint_{C_{z_2}} dz_1 c \frac{\partial \Lambda}{z_{12}} \\
&= c\partial\Lambda.
\end{aligned} \tag{2.29}$$

Now, we will verify the conditions the polynomial P should satisfy to $U(P,k,z)$ have conformal weight $h = 1$, for $n = 0, 1, 2, 3$.

n=0. The simplest vertex operator is the tachyonic one, with $P = 1$,

$$U_T(k,z) =: e^{ik\cdot x} :, \qquad (2.30)$$
$$k^2 = 2,$$

with conformal weight $h = \frac{k^2}{2}$. This gives rise to an unintegrated tachyon vertex

$$V_T(k,z) = : ce^{ik\cdot x} : . \qquad (2.31)$$

n=1. Considering the massless level $n = 1$, one can write the general combination 2.24 with

$$P = \xi_m(k)\partial x^m. \qquad (2.32)$$

The cubic pole of the OPE between $U(P,k,z)$ and T_{matter} is proportional to $\xi \cdot k$; it, therefore, has conformal weight $h = 1$ if $\xi \cdot k = 0$. So, BRST invariance implies the transversality condition

$$[Q_{BRST}, Pe^{ik\cdot x}] = 0 \Rightarrow \xi \cdot k = 0, \qquad (2.33)$$

and the gauge invariance 2.28 with $\Lambda = e^{ik\cdot x}$ implies the redefinition $\delta\xi_m = ik_m$.

n=2. Now, considering the massive level $n = 2$, one can write the polynomial as

$$P = \xi^1_{mn}\partial x^m \partial x^n + \xi^2_m \partial^2 x^m, \qquad (2.34)$$

and $: Pe^{ik\cdot x} :$ has conformal weight $n = 1$ if the quartic and cubic pole with energy-momentum tensor T_{matter} vanishes. Indeed, from the OPE

$$\begin{aligned} T_{matter} \; : Pe^{ik\cdot x} := &-\frac{1}{z_{12}^4}\left(\delta^{mn}\xi^1_{mn} + 2ik^m\xi^2_m\right)e^{ik\cdot x} \\ &+\frac{1}{z_{12}^3}\left(-ik^{(m}\partial x^{n)}\xi^1_{mn} + 2\partial x^m \xi^2_m\right)e^{ik\cdot x} \\ &+\frac{2+\frac{k^2}{2}}{z_{12}^2}Pe^{ik\cdot x} + \frac{1}{z_{12}}\partial\left(Pe^{ik\cdot x}\right), \end{aligned} \qquad (2.35)$$

the BRST invariance of $:Pe^{ik\cdot x}:$ requires the following polarization constraints

$$\begin{aligned} \xi^1_{mn}\delta^{mn} + 2ik\cdot\xi^2 = 0 \\ -\frac{1}{2}\xi^1_{(mn)}ik^n + \xi^2_m = 0, \end{aligned} \tag{2.36}$$

where $\xi_{(m_1 \dots m_k)} = \xi_{m_1\dots m_k} + (\textit{all symmetric permutations})$.

The gauge invariance 2.28 with

$$\Lambda_1 =: \lambda_m \partial x^m e^{ik\cdot x} :, \tag{2.37}$$

implies the redefinition

$$\delta\xi^1_{mn} = \frac{1}{2}\lambda_{(m}ik_{n)}, \tag{2.38}$$

$$\delta\xi^2_m = \lambda_m, \qquad \lambda_m k^m = 0. \tag{2.39}$$

And from the combination

$$\Lambda_2 = \lambda\big(2 : bce^{ik\cdot x} : -3 : ik\cdot\partial x e^{ik\cdot x} : \big), \tag{2.40}$$

parameterized by a number λ, one has the following spurious state

$$\delta\xi^1_{mn} = \lambda\delta_{mn} - 3\lambda ik_m ik_n \tag{2.41}$$

$$\delta\xi^2_m = -5\lambda ik_m. \tag{2.42}$$

The first positive mass level therefore has $\frac{26\cdot 27}{2} - 27 = 324$ physical degrees of freedom ($d.o.f.$) represented by a symmetric 2-tensor ξ^1_{mn} ($\frac{26\cdot 27}{2}$ $d.o.f.$), such that

$$k^m\xi^1_{mn} = 0 \qquad \big(-26\; d.o.f.\big), \tag{2.43}$$

$$\delta^{mn}\xi^1_{mn} = 0 \qquad \big(-1\; d.o.f.\big). \tag{2.44}$$

n=3. Considering the massive level $n = 3$, one can write the polynomial as

$$P = \xi^1_{mnp}\partial x^m \partial x^n \partial x^p + \xi^2_{mn}\partial^2 x^m \partial x^n + \xi^3_m \partial^3 x^m, \tag{2.45}$$

and $:Pe^{ik\cdot x}:$ has conformal weight $n=1$ if the order 5,4,3 poles with energy-momentum tensor T_{matter} vanishes,

$$
\begin{aligned}
T_{matter}:\xi^1_{mnp}\partial x^m\partial x^n\partial x^p e^{ik\cdot x}:&\sim\left(\frac{-3}{z_{12}^4}\delta^{mn}\xi^1_{mnp}\partial x^p-\frac{1}{z_{12}^3}ik^m\xi^1_{mnp}\partial x^n\partial^p\right)e^{ik\cdot x},\\
T_{matter}:\xi^2_{mn}\partial^2x^m\partial x^n e^{ik\cdot x}:&\sim\left(\frac{2}{z_{12}^3}\xi^2_{mn}\partial x^m\partial x^n-\frac{ik^n}{z_{12}^3}\xi^2_{mn}\partial^2 x^m\right)e^{ik\cdot x}\\
&+\left(-\frac{2ik^m}{z_{12}^4}\xi^2_{mn}\partial x^n-\frac{2}{z_{12}^5}\xi^2_{mn}\delta^{mn}\right)e^{ik\cdot x},\\
T_{matter}:\xi^3_m\partial^3x^m e^{ik\cdot x}:&\sim\left(\frac{6}{z_{12}^4}(\partial x^m+z_{12}\partial^2x^m)\xi^3_m-\frac{6ik^m}{z_{12}^5}\xi^3_m\right)e^{ik\cdot x}.
\end{aligned}
\tag{2.46}
$$

So the BRST invariance of $:Pe^{ik\cdot x}:$ implies the following polarization constraints

$$3ik^m\xi^3_m+\delta^{mn}\xi^2_{mn}=0,\tag{2.47}$$

$$3\delta^{mn}\xi^1_{mnp}+2ik^m\xi^2_{mp}-6\xi_p=0,\tag{2.48}$$

$$-ik^n\xi^2_{mn}+6\xi^3_m=0,\tag{2.49}$$

$$-3ik^m\xi^1_{mnp}+\xi^2_{(mp)}=0.\tag{2.50}$$

Again, spurious states can be found as exact states under the BRST charge 2.13. For example, the gauge invariance 2.28 with

$$\Lambda=\Gamma_m:\partial^2x^m e^{ik\cdot x}:+\Lambda_{mn}:\partial x^m\partial x^n e^{ik\cdot x}:,\tag{2.51}$$

with $\Gamma_m,\ \Lambda_{mn}$ satisfying 2.36, implies the redefinition

$$\delta\xi^1_{mnp}=\frac{1}{3!}ik_{(m}\Lambda_{np)},\tag{2.52}$$

$$\delta\xi^2_{mn}=2\Lambda_{mn}+\Gamma_m ik_n,\tag{2.53}$$

$$\delta\xi^3_m=\Gamma_m.\tag{2.54}$$

One can identify the ξ^3_m and $\xi^2_{(mn)}$ as spurious degrees of freedom [25]. So in the physical gauge of the $n=3$ level one has an anti-symmetric 2-tensor ξ^2_{mn} ($\frac{25\cdot 26}{2}$ $d.o.f$), and a symmetric 3-tensor ξ^1_{mnp} ($\frac{28\cdot 27\cdot 26}{3!}$ $d.o.f$), such that [26]

$$k^m\xi^2_{mn}=0,\qquad\qquad(-25\ d.o.f.)\tag{2.55}$$

$$k^m\xi^1_{mnp}=0,\ \delta^{mn}\xi^1_{mnp}=0,\qquad\qquad\left(-\frac{26\cdot 27}{2}-25\ d.o.f.\right),\tag{2.56}$$

It therefore gives the $\frac{26\cdot 25}{2} + \frac{28\cdot 27\cdot 26}{6} - 25 - \frac{27\cdot 26}{2} - 25 = 3200$ physical degrees of freedom of the $n = 3$ mass level.

Another method to construct massive vertex operators, which is the main computation of this dissertation in chapter 3, relies on the fact that one can recover all theory higher mass resonances using the operator algebra of string theory primary fields [5]. Since unintegrated vertex operators of mass $m^2 = \frac{n-1}{\alpha'} = 2(n-1)$ should be constructed from combinations $P^{(n)} = P(\partial x, \partial^2 x, ..., \partial^{(n)})$ with conformal weight n, one can define the unintegrated vertex operator corresponding to the massive states of the open bosonic string as

$$V^{(12)}_{(m^2=\frac{n-1}{\alpha'})}(\bar{k}, z_2) = \oint_{C_{z_2}} dz_1 U^1_T(k_1, z_1) V^2_T(k_2, z_2). \tag{2.57}$$

$$\bar{k}^2 = 2(1-n) \tag{2.58}$$

where $\bar{k} = k_1 + k_2$, and U^1_T (V^2_T) is the tachyon integrated (unintegrated) vertex defined in equation 2.30 (2.31). The first-order pole gives a massive unintegrated vertex in the cohomology of Q_{BRST},

$$\begin{aligned}[Q_{BRST}, V^{(12)}] &= \oint_{C_{z_2}} dz_1 \Big([Q_{BRST}, U^1_T] V^2_T + U^1_T [Q_{BRST}, V^2_T] \Big) \\ &= \oint_{C_{z_2}} dz_1 \partial_1 V^1_T V^2_T \\ &= 0\end{aligned} \tag{2.59}$$

so one can write

$$V^{(12)}_{(m^2=\frac{n-1}{\alpha'})}(\bar{k}, z_2) =: c\left(\frac{\partial^{(n)}}{(n)!} e^{ik_1\cdot x}\right) e^{ik_2\cdot x} :. \tag{2.60}$$

And for $n = 1, 2, 3$ one can expand 2.60 as

$$V^{(12)}_{m^2=0} =: c(ik_1\cdot\partial x) e^{i\bar{k}\cdot x} :, \qquad \bar{k}^2 = 0 \tag{2.61}$$

$$V^{(12)}_{m^2=2} =: c\frac{1}{2}\big((\partial x\cdot ik_1)^2 + \partial^2 x\cdot ik_1\big) e^{i\bar{k}\cdot x} :, \qquad \bar{k}^2 = -2 \tag{2.62}$$

$$V^{(12)}_{m^2=4} =: c\frac{1}{3!}\big((ik_1\cdot\partial x)^3 + 3(ik_1\cdot\partial x)(ik_1\cdot\partial^2 x) + (ik_1\cdot\partial^3 x)\big) e^{i\bar{k}\cdot x} :, \quad \bar{k}^2 = -4 \tag{2.63}$$

The only particles one can obtain from 2.58 are those that couple with tachyons. In particular, there are no antisymmetric polarizations [27]. Nevertheless, one can also consider the physical content coming from the residue of the poles of massless particles 2.32,

$$V =: c\partial x \cdot A : \tag{2.64}$$

where $A_m = \xi_m e^{ik\cdot x}$, and the respective integrated vertex

$$U =: \partial x \cdot A : . \tag{2.65}$$

As an example of 2.22 one can compute, omitting normal ordering simbol $: \cdot :$,

$$\begin{aligned}
[Q_{BRST}, U] &= -\frac{1}{2}\oint_{C_{z_2}} dz_1 \big(c\partial x\cdot\partial x\big)(z_1)\big(\partial x\cdot A\big)(z_2)\\
&= -\frac{1}{2}\oint_{C_{z_2}} dz_1 \bigg(-\frac{2}{z_{12}^2}c(\partial x\cdot\xi) - \frac{2}{z_{12}}c(\partial x\cdot ik)(\partial x\cdot\xi)\bigg)e^{ik\cdot x}\\
&= \bigg(\partial c(\partial x\cdot\xi) + c(\partial^2 x\cdot\xi) + (c\partial x\cdot ik)(\partial x\cdot\xi)\bigg)e^{ik\cdot x}\\
&= \partial\big(c(\partial x\cdot\xi)e^{ik\cdot x}\big),
\end{aligned}$$

$$\Rightarrow [Q_{BRST}, U] = \partial\big(cU\big) \tag{2.66}$$

with a similar computation for

$$[Q_{BRST}, V] = 0. \tag{2.67}$$

So one can define an unintegrated vertex operator with mass $m^2 = \frac{n-1}{\alpha'}$ as

$$V^{(12)}(z_2) = \oint_{C_{z_2}} dz_1 U^{(1)}(z_1)V^{(2)}(z_2), \tag{2.68}$$

$$\bar{k}^2 \equiv (k_1 + k_2)^2 = -\frac{n-1}{\alpha'}. \tag{2.69}$$

And from the first order pole of the OPE $\big(\partial x\cdot A^{(1)}\big)(z_1)\big(\partial x\cdot A^{(2)}\big)(z_2)$, one obtains for $n = 2$

$$V^{(12)} = c\left[\partial x^m \partial x^n \frac{1}{2}\zeta^1_{(mn)} + \partial^2 x^m \zeta^2_m\right]e^{i\bar{k}\cdot x},\ \bar{k}^2 = -2 \tag{2.70}$$

$$\zeta^1_{(mn)} = +\xi^1_{(m}\xi^2_{n)} - (ik_2\cdot\xi_1)ik^1_{(m}\xi^2_{n)} + (ik_1\cdot\xi_2)ik^1_{(m}\xi^1_{n)} \\ - \big(\xi_1\cdot\xi_2 + (ik_1\cdot\xi_2)(ik_2\cdot\xi_1)\big)ik^1_m ik^1_n, \tag{2.71}$$

$$\zeta^2_m = -\frac{1}{2}(\xi_1\cdot\xi_2)ik^1_m + (ik_1\cdot\xi_2)\xi^1_m - \frac{1}{2}(ik^1\cdot\xi_2)(ik_2\cdot\xi_1)ik^1_m, \tag{2.72}$$

which satisfy the condition 2.36, implying BRST invariance $[Q_{BRST}, V^{(12)}] = 0$.

