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TWO LOOP SUPER - WZNW MODEL WITH REDUCTIONS TO
SUPERSYMMETRIC AND FERMIONIC INTEGRABLE MODELS

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Two Loop Super-WZNW Model with Reductions to Supersymmetric and
Fermionic Integrable models.

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Resumo

Nessa tese se constrói o funcional de ação dos modelos de Toda supersimétricos afins e se deduzem as transformações de supersimetria dos modelos desde o ponto de vista de fluxos de simetria Fermionicos. A integrabilidade é definida em termos de um problema de fatorização de Riemann-Hilbert estendido que unifica parâmetros de evolução Abelianos e não Abelianos. Se introduzem modelos integráveis puramente Fermionicos definidos em supercosets onde toda a parte Bosonica é completamente eliminada. Vários exemplos são considerados.

Palavras Chaves: Integrabilidade em Teória de Campos, Modelos de Toda Supersimétricos, Superalgebras Torcidas Afins, Modelos Integráveis Fermionicos, Fluxos de Supersimetria, Hierarquias Integráveis.

Áreas de conhecimento: Sistemas Integráveis, Física Matemática.

Abstract

In this thesis we construct the action functional for the supersymmetric affine Toda models and deduce the supersymmetry transformations of the models from the point of view of Fermionic symmetry flows. The underlying Integrability is defined in terms of an extended Riemann-Hilbert factorization problem which unifies Abelian and non-Abelian evolution parameters. We also introduce purely Fermionic Integrable models defined on supercosets where the Bosonic part is completely eliminated. Several examples are considered.

Keywords: Integrable Field Theory, Supersymmetric Toda Models, Twisted Affine Superalgebras, Fermionic Integrable Models, Supersymmetry Flows, Integrable Hierarchies.

Fields of knowledge: Integrable Systems, Mathematical Physics.

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Chapter 1

Introduction.

It is well known that Bosonic Toda models are underlined by Lie algebras and that they provide some sort of field theoretic realization of them. They are relevant to particle physics because describe massive integrable perturbations of two-dimensional conformal field theories, allow soliton configurations in their spectrum and are useful laboratories to develop new methods relevant to the study of non-perturbative aspects of quantum field theory.

A natural step when having a Bosonic field theory is to try to incorporate Fermions and to construct its supersymmetry extension. In the case of Bosonic Toda models this is a not an easy task because we want to preserve the integrability, which is the main property of these kind of theories. Integrability is a consequence of the existence of an infinite number of conserved Bosonic Hamiltonians which depend strongly on the Lie algebraic input data defining the model itself. Each Hamiltonian generate a Bosonic (even) symmetry flow and due that Supersymmetry is a symmetry, it is natural to expect the presence of conserved supercharges generating Fermionic (odd) symmetry flows and that the supersymmetric extension is related not to a Lie algebra but to a superalgebra. See [6] for an example of how Bosonic $\mathcal{W}$ -symmetries are not preserved by supersymmetrization. By definition, a supersymmetry is a symmetry where the application of two successive odd transformations close into a even one. If there is an infinite number of even flows then we have to incorporate the same number of odd flows in order to close the flow super-

algebra. Hence, the set of fields $\mathcal{F}$ will depend on an infinite number of even and odd variables $\mathcal{F} = \mathcal{F}(t_{\pm 1/2}, t_{\pm 1}, t_{\pm 3/2}, t_{\pm 3}, \dots)$. See [16] for a first example of this 'flow approach' applied to the KP Hierarchy. The main motivation for this thesis was to construct the action functional for the supersymmetric affine Toda models and to study the supersymmetry of these models within this 'flow approach'. The main goal is still to construct a 'supertau' function formulation, to solve the relativistic hierarchy by using super-Vertex operators and to set the ground for a quantization of the full supersymmetric theory. Although there is nothing quantum in this thesis, all the structures used have a well defined quantum counterpart or are susceptible of generalizations.

Several authors have studied the problem of constructing supersymmetric extensions of integrable hierarchies. On one side, for the Toda lattice most of them use superfields as a natural way to supersymmetrize Lax operators while preserving integrability or to obtain a manifestly supersymmetric Hamiltonian reduction of Super WZNW models. See for example [1], [2],[5],[13]. The common conclusion is that only Lie superalgebras (classical or affine) with a purely Fermionic simple root system allow supersymmetric integrable extensions, otherwise supersymmetry is broken. On the other side, there are several supersymmetric formulations of the Drinfeld-Sokolov method for constructing integrable hierarchies in which the algebraic Dressing method and the 'flow approach' were gradually developed and worked out in several examples. See for example [3],[4],[7], [15]. The main goal of these works is the construction of an infinite set of Fermionic symmetry flows but a clear relation between the conserved supercharges and the corresponding field transformation is still obscure. In [10], Fermionic fields were coupled to the Toda fields in a supersymmetric way in the spirit of generalized Toda models coupled to matter fields introduced in [19] and further analyzed in [20]. This coupling was performed on-shell but only half of the supersymmetric sector was analyzed. An important result of this paper was the introduction of a 'reductive' automorphism τ_{red} devised to remove the non-locality of the lowest supersymmetry flows, as a consequence, it was shown that it is not strictly necessary to start with an affine superalgebra with a purely Fermionic simple root system in order to get an integrable

supersymmetric extension of a Bosonic model. The complementary off-shell Hamiltonian reduction was developed in [11] by using a two-Loop super-WZNW model where the action functional for the supersymmetric Leznov-Saveliev equations of motion was constructed in principle, for any superalgebra endowed with a half-integer gradation and invariant under certain τ_{red} . It was also shown that several known purely Fermionic integrable models belong to the family of perturbed WZNW on supercosets where the Bosonic part is fully gauged away. The full set of 2D supersymmetry transformations were derived in [12] in a purely algebraic way by using the algebraic dressing technique and the supercharges were derived in two different ways, as an identity consequence of the Riemann-Hilbert problem and by applying the Noether procedure to the action functional of these models. It was also shown that the number of supercharges is equal to the dimension of the Fermionic kernel of a given semisimple element $E \in \hat{\mathfrak{g}}$ which defines both, the physical degrees of freedom and the symmetries of the model.

Chapter 2

Lax Operators and Supersymmetry Flows.

The core of the flow approach we follow relies in the algebraic Dressing technique used to unify symmetry flows (Isospectral and Non-abelian) of integrable hierarchies related to Affine Lie algebras. The Riemann-Hilbert factorization problem allows us to define an Integrable structure and a related hierarchy of non-linear partial differential equations. This also provides the explicit form of the Lax operators from which the equations of motion can be written as a zero curvature representation. This chapter follows references [11] and [12].

2.1 Relativistic sector of the Extended mKdV Hierarchy.

Consider an Affine Lie superalgebra $\widehat{\mathfrak{g}} = \bigoplus_{i \in \mathbb{Z}/2 = -\infty}^{+\infty} \widehat{\mathfrak{g}}_i$ with a half-integer gradation induced by an operator Q , i.e $[Q, \widehat{\mathfrak{g}}_i] = i\widehat{\mathfrak{g}}_i$. We have the following

Definition 1 *The relativistic sector of the extended mKdV hierarchy is defined by*

the following evolution equations

$$\begin{aligned} \partial_+ \Theta &= \left(\Theta E_+^{(+1)} \Theta^{-1} \right)_- \Theta, \quad \partial_+ \Pi = - \left(\Theta E_+^{(+1)} \Theta^{-1} \right)_+ \Pi \\ \partial_- \Theta &= \left(\Pi E_-^{(-1)} \Pi^{-1} \right)_- \Theta, \quad \partial_- \Pi = - \left(\Pi E_-^{(-1)} \Pi^{-1} \right)_+ \Pi, \end{aligned} \quad (2.1.1)$$

for the two Isospectral times $t_{\pm 1} = \pm x^{\pm}$ associated to the degree ± 1 constant elements $E_{\pm}^{(\pm 1)} \in \widehat{\mathfrak{g}}$. The $(*)_{\pm}$ denote projections onto grades ≥ 0 and $\leq -1/2$.

The Θ and Π are the Dressing matrices entering an extended Riemann Hilbert factorization problem given by

$$\exp \left(- \sum_{n=1}^{+\infty} E^{(n)} t_{+n} \right) g \exp \left(\sum_{n=1}^{+\infty} E^{(-n)} t_{-n} \right) = \Theta^{-1}(t) \Pi(t), \quad (2.1.2)$$

defining the extended mKdV integrable structure we are dealing with. They are expanded as

$$\begin{aligned} \Theta &= \exp \left(\psi^{(-1/2)} + \psi^{(-1)} + \psi^{(-3/2)} + \dots \right) \\ \Pi &= BM, \quad M = \exp \left(-\psi^{(+1/2)} - \psi^{(+1)} - \psi^{(+3/2)} - \dots \right), \end{aligned} \quad (2.1.3)$$

where $B = \exp \widehat{\mathfrak{g}}_0 \in \widehat{G}_0$ and g is a constant matrix. The evolution equations (2.1.1) are consequence of (2.1.2) after applying $\partial_{\pm}$ derivations. The constant semisimple operators $ad E^{(\pm 1)}$ split the superalgebra $\widehat{\mathfrak{g}} = \mathcal{K} + \mathcal{M}$ into kernel and image subspaces obeying $[\mathcal{K}, \mathcal{K}] \subset \mathcal{K}$, $[\mathcal{K}, \mathcal{M}] \subset \mathcal{M}$ and we also assume that $[\mathcal{M}, \mathcal{M}] \subset \mathcal{K}$ holds, which means that $\widehat{G}/K$ is a symmetric space.

The Lax covariant derivative ($L = d + A^L$) extracted from (2.1.1) has a Lax connection ($A^L = A_-^L dx^- + A_+^L dx^+$) given by

$$\begin{aligned} L_- &= \partial_- + A_-^L, \quad A_-^L = -B \left(E_-^{(-1)} + \psi_-^{(-1/2)} \right) B^{-1} \\ L_+ &= \partial_+ + A_+^L, \quad A_+^L = -\partial_+ B B^{-1} + \psi_+^{(+1/2)} + E_+^{(+1)}, \end{aligned} \quad (2.1.4)$$

where

$$\psi_{\pm}^{(\pm 1/2)} = \pm \left[\psi^{(\mp 1/2)}, E_{\pm}^{(\pm 1)} \right]. \quad (2.1.5)$$

Similarly we have:

Definition 2 *The gauge-equivalent relativistic sector is defined by the evolution equations*

$$\begin{aligned} \partial_+ \Theta' &= \left(\Theta' E_+^{(+1)} \Theta'^{-1} \right)_- \Theta' , \quad \partial_+ \Pi' = - \left(\Theta' E_+^{(+1)} \Theta'^{-1} \right)_+ \Pi' \\ \partial_- \Theta' &= \left(\Pi' E_-^{(-1)} \Pi'^{-1} \right)_- \Theta' , \quad \partial_- \Pi' = - \left(\Pi' E_-^{(-1)} \Pi'^{-1} \right)_+ \Pi' , \end{aligned} \quad (2.1.6)$$

where $\Theta' = B^{-1} \Theta$ and $\Pi' = M$. The $(*)_{\pm}$ denote projections onto grades $\geq +1/2$ and ≤ 0 .

The Lax covariant derivative extracted from (2.1.6) has a Lax connection

$$\begin{aligned} L'_- &= \partial_- + A_-^L , \quad A_-^L = B^{-1} \partial_- B - \psi_-^{(-1/2)} - E_-^{(-1)} \\ L'_+ &= \partial_+ + A_+^L , \quad A_+^L = B^{-1} \left(E_+^{(+1)} + \psi_+^{(+1/2)} \right) B \end{aligned} \quad (2.1.7)$$

and it is related to (2.1.4) by a gauge transformation $L \rightarrow L'$ with $g = B^{-1}$, where $(A'^L = g A^L g^{-1} - dg g^{-1})$. Clearly, the two definitions are equivalent.

The constant part of the Lax connection is given by

$$E^{(\pm 1)} = E_{\pm}^{(\pm 1)} dx^{\pm} \in \Omega_B(\Sigma) \otimes \hat{\mathfrak{g}}_{(\pm 1)}$$

and they change under coordinate transformations because of their $dx^{\pm}$ basis. We also have that

$$\psi_{\pm}^{(\pm 1/2)} dx^{\pm} = \Omega_F(\Sigma) \otimes \hat{\mathfrak{g}}_{(\pm 1/2)}$$

are Fermionic 1-forms. Thus, A^L is a superalgebra-valued 1-form.

The equations of motion are defined by the zero curvature of A^L , namely $[L_+, L_-] = 0$ and leads to a system of non-linear differential equations in which the derivatives $\partial_{\pm}$ appears with the same power. Hence the name relativistic.

In the definitions of the Lax operators above we actually have

$$\begin{aligned} -\partial_+ B B^{-1} &= A_+^{(0)} + Q_+^{(0)} \\ -B^{-1} \partial_- B &= A_-^{(0)} + Q_-^{(0)} , \end{aligned} \quad (2.1.8)$$

where

$$\begin{aligned} A_{\pm}^{(0)} &= \pm \left[\psi^{(\mp 1)}, E_{\pm}^{(\pm 1)} \right] \in \mathcal{M}_B^{(0)} \\ Q_{\pm}^{(0)} &= \frac{1}{2} \left[\psi^{(\mp 1/2)}, \left[\psi^{(\mp 1/2)}, E_{\pm}^{(\pm 1)} \right] \right] \in \mathcal{K}_B^{(0)} . \end{aligned} \quad (2.1.9)$$

The existence of the Fermion bilinear $Q_{\pm}^{(0)}$ in the definition of B imply the existence of non-local Fermionic symmetry flows as shown in [10] and that the integrable models will have gauge symmetries as can be deduced from the off-shell formulation of the system (2.1.4) done in [11]. The presence of a $\mathcal{K}_B^{(0)} \neq \emptyset$ translates into the existence of flat directions of the Toda potential which takes the models out of the mKdV hierarchy. Thus, we demand full cancelation of $Q_{+}^{(0)}$. Another reason why we demand $Q_{\pm}^{(0)} = 0$, is to get a well defined relation between the Dressing matrix Θ and the potential A in the spirit of [15], which means that the dynamical fields are describe entirely in terms of the image part of the algebra $\mathcal{M}$. The kernel part $\mathcal{K}$ is responsible only for the symmetries of the model.

Remark 3 *Flat directions in the Toda potential $V_B = \langle E_+^{(+1)} B E_-^{(-1)} B^{-1} \rangle$ imply the existence of soliton solutions with internal symmetries and Noether charges, e.g the electrically charged solitons of the complex Sine-Gordon model. This model is known to be a Singular Toda model and belongs to the relativistic sector of the AKNS Hierarchy [14] instead of the mKdV. Then, by restricting to superalgebras in which $Q_{\pm}^{(0)} = 0$ we have local flows and models inside the mKdV hierarchy because B is parametrized only by elements in the image $\mathcal{M}_B^{(0)}$. Furthermore, in the Hamiltonian reduction of a 2-Loop WZNW model this condition truncates the potential expansion up to second term, rendering the model integrable and supersymmetric [11].*

In what follows parametrize

$$B \equiv g \exp[\eta Q] \exp[\nu C]$$

as the exponential of $\mathcal{M}_B^{(0)}$, provided we have a subalgebra solution to the algebraic conditions $Q_{\pm}^{(0)} = 0$. The model is then defined on a reduced group manifold and (2.1.8) is correctly parametrized by $\mathcal{M}_B^{(0)}$ i.e $-\partial_+ B B^{-1} = A_+^{(0)}$ and $-B^{-1} \partial_- B = A_-^{(0)}$.

The zero curvature ($F_L = 0$) of (2.1.4) gives the Supersymmetric version of the Leznov-Saveliev equations [10] (see also eq (3.1.4) below)

$$\begin{aligned} 0 &= F_{+-}^{(+1/2)} = -\partial_- \psi_+^{(+1/2)} + [B\psi_-^{(-1/2)} B^{-1}, E_+^{(+1)}] \\ 0 &= F_{+-}^{(0)} = \partial_- (\partial_+ B B^{-1}) - [E_+^{(+1)}, B E_-^{(-1)} B^{-1}] - [\psi_+^{(+1/2)}, B\psi_-^{(-1/2)} B^{-1}] \\ 0 &= F_{+-}^{(-1/2)} = B \left(-\partial_+ \psi_-^{(-1/2)} + [E_-^{(-1)}, B^{-1} \psi_+^{(+1/2)} B] \right) B^{-1}. \end{aligned} \quad (2.1.10)$$

Written more explicitly in the form

$$\begin{aligned} \partial_- \psi_+^{(+1/2)} &= e^{-\eta/2} [g\psi_-^{(-1/2)} g^{-1}, E_+^{(+1)}] \\ \partial_- (\partial_+ g g^{-1}) + \partial_- \partial_+ \nu C &= e^{-\eta} [E_+^{(+1)}, g E_-^{(-1)} g^{-1}] + e^{-\eta/2} [\psi_+^{(+1/2)}, g\psi_-^{(-1/2)} g^{-1}] \\ \partial_+ \psi_-^{(-1/2)} &= e^{-\eta/2} [E_-^{(-1)}, g^{-1} \psi_+^{(+1/2)} g] \\ \partial_- \partial_+ \eta Q &= 0, \end{aligned}$$

the linearized equations of motion with $\eta = \eta_0$, $\eta_0 \in \mathbb{R}$ may be written in the Klein-Gordon form

$$\begin{aligned} (\partial_+ \partial_- + \hat{m}^2) \circ (\Xi) &= 0 \\ \partial_+ \partial_- \nu - \Lambda &= 0 \end{aligned}$$

for $\Xi = \psi^{(\pm 1/2)}$ and $\log g$, where $\hat{m}^2$ is the mass operator

$$\hat{m}^2(\Xi) = e^{-\eta_0} \left(ad E_-^{(-1)} \circ ad E_+^{(+1)} \right) \circ (\Xi) = m^2 I \Xi.$$

We have used $e^{-\eta_0} [E_+^{(+1)}, E_-^{(-1)}] = \Lambda C$. Then, the Higgs-like field η_0 sets the mass scale of the theory. The massless limit corresponds to $\eta_0 \rightarrow \infty$. Note that all fields have the same mass which is what we expect in a supersymmetric theory. Taking $\eta = \eta_0$, the free Fermion equations of motion reads

$$\partial_{\pm} \psi^{(\pm 1/2)} = \mp \hat{m}^{\pm} (\psi^{(\mp 1/2)}),$$

where $\hat{m}^{\pm}(\psi) = e^{-\eta_0/2} ad E_{\pm}^{(\pm 1)} \circ (\psi)$. This equations shows that Fermions of opposite 'chirality' are mixed by mass terms and that in the massless limit they decouple.

2.2 Non-Abelian Flows: The Odd Lax pairs $L_{\pm 1/2}$.

Here we deduce the two lowest odd degree Fermionic Lax operators giving the $\pm 1/2$ flows, which are the ones we are mainly concerned in this thesis

Definition 4 *The Non-Abelian evolution equations are defined by*

$$\begin{aligned}\delta_{K^+}\Theta &= (\Theta K^+ \Theta^{-1})_- \Theta, & \delta_{K^+}\Pi &= -(\Theta K^+ \Theta^{-1})_+ \Pi \\ \delta_{K^-}\Theta &= -(\Pi K^- \Pi^{-1})_- \Theta, & \delta_{K^-}\Pi &= (\Pi K^- \Pi^{-1})_+ \Pi,\end{aligned}\quad (2.2.1)$$

for some positive-negative degree generators K^+ and K^- in the kernel of E . Equivalently, we have*

$$\begin{aligned}\delta_{K^+}\Theta' &= (\Theta' K^+ \Theta'^{-1})_- \Theta', & \delta_{K^+}\Pi' &= -(\Theta' K^+ \Theta'^{-1})_+ \Pi' \\ \delta_{K^-}\Theta' &= -(\Pi' K^- \Pi'^{-1})_- \Theta', & \delta_{K^-}\Pi' &= (\Pi' K^- \Pi'^{-1})_+ \Pi'.\end{aligned}\quad (2.2.2)$$

Proposition 5 *The flows (2.2.1) and (2.2.2) satisfy*

$$\begin{aligned}\left[\delta_{K_i^\pm}, \delta_{K_j^\pm}\right](*) &= \delta_{[K_i^\pm, K_j^\pm]}(*) = (f_\pm)_{ij}^k \delta_{K_k^\pm}(*) \\ \left[\delta_{K_i^+}, \delta_{K_j^-}\right](*) &= 0,\end{aligned}$$

where $(*) = \Theta, \Pi, \Theta', \Pi'$. Thus, the map $\delta : \mathcal{K} \rightarrow \delta_{\mathcal{K}}$ is an Isomorphism when restricted to the positive/negative parts of the kernel algebra $\mathcal{K}_\pm$.

Proof. The proof is straightforward. ■

The last relation above means that the symmetries generated by elements in $\mathcal{K}_\pm$ commute themselves. This can be traced back to be a consequence of the second Lie structure induced on $\widehat{\mathfrak{g}}$ by the action of the Dressing group which introduces a classical super r -matrix $R = \frac{1}{2}(\mathcal{P}_+ - \mathcal{P}_-)$ defined in terms of the projections $\mathcal{P}_+$ and $\mathcal{P}_-$ of $\widehat{\mathfrak{g}} = \widehat{\mathfrak{g}}_+ + \widehat{\mathfrak{g}}_-$ along the positive/negative subalgebras $\widehat{\mathfrak{g}}_\pm$ [4]. In particular, this imply the commutativity of 2D supersymmetry transformations c.f (2.3.5), as expected.

*Recall that in (2.2.1) and (2.2.2) the projections are different.

The $\pm 1/2$ flows are derived from the evolution equations

$$\begin{aligned}\partial_{+1/2}\Theta &= (\Theta D^{(+1/2)}\Theta^{-1})_- \Theta, \quad \partial_{+1/2}\Pi = -(\Theta D^{(+1/2)}\Theta^{-1})_+ \Pi \\ \partial_{-1/2}\Theta &= -(\Pi D^{(-1/2)}\Pi^{-1})_- \Theta, \quad \partial_{-1/2}\Pi = (\Pi D^{(-1/2)}\Pi^{-1})_+ \Pi,\end{aligned}$$

giving rise to the Dressing expressions

$$\begin{aligned}\Theta (\partial_{+1/2} + D^{(+1/2)}) \Theta^{-1} &= \partial_{+1/2} + D^{(0)} + D^{(+1/2)} = L_{+1/2} \\ \Pi (\partial_{-1/2} + D^{(-1/2)}) \Pi^{-1} &= \partial_{-1/2} + BD^{(-1/2)}B^{-1} = L_{-1/2},\end{aligned}$$

where $D^{(0)} = [\psi^{(-1/2)}, D^{(+1/2)}]$. The derivation of $L_{-1/2}$ follows exactly the same lines for the derivation of $L_{+1/2}$ done in [10]. At this point we have four Lax operators $L_{\pm 1/2}$ and $L_{\pm 1}$. The graded subspace decomposition of the relations $[L_{\pm 1/2}, L_{+1}] = [L_{\pm 1/2}, L_{-1}] = 0$ imply $D^{(0)} = -\partial_{+1/2}BB^{-1}$. The compatibility of these system of four Lax operators provides the 2d supersymmetry transformations among the field component. Indeed, using the equations of motion we get the transformations (2.3.1) and (2.3.2) below.

Finally, the odd Lax operators reads

$$L_{+1/2} = \partial_{+1/2} - \partial_{+1/2}BB^{-1} + D^{(+1/2)} \quad (2.2.3)$$

$$L_{-1/2} = \partial_{-1/2} + BD^{(-1/2)}B^{-1}. \quad (2.2.4)$$

The operator $L_{+1/2}$ was already constructed in [10] and the $L_{-1/2}$ is the novelty here.

