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Construction of type-II Bäcklund transformation for the mKdV hierarchy

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Abstract
From an algebraic construction of the mKdV hierarchy we observe that the space component of the Lax operator plays the role of a universal algebraic object. This fact induces the universality of a gauge transformation that relates two field configurations of a given member of the hierarchy. Such gauge transformation generates the Bäcklund transformation (BT). In this paper we propose a systematic construction of BT for the entire mKdV hierarchy from the known type-II BT of the sinh-Gordon theory. We explicitly construct the BT of the first few integrable models associated to positive and negative grade-time evolutions. Solutions of these transformations for several cases describing the transition from vacuum–vacuum and the vacuum to one-soliton solutions which determines the value for the auxiliary field and the Bäcklund parameter respectively, independently of the model. The same follows for the scattering of two one-soliton solutions. The resultant delay is determined by a condition independent of the model considered.

Keywords: integrable hierarchies, solitons, Bäcklund transformation

1. Introduction

Bäcklund transformation (BT) was introduced a long time ago and relates two solutions of certain non-linear equation of motion in a lower order differential equations. BT appears as a peculiar feature of integrable models and generates a construction of an infinite sequence of soliton solutions from a non-linear superposition principle (see [1] for a review).

More recently BT has been employed to describe integrable defects in the sense that two solutions of an integrable model may be interpolated by a defect at certain spatial position. The integrability is preserved if, at the defect position, the two solutions are described by a BT. The formulation is such that, canonical linear momentum and energy are no longer conserved and modifications to ensure its conservation need to be added to take into account the contribution of the defect [2]. A few well known (relativistic) integrable models as the sine (sinh)-Gordon (SG) [4], Lund-Regge [3] and other (non-relativistic) models as non-linear Schroedinger (NLS), mKdV, etc have been studied within such context [4].

The first type of BT involves only the fields of the theory and is called type I. In particular, it may be observed that the *space component* of the type I BT for the mKdV and SG equations coincides for their corresponding fields [1]. On the other hand, these two equations are known to belong to the same integrable hierarchy which is characterized by a common (space) Lax operator. In fact, an infinite number of integrable equations of motion may be systematically constructed from the same Lax operator and the set of equations are known as an integrable hierarchy (see for instance [7]). Each equation of motion describes the time evolution of a field according to some integer graded object of a Lie algebraic origin. In [6] it was observed that all integrable equations within the same hierarchy share the same space component of the type I BT. They differ by the time component BT which was constructed from the equations of motion.

More recently a new type of BT involving auxiliary fields was shown to be compatible with the equations of motion for the sine (sinh)-Gordon and Tzitzeica models [5]. These are known as type II BT. In [8] it was shown that they may be constructed from gauge transformation relating two field configurations of the same equation of motion.

The purpose of this paper is to extend such results to type II BT to all integrable equations of the mKdV hierarchy. Here we observe that the space component of the type II BT remains the samewithin the hierarchy and have developed a systematic construction for the time component for several cases.

In section 2 we review the construction of integrable hierarchies from the algebraic approach. We show the mKdV and SG models represent the first examples of the positive ($t_N = t_3$) and negative ($t_N = t_{-1}$) sub-hierarchies respectively. Higher (lower) grade examples are also developed. In section 3 we discuss the type II BT for the SG models proposed in [5] and present a gauge matrix K that interpolates between two SG solutions [8]. Such gauge matrix acts on a two dimensional gauge potentials of the zero curvature representation.

In section 4, assuming K to be *universal within all models of the hierarchy*, we extend the construction to the time component BT for the positive sub-hierarchy. We consider explicitly the mKdV ($t_N = t_3$) and the next non-trivial equation for $t_N = t_5$. We show that the compatibility of the two components of the BT indeed generate the correct equations of motion. In section 5 we develop the formalism for the negative grade sub-hierarchy and develop cases for t_N, $N = -3, -5$. Again K plays the role of universal object which leads to a compatible pair of BT.

Finally, in section 6 we discuss the first three simplest solutions for the BT. The first solution consists of two vacuum solutions which fixes a condition for the auxiliar field Λ. The second describes the transition between the vacuum and one-soliton solutions which establishes a condition for the Bäcklund parameter.We have considered several cases where t_N, $N = -1, \pm 3, \pm 5$ and in all of them, the very same condition arises. The third solution consists of the scattering of two one-soliton solutions where the delay (phase shift) R is determined. Again considering the cases where $t_N = -1, \pm 3, \pm 5$ we found the very same solution for R. These examples indicate an universality of Bäcklund parameters and delays within the different models of the hierarchy.