Now, one can use the gauge invariance 2.40, with $\lambda = -\frac{1}{20}$, to impose traceless condition 2.44,

$$\zeta^{2'}_m = \zeta^2_m + \delta\zeta^2_m, \tag{2.73}$$

$$\delta\zeta^2_m = \frac{1}{4}\big(\xi_1\cdot\xi_2 - (ik_1\cdot\xi_2)(ik_2\cdot\xi_1)\big)i\bar{k}_m, \tag{2.74}$$

and the gauge invariance 2.37 with $\lambda_m = \zeta^{2'}_m$, which gives the following transverse traceless tensor

$$\frac{1}{2}\zeta^{1'}_{(mn)} = \frac{1}{2}\zeta^1_{(mn)} + \delta\zeta^1_{(mn)}, \tag{2.75}$$

$$\delta\zeta^1_{(mn)} = +\left(\frac{1}{4}i\bar{k}_{(m}ik^1_{n)} + \frac{1}{4}i\bar{k}_m i\bar{k}_n + \frac{1}{20}(\delta_{mn} - 3i\bar{k}_m i\bar{k}_n)\right)(ik_1\cdot\xi_2)(ik_2\cdot\xi_1) \\ + \left(\frac{1}{4}i\bar{k}_{(m}ik^1_{n)} - \frac{1}{4}i\bar{k}_m i\bar{k}_n - \frac{1}{20}(\delta_{mn} - 3i\bar{k}_m i\bar{k}_n)\right)(\xi_1\cdot\xi_2) \\ - \frac{1}{2}(ik_1\cdot\xi_2)\left(i\bar{k}_{(m}\xi^1_{n)}\right) \tag{2.76}$$

This procedure can be generalized to any level of the bosonic string if one start with the lower level operators carrying the relevant quantum numbers [28]. Still, it is not possible to obtain fermionic states here. Therefore, the application of this method will be saved for the next chapter, where it will be discussed in the context of open superstrings.

2.2 RNS superstring

In this section, we will review the field content of the first massive states of the open superstring using the RNS formalism. The supersymmetric partners of the b, c ghost system described in the previous section require the notion of picture changing. Furthermore, the spectrum is supersymmetric and free of tachyons only after the GSO projection, which will be described in more detail below.

In the case of superstring in RNS formalism, after choosing superconformal gauge [5], one has the following gauge fixed matter action:

$$S_{matter} = \int d^2zd^2\theta \frac{1}{2}\bar{D}\mathbf{X}^m D\mathbf{X}^m = \frac{1}{2}\int d^2z\big(\partial x\cdot\bar{\partial}x - \psi\cdot\bar{\partial}\psi - \bar{\psi}\cdot\partial\bar{\psi}\big) \tag{2.77}$$

where $D_{(i)} = \partial_{\theta_i} + \theta\partial_{z_i}$. As in the bosonic case, we have a path-integral over a conjugate pair of ghosts with action:

$$S_{ghost} = \int d^2zd^2\theta B_{z\theta}\bar{D}C^z = \int d^2z\big(\beta\bar{\partial}\gamma + b\bar{\partial}c\big), \tag{2.78}$$

From the equations of motion of 2.77, one can split the embedding field $\mathbf{X}^m$ into holomorphic and anti-holomorphic components. The holomorphic matter and ghost superfields can be written as

$$\mathbf{X}^\mu = \theta\psi^\mu(z) + x^\mu(z), \tag{2.79}$$

$$B_{z\theta} = \beta_{z\theta} + \theta b_{zz},\ C^z = c^z + \theta\gamma^\theta. \tag{2.80}$$

The correlation functions involving the left and right moving sectors decouple, so one can focus on the holomorphic sector, which has the following correlation functions

$$\langle\mathbf{X}^m(z_1)\mathbf{X}^n(z_2)\rangle = -\eta^{mn}ln(z_{12}), \qquad \langle B(z_1,\theta_1)C(z_2,\theta_2)\rangle = \frac{\theta_{12}}{z_{12}}, \tag{2.81}$$

where $z_{ij} \equiv z_i - z_j - \theta_i\theta_j$.

The action $S_{matter} + S_{ghost}$ has a residual superconformal symmetry, and its associated super energy-momentum tensor is composed of a general combination of matter and ghost superfields with conformal dimension $\frac{3}{2}$

$$T(z,\theta) = T_{matter}(z,\theta) + T_{ghost}(z,\theta) \tag{2.82}$$

$$T_{matter}(z,\theta) = A_1\big(D\mathbf{X}^\mu D^2\mathbf{X}^\mu\big) \tag{2.83}$$

$$T_{ghost}(z,\theta) = A_2\big(C(D^2B)\big) + A_3\big((DC)(DB)\big) + A_4\big((D^2C)B\big). \tag{2.84}$$

Similarly to bosonic case 2.17, one can define a superconformal primary field $\phi(z_i,\theta_i)$ of conformal dimension h as a superfield with the following operator product expansion

$$T(z_j,\theta_j)\phi(z_i,\theta_i) \sim h\frac{\theta_{ij}}{z_{ji}^2}\phi(z_i,\theta_i) + \frac{1/2}{z_{ji}}D_i\phi + \frac{\theta_{ij}}{z_{ji}}\partial_i\phi, \tag{2.85}$$

where $\theta_{ij} = \theta_i - \theta_j$, so this implies the following set of 2.82 coeficients

$$A_1 = -\frac{1}{2},\ A_2 = -1,\ A_3 = \frac{1}{2},\ A_4 = -\frac{3}{2}. \tag{2.86}$$

For example, we have

$$\begin{aligned} T(z_1,\theta_1)\mathbf{X}^m(z_2,\theta_2) &\sim \frac{1}{2}\big(D\mathbf{X}^m\big)(z_1,\theta_1)\partial_1 ln(z_{12}) + \frac{1}{2}\big(\partial\mathbf{X}^m\big)(z_1,\theta_1)D_1 ln(z_{12}) \\ &\sim \frac{1}{2}\big(D\mathbf{X}^m\big)(z_1,\theta_1)\frac{1}{z_{12}} + \frac{1}{2}\big(\partial\mathbf{X}^m\big)(z_1,\theta_1)\frac{\theta_{12}}{z_{12}} \\ &\sim \frac{1/2}{z_{12}}\big(D\mathbf{X}^m\big)(z_2,\theta_2) + \frac{\theta_{12}}{z_{12}}\big(\partial\mathbf{X}^m\big)(z_2,\theta_2), \end{aligned} \tag{2.87}$$

and for the ghost superfields, using $D^2_{(i)} = \partial_{z_i}$,

$$\begin{aligned} T(z_1,\theta_1)C(z_2,\theta_2) &\sim -C(z_1,\theta_1)\partial_{z_1}\big(\frac{\theta_{12}}{z_{12}}\big) + \frac{1}{2}(D_{(1)}C)D_{(1)}\big(\frac{\theta_{12}}{z_{12}}\big) - \frac{3}{2}(\partial_{z_1}C)\big(\frac{\theta_{12}}{z_{12}}\big) \\ &\sim C(z_1,\theta_1)\frac{\theta_{12}}{z_{12}^2} + \frac{\frac{1}{2}}{z_{12}}(D_{(1)}C) - \frac{3}{2}(\partial_{z_2}C)\big(\frac{\theta_{12}}{z_{12}}\big) \\ &\sim -\frac{\theta_{12}}{z_{12}^2}(C + z_{12}\partial C)(z_2,\theta_2) + \frac{\frac{1}{2}}{z_{12}}(D_{(2)}C + \theta_{12}D^2_{(2)}C) + \frac{3}{2}\frac{\theta_{12}}{z_{12}}(\partial_{z_2}C) \\ &\sim (-1)\frac{\theta_{12}}{z_{12}^2}C(z_2,\theta_2) + \frac{\frac{1}{2}}{z_{12}}D_{(2)}C + \frac{\theta_{12}}{z_{12}}(\partial_{z_2}C), \end{aligned} \tag{2.88}$$

and

$$T(z_1,\theta_1)B(z_2,\theta_2) \sim \frac{3}{2}\frac{\theta_{12}}{z_{12}^2}B(z_2,\theta_2) + \frac{\frac{1}{2}}{z_{12}}D_{(2)}B + \frac{\theta_{12}}{z_{12}}(\partial_{z_2}B), \tag{2.89}$$

where we have expanded the superfields using Taylor series in superspace,

$$V(z_1,\theta_1) = \sum_{n=0} z_{12}^n\partial_{z_2}^n\big(V + \theta_{12}D_{(2)}V\big)(z_2,\theta_2). \tag{2.90}$$

The superconformal algebra deduced from $T(z,\theta) = T_{z\theta}(z) + \theta T_{zz}(z)$ expansion

2.82,2.86, is

$$T_{zz}(z_1)T_{zz}(z_2) \sim \frac{\frac{1}{2}c}{(z_1-z_2)^4} + \frac{2T_{zz}(z_2)}{(z_1-z_2)^2} + \frac{\partial T_{zz}(z_2)}{(z_1-z_2)} \tag{2.91}$$

$$T_{z\theta}(z_1)T_{z\theta}(z_2) \sim \frac{\frac{c}{6}}{(z_1-z_2)^3} + \frac{\frac{1}{2}T_{zz}(z_2)}{(z_1-z_2)} \tag{2.92}$$

$$T_{zz}(z_1)T_{z\theta}(z_2) \sim \frac{\frac{3}{2}T_{z\theta}(z_2)}{(z_1-z_2)^2} + \frac{\partial T_{z\theta}(z_2)}{(z_1-z_2)}, \quad c = \frac{3D}{2} - 15, \tag{2.93}$$

and the transformation of superconformal field $\phi_h(z,\theta) = \phi_h(z) + \theta\phi_{h+\frac{1}{2}}(z)$ components under 2.85 is

$$T_{zz}(z_1)\phi_h(z_2) \sim \frac{h\phi_h}{(z_1-z_2)^2} + \frac{\partial\phi_h(z_2)}{(z_1-z_2)} \tag{2.94}$$

$$T_{zz}(z_1)\phi_{h+\frac{1}{2}}(z_2) \sim \frac{(h+\frac{1}{2})\phi_{h+\frac{1}{2}}}{(z_1-z_2)^2} + \frac{\partial\phi_{h+\frac{1}{2}}(z_2)}{(z_1-z_2)} \tag{2.95}$$

$$T_{z\theta}(z_1)\phi_h(z_2) \sim \frac{\frac{1}{2}\phi_{h+\frac{1}{2}}}{(z_1-z_2)} \tag{2.96}$$

$$T_{z\theta}(z_1)\phi_{h+\frac{1}{2}}(z_2) \sim \frac{h\phi_h}{(z_1-z_2)^2} + \frac{\frac{1}{2}\partial\phi_h}{(z_1-z_2)} \tag{2.97}$$

2.2.1 Hilbert Space

Matter Sector

There is a subtlety related to the appearance of two-dimensional spinor fields in the RNS formalism: the fermionic components are allowed to be double-valued. There exist two possible periodic conditions,

$$\phi_f(e^{2\pi i}z) = e^{2\pi i\nu}\phi_f(z), \tag{2.98}$$

$\nu = 0, \frac{1}{2}$ that correspond to Neveu-Schwarz (NS) or Ramond (R) fields, so the Laurent expansion on the complex plane has half-odd integer modes for NS fields and integer modes for R fields. In particular, the energy-momentum tensor 2.82 has the general component structure:

$$T_{matter}(z,\theta) = G(z) + \theta T_m(z) = \sum_r \frac{1}{z^{r+\frac{3}{2}}}\frac{1}{2}G_r + \theta\sum_{n\in\mathbb{Z}}\frac{1}{z^{n+2}}L_n, \tag{2.99}$$

with $r \in \mathbb{Z} + \frac{1}{2}$ in the NS sector and in the R sector $r \in \mathbb{Z}$. The Hilbert space is then constructed by the standard procedure in conformal field theory of applying combinations of raising operators L_{-n} and G_{-r}, with $n, r > 0$ in superfields highest weight states. The highest weight state satisfies $L_n|h\rangle = G_{n\geq 0}|h\rangle = 0$ and $L_0|h\rangle = h|h\rangle$. The matter superfield in components is

$$D\mathbf{X}^m = \left(\sum_r \frac{\psi_r^m}{z^{r+\frac{1}{2}}}\right) + \theta\left(\sum_n \frac{\alpha_n^m}{z^{n+1}}\right) \tag{2.100}$$

then, in the NS sector, the vacuum is annihilated by ψ_r^m, for $r \in \mathbb{N} - 1/2$, while the R ground-state is degenerated and forms a representation of a gamma matrix algebra. This follows from the mode commutations deduced from

$$\mathbf{X}^m(z_1, \theta_1)\mathbf{X}^n(z_2, \theta_2) \sim -\eta^{mn} ln(z_{12}). \tag{2.101}$$

Ground states in the R sector have conformal dimension $\frac{D}{16}$, and one can construct a spin field $S_\alpha(z)$, which creates these R ground states from NS vacuum [5]. Using a bosonization procedure, with a set of five free chiral Bosons H_i, such that

$$H_i(z_1)H_j(z_2) \sim -\delta_{ij} \ln z_{12} \tag{2.102}$$

one can write the dimension $h = \frac{1}{2}$ vector representation and dimension $h = \frac{10}{16}$ chiral spinor representation using Weyl-Cartan basis [20]

$$\frac{1}{\sqrt{2}}(\psi^{2j} \pm i\psi^{2j+1}) = e^{\pm i e_j \cdot H_j}, \tag{2.103}$$

$$S_\alpha = e^{i\vec{\alpha} \cdot H_j}, \tag{2.104}$$

where e_j is a unit vector in the jth direction (e.g. $e_0 = (1, 0, 0, 0, 0)$), and the lower index α correspond to a chiral spinor weight $\vec{\alpha} = (\pm\frac{1}{2}, \pm\frac{1}{2}, \pm\frac{1}{2}, \pm\frac{1}{2}, \pm\frac{1}{2})$ with even number of minuses resulting in 2^4 possibilities. They satisfy the correct Lorentz transformations under the current $j^{mn}(z) \equiv \psi^m\psi^n(z)$, and therefore obey operator product expansions with rational powers of z_{12} as

$$\psi^m(z_1)S_\alpha(z_2) \sim \frac{1}{\sqrt{2}z_{12}^{1/2}}\gamma^m_{\alpha\beta}S^\beta + ... \tag{2.105}$$

$$S_\alpha(z_1)S^\beta(z_2) \sim \frac{1}{z_{12}^{5/4}}\delta_\alpha^{\ \beta} + \frac{1}{4z_{12}^{1/4}}(\gamma_{mn})_\alpha^{\ \beta}\psi^m\psi^n + ... \tag{2.106}$$

$$S_\alpha(z_1)S_\beta(z_2) \sim \frac{1}{\sqrt{2}z_{12}^{3/4}}\gamma^m_{\alpha\beta}\psi_m(z_2) + ..., \tag{2.107}$$

For example, consider the first OPE, with $m = (0, ..., \pm 1, ..., 0)$, and using 2.103, up to order $O(z_{12}^{1/2})$ one has

$$e^{i\vec{m}\cdot H(z_1)}e^{i\vec{\alpha}\cdot H} \sim z_{12}^{\vec{m}\cdot\vec{\alpha}}(1 + z_{12}i\vec{m}\cdot\partial H(z_2) + ...)e^{i(\vec{m}+\vec{\alpha})\cdot H}. \tag{2.108}$$

In the case $\vec{m}\cdot\vec{\alpha} = -\frac{1}{2}$, the resulting vector $\vec{\beta} = \vec{m} + \vec{\alpha}$ correspond to a spin-field with opposite chirality, $S^\beta = e^{i\vec{\beta}\cdot H}$, so one can write

$$\psi^m(z_1)S_\alpha(z_2) \sim \sum_\beta \frac{\delta(\vec{m} + \vec{\alpha} - \vec{\beta})}{z_{12}^{1/2}}S^\beta, \tag{2.109}$$

and equation 2.105 follows from the gamma matrices representation in Weyl-Cartan basis $\frac{1}{\sqrt{2}}\gamma^m_{\alpha\beta} = \delta(\vec{m} + \vec{\alpha} - \vec{\beta})$. We are omitting Jordan-Wigner cocycle factors, which are essential to ensure the correct statistics of the exponentials 2.103 and 2.104. For further details, one can consult [20].