Note that in (2.2.3) and (2.2.4) are in different gauges. This is the key idea for introducing the Toda potential in the supersymmetry transformations.

2.3 Local Supersymmetry Flows $\partial_{\pm 1/2}$.

The equations (2.1.10) are invariant under a pair of Non-Abelian Fermionic flows $\partial = \partial_{-1/2} + \partial_{+1/2}$ as a consequence of the compatibility relations $[L_{\pm 1/2}, L_+] = [L_{\pm 1/2}, L_-] = 0$ supplemented by the equations of motion $[L_+, L_-] = 0$ and the

Jacobi identity. They are generated by the elements in the Fermionic kernel $\mathcal{K}_F^{(\pm 1/2)}$ and are explicitly given by

$$\begin{aligned}\partial_{+1/2}\psi_-^{(-1/2)} &= [E_-^{(-1)}, B^{-1}D^{(+1/2)}B] \\ \partial_{+1/2}BB^{-1} &= [D^{(+1/2)}, \psi_-^{(-1/2)}] \\ \partial_{+1/2}\psi_+^{(+1/2)} &= [\partial_{+1/2}BB^{-1}, \psi_+^{(+1/2)}] - [\partial_+BB^{-1}, D^{(+1/2)}].\end{aligned}\tag{2.3.1}$$

and

$$\begin{aligned}\partial_{-1/2}\psi_-^{(-1/2)} &= -[B^{-1}\partial_{-1/2}B, \psi_-^{(-1/2)}] - [B^{-1}\partial_-B, D^{(-1/2)}] \\ B^{-1}\partial_{-1/2}B &= [D^{(-1/2)}, \psi_+^{(+1/2)}] \\ \partial_{-1/2}\psi_+^{(+1/2)} &= [E_+^{(+1)}, BD^{(-1/2)}B^{-1}].\end{aligned}\tag{2.3.2}$$

The physical degrees of freedom are parametrized by the image part $\mathcal{M}$. To guarantee that the variations of the fields remain inside $\mathcal{M}$ we have to check that the kernel components of the above transformations vanishes, i.e

$$[\partial_{+1/2}BB^{-1}, \psi_+^{(+1/2)}] \in \mathcal{K} = 0, \quad [B^{-1}\partial_{-1/2}B, \psi_-^{(-1/2)}] \in \mathcal{K} = 0.\tag{2.3.3}$$

We do not have a general proof of this condition but we will see below in the examples that $Q_\pm^{(0)} = 0$ imply (2.3.3).

The Lax operators (2.2.3) (2.2.4) generating the odd flows (2.3.1) (2.3.2) are related to 2D supersymmetry transformations of the type

$$N = (N_+, N_-),\tag{2.3.4}$$

where[†] $N_\pm = \dim \mathcal{K}_F^{(\pm 1/2)}$. As the map $D^{(\pm 1/2)} \rightarrow \partial_{\pm 1/2}$ obeys

$$\begin{aligned}[\partial_{\pm 1/2}, \partial_{\pm 1/2}] \Theta &= \partial_{[D^{\pm 1/2}, D^{\pm 1/2}]} \Theta \sim \partial_\pm \Theta \\ [\partial_{+1/2}, \partial_{-1/2}] \Theta &= 0,\end{aligned}\tag{2.3.5}$$

[†]We consider only constant elements $E_\pm^{(\pm 1)}$ which gives rise to isomorphic subspaces $\mathcal{K}_F^{(+1/2)} \simeq \mathcal{K}_F^{(-1/2)}$ and to $N_+ = N_-$.

we see that two Fermionic transformations close into derivatives, which is by definition a Supersymmetry. This is the case provided $(F^{(\pm 1/2)})^2 \sim E_{\pm}^{(\pm 1)}$ for $F^{(\pm 1/2)} \in \mathcal{K}_F^{(\pm 1/2)}$, which is significant for the supersymmetry structure of the models, see for example [3]. Furthermore, all the Non-Abelian odd flows close into the Isospectral even flows. The central and gradation fields do not transform and are not truly degrees of freedom of the model.

2.4 Recursion operators and Higher Odd Flows.

Trying to compute explicit expressions for odd Lax operators generating higher degree supersymmetry flows is considerably more involved than the $\pm 1/2$ cases. Hence we try another approach by using the Dressing map $\mathcal{K} \rightarrow \delta_{\mathcal{K}}$ from the Kernel algebra to the flow algebra

$$[\delta_{\mathcal{K}}, \delta_{\mathcal{K}'}] \Theta = \delta_{[\mathcal{K}, \mathcal{K}']} \Theta.$$

From the relations

$$[\delta_{K^{(\pm 1)}}, \delta_{F^{(\pm 1/2)}}] \Theta = \delta_{[K^{(\pm 1)}, F^{(\pm 1/2)}]} \Theta = \delta_{F^{(\pm 3/2)}} \Theta$$

we infer the following behavior

$$\begin{aligned} \delta_{F^{(\pm n \pm 1/2)}} &= ad_{\delta_{K^{(\pm 1)}}}^n (\delta_{F^{(\pm 1/2)}}) = (\mathcal{R}^{\pm 1})^n (\delta_{F^{(\pm 1/2)}}) \\ \mathcal{R}^{\pm 1}(\ast) &= ad_{\delta_{K^{(\pm 1)}}}(\ast) = [\delta_{K^{(\pm 1)}}, \ast] \end{aligned} \quad (2.4.1)$$

in terms of the recursion operators $\mathcal{R}^{\pm 1}$. The power of this formalism is not to reproduce the well known supersymmetry transformations but to offer a method to construct systematically all the Higher odd symmetry flows present in the integrable model in terms of its underlying flow symmetry structure. We have to mention that the use of super pseudo-differential operators, i.e scalar Lax operators, seems to be more appropriated for this purpose.

From this analysis, we have the following chains of increasing supersymmetry transformations

$$\begin{aligned} \partial_{+1/2} &\xrightarrow{\delta_{K^+}} \partial_{+3/2} \xrightarrow{\delta_{K^+}} \partial_{+5/2} \xrightarrow{\delta_{K^+}} \partial_{+7/2} \dots \\ \dots \partial_{-7/2} &\xleftarrow{\delta_{K^-}} \partial_{-5/2} \xleftarrow{\delta_{K^-}} \partial_{-3/2} \xleftarrow{\delta_{K^-}} \partial_{-1/2}, \end{aligned} \quad (2.4.2)$$

where the ones corresponding to $\partial_{\pm 1/2}$ are the starting points. The variations $\delta_{K\pm}$ are also given by (2.2.1). For example, for a degree +1 element $K^{(+1)}$ we have

$$\delta_{K^{(+1)}} \psi_+^{(+1/2)} = - \left[E_+^{(+1)}, (\Phi K^{(+1)} \Phi^{-1})_{-1/2} |_{\mathcal{M}} \right] \quad (2.4.3)$$

$$\delta_{K^{(+1)}} (\partial_+ B B^{-1}) = + \left[E_+^{(+1)}, (\Phi K^{(+1)} \Phi^{-1})_{-1} \right] + \left[\psi_+^{(+1/2)}, (\Phi K^{(+1)} \Phi^{-1})_{-1/2} |_{\mathcal{K}} \right]$$

$$\delta_{K^{(+1)}} \psi_-^{(-1/2)} = - \left[E_-^{(-1)}, B^{-1} (\Phi K^{(+1)} \Phi^{-1})_{+1/2} B |_{\mathcal{M}} \right].$$

The Dressing matrix Φ factorizes as $\Phi = US$, where $U \in \mathcal{M}$ is local and $S \in \mathcal{K}$ is non-local in the fields [10], splitting the Dressing of the vacuum Lax operators $(L_{\pm} = \Theta L_{\pm}^V \Theta^{-1})$ as a two step process. A U and a S rotation given respectively by

$$U^{-1} L_{\pm} U = \partial_{\pm} + E_{\pm}^{(\pm 1)} + K^{(\mp 1)} \quad (2.4.4)$$

$$S^{-1} \left(\partial_{\pm} + E_{\pm}^{(\pm 1)} + K^{(\mp 1)} \right) S = \partial_{\pm} + E_{\pm}^{(\pm 1)} = L_{\pm}^V, \quad (2.4.5)$$

where $K^{(\mp 1)} \in \mathcal{K}$ involves expansions on negative-positive degrees. The components $\psi^{(i)}$, $i = -1/2, -3/2, \dots$ of U are extracted from projecting (2.4.4) along $\mathcal{M}$ and the components $s^{(i)}$, $i = -1/2, -3/2, \dots$ by projecting (2.4.5) along $\mathcal{K}$. This allows to compute (2.4.3). The Higher supersymmetry transformations are inevitably non-local because of the presence of the kernel part S appearing in the transformations $\delta_{K^{(\pm 1)}}$ used to construct them. Thus, the best we can do is to restrict ourselves to the reduced manifold (defined by $Q_{\pm}^{(0)} = 0$) in which $\partial_{\pm 1/2}$ are local. Using (2.2.1) we can see that

$$[\delta_{K^{(+1)}}, \partial_{-1/2}] \Theta = 0.$$

This means we cannot link $\partial_{-1/2}$ and $\partial_{+1/2}$ through a $\delta_{K^{(+1)}}$ flow, reflecting the chiral independence of the $\partial_{\pm 1/2}$ transformations. This is why in (2.4.2) the sectors are treated separately.

2.5 Generalized Relativistic Current Identities.

In this section we derive an infinite set of identities associated to the flows generated by $\mathcal{K}_F^{(\pm 1/2)}$. The word relativistic is used in the sense that each $t_{\pm n}$ is coupled to its opposite counterpart $t_{\mp n}$.

Proposition 6 *The infinite set of Fermionic local currents defined by*

$$\begin{aligned} J_{+n}^{(+1/2)} &= \langle D^{(+1/2)} \Theta E^{(+n)} \Theta^{-1} \rangle, & J_{-n}^{(+1/2)} &= \langle D^{(+1/2)} \Pi E^{(-n)} \Pi^{-1} \rangle \\ J_{+n}^{(-1/2)} &= \langle D^{(-1/2)} \Theta' E^{(+n)} \Theta'^{-1} \rangle, & J_{-n}^{(-1/2)} &= \langle D^{(+1/2)} \Pi' E^{(-n)} \Pi'^{-1} \rangle, \end{aligned} \quad (2.5.1)$$

satisfy the following identities

$$\partial_{+n} J_{-n}^{(\pm 1/2)} + \partial_{-n} J_{+n}^{(\pm 1/2)} = 0. \quad (2.5.2)$$

The $D^{(\pm 1/2)} \in \mathcal{K}_F^{(\pm 1/2)}$ are the generators of the Fermionic kernel.

Proof. The proof is extremely simple and is based only on the relations (2.2.1) and (2.2.2). Start with

$$\begin{aligned} \partial_{+n} J_{-m}^{(+1/2)} &= \left\langle D^{(+1/2)} \left[(\Pi E^{(-m)} \Pi^{-1}), (\Theta E^{(+n)} \Theta^{-1})_+ \right] \right\rangle \\ \partial_{-n} J_{+m}^{(+1/2)} &= - \left\langle D^{(+1/2)} \left[(\Pi E^{(-n)} \Pi^{-1})_-, (\Theta E^{(+m)} \Theta^{-1}) \right] \right\rangle \end{aligned}$$

to get

$$\begin{aligned} \partial_{+n} J_{-m}^{(+1/2)} + \partial_{-n} J_{+m}^{(+1/2)} &= \left\langle D^{(+1/2)} \left[(\Pi E^{(-m)} \Pi^{-1})_-, (\Theta E^{(+n)} \Theta^{-1})_+ \right] \right\rangle - \\ &\quad - \left\langle D^{(+1/2)} \left[(\Pi E^{(-n)} \Pi^{-1})_-, (\Theta E^{(+m)} \Theta^{-1})_+ \right] \right\rangle. \end{aligned}$$

This sum vanishes for $m = n$. For $J^{(-1/2)}$ the proof is analogous. Note that the use of different gauges is crucial for the proof. ■

These identities mixes the two sectors corresponding to positive and negative Isospectral times in a relativistic manner. They can be written in a covariant form $\eta^{ij} \frac{\partial}{\partial t_i} J_j^{(\pm 1/2)} = 0$ if we define a constant metric $\eta = \eta_{ij} dt_i dt_j$ for each pair of positive and negative times.

Consider now the lowest Isospectral flows $t_{\pm 1} = \pm x^{\pm}$. The current components (2.5.1) are given by

$$\begin{aligned} J_+^{(+1/2)} &= - \left\langle D^{(+1/2)} \left[\psi^{(-1/2)}, \partial_+ B B^{-1} \right] \right\rangle, & J_-^{(+1/2)} &= \left\langle D^{(+1/2)} B \psi_-^{(-1/2)} B^{-1} \right\rangle \\ J_+^{(-1/2)} &= \left\langle D^{(-1/2)} B^{-1} \psi_+^{(+1/2)} B \right\rangle, & J_-^{(-1/2)} &= \left\langle D^{(-1/2)} \left[\psi^{(+1/2)}, B^{-1} \partial_- B \right] \right\rangle. \end{aligned} \quad (2.5.3)$$

Then, there are $\dim \mathcal{K}_F^{(\pm 1/2)}$ associated relativistic conservation laws given by $\partial_+ J_-^{(\pm 1/2)} - \partial_- J_+^{(\pm 1/2)} = 0$. More explicitly, we have

$$\begin{aligned} \partial_- \left(\left[\psi^{(-1/2)}, \partial_+ B B^{-1} \right] |_{\mathcal{K}} \right) + \partial_+ \left(B \psi_-^{(-1/2)} B^{-1} |_{\mathcal{K}} \right) &= 0 \\ -\partial_- \left(B^{-1} \psi_+^{(+1/2)} B |_{\mathcal{K}} \right) + \partial_+ \left(\left[\psi^{(+1/2)}, B^{-1} \partial_- B \right] |_{\mathcal{K}} \right) &= 0. \end{aligned} \quad (2.5.4)$$

This time, the identities provide supercharge conservation laws due to the fact that the flows $t_{\pm 1}$ are identified with the light-cone coordinates $x^{\pm} = \frac{1}{2}(x^0 \pm x^1)$. It is not clear if the identities associated to the higher flows $t_{\pm n}$, $n \geq +1$ provide new conserved quantities because one is not supposed to impose boundary conditions along these directions. For higher times they are taken as simple identities consequence of the flow algebra.

Now that we have $\dim \mathcal{K}_F^{(\pm 1/2)}$ supercurrents associated to $\mathcal{K}_F^{(\pm 1/2)}$, let's compute their corresponding supercharges by the Noether procedure in order to check that they really generate the supersymmetry transformations (2.3.1) and (2.3.2).

2.6 Supercharges for the SUSY Flows $\partial_{\pm 1/2}$.

The action for the Affine supersymmetric Toda models[†] was deduced in [11] and it is given by

$$\begin{aligned} S_{Toda}[B, \psi] &= S_{WZNW}[B] - \frac{k}{4\pi} \int_{\Sigma} \left\langle \psi_+^{(+1/2)} \partial_- \psi^{(-1/2)} + \psi_-^{(-1/2)} \partial_+ \psi^{(+1/2)} \right\rangle \\ &\quad + \frac{k}{2\pi} \int_{\Sigma} \left\langle E_+^{(+1)} B E_-^{(-1)} B^{-1} + \psi_+^{(+1/2)} B \psi_-^{(-1/2)} B^{-1} \right\rangle. \end{aligned} \quad (2.6.1)$$

This corresponds to the situation when we restrict to the sub-superalgebras solving the condition $Q_{\pm}^{(0)} = 0$. In this case the potential ends at the second term providing a Yukawa-type term turning the model integrable and supersymmetric.

[†]The light-cone notation used for the flat Minkowski space Σ is $x^{\pm} = \frac{1}{2}(x^0 \pm x^1)$, $\partial_{\pm} = \partial_0 \pm \partial_1$, $\eta_{+-} = \eta_{-+} = 2$, $\eta^{+-} = \eta^{-+} = \frac{1}{2}$, $\epsilon_{+-} = -\epsilon_{-+} = 2$, $\epsilon^{-+} = -\epsilon^{+-} = \frac{1}{2}$ corresponding to the metric $\eta_{00} = 1$, $\eta_{11} = -1$ and antisymmetric symbol $\epsilon_{10} = -\epsilon_{01} = +1$. A coupling constant is introduced by setting $E_{\pm}^{(\pm 1)} \rightarrow \mu E_{\pm}^{(\pm 1)}$ and $\psi^{(\pm 1/2)} \rightarrow \mu^{-1/2} \psi^{(\pm 1/2)}$.

An arbitrary variation of the action is given by

$$\begin{aligned} \frac{2\pi}{k} \delta S_{Toda}[B, \psi] &= \int_{\Sigma} \left\langle (\delta B B^{-1}) \left\{ \partial_- (\partial_+ B B^{-1}) - [E_+^{(+1)}, B E_-^{(-1)} B^{-1}] - [\psi_+^{(+1/2)}, B \psi_-^{(-1/2)} B^{-1}] \right\} \right\rangle \\ &\quad + \int_{\Sigma} \left\langle \delta \psi_-^{(-1/2)} \left\{ -\partial_+ \psi_+^{(+1/2)} + B^{-1} \psi_+^{(+1/2)} B \right\} \right\rangle + \\ &\quad + \int_{\Sigma} \left\langle \delta \psi_+^{(+1/2)} \left\{ -\partial_- \psi_-^{(-1/2)} + B \psi_-^{(-1/2)} B^{-1} \right\} \right\rangle \end{aligned} \quad (2.6)$$

and the equations of motions are exactly the super Leznov-Saveliev equations (2.1.10), i.e $[L_+, L_-] = 0$.

Taking $\delta \rightarrow \partial = \partial_{-1/2} + \partial_{+1/2}$, using (2.3.1), (2.3.2) and considering $D^{(\pm 1/2)}$ as functions of the coordinates $x^{\pm}$, we have the supersymmetric variation of the action

$$\begin{aligned} \frac{2\pi}{k} \delta S_{Toda} &= \int_{\Sigma} \left\langle \partial_- D^{(-1/2)} \left(B^{-1} \psi_+^{(+1/2)} B \right) - \partial_+ D^{(-1/2)} \left[\psi_+^{(+1/2)}, B^{-1} \partial_- B \right] \right\rangle - \\ &\quad - \int_{\Sigma} \left\langle \partial_- D^{(+1/2)} \left[\psi_-^{(-1/2)}, \partial_+ B B^{-1} \right] + \partial_+ D^{(+1/2)} \left(B \psi_-^{(-1/2)} B^{-1} \right) \right\rangle. \end{aligned}$$

This allows to obtain two conservation laws

$$0 = \int_{\Sigma} \left\langle D^{(\mp 1/2)} \left(\partial_- j_+^{(\pm 1/2)} + \partial_+ j_-^{(\pm 1/2)} \right) \right\rangle,$$

which are exactly the ones derived by using the extended Riemann-Hilbert approach (2.5.2) for the lowest flows (2.5.4). Then, there are $\dim \mathcal{K}_F$ supercurrents and supercharges given by

$$\begin{aligned} \text{Flow } \partial_{+1/2} &: j_+^{(-1/2)} = [\psi_-^{(-1/2)}, \partial_+ B B^{-1}]|_{\mathcal{K}}, \quad j_-^{(-1/2)} = B \psi_-^{(-1/2)} B^{-1}|_{\mathcal{K}} \\ Q_+ &= \int dx^1 \left([\psi_-^{(-1/2)}, \partial_+ B B^{-1}] + B \psi_-^{(-1/2)} B^{-1} \right) |_{\mathcal{K}} \end{aligned} \quad (2.6.3)$$

$$\begin{aligned} \text{Flow } \partial_{-1/2} &: j_+^{(+1/2)} = -B^{-1} \psi_+^{(+1/2)} B|_{\mathcal{K}}, \quad j_-^{(+1/2)} = [\psi_+^{(+1/2)}, B^{-1} \partial_- B]|_{\mathcal{K}} \\ Q_- &= \int dx^1 \left([\psi_+^{(+1/2)}, B^{-1} \partial_- B] - B^{-1} \psi_+^{(+1/2)} B \right) |_{\mathcal{K}}. \end{aligned} \quad (2.6.4)$$

The variation above is the same when (2.3.3) are zero or not, this is because all the fields are defined in $\mathcal{M}$ and the kernel part does not affect the variation at all.

These two ways of extracting the supercharges show a deep relation between the Dressing formalism and the Hamiltonian reduction.

Now specialize the construction done above to the simplest toy examples. The Supercharges are computed from the general formulas (2.6.3) and (2.6.4). We want to emphasize that the sub-superalgebras solving the condition $Q_{\pm}^{(0)} = 0$ have no Bosonic kernel $\mathcal{K}_B$ of degree zero in consistency with the absence of positive even Isospectral flows t_{+2n} in the mKdV Hierarchy.

2.7 Examples.

These examples show how the superspace notion of supersymmetry can be embedded consistently into the infinite-dimensional flow approach. The usual SUSY transformations corresponds to the flow algebra spanned by the times $(t_{-1}, t_{-1/2}, t_{+1/2}, t_{+1})$. We can have several pairs of odd times $t_{\pm 1/2}$ depending on the dimension of $\mathcal{K}_F^{(\pm 1/2)}$.

2.7.1 The N=(1,1) sinh-Gordon model Reloaded.

Take the $sl(2, 1)_{[1]}^{(2)}$ superalgebra invariant under τ_{red} (see Appendix A). The Lagrangian is

$$\begin{aligned} L &= -\frac{k}{2\pi} [\partial_+ \phi \partial_- \phi + \psi_- \partial_+ \psi_- + \psi_+ \partial_- \psi_+ - V] \\ V &= 2\mu^2 \cosh[2\phi] + 4\mu \psi_+ \psi_- \cosh[\phi] \end{aligned} \quad (2.7.1)$$

and the equations of motion are

$$\begin{aligned} \partial_+ \partial_- \phi &= -2\mu^2 \sinh[2\phi] - 2\mu \psi_+ \psi_- \sinh[\phi] \\ \partial_- \psi_+ &= 2\mu \psi_- \cosh[\phi] \\ \partial_+ \psi_- &= -2\mu \psi_+ \cosh[\phi]. \end{aligned} \quad (2.7.2)$$

With $D^{(+1/2)} = \epsilon_- F_2^{(+1/2)}$, $D^{(-1/2)} = \epsilon_+ F_1^{(-1/2)}$ and $\psi_{\pm} \rightarrow \frac{1}{2}\psi_{\pm}$, the supersym-

metry flows are

$$\begin{aligned}\partial_{\pm 1/2}\phi &= \pm\epsilon_{\mp}\psi_{\pm} \\ \partial_{\pm 1/2}\psi_{\pm} &= \mp\epsilon_{\mp}\partial_{\pm}\phi \\ \partial_{\pm 1/2}\psi_{\mp} &= 2\mu\epsilon_{\mp}\sinh[\phi],\end{aligned}\tag{2.7.3}$$

where we used the parametrizations

$$\begin{aligned}B &= \exp[\phi H_1], \quad \psi^{(+1/2)} = \psi_- G_1^{(+1/2)}, \quad \psi^{(-1/2)} = \psi_+ G_2^{(-1/2)} \\ \psi_-^{(-1/2)} &= 2\psi_- G_2^{(-1/2)}, \quad \psi_+^{(+1/2)} = -2\psi_+ G_1^{(+1/2)}.\end{aligned}$$

We can check (2.3.5) by applying (2.7.3) twice giving

$$\begin{aligned}[\partial_{\pm 1/2}, \partial_{\pm 1/2}] &= 2\epsilon_{\mp}\epsilon'_{\mp}\partial_{\pm} \\ [\partial_{+1/2}, \partial_{-1/2}] &= 0.\end{aligned}$$

Then, we have two real supercharges $N = (1, 1)$ because $\dim\mathcal{K}_F^{(\pm 1/2)} = 1$ given by

$$\begin{aligned}\partial_{\pm 1/2} &: Q^{(\mp 1/2)} = Q_{\pm}^1 F_{1,2}^{(\mp 1/2)} \\ Q_{\pm}^1 &= \int dx^1 (\psi_{\pm}\partial_{\pm}\phi \mp \psi_{\mp}h'(\phi)),\end{aligned}$$

where $h(\phi) = 2\mu\cosh[\phi]$ and $h'(\phi)$ is its functional derivative.