Finally, the RNS superstring spectrum will be the projection onto the subspace of states of even Fermion number. We consider an operator $\Gamma = (-)^F$ that anticommutes with fermionic field components and commutes with the Bosonic parts. If one assigns odd parity to the NS ground state, then the lowest energy ground states in the NS sector will compose a massless vector $\psi^\mu_{-\frac{1}{2}}|0\rangle$. And if the GSO operator projects onto the positive chiral part of the R sector, such that the R ground-states form a Weyl-Majorana spinor, then R and NS ground-sates in fact compose a supermultiplet [5]. After discussing the Ghost sector and the BRST conditions, we will define the GSO operator and show that it projects to a spectrum without tachyonic states.

Ghost Sector

To analyze the new features provided by appearance of commuting ghosts, it is useful to study the action 2.78 from the point of view of general $(\mathbf{b}, \mathbf{c})$-systems, where $\mathbf{b}$ and $\mathbf{c}$ are conjugate fields, with dimension λ and $1-\lambda$ respectively. Both fields are Bose ($\varepsilon = -1$) or Fermi ($\varepsilon = 1$). From 2.78 and 2.82 one can write the stress-tensor as:

$$T_{\mathbf{bc}} = -\lambda \mathbf{b}\partial\mathbf{c} + (1-\lambda)\partial\mathbf{b}\mathbf{c}, \tag{2.110}$$

which has central charge $c = \varepsilon(1-3Q^2)$, where $Q = \varepsilon(1-2\lambda)$.

There is a further $U(1)$ symmetry on the $(\mathbf{b}, \mathbf{c})$-system that restricts the operators that produce non-vanishing expectation values. The ghost current is $\mathbf{j} = -\mathbf{bc}$, which counts ghost number charge

$$\mathbf{j}(z)\mathcal{O}(w) \sim \frac{q\mathcal{O}(w)}{z-w} \tag{2.111}$$

and is not strictly a quasi-primary field,

$$T(z_1)\mathbf{j}(z_2) \sim \frac{Q}{z_{12}^3} + \frac{\mathbf{j}(z_2)}{z_{12}^2} + \frac{\partial\mathbf{j}(z_2)}{z_{12}}. \tag{2.112}$$

This ghost-number anomaly implies that only operators with ghost charge $-Q$ will produce non-vanishing expectation values [5].

One can bosonize the ghost current defining:

$$\mathbf{j}_\phi(z) = \varepsilon\partial\phi \tag{2.113}$$

$$\phi(z)\phi(w) \sim \varepsilon ln(z-w), \tag{2.114}$$

we can also bosonize the Fermi $(\mathbf{b}, \mathbf{c})$-system, defining

$$T_\phi(z) = \frac{\varepsilon}{2}\left(\mathbf{j}_\phi^2(z) - Q\partial\mathbf{j}_\phi(z)\right), \tag{2.115}$$

with central charge $c_j = 1 - \varepsilon 3Q^2$. For the Bose statistics $(\mathbf{b}, \mathbf{c})$-system, the central charge:

$$c = c_j - 2, \tag{2.116}$$

requires the introduction of an auxiliary linear fermi (η, ξ)-system with $\lambda = 1$, nonsingular with respect to ϕ, whose energy-momentum tensor has central charge $c_{\eta\xi} = -2$.

In all, one can recover all OPE's stated in this subsection with the following bosonization procedure:

$$\mathbf{b}(z) = e^{-\phi(z)}, \quad \mathbf{c}(z) = e^{\phi(z)}, \tag{2.117}$$

for fermi systems, and

$$\mathbf{b}(z) = e^{-\phi(z)}\partial\xi(z), \quad \mathbf{c}(z) = e^{\phi(z)}\eta(z), \tag{2.118}$$

for Bose systems. For example, one has

$$T_\phi(z_1)e^{q\phi(z_2)} \sim \frac{\varepsilon}{2}(\frac{1}{2}\partial\phi\partial\phi - Q\varepsilon\partial^2\phi)e^{q\phi(z_2)} \tag{2.119}$$

$$\sim \frac{\varepsilon}{2}(2\partial\phi\frac{\varepsilon q}{z_{12}} + \frac{q^2\varepsilon^2}{z_{12}^2})e^{q\phi(z_2)} - \frac{Q\varepsilon^2}{2}\frac{(-q\varepsilon)}{z_{12}^2}e^{q\phi(z_2)} \tag{2.120}$$

$$\sim \frac{\frac{1}{2}\varepsilon q(q+Q)}{z_{12}^2}e^{q\phi(z_2)} + \frac{1}{z_{12}}\partial e^{q\phi(z_2)}, \tag{2.121}$$

and the $\frac{1}{z^4}$ pole of $T_\phi T_\phi$ is

$$T_\phi(z_1)T_\phi(z_2) \sim \frac{\frac{1}{2}(1-3\varepsilon Q^2)}{z_{12}^4} + \mathcal{O}(\frac{1}{z_{12}^2}). \tag{2.122}$$

One can construct the Bose sea-levels defining a set of infinite inequivalent vacua $|q\rangle$ of ghost charge q and dimension $\frac{1}{2}\varepsilon q(q+Q)$:

$$|q\rangle = e^{q\phi(0)}|0\rangle, \tag{2.123}$$

and analysing product expansion of $e^{q\phi}$ with $\mathbf{b}$ and $\mathbf{c}$, by holomorphicity one obtains:

$$\mathbf{b}_n|q\rangle = 0, n > \varepsilon q - \lambda, \quad \mathbf{c}_n|q\rangle = 0, \ n \geq \varepsilon q + \lambda, \tag{2.124}$$

with $q \in \mathbb{Z} + \frac{1}{2}$ for R sector and $q \in \mathbb{Z}$ for NS sector. In particular, for (β, γ) system with $\lambda = \frac{3}{2}$ in the NS sector, where $\gamma(z) = \sum_{r\in\mathbb{Z}+\frac{1}{2}}\frac{\gamma_r}{z^{r-\frac{1}{2}}}$, one has the canonical NS vacuum,

$$|-1\rangle_{(\beta,\gamma)} = e^{-\phi_{\beta\gamma}(0)}|0\rangle, \tag{2.125}$$

and for (β, γ) system with $\lambda = \frac{3}{2}$ in the R sector, where $\gamma(z) = \sum_{n\in\mathbb{Z}}\frac{\gamma_n}{z^{n-\frac{1}{2}}}$, one can define the canonical R vacuum as

$$|-\frac{1}{2}\rangle_{(\beta,\gamma)} = e^{-\frac{1}{2}\phi_{\beta\gamma}(0)}|0\rangle. \tag{2.126}$$

And as in the bosonic case, for the (b,c) system with $\lambda = 2$ and $c(z) = \sum_{n\in\mathbb{Z}} \frac{c_n}{z^{n-1}}$ we have the following highest weight vacuum state

$$|1\rangle_{(b,c)} = c(0)|0\rangle. \tag{2.127}$$

2.2.2 BRST Conditions, Picture Changing and GSO projection

The BRST charge will be constructed the same way as in the bosonic case, and the conditions that the operators must satisfy in order to be in the cohomology of Q_{BRST} will then be derived. Following the procedure of 2.11, one can write

$$Q_{BRST} = \oint dz d\theta J_{BRST}, \tag{2.128}$$

with BRST supercurrent

$$J_{BRST} = -C^z(T_{matter} + \frac{1}{2}T_{ghost}) + \frac{3}{4}D(C^z(DC)B_{z\theta}) \tag{2.129}$$

$$= +\frac{1}{2}CD\mathbf{X}\partial_z\mathbf{X} - C^z DC^z DB_{z\theta} + \frac{3}{4}DC^z DC^z B_{z\theta}, \tag{2.130}$$

it is possible to verify that J_{BRST} is conformal primary if the spacetime dimension is $D = 10$, and the BRST charge is indeed nilpotent.

To construct vertex operators satisfying

$$[Q_{BRST}, \mathcal{V}] = 0, \tag{2.131}$$

it is useful to write the BRST charge in the following way,

$$Q_{BRST} = Q_0 + Q_1 + Q_2, \tag{2.132}$$

$$Q_0 = \oint dz(c\tilde{T}), \tag{2.133}$$

$$Q_1 = \oint dz\, \gamma\frac{1}{2}\psi^m\partial x_m = -\oint dz\, e^{\phi_{\beta\gamma}}\eta G, \tag{2.134}$$

$$Q_2 = -\oint dz\, \gamma^2\frac{b}{4} = -\oint dz\, e^{2\phi_{\beta\gamma}}\eta\partial\eta\frac{b}{4}, \tag{2.135}$$

where (2.99,2.110)

$$\begin{aligned}\tilde{T} &= T_m + T_{\beta\gamma} + \partial c b \\ &= -\frac{1}{2}(\partial x^\mu \partial x_\mu + \partial\psi_\mu \psi^\mu) - \frac{1}{2}\gamma\partial\beta - \frac{3}{2}\beta\partial\gamma + \partial c b,\end{aligned} \tag{2.136}$$

because one can analyze the implications of physical states being annihilated by Q_0,Q_1,Q_2 separately.

Q_0 condition. As in the bosonic case, one can define combinations

$$U(z) = P_h(\partial x^m, \psi^m, \beta, \gamma)e^{ik\cdot x} \tag{2.137}$$

with matter fields and (β, γ) ghosts, as suggested by 2.136. So, we will consider the conformal weight with respect to $\tilde{T}$ in this section. Similarly to 2.22, BRST invariance implies that U_h should have conformal weight $h(U) = 1$,

$$Q_0\, U = \partial(cU), \tag{2.138}$$

with $c(z)U_{h=1}(z)$ as an unintegrated representation of the same state. Using the mass-shell condition $m^2 = -k^2$ and 2.137 one has

$$Q_0\, (cU)(z_2) = \oint_{C_{z_2}} dz_1 \left[c(z_1)\left(\frac{c(1-h+\frac{m^2}{2})U}{z_{12}^2} + \frac{cU}{z_{12}} \right) + c\partial c \frac{U}{z_{12}} \right] \tag{2.139}$$

$$\Rightarrow Q_0\, (cU)(z_2) = (1 - h + \frac{m^2}{2})c\partial c U \tag{2.140}$$

$$\Rightarrow m^2 = 2(h-1),\;\; 2\alpha' = 1 \tag{2.141}$$

and the mass is determined from the P_h conformal weight in 2.137.

Q_1 condition. To analyse the constraint $Q_1|phys\rangle = 0$ on vertex operators, one can rewrite the primary 2.137 associated with the canonial vacuum of NS sector 2.125,

$$U(z) = U_h e^{-\phi_{\beta\gamma}}, \tag{2.142}$$

where U_h has conformal weight $h = \frac{1}{2}$ because Q_0 invariance. Vanishing of $Q_1 U(z)$

$$Q_1\, U(z_2) = \oint_{C_{z_2}} dz_1\, \eta(z_2) z_{12} (G(z_1)U_h(z_2)) \tag{2.143}$$

implies that the pole with $G\ U_h$ OPE is of order $\mathcal{O}(\frac{1}{z_{12}})$, so U_h is the lowest component of a superconformal primary field of weight $h = \frac{1}{2}$ satisfying 2.96.

In the R sector one has

$$U(z) = U_h e^{-\frac{1}{2}\phi_{\beta\gamma}}, \tag{2.144}$$

implying U_h with conformal weight $h = \frac{5}{8}$ to ensure Q_0 invariance. The Q_1 invariance implies that U_h is equivalent to a highest weight state of super-Virasoro algebra. If one indeed imposes $Q_1\ U_h = 0$, and uses 2.134,2.144,

$$Q_1\ U(z_2) = \oint_{C_{z_2}} dz_1 \eta(z_1) e^{\phi_{\beta\gamma}(z_1)} G(z_1) U_h(z_2) e^{-\frac{1}{2}\phi_{\beta\gamma}(z_2)}, \tag{2.145}$$

$$= \oint_{C_{z_2}} dz_1 \eta(z_2) z_{12}^{1/2} e^{\frac{1}{2}\phi_{\beta\gamma}(z_2)} \big(G(z_1) U_h(z_2)\big), \tag{2.146}$$

then U_h is annihilated by $G_{n\geq 0}$.

Q_2 condition. Finaly, for the vertices 2.142 and 2.144 of NS and R sectors one has

$$\begin{aligned} Q_2\ U(z_2) &= \oint_{C_{z_2}} dz_1 e^{2\phi_{\beta\gamma}} \eta\partial\eta \frac{b}{4} U_h(z_2) e^{q\phi_{\beta\gamma}(z_2)}, \\ &= \oint_{C_{z_2}} dz_1 \frac{1}{z_{12}^{2q}} \big(\eta\partial\eta \frac{1}{4} b\big) \big(e^{(2+q)\phi_{\beta\gamma}} U_h\big)(z_2) \end{aligned} \tag{2.147}$$

which vanishes identically for $q < \frac{1}{2}$.

RNS pictures

Before starting the discussion of massless and first massive level vertex operators, we will briefly discuss the different pictures of vertex operators. The ghost picture is the charge under the anomalous $U(1)$ symmetry (2.111), and to write non-vanishing expectation values in general, one has to represent the same physical state using operators in different pictures. One can create equivalent vertex using a picture changing operator $P_+(z) \equiv \{Q_{BRST}, \xi(z)\}$. This can be done in the following way,

$$\mathcal{U}^{(n+1)} = -2[Q_{BRST}, \xi\mathcal{U}^{(n)}], \tag{2.148}$$

where the upper index between parentheses will refer to the vertex (β, γ) ghost number. $\mathcal{V}_{(n+1)}$ is not BRST trivial, because only $\partial\xi$ is used to construct irreducible representations of the ghost algebra 2.118. As an example, the Q_2 part of the 2.148

acting on 2.144 gives rise to a term that is essential to BRST invariance,

$$\begin{aligned}[-2Q_2, \xi U_h e^{-\frac{1}{2}\phi_{\beta\gamma}}] &= -\frac{1}{2}\oint_{C_{z_2}} dz_1 e^{2\phi}\eta\partial\eta b(\xi U_h e^{-\frac{1}{2}\phi}) \\ &= -\frac{1}{2}\oint_{C_{z_2}} dz_1 [\frac{1}{z_{12}^2}\eta b + \frac{1}{z_{12}}\partial\eta b] U_h z_{12}^{(-2\cdot -\frac{1}{2})} e^{\frac{3}{2}\phi_{\beta\gamma}} \\ &= -\frac{1}{2}\eta b U_h e^{\frac{3}{2}\phi_{\beta\gamma}}, \end{aligned} \tag{2.149}$$

as we will see in the next subsection with the massless vertex of the R sector with picture $q = \frac{1}{2}$.

GSO projection

Besides BRST invariance, only GSO-projected states should be considered, as commented below 2.109. Using the bosonizations in 2.103 and 2.118, one can show that the picture changing operation 2.148 does not change the parity assignments defined by the following GSO operator

$$F = \oint dz(\sum_{j=0}^{4} i\partial H_j - \partial\phi_{\beta\gamma}), \tag{2.150}$$

which assign odd parity to NS canonical vacuum and all its even fermion number excitations as well as states created by the spin field S^α, as a result of

$$\begin{aligned} F(e^{i\vec{u}\cdot H} e^{q\phi_{\beta\gamma}}) &= \oint_{C_{z_2}} dz_1 (\sum_{j=0}^{4} (i\partial H_j)(z_1) e^{iu_j H_j}(z_2) e^{i\sum_{k\neq j} u_k H_k} e^{q\phi_{\beta\gamma}} - e^{i\vec{u}\cdot H}\partial\phi_{\beta\gamma}(z_1) e^{q\phi_{\beta\gamma}(z_2)}) \\ &= \oint_{C_{z_2}} dz_1 (\sum_{j=0}^{4} \frac{u_j}{z_{12}} e^{i\vec{u}\cdot H} e^{q\phi_{\beta\gamma}} - e^{i\vec{u}\cdot H}\frac{-q}{z_{12}} e^{q\phi_{\beta\gamma}}) \\ &= (\sum_{j=0}^{4} u_j + q)(e^{i\vec{u}\cdot H} e^{q\phi_{\beta\gamma}}), \end{aligned} \tag{2.151}$$

where we used 2.102, 2.114. In particular, the tachyonic state $T(z) = ce^{ik\cdot x}e^{-\phi_{\beta\gamma}}$, with negative mass (see 2.141) is allowed from the BRST invariance, but satisfies $(-1)^F\ T(z) = -T(z)$. It is, therefore, ruled out from GSO projection.

2.2.3 RNS Vertex Operators

Now we will write the massless vertex operators of NS and R sectors in the canonical ghost picture and one picture above. BRST invariance will be examined at the massless and first massive levels, as well as the existence of spurious states.

NS sector $(m^2 = 0)$ **.** If one considers the canonical ghost picture $q = -1$ in the NS sector, one can write the integrated vertex (2.137) as

$$U^{(-1)}(\xi_m, k; z) = \xi_m \psi^m e^{-\phi_{\beta\gamma}} e^{ik\cdot x}, \tag{2.152}$$

and BRST invariance implies transverse polarizations (ε_m),

$$Q_{BRST}\, U^{(-1)}(z_2) = \frac{1}{2}\eta(ik\cdot\varepsilon)e^{ik\cdot x} \Rightarrow \varepsilon_m k^m = 0. \tag{2.153}$$

Longitudinal polarizations $\varepsilon_m \sim k_m$ are BRST exact,

$$Q_{BRST}\left(2ie^{-2\phi_{\beta\gamma}}\partial\xi e^{ik\cdot x}\right) = k_m \psi^m e^{-\phi_{\beta\gamma}} e^{ik\cdot x}, \tag{2.154}$$

and $U^{(-1)}(k_m, z)$ therefore correspond to spurious states.

One can raise the picture of 2.152 using 2.148,

$$U^{(0)}(\varepsilon_m, z) = \left[\varepsilon\cdot x + (ik\cdot\psi)(\varepsilon\cdot\psi)\right]e^{ik\cdot x}, \tag{2.155}$$

and the integrated one is

$$V^{(0)}(\varepsilon_m, z) = cU^{(0)}(\varepsilon_m, z) - \frac{1}{2}\gamma(\varepsilon\cdot\psi)e^{ik\cdot x}, \tag{2.156}$$

where the second term of right-hand side comes from $[-2Q_2, \xi c U^{(-1)}]$, ensuring BRST invariance of $V^{(0)}$. So, the massless NS sector has 8 physical degrees of freedom corresponding to the transverse polarization ε_m.

R sector $(m^2 = 0)$ **.** To construct the unintegrated vertex in the canonical $q = -\frac{1}{2}$ ghost picture in the R sector, one can contract the spin-field with a spinor polarization u^α,

$$U^{(-\frac{1}{2})}(k, u^\alpha; z) = u^\alpha S_\alpha e^{-\frac{1}{2}\phi_{\beta\gamma}} e^{ik\cdot x}, \tag{2.157}$$

satisfying the massless Dirac equation as a consequence of BRST invariance,

$$Q_{BRST}U^{(-\frac{1}{2})} = 0 \Rightarrow k_m \gamma^m_{\alpha\beta} u^\beta = 0, \tag{2.158}$$

where the OPE 2.105 has been used. There are no spurious states and the Dirac equation implies that u^α is a Majorana-Weyl spinor with 8 degrees of freedom. Finally, one can consider the picture $q = +1/2$ operator

$$[-2Q_1, \xi U^{(-\frac{1}{2})}] = -\oint_{C_{z_2}} dz_1 \big(e^\phi \eta \psi \cdot \partial x\big)(z_1) \big(u^\alpha S_\alpha e^{-\frac{1}{2}} e^{ik\cdot x}\big)(z_2), \tag{2.159}$$

$$= \big(\partial x_m + \frac{1}{4}(ik\cdot\psi)\psi_m\big) u^\alpha \gamma^m_{\alpha\beta} S^\beta e^{\frac{1}{2}\phi_{\beta\gamma}} e^{ik\cdot x} \tag{2.160}$$

and using the subleading terms of 2.105,

$$\psi^m S_\alpha \sim \frac{1}{z_{12}^{1/2}} \gamma^m_{\alpha\beta} S^\beta + \frac{z_{12}^{1/2}}{4}\left(\psi^m \psi_n \gamma^n_{\alpha\beta} S^\beta + \frac{1}{18}\psi^n \psi^n \gamma^m_{\alpha\delta}(\gamma_{np})^\delta_\beta S^\beta\right), \tag{2.161}$$

and the term coming from 2.149, one has

$$U^{(\frac{1}{2})}(k, u^\alpha; z) = \frac{1}{\sqrt{2}}\big(\partial x_m + \frac{1}{4}(ik\cdot\psi)\psi_m\big) u^\alpha \gamma^m_{\alpha\beta} S^\beta e^{\frac{1}{2}\phi_{\beta\gamma}} e^{ik\cdot x} - \frac{1}{2}\eta b u^\alpha S_\alpha e^{\frac{3}{2}\phi_{\beta\gamma}} e^{ik\cdot x}. \tag{2.162}$$

NS sector $(m^2 = 2)$**.** For the $m^2 = 2$ mass level of open superstring in the NS sector one can write the following combination for P_h with conformal weight $h = \frac{3}{2}$, in the $q = -1$ picture,

$$U^{(-1)}(B_{mnp}, G_{mn}, D_m; z) = \big(B_{mnp}\psi^m \psi^n \psi^p + G_{mn}\partial x^m \psi^n + D_m \partial\psi^m + F_m \partial\phi_{\beta\gamma}\psi^m\big) e^{-\phi_{\beta\gamma}} e^{ik\cdot x}, \tag{2.163}$$

and the last term can be eliminated if we add a total derivative $\partial\big(F_m \psi^m e^{ik\cdot x} e^{-\phi_{\beta\gamma}}\big)$ to $U^{(-1)}$, which does not change the resulting vertex operator and produces the transformations

$$\delta F_m = -\xi_m, \tag{2.164}$$

$$\delta G_{mn} = ik_m \xi_n, \tag{2.165}$$

$$\delta D_m = \xi_m. \tag{2.166}$$

The vertex can therefore be written as 2.142 with

$$U_h = \left(B_{mnp}\psi^m\psi^n\psi^p + G_{mn}\partial x^m\psi^n + D_m\partial\psi^m\right)e^{ik\cdot x}. \tag{2.167}$$

As was shown in 2.143, BRST invariance implies the vanishing of the cubic and quadratic poles of GU_h OPE,

$$\begin{aligned} -2G(z_1)U_h(z_2) \sim &-\frac{1}{z_{12}^3}\left(G_{mn}\delta^{mn} + ik^m D_m\right)e^{ik\cdot x} \\ &-\frac{1}{z_{12}^2}\left((3B_{mnp}ik^m + G_{np})\psi^n\psi^p + (ik^n G_{mn} - D_m)\partial x^m\right)e^{ik\cdot x}, \end{aligned} \tag{2.168}$$

so 2.167 polarizations satisfy the following equations,

$$\begin{aligned} 3ik^m B_{mnp} + \frac{1}{2}G_{[np]} &= 0, \\ ik^n G_{mn} - D_m &= 0, \\ \delta^{mp}G^{mp} + ik^m D_m &= 0, \end{aligned} \tag{2.169}$$

in order to be BRST invariant.

One can identify spurious solutions to 2.169 if one obtains them from BRST exact terms. As an example, if one considers a transverse antisymmetric tensor C_{mn}, the BRST exact term

$$\begin{aligned} [2Q_{BRST}, e^{-2\phi_{\beta\gamma}}C_{mn}\psi^m\psi^n\partial\xi e^{ik\cdot x}] &= \oint_{C_{z_2}} dz_1\left(e^{\phi_{\beta\gamma}}\eta\psi\cdot\partial x\right)\left(e^{-2\phi_{\beta\gamma}}C_{mn}\psi^m\psi^n\partial\xi e^{ik\cdot x}\right) \\ &= \left(2C_{mn}\partial x^m\psi^n - \frac{1}{3!}ik_{[m}C_{np]}\psi^m\psi^n\psi^p\right)e^{-\phi_{\beta\gamma}}e^{ik\cdot x}, \end{aligned} \tag{2.170}$$

gives the spurious solution to 2.169 [21]

$$B_{mnp} = -\frac{1}{3!}ik_{[m}C_{np]}, \qquad G_{mn} = C_{mn}, \qquad D_m = 0, \tag{2.171}$$

$$\delta^{mn}C_{mn} = 0 \qquad k^m C_{mn} = 0, \tag{2.172}$$

so one can gauge away the antisymmetric part of G_{mn}. Similarly one can identify the trace $\delta^{mn}C_{mn}$ and the vector D_m as spurious [21], so the physical solution to 2.169 is a symmetric tensor g_{mn} ($\frac{10\cdot 11}{2}$ $d.o.f.$) and a three form b_{mnp} ($\frac{10!}{3!7!}$ $d.o.f.$), satisfying

$$
\begin{aligned}
\delta^{mn} g_{mn} &= 0 && (-1\ d.o.f.) \\
k^m g_{mn} &= 0 && (-10\ d.o.f.) \qquad (2.173) \\
k^m b_{mnp} &= 0 && (-\frac{9!}{2!7!}\ d.o.f.) \qquad (2.174)
\end{aligned}
$$

And the NS sector at the first massive level has 128 degrees of freedom, corresponding to $\frac{10 \cdot 11}{2} - 10 - 1 = 44$ degrees of freedom of a spin-two tensor g_{mn} and $\frac{10!}{3!7!} - \frac{9!}{2!7!} = 84$ degrees of freedom of the transverse antisymmetric tensor b_{mnp}.

R sector $(m^2 = 2)$**.** The interaction of fermions with spin-fields complicates the computations in the R sector. But the reasoning is the same as for all previous cases. One has to identify all possible terms in the vertex, impose BRST invariance to constrain the polarizations, and determine the spurious states.

We will comment on the results derived in [21] for the massive vertex. In the canonical picture, where $h(e^{-\frac{1}{2}\phi_{\beta\gamma}} e^{ik \cdot x}) = \frac{3}{8}$, one can write the following combination

$$
U^{(-\frac{1}{2})}(u, v, z) = \left(u_m^\alpha \partial x^m + v_\beta^m \psi_m \psi_n (\gamma^n)^{\beta\alpha}\right) S_\alpha e^{-\frac{1}{2}\phi_{\beta\gamma}} e^{ik \cdot x}, \qquad (2.175)
$$

so the BRST condition implies the following set of equations for (u_m^α, v_β^m),

$$
\begin{aligned}
v_\beta^m &= -\frac{1}{8} u^{m\delta} k_n \gamma^n_{\delta\alpha} + \frac{1}{36} k^n u_n^\delta \gamma^m_{\delta\beta} \\
u_m^\alpha \gamma^m_{\alpha\beta} &= k^n k_m u_n^\alpha \gamma^m_{\alpha\beta}, \qquad (2.176)
\end{aligned}
$$

this correspond to $10 \cdot 16 + 16 = 176$ independent relations, so the number of states is constrained to $2 \cdot (16 \cdot 10) - 176 = 144$. From supersymmetry, we know there are 16 spurious states, so one can use the gauge freedom to fix the trace u_m^α, and then the physical solution to 2.176 correspond to a transverse traceless vector-spinor ψ_m^α

$$
\begin{aligned}
\gamma^m_{\alpha\beta} \psi_m^\alpha &= 0 && (-16 d.o.f.) \\
k_m \psi_\alpha^m &= 0 && (-16 d.o.f.), \qquad (2.177)
\end{aligned}
$$

with $10 \cdot 16 - 32 = 128$ degrees of freedom. The vertex is now written as

$$U^{(-\frac{1}{2})}(u,z) = \big(\psi^{\alpha}_{m}\partial x^{m} - \frac{1}{8}\psi^{m\delta}k_{p}\gamma^{p}_{\delta\beta}\psi_{m}\psi_{n}(\gamma^{n})^{\beta\alpha}\big)S_{\alpha}e^{-\psi_{\beta\gamma}}e^{ik\cdot x}. \tag{2.178}$$

This vector-spinor ψ^{α}_{m} 2.177, the spin-two tensor g_{mn} 2.173 and the antisymmetric tensor b_{mnp} 2.174 compose the spin-2 massive supermultiplet,

$$(\gamma^{m})_{\beta\alpha}\psi^{\alpha}_{m} = 0,\ \ k^{m}g_{mn} = 0,\ \ k_{m}\psi^{m\alpha} = 0,\ \ k^{m}b_{mnp} = 0. \tag{2.179}$$

In the next chapter, we will present it in superspace using $d = 10$ super-Yang-Mills fields.

Chapter 3

Massive vertex in Pure Spinor formalism

In this chapter, after a brief review of pure spinor formalism, the unintegrated vertex operator at the first mass level will be computed from the OPE between two massless vertices, and its BRST invariance will be verified. In section 3.1, the gauge symmetries are used to find a gauge where the vertex operator superfields are related to the usual supergravity superfields [10], which describe the spin-2 massive multiplet. This computation is an original work published in [17]. For details of pure spinor formalism at tree level, one can consult [8], [29].