Now rotate the Fermions by the phase $\exp(i\pi/4)$ in order to write (2.7.1) in a more familiar form

$$\begin{aligned}L &= -\frac{k}{\pi} \left[\frac{1}{2}\partial_+\phi\partial_-\phi + \frac{i}{2}\psi_-\partial_+\psi_- + \frac{i}{2}\psi_+\partial_-\psi_+ - V \right] \\ V &= \frac{1}{2} (h'(\phi))^2 + ih''(\phi)\psi_+\psi_- + \mu^2.\end{aligned}$$

which is known to be invariant under the $N = (1, 1)$ superspace transformations for a real Bosonic superfield. The area term comes from squaring $h'(\phi) = 2\mu\sinh[\phi]$.

With the Dirac brackets[§] $\{\phi, \partial_t \phi\} = 1$, $\{\psi_\pm, \psi_\pm\} = -i$ and $Q_\pm^1 f = \{Q_\pm^1, f\}$ we have, after replacing $Q_\pm^1 \rightarrow iQ_\pm^1$, the action of the supercharges on the field components

$$\begin{aligned} Q_\pm^1 \phi &= -i\psi_\pm \\ Q_\pm^1 \psi_\pm &= +\partial_\pm \phi \\ Q_\pm^1 \psi_\mp &= \mp h'(\phi). \end{aligned}$$

Finally, the total flow can be written as $\partial = \partial_{+1/2} + \partial_{-1/2} = -i\epsilon_- Q_+^1 + i\epsilon_+ Q_-^1$

$$\begin{aligned} \partial \phi &= +i\epsilon_- \psi_+ - i\epsilon_+ \psi_- \\ \partial \psi_\pm &= \mp \epsilon_\mp \partial_\pm \phi + \epsilon_\pm h'(\phi), \end{aligned}$$

which are the ordinary $N = (1, 1)$ supersymmetry transformations.

Now, construct the $\partial_{+3/2}$ transformations starting from $\partial_{+1/2}$ by applying $\delta_{K^{(+1)}}$ as shown in (2.4.2). Following [10] we find

$$\begin{aligned} \delta_{K^{(+1)}} \phi &= -\frac{1}{2} \alpha \psi_+ Q(x^+) \\ \delta_{K^{(+1)}} \psi_+ &= +\frac{1}{2} \alpha (\partial_+ \psi_+ - \partial_+ \phi Q(x^+)) \\ \delta_{K^{(+1)}} \psi_- &= -\frac{1}{2} \alpha (\psi_+ h(\phi) - h'(\phi) Q(x^+)), \end{aligned} \tag{2.7.4}$$

where $Q(x^+) = \int^{x^+} (\psi_+ \partial_+ \phi)$. From the relation $[K_2^{(+1)}, F_2^{(+1/2)}] = F_1^{(+3/2)}$ we

[§]The Poisson brackets are defined by

$$\{A, B\}_{PB} = -(-1)^{\epsilon_A \epsilon_B} \left(\frac{\partial A}{\partial f} \frac{\partial B}{\partial \pi_f} - (-1)^{\epsilon_A \epsilon_B} \frac{\partial A}{\partial \pi_f} \frac{\partial B}{\partial f} \right)$$

where $\epsilon = 1, 0$ for Bosonic-Fermionic quantities and $\pi_f = \frac{\partial L}{\partial(\partial_t f)}$. The Dirac bracket is defined by

$$\{A, B\}_{DB} = \{A, B\}_{PB} - \{A, \phi_i\}_{PB} (C^{-1})_{ij} \{\phi_j, B\}_{PB}$$

where $C_{ij} = \{\phi_i, \phi_j\}_{PB}$ and ϕ_i are the second class constraints.

have $\partial_{+3/2} = [\delta_{K^{(+1)}}, \partial_{+1/2}]$ and the transformations are given by

$$\begin{aligned}\partial_{+3/2}\phi &= \epsilon_- \left(\frac{1}{2}\partial_+\psi_+ - \partial_+\phi Q(x^+) + \frac{1}{2}\psi_+ H(x^+) \right) \\ \partial_{+3/2}\psi_+ &= \epsilon_- \left(\frac{1}{2}\partial_+^2\phi + \partial_+\psi_+ Q(x^+) - \frac{1}{2}\partial_+\phi H(x^+) \right) \\ \partial_{+3/2}\psi_- &= \epsilon_- \left(-\frac{1}{2}\partial_+\phi h(\phi) - \psi_+ Q(x^+) h(\phi) + \frac{1}{2}h'(\phi) H(x^+) \right)\end{aligned}\quad (2.7.5)$$

where $H(x^+) = \int^{x^+} ((\partial_+\phi)^2 + \psi_+ \partial_+\psi_+)$. We also find the variations

$$\begin{aligned}\partial_{+1/2}Q(x^+) &= -\epsilon_- H(x^+) \\ \partial_{+3/2}Q(x^+) &= \epsilon_- \left(\frac{1}{4}(\partial_+\phi)^2 - \frac{1}{2}\psi_+ \partial_+\psi_+ + \partial_+\phi \psi_+ Q(x^+) - \frac{1}{4}H^2(x^+) \right) \\ \partial_{+3/2}H(x^+) &= \epsilon_- \left(\partial_+\phi \partial_+\psi_+ - \frac{1}{2}\partial_+^2\phi \psi_+ + \frac{1}{2}\partial_+\phi \psi_+ H(x^+) - Q(x^+) \partial_+ H(x^+) \right)\end{aligned}$$

Applying $\partial_{+3/2}$ twice we get a local flow description of the Hierarchy for t_{+1} and t_{+3} in terms of the sinh-Gordon variables used to described it in terms of t_{-1} and t_{+1} (2.7.2)

$$\begin{aligned}4\partial_{+3}\phi &= \partial_+^3\phi - 2(\partial_+\phi)^3 - 3\partial_+\phi\psi_+\partial_+\psi_+ \\ 4\partial_{+3}\psi_+ &= \partial_+^3\psi_+ - 3\partial_+\phi\partial_+(\partial_+\phi\psi_+) \\ 4\partial_{+3}\psi_- &= (2(\partial_+\phi)^2\psi_+ - \partial_+^2\psi_+)h(\phi),\end{aligned}\quad (2.7.6)$$

where $[\partial_{+3/2}, \partial'_{+3/2}](*) = -2\epsilon_1\epsilon_2\partial_{+3}(*)$ in agreement with $\{F_1^{(+3/2)}, F_1^{(+3/2)}\} = -2E_+^{(+3)}$. Introducing[¶] $u = u' = \partial_+\phi$ we recover the super mKdV equations

$$\begin{aligned}4\partial_{+3}u &= u''' - 6u^2u' - 3\psi_+(u\psi_+)' \\ 4\partial_{+3}\psi_+ &= \psi_+''' - 3u(u\psi_+)' \\ 4\partial_{+3}\psi_- &= (2u^2\psi_+ - \psi_+'')h(\partial^{-1}u).\end{aligned}$$

The ψ_- equation is non-local and is a remnant of the negative part of the Hierarchy. The supersymmetry has to be reduced to the usual $N = (1, 0)$ in order to have

[¶]In this case the space variable x is described by t_{+1} .

a local description in terms of the mKdV variables u and ψ_+ . Note that $Q(x^+)$ and $H(x^+)$ resembles to some components of supercharge and the stress tensor. The positive and negative parts of the extended mKdV hierarchy carries exactly the same information when considered separately and to obtain relativistic equations we combine their Lax pairs in different gauges. The potential couples the two sectors. We check this explicitly by considering (t_{+1}, t_{+3}) , (t_{-1}, t_{-3}) and compute $L_{\pm 3}$ in order to construct a 1-soliton solution which solves any equation of the hierarchy mixing the four times $(t_{+1}, t_{+3}, t_{-1}, t_{-3})$.

The Dressing technique suggest the following forms for the Lax operators

$$\begin{aligned} L_{+3} &= \partial_{+3} + D^{(0)} + D^{(+1/2)} + D^{(+1)} + D^{(+3/2)} + D^{(+2)} + D^{(+5/2)} + E^{(+3)} \\ L_{-3} &= \partial_{-3} - B \left(D^{(-1/2)} + D^{(-1)} + D^{(-3/2)} + D^{(-2)} + D^{(-5/2)} + E^{(-3)} \right) B \end{aligned} \quad (2.7.7)$$

From $[L_+, L_{+3}] = 0$ we get, following [10], the solution

$$\begin{aligned} D^{(+5/2)} &= (-2\psi_+) G_1^{(+5/2)} \\ D^{(+2)} &= (-\partial_+ \phi) M_1^{(+2)} \\ D^{(+3/2)} &= (-\psi_+ \partial_+ \phi) F_1^{(+3/2)} + (\partial_+ \psi_+) G_2^{(+3/2)} \\ D^{(+1)} &= (-2\psi_+ \partial_+ \psi_+) K_1^{(+1)} + \left(\frac{1}{2} (\partial_+ \phi)^2 + 2\psi_+ \partial_+ \psi_+ \right) K_2^{(+1)} + \left(\frac{1}{2} \partial_+^2 \phi \right) M_2^{(+1)} \\ D^{(+1/2)} &= -\frac{1}{2} (\partial_+^2 \phi \psi_+ - \partial_+ \phi \partial_+ \psi_+) F_2^{(+1/2)} + \left(-\frac{1}{2} \partial_+^2 \psi_+ + \psi_+ (\partial_+ \phi)^2 \right) G_1^{(+1/2)} \\ D^{(0)} &= \left(-\frac{1}{4} \partial_+^3 \phi + \frac{1}{2} (\partial_+ \phi)^3 + 3\partial_+ \phi \psi_+ \partial_+ \psi_+ \right) M_1^{(0)} \end{aligned}$$

The equation of motion are given by the degree zero component and are, as expected, given the first two equations of (2.7.6), after taking $\psi_+ \rightarrow \frac{1}{2}\psi_+$. Now, performing a gauge transformation with B^{-1} in order to eliminate the B conjugation on L_{-3} and L_- we change to

$$\begin{aligned} L_- &= \partial_- + B^{-1} \partial_- B - \left(E_-^{(-1)} + \psi_-^{(-1/2)} \right) \\ L_{-3} &= \partial_{-3} + B^{-1} \partial_{-3} B - \left(D^{(-1/2)} + D^{(-1)} + D^{(-3/2)} + D^{(-2)} + D^{(-5/2)} + E^{(-3)} \right) \end{aligned}$$

and from $[L_-, L_{-3}] = 0$ we get the solutions

$$\begin{aligned}
D^{(-5/2)} &= (2\psi_-) G_2^{(-5/2)} \\
D^{(-2)} &= (-\partial_- \phi) M_1^{(-2)} \\
D^{(-3/2)} &= (\psi_- \partial_- \phi) F_2^{(-3/2)} + (\partial_- \psi_-) G_1^{(-3/2)} \\
D^{(-1)} &= (-2\psi_- \partial_- \psi_-) K_1^{(-1)} + \left(\frac{1}{2} (\partial_- \phi)^2 + 2\psi_- \partial_- \psi_- \right) K_2^{(-1)} + \left(-\frac{1}{2} \partial_-^2 \phi \right) M_2^{(-1)} \\
D^{(-1/2)} &= -\frac{1}{2} (\partial_-^2 \phi \psi_- - \partial_- \phi \partial_- \psi_-) F_1^{(-1/2)} + \left(\frac{1}{2} \partial_-^2 \psi_- - (\partial_- \phi)^2 \psi_- \right) G_2^{(-1/2)}.
\end{aligned}$$

The equations of motion are now

$$\begin{aligned}
4\partial_{-3}\phi &= \partial_-^3 \phi - 2(\partial_- \phi)^3 - 3\partial_- \phi \psi_- \partial_- \psi_- \\
4\partial_{-3}\psi_- &= \partial_-^3 \psi_- - 3\partial_- \phi \partial_- (\partial_- \phi \psi_-)
\end{aligned}$$

after taking $\psi_- \rightarrow \frac{1}{2}\psi_-$. With^{||} $v = \partial_- \phi$ we get

$$\begin{aligned}
4\partial_{-3}v &= v''' - 6v^2v - 3\psi_- (v\psi_-)' \\
4\partial_{-3}\psi_- &= \psi_-''' - 3v (v\psi_-)'.
\end{aligned}$$

From the solution of the equation (B.0.1) $D^{(0)} = -\partial_{+3} B B^{-1}$ we confirm that $4\partial_{+3}\phi = \partial_+^3 \phi - 2(\partial_+ \phi)^3 - 3\partial_+ \phi \psi_+ \partial_+ \psi_+$, after taking $\psi_+ \rightarrow \frac{1}{2}\psi_+$. This is the $N = (0, 1)$ mKdV equation with opposite chirality.

The simplest example of a relativistic equation is provided by the sinh-Gordon model which corresponds to the lowest $t_{\pm 1}$ times. Now compute $[L_{+3}, L_{-3}] = 0$ with L_{-3} in the form (2.7.7), i.e in its original gauge. The relativistic system of equations is given in the Appendix B, eq (B.0.2), and is a generalization to the $t_{\pm 3}$ times. The two sectors of the extended hierarchy are identical, thus we complete the system (2.7.8) to get

$$\begin{aligned}
4\partial_{-3}\phi &= \partial_-^3 \phi - 2(\partial_- \phi)^3 - 3\partial_- \phi \psi_- \partial_- \psi_- \\
4\partial_{-3}\psi_- &= \partial_-^3 \psi_- - 3\partial_- \phi \partial_- (\partial_- \phi \psi_-) \\
4\partial_{-3}\psi_+ &= (2(\partial_- \phi)^2 \psi_- - \partial_-^2 \psi_-) h(\phi)
\end{aligned} \tag{2.7.8}$$

^{||}In this case the space variable x is described by t_{-1} .

and due to this symmetric behavior we can easily read off the $\partial_{-3/2}$ transformations from (2.7.5) simply by replacing all + subindexes by - subindexes.

Now consider for simplicity the Bosonic limit of the extended equations (B.0.2). It is given by

$$\begin{aligned}
\partial_{+3}\partial_{-3}\phi &= -2\sinh[2\phi] + \frac{1}{2}((\partial_+\phi)^2(\partial_-\phi)^2 + \partial_+^2\phi\partial_-^2\phi)\sinh[2\phi] - \\
&\quad - \frac{1}{2}(\partial_+^2\phi(\partial_-\phi)^2 + (\partial_+\phi)^2\partial_-^2\phi)\cosh[2\phi] \\
\partial_{\pm 3}(\partial_{\mp}\phi) &= (\partial_{\pm}\phi)^2\sinh[2\phi] - \partial_{\pm}^2\phi\cosh[2\phi] \\
\partial_{\pm 3}(\partial_{\mp}\phi)^2 &= 2\partial_{\mp}\phi(\partial_{\pm}\phi)^2\sinh[2\phi] - 2\partial_{\mp}\phi\partial_{\pm}^2\phi\cosh[2\phi] \\
\partial_{\pm 3}(\partial_{\mp}^2\phi) &= 4\partial_{\pm}\phi + 2\partial_{\mp}\phi(\partial_{\pm}\phi)^2\cosh[2\phi] - 2\partial_{\mp}\phi\partial_{\pm}^2\phi\sinh[2\phi]
\end{aligned} \tag{2.7.9}$$

and describe the behavior of the ϕ -‘descendants’ $\partial_{\pm}\phi$, $(\partial_{\pm}\phi)^2$ and $\partial_{\pm}^2\phi$ in terms of opposite $t_{\mp 3}$ times. The equation (2.7.9) can be obtained alternatively by using the basic non-linear relations $\partial_{\pm 3}\phi = \frac{1}{4}\partial_{\pm}^3\phi - \frac{1}{2}(\partial_{\pm}\phi)^3$, and the lowest relativistic equation $\partial_+\partial_-\phi = -2\sinh[2\phi]$. The whole set of equations associated to the times $(t_{-3}, t_{-1}, t_{+1}, t_{+3})$ is completed by the equations extracted from the relations $[L_+, L_{-3}] = 0$ and $[L_-, L_{+3}] = 0$. They are given by the sinh-Gordon equation itself, the equations on the third line of (2.7.9) and

$$\begin{aligned}
4\partial_{\mp}\phi &= \partial_{\pm}(\partial_{\mp}\phi)^2\sinh[2\phi] - \partial_{\pm}\partial_{\mp}^2\phi\cosh[2\phi] \\
0 &= \partial_{\pm}\partial_{\mp}^2\phi\sinh[2\phi] - \partial_{\pm}(\partial_{\mp}\phi)^2\cosh[2\phi].
\end{aligned} \tag{2.7.10}$$

Now take $K^{\pm(n)} = E^{(\pm n)}$ in (2.2.1) and conjugate them with the grading operator Q as follows

$$E'^{(\pm n)} = \exp(\alpha Q)E^{(\pm n)}\exp(-\alpha Q) = \exp(\pm n)E^{(\pm n)} = \Lambda^{\pm n}E^{(\pm n)},$$

where $\Lambda = \exp(\alpha)$. The equations are invariant under these rescalings and the Lorentz transformations, i.e $x'^{\pm} = \Lambda^{\pm 1}x^{\pm}$ can be generalized to the whole set of flows by taking $t'_{\pm n} = \Lambda^{\pm n}t_{\pm n}$.

To find the 1-Soliton solution of the equation (2.7.9) we use the Dressing method, see [17] for details. The four vacuum Lax operators involved are

$$L_{\pm} = \partial_{\pm} \pm E_{\pm}^{(\pm 1)}, \quad L_{\pm 3} = \partial_{\pm 3} \pm E^{(\pm 3)}$$

and the zero curvature conditions imply they are pure gauge $A_i^V = T_0^{-1} \partial_i T_0$. Hence we have as usual

$$\begin{aligned} A_+^V &= E_+^{(+1)} - x^- C, \quad A_-^V = -E_-^{(-1)} \\ A_{+3}^V &= E^{(+3)} - 3t_{-3} C, \quad A_{-3}^V = -E^{(-3)} \\ T_0 &= \exp \left(t_{+3} E^{(+3)} + x^+ E_+^{(+1)} \right) \exp \left(-x^- E_-^{(-1)} - t_{-3} E^{(-3)} \right), \end{aligned}$$

where we have used $[E^{(m)}, E^{(n)}] = \frac{1}{2}(m-n)\delta_{m+n,0}C$ for (m, n) odd integers. The Dressing of a vacuum Lax connections A_i^V is the gauge transformation $A_i = (\Theta_\pm)^{-1} A_i^V \Theta_\pm + (\Theta_\pm)^{-1} \partial_i \Theta_\pm$, satisfying $\Theta_- \Theta_+^{-1} = T_0^{-1} g T_0$, where g is an arbitrary constant group element. Assuming that $\Theta_-^{-1} = e^{p(-1)} e^{p(-2)} \dots$, $\Theta_+^{-1} = e^{q(0)} e^{q(1)} e^{q(2)} \dots$, where $p(-i)$ and $q(i)$ are linear combinations of grade $(-i)$ and $(+i)$ respectively, the zero grade component of A_i leads to the solution

$$e^{q(0)} = B^{-1} e^{-\nu C}, \quad B = \exp(\phi H).$$

From this we have

$$\langle \lambda' | B^{-1} | \lambda \rangle e^{-\nu} = \langle \lambda' | T_0^{-1} g T_0 | \lambda \rangle,$$

where $|\lambda\rangle$ and $\langle \lambda'|$ are annihilated by the grade $(-i)$ and (i) generators. Taking the highest weight states $|\lambda_i\rangle$, $i = 0, 1$ of $sl(2)^{(1)}$ we get the following tau functions

$$\begin{aligned} \tau_1 &= e^{-\phi-\nu} = \langle \lambda_1 | T_0^{-1} g T_0 | \lambda_1 \rangle \\ \tau_0 &= e^{-\nu} = \langle \lambda_0 | T_0^{-1} g T_0 | \lambda_0 \rangle. \end{aligned}$$

The solitonic specialization corresponds to the situation when g is given by the exponential of an eigenvalue $F(z)$, $z \in \mathbb{C}$ of the operator $adE^{(k)}$. In this case we have

$$[E^{(n)}, F(z)] = -2z^n F(z)$$

where $F(z)$ is the vertex operator

$$F(z) = \sum_{n=-\infty}^{+\infty} \left[\left(M_1^{(2n)} - \frac{1}{2} \delta_{n,0} C \right) z^{-2n} + M_2^{(2n+1)} z^{-2n-1} \right].$$

With the following result

$$\begin{aligned} T_0^{-1}gT_0 &= \exp\{\rho(z)F(z)\} \\ \rho(\gamma) &= \exp 2\{-t_{-3}z^{-3} - x^-z^{-1} + x^+z + t_{+3}z^3\}, \end{aligned} \quad (2.7.11)$$

we get the 1-soliton solution depending on the first four times of the hierarchy

$$\phi(t_{-3}, x^-, x^+, t_{+3}) = -\log\left(\frac{\tau_1}{\tau_0}\right) \quad (2.7.12)$$

where $\tau_1 = 1 + \frac{b}{2}\rho(z)$, $\tau_0 = 1 - \frac{b}{2}\rho(z)$. The field (2.7.12) is a simultaneous solution of the Bosonic limits (with $\mu = 1$) of the equations (2.7.2), (2.7.6), (2.7.8) and the whole set (2.7.9) and (2.7.10). The interesting point is that the second and third terms of the RHS of (2.7.9) cancel each other and the field ϕ has to obey

$$\partial_{+3}\partial_{-3}\phi = -2\sinh[2\phi],$$

which is the case. For higher grade times the extension in (2.7.11) is direct. At this point we can notice that each chirality of the extended hierarchy is attached separately to the poles $z = +\infty$ (positive flows) and $z = 0$ (negative flows) of the spectral parameter z , see [24]. This is exactly the pole structure entering the definition of the sinh-Gordon Lax pair.