The pure spinor formalism for the open string has the following action

$$S_{PS} = \frac{1}{\pi}\int d^2z\left(\frac{1}{2}\partial x^m\bar{\partial}x_m + p_\alpha\bar{\partial}\theta^\alpha - w_\alpha\bar{\partial}\lambda^\alpha\right), \tag{3.1}$$

where $m = 0, ..., 9$, and $\alpha = 1, ..., 16$ are the vector and spinorial indices of $SO(10)$, together with a nilpotent BRST operator

$$Q = \oint dz\lambda^\alpha d_\alpha, \tag{3.2}$$

with the GS constraint defined as

$$d_\alpha = p_\alpha - \frac{1}{2}\partial x^m(\gamma_m\theta)_\alpha - \frac{1}{8}(\theta\gamma^m\partial\theta)(\gamma_m\theta)_\alpha,$$

and the field λ^α satisfying the pure spinor property $\lambda^\alpha\gamma^m_{\alpha\beta}\lambda^\beta = 0$. The worldsheet variables θ^α, λ^α have conformal weight $h = 0$ and their conjugate pairs p_α, w_α have conformal weight $h = 1$. There is a ghost current $J = w_\alpha\lambda^\alpha$ that can be used to define the ghost number of pure spinor operators.

The integrated and unintegrated vertex operators are [8]

$$U(z) =: \Pi^m A_m : + : \partial\theta^\alpha A_\alpha : + : d_\alpha W^\alpha : + : \frac{1}{2}N^{mn}F_{mn} :, \tag{3.3}$$

$$V(z) = \lambda^\alpha A_\alpha, \tag{3.4}$$

with supersymmetric momentum $\Pi^m = \partial x^m + \frac{1}{2}(\theta\gamma^m\partial\theta)$, the Lorentz current $N^{mn} = \frac{1}{2}w\gamma^{mn}\lambda$ and superfields $[A_m, A_\alpha, W^\alpha, F_{mn}]$ built out of A_α,

$$W^\alpha = \frac{1}{10}(\gamma^m)^{\alpha\beta}(D_\beta A_m - \partial_m A_\beta) \tag{3.5}$$

$$A_m = \frac{1}{8}\gamma_m^{\alpha\beta}D_\alpha A_\beta \tag{3.6}$$

$$F_{mn} = \frac{1}{8}(\gamma_{mn})^\alpha{}_\beta D_\alpha W^\beta, \tag{3.7}$$

and their super Yang-Mills equations implies the onshell condition $Q \cdot V = 0$ and the descent relation $Q \cdot U = \partial V$. The superfields 3.5, 3.6 and 3.7 are expanded in θ using the $\theta^\alpha A_\alpha = 0$ gauge as [30, 18, 31]

$$A_\alpha(x,\theta) = \frac{1}{2}\xi_m(\gamma^m\theta)_\alpha e^{ik\cdot x} - \frac{1}{3}\chi^\beta(\gamma^m\theta)_\beta(\gamma_m\theta)_\alpha e^{ik\cdot x} + \cdots, \tag{3.8}$$

where ξ_m and χ^α are the gluon and gluino polarizations, respectively. The normal ordering $: \cdot :$ prescription is defined as [24]

$$: A(z)B(w) :\equiv \oint \frac{dz}{z-w}A(z)B(w). \tag{3.9}$$

The relevant OPEs for subsequent computations are

$$\begin{aligned}
x^m(z,\bar{z})x^n(w,\bar{w}) &\sim -\delta^{mn}\ln|z-w|^2, & d_\alpha(z)\theta^\beta(w) &\sim \frac{\delta_\alpha^\beta}{z-w},\\
d_\alpha(z)d_\beta(w) &\sim -\frac{\gamma^m_{\alpha\beta}\Pi_m(w)}{z-w}, & \Pi^m(z)\Pi^n(w) &\sim -\frac{\delta^{mn}}{(z-w)^2},\\
d_\alpha(z)\Pi^m(w) &\sim \frac{(\gamma^m\partial\theta(w))_\alpha}{z-w}, & \Pi^m(z)V(w) &\sim -\frac{\partial^m V(w)}{z-w},\\
N^{mn}(z)\lambda^\alpha(w) &\sim \frac{1}{2}\frac{(\gamma^{mn})^\alpha{}_\beta\lambda^\beta(w)}{z-w}, & d_\alpha(z)V(w) &\sim D_\alpha V,
\end{aligned} \tag{3.10}$$

where $V(w) = V(\theta)e^{ik\cdot x}$ is a superfield, $D_\alpha = \frac{\partial}{\partial\theta^\alpha} + \frac{1}{2}(\gamma^m\theta)\partial_m$ is the supersymmetric derivative, and $\partial_m \equiv \frac{\partial}{\partial x^m}$.

The operator algebra of string theory primary fields can be used to recover all theory higher mass resonances [5]. Since unintegrated vertex operators of mass $m^2 = 2n$ should be constructed from combinations of $[\Pi^m, d_\alpha, \theta^\alpha, N^{mn}, J, \lambda^\alpha]$ with ghost number 1 and conformal weight n, one can define the unintegrated vertex operator corresponding to the first massive state as

$$V^{(12)}_{m^2=2} \equiv \oint dz_1\, U^{(1)}(z_1) V^{(2)}(z_2), \tag{3.11}$$

$$(k_1 + k_2)^2 = -2, \tag{3.12}$$

where $U^{(1)}$ and $V^{(2)}$ are integrated and unintegrated massless vertex operators, respectively. The onshell and descent relations of $V^{(2)}$ and $U^{(1)}$ implies $Q \cdot V^{(12)}_{m^2=2} = 0$.

To write 3.11 in terms of super Yang-Mills superfields, first consider the OPE between the first term of 3.3 with 3.4,

$$: \Pi^m A^1_m(z_1) :: \lambda^\alpha A^2_\alpha(z_2) := + \frac{1}{z_{12}} : \Pi^m A^1_m \lambda^\alpha A^2_\alpha(z_2) : - \frac{1}{z_{12}} : \partial A^1_m \lambda^\alpha \partial^m A^2_\alpha(z_2) : . \tag{3.13}$$

Using the equation $\partial K = \partial\theta^\alpha D_\alpha K + \Pi^m (ik_m) K$ on 3.13, one has

$$\begin{aligned} \oint_{z_2} dz_1 : \Pi^m A^1_m(z_1) : \lambda^\alpha A^2_\alpha(z_2) = &: \Pi^m \lambda^\alpha A^1_m A^2_\alpha : \\ &- : \partial\theta^\beta \lambda^\alpha D_\beta A^1_m \partial^m A^2_\alpha : \\ &- : \Pi^m \lambda^\alpha (ik^1_m) A^1_n \partial^n A^2_\alpha : \;. \end{aligned} \tag{3.14}$$

Considering the other terms of 3.3, one obtains

$$\oint_{z_2} dz_1 : \partial\theta^\beta A^1_\beta(z_1) : \lambda^\alpha A^2_\alpha(z_2) = : \partial\theta^\beta \lambda^\alpha A^1_\beta A^2_\alpha :, \tag{3.15}$$

$$\begin{aligned} \oint_{z_2} dz_1 : d_\beta W^{1\beta}(z_1) : \lambda^\alpha A^2_\alpha(z_2) =&: d_\beta \lambda^\alpha W^{1\beta} A^2_\alpha : \\ &- : \partial\theta^\beta \lambda^\alpha D_\beta W^{1\xi} D_\xi A^2_\alpha : \\ &- : \Pi^m \lambda^\alpha (ik^1_m) W^{1\xi} D_\xi A^2_\alpha :, \end{aligned} \tag{3.16}$$

$$\begin{aligned} \oint_{z_2} dz_1 : \frac{1}{2} N^{mn} F^1_{mn}(z_1) : \lambda^\alpha A^2_\alpha(z_2) =&: N^{mn} \lambda^\alpha \left(\frac{1}{2} F^1_{mn} A^2_\alpha\right) : \\ &- : \partial\theta^\beta \lambda^\alpha D_\beta \left(\frac{1}{4} F^1_{pq} (\gamma^{pq})_\alpha{}^\xi\right) A^2_\xi \\ &- : \Pi^m \lambda^\alpha \left(ik^1_m \frac{1}{4} (\gamma^{pq})_\alpha{}^\xi F^1_{pq} A^2_\xi\right) : . \end{aligned} \tag{3.17}$$

The vertex operator can therefore be written as

$$V^{(12)}_{m^2=2} =: \partial\theta^\beta \lambda^\alpha \bar{B}_{\alpha\beta} :+ : \Pi^m \lambda^\alpha \bar{H}_{m\alpha} : + : d_\beta \lambda^\alpha \bar{C}^\beta_\alpha : + : \frac{1}{2} N^{mn} \lambda^\alpha \bar{F}_{mn\alpha} :, \tag{3.18}$$

with

$$\bar{B}_{\alpha\beta} = -(\gamma^m W^1)_\beta (ik^2_m) A^2_\alpha - D_\beta W^{1\xi} D_\xi A^2_\alpha - D_\beta D_\alpha W^{1\xi} A^2_\xi \tag{3.19}$$

$$\bar{H}_{m\alpha} = A^1_m A^2_\alpha - (ik^1_m)\big(A^1_n ik^n_2 A^2_\alpha + W^{1\xi} D_\xi A^2_\alpha + D_\alpha W^{1\xi} A^2_\xi\big), \tag{3.20}$$

$$\bar{C}^\beta_\alpha = W^{1\beta} A^2_\alpha, \tag{3.21}$$

$$\bar{F}_{mn\alpha} = F^1_{mn} A^2_\alpha. \tag{3.22}$$

It is BRST invariant by construction, as one can see by applying the onshell condition and the descent relation for $U^{(1)}$ and $V^{(2)}$. But one can check how BRST charge acts on each term of 3.18,

$$Q : \Pi^m \lambda^\alpha A^1_m A^2_\alpha : =+ : (\gamma^m \partial\theta)_\alpha \lambda^\alpha \lambda^\beta A^1_m A^2_\beta : + : \Pi^m \lambda^\alpha \lambda^\beta (D_\alpha A^1_m) A^2_\beta : \tag{3.23}$$

$$\begin{aligned} Q : \Pi^n (ik^1_n) A^1_m \lambda^\alpha \partial^m A^2_\alpha :=+ \ & : (\gamma^n \partial\theta)_\alpha \lambda^\alpha \lambda^\beta (ik^1_n)(ik^{2m}) A^1_m A^2_\beta : \\ & + : \Pi^n \lambda^\beta \lambda^\alpha (ik^1_n)(ik^{2m}) D_\alpha A^1_m A^2_\beta : \end{aligned} \tag{3.24}$$

$$\begin{aligned} Q : ik_{1m} \Pi^m W^{1\alpha} \lambda^\beta D_\alpha A^2_\beta :=+ & : \lambda^\xi (\gamma^m \partial\theta)_\xi \lambda^\beta (ik^1_m) W^{1\alpha} D_\alpha A^2_\beta : \\ & + : \Pi^m \lambda^\beta (ik^1_m) \lambda^\xi D_\xi W^{1\alpha} D_\alpha A^2_\beta : \\ & - : \Pi^m \lambda^\beta (ik^1_m) W^{1\alpha} \lambda^\xi D_\xi D_\alpha A^2 : \end{aligned} \tag{3.25}$$

$$\begin{aligned} Q : ik_{1p} \Pi^p \lambda^\alpha F^1_{mn} (\gamma^{mn})^\beta{}_\alpha A^2_\beta :=+ & : (\gamma^p \partial\theta)_\alpha \lambda^\alpha \lambda^\xi (ik^1_p) F^1_{mn} (\gamma^{mn})^\beta{}_\xi A^2_\beta : \\ & + : \Pi^p \lambda^\xi (ik^1_p) \lambda^\alpha D_\alpha F^1_{mn} (\gamma^{mn})^\beta{}_\xi A^2_\beta : \\ & + : \Pi^p \lambda^\xi (ik^1_p) F^1_{mn} (\gamma^{mn})^\beta{}_\xi \lambda^\alpha D_\alpha A^2_\beta : \end{aligned} \tag{3.26}$$

$$\begin{aligned} Q: d_\alpha\lambda^\beta W^{1\alpha}A^2_\beta :=& - :\Pi_m\lambda^\xi\lambda^\beta\gamma^m_{\xi\alpha}W^{1\alpha}A^2_\beta: \\ &+ :\partial\lambda^\xi\lambda^\beta\gamma^m_{\xi\alpha}(ik^{12}_m)W^{1\alpha}A^2_\beta: \\ &- :d_\alpha\lambda^\xi\lambda^\beta D_\xi W^{1\alpha}A^2_\beta: \end{aligned} \tag{3.27}$$

$$\begin{aligned} Q: -\partial\theta^\xi\lambda^\beta D_\xi W^{1\alpha}D_\alpha A^2_\beta :=& - :\partial\lambda^\alpha\lambda^\beta D_\alpha W^{1\xi}D_\xi A^2_\beta: \\ &+ :\partial\theta^\alpha\lambda^\beta\lambda^\xi D_\xi\big(D_\alpha W^{1\gamma}D_\gamma A^2_\beta\big): \end{aligned} \tag{3.28}$$

$$\begin{aligned} Q: -\partial\theta^\alpha\lambda^\beta D_\alpha A^{(1)}_m\partial^m A^{(2)}_\beta :=& - :\partial\lambda^\alpha\lambda^\beta(\gamma_m W^1)_\alpha(ik^m_2)A^2_\beta: \\ &- :\partial\lambda^\alpha\lambda^\beta A^1_\alpha A^2_\beta: \\ &+ :\partial\theta^\alpha\lambda^\xi\lambda^\beta\gamma_{m\alpha\gamma}D_\xi W^{1\gamma}(ik^m_2)A^2_\beta: \\ &+ :\partial\theta^\alpha\lambda^\xi\lambda^\beta D_\xi A^1_\alpha A^2_\beta: \end{aligned} \tag{3.29}$$

$$Q: \partial\theta^\alpha\lambda^\beta A^1_\alpha A^2_\beta := + \ :\partial\lambda^\alpha\lambda^\beta A^1_\alpha A^2_\beta: - :\partial\theta^\alpha\lambda^\beta\lambda^\xi D_\xi A^1_\alpha A^2_\beta: \tag{3.30}$$

$$\begin{aligned} Q: \partial\theta^\alpha D_\alpha F^1_{mn}(\gamma^{mn}\lambda)^\beta A^2_\beta :=& - :\partial\lambda^\alpha\lambda^\xi D_\alpha F^1_{mn}(\gamma^{mn})^\beta{}_\xi A^2_\beta: \\ &- :\partial\theta^\alpha\lambda^\delta\lambda^\xi D_\xi(D_\alpha F^1_{mn})(\gamma^{mn})^\beta{}_\delta A^2_\beta: \\ &+ :\partial\theta^\alpha\lambda^\xi\lambda^\gamma D_\alpha F^1_{mn}(\gamma^{mn})^\beta{}_\xi D_\gamma A^2_\beta: \end{aligned} \tag{3.31}$$

$$\begin{aligned} Q: \frac{1}{2}N^{mn}\lambda^\beta F^1_{mn}A^2_\beta :=& -\frac{1}{4} :(\gamma^{mn})^\alpha{}_\xi\partial\lambda^\xi\lambda^\beta D_\alpha(F^1_{mn}A^2_\beta): \\ &-\frac{1}{4} :(\gamma^{mn})^\alpha{}_\xi d_\alpha\lambda^\xi\lambda^\beta F^1_{mn}A^2_\beta: \\ &+\frac{1}{2} :N^{mn}\lambda^\beta\lambda^\alpha D_\alpha F^1_{mn}A^2_\beta:. \end{aligned} \tag{3.32}$$