We end by giving the Fermionic currents for the times $t_{\pm 3}$ and $D^{(+1/2)}$. After a lengthy computation, the equation (2.5.2) become $\partial_{+3}J_{-3}^{(+1/2)} + \partial_{-3}J_{+3}^{(+1/2)} = 0$ with current components

$$\begin{aligned} -\frac{1}{8}j_{+3}^{(+1/2)} &= \partial_{-}\phi\partial_{-}\psi_{-} - \psi_{-}\partial_{-}^2\phi \\ -\frac{1}{8}j_{-3}^{(+1/2)} &= -2\psi_{+}\sinh[\phi] - \left(\frac{1}{2}\partial_{+}^2\psi_{+}\partial_{+}\phi - \psi_{+}(\partial_{+}\phi)^3 - \frac{1}{2}\partial_{+}\psi_{+}\partial_{+}^2\phi\right)\cosh[\phi]. \end{aligned}$$

2.7.2 The N=(2,2) Landau-Ginzburg Toda model.

Take the $psl(2, 2)_{[3]}^{(2)}$ superalgebra (see Appendix A). The Lagrangian is given by

$$L = -\frac{k}{2\pi} \left[\begin{aligned} & \partial_+ \phi_1 \partial_- \phi_1 + \partial_+ \phi_3 \partial_- \phi_3 + \\ & \psi_1 \partial_+ \psi_1 + \psi_3 \partial_+ \psi_3 + \bar{\psi}_2 \partial_- \bar{\psi}_2 + \bar{\psi}_4 \partial_- \bar{\psi}_4 - V \end{aligned} \right]$$

$$V = \left(\begin{aligned} & 2 \cosh[2\phi_1] - 2 \cos[2\phi_3] - 4 (\psi_1 \bar{\psi}_2 + \psi_3 \bar{\psi}_4) \cosh[\phi_1] \cos[\phi_3] + \\ & + 4 (\psi_1 \bar{\psi}_4 - \psi_3 \bar{\psi}_2) \sinh[\phi_1] \sin[\phi_3] \end{aligned} \right).$$

Taking $D^{(+1/2)} = D_1^{(+1/2)} + D_2^{(+1/2)}$ with $D_1^{(+1/2)} = \epsilon_1 F_1^{(+1/2)}$, $D_2^{(+1/2)} = \epsilon_2 F_4^{(+1/2)}$ and $D^{(-1/2)} = D_1^{(-1/2)} + D_2^{(-1/2)}$ with $D_1^{(-1/2)} = \bar{\epsilon}_1 F_2^{(-1/2)}$, $D_2^{(-1/2)} = \bar{\epsilon}_2 F_3^{(-1/2)}$, the supersymmetry transformations (2.3.1) and (2.3.2) with $\psi_i, \bar{\psi}_i \rightarrow \frac{1}{2}\psi_i, \frac{1}{2}\bar{\psi}_i$ are

$$\begin{aligned} \partial_{+1/2} \phi_1 &= +2 (\epsilon_1 \bar{\psi}_4 - \epsilon_2 \bar{\psi}_2) & , & \quad \partial_{+1/2} \phi_3 = -2 (\epsilon_2 \bar{\psi}_4 + \epsilon_1 \bar{\psi}_2) & (2.7.13) \\ \partial_{+1/2} \bar{\psi}_2 &= +(\epsilon_2 \partial_+ \phi_1 + \epsilon_1 \partial_+ \phi_3) & , & \quad \partial_{+1/2} \bar{\psi}_4 = -(\epsilon_1 \partial_+ \phi_1 - \epsilon_2 \partial_+ \phi_3) \\ \partial_{+1/2} \psi_1 &= -2\epsilon_1 \cosh[\phi_1] \sin[\phi_3] - 2\epsilon_2 \sinh[\phi_1] \cos[\phi_3] \\ \partial_{+1/2} \psi_3 &= +2\epsilon_1 \sinh[\phi_1] \cos[\phi_3] - 2\epsilon_2 \cosh[\phi_1] \sin[\phi_3] \\ \partial_{-1/2} \phi_1 &= +2 (\bar{\epsilon}_1 \psi_3 - \bar{\epsilon}_2 \psi_1) & , & \quad \partial_{-1/2} \phi_3 = +2 (\bar{\epsilon}_1 \psi_1 + \bar{\epsilon}_2 \psi_3) \\ \partial_{-1/2} \psi_1 &= +(\bar{\epsilon}_2 \partial_- \phi_1 - \bar{\epsilon}_1 \partial_- \phi_3) & , & \quad \partial_{-1/2} \psi_3 = -(\bar{\epsilon}_1 \partial_- \phi_1 + \bar{\epsilon}_2 \partial_- \phi_3) \\ \partial_{-1/2} \bar{\psi}_2 &= -2\bar{\epsilon}_1 \cosh[\phi_1] \sin[\phi_3] + 2\bar{\epsilon}_2 \sinh[\phi_1] \cos[\phi_3] \\ \partial_{-1/2} \bar{\psi}_4 &= -2\bar{\epsilon}_1 \sinh[\phi_1] \cos[\phi_3] - 2\bar{\epsilon}_2 \cosh[\phi_1] \sin[\phi_3]. \end{aligned}$$

We can check (2.3.5) by applying (2.7.13) twice giving

$$\begin{aligned} [\partial_{+1/2}, \partial'_{+1/2}] &= +4 (\epsilon_1 \epsilon'_1 + \epsilon_2 \epsilon'_2) \partial_+ \\ [\partial_{-1/2}, \partial'_{-1/2}] &= +4 (\bar{\epsilon}_1 \bar{\epsilon}'_1 + \bar{\epsilon}_2 \bar{\epsilon}'_2) \partial_- \\ [\partial_{+1/2}, \partial_{-1/2}] &= 0. \end{aligned}$$

We have four real supercharges $N = (2, 2)$ because $\dim \mathcal{K}_F^{(\pm 1/2)} = 2$. They are

extracted from (2.6.3) and (2.6.4)

$$\text{Flow } \partial_{+1/2} : Q^{(-1/2)} = Q_+^1 F_2^{-1/2} + Q_+^2 F_3^{-1/2}$$

$$Q_+^1 = \int dx^1 (\bar{\psi}_2 \partial_+ \phi_3 - \bar{\psi}_4 \partial_+ \phi_1 - 2\psi_1 \cosh[\phi_1] \sin[\phi_3] + 2\psi_3 \sinh[\phi_1] \cos[\phi_3])$$

$$Q_+^2 = \int dx^1 (\bar{\psi}_4 \partial_+ \phi_3 + \bar{\psi}_2 \partial_+ \phi_1 - 2\psi_1 \sinh[\phi_1] \cos[\phi_3] - 2\psi_3 \cosh[\phi_1] \sin[\phi_3])$$

$$\text{Flow } \partial_{-1/2} : Q^{(+1/2)} = Q_-^1 F_1^{+1/2} + Q_-^2 F_4^{+1/2}$$

$$Q_-^1 = \int dx^1 (\psi_1 \partial_- \phi_3 + \psi_3 \partial_- \phi_1 + 2\bar{\psi}_2 \cosh[\phi_1] \sin[\phi_3] + 2\bar{\psi}_4 \sinh[\phi_1] \cos[\phi_3])$$

$$Q_-^2 = \int dx^1 (\psi_3 \partial_- \phi_3 - \psi_1 \partial_- \phi_1 - 2\bar{\psi}_2 \sinh[\phi_1] \cos[\phi_3] + 2\bar{\psi}_4 \cosh[\phi_1] \sin[\phi_3]) .$$

Now introduce the complex fields

$$\psi_- = \psi_1 + i\psi_3, \psi_+ = -\bar{\psi}_2 + i\bar{\psi}_4, \phi = \phi_3 + i\phi_1$$

and the superpotential $W(\phi) = 2\mu \cos \phi$ in order to write the lagrangian in a more familiar form. Then,

$$L = -\frac{k}{2\pi} [\partial_+ \phi \partial_- \phi^* + \psi_-^* \partial_+ \psi_- + \psi_+^* \partial_- \psi_+ - V] \quad (2.7.14)$$

$$V = |W'(\phi)|^2 - [W''(\phi) \psi_- \psi_+ + W^{**}(\phi^*) \psi_-^* \psi_+^*],$$

where $|W'(\phi)|^2 = 2\mu^2 \cosh[2\phi_1] - 2\mu^2 \cos[2\phi_3]$. This Lagrangian is invariant under the common $N = (2, 2)$ superspace transformations for a complex chiral Bosonic superfield. In terms of the new complex fields we have for (2.7.13) with $\epsilon_- = -(\epsilon_1 + i\epsilon_2)$ and $\epsilon_+ = \bar{\epsilon}_1 - i\bar{\epsilon}_2$ that

$$\begin{aligned} \partial_{+1/2} \phi &= -2\epsilon_- \psi_+ & \partial_{-1/2} \phi &= +2\epsilon_+ \psi_- \\ \partial_{+1/2} \psi_+ &= +\epsilon_-^* \partial_+ \phi & \partial_{-1/2} \psi_+ &= -i\epsilon_+ W'^*(\phi^*) \\ \partial_{+1/2} \psi_- &= -i\epsilon_- W'^*(\phi^*) & \partial_{-1/2} \psi_- &= -\epsilon_+^* \partial_- \phi \end{aligned} \quad (2.7.15)$$

plus their conjugates. Define now the following complex combinations of the supercharges $\frac{1}{2}Q_\pm = \mp Q_\pm^1 + iQ_\pm^2$

$$Q_\pm = 2 \int dx^1 (\psi_\pm \partial_\pm \phi^* \mp i\psi_\mp^* W'^*(\phi^*))$$

$$\bar{Q}_\pm = 2 \int dx^1 (\psi_\pm^* \partial_\pm \phi \pm i\psi_\mp W'(\phi)) .$$

Now, with the brackets $\{\phi, \partial_t \phi^*\}_B = \{\phi^*, \partial_t \phi\}_B = +1$ and $\{\psi_+, \psi_+^*\}_B = \{\psi_-, \psi_-^*\}_B = +1/2$ we have with $Q_\pm f = \{Q_\pm, f\}_B$

$$\begin{aligned} Q_+ \phi &= -2\psi_+ & , & \quad \bar{Q}_+ \phi = 0 \\ Q_+ \psi_+ &= 0 & , & \quad \bar{Q}_+ \psi_+ = \partial_+ \phi \\ Q_+ \psi_- &= -iW'^*(\phi^*) & , & \quad \bar{Q}_+ \psi_- = 0 \\ Q_- \phi &= -2\psi_- & , & \quad \bar{Q}_- \phi = 0 \\ Q_- \psi_+ &= iW'^*(\phi^*) & , & \quad \bar{Q}_- \psi_+ = 0 \\ Q_- \psi_- &= 0 & , & \quad \bar{Q}_- \psi_- = \partial_- \phi \end{aligned}$$

plus their conjugates.

Finally, the total variation becomes $\partial = \partial_{+1/2} + \partial_{-1/2} = \epsilon_- Q_+ - \epsilon_+ Q_- + \epsilon_-^* \bar{Q}_+ - \epsilon_+^* \bar{Q}_-$ with

$$\begin{aligned} \partial \phi &= +2\epsilon_+ \psi_- - 2\epsilon_- \psi_+ & , & \quad \partial \phi^* = +2\epsilon_+^* \psi_-^* - 2\epsilon_-^* \psi_+^* \\ \partial \psi_+ &= +\epsilon_-^* \partial_+ \phi - i\epsilon_+ W'^*(\phi^*) & , & \quad \partial \psi_+^* = +\epsilon_- \partial_+ \phi^* + i\epsilon_+^* W'(\phi) \\ \partial \psi_- &= -\epsilon_+^* \partial_- \phi - i\epsilon_- W'^*(\phi^*) & , & \quad \partial \psi_-^* = -\epsilon_+ \partial_- \phi^* + i\epsilon_-^* W'(\phi) \end{aligned}$$

which are the usual $N = (2, 2)$ supersymmetry transformations. As in the case of the $N = (1, 1)$ model, we expect the existence of higher Fermionic supercharges for this model.

Remark 7 *The action (2.7.14) is a Landau-Ginzburg model on a flat non-compact trivial Calabi-Yau manifold X , i.e $X = \mathbb{C}$. As is well known this model is B -twistable. It would be interesting to see the relation of its chiral ring and the chiral ring of a twisted (possibly the A twist) topological version of a superstring on $AdS^2 \times S^2$ where the action (2.7.14) is extracted as a Polhmeyer reduction [18]. This would be in principle, a simple way to test up to what point the Polhmeyer reduction can be understood as a full equivalence of quantum field theories, at least at the level of ground states, BPS solitons etc. It would be also interesting to trace back the role played by the extended conformal symmetries of $\mathcal{W}$ -type in terms of sigma model variables.*

Remark 8 *In the relativistic sector of the AKNS hierarchy which is associated to the Homogeneous gradation, the Toda potential has a symmetry ,i.e $\mathcal{K}_B^{(0)} \neq \emptyset$. In this case further reduction of the model can be performed, eliminating the flat directions. This is done by coupling to a quadratic gauge field A and is equivalent to the introduction of singular metrics defining Singular Toda models. The minimal coupling of the fermions with the gauge field results in terms proportional to $Q_{\pm}^{(0)}$. After integration of A , the action will have a potential four-Fermion term of the form*

$$V = \left\langle Q_+^{(0)} B Q_-^{(0)} B^{-1} \right\rangle \sim f(B) \psi_+^1 \psi_+^2 \psi_-^1 \psi_-^2,$$

which is possibly related to the curvature of the background metric. This Hierarchy is relevant from the point of view of Polhmeyer reductions where reductions to models inside this Hierarchy are quite common [18].

Chapter 3

Supersymmetric Affine Toda Field Theory Action.

We now show how the action functional governing the equations of motion of the Supersymmetric Affine Toda models can be derived from a gauged 2- loop WZNW action where the fundamental matrix fields takes values on a Twisted Affine Lie superalgebra $\mathfrak{g}^{(2)}$. A systematic construction of the action for a class of supersymmetric integrable models is discussed and explicit examples of the $N = (1, 1)$ and $N = (2, 2)$ super sinh(sine)-Gordon models are discussed. This chapter follows reference [11].

3.1 Hamiltonian reduction of the 2- loop WZNW model.

In order to introduce the field content we identify the $\mathbb{Z}_2$ statistics of the fields with the $\mathbb{Z}_2$ structure of the superalgebra $\mathfrak{g}^{(2)} = \widehat{\mathfrak{g}}_0 \oplus \widehat{\mathfrak{g}}_1$. This means that Bosonic-Fermionic fields will parametrize respectively the even-odd part of the superalgebra with $\Phi_B \in F \otimes \widehat{\mathfrak{g}}_0$ and $\Psi_F \in F \otimes \widehat{\mathfrak{g}}_1$ given by

$$\Phi_B \equiv \phi_i \otimes \widehat{\mathfrak{g}}_0^i, \quad \Psi_F \equiv \psi_k \otimes \widehat{\mathfrak{g}}_1^k, \quad (3.1.1)$$

where F is the field space depending on the whole set of flow parameters $(t_{\pm n}, t_{\pm n/2})$.

We assume that an arbitrary group element is written in terms of a generalized Gauss-type decomposition form of subspaces with a well defined degree. An element g of the supergroup $\widehat{G}$ is then written as follows

$$g = K_{<} \Gamma K_{>} \quad , \quad \Gamma = \Phi B \Psi, \quad (3.1.2)$$

where $K_{<} = \prod_{i \in \mathbb{Z}/2 = -\infty}^{-1} \exp[\widehat{\mathfrak{g}}_i]$, $K_{>} = \prod_{i \in \mathbb{Z}/2 = +1}^{+\infty} \exp[\widehat{\mathfrak{g}}_i]$ and $\Phi = \exp[\psi^{(-1/2)}]$, $B \in G_0$ and $\Psi = \exp[\psi^{(+1/2)}]$. Now consider the following

Proposition 9 *The Affine Super Toda Field theory action for sub-superalgebras satisfying the condition $Q_{0\pm}^2 = 0$ is given by*

$$\begin{aligned} S_{Toda}[B, \psi] = & S_{WZNW}[B] - \frac{k}{4\pi} \int_{\Sigma} \left\langle \psi_+^{(+1/2)} \partial_- \psi_-^{(-1/2)} + \psi_-^{(-1/2)} \partial_+ \psi_+^{(+1/2)} \right\rangle \\ & + \frac{k}{2\pi} \int_{\Sigma} \left\langle E_+^{(+1)} B E_-^{(-1)} B^{-1} + \psi_+^{(+1/2)} B \psi_-^{(-1/2)} B^{-1} \right\rangle. \end{aligned} \quad (3.1.3)$$

Proof. See Appendix C. ■

The arbitrary variation of this action is given by (2.6.2) and the equations of motion corresponds to the supersymmetric version of the Leznov-Saveliev found before in (2.1.10) in terms of Lax operators.

Note from the action (3.1.3) that the Fermionic kernel fields decouple completely and have no dynamics at all. This allows to us promote them to be the constant Grassmann parameters appearing in the global supersymmetry transformations written in (2.3.1) and (2.3.2). This is consistent with the role played by the Fermionic kernel $\mathcal{K}_F^{(\pm 1/2)}$ in defining odd symmetries within the flow approach.

We have obtained in an alternative way the supersymmetric version of the Leznov-Saveliev equations

$$\partial_- (\partial_+ B B^{-1}) = [E_+, B E_- B^{-1}], \quad (3.1.4)$$

supplemented by the conditions $Q_{0\pm}^2 = 0$. This shows the field-independence of the conditions $Q_{0\pm}^2 = 0$. They does not appear in the Hamiltonian reduction and this reflects its pure-algebraic nature. Equations (3.1.4) can also be derived from a 2

Loop WZNW action by considering $\mathbb{Z}$ -gradations and Affine lie algebras $\mathfrak{g}^{(1)}$, as done in [21]. The corresponding action is

$$S_{Toda}[B] = S_{WZNW}[B] + \frac{k}{2\pi} \int_{\Sigma} \left\langle E_+^{(+1)} B E_-^{(-1)} B^{-1} \right\rangle \quad (3.1.5)$$

and from it we can obtain any Bosonic Toda model, i.e, Conformal or Affine. Hence, the action (3.1.3) is the supersymmetric extension of the Bosonic action (3.1.5) deduced in [21].

The equations of motion (2.1.10) are of the same form as the ones introduced in [19] where all the matter fields were Bosonic and $\mathbb{Z}$ - gradations considered with the difference that no analogue for the equations (2.1.9) appears. Here the matter fields are Fermionic and given by $\psi^{(\pm 1/2)}$. We also emphasize on the locality of the action (3.1.3) in contrast to what has been argued in the literature [19], [20], in the sense that the connection between the action and the equations of motion involves a non-local field transformation. Here we can see that the transformation is not only local but involves just a degree change c.f (2.1.5), this means that the transformation is not at the field level but rather at an algebraic one.

Remark 10 *The explicit local expression of the action (3.1.3) allows now to a direct computation of the energy momentum tensor, instead of using a constrained Sugawara form for the "free" currents of the 2-Loop WZNW model. We expect to have not only conformal symmetry but an enhance conformal symmetry, leading to the existence of super $\mathcal{W}$ -algebras as usual when dealing with Toda models, see for example [22] and [23].*

3.2 Examples.

Here we show how to construct the Lagrangians for integrable models with $N = (1, 1)$ and $N = (2, 2)$ supersymmetry.

3.2.1 The $sl(2, 1)$ models with $N = (1, 1)$.

The super sinh-Gordon model was already constructed in section (3.7.1). To get the super sine-Gordon model we use the following parametrization for the superalgebra $sl(2, 1)_{[2]}^{(2)}$ (see Appendix B) which is given by

$$\begin{aligned} B &= \exp[\phi M_2^{(0)}], \quad \psi^{(+1/2)} = \psi_- G_1^{(+1/2)}, \quad \psi^{(-1/2)} = \psi_+ G_2^{(-1/2)} \\ \psi_-^{(-1/2)} &= 2\psi_- G_2^{(-1/2)}, \quad \psi_+^{(+1/2)} = -2\psi_+ G_1^{(+1/2)}. \end{aligned}$$

The Lagrangian density is

$$\begin{aligned} L &= \frac{k}{2\pi} [\partial_+ \phi \partial_- \phi + \psi_- \partial_+ \psi_- + \psi_+ \partial_- \psi_+ + V] \\ V &= 2\mu^2 \cos[2\phi] + 4\mu \psi_+ \psi_- \cos[\phi] \end{aligned} \tag{3.2.1}$$

and the equations of motion given by the $N = 1$ sine-Gordon equations. A similar analysis of the supersymmetry properties of this model follows the same lines of section (3.7.1).

The two Lax pairs for the $sl(2, 1)_{[1]}^{(2)}$ and $sl(2, 1)_{[2]}^{(2)}$ models are explicitly given by

$$\begin{aligned}
A_{+[1]} &= \lambda^0 \begin{pmatrix} -\partial_+\phi & 1 & \bar{\psi} \\ 0 & \partial_+\phi & 0 \\ 0 & -\bar{\psi} & 0 \end{pmatrix} + \lambda^{+1/2} \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & -\bar{\psi} \\ \bar{\psi} & 0 & 2 \end{pmatrix} + \lambda \begin{pmatrix} 0 & 0 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} \\
A_{-[1]} &= \lambda^{-1} \begin{pmatrix} 0 & -e^{2\phi} & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} + \lambda^{-1/2} \begin{pmatrix} -1 & 0 & \psi e^\phi \\ 0 & -1 & 0 \\ 0 & \psi e^\phi & -2 \end{pmatrix} + \\
&\quad + \lambda^0 \begin{pmatrix} 0 & 0 & 0 \\ -e^{-2\phi} & 0 & -\psi e^{-\phi} \\ -\psi e^{-\phi} & 0 & 0 \end{pmatrix} \\
A_{+[2]} &= \lambda^{-1/2} \begin{pmatrix} 0 & -\partial_+\phi & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} + \lambda^0 \begin{pmatrix} 0 & 1 & \bar{\psi} \\ 0 & 0 & 0 \\ 0 & -\bar{\psi} & 0 \end{pmatrix} + \\
&\quad + \lambda^{1/2} \begin{pmatrix} 1 & 0 & 0 \\ \partial_+\phi & 1 & -\bar{\psi} \\ \bar{\psi} & 0 & 2 \end{pmatrix} + \lambda \begin{pmatrix} 0 & 0 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} \\
A_{-[2]} &= \lambda^{-1} \begin{pmatrix} 0 & -\cos[2\phi] & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} + \\
&\quad + \lambda^{-1/2} \begin{pmatrix} -(1 + \sin[2\phi]) & 0 & \psi(\cos[2\phi] - \sin[2\phi]) \\ 0 & -1 + \sin[2\phi] & 0 \\ 0 & \psi(\cos[2\phi] + \sin[2\phi]) & 0 \end{pmatrix} + \\
&\quad + \lambda^0 \begin{pmatrix} 0 & 0 & 0 \\ -\cos[2\phi] & 0 & -\psi(\cos[2\phi] + \sin[2\phi]) \\ 0 & \psi(\cos[2\phi] + \sin[2\phi]) & 0 \end{pmatrix},
\end{aligned}$$

The twisted superalgebra $sl(2, 1)^{(2)}$ supports the two models as the two solutions found to the conditions $Q_\pm^{(0)} = 0$. One model is the field analytic continuation of the other but only at the action level and not in their Lax connections which is where the models are truly specified in terms of their analytical properties [24]. We are

using a two-valued spectral parameter $\lambda^{\pm 1/2}$ due to the gradation used in contrast to the single-valued Lax pairs usually considered so this $\lambda^{\pm 1/2}$ plays a role. The Lax connections are analytically different and have cuts, this opens the question of having a formal Zakharov- Shabat like construction involving a two-valued spectral parameter.

The Affine superalgebra $sl(2, 1)^{(1)}$ does not admit a purely Fermionic simple root system (the longest root being Bosonic). This means that the usual criteria [1], [2] that only Contragredient Lie superalgebras with a purely Fermionic simple root system admit supersymmetric integrable extensions does not apply in our approach. The $sl(2, 1)^{(1)}$ provides a counter-example for this statement, from it we could construct the $N = 1$ models. We expect this construction help us to have a better understanding of the intricate relation between supersymmetric Toda models and Lie superalgebras.

3.2.2 The $psl(2, 2)$ models with $N = (2, 2)$.

There are four superalgebras satisfying the condition $Q_{\pm}^{(0)} = 0$ (see Appendix A). The model associated to $psl(2, 2)_{[3]}^{(2)}$ was already studied in section (3.7.2). Here we write down the Lagrangians corresponding to the other solutions.