Collecting each ghost number 2 component proportional to $\partial\theta^\xi\lambda^\alpha\lambda^\beta$, $\Pi^m\lambda^\alpha\lambda^\beta$, $\partial\lambda^\alpha\lambda^\beta$, $d_\xi\lambda^\alpha\lambda^\beta$ and $N^{mn}\lambda^\alpha\lambda^\beta$, one can see that the BRST variation of $V^{(12)}_{m^2=2}$ vanishes. For example, the terms proportional to $d_\xi\lambda^\alpha\lambda^\beta$ in 3.27 and 3.32 cancel each other. Using the equation of motion $ik^1_m(\gamma^m W^1)_\alpha = 0$ and the pure spinor identity

$(\lambda\gamma^n)_\alpha(\lambda\gamma_n)_\beta = 0$, one can show that the following constraint [10]

$$:N^{mn}\lambda^\beta\lambda^\alpha:(\gamma_m)_{\alpha\gamma} = \frac{1}{2}:J\lambda^\beta\lambda^\alpha:\gamma^n_{\alpha\gamma} + \frac{5}{2}\lambda^\beta\partial\lambda^\alpha\gamma^n_{\alpha\gamma} + \frac{1}{2}\lambda^\delta\partial\lambda^\alpha(\gamma^{sn})_\delta^{\ \beta}(\gamma_s)_{\alpha\gamma}, \tag{3.33}$$

implies that the last term of 3.32 can be written as

$$:\frac{1}{2}N^{mn}\lambda^\alpha\lambda^\beta D_\alpha F^1_{mn}A^2_\beta: = :\frac{1}{4}\partial\lambda^\alpha\lambda^\delta(\gamma^{ns})_\delta^{\ \beta}D_\alpha F^1_{ns}A^2_\beta:, \tag{3.34}$$

and therefore cancels all other terms proportional to $\partial\lambda^\alpha\lambda^\beta$.[1]

Physical information of 3.18 is obtained through a gauge fixing procedure wherein the massive vertex operator superfields are related to the spin-2 massive supermultiplet in 10 dimensions. This multiplet comprises a traceless symmetric tensor denoted as g_{mn}, a three form b_{mnp} and a spin-$\frac{3}{2}$ field $\psi_{m\alpha}$, all satisfying

$$(\gamma^m)^{\beta\alpha}\psi_{m\alpha} = 0, \ \ \partial^m g_{mn} = 0, \ \ \partial^m\psi_{m\alpha} = 0, \ \ \partial^m b_{mnp} = 0. \tag{3.35}$$

3.1 Gauge transformations

In this section, the operator vertex 3.18 will be gauge fixed following the procedure of [10] to a gauge where

$$B_{\alpha\beta} = \gamma^{mnp}_{\alpha\beta}B_{mnp}, \qquad \partial^m B_{mnp} = 0, \tag{3.36}$$

$$\gamma^{m\alpha\beta}H_{m\beta} = 0, \qquad \partial^m H_{m\alpha} = 0, \tag{3.37}$$

$$C^{\ \beta}_\alpha = (\gamma^{mnpq})^\beta_{\ \alpha}C_{mnpq}, \qquad \gamma^{m\alpha\beta}F_{\alpha mn} = 0. \tag{3.38}$$

In this gauge, one can check that the $\theta = 0$ components of the superfields B_{mnp}, $G_{mn} \equiv D_\alpha\gamma^{\alpha\beta}_{(m}H_{n)\beta}$ and $-\frac{1}{72}H_{m\alpha}$ are $b_{mnp}, g_{mn}, \psi_{m\alpha}$ respectively [10].

The operator vertex 3.18 is gauge invariant by

$$V^{(12)}_{m^2=2} \longrightarrow V^{(12)}_{m^2=2} + Q\Omega, \tag{3.39}$$

[1] I would like to thank Carlos Mafra for correcting an error in an earlier version of this computation.

where

$$\Omega =+ : \partial\theta^\alpha \Omega_{1\alpha} : + : d_\alpha \Omega_2^\alpha : + : \Pi^m \Omega_{3m} : + : J\Omega_4 : + : N^{mn}\Omega_{5mn} : . \tag{3.40}$$

Using the OPEs 3.10, one finds

$$\begin{aligned} Q\Omega =&: \partial\lambda^\alpha \Big(\Omega_{1\alpha} + \gamma^m_{\alpha\xi}\partial_m \Omega_2^\xi - D_\alpha\Omega_4 - \frac{1}{2}(\gamma^{mn})^\beta{}_\alpha D_\beta \Omega_{5mn}\Big) : \\ &+ : \partial\theta^\beta\lambda^\alpha \Big(- D_\alpha\Omega_{1\beta} + \gamma^m_{\alpha\beta}\Omega_{3m}\Big) : \\ &+ : \Pi^m\lambda^\alpha \Big(- \gamma_{m\alpha\xi}\Omega_2^\xi + D_\alpha\Omega_{3m}\Big) : \\ &+ : d_\beta\lambda^\alpha \Big(- D_\alpha\Omega_2^\beta - \delta^\beta_\alpha\Omega_4 - \frac{1}{2}(\gamma^{mn})^\beta{}_\alpha\Omega_{5mn}\Big) : \\ &+ : N^{mn}\lambda^\alpha \Big(D_\alpha\Omega_{5mn}\Big) : \\ &+ : J\lambda^\alpha \Big(D_\alpha\Omega_4\Big) :, \end{aligned} \tag{3.41}$$

so the vertex operator superfields have the following variations

$$\delta\bar{B}_{\alpha\beta} = -D_\alpha\Omega_{1\beta} + \gamma^m_{\alpha\beta}\Omega_{3m}, \tag{3.42}$$

$$\delta\bar{H}_{m\alpha} = -\gamma_{m\alpha\xi}\Omega_2^\xi + D_\alpha\Omega_{3m}, \tag{3.43}$$

$$\delta\bar{C}^\beta_\alpha = -D_\alpha\Omega_2^\beta - \delta^\beta_\alpha\Omega_4 - \frac{1}{2}(\gamma^{mn})^\beta{}_\alpha\Omega_{5mn}, \tag{3.44}$$

$$\delta\bar{F}_{mn\alpha} = D_\alpha\Omega_{5mn}. \tag{3.45}$$

There are additional terms proportional to $\partial\lambda^\alpha$ and $J\lambda^\alpha$ coming from the gauge transformation 3.39,

$$\bar{G}_\alpha \equiv \Omega_{1\alpha} + \gamma^m_{\alpha\xi}\partial_m\Omega_2^\xi - D_\alpha\Omega_4 - \frac{1}{2}(\gamma^{mn})^\beta{}_\alpha D_\beta\Omega_{5mn}, \tag{3.46}$$

$$\bar{E}_\alpha \equiv D_\alpha\Omega_4, \tag{3.47}$$

and the following constraint [10]

$$: N^{mn}\lambda^\alpha\gamma_{m\alpha\beta} : -\frac{1}{2} : J\lambda^\alpha\gamma^n_{\alpha\beta} : -2\partial\lambda^\alpha\gamma^n_{\alpha\beta} = 0 \tag{3.48}$$

implies that 3.18 is invariant under the field redefinition

$$\delta_\Lambda \bar{G}_\alpha = -4\gamma^n_{\alpha\xi}\Lambda^\xi_n, \tag{3.49}$$
$$\delta_\Lambda F_{\alpha mn} = \gamma_{m\alpha\xi}\Lambda^\xi_n - \gamma_{n\alpha\xi}\Lambda^\xi_m, \tag{3.50}$$
$$\delta_\Lambda \bar{E}_\alpha = -\gamma^n_{\alpha\xi}\Lambda^\xi_n. \tag{3.51}$$

Finally, after the gauge-fixing procedure 3.36, 3.37, 3.38, all vertex operator superfields will be expressed in terms of d=10 Yang-Mills superfields and will satisfy the equations:

$$H_{m\alpha} = \frac{3}{7}(\gamma^{st})_\alpha{}^\beta D_\beta B_{mst}, \tag{3.52}$$
$$C^\alpha_\beta = \frac{1}{4}(\gamma^{mnpq})^\alpha{}_\beta \partial_m B_{npq}, \tag{3.53}$$
$$F_{mn\alpha} = \frac{1}{16}(6\mathcal{H}_{mn\alpha} - (\gamma_{p[m})_\alpha{}^\beta \mathcal{H}_{n]p\beta}), \tag{3.54}$$
$$E_\alpha = 0, \tag{3.55}$$
$$G_\alpha = 0 \tag{3.56}$$

where $\mathcal{H}_{mn\alpha} \equiv \partial_{[m}H_{n]\alpha}$. The above equations and $(\partial^m\partial_m - 2)V^{(12)}_{m^2=2} = 0$ imply that 3.18 describes a massive spin-two multiplet with $(mass)^2 = 2$ [10].

3.1.1 Fixing B and H

In this subsection, the 42 degrees of freedom of $\Omega_{1\beta}, \Omega^\xi_2, \Omega_{3m}$ will be used to impose the following constraints on $\bar{B}_{\alpha\beta}$ and $\bar{H}_{m\beta}$

$$B_{\alpha\beta} = \gamma^{mnp}_{\alpha\beta} B_{mnp}, \tag{3.57}$$
$$H_{m\beta}\gamma^{m\beta\alpha} = 0. \tag{3.58}$$

Using Super Yang-Mills equations of motion 3.5, 3.6, 3.7, and Fierz decomposition A.19 the bi-spinor 3.19 can be written as

$$\bar{B}_{\alpha\beta} \equiv \gamma^{m_1}_{\alpha\beta}\bar{B}_{m_1} + \gamma^{m_1m_2m_3}_{\alpha\beta}\bar{B}_{m_1m_2m_3} + \gamma^{m_1m_2m_3m_4m_5}_{\alpha\beta}\bar{B}_{m_1m_2m_3m_4m_5}, \tag{3.59}$$

where

$$B_{m_1} = -\frac{1}{2}W^1\gamma_{m_1}W^2 - F^1_{m_1 m}A^2_m - (ik^1_{m_1})W^{1\xi}A^2_\xi + \frac{\gamma^{\alpha\beta}_{m_1}}{16}D_\alpha\Big((\gamma^m W^1)_\beta A^2_m + D_\beta W^{1\xi}A^2_\xi\Big), \tag{3.60}$$

$$B_{m_1 m_2 m_3} = \frac{1}{24}W^1\gamma_{m_1 m_2 m_3}W^2 + \frac{\gamma^{\alpha\beta}_{m_1 m_2 m_3}}{96}D_\alpha\Big((\gamma^m W^1)_\beta A^2_m + D_\beta W^{1\xi}A^2_\xi\Big), \tag{3.61}$$

$$B_{m_1 m_2 m_3 m_4 m_5} = \frac{\gamma^{\alpha\beta}_{m_1 m_2 m_3 m_4 m_5}}{3840}D_\alpha\Big((\gamma^m W^1)_\beta A^2_m + D_\beta W^{1\xi}A^2_\xi\Big). \tag{3.62}$$

To obtain the algebraic condition 3.57, one can choose

$$\Omega'_{1\gamma} = (\gamma^m W^1)_\gamma A^2_m + D_\gamma W^{1\xi}A^2_\xi, \tag{3.63}$$

$$\Omega'_{3m} = \frac{1}{2}(W^1\gamma_m W^2) + F^1_{mn}A^{2n} + (ik^1_m)W^{1\xi}A^2_\xi, \tag{3.64}$$

and 3.58 is therefore implied by,

$$\Omega_2^{'\beta} = \frac{1}{10}\big[-7D_\xi W^{1\beta}W^{2\xi} - 10(ik^1_n)W^{1\beta}A^{2n} + 3W^{1\xi}D_\xi W^{2\beta}\big]. \tag{3.65}$$

In this gauge, $B'_{mnp} = \frac{1}{96}\gamma^{\alpha\beta}_{mnp}(\bar{B}_{\alpha\beta} + \delta\bar{B}_{\alpha\beta})$ is

$$B'_{mnp} = \frac{1}{24}W^1\gamma_{mnp}W^2, \tag{3.66}$$

and $H'_{m\alpha} = \bar{H}_{m\alpha} + \delta\bar{H}_{m\alpha}$ is

$$H'_{m\alpha} = \left(-\frac{8}{20}\gamma^p_{\alpha\xi}\delta^q_m - \frac{1}{20}\gamma^{mpq}_{\alpha\xi}\right)\left(F^1_{pq}W^{2\xi} + F^2_{pq}W^{1\xi}\right), \tag{3.67}$$

which is traceless, as one can verify by using $\gamma^{m\beta\alpha}(\gamma_{pqm})_{\alpha\xi} = 8(\gamma_{pq})^{\ \beta}_{\xi}$.

To understand the relation between 3.66 and 3.67, one can define the tensor

$$H^{B'}_{m\alpha} := (\gamma^{np})_\alpha^{\ \beta}D_\beta B'_{mnp}. \tag{3.68}$$

It can be expressed from 3.66 as

$$H^{B'}_{m\alpha} = \left(-\frac{10}{12}\gamma^{p}_{\alpha\xi}\delta^{q}_{m} - \frac{2}{12}\gamma^{mpq}_{\alpha\xi} \right) \left(F^{1}_{pq}W^{2\xi} + F^{2}_{pq}W^{1\xi} \right), \tag{3.69}$$

and has a non-vanishing trace

$$F^{\beta} \equiv \gamma^{m\beta\alpha}H^{B'}_{m\alpha} = 2D_{\xi}\big(W^{1[\beta}W^{2\xi]}\big). \tag{3.70}$$

It will be useful to note that the traceless part $(H^{B'})^{(0)}_{m\alpha} \equiv H^{B'}_{m\alpha} - (\gamma_m)_{\alpha\xi}\left(\frac{1}{10}F^{\xi}\right)$ of 3.68 satisfies the relation

$$H'_{s\alpha} = \frac{3}{7}(H^{B'})^{(0)}_{s\alpha}. \tag{3.71}$$

Nevertheless, the expression 3.66 for B'_{mnp} does not satisfy the transversality condition. This is a necessary condition to remove the extra degrees of freedom at the zeroth order in θ expansion of B'_{mnp} and $H'_{m\alpha}$ [32].

3.1.2 Additional gauge-fixing

In this subsection, it will be shown that $\partial^{m}B_{mnp} = 0$, when $\Omega_{1\beta}$ is written as

$$\Omega_{1\beta} = \Omega'_{1\beta} + D_{\beta}\Lambda. \tag{3.72}$$

In this gauge, $B_{\alpha\beta}$ and $H_{m\alpha}$ are related as 3.52.

The additional contribution $\Omega^{(1)}_{1\beta} = D_{\beta}\Lambda$ does not change the five-form part of $B_{\alpha\beta}$ because of the identity $\gamma^{\alpha\beta}_{mnpqr}D_{\alpha}D_{\beta} = 0$. So the previous subsection gauge fixing leaves gauge invariances parameterized by $\Omega^{(1)}_{1\beta}$. After this additional gauge-fixing, the resulting B_{mnp} is

$$B_{mnp} = \frac{1}{24}W^{1}\gamma_{mnp}W^{2} - \frac{1}{96}\gamma^{\alpha\beta}_{mnp}D_{\alpha}\Omega^{(1)}_{1\beta}. \tag{3.73}$$

To obtain Λ in terms of SYM superfields, $H^{B}_{m\alpha} := (\gamma^{np})_{\alpha}{}^{\beta}D_{\beta}B_{mnp}$ will be required to satisfy $\gamma^{m\alpha\beta}H^{B}_{m\alpha} = 0$. Indeed, if $H^{B}_{m\alpha}$ is assumed to be traceless, 3.70 implies that

$$(\gamma^{mst})^{\beta\xi}D_{\xi}\left(-\frac{1}{96}\gamma^{\delta\alpha}_{mst}D_{\delta}\Omega^{(1)}_{1\alpha} \right) = -2D_{\xi}\left(W^{1[\beta}W^{2\xi]} \right). \tag{3.74}$$

Hitting both sides of 3.74 with D_β, one finds that

$$\frac{1}{96}(D\gamma^{mnp}D)(D\gamma_{mnp}D)\Lambda = 2D_\beta D_\xi\Big(W^{1[\beta}W^{2\xi]}\Big). \tag{3.75}$$

But $(D\gamma^{mnp}D)(D\gamma_{mnp}D) = 96\cdot 48$ at the first massive level, then Λ is given by

$$\Lambda = -\frac{1}{6}F^1_{mn}F^2_{mn}, \tag{3.76}$$

and the additional gauge fixing $\Omega^{(1)}_{1\beta}$ is

$$\Omega^{(1)}_{1\beta} = -\frac{1}{3}\Big[ik^1_m(\gamma_n W^1)_\beta F^2_{mn} + (1\leftrightarrow 2)\Big]. \tag{3.77}$$

In the gauge $\gamma_m^{\alpha\beta}B_{\alpha\beta} = 0$, $\gamma^{m\alpha\beta}H_{m\alpha} = 0$, one has

$$\Omega_{3m} \equiv \Omega'_{3m} + \Omega^{(1)}_{3m} = \frac{1}{2}(W^1\gamma_m W^2) + F^1_{mn}A^{2n} + (ik^1_m)W^{1\xi}A^2_\xi + \frac{1}{2}\partial_m\Lambda, \tag{3.78}$$

and

$$\Omega^\beta_2 \equiv \Omega'^\beta_2 + \Omega^{(1)\beta}_2 = -\partial^m(W^{1\beta}A^2_m) - \frac{2}{3}D_\alpha W^{1\beta}W^{2\alpha} + \frac{1}{3}W^{1\alpha}D_\alpha W^{2\beta}, \tag{3.79}$$

and B_{mnp} is transverse to $k_1 + k_2$ because of

$$\partial^m\Big(-\frac{1}{96}\gamma^{\alpha\beta}_{mnp}D_\alpha D_\beta\Lambda\Big) = -\partial^m(\frac{1}{24}W^1\gamma_{mnp}W^2). \tag{3.80}$$

To demonstrate 3.52, one can write

$$H_{m\alpha} \equiv H'_{m\alpha} + \delta H_{m\alpha} \tag{3.81}$$
$$H^B_{m\alpha} \equiv H^{B'}_{m\alpha} + \delta H^{B'}_{m\alpha}, \tag{3.82}$$

where $\delta H_{m\alpha} = -(\gamma_m)_{\alpha\xi}\Omega^{(1)\xi}_2 + D_\alpha\Omega^{(1)}_{3m}$ is the variation of 3.67,

$$\delta H_{m\alpha} = -\frac{1}{3\cdot 10}(\gamma_m)_{\alpha\xi}(D_\beta W^{1\xi}W^{2\beta} + W^{1\beta}D_\beta W^{2\xi}) + D_\alpha(\frac{1}{2}\partial_m\Lambda), \tag{3.83}$$

and $\delta H^{B'}_{m\alpha}$ is the variation of 3.68

$$\delta H^{B'}_{m\alpha} = (\gamma^{st})^{\beta}_{\alpha} D_{\beta}\big(-\frac{1}{96}\gamma^{\gamma\delta}_{mst} D_{\gamma} D_{\delta}\Lambda\big), \tag{3.84}$$

which is implied by 3.68, 3.73 and 3.82. Using 3.71, one can write a statement equivalent to 3.52,

$$\delta H_{m\alpha} = \frac{3}{7}\big(\delta H^{B'}_{m\alpha} + \frac{1}{10}(\gamma_m)_{\alpha\beta} F^{\beta}\big), \tag{3.85}$$

with F^{β} defined in 3.70. One finds from the identity

$$(\gamma^{st})_{\alpha}^{\ \beta}(\gamma_{mst})^{\gamma\delta} D_{\beta} D_{\gamma} D_{\delta} = -72\partial_m D_{\alpha} + 40(\gamma_{mt})_{\alpha}^{\ \beta}\partial^t, \tag{3.86}$$

that the variation 3.84 is

$$\delta H^{B'}_{m\alpha} = \frac{7}{6}\partial_m D_{\alpha}\Lambda - \frac{5}{36}(\gamma_m)_{\alpha\beta} F^{\beta}, \tag{3.87}$$

and 3.85 is therefore satisfied,

$$\delta H^{B'}_{m\alpha} + \frac{1}{10}(\gamma_m)_{\alpha\beta} F^{\beta} = \frac{7}{3}\left[\frac{1}{2}\partial_m D_{\alpha}\Lambda - \frac{1}{60}(\gamma_m)_{\alpha\beta} F^{\beta}\right]. \tag{3.88}$$

So it has been proven that in the gauge 3.57, 3.58 and $\partial^m B_{mnp} = 0$, the equation 3.52 is satisfied.

In this gauge, the superfield $H_{m\alpha}$ is

$$H_{m\alpha} = \left[\partial^m\partial^n \frac{1}{6}\gamma^p_{\alpha\beta}\delta^q_n - \frac{10}{24}\gamma^p_{\alpha\beta}\delta^q_m - \frac{1}{24}(\gamma_{mpq})_{\alpha\beta}\right]\big(F^{1pq}W^{2\beta} + F^{2pq}W^{1\beta}\big), \tag{3.89}$$

3.1.3 Fixing C

In this subsection, the 46 degrees of freedom of Ω_4 and Ω_{5mn} will be used to impose the algebraic constraint

$$C^{\beta}_{\alpha} = (\gamma^{mnpq})^{\beta}_{\ \alpha} C_{mnpq}. \tag{3.90}$$

From the Fierz decomposition A.20, one finds

$$\Omega_4 = -\frac{1}{24} F^1_{mn} F^{2mn}, \tag{3.91}$$

$$\Omega_{5pq} = \frac{1}{16} (\gamma_{pq})_\beta^{\ \alpha} [W^{1\beta} A^2_\alpha - D_\alpha \Omega^\beta_2]. \tag{3.92}$$

Using 3.79, Ω_{5mn} is

$$\Omega_{5mn} = \frac{1}{2} F^1_{mn} (ik^1 \cdot A^2) + \frac{1}{4} \partial_{[m} W^1 \gamma_{n]} W^2 - \frac{1}{8} \partial^r W^1 \gamma_{mnr} W^2 + \frac{1}{4} F^1_{p[m} F^2_{n]p}. \tag{3.93}$$

The $\gamma^{(4)}$ component of $C_\alpha^{\ \beta}$ is

$$C^{mnpq} = \frac{(\gamma^{mnpq})_\beta^{\ \alpha}}{384} \left(\bar{C}_\alpha^\beta - D_\alpha \Omega^\beta_2 - \delta^\beta_\alpha \Omega_4 - \frac{(\gamma^{pq})^\beta_{\ \alpha}}{2} \Omega_{5pq} \right), \tag{3.94}$$

one therefore obtains from 3.79, 3.91, 3.93 that

$$C^{mnpq} = \frac{1}{96 \cdot 12} F^1_{[mn} F^2_{pq]} + \frac{1}{96 \cdot 36} \partial_{[m} W^1 \gamma_{npq]} W^2. \tag{3.95}$$

Finally, the equation

$$-\frac{1}{96} \partial_{[m} \gamma^{\alpha\beta}_{npq]} D_\alpha D_\beta \Lambda = \frac{1}{12} F^1_{[mn} F^2_{pq]} - \frac{1}{72} \partial_{[m} W^1 \gamma_{npq]} W^2 \tag{3.96}$$

implies that 3.73 and 3.95 are related as

$$C_{mnpq} = \frac{1}{96} \partial_{[m} B_{npq]}. \tag{3.97}$$

3.1.4 Fixing F

In this subsection, the gauge invariance 3.50 with

$$\Lambda^\beta_n = (\gamma_n)^{\beta\alpha} \left(\frac{1}{10} (\gamma^n)_{\alpha\xi} \Lambda^\xi_n\right) + \Lambda^{(0)\beta}_n \tag{3.98}$$

will be used to impose the following algebraic constraint

$$\gamma^{m\beta\alpha} [\frac{1}{2} \bar{F}_{mn\alpha} + \delta \bar{F}_{mn\alpha} + \delta_\Lambda \bar{F}_{mn\alpha}] = 0, \tag{3.99}$$

$$\bar{E}_\alpha + \delta_\Lambda E_\alpha = 0. \tag{3.100}$$

To obtain 3.100, the trace part of 3.98 should be

$$\gamma^n_{\alpha\beta}\Lambda^\beta_n = D_\alpha\Omega_4, \tag{3.101}$$

so the constraints 3.99, 3.100 imply

$$\Lambda^\beta_n = -\frac{1}{8}\gamma^{m\beta\alpha}\big[\frac{1}{2}F^1_{mn}A^2_\alpha + D_\alpha\Omega_{5mn}\big] - \frac{1}{8}\gamma^{\beta\alpha}_n D_\alpha\Omega_4. \tag{3.102}$$

In this gauge, $\frac{1}{2}F_{mn\alpha}$ can be written as

$$\frac{1}{2}F_{mn\alpha} = \frac{6}{8}\left(\frac{1}{2}F^1_{mn}A^2_\alpha + D_\alpha\Omega_{5mn}\right) - \frac{1}{16}(\gamma_{mn})_\alpha{}^\beta D_\beta\Lambda - \frac{1}{8}(\gamma_{p[m})_\alpha{}^\beta\big(\frac{1}{2}F^1_{n]p}A^2_\beta + D_\beta\Omega_{5n]p}\big). \tag{3.103}$$

Using the equation $\gamma_{p[m}\gamma_{n]p} = 16\gamma_{mn}$, one obtains

$$F_{mn\alpha} = \frac{1}{8}\big(6\mathcal{F}_{mn\alpha} - (\gamma_{p[m}\mathcal{F}_{n]p})_\alpha\big), \tag{3.104}$$

where

$$\mathcal{F}_{mn\alpha} = F^1_{mn}A^2_\alpha + 2D_\alpha\Omega_{5mn} + \frac{1}{10}(\gamma_{mn})_\alpha{}^\beta D_\beta\Lambda. \tag{3.105}$$

To show the relation 3.54, one can add $\mathcal{F}^{(0)}_{mn\alpha}$ to $\mathcal{F}_{mn\alpha}$, such that

$$6\mathcal{F}^{(0)}_{mn\alpha} = \gamma_{p[m}\mathcal{F}^{(0)}_{n]p\alpha}. \tag{3.106}$$

So one can define the following tensors

$$\begin{aligned}
\mathcal{A}^{W^1}_{mn\alpha} &= \partial_p(\gamma_{mnp}W^1)_\alpha F^2_{pq}, & \mathcal{A}^{W^2}_{mn\alpha} &= \partial_p(\gamma_{mnp}W^2)_\alpha F^1_{pq},\\
\mathcal{B}^{W^1}_{mn\alpha} &= \partial_r(\gamma_{[m}W^1)_\alpha F^2_{n]r}, & \mathcal{B}^{W^2}_{mn\alpha} &= \partial_r(\gamma_{[m}W^2)_\alpha F^1_{n]r}\\
\mathcal{M}^{(k^iW^1)}_{mn\alpha} &= ik^i_{[m}(\gamma_{n]pq}W^1)_\alpha F^2_{pq}, & \mathcal{M}^{(k^iW^2)}_{mn\alpha} &= ik^i_{[m}(\gamma_{n]pq}W^2)_\alpha F^1_{pq},\\
\mathcal{N}^{(k^iW^1)}_{mn\alpha} &= ik^i_{[m}F^2_{n]r}(\gamma^r W^1)_\alpha, & \mathcal{N}^{(k^iW^2)}_{mn\alpha} &= ik^i_{[m}F^1_{n]r}(\gamma^r W^2)_\alpha,
\end{aligned} \tag{3.107}$$

whose combinations

$$\mathcal{R}^{W^i}_{mn\alpha} = \mathcal{A}^{W^i}_{mn\alpha} + 4\mathcal{B}^{W^i}_{mn\alpha}, \tag{3.108}$$
$$\mathcal{S}^{W^i}_{mn\alpha} = 2\mathcal{N}^{(k^iW^i)}_{mn\alpha} + \mathcal{M}^{(k^iW^i)}_{mn\alpha} - 4\mathcal{B}^{W^i}_{mn\alpha}, \tag{3.109}$$
$$\mathcal{T}^{W^1}_{mn\alpha} = 2\mathcal{N}^{(k^2W^1)}_{mn\alpha} + \mathcal{M}^{(k^2W^1)}_{mn\alpha}, \tag{3.110}$$
$$\mathcal{T}^{W^2}_{mn\alpha} = 2\mathcal{N}^{(k^1W^2)}_{mn\alpha} + \mathcal{M}^{(k^1W^2)}_{mn\alpha}, \tag{3.111}$$

satisfy the relation 3.106. Expanding 3.105, it is straightforward to check that

$$\frac{1}{2}\mathcal{H}_{mn\alpha} = \mathcal{F}_{mn\alpha} + \mathcal{F}^{(0)}_{mn\alpha}, \tag{3.112}$$

where

$$\mathcal{F}^{(0)}_{mn\alpha} = \frac{1}{30}\mathcal{R}^{W^1}_{mn\alpha} + \frac{1}{30}\mathcal{R}^{W^2}_{mn\alpha} + \frac{1}{6}\mathcal{S}^{W^1}_{mn\alpha} - \frac{1}{12}\mathcal{S}^{W^2}_{mn\alpha} + \frac{1}{24}\mathcal{T}^{W^1}_{mn\alpha} - \frac{5}{24}\mathcal{T}^{W^2}_{mn\alpha},$$

thus 3.54 holds.