For $psl(2, 2)_{[1]}^{(2)}$ we have the parametrization

$$\begin{aligned} B &= \exp[\phi_1 M_1^{(0)} + \phi_3 M_3^{(0)}] \\ \psi^{(+1/2)} &= \psi_1 G_1^{(+1/2)} + \psi_3 G_3^{(+1/2)}, \quad \psi^{(-1/2)} = \bar{\psi}_2 G_2^{(-1/2)} + \bar{\psi}_4 G_4^{(-1/2)} \end{aligned}$$

and the Lagrangian density of the $N = (2, 2)$ sinh-Gordon model (see for instance [28] and [29])

$$\begin{aligned} L &= -\frac{k}{2\pi} \left[\begin{aligned} &\partial_+ \phi_1 \partial_- \phi_1 - \partial_+ \phi_3 \partial_- \phi_3 + \\ &+ \psi_1 \partial_+ \psi_1 + \bar{\psi}_2 \partial_- \bar{\psi}_2 - \psi_3 \partial_+ \psi_3 - \bar{\psi}_4 \partial_- \bar{\psi}_4 - V \end{aligned} \right] \quad (3.2.2) \\ V &= \left(\begin{aligned} &2 \cosh[2\phi_1] - 2 \cosh[2\phi_3] - 4 (\psi_1 \bar{\psi}_2 - \psi_3 \bar{\psi}_4) \cosh[\phi_1] \cosh[\phi_3] - \\ &- 4 (\psi_1 \bar{\psi}_4 - \psi_3 \bar{\psi}_2) \sinh[\phi_1] \sinh[\phi_3] \end{aligned} \right). \end{aligned}$$

For $psl(2, 2)_{[2]}^{(2)}$ we have the parametrization

$$\begin{aligned} B &= \exp[\phi_1 M_2^{(0)} + \phi_3 M_4^{(0)}] \\ \psi^{(+1/2)} &= \psi_1 G_1^{(+1/2)} + \psi_3 G_3^{(+1/2)} \quad , \quad \psi^{(-1/2)} = \bar{\psi}_2 G_2^{(-1/2)} + \bar{\psi}_4 G_4^{(-1/2)} \end{aligned}$$

and the Lagrangian density for the $N = (2, 2)$ sine-Gordon model

$$\begin{aligned} L &= -\frac{k}{2\pi} \left[\begin{aligned} &-\partial_+ \phi_1 \partial_- \phi_1 + \partial_+ \phi_3 \partial_- \phi_3 + \\ &+ \psi_1 \partial_+ \psi_1 + \bar{\psi}_2 \partial_- \bar{\psi}_2 - \psi_3 \partial_+ \psi_3 - \bar{\psi}_4 \partial_- \bar{\psi}_4 - V \end{aligned} \right] \quad (3.2.3) \\ V &= \left(\begin{aligned} &2 \cos[2\phi_1] - 2 \cos[2\phi_3] - 4 (\psi_1 \bar{\psi}_2 - \psi_3 \bar{\psi}_4) \cos[\phi_1] \cos[\phi_3] + \\ &+ 4 (\psi_1 \bar{\psi}_4 - \psi_3 \bar{\psi}_2) \sin[\phi_1] \sin[\phi_3] \end{aligned} \right). \end{aligned}$$

For $psl(2, 2)_{[4]}^{(2)}$ we have essentially the same model as the one obtained with $psl(2, 2)_{[3]}^{(2)}$ with the replacements $\sinh \rightarrow \sin$, $\cosh \rightarrow \cos$ and some sign changes.

3.2.3 Interpolating sine-sinh-Gordon model.

When solving the conditions $Q_{\pm}^{(0)} = 0$ we have found four different Affine subalgebras where models with a local supersymmetry flow can be defined. We have that the degree zero fields in B are respectively associated to the generators $(M_1^{(0)}, M_3^{(0)})$, $(M_2^{(0)}, M_4^{(0)})$, $(M_1^{(0)}, M_4^{(0)})$ and $(M_2^{(0)}, M_3^{(0)})$. If we consider the combinations $(M_1^{(0)}, M_2^{(0)})$ and $(M_3^{(0)}, M_4^{(0)})$ we cannot closed in any subalgebra because the affine indexes are inconsistent for the G and F generators and no new supersymmetric model appear here. However, we can still have a Bosonic integrable model associated to the subalgebras $\{M_1^{(0)}, M_2^{(0)}, K_2^{(0)}\}$ and $\{M_3^{(0)}, M_4^{(0)}, K_3^{(0)}\}$ which are isomorphic. Then, considering the first subalgebra with the following parametrization

$$B = \exp[\phi_1 M_1^{(0)} + \phi_2 M_2^{(0)} + \eta Q + \nu C],$$

we get the following Lagrangian density

$$\begin{aligned} L &= -\frac{k}{2\pi} [\partial_+ \phi_1 \partial_- \phi_1 - \partial_+ \phi_2 \partial_- \phi_2 + \partial_+ \eta \partial_- \nu + \partial_+ \nu \partial_- \eta - V] \quad (3.2.4) \\ V &= 4 \exp[-\eta] \sinh^2 \left(\left| \vec{\phi} \right|^2 \right), \end{aligned}$$

where $\left| \vec{\phi} \right|^2 = (\phi_1^2 - \phi_2^2)^{1/2} = (\Phi_+ \Phi_-)^{1/2}$ with $\Phi_{\pm} = \phi_1 \pm \phi_2$. The Lax pair for this model is

$$\begin{aligned} A_+ &= E_+^{(+1)} - \frac{1}{2} (\partial_+ \Phi_+ + \partial_+ \Phi_-) M_1^{(0)} - \frac{1}{2} (\partial_+ \Phi_+ - \partial_+ \Phi_-) M_2^{(0)} - \partial_+ \eta Q - \partial_+ \nu C \\ A_- &= -\exp[-\eta] E_-^{(-1)} + \frac{1}{2} \left(\frac{\exp[-\eta] \sinh \left[2 (\Phi_+ \Phi_-)^{1/2} \right]}{(\Phi_+ \Phi_-)^{1/2}} \right) \left[(\Phi_+ - \Phi_-) M_1^{(0)} + (\Phi_+ + \Phi_-) M_2^{(0)} \right] + \\ &\quad + \exp[-\eta] \left(\cosh \left[2 (\Phi_+ \Phi_-)^{1/2} \right] - 1 \right) K_2^{(0)} \end{aligned}$$

and leads to the equations of motion

$$\begin{aligned} \partial_+ \partial_- \Phi_+ &= 2 \exp[-\eta] \left(\frac{\Phi_+}{\Phi_-} \right)^{1/2} \sinh \left[2 (\Phi_+ \Phi_-)^{1/2} \right] \\ \partial_+ \partial_- \Phi_- &= 2 \exp[-\eta] \left(\frac{\Phi_-}{\Phi_+} \right)^{1/2} \sinh \left[2 (\Phi_+ \Phi_-)^{1/2} \right] \\ \partial_+ \partial_- \nu &= -2 \exp[-\eta] \sinh^2 \left[(\Phi_+ \Phi_-)^{1/2} \right] \\ \partial_+ \partial_- \eta &= 0. \end{aligned} \tag{3.2.5}$$

The interesting feature is that if we take $\phi_2 \rightarrow 0$ we get the Affine sinh-Gordon model

$$\begin{aligned} \partial_+ \partial_- \phi_1 &= 2 \exp[-\eta] \sinh[2\phi_1] \\ \partial_+ \partial_- \nu &= -\exp[-\eta] (\cosh[2\phi_1] - 1) \\ \partial_+ \partial_- \eta &= 0 \end{aligned}$$

and if we take $\phi_1 \rightarrow 0$ we get the Affine sine-Gordon model

$$\begin{aligned} \partial_+ \partial_- \phi_2 &= 2 \exp[-\eta] \sin[2\phi_2] \\ \partial_+ \partial_- \nu &= -\exp[-\eta] (\cos[2\phi_2] - 1) \\ \partial_+ \partial_- \eta &= 0. \end{aligned}$$

Then, the (3.2.4) is a truly interpolating integrable extension of these two well known models. This model is singular in the field space of solutions at $\left| \vec{\phi} \right|^2 = 0$ but not

in the potential term V and it would be interesting to see the behavior of a solution for this equation because in principle, we would have a soliton solution which is, in some sense, Hydrodynamical and Topological at the same time.

Chapter 4

Fermionic Integrable Perturbations.

In this chapter we are not interested in study susy flows but to perturb the constructed model by Fermionic terms in the potentials. We consider a more refine Integrability condition than the one introduce in [11] which apply to Axial and Vector gauged models, opening the possibility to study T-duality. Pure Fermionic theories arises for cosets $sl(p, 1)/sl(p) \otimes u(1)$ when a maximal kernel condition is fulfilled. The integrability condition of such models is discussed and it is shown that the simplest example when $p = 2$ leads to the constrained Bukhvostov-Lipatov, Thirring, scalar massive and pseudo-scalar massless Gross-Neveu models. This chapter follows reference [11].

4.1 Integrability Conditions for $Q_{\pm}^{(0)} \neq 0$.

We now discuss the construction of pure fermionic theories by considering the maximal Kernel condition i.e., $\mathcal{M}_B = \emptyset$. The Bosonic fields in this cases lie in $\mathcal{K}_B$ and may be gauged away by considering the coset G/H_0 , where H_0 is the degree zero subgroup associated to $\mathcal{K}_B$. As a general prototype, consider the super algebra $sl(p, 1)$ with homogeneous gradation $Q = d$ and $E^+ = \lambda_p \cdot H^{(1)}$ (where λ_p is the

p - th fundamental weight) such that

$$\mathcal{K}_B = sl(p) \otimes u(1), \quad \mathcal{M}_B = \mathcal{O}$$

and Fermionic subspace generated by $\{E_{\alpha_p+\dots+\alpha_i}, E_{-(\alpha_p+\dots+\alpha_i)}, i = 1, 2, \dots, p\}$. Now perturb the action (2.6.1) with the following four Fermion potential term $V = \lambda \langle Q_+^{(0)} B Q_-^{(0)} B^{-1} \rangle$. To study the Lagrangian formulation, introduce the Axial/Vector gauged model

$$S = S_{WZNW}[B, A]_{G_0/H_0} - \frac{k}{4\pi} \int_{\Sigma} \langle \psi_+^{(+1/2)} \partial_- \psi_-^{(-1/2)} + \psi_-^{(-1/2)} \partial_+ \psi_+^{(+1/2)} \rangle \quad (4.1.1) \\ + \frac{k}{2\pi} \int_{\Sigma} \langle E_+^{(+1)} B E_-^{(-1)} B^{-1} \rangle + \langle \psi_+^{(+1/2)} B \psi_-^{(-1/2)} B^{-1} \rangle + \lambda \langle Q_+^{(0)} B Q_-^{(0)} B^{-1} \rangle,$$

with

$$S_{WZNW}[B, A]_{G_0/H_0} = S_{WZNW}[B]_{G_0} - \frac{k}{2\pi} \int_{\Sigma} \left\langle \eta A_- \left(\partial_+ B B^{-1} + Q_+^{(0)} \right) + A_+ \left(B^{-1} \partial_- B + Q_-^{(0)} \right) \right. \\ \left. + A_-^0 A_+^0 + \eta A_- B A_+ B^{-1} \right\rangle \\ S_{WZNW}[B]_{G_0} = S_{WZNW}[g] - x \frac{k}{4\pi} \int_{\Sigma} (\partial_+ \eta \partial_- \nu + \partial_+ \nu \partial_- \eta) \\ S_{WZNW}[g] = -\frac{k}{4\pi} \left(\int_{\Sigma} \langle g^{-1} \partial_+ g g^{-1} \partial_- g \rangle - \frac{1}{3} \int_B \langle (g^{-1} dg)^3 \rangle \right).$$

The action is invariant under the transformations

$$\begin{aligned} B' &= \Gamma_- B \Gamma_+ \\ \psi^{(-1/2)'} &= \Gamma_- \psi^{(-1/2)} \Gamma_-^{-1} \\ \psi^{(+1/2)'} &= \Gamma_+^{-1} \psi^{(+1/2)} \Gamma_+ \\ A_-'^0 &= A_-^0 - \eta \gamma_0^{-1} \partial_- \gamma_0 \\ A_+'^0 &= A_+^0 - \gamma_0'^{-1} \partial_+ \gamma_0' \\ A_-' &= \Gamma_- A_- \Gamma_-^{-1} - \eta \partial_- \Gamma_- \Gamma_-^{-1} \\ A_+' &= \Gamma_+^{-1} A_+ \Gamma_+ - \Gamma_+^{-1} \partial_+ \Gamma_+ \\ D_- &= \partial_- + \eta [A_-,] \\ D_+ &= \partial_+ - [A_+,], \end{aligned} \quad (4.1.2)$$

where $\Gamma_- = \gamma_0 \gamma_-$ and $\Gamma_+ = \gamma_+ \gamma'_0$ are both in Ker_B^0 and $A_\pm = A_\pm^0 + N_\pm^0 \in Ker_B^0$ are the gauge fields. The $D_\pm$ are the covariant derivatives $D'_- \psi^{(-1/2)'} = \Gamma_- \left(D_- \psi^{(-1/2)} \right) \Gamma_-^{-1}$, $D'_+ \psi^{(+1/2)'} = \Gamma_+^{-1} \left(D_+ \psi^{(+1/2)} \right) \Gamma_+$. Take $\eta = 1$ in the Axial gauge $\gamma'_0 = \gamma_0$ and $\eta = -1$ in the Vector gauge $\gamma'_0 = \gamma_0^{-1}$.

The classical integrability of the perturbed action (4.1.1) is guaranteed by the following

Proposition 11 *The action (4.1.1) with $B \in \mathcal{K}_B^{(0)}$ is Integrable when $\lambda = +1/2$ and*

$$\begin{aligned} 0 &= F_{+-}^A - D_- (\eta A_+^0 + A_+) \\ 0 &= F_{+-}^A - D_+ (A_-^0 + \eta A_-) \\ 0 &= \left[Q_+^{(0)}, B Q_-^{(0)} B^{-1} \right]. \end{aligned} \tag{4.1.3}$$

Proof. See Appendix C. ■

Note that in this extreme case the equation (2.1.8) is given by $-\partial_+ B B^{-1} = Q_+^{(0)}$, $-B^{-1} \partial_- B = Q_-^{(0)}$, which is some sort of 'classical Bosonization' formula. It corresponds to the A_+ equation of motion in the gauge $A = 0$. We gauge away the Toda field $B \in H_0$ in order to preserve integrability, i.e to get $F_{+-}^{(\pm 1)} = 0$. This can be done in the Axial case $\eta = 1$ by choosing a gauge in which $B' = I$. In this gauge the integrability conditions reduces essentially to the commutativity of the currents $Q_\pm^{(0)}$, i.e to $\left[Q_+^{(0)}, Q_-^{(0)} \right] = 0$. The Vector gauge $\eta = -1$ is not considered at all but it is included just to open the possibility of study T-duality in the usual Axial-Vector sense.

Note that by taking all Fermions to zero we have a Topological WZNW model on the coset H_0/H_0 .

4.2 Examples.

As an illustration consider the simplest case where $\mathfrak{g}^{(2)} = sl(2, 1)^{(2)}$ in the Homogeneous gradation $Q = d$. Let us parametrize

$$\begin{aligned}\psi^{(+1/2)} &= \psi_1 E_{+\alpha_2}^{(+1/2)} + \psi_2 E_{+\alpha_1+\alpha_2}^{(+1/2)} + \psi_3 E_{-\alpha_2}^{(+1/2)} + \psi_4 E_{-\alpha_1-\alpha_2}^{(+1/2)} \\ \psi^{(-1/2)} &= \bar{\psi}_1 E_{+\alpha_2}^{(-1/2)} + \bar{\psi}_2 E_{+\alpha_1+\alpha_2}^{(-1/2)} + \bar{\psi}_3 E_{-\alpha_2}^{(-1/2)} + \bar{\psi}_4 E_{-\alpha_1-\alpha_2}^{(-1/2)}\end{aligned}\quad (4.2.1)$$

from where we evaluate

$$\begin{aligned}Q_-^{(0)} &= (\psi_2 \psi_4) H_1^{(0)} + (\psi_1 \psi_3 + \psi_2 \psi_4) H_2^{(0)} + (\psi_2 \psi_3) E_{+\alpha_1}^{(0)} + (\psi_1 \psi_4) E_{-\alpha_1}^{(0)} \\ Q_-^{(+1/2)} &= \frac{1}{3} [\psi^{(+1/2)}, Q_-^{(0)}] = \frac{2}{3} (\psi_1 \psi_2 \psi_4) E_{+\alpha_2}^{(+1/2)} \\ &\quad - \frac{2}{3} (\psi_1 \psi_2 \psi_3) E_{+\alpha_1+\alpha_2}^{(+1/2)} + \frac{2}{3} (\psi_2 \psi_3 \psi_4) E_{-\alpha_2}^{(+1/2)} \\ Q_-^{(+1)} &= \frac{1}{4} [\psi^{(+1/2)}, Q_-^{(+1/2)}] = \frac{1}{6} (\psi_1 \psi_2 \psi_3 \psi_4) (H_1^{(+1)} + 3H_2^{(+1)})\end{aligned}\quad (4.2.2)$$

and similar for $Q_+^{(0)}$, $Q_+^{(-1/2)}$ and $Q_+^{(-1)}$ in terms $\bar{\psi}_i$ fields. The potential term in the Lagrangian decomposes according to the number of fermions involving the following individual contributions,

$$\begin{aligned}\langle Q_+^{(0)}, Q_-^{(0)} \rangle &= -(\bar{\psi}_2 \bar{\psi}_4) (\psi_1 \psi_3) - (\bar{\psi}_1 \bar{\psi}_3) (\psi_2 \psi_4) \\ &\quad + (\bar{\psi}_2 \bar{\psi}_3) (\psi_1 \psi_4) + (\bar{\psi}_1 \bar{\psi}_4) (\psi_2 \psi_3) \\ \langle Q_+^{(-1/2)} Q_-^{(+1/2)} \rangle &= -\frac{4}{9} ((\bar{\psi}_1 \bar{\psi}_2 \bar{\psi}_4) (\psi_2 \psi_3 \psi_4) - (\bar{\psi}_2 \bar{\psi}_3 \bar{\psi}_4) (\psi_1 \psi_2 \psi_4)) \\ \langle Q_+^{(-1)} Q_-^{(+1)} \rangle &= -\frac{1}{9} (\bar{\psi}_1 \bar{\psi}_2 \bar{\psi}_3 \bar{\psi}_4) (\psi_1 \psi_2 \psi_3 \psi_4).\end{aligned}\quad (4.2.3)$$

The Lagrangian density with normalized coupling constants * then becomes

$$\begin{aligned}
-\frac{2\pi}{k}L_{p=2} = & \psi_1\partial_+\psi_3 - \bar{\psi}_1\partial_-\bar{\psi}_3 + \psi_2\partial_+\psi_4 - \bar{\psi}_2\partial_-\bar{\psi}_4 \\
& - \bar{\psi}_1\psi_3 + \bar{\psi}_3\psi_1 - \bar{\psi}_2\psi_4 + \bar{\psi}_4\psi_2 + \bar{\psi}_2\bar{\psi}_4\psi_1\psi_3 \\
& + \bar{\psi}_1\bar{\psi}_3\psi_2\psi_4 - \bar{\psi}_2\bar{\psi}_3\psi_1\psi_4 - \bar{\psi}_1\bar{\psi}_4\psi_2\psi_3 \\
& - \frac{4}{9}\bar{\psi}_1\bar{\psi}_2\bar{\psi}_4\psi_2\psi_3\psi_4 + \frac{4}{9}\bar{\psi}_2\bar{\psi}_3\bar{\psi}_4\psi_1\psi_2\psi_4 \\
& - \frac{1}{9}\bar{\psi}_1\bar{\psi}_2\bar{\psi}_3\bar{\psi}_4\psi_1\psi_2\psi_3\psi_4
\end{aligned} \tag{4.2.4}$$

which can be put in the Dirac form

$$\begin{aligned}
L = & \bar{\Psi}_D (i\gamma^\mu\partial_\mu - 1) \Psi_D + \bar{\Phi}_D (i\gamma^\mu\partial_\mu - 1) \Phi_D \\
& + \frac{5}{4} (\bar{\Psi}_D\gamma^\mu\Psi_D) (\bar{\Phi}_D\gamma_\mu\Phi_D) + \frac{5}{4} (\bar{\Psi}_D\Psi_D) (\bar{\Phi}_D\Phi_D) \\
& - \frac{5}{4} (\bar{\Psi}_D\gamma^5\Psi_D) (\bar{\Phi}_D\gamma^5\Phi_D) - \frac{1}{9} (\bar{\Phi}_D\gamma^\mu\Phi_D) (\bar{\Phi}_D\gamma_\mu\Phi_D) (\bar{\Psi}_D\Psi_D) \\
& + \frac{1}{144} (\bar{\Phi}_D\gamma^\mu\Phi_D) (\bar{\Phi}_D\gamma_\mu\Phi_D) (\bar{\Psi}_D\gamma^\nu\Psi_D) (\bar{\Psi}_D\gamma_\nu\Psi_D)
\end{aligned}$$

where the complex Dirac spinor components are defined by $\Psi_{D1} = i\psi_3$, $\Psi_{D2} = -\bar{\psi}_3$, $\Psi_{D1}^+ = -\psi_1$, $\Psi_{D2}^+ = -i\bar{\psi}_1$ and $\Phi_{D1} = i\psi_4$, $\Phi_{D2} = -\bar{\psi}_4$, $\Phi_{D1}^+ = -\psi_2$, $\Phi_{D2}^+ = -i\bar{\psi}_2$.

The constraints use below in the examples are such that cancel the higher grade currents $Q_{\mp}^{(\pm 1/2)}$, $Q_{\mp}^{(\pm 1)}$ and satisfy the integrability condition $[Q_+^{(0)}, Q_-^{(0)}] = 0$ allowing to use the proposition (4.1.3). This construction unveil the algebraic structure behind the classical integrability of the well known models treated below.

4.2.1 Constrained Bukhvostov-Lipatov model.

Take the constraints $\psi_2\psi_3 = \psi_1\psi_4 = 0$ and $\bar{\psi}_2\bar{\psi}_3 = \bar{\psi}_1\bar{\psi}_4 = 0$. Under such conditions we find from eqns. (4.2.2) and (4.2.3),

$$\begin{aligned}
Q_-^{(0)} &= \psi_2\psi_4 H_1^{(0)} + (\psi_1\psi_3 + \psi_2\psi_4) H_2^{(0)} \\
Q_+^{(0)} &= \bar{\psi}_2\bar{\psi}_4 H_1^{(0)} + (\bar{\psi}_1\bar{\psi}_3 + \bar{\psi}_2\bar{\psi}_4) H_2^{(0)}
\end{aligned} \tag{4.2.5}$$

*Coupling constants g and μ may be introduced by $\psi^{(\pm 1/2)} \rightarrow g\psi^{(\pm 1/2)}$ and $E_{\pm}^{(\pm 1)} \rightarrow \mu E_{\pm}^{(\pm 1)}$.

and

$$\langle Q_+^{(0)}, Q_-^{(0)} \rangle = -\bar{\psi}_2 \bar{\psi}_4 \psi_1 \psi_3 - \bar{\psi}_1 \bar{\psi}_3 \psi_2 \psi_4.$$

The Lagrangian density becomes [27],

$$\begin{aligned} L = & \bar{\Psi}_D (i\gamma^\mu \partial_\mu - 1) \Psi_D + \bar{\Phi}_D (i\gamma^\mu \partial_\mu - 1) \Phi_D \\ & + \frac{5}{4} (\bar{\Psi}_D \gamma^\mu \Psi_D) (\bar{\Phi}_D \gamma_\mu \Phi_D). \end{aligned} \quad (4.2.6)$$

4.2.2 Thirring model.

Take now the constraints $\psi_1 = -\psi_4$, $\psi_2 = -\psi_3$, $\bar{\psi}_1 = \bar{\psi}_4$ and $\bar{\psi}_2 = \bar{\psi}_3$. We find

$$Q_-^{(0)} = \psi_2 \psi_4 H_1^{(0)}, \quad Q_+^{(0)} = \bar{\psi}_2 \bar{\psi}_4 H_1^{(0)}$$

yielding the Lagrangian density

$$L = \bar{\Psi}_D (i\gamma^\mu \partial_\mu - 1) \Psi_D + (\bar{\Psi}_D \gamma^\mu \Psi_D) (\bar{\Psi}_D \gamma_\mu \Psi_D).$$

which correspond to the Thirring model. From the constraints above we parametrize the Fermion fields as

$$\begin{aligned} \psi^{(+1/2)} &= \psi_1 g_1^{(+1/2)} + \psi_2 g_2^{(+1/2)} \\ \psi^{(-1/2)} &= \bar{\psi}_1 g_1^{(-1/2)} + \bar{\psi}_2 g_2^{(-1/2)}, \end{aligned}$$

where

$$\begin{aligned} g_1^{(+1/2)} &= E_{\alpha_2}^{(+1/2)} - E_{-\alpha_1 - \alpha_2}^{(+1/2)}, \quad g_2^{(+1/2)} = -E_{-\alpha_2}^{(+1/2)} + E_{\alpha_1 + \alpha_2}^{(+1/2)} \\ g_1^{(-1/2)} &= E_{\alpha_2}^{(-1/2)} + E_{-\alpha_1 - \alpha_2}^{(-1/2)}, \quad g_2^{(-1/2)} = E_{-\alpha_2}^{(-1/2)} + E_{\alpha_1 + \alpha_2}^{(-1/2)} \\ f_1^{(+1/2)} &= E_{\alpha_2}^{(+1/2)} + E_{-\alpha_1 - \alpha_2}^{(+1/2)}, \quad f_2^{(+1/2)} = -E_{-\alpha_2}^{(+1/2)} - E_{\alpha_1 + \alpha_2}^{(+1/2)} \\ f_1^{(-1/2)} &= E_{\alpha_2}^{(-1/2)} - E_{-\alpha_1 - \alpha_2}^{(-1/2)}, \quad f_2^{(-1/2)} = E_{-\alpha_2}^{(-1/2)} - E_{\alpha_1 + \alpha_2}^{(-1/2)}. \end{aligned}$$

We have introduced the orthogonal complements f 's for the g 's under $\langle * \rangle$. These elements together the Bosonic ones $K_1^{(\pm 1)}$, $K_2^{(\pm 1)}$, $M_1^{(0)}$, $M_2^{(\pm 1)}$ (as defined in (A.0.2)) close into the subalgebra where the Thirring model is defined. A similar situation occurs for the models below.

4.2.3 Pseudo-scalar, massless Gross-Neveu model.

Consider the constraints $\bar{\psi}_2 = \bar{\psi}_4$, $\bar{\psi}_1 = \bar{\psi}_3$, $\psi_2 = \psi_4$ and $\psi_1 = \psi_3$. We get

$$Q_-^{(0)} = -\psi_1\psi_2H_1^{(0)}, \quad Q_+^{(0)} = -\bar{\psi}_1\bar{\psi}_2H_1^{(0)}$$

yielding the Gross-Neveu model

$$L = \bar{\Psi}_D (i\gamma^\mu \partial_\mu) \Psi_D + \bar{\Phi}_D (i\gamma^\mu \partial_\mu) \Phi_D - (\bar{\Psi}_D \gamma^5 \Psi_D) (\bar{\Phi}_D \gamma^5 \Phi_D).$$

4.2.4 Scalar, massive Gross-Neveu model.

Consider $\bar{\psi}_2 = \bar{\psi}_4$, $\bar{\psi}_1 = \bar{\psi}_3$, $\psi_2 = -\psi_4$ and $\psi_1 = -\psi_3$. We find

$$Q_-^{(0)} = \psi_1\psi_2(E_{+\alpha_1}^{(0)} - E_{-\alpha_1}^{(0)}), \quad Q_+^{(0)} = \bar{\psi}_1\bar{\psi}_2(E_{+\alpha_1}^{(0)} - E_{-\alpha_1}^{(0)})$$

yielding the Gross-Neveu model

$$L = \bar{\Psi}_D (i\gamma^\mu \partial_\mu - 1) \Psi_D + \bar{\Phi}_D (i\gamma^\mu \partial_\mu - 1) \Phi_D - (\bar{\Psi}_D \Psi_D) (\bar{\Phi}_D \Phi_D).$$

Chapter 5

Conclusions and Future Directions.

Based on the gauged WZNW model, we have established a general framework for constructing systematically the action for a class of $N = 1, 2$ supersymmetric relativistic integrable models of sinh(sine)-Gordon type. It is important to stress that the field content of the theory is established by the group theoretic structure of a coset $G/\mathcal{K}$ and the latter by the decomposition of a twisted affine Kac-Moody superalgebra.

Another important achievement is the construction of pure Fermionic theories by considering the coset $sl(p, 1)/sl(p) \otimes U(1)$ where all Bosonic fields lie in the maximal kernel subalgebra $\mathcal{K} = sl(p) \otimes U(1) \in sl(p, 1)$. General integrability conditions were discussed and explicit examples for $p = 2$ were constructed.

We have shown how to embed the common notion of superspace into the flow algebra spanned by set of times $(t_{-1}, t_{-1/2}, t_{+1/2}, t_{+1}) \subset (t_{\pm 1/2}, t_{\pm 1}, t_{\pm 3/2}, \dots)$ and also shown that the higher supersymmetry flows are essentially non-local. In particular, when the $(t_{-1}, t_{-1/2}, t_{+1/2}, t_{+1})$ flows are supplemented by the conditions $Q_{\pm}^{(0)} = 0$, we remove the non-local part of the SUSY transformations $\partial_{\pm 1/2}$ and the integrable model is then restricted to a reduced manifold spanned by the invariant subalgebra of a reductive automorphism τ_{red} . The reduction imply a well defined connection between the Dressing elements and physical degrees of freedom c.f (2.1.8), as well as

the right terms in the potential appearing in the action functional by truncating it at the second term. We do not supersymmetrized the field 'angles' that parametrize the group elements $\widehat{G}$ as is usually done in the literature.

What remain to be done is a general proof of the statement that the introduction of τ_{red} is indeed responsible for the locality ($Q_{\pm}^{(0)} = 0$) of the lowest supersymmetry flows ($\partial_{\pm 1/2}$) and the explicit construction of it for bigger superalgebras in the A-series (or other series). The main idea is to set the ground for a super tau-function formulation which we expect to be a natural generalization of the one introduced in [9] for Bosonic Affine Toda models, where an infinite number of conserved charges $Q_n^{\pm}$ where written in terms of the boundary values B of a single tau function τ , in the form $Q_n^{\pm} \sim \partial_{\pm n} \log \tau|_B$. As we are using a Fermionic version of the Toda models coupled to matter fields constructed in [20], it is possible that one has to consider a matrix of tau functions τ_{mn} when solving the whole system (2.1.10). In this case, the interpretation of the single τ function as a classical limit of the partition function of some quantum integrable system will change or will have to be modified in an appropriate way. The point is that quantization can be done, in principle, by quantizing τ_{mn} , i.e by using a quantum group of Dressing transformations.

Appendix A

Superalgebras Solving the Condition $Q_{\pm}^{(0)} = 0$.

The dynkin diagrams corresponds to a distinguished simple root system constructed from the following basis (e_i, ζ_k) of a real pseudo-euclidean $(m+n)$ -dimensional space:

$$e_i \cdot e_j = \delta_{ij}, \quad \zeta_k \cdot \zeta_l = -\delta_{kl}, \quad e_i \cdot \zeta_k = 0,$$

where $\bigcirc$ is a white simple root ($\alpha \cdot \alpha \neq 0$) and $\bigotimes$ is a grey simple root ($\alpha \cdot \alpha = 0$).

The non-degenerate Cartan matrix is given by $K_{ij} = (\alpha_i, \alpha_j)$. Define also

$$\lambda_i = [K^{-1}]_{ij} \alpha_j, \quad (\text{A.0.1})$$

The commutation relations obey:

$$\begin{aligned} [H_i, E_{\pm\alpha_j}] &= \pm K_{ij} E_{\pm\alpha_j} \\ [E_{\alpha_i}, E_{-\alpha_j}] &= \delta_{ij} H_i \\ [H_i, H_j] &= 0. \end{aligned}$$

Remark 12 *These Dynkin diagrams are not the ones corresponding to the Twisted Affine Lie superalgebras $\mathfrak{g}^{(2)}$ considered above, they correspond to classical Lie superalgebras and we consider them because the information they carry is enough for clarifying our purposes.*

A.0.5 The Superalgebra $sl(2, 1)$.

The dynking diagram is $\odot \leftrightarrow \otimes$ with simple roots:

$$\alpha_1 = e_1 - e_2, \quad \alpha_2 = e_2 - \zeta_1$$

and cartan matrix :

$$[K] = \begin{pmatrix} 2 & -1 \\ -1 & 0 \end{pmatrix}, \quad [K^{-1}] = \begin{pmatrix} 0 & -1 \\ -1 & -2 \end{pmatrix}.$$

For this superalgebra we define the gradation $Q = 2d + \frac{1}{2}H_1$ and the constant semisimple elements $E_{\pm}^{(\pm 1)} = \lambda_2.H^{(\pm 1/2)} - \left(E_{\pm\alpha_1}^{(0)} + E_{\mp\alpha_1}^{(\pm 1)}\right)$.

For simplicity we consider the reductive automorphism τ_{red} for this case only. Following [10], take the following extension of the cartan involution on the finite dimensional lie superalgebra $\mathfrak{g}$ given by

$$\begin{aligned} \sigma(E_{\alpha}) &= -E_{-\alpha}, \quad \sigma(H) = -H \quad \text{for } \alpha \in \text{Bosonic} \\ \sigma(E_{\alpha}) &= iE_{-\alpha} \quad \text{for } \alpha \in \text{Fermionic}. \end{aligned}$$

This automorphism is compatible with the Bosonic/Fermionic anti/commutation relations. Now, consider the extension of σ to the infinite-dimensional Loop superalgebra $\tilde{\mathfrak{g}}$ defined by

$$\begin{aligned} \hat{\sigma}(E_{\alpha}^{(n)}) &= -(-1)^{q(E_{\alpha}^{(n)})} E_{-\alpha}^{(n+\eta_{E_{\alpha}})} \quad \text{for } \alpha \in \text{Bosonic} \\ \hat{\sigma}(E_{\alpha}^{(n)}) &= i(-1)^{q(E_{\alpha}^{(n)})} E_{-\alpha}^{(n+\eta_{E_{\alpha}})} \quad \text{for } \alpha \in \text{Fermionic}, \end{aligned}$$

where $q(E_{\alpha}^{(n)})$ is the grade of $E_{\alpha}^{(n)}$, i.e $[Q, E_{\alpha}^{(n)}] = q(E_{\alpha}^{(n)})E_{\alpha}^{(n)}$ and $\eta_{E_{\alpha}}$ is defined by $[\alpha.H, E_{\alpha}] = \eta_{E_{\alpha}}E_{\alpha}$. The term $(-1)^{q(E_{\alpha}^{(n)})}$ corresponds to the mapping $\lambda \rightarrow -\lambda$ used in the case of the Homogeneous gradation. The elements in the decomposition of $\tilde{\mathfrak{g}} = \text{Ker}(adE_+) \oplus \text{Im}(adE_+)$ which are invariant under $\hat{\sigma}$ define the subalgebra $sl(2, 1)_{[1]}^{(2)}$ given below. Now by taking the special value $\lambda = 1$ and exchanging $F_1 \rightarrow F_2$ we can put back the affine indexes consistently and close the relations into another sub-superalgebra, defined by $sl(2, 1)_{[2]}^{(2)}$ below. Then, we have the invariant superalgebras

$sl(2, 1)_{[1]}^{(2)} :$

$$\begin{aligned}
\mathcal{K}_{Bose} &= \left\{ K_1^{(2n+1)} = \lambda_2 H^{(n+1/2)} , K_2^{(2n+1)} = E_{\alpha_1}^{(n)} + E_{-\alpha_1}^{(n+1)} \right\} \\
\mathcal{M}_{Bose} &= \left\{ M_1^{(2n)} = H_1^{(n)} , M_2^{(2n+1)} = E_{\alpha_1}^{(n)} - E_{-\alpha_1}^{(n+1)} \right\} \\
\mathcal{K}_{Fermi} &= \left\{ \begin{aligned} F_1^{(2n+3/2)} &= \left(E_{\alpha_2}^{(n+1)} + E_{\alpha_1+\alpha_2}^{(n+1/2)} \right) + \left(E_{-\alpha_2}^{(n+1/2)} + E_{-\alpha_1-\alpha_2}^{(n+1)} \right) \\ F_2^{(2n+1/2)} &= \left(E_{\alpha_2}^{(n+1/2)} + E_{\alpha_1+\alpha_2}^{(n)} \right) - \left(E_{-\alpha_2}^{(n)} + E_{-\alpha_1-\alpha_2}^{(n+1/2)} \right) \end{aligned} \right\} \\
\mathcal{M}_{Fermi} &= \left\{ \begin{aligned} G_1^{(2n+1/2)} &= \left(E_{\alpha_2}^{(n+1/2)} - E_{\alpha_1+\alpha_2}^{(n)} \right) + \left(E_{-\alpha_2}^{(n)} - E_{-\alpha_1-\alpha_2}^{(n+1/2)} \right) \\ G_2^{(2n+3/2)} &= \left(E_{\alpha_2}^{(n+1)} - E_{\alpha_1+\alpha_2}^{(n+1/2)} \right) - \left(E_{-\alpha_2}^{(n+1/2)} - E_{-\alpha_1-\alpha_2}^{(n+1)} \right) \end{aligned} \right\} .
\end{aligned} \tag{A.0.2}$$

$sl(2, 1)_{[2]}^{(2)} :$

$$\begin{aligned}
\mathcal{K}_{Bose} &= \left\{ K_1^{(2n+1)} = \lambda_2 H^{(n+1/2)} , K_2^{(2n+1)} = E_{\alpha_1}^{(n)} + E_{-\alpha_1}^{(n+1)} \right\} \\
\mathcal{M}_{Bose} &= \left\{ M_1^{(2n+1)} = H_1^{(n+1/2)} , M_2^{(2n)} = E_{\alpha_1}^{(n-1/2)} - E_{-\alpha_1}^{(n+1/2)} \right\} \\
\mathcal{K}_{Fermi} &= \left\{ \begin{aligned} F_1^{(2n+1/2)} &= \left(E_{\alpha_2}^{(n+1/2)} + E_{\alpha_1+\alpha_2}^{(n)} \right) + \left(E_{-\alpha_2}^{(n)} + E_{-\alpha_1-\alpha_2}^{(n+1/2)} \right) \\ F_2^{(2n+3/2)} &= \left(E_{\alpha_2}^{(n+1)} + E_{\alpha_1+\alpha_2}^{(n+1/2)} \right) - \left(E_{-\alpha_2}^{(n+1/2)} + E_{-\alpha_1-\alpha_2}^{(n+1)} \right) \end{aligned} \right\} \\
\mathcal{M}_{Fermi} &= \left\{ \begin{aligned} G_1^{(2n+1/2)} &= \left(E_{\alpha_2}^{(n+1/2)} - E_{\alpha_1+\alpha_2}^{(n)} \right) + \left(E_{-\alpha_2}^{(n)} - E_{-\alpha_1-\alpha_2}^{(n+1/2)} \right) \\ G_2^{(2n+3/2)} &= \left(E_{\alpha_2}^{(n+1)} - E_{\alpha_1+\alpha_2}^{(n+1/2)} \right) - \left(E_{-\alpha_2}^{(n+1/2)} - E_{-\alpha_1-\alpha_2}^{(n+1)} \right) \end{aligned} \right\} .
\end{aligned} \tag{A.0.3}$$

The above subalgebras satisfy the condition $Q_{\pm}^{(0)} = 0$, because the degree zero part of the Bosonic Kernel is empty $\mathcal{K}_{Bose}^{(0)} = \emptyset$.

A.0.6 The Superalgebra $sl(2, 2)$.

The dynkin diagram is $\odot \leftrightarrow \otimes \leftrightarrow \odot$ with simple roots

$$\alpha_1 = e_1 - e_2, \alpha_2 = e_2 - \zeta_1, \alpha_3 = \zeta_1 - \zeta_2.$$

The cartan matrix is singular. For this superalgebra we define the gradation $Q = d + \frac{1}{2}(H_1 + H_3)$ and the constant semisimple element $E_{\pm} = (E_{\alpha_1} + E_{-\alpha_1}) + (E_{\alpha_3} + E_{-\alpha_3}) + I$. We have the sub-superalgebras

$$sl(2, 2)_{[1]}^{(2)} :$$

$$\begin{aligned}
\mathcal{K}_B &= \left\{ K_1^{(2n+1)} = I^{(2n+1)}, \quad K_2^{(2n+1)} = E_{\alpha_1}^{(2n)} + E_{-\alpha_1}^{(2n+2)}, \quad K_3^{(2n+1)} = E_{\alpha_3}^{(2n+2)} + E_{-\alpha_3}^{(2n)} \right\} \\
\mathcal{M}_B &= \left\{ \begin{aligned} M_1^{(2n)} &= H_1^{(2n)}, \quad M_2^{(2n+1)} = E_{\alpha_1}^{(2n)} - E_{-\alpha_1}^{(2n+2)}, \\ M_3^{(2n)} &= H_3^{(2n)}, \quad M_4^{(2n+1)} = E_{\alpha_3}^{(2n+2)} - E_{-\alpha_3}^{(2n)} \end{aligned} \right\} \quad (\text{A.0.4}) \\
\mathcal{K}_F &= \left\{ \begin{aligned} F_1^{(2n+3/2)} &= \left(E_{\alpha_2}^{(2n+3/2)} + E_{\alpha_1+\alpha_2+\alpha_3}^{(2n+3/2)} \right) + \left(E_{\alpha_1+\alpha_2}^{(2n+1/2)} + E_{\alpha_2+\alpha_3}^{(2n+5/2)} \right) + \\ &\quad + \left(E_{-\alpha_2}^{(2n+3/2)} + E_{-\alpha_1-\alpha_2-\alpha_3}^{(2n+3/2)} \right) + \left(E_{-\alpha_1-\alpha_2}^{(2n+5/2)} + E_{-\alpha_2-\alpha_3}^{(2n+1/2)} \right) \\ F_2^{(2n+1/2)} &= \left(E_{\alpha_2}^{(2n+1/2)} + E_{\alpha_1+\alpha_2+\alpha_3}^{(2n+1/2)} \right) + \left(E_{\alpha_1+\alpha_2}^{(2n-1/2)} + E_{\alpha_2+\alpha_3}^{(2n+3/2)} \right) - \\ &\quad - \left(E_{-\alpha_2}^{(2n+1/2)} + E_{-\alpha_1-\alpha_2-\alpha_3}^{(2n+1/2)} \right) - \left(E_{-\alpha_1-\alpha_2}^{(2n+3/2)} + E_{-\alpha_2-\alpha_3}^{(2n-1/2)} \right) \\ F_3^{(2n+3/2)} &= \left(E_{\alpha_2}^{(2n+3/2)} + E_{\alpha_1+\alpha_2+\alpha_3}^{(2n+3/2)} \right) - \left(E_{\alpha_1+\alpha_2}^{(2n+1/2)} + E_{\alpha_2+\alpha_3}^{(2n+5/2)} \right) + \\ &\quad + \left(E_{-\alpha_2}^{(2n+3/2)} + E_{-\alpha_1-\alpha_2-\alpha_3}^{(2n+3/2)} \right) - \left(E_{-\alpha_1-\alpha_2}^{(2n+5/2)} + E_{-\alpha_2-\alpha_3}^{(2n+1/2)} \right) \\ F_4^{(2n+1/2)} &= - \left(E_{\alpha_2}^{(2n+1/2)} + E_{\alpha_1+\alpha_2+\alpha_3}^{(2n+1/2)} \right) + \left(E_{\alpha_1+\alpha_2}^{(2n-1/2)} + E_{\alpha_2+\alpha_3}^{(2n+3/2)} \right) + \\ &\quad + \left(E_{-\alpha_2}^{(2n+1/2)} + E_{-\alpha_1-\alpha_2-\alpha_3}^{(2n+1/2)} \right) - \left(E_{-\alpha_1-\alpha_2}^{(2n+3/2)} + E_{-\alpha_2-\alpha_3}^{(2n-1/2)} \right) \end{aligned} \right\} \\
\mathcal{M}_F &= \left\{ \begin{aligned} G_1^{(2n+1/2)} &= \left(E_{\alpha_1+\alpha_2}^{(2n-1/2)} - E_{\alpha_2+\alpha_3}^{(2n+3/2)} \right) + \left(E_{\alpha_2}^{(2n+1/2)} - E_{\alpha_1+\alpha_2+\alpha_3}^{(2n+1/2)} \right) + \\ &\quad + \left(E_{-\alpha_1-\alpha_2}^{(2n+3/2)} - E_{-\alpha_2-\alpha_3}^{(2n-1/2)} \right) + \left(E_{-\alpha_2}^{(2n+1/2)} - E_{-\alpha_1-\alpha_2-\alpha_3}^{(2n+1/2)} \right) \\ G_2^{(2n+3/2)} &= \left(E_{\alpha_1+\alpha_2}^{(2n+1/2)} - E_{\alpha_2+\alpha_3}^{(2n+5/2)} \right) + \left(E_{\alpha_2}^{(2n+3/2)} - E_{\alpha_1+\alpha_2+\alpha_3}^{(2n+3/2)} \right) - \\ &\quad - \left(E_{-\alpha_1-\alpha_2}^{(2n+5/2)} - E_{-\alpha_2-\alpha_3}^{(2n+1/2)} \right) - \left(E_{-\alpha_2}^{(2n+3/2)} - E_{-\alpha_1-\alpha_2-\alpha_3}^{(2n+3/2)} \right) \\ G_3^{(2n+1/2)} &= \left(E_{\alpha_1+\alpha_2}^{(2n-1/2)} - E_{\alpha_2+\alpha_3}^{(2n+3/2)} \right) - \left(E_{\alpha_2}^{(2n+1/2)} - E_{\alpha_1+\alpha_2+\alpha_3}^{(2n+1/2)} \right) + \\ &\quad + \left(E_{-\alpha_1-\alpha_2}^{(2n+3/2)} - E_{-\alpha_2-\alpha_3}^{(2n-1/2)} \right) - \left(E_{-\alpha_2}^{(2n+1/2)} - E_{-\alpha_1-\alpha_2-\alpha_3}^{(2n+1/2)} \right) \\ G_4^{(2n+3/2)} &= - \left(E_{\alpha_1+\alpha_2}^{(2n+1/2)} - E_{\alpha_2+\alpha_3}^{(2n+5/2)} \right) + \left(E_{\alpha_2}^{(2n+3/2)} - E_{\alpha_1+\alpha_2+\alpha_3}^{(2n+3/2)} \right) + \\ &\quad + \left(E_{-\alpha_1-\alpha_2}^{(2n+5/2)} - E_{-\alpha_2-\alpha_3}^{(2n+1/2)} \right) - \left(E_{-\alpha_2}^{(2n+3/2)} - E_{-\alpha_1-\alpha_2-\alpha_3}^{(2n+3/2)} \right) \end{aligned} \right\}.
\end{aligned}$$