The gauge parameter Λ^β_n degrees of freedom are sufficient to enforce both conditions 3.99 and 3.100. Indeed, the following spinor

$$\tilde{\Lambda}^{(0)\beta}_n = -\frac{1}{8}\gamma^{\beta\alpha}_m\left(\frac{1}{2}F^1_{mn}A^2_\alpha + D_\alpha\Omega_{5mn}\right) - \gamma^{\beta\alpha}_n\left(\frac{9}{160}D_\alpha\Lambda\right), \tag{3.113}$$

should be exactly the traceless part of 3.98, as one can see by subtracting $(\gamma_n)^{\beta\xi}\left(\frac{1}{10}\gamma^m_{\xi\alpha}\Lambda^\alpha_m\right)$ from 3.102. The gamma matrix expression A.9 and super Yang-Mills equations of motion implies that

$$\gamma^n_{\xi\beta}\tilde{\Lambda}^{(0)\beta}_n = -\frac{3}{8}D_\xi\Lambda - \frac{3}{16}\gamma^n_{\xi\beta}(W^{1\beta}A^2_n + \partial_n\Omega^\beta_2). \tag{3.114}$$

After expressing 3.114 in terms of SYM superfields, one obtains

$$\gamma^n_{\xi\beta}\tilde{\Lambda}^{(0)\beta}_n = \frac{\partial^m}{32}\Big(2(\gamma_nW^2)_\xi F^1_{mn} - (\gamma_nW^1)_\xi F^2_{mn}\Big) - \frac{\partial_n}{2\cdot 32}\Big(2(\gamma^{pqn}W^2)_\xi F^1_{pq} - (\gamma^{pqn}W^1)_\xi F^2_{pq}\Big), \tag{3.115}$$

which vanishes by expanding the second term of the right-hand side with equation A.10.

Finally, it will be shown that $\bar{G}_\alpha + \delta_\Lambda G_\alpha = 0$. Using 3.49, G_α can be written as

$$\bar{G}_\alpha + \delta_\Lambda G_\alpha = \Omega'_{1\alpha} + \gamma^m_{\alpha\beta}\partial_m \Omega^\beta_2 - D_\alpha \Omega_4 - \frac{1}{2}(\gamma^{mn})^\beta{}_\alpha D_\beta \Omega_{5mn}, \tag{3.116}$$

and performing a computation similar to 3.114, one has

$$\bar{G}_\alpha + \delta_\Lambda G_\alpha = \frac{1}{2}D_\alpha \Lambda + \frac{1}{4}\gamma^m_{\xi\alpha}(W^{1\xi}A^2_m + \partial_m \Omega^\xi_2) = 0. \tag{3.117}$$

This is the equation 3.56. So the vertex operator 3.18 has been fixed to the gauge 3.36,3.37,3.37, where it can be written as

$$V^{(12)}_{m^2=2} =: \partial\theta^\beta \lambda^\alpha (\gamma^{mnp}_{\alpha\beta} B_{mnp}) :+ : \Pi^m \lambda^\alpha H_{m\alpha} : + : d_\beta \lambda^\alpha C^\beta_\alpha : + : \frac{1}{2} N^{mn} \lambda^\alpha F_{mn\alpha} :, \tag{3.118}$$

with

$$B_{mnp} = \frac{1}{36} W^1 \gamma_{mnp} W^2 - \frac{1}{36} ik^1_{[m} ik^2_n W^1 \gamma_{p]} W^2 + \frac{1}{72}\partial^r (F^1_{r[m} F^2_{np]} + F^2_{r[m} F^1_{np]}) \tag{3.119}$$

$$H_{m\alpha} = \frac{3}{7}(\gamma^{st})^\beta_\alpha D_\beta B_{mst}, \tag{3.120}$$

$$C^\alpha_\beta = \frac{1}{4}(\gamma^{mnpq})^\alpha{}_\beta \partial_m B_{npq}, \tag{3.121}$$

$$F_{mn\alpha} = \frac{1}{16}(6\mathcal{H}_{mn\alpha} - (\gamma_{p[m})_\alpha{}^\beta \mathcal{H}_{n]p\beta}), \quad \mathcal{H}_{mn\alpha} \equiv \partial_{[m} H_{n]\alpha} \tag{3.122}$$

and therefore gives a SYM realization of the massive spin-two multiplet of mass $(mass)^2 = 2$.

Chapter 4

Conclusions

In this dissertation, we reviewed the construction of vertex operators in the bosonic open string and in the RNS formalism for the open superstring, and then show how to compute massive vertex operators using the operator product expansions of massless vertices in the pure spinor formalism. These operators correspond to physical states in the cohomology of a BRST operator. The method for obtaining massive vertex as resonances of massless ones was first presented in section 2.1. The RNS superstring was presented in section 2.2, and it has a supersymmetric spectrum free of tachyonic modes after the GSO projection. The vertex operators up to the first excited state were constructed, and the conditions for BRST invariance of each vertex were described in both NS and R sectors, as well as its polarizations after fixing the exact BRST states.

In chapter 3, the unintegrated vertex operator of the open superstring at the first massive level was computed by expanding the operator product between a massless integrated vertex operator and a massless unintegrated vertex operator, using the pure spinor formalism. In principle, the method for deriving massive vertex operators as resonances of massless operators allows for the description of all spectrum states accessible through the collision between the lowest-level particles.

This procedure yields expressions that are automatically BRST invariant for unintegrated vertex operators and, therefore, differs from the construction in [10], where the starting point involves arbitrary superfields and thus requires the imposition of $Q \cdot V = 0$. This advantage is also emphasized in [16], where the massive vertex operators are extracted from the scattering amplitudes of massless particles via factorization, using the RNS formalism, and the conformal invariance condition of the massive operators is guaranteed by the fact that the residual amplitudes contain only physical singularities.

In contrast to the RNS formalism presented in Chapter 2, where the manifest Lorentz invariance must be broken to perform calculations with the spin field for Ramond states, the vertex constructed in Chapter 3 allows for manifestly super-Poincaré invariant computations of superstring scattering amplitudes.

As an application of the method, one can compute amplitudes with massive states in terms of amplitudes of massless states. In particular, a map can be defined from a massive state with momentum $\bar{k}^2 = -\frac{1}{\alpha'}$ to two asymptotic massless states, subject to the constraint $k_1 \cdot k_2 = -\frac{1}{2\alpha'}$. As the massive polarizations can be extracted from the $\theta = 0$ components of the superfields [10],

$$\begin{aligned} b_{mnp} &\equiv B_{mnp}\Big|_{\theta=0}, \\ g_{mn} &\equiv D_\alpha \gamma^{\alpha\beta}_{(m} H_{n)\beta}\Big|_{\theta=0}, \\ \psi_{m\alpha} &\equiv -\frac{1}{72} H_{m\alpha}\Big|_{\theta=0}, \end{aligned}$$

then 3.118 provides the mapping between the massive polarizations and the product of the massless polarizations. For example, expanding B_{mnp} using the main result 3.119 and 3.8, one has the following map

$$b_{mnp} = \frac{\alpha'}{18}\left(ik^1_{[m}\xi^1_n\xi^2_{p]} + ik^2_{[m}\xi^2_n\xi^1_{p]} - 2\alpha'(ik^1\cdot\xi^2)ik^1_{[m}\xi^1_n ik^2_{p]} - 2\alpha'(ik^2\cdot\xi^1)ik^2_{[m}\xi^2_n ik^1_{p]}\right),$$

considering only the bosonic sector. Using this map, one can compute the expansion in components of amplitudes with massive states by writing $(B_{\alpha\beta}, H_{m\alpha}, C^\alpha{}_\beta, F_{mn\alpha})$ as a function of the massless super-Yang-Mills superfields, and in [19] the three-point amplitude with one massive and two massless bosonic external states is mapped to the α'^2 correction to the massless four-point amplitude [33].

To calculate amplitudes with more than three asymptotic states, integrated vertex operators whose conformal weight is $h = 1$ and whose ghost number is 0 must be introduced. The superspace representation of the integrated operator for the first massive state was obtained in [14] through the descent relation. The method outlined in Chapter 3 can similarly be employed to derive the integrated vertex and express it with massless super-Yang-Mills superfields. For integrated massless vertex operators $U^{(1)}$ and $U^{(2)}$, the contour

$$U^{(12)}(w) \equiv \oint_{C_w} dz U^{(1)}(z) U^{(2)}(w)$$

has conformal weight 1, ghost number 0, and satisfies the condition $Q \cdot U^{(12)} = \partial V^{(12)}$, where $V^{(12)}$ is defined in 3.11. Consequently, it can be identified as the integrated vertex operator at the first massive level of the open superstring. Finally, one can consider the vertex operators for the closed string as the holomorphic square of the open string vertex, and then massive amplitudes with closed strings can also be computed [34].

Appendix A

Conventions and Gamma Matrix formulas

The gamma matrices satisfy

$$(\gamma^m)^{\alpha\sigma}\,\gamma^n_{\sigma\beta} + (\gamma^n)^{\alpha\sigma}\,\gamma^m_{\sigma\beta} = 2\delta^{mn}\delta^\alpha_\beta, \tag{A.1}$$

and the antisymmetrization is represented by square brackets, for instance:

$$\gamma^{m_1...m_k} \equiv \frac{1}{k!}\gamma^{[m_1}...\gamma^{m_k]} \equiv \frac{1}{k!}\big(\gamma^{m_1}...\gamma^{m_k} + \text{all antisymmetric permutations}\big). \tag{A.2}$$

There are the following important identities,

$$\gamma^m_{\alpha(\beta}\gamma^m_{\gamma\delta)} = 0 \tag{A.3}$$

$$\gamma^{mnp}_{\alpha[\beta}\gamma^{mnp}_{\gamma\delta]} = 0 \tag{A.4}$$

$$\gamma^{\alpha\beta}_{mnp}\gamma^{mnp}_{\gamma\delta} = 48\left(\delta^\alpha_\gamma\delta^\beta_\delta - \delta^\beta_\gamma\delta^\alpha_\delta\right) \tag{A.5}$$

$$\gamma^{mnp}_{\alpha\beta}\gamma^{mnp}_{\gamma\delta} = 12\left(\gamma^m_{\alpha\delta}\gamma^m_{\beta\gamma} - \gamma^m_{\alpha\gamma}\gamma^m_{\beta\delta}\right) \tag{A.6}$$

$$\gamma^m_{\alpha\beta}\gamma^m_{\delta\sigma} = -\frac{1}{2}\gamma^m_{\alpha\delta}\gamma^m_{\beta\sigma} - \frac{1}{24}\gamma^{mnp}_{\alpha\delta}\gamma^{mnp}_{\beta\sigma}, \tag{A.7}$$

$$\gamma^{mnp}_{\alpha\beta}\gamma^{mnp}_{\delta\sigma} = -12\gamma^m_{\alpha\beta}\gamma^m_{\delta\sigma} - 24\gamma^m_{\alpha\delta}\gamma^m_{\beta\sigma}, \tag{A.8}$$

$$(\gamma^{mn})_\alpha{}^\delta\,(\gamma_{mn})_\beta{}^\sigma = -8\delta^\sigma_\alpha\delta^\delta_\beta - 2\delta^\delta_\alpha\delta^\sigma_\beta + 4\gamma^m_{\alpha\beta}\gamma^{\delta\sigma}_m, \tag{A.9}$$

$$\gamma^{m_1...m_k} = \gamma^{m_1}\gamma^{m_2...m_k} - \frac{1}{(k-2)!}\delta^{m_1[m_2}\gamma^{m_3...m_k]},\ \ k = 2, ..., 5; \tag{A.10}$$

$$\gamma^m\gamma^{n_1...n_k}\gamma_m = (-1)^k(10-2k)\gamma^{n_1...n_k}, k = 2, ..., 5; \tag{A.11}$$

$$\gamma^{st}\gamma_{mnpqr}\gamma^{st} = 10\gamma_{mnpqr}, \tag{A.12}$$
$$\gamma^{stu}\gamma_{mnpqr}\gamma^{stu} = 0, \tag{A.13}$$
$$\gamma^{stuv}\gamma_{mnpqr}\gamma^{stuv} = 240\gamma_{mnpqr}, \tag{A.14}$$

$$\gamma^{st}\gamma_{mnpq}\gamma^{st} = 6\gamma_{mnpq}, \tag{A.15}$$
$$\gamma^{stu}\gamma_{mnpq}\gamma^{stu} = 48\gamma_{mnpq}, \tag{A.16}$$
$$\gamma^{stuv}\gamma_{mnpq}\gamma^{stuv} = 48\gamma_{mnpq}, \tag{A.17}$$

$$\gamma^{st}\gamma_{mnp}\gamma^{st} = -6\gamma_{mnp}. \tag{A.18}$$

The bispinors Fierz decompositions are

$$\chi^{\alpha}\psi^{\beta} = \frac{1}{16}\gamma_m^{\alpha\beta}\left(\chi\gamma^m\psi\right) + \frac{1}{3!16}\gamma_{mnp}^{\alpha\beta}\left(\chi\gamma^{mnp}\psi\right) + \frac{1}{5!16}\left(\frac{1}{2}\right)\gamma_{mnpqr}^{\alpha\beta}\left(\chi\gamma^{mnpqr}\psi\right). \tag{A.19}$$
$$\chi_{\alpha}\psi^{\beta} = \frac{1}{16}\delta_{\alpha}^{\beta}(\chi\psi) - \frac{1}{2!16}\left(\gamma_{mn}\right)_{\alpha}^{\beta}\left(\chi\gamma^{mn}\psi\right) + \frac{1}{4!16}\left(\gamma_{mnpq}\right)_{\alpha}^{\beta}\left(\chi\gamma^{mnpq}\psi\right). \tag{A.20}$$

And trace relations are given by

$$\mathrm{Tr}\left(\gamma^{m_1\ldots m_k}\gamma_{n_1\ldots n_k}\right) = +16\cdot k!\delta_{n_1\ldots n_k}^{m_1\ldots m_k}, \quad k = 1,4; \tag{A.21}$$
$$\mathrm{Tr}\left(\gamma^{m_1\ldots m_k}\gamma_{n_1\ldots n_k}\right) = -16\cdot k!\delta_{n_1\ldots n_k}^{m_1\ldots m_k}, \quad k = 2,3; \tag{A.22}$$
$$\mathrm{Tr}\left(\gamma^{m_1\ldots m_5}\gamma_{n_1\ldots n_5}\right) = 16\cdot 5!\delta_{n_1\ldots n_5}^{m_1\ldots m_5} + 16\epsilon^{m_1\ldots m_5}{}_{n_1\ldots n_5}. \tag{A.23}$$

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