$$sl(2, 2)_{[2]}^{(2)} :$$

$$\begin{aligned}
\mathcal{K}_B &= \left\{ K_1^{(2n+1)} = I^{(2n+1)}, \quad K_2^{(2n+1)} = E_{\alpha_1}^{(2n)} + E_{-\alpha_1}^{(2n+2)}, \quad K_3^{(2n+1)} = E_{\alpha_3}^{(2n+2)} + E_{-\alpha_3}^{(2n)} \right\} \\
\mathcal{M}_B &= \left\{ \begin{aligned} M_1^{(2n+1)} &= H_1^{(2n+1)}, \quad M_2^{(2n)} = E_{\alpha_1}^{(2n-1)} - E_{-\alpha_1}^{(2n+1)}, \\ M_3^{(2n+1)} &= H_3^{(2n+1)}, \quad M_4^{(2n)} = E_{\alpha_3}^{(2n+1)} - E_{-\alpha_3}^{(2n-1)} \end{aligned} \right\} \quad (\text{A.0.5}) \\
\mathcal{K}_F &= \left\{ \begin{aligned} F_1^{(2n+1/2)} &= \left(E_{\alpha_2}^{(2n+1/2)} + E_{\alpha_1+\alpha_2+\alpha_3}^{(2n+1/2)} \right) + \left(E_{\alpha_1+\alpha_2}^{(2n-1/2)} + E_{\alpha_2+\alpha_3}^{(2n+3/2)} \right) + \\ &\quad + \left(E_{-\alpha_2}^{(2n+1/2)} + E_{-\alpha_1-\alpha_2-\alpha_3}^{(2n+1/2)} \right) + \left(E_{-\alpha_1-\alpha_2}^{(2n+3/2)} + E_{-\alpha_2-\alpha_3}^{(2n-1/2)} \right) \\ F_2^{(2n+3/2)} &= \left(E_{\alpha_2}^{(2n+3/2)} + E_{\alpha_1+\alpha_2+\alpha_3}^{(2n+3/2)} \right) + \left(E_{\alpha_1+\alpha_2}^{(2n+1/2)} + E_{\alpha_2+\alpha_3}^{(2n+5/2)} \right) - \\ &\quad - \left(E_{-\alpha_2}^{(2n+3/2)} + E_{-\alpha_1-\alpha_2-\alpha_3}^{(2n+3/2)} \right) - \left(E_{-\alpha_1-\alpha_2}^{(2n+5/2)} + E_{-\alpha_2-\alpha_3}^{(2n+1/2)} \right) \\ F_3^{(2n+1/2)} &= \left(E_{\alpha_2}^{(2n+1/2)} + E_{\alpha_1+\alpha_2+\alpha_3}^{(2n+1/2)} \right) - \left(E_{\alpha_1+\alpha_2}^{(2n-1/2)} + E_{\alpha_2+\alpha_3}^{(2n+3/2)} \right) + \\ &\quad + \left(E_{-\alpha_2}^{(2n+1/2)} + E_{-\alpha_1-\alpha_2-\alpha_3}^{(2n+1/2)} \right) - \left(E_{-\alpha_1-\alpha_2}^{(2n+3/2)} + E_{-\alpha_2-\alpha_3}^{(2n-1/2)} \right) \\ F_4^{(2n+3/2)} &= - \left(E_{\alpha_2}^{(2n+3/2)} + E_{\alpha_1+\alpha_2+\alpha_3}^{(2n+3/2)} \right) + \left(E_{\alpha_1+\alpha_2}^{(2n+1/2)} + E_{\alpha_2+\alpha_3}^{(2n+5/2)} \right) + \\ &\quad + \left(E_{-\alpha_2}^{(2n+3/2)} + E_{-\alpha_1-\alpha_2-\alpha_3}^{(2n+3/2)} \right) - \left(E_{-\alpha_1-\alpha_2}^{(2n+5/2)} + E_{-\alpha_2-\alpha_3}^{(2n+1/2)} \right) \end{aligned} \right\} \\
\mathcal{M}_F &= \left\{ \begin{aligned} G_1^{(2n+1/2)} &= \left(E_{\alpha_1+\alpha_2}^{(2n-1/2)} - E_{\alpha_2+\alpha_3}^{(2n+3/2)} \right) + \left(E_{\alpha_2}^{(2n+1/2)} - E_{\alpha_1+\alpha_2+\alpha_3}^{(2n+1/2)} \right) + \\ &\quad + \left(E_{-\alpha_1-\alpha_2}^{(2n+3/2)} - E_{-\alpha_2-\alpha_3}^{(2n-1/2)} \right) + \left(E_{-\alpha_2}^{(2n+1/2)} - E_{-\alpha_1-\alpha_2-\alpha_3}^{(2n+1/2)} \right) \\ G_2^{(2n+3/2)} &= \left(E_{\alpha_1+\alpha_2}^{(2n+1/2)} - E_{\alpha_2+\alpha_3}^{(2n+5/2)} \right) + \left(E_{\alpha_2}^{(2n+3/2)} - E_{\alpha_1+\alpha_2+\alpha_3}^{(2n+3/2)} \right) - \\ &\quad - \left(E_{-\alpha_1-\alpha_2}^{(2n+5/2)} - E_{-\alpha_2-\alpha_3}^{(2n+1/2)} \right) - \left(E_{-\alpha_2}^{(2n+3/2)} - E_{-\alpha_1-\alpha_2-\alpha_3}^{(2n+3/2)} \right) \\ G_3^{(2n+1/2)} &= \left(E_{\alpha_1+\alpha_2}^{(2n-1/2)} - E_{\alpha_2+\alpha_3}^{(2n+3/2)} \right) - \left(E_{\alpha_2}^{(2n+1/2)} - E_{\alpha_1+\alpha_2+\alpha_3}^{(2n+1/2)} \right) + \\ &\quad + \left(E_{-\alpha_1-\alpha_2}^{(2n+3/2)} - E_{-\alpha_2-\alpha_3}^{(2n-1/2)} \right) - \left(E_{-\alpha_2}^{(2n+1/2)} - E_{-\alpha_1-\alpha_2-\alpha_3}^{(2n+1/2)} \right) \\ G_4^{(2n+3/2)} &= - \left(E_{\alpha_1+\alpha_2}^{(2n+1/2)} - E_{\alpha_2+\alpha_3}^{(2n+5/2)} \right) + \left(E_{\alpha_2}^{(2n+3/2)} - E_{\alpha_1+\alpha_2+\alpha_3}^{(2n+3/2)} \right) + \\ &\quad + \left(E_{-\alpha_1-\alpha_2}^{(2n+5/2)} - E_{-\alpha_2-\alpha_3}^{(2n+1/2)} \right) - \left(E_{-\alpha_2}^{(2n+3/2)} - E_{-\alpha_1-\alpha_2-\alpha_3}^{(2n+3/2)} \right) \end{aligned} \right\}.
\end{aligned}$$

$sl(2, 2)_{[3]}^{(2)} :$

$$\mathcal{K}_B = \left\{ K_1^{(2n+1)} = I^{(2n+1)}, \quad K_2^{(2n+1)} = E_{\alpha_1}^{(2n)} + E_{-\alpha_1}^{(2n+2)}, \quad K_3^{(2n+1)} = E_{\alpha_3}^{(2n+2)} + E_{-\alpha_3}^{(2n)} \right\}$$

$$\mathcal{M}_B = \left\{ \begin{array}{l} M_1^{(2n)} = H_1^{(2n)}, \quad M_2^{(2n+1)} = E_{\alpha_1}^{(2n)} - E_{-\alpha_1}^{(2n+2)}, \\ M_3^{(2n+1)} = H_3^{(2n+1)}, \quad M_4^{(2n)} = E_{\alpha_3}^{(2n+1)} - E_{-\alpha_3}^{(2n-1)} \end{array} \right\} \quad (\text{A.0.6})$$

$$\mathcal{K}_F = \left\{ \begin{array}{l} F_1^{(2n+1/2)} = \left(E_{\alpha_2}^{(2n+1/2)} + E_{\alpha_1+\alpha_2+\alpha_3}^{(2n+1/2)} \right) + \left(E_{\alpha_1+\alpha_2}^{(2n-1/2)} + E_{\alpha_2+\alpha_3}^{(2n+3/2)} \right) + \\ \quad + \left(E_{-\alpha_2}^{(2n+1/2)} + E_{-\alpha_1-\alpha_2-\alpha_3}^{(2n+1/2)} \right) + \left(E_{-\alpha_1-\alpha_2}^{(2n+3/2)} + E_{-\alpha_2-\alpha_3}^{(2n-1/2)} \right) \\ F_2^{(2n+3/2)} = \left(E_{\alpha_2}^{(2n+3/2)} + E_{\alpha_1+\alpha_2+\alpha_3}^{(2n+3/2)} \right) + \left(E_{\alpha_1+\alpha_2}^{(2n+1/2)} + E_{\alpha_2+\alpha_3}^{(2n+5/2)} \right) - \\ \quad - \left(E_{-\alpha_2}^{(2n+3/2)} + E_{-\alpha_1-\alpha_2-\alpha_3}^{(2n+3/2)} \right) - \left(E_{-\alpha_1-\alpha_2}^{(2n+5/2)} + E_{-\alpha_2-\alpha_3}^{(2n+1/2)} \right) \\ F_3^{(2n+3/2)} = \left(E_{\alpha_2}^{(2n+3/2)} + E_{\alpha_1+\alpha_2+\alpha_3}^{(2n+3/2)} \right) - \left(E_{\alpha_1+\alpha_2}^{(2n+1/2)} + E_{\alpha_2+\alpha_3}^{(2n+5/2)} \right) + \\ \quad + \left(E_{-\alpha_2}^{(2n+3/2)} + E_{-\alpha_1-\alpha_2-\alpha_3}^{(2n+3/2)} \right) - \left(E_{-\alpha_1-\alpha_2}^{(2n+5/2)} + E_{-\alpha_2-\alpha_3}^{(2n+1/2)} \right) \\ F_4^{(2n+1/2)} = - \left(E_{\alpha_2}^{(2n+1/2)} + E_{\alpha_1+\alpha_2+\alpha_3}^{(2n+1/2)} \right) + \left(E_{\alpha_1+\alpha_2}^{(2n-1/2)} + E_{\alpha_2+\alpha_3}^{(2n+3/2)} \right) + \\ \quad + \left(E_{-\alpha_2}^{(2n+1/2)} + E_{-\alpha_1-\alpha_2-\alpha_3}^{(2n+1/2)} \right) - \left(E_{-\alpha_1-\alpha_2}^{(2n+3/2)} + E_{-\alpha_2-\alpha_3}^{(2n-1/2)} \right) \end{array} \right\}$$

$$\mathcal{M}_F = \left\{ \begin{array}{l} G_1^{(2n+1/2)} = \left(E_{\alpha_1+\alpha_2}^{(2n-1/2)} - E_{\alpha_2+\alpha_3}^{(2n+3/2)} \right) + \left(E_{\alpha_2}^{(2n+1/2)} - E_{\alpha_1+\alpha_2+\alpha_3}^{(2n+1/2)} \right) + \\ \quad + \left(E_{-\alpha_1-\alpha_2}^{(2n+3/2)} - E_{-\alpha_2-\alpha_3}^{(2n-1/2)} \right) + \left(E_{-\alpha_2}^{(2n+1/2)} - E_{-\alpha_1-\alpha_2-\alpha_3}^{(2n+1/2)} \right) \\ G_2^{(2n+3/2)} = \left(E_{\alpha_1+\alpha_2}^{(2n+1/2)} - E_{\alpha_2+\alpha_3}^{(2n+5/2)} \right) + \left(E_{\alpha_2}^{(2n+3/2)} - E_{\alpha_1+\alpha_2+\alpha_3}^{(2n+3/2)} \right) - \\ \quad - \left(E_{-\alpha_1-\alpha_2}^{(2n+5/2)} - E_{-\alpha_2-\alpha_3}^{(2n+1/2)} \right) - \left(E_{-\alpha_2}^{(2n+3/2)} - E_{-\alpha_1-\alpha_2-\alpha_3}^{(2n+3/2)} \right) \\ G_3^{(2n+3/2)} = \left(E_{\alpha_1+\alpha_2}^{(2n+1/2)} - E_{\alpha_2+\alpha_3}^{(2n+5/2)} \right) - \left(E_{\alpha_2}^{(2n+3/2)} - E_{\alpha_1+\alpha_2+\alpha_3}^{(2n+3/2)} \right) + \\ \quad + \left(E_{-\alpha_1-\alpha_2}^{(2n+5/2)} - E_{-\alpha_2-\alpha_3}^{(2n+1/2)} \right) - \left(E_{-\alpha_2}^{(2n+3/2)} - E_{-\alpha_1-\alpha_2-\alpha_3}^{(2n+3/2)} \right) \\ G_4^{(2n+1/2)} = - \left(E_{\alpha_1+\alpha_2}^{(2n-1/2)} - E_{\alpha_2+\alpha_3}^{(2n+1/2)} \right) + \left(E_{\alpha_2}^{(2n+1/2)} - E_{\alpha_1+\alpha_2+\alpha_3}^{(2n+1/2)} \right) + \\ \quad + \left(E_{-\alpha_1-\alpha_2}^{(2n+3/2)} - E_{-\alpha_2-\alpha_3}^{(2n-1/2)} \right) - \left(E_{-\alpha_2}^{(2n+1/2)} - E_{-\alpha_1-\alpha_2-\alpha_3}^{(2n+1/2)} \right) \end{array} \right\}.$$

$sl(2, 2)_{[4]}^{(2)} :$

$$\begin{aligned}
\mathcal{K}_B &= \left\{ K_1^{(2n+1)} = I^{(2n+1)}, \quad K_2^{(2n+1)} = E_{\alpha_1}^{(2n)} + E_{-\alpha_1}^{(2n+2)}, \quad K_3^{(2n+1)} = E_{\alpha_3}^{(2n+2)} + E_{-\alpha_3}^{(2n)} \right\} \\
\mathcal{M}_B &= \left\{ \begin{aligned} M_1^{(2n+1)} &= H_1^{(2n+1)}, \quad M_2^{(2n)} = E_{\alpha_1}^{(2n-1)} - E_{-\alpha_1}^{(2n+1)}, \\ M_3^{(2n)} &= H_3^{(2n+1)}, \quad M_4^{(2n+1)} = E_{\alpha_3}^{(2n+2)} - E_{-\alpha_3}^{(2n)} \end{aligned} \right\} \quad (\text{A.0.7}) \\
\mathcal{K}_F &= \left\{ \begin{aligned} F_1^{(2n+3/2)} &= \left(E_{\alpha_2}^{(2n+3/2)} + E_{\alpha_1+\alpha_2+\alpha_3}^{(2n+3/2)} \right) + \left(E_{\alpha_1+\alpha_2}^{(2n+1/2)} + E_{\alpha_2+\alpha_3}^{(2n+5/2)} \right) + \\ &\quad + \left(E_{-\alpha_2}^{(2n+3/2)} + E_{-\alpha_1-\alpha_2-\alpha_3}^{(2n+3/2)} \right) + \left(E_{-\alpha_1-\alpha_2}^{(2n+5/2)} + E_{-\alpha_2-\alpha_3}^{(2n+1/2)} \right) \\ F_2^{(2n+1/2)} &= \left(E_{\alpha_2}^{(2n+1/2)} + E_{\alpha_1+\alpha_2+\alpha_3}^{(2n+1/2)} \right) + \left(E_{\alpha_1+\alpha_2}^{(2n-1/2)} + E_{\alpha_2+\alpha_3}^{(2n+3/2)} \right) - \\ &\quad - \left(E_{-\alpha_2}^{(2n+1/2)} + E_{-\alpha_1-\alpha_2-\alpha_3}^{(2n+1/2)} \right) - \left(E_{-\alpha_1-\alpha_2}^{(2n+3/2)} + E_{-\alpha_2-\alpha_3}^{(2n-1/2)} \right) \\ F_3^{(2n+1/2)} &= \left(E_{\alpha_2}^{(2n+1/2)} + E_{\alpha_1+\alpha_2+\alpha_3}^{(2n+1/2)} \right) - \left(E_{\alpha_1+\alpha_2}^{(2n-1/2)} + E_{\alpha_2+\alpha_3}^{(2n+3/2)} \right) + \\ &\quad + \left(E_{-\alpha_2}^{(2n+1/2)} + E_{-\alpha_1-\alpha_2-\alpha_3}^{(2n+1/2)} \right) - \left(E_{-\alpha_1-\alpha_2}^{(2n+3/2)} + E_{-\alpha_2-\alpha_3}^{(2n-1/2)} \right) \\ F_4^{(2n+3/2)} &= - \left(E_{\alpha_2}^{(2n+3/2)} + E_{\alpha_1+\alpha_2+\alpha_3}^{(2n+3/2)} \right) + \left(E_{\alpha_1+\alpha_2}^{(2n+1/2)} + E_{\alpha_2+\alpha_3}^{(2n+5/2)} \right) + \\ &\quad + \left(E_{-\alpha_2}^{(2n+3/2)} + E_{-\alpha_1-\alpha_2-\alpha_3}^{(2n+3/2)} \right) - \left(E_{-\alpha_1-\alpha_2}^{(2n+5/2)} + E_{-\alpha_2-\alpha_3}^{(2n+1/2)} \right) \\ G_1^{(2n+1/2)} &= \left(E_{\alpha_1+\alpha_2}^{(2n-1/2)} - E_{\alpha_2+\alpha_3}^{(2n+3/2)} \right) + \left(E_{\alpha_2}^{(2n+1/2)} - E_{\alpha_1+\alpha_2+\alpha_3}^{(2n+1/2)} \right) + \\ &\quad + \left(E_{-\alpha_1-\alpha_2}^{(2n+3/2)} - E_{-\alpha_2-\alpha_3}^{(2n-1/2)} \right) + \left(E_{-\alpha_2}^{(2n+1/2)} - E_{-\alpha_1-\alpha_2-\alpha_3}^{(2n+1/2)} \right) \\ G_2^{(2n+3/2)} &= \left(E_{\alpha_1+\alpha_2}^{(2n+1/2)} - E_{\alpha_2+\alpha_3}^{(2n+5/2)} \right) + \left(E_{\alpha_2}^{(2n+3/2)} - E_{\alpha_1+\alpha_2+\alpha_3}^{(2n+3/2)} \right) - \\ &\quad - \left(E_{-\alpha_1-\alpha_2}^{(2n+5/2)} - E_{-\alpha_2-\alpha_3}^{(2n+1/2)} \right) - \left(E_{-\alpha_2}^{(2n+3/2)} - E_{-\alpha_1-\alpha_2-\alpha_3}^{(2n+3/2)} \right) \\ G_3^{(2n+3/2)} &= \left(E_{\alpha_1+\alpha_2}^{(2n+1/2)} - E_{\alpha_2+\alpha_3}^{(2n+5/2)} \right) - \left(E_{\alpha_2}^{(2n+3/2)} - E_{\alpha_1+\alpha_2+\alpha_3}^{(2n+3/2)} \right) + \\ &\quad + \left(E_{-\alpha_1-\alpha_2}^{(2n+5/2)} - E_{-\alpha_2-\alpha_3}^{(2n+1/2)} \right) - \left(E_{-\alpha_2}^{(2n+3/2)} - E_{-\alpha_1-\alpha_2-\alpha_3}^{(2n+3/2)} \right) \\ G_4^{(2n+1/2)} &= - \left(E_{\alpha_1+\alpha_2}^{(2n-1/2)} - E_{\alpha_2+\alpha_3}^{(2n+1/2)} \right) + \left(E_{\alpha_2}^{(2n+1/2)} - E_{\alpha_1+\alpha_2+\alpha_3}^{(2n+1/2)} \right) + \\ &\quad + \left(E_{-\alpha_1-\alpha_2}^{(2n+3/2)} - E_{-\alpha_2-\alpha_3}^{(2n-1/2)} \right) - \left(E_{-\alpha_2}^{(2n+1/2)} - E_{-\alpha_1-\alpha_2-\alpha_3}^{(2n+1/2)} \right) \end{aligned} \right\}
\end{aligned}$$

Appendix B

Relativistic Equations for the $t_{\pm 3}$ Times.

The decomposition of the zero curvature equation $[L_{+3}, L_{-3}] = 0$ on the different graded subspaces reads

$$\begin{aligned}
 \partial_{\pm 3} D^{(\mp 5/2)} &= \pm [E^{(\mp 3)}, B^{\mp 1} D^{(\pm 1/2)} B^{\pm 1}] \\
 \partial_{\pm 3} D^{(\mp 2)} &= \pm [D^{(\mp 5/2)}, B^{\mp 1} D^{(\pm 1/2)} B^{\pm 1}] \pm [E^{(\mp 3)}, B^{\mp 1} D^{(\pm 1)} B^{\pm 1}] \\
 \partial_{\pm 3} D^{(\mp 3/2)} &= \pm [D^{(\mp 2)}, B^{\mp 1} D^{(\pm 1/2)} B^{\pm 1}] \pm [D^{(\mp 5/2)}, B^{\mp 1} D^{(\pm 1)} B^{\pm 1}] \pm \\
 &\quad \pm [E^{(\mp 3)}, B^{\mp 1} D^{(\pm 3/2)} B^{\pm 1}] \\
 \partial_{\pm 3} D^{(\mp 1)} &= \pm [D^{(\mp 3/2)}, B^{\mp 1} D^{(\pm 1/2)} B^{\pm 1}] \pm [D^{(\mp 2)}, B^{\mp 1} D^{(\pm 1)} B^{\pm 1}] \pm \\
 &\quad \pm [D^{(\mp 5/2)}, B^{\mp 1} D^{(\pm 3/2)} B^{\pm 1}] \pm [E^{(\mp 3)}, B^{\mp 1} D^{(\pm 2)} B^{\pm 1}] \\
 \partial_{\pm 3} D^{(\mp 1/2)} &= \pm [D^{(\mp 1)}, B^{\mp 1} D^{(\pm 1/2)} B^{\pm 1}] \pm [D^{(\mp 3/2)}, B^{\mp 1} D^{(\pm 1)} B^{\pm 1}] \pm \\
 &\quad \pm [D^{(\mp 2)}, B^{\mp 1} D^{(\pm 3/2)} B^{\pm 1}] \pm [D^{(\mp 5/2)}, B^{\mp 1} D^{(\pm 2)} B^{\pm 1}] \pm \\
 &\quad \pm [E^{(\mp 3)}, B^{\mp 1} D^{(\pm 5/2)} B^{\pm 1}] \\
 \partial_{-3} (\partial_{+3} B B^{-1}) &= [D^{(+1/2)}, B D^{(-1/2)} B^{-1}] + [D^{(+1)}, B D^{(-1)} B^{-1}] + \\
 &\quad + [D^{(+3/2)}, B D^{(-3/2)} B^{-1}] + [D^{(+2)}, B D^{(-2)} B^{-1}] + \\
 &\quad + [D^{(+5/2)}, B D^{(-5/2)} B^{-1}] + [E^{(+3)}, B E^{(-3)} B^{-1}],
 \end{aligned}$$

where we have replaced the solution $D^{(0)} = -\partial_{+3}BB^{-1}$ for the degree -3 equation

$$0 = [D^{(0)} + \partial_{+3}BB^{-1}, BE^{(+3)}B^{-1}]. \quad (\text{B.0.1})$$

For the $sl(2, 1)_{[1]}^{(2)}$ algebra set

$$\begin{aligned} D^{(+5/2)} &= \psi_+^1 G_1^{(+5/2)} & , D^{(-5/2)} &= \psi_-^1 G_2^{(-5/2)} \\ D^{(+2)} &= \phi_+^1 M_1^{(+2)} & , D^{(-2)} &= \phi_-^1 M_1^{(-2)} \\ D^{(+3/2)} &= \psi_+^2 F_1^{(+3/2)} + \psi_+^3 G_2^{(+3/2)} & , D^{(-3/2)} &= \psi_-^2 F_2^{(-3/2)} + \psi_-^3 G_1^{(-3/2)} \\ D^{(+1)} &= \phi_+^2 K_1^{(+1)} + \phi_+^3 K_2^{(+1)} + \phi_+^4 M_2^{(+1)} & , D^{(-1)} &= \phi_-^2 K_1^{(-1)} + \phi_-^3 K_2^{(-1)} + \phi_-^4 M_2^{(-1)} \\ D^{(+1/2)} &= \psi_+^4 F_2^{(+1/2)} + \psi_+^5 G_1^{(+1/2)} & , D^{(-1/2)} &= \psi_-^4 F_1^{(-1/2)} + \psi_-^5 G_2^{(-1/2)} \\ D^{(0)} &= \phi_+^5 M_1^{(0)} \end{aligned}$$

then, the above equations of motion are

$$\begin{aligned}
\partial_{\pm 3} \psi_{\mp}^1 &= 2\psi_{\pm}^4 \sinh[\phi] \pm 2\psi_{\pm}^5 \cosh[\phi] & (\text{B.0.2}) \\
\partial_{\pm 3} \phi_{\mp}^1 &= 2\psi_{\mp}^1 \psi_{\pm}^4 \cosh[\phi] \pm 2\psi_{\mp}^1 \psi_{\pm}^5 \sinh[\phi] - 2\phi_{\pm}^3 \sinh[2\phi] \pm 2\phi_{\pm}^4 \cosh[2\phi] \\
\partial_{\pm 3} \psi_{\mp}^2 &= -\phi_{\mp}^1 \psi_{\pm}^4 \sinh[\phi] \mp \phi_{\mp}^1 \psi_{\pm}^5 \cosh[\phi] + \psi_{\mp}^1 \phi_{\pm}^3 \sinh[2\phi] \mp \psi_{\mp}^1 \phi_{\pm}^4 \cosh[2\phi] \\
\partial_{\pm 3} \psi_{\mp}^3 &= \mp (\phi_{\mp}^1 \psi_{\pm}^4 - 2\psi_{\pm}^3) \cosh[\phi] - (\phi_{\mp}^1 \psi_{\pm}^5 - 2\psi_{\pm}^2) \sinh[\phi] \pm \phi_{\pm}^3 \psi_{\mp}^1 \cosh[2\phi] - \\
&\quad - \psi_{\mp}^1 \phi_{\pm}^4 \sinh[2\phi] \pm \psi_{\mp}^1 \phi_{\pm}^2 \\
\partial_{\pm 3} \phi_{\mp}^2 &= 2(-\psi_{\mp}^2 \psi_{\pm}^4 + \psi_{\mp}^3 \psi_{\pm}^5 - \psi_{\mp}^1 \psi_{\pm}^3) \cosh[\phi] \pm 2(-\psi_{\mp}^2 \psi_{\pm}^5 + \psi_{\mp}^3 \psi_{\pm}^4 - \psi_{\mp}^1 \psi_{\pm}^2) \sinh[\phi] \\
\partial_{\pm 3} \phi_{\mp}^3 &= 2(-\psi_{\mp}^2 \psi_{\pm}^4 - \psi_{\mp}^3 \psi_{\pm}^5 + \psi_{\mp}^1 \psi_{\pm}^3) \cosh[\phi] \mp 2(\psi_{\mp}^2 \psi_{\pm}^5 + \psi_{\mp}^3 \psi_{\pm}^4 - \psi_{\mp}^1 \psi_{\pm}^2) \sinh[\phi] - \\
&\quad - 2\phi_{\mp}^1 \phi_{\pm}^3 \sinh[2\phi] \pm 2\phi_{\mp}^1 \phi_{\pm}^4 \cosh[2\phi] \\
\partial_{\pm 3} \phi_{\mp}^4 &= \pm 2(\psi_{\mp}^2 \psi_{\pm}^4 + \psi_{\mp}^3 \psi_{\pm}^5 - \psi_{\mp}^1 \psi_{\pm}^3) \sinh[\phi] + 2(\psi_{\mp}^2 \psi_{\pm}^5 + \psi_{\mp}^3 \psi_{\pm}^4 - \psi_{\mp}^1 \psi_{\pm}^2) \cosh[\phi] \pm \\
&\quad \pm 2\phi_{\mp}^1 \phi_{\pm}^3 \cosh[2\phi] - 2\phi_{\mp}^1 \phi_{\pm}^4 \sinh[2\phi] \pm 2\phi_{\pm}^1 \\
\partial_{\pm 3} \psi_{\mp}^4 &= \mp (\phi_{\mp}^2 \psi_{\pm}^4 - \phi_{\mp}^3 \psi_{\pm}^4 - \phi_{\mp}^4 \psi_{\pm}^5 + \phi_{\mp}^1 \psi_{\pm}^3) \cosh[\phi] + \\
&\quad + (\phi_{\mp}^4 \psi_{\pm}^4 - \phi_{\mp}^2 \psi_{\pm}^5 + \phi_{\mp}^3 \psi_{\pm}^5 - \phi_{\mp}^1 \psi_{\pm}^2) \sinh[\phi] \mp \\
&\quad \mp (\phi_{\pm}^3 \psi_{\mp}^2 + \phi_{\pm}^4 \psi_{\mp}^3) \cosh[2\phi] + \\
&\quad + (\psi_{\mp}^2 \phi_{\pm}^4 + \psi_{\mp}^3 \phi_{\pm}^3) \sinh[2\phi] \pm \psi_{\mp}^1 \phi_{\pm}^1 \pm \psi_{\mp}^2 \phi_{\pm}^2 \\
\partial_{\pm 3} \psi_{\mp}^5 &= \mp (\phi_{\mp}^4 \psi_{\pm}^4 + \phi_{\mp}^2 \psi_{\pm}^5 + \phi_{\mp}^3 \psi_{\pm}^5 + \phi_{\mp}^1 \psi_{\pm}^2 - 2\psi_{\pm}^1) \cosh[\phi] - \\
&\quad - (\phi_{\mp}^2 \psi_{\pm}^4 + \phi_{\mp}^3 \psi_{\pm}^4 + \phi_{\mp}^4 \psi_{\pm}^5 + \phi_{\mp}^1 \psi_{\pm}^3) \sinh[\phi] - \\
&\quad - (\psi_{\mp}^2 \phi_{\pm}^3 + \psi_{\mp}^3 \phi_{\pm}^4) \sinh[2\phi] \pm (\psi_{\mp}^2 \phi_{\pm}^4 + \psi_{\mp}^3 \phi_{\pm}^3) \cosh[2\phi] \pm \psi_{\mp}^3 \phi_{\pm}^2 \\
\partial_{+3} \partial_{-3} \phi &= 2(\psi_{+}^4 \psi_{-}^4 - \psi_{+}^5 \psi_{-}^5 - \psi_{+}^2 \psi_{-}^2 + \psi_{+}^3 \psi_{-}^3 - \psi_{+}^1 \psi_{-}^1) \sinh[\phi] + \\
&\quad + 2(\psi_{+}^4 \psi_{-}^5 + \psi_{+}^4 \psi_{-}^5 - \psi_{+}^2 \psi_{-}^3 - \psi_{+}^2 \psi_{-}^3) \cosh[\phi] - \\
&\quad - 2(1 - \phi_{+}^3 \phi_{-}^3 + \phi_{+}^4 \phi_{-}^4) \sinh[2\phi] - 2(\phi_{+}^3 \phi_{-}^4 - \phi_{-}^3 \phi_{+}^4) \cosh[2\phi].
\end{aligned}$$

The generalize equations of motion are found by replacing above the solutions

$$\psi_{\pm}^1 = \mp 2\psi_{\pm} \quad , \quad \psi_{\pm}^2 = \mp \psi_{\pm} \partial_{\pm} \phi \quad , \quad \psi_{\pm}^3 = \partial_{\pm} \psi_{\pm} \quad , \quad \psi_{\pm}^4 = -\frac{1}{2} (\psi_{\pm} \partial_{\pm}^2 \phi - \partial_{\pm} \phi \partial_{\pm} \psi_{\pm})$$

$$\psi_{\pm}^5 = \mp \frac{1}{2} \partial_{\pm}^2 \psi_{\pm} \pm \psi_{\pm} (\partial_{\pm} \phi)^2 \quad , \quad \phi_{\pm}^1 = -\partial_{\pm} \phi \quad , \quad \phi_{\pm}^2 = -2\psi_{\pm} \partial_{\pm} \psi_{\pm}$$

$$\phi_{\pm}^3 = \frac{1}{2} (\partial_{\pm} \phi)^2 + 2\psi_{\pm} \partial_{\pm} \psi_{\pm} \quad , \quad \phi_{\pm}^4 = \pm \frac{1}{2} \partial_{\pm}^2 \phi$$

$$\phi_{+}^5 = -\frac{1}{4} \partial_{+}^3 \phi + \frac{1}{2} (\partial_{+} \phi)^3 + 3\partial_{+} \phi \psi_{+} \partial_{+} \psi_{+}.$$

Appendix C

Proof of the Main Propositions.

Here we show the deduction of the action given announce in proposition (3.1.3) and give the proof of the integrability conditions (4.1.3). The are given respectively by **Proof.** Write the WZNW action invariant under the following local gauge transformations

$$\begin{aligned} g &\rightarrow g' = \alpha g \beta, \quad \alpha \in K_{<} \text{ and } \beta \in K_{>} \\ A'_+ &= \beta^{-1} A_+ \beta + \partial_+ \beta^{-1} \beta \\ A'_- &= \alpha A_- \alpha^{-1} + \alpha \partial_- \alpha^{-1}, \end{aligned}$$

which is given by

$$\begin{aligned} S[g, A_{\pm}] &= S_{WZNW}[g] - \\ &\quad - \frac{k}{2\pi} \int \langle A_- (\partial_+ g g^{-1} - E_+) + A_+ (g^{-1} \partial_- g - E_-) + A_- g A_+ g^{-1} \rangle, \end{aligned}$$

where $A_+ \in \widehat{\mathfrak{g}}_{\geq+1}$, $A_- \in \widehat{\mathfrak{g}}_{\leq-1}$. Now take $\alpha = K_{<}^{-1}$ and $\beta = K_{>}^{-1}$, which is equivalent to the elimination of an infinite number of fields (First class constraints). We get

$$\begin{aligned} S[\Gamma, A'_{\pm}] &= S_{WZNW}[\Gamma] - \\ &\quad - \frac{k}{2\pi} \int \langle A'_- (\partial_+ \Gamma \Gamma^{-1} - E_+) + A'_+ (\Gamma^{-1} \partial_- \Gamma - E_-) + A'_- \Gamma A'_+ \Gamma^{-1} \rangle. \end{aligned}$$

To handle the action $S_{WZNW}[\Gamma]$, we use the Polyakov-Wiegmann identity* to obtain

$$S_{WZNW}[\Gamma] = S_{WZNW}[B] - \frac{k}{2\pi} \int \langle (\Phi^{-1} \partial_- \Phi) B (\partial_+ \Psi \Psi^{-1}) B^{-1} \rangle \quad (C.0.1)$$

and for the second term we look for the non-zero contributions to the following inner product

$$I = \langle A_- (\partial_+ \Gamma \Gamma^{-1} - E_+) + A_+ (\Gamma^{-1} \partial_- \Gamma - E_-) + A_- \Gamma A_+ \Gamma^{-1} \rangle.$$

Analizing term by term we have

$$\begin{aligned} \langle A_- ((\partial_+ \Gamma \Gamma^{-1})|_{\geq +1} - E_+) \rangle &= \langle A_- (\Phi B \partial_+ \Psi \Psi^{-1} B^{-1} \Phi^{-1} - E_+) \rangle \\ \langle A_+ ((\Gamma^{-1} \partial_- \Gamma)|_{\leq -1} - E_-) \rangle &= \langle A_+ (\Psi^{-1} B^{-1} \Phi^{-1} \partial_- \Phi B \Psi - E_-) \rangle. \end{aligned}$$

Now, solve the gauge field equations of motion to obtain

$$\begin{aligned} A_+ &= \Gamma^{-1} E_+ \Gamma - \Psi^{-1} \partial_+ \Psi \\ A_- &= \Gamma E_- \Gamma^{-1} - \partial_- \Phi \Phi^{-1} \end{aligned}$$

and put them back in (C.0.2) to get

$$\begin{aligned} I &= -\langle A_- \Gamma A_+ \Gamma^{-1} \rangle \\ &= -\langle (\Phi^{-1} E_+ \Phi) B (\Psi E_- \Psi^{-1}) B^{-1} \rangle - \langle (\Phi^{-1} \partial_- \Phi) B (\partial_+ \Psi \Psi^{-1}) B^{-1} \rangle + \\ &\quad + \langle (\Phi^{-1} E_+ \Phi) (\Phi^{-1} \partial_- \Phi) + (\Psi E_- \Psi^{-1}) (\partial_+ \Psi \Psi^{-1}) \rangle. \end{aligned} \quad (C.0.2)$$

*This is given by:

$$\begin{aligned} S_{WZNW}[ABC] &= S_{WZNW}[A] + S_{WZNW}[B] + S_{WZNW}[C] - \\ &\quad - \frac{k}{2\pi} \int \left\langle \begin{aligned} &(A^{-1} \partial_- A) (\partial_+ B B^{-1}) + (B^{-1} \partial_- B) (\partial_+ C C^{-1}) + \\ &+ (A^{-1} \partial_- A) B (\partial_+ C C^{-1}) B^{-1} \end{aligned} \right\rangle, \end{aligned}$$

where

$$S_{WZNW}[g] = -\frac{k}{4\pi} \int \langle (g^{-1} \partial_+ g) (g^{-1} \partial_- g) \rangle + \frac{k}{24\pi} \int \epsilon^{IJK} \langle (g^{-1} \partial_I g) (g^{-1} \partial_J g) (g^{-1} \partial_K g) \rangle$$

Notice that the term involving the two derivatives in (C.0.1) and (C.0.2) cancel each other leaving only linear terms in the derivatives which are the correct kinetic terms expected for Fermionic fields. Finally, we have

$$S[\Phi, B, \Psi] = S_{WZNW}[B] + \frac{k}{2\pi} \int \langle (\Phi^{-1} E_+ \Phi) B (\Psi E_- \Psi^{-1}) B^{-1} \rangle - \quad (C.0.3)$$

$$- \frac{k}{2\pi} \int \langle (\Phi^{-1} E_+ \Phi) (\Phi^{-1} \partial_- \Phi) + (\Psi E_- \Psi^{-1}) (\partial_+ \Psi \Psi^{-1}) \rangle.$$

Guided by the meaning found above for the conditions (2.1.9), i.e $Q_{0\pm}^2 = 0$ we impose them here $(\Phi^{-1} E_+ \Phi)|_0 = (\Psi E_- \Psi^{-1})|_0 = 0$. If (2.1.9) does not hold the action would have, in principle, an infinite number of contributions[†] coming from the Baker-Hausdorff expansions of the terms $(\Phi^{-1} E_+ \Phi)$ and $(\Psi E_- \Psi^{-1})$ appearing in the potential term of the reduced action. Thus, we restrict to the subalgebras that fulfill the conditions (2.1.9). These constraints do not invalidate the reduction due to the fact that we only need to know a priori the grades of the matrix fields involved and not their number of component degrees of freedom. Performing one further manipulation we finally get the action (3.1.3). ■

[†]The expansion will truncate due to the Grasmannian character of the fields appearing in $\psi^{(\pm 1/2)}$ but a more subtle point is that even when $\psi^{(\pm 1/2)}$ has degree $\pm 1/2$, the conjugations will spread over the whole algebra again, i.e, an expansion in all $\hat{\mathfrak{g}}_i$. These terms, breaks the integrability of the model.

Proof. The variation of the action (4.1.1) respect all fields is given by

$$\begin{aligned}
\frac{2\pi}{k}\delta S = & \int_{\Sigma} \left\langle \delta B B^{-1} \left\{ D_{-}(D_{+} B B^{-1}) + F_{+-}^A - [E_{+}^{(+1)}, B E_{-}^{(-1)} B^{-1}] - \right. \right. \\
& \left. \left. - [\psi_{+}^{(+1/2)}, B \psi_{-}^{(-1/2)} B^{-1}] - \lambda [Q_{+}^{(0)}, B Q_{-}^{(0)} B^{-1}] \right\} \right\rangle + \\
& + \int_{\Sigma} \left\langle \delta \psi^{(+1/2)} \left\{ D_{+} \psi_{-}^{(-1/2)} - [E_{-}^{(-1)}, B^{-1} \psi_{+}^{(+1/2)} B] - \right. \right. \\
& \left. \left. - \lambda [\psi_{-}^{(-1/2)}, B^{-1} Q_{+}^{(0)} B] + \right. \right. \\
& \left. \left. + \frac{\lambda}{2} [\psi^{(+1/2)}, [E_{-}^{(-1)}, B^{-1} Q_{+}^{(0)} B]] \right\} \right\rangle \\
& + \int_{\Sigma} \left\langle \delta \psi^{(-1/2)} \left\{ -D_{-} \psi_{+}^{(+1/2)} + [B \psi_{-}^{(-1/2)} B^{-1}, E_{+}^{(+1)}] + \right. \right. \\
& \left. \left. + \lambda [B Q_{-}^{(0)} B^{-1}, \psi_{+}^{(+1/2)}] + \right. \right. \\
& \left. \left. + \frac{\lambda}{2} [\psi^{(-1/2)}, [E_{+}^{(+1)}, B Q_{-}^{(0)} B^{-1}]] \right\} \right\rangle + \\
& - \int_{\Sigma} \left\langle \eta \delta A_{-} \left\{ D_{+} B B^{-1} + Q_{+}^{(0)} + \eta A_{+}^0 + A_{+} \right\} + \right. \\
& \left. + \delta A_{+} \left\{ B^{-1} D_{-} B + Q_{-}^{(0)} + A_{-}^0 + \eta A_{-} \right\} \right\rangle, \tag{C.0.4}
\end{aligned}$$

where $F_{+-}^A = [\partial_{+} - A_{+}, \partial_{-} + \eta A_{-}] = \eta \partial_{+} A_{-} + \partial_{-} A_{+} - \eta [A_{+}, A_{-}]$. Consider now the components of the curvature F_L^A of the following Lax-Gauge connection

$$\begin{aligned}
A_{-}^{Lax}(A) &= \eta A_{-} - B \left(E_{-}^{(-1)} + \psi_{-}^{(-1/2)} + X Q_{-}^{(0)} \right) B^{-1} \\
A_{+}^{Lax}(A) &= -A_{+} - D_{+} B B^{-1} + \left(E_{+}^{(+1)} + \psi_{+}^{(+1/2)} + Y Q_{+}^{(0)} \right), \tag{C.0.5}
\end{aligned}$$

given by

$$\begin{aligned}
F_{+-}^{(+1)} &= -X [E_{+}^{(+1)}, B Q_{-}^{(0)} B^{-1}] \\
F_{+-}^{(+1/2)} &= -D_{-} \psi_{+}^{(+1/2)} + [B \psi_{-}^{(-1/2)} B^{-1}, E_{+}^{(+1)}] + X [B Q_{-}^{(0)} B^{-1}, \psi_{+}^{(+1/2)}] \\
F_{+-}^{(0)} &= D_{-} (D_{+} B B^{-1}) + F_{+-}^A - [E_{+}^{(+1)}, B E_{-}^{(-1)} B^{-1}] - [\psi_{+}^{(+1/2)}, B \psi_{-}^{(-1/2)} B^{-1}] - \\
&\quad - XY [Q_{+}^{(0)}, B Q_{-}^{(0)} B^{-1}] - (Y D_{-} Q_{+}^{(0)} + X B D_{+} Q_{-}^{(0)} B^{-1}) \\
F_{+-}^{(-1/2)} &= -B \left(D_{+} \psi_{-}^{(-1/2)} - [E_{-}^{(-1)}, B^{-1} \psi_{+}^{(+1/2)} B] - Y [\psi_{-}^{(-1/2)}, B^{-1} Q_{+}^{(0)} B] \right) B^{-1} \\
F_{+-}^{(-1)} &= -Y B [E_{-}^{(-1)}, B^{-1} Q_{+}^{(0)} B] B^{-1}.
\end{aligned}$$

We want to relate the variation (C.0.4) with the zero curvature representation of (C.0.5). Let's study the curvature first.

We have $F_{+-}^{(\pm 1)} = 0$ because $B \in \mathcal{K}$, the zero curvature $F_L^A = 0$ imply

$$\begin{aligned} D_- \psi_+^{(+1/2)} &= [B \psi_-^{(-1/2)} B^{-1}, E_+^{(+1)}] + X [B Q_-^{(0)} B^{-1}, \psi_+^{(+1/2)}] \\ D_- (D_+ B B^{-1}) + F_{+-}^A &= [E_+^{(+1)}, B E_-^{(-1)} B^{-1}] + [\psi_+^{(+1/2)}, B \psi_-^{(-1/2)} B^{-1}] + \\ &\quad + XY [Q_+^{(0)}, B Q_-^{(0)} B^{-1}] + (Y D_- Q_+^{(0)} + X B D_+ Q_-^{(0)} B^{-1}) \\ D_+ \psi_-^{(-1/2)} &= [E_-^{(-1)}, B^{-1} \psi_+^{(+1/2)} B] + Y [\psi_-^{(-1/2)}, B^{-1} Q_+^{(0)} B]. \end{aligned}$$

On the other side, the $\psi^{(\pm 1/2)}$ equations of motion provide the identity

$$(Y D_- Q_+^{(0)} + X B D_+ Q_-^{(0)} B^{-1}) = -(X + Y) [\psi_+^{(+1/2)}, B \psi_-^{(-1/2)} B^{-1}] - 2XY [Q_+^{(0)}, B Q_-^{(0)} B^{-1}]$$

but $F_{+-}^{(0)} = 0$ gives

$$\begin{aligned} (Y D_- Q_+^{(0)} + X B D_+ Q_-^{(0)} B^{-1}) &= D_- (D_+ B B^{-1}) + F_{+-}^A - [E_+^{(+1)}, B E_-^{(-1)} B^{-1}] - \\ &\quad - [\psi_+^{(+1/2)}, B \psi_-^{(-1/2)} B^{-1}] - XY [Q_+^{(0)}, B Q_-^{(0)} B^{-1}]. \end{aligned}$$

Then, the self-consistency of this system of equations imply that

$$\begin{aligned} 0 &= D_- (D_+ B B^{-1}) + F_{+-}^A + (X + Y - 1) [\psi_+^{(+1/2)}, B \psi_-^{(-1/2)} B^{-1}] + (C.0.6) \\ &\quad + XY [Q_+^{(0)}, B Q_-^{(0)} B^{-1}] + [E_+^{(+1)}, B E_-^{(-1)} B^{-1}]. \end{aligned}$$

Now let's study the variation (C.0.4) from down to up. In this situation the A equations of motion imply

$$Q_-^{(0)} = -B^{-1} D_- B - (A_-^0 + \eta A_-) \quad (C.0.7)$$

$$Q_+^{(0)} = -D_+ B B^{-1} - (\eta A_+^0 + A_+) \quad (C.0.8)$$

from which we get the identity ($[F_{+-}^A, B] B^{-1} = [D_+, D_-] B B^{-1}$)

$$\begin{aligned} &D_- Q_+^{(0)} - B D_+ Q_-^{(0)} B^{-1} \quad (C.0.9) \\ &= [F_{+-}^A, B] B^{-1} + B D_+ (A_-^0 + \eta A_-) B^{-1} - D_- (\eta A_+^0 + A_+). \end{aligned}$$

The $\psi^{(\pm 1/2)}$ equations of motion can be written as $F_{+-}^{(+1/2)} = F_{+-}^{(-1/2)} = 0$ if $\lambda = X = Y$. Finally, to write the B equations of motion as $F_{+-}^{(0)} = 0$ we need to impose the following integrability condition

$$\begin{aligned} 0 &= D_- Q_+^{(0)} + B D_+ Q_-^{(0)} B^{-1} \\ &= F_{+-}^A - D_- (\eta A_+^0 + A_+) + B (F_{+-}^A - D_+ (A_-^0 + \eta A_-)) B^{-1} + \\ &\quad + 2(2\lambda - 1) [\psi_+^{(+1/2)}, B \psi_-^{(-1/2)} B^{-1}] + \\ &\quad + 2\lambda^2 [Q_+^{(0)}, B Q_-^{(0)} B^{-1}] + 2 [E_+^{(+1)}, B E_-^{(-1)} B^{-1}], \end{aligned}$$

where we have used (C.0.7), (C.0.8), (C.0.9) and (C.0.6). We will demand cancelation of each term independently, this gives the conditions (4.1.3) and the result follows. ■

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