2. The mKdV hierarchy

In [7] it is explained how the mKdV hierarchy can be constructed from a basic *universal object* known as Lax operator L defined as

$$L = \partial_x + E^{(1)} + A_0, \tag{2.1}$$

where the fundamental Lie algebraic elements $E^{(1)} = E_{\alpha}^{(0)} + E_{-\alpha}^{(1)}$ and $A_0 = v(x, t)h^{(0)}$ are obtained from a decomposition of an affine (centerless) $\hat{\mathcal{G}} = \hat{sl}(2)$ Kac Moody algebra

$$\begin{aligned} &\left[h^{(m)}, E_{\pm\alpha}^{(n)}\right] = \pm\, 2E_{\pm\alpha}^{(m+n)}, \\ &\left[E_{\alpha}^{(m)}, E_{-\alpha}^{(n)}\right] = h^{(m+n)} \qquad \left[d, T_i^{(n)}\right] = nT_i^{(n)} \end{aligned} \tag{2.2}$$

into graded subspaces, $\hat{\mathcal{G}} = \oplus_a \mathcal{G}_a$, $a \in Z$. Here $T_i^{(n)}$ denotes either $h^{(n)}$ or $E_{\pm\alpha}^{(n)}$. For the mKdV hierarchy, such decomposition is induced by the grading operator (principal gradation) $Q = 2d + 1/2h^{(0)}$ such that the even and odd subspaces are respectively given by

$$\mathcal{G}_{2n} = \left\{h^{(n)}\right\} \qquad \mathcal{G}_{2n+1} = \left\{E_a^{(n)} + E_{-\alpha}^{(n+1)}, E_a^{(n)} - E_{-\alpha}^{(n+1)}\right\} \tag{2.3}$$

i.e.,

$$\left[Q, \ \mathcal{G}_{2n+1}\right] = (2n+1)\mathcal{G}_{2n+1}, \qquad \left[Q, \ \mathcal{G}_{2n}\right] = 2n\mathcal{G}_{2n} \tag{2.4}$$

and $[\mathcal{G}_a, \mathcal{G}_b] \in \mathcal{G}_{a+b}$. Furthermore, the choice of the constant grade one element $E^{(1)}$, defines the Kernel, i.e., $\mathcal{K} = \{\ \forall x \in \hat{\mathcal{G}}, \quad [x, E^{(1)}] = 0\}$ and induces a second decomposition of the affine algebra $\hat{\mathcal{G}} = \mathcal{K} \oplus \mathcal{M}$. It follows that for the mKdV hierarchy the Kernel has grade $2n+1$ and is generated by the combination

$$\mathcal{K} = \mathcal{K}_{2n+1} = \left\{E_a^{(n)} + E_{-\alpha}^{(n+1)}\right\}. \tag{2.5}$$

$\mathcal{M}$ is its complement and satisfies

$$[\mathcal{K}, \mathcal{K}] \subset \mathcal{K}, \qquad [\mathcal{K}, \mathcal{M}] \subset \mathcal{M}, \qquad [\mathcal{M}, \mathcal{M}] \subset \mathcal{K}.$$

In general the Lax operator (2.1) is constructed systematically from $A_0 \in \mathcal{M}_0 = \mathcal{G}_0 \cap \mathcal{M}$ and is parametrized by the physical fields of the theory.

Equations of motion for the *positive mKdV sub-hierarchy* are given by the zero curvature representation

$$\left[\partial_x + E^{(1)} + A_0, \quad \partial_{t_N} + D^{(N)} + D^{(N-1)} + \cdots + D^{(0)}\right] = 0. \tag{2.6}$$

The solution of equation (2.6) may be systematically constructed by considering $D^{(j)} \in \mathcal{G}_j$ and decomposed according to the graded structure as

$$\left[E^{(1)}, D^{(N)}\right] = 0 \tag{2.7}$$

$$\begin{aligned} \left[E^{(1)}, D^{(N-1)}\right] + \left[A_0, D^{(N)}\right] + \partial_x D^{(N)} &= 0 \\ \vdots &= \vdots \\ \left[A_0, D^{(0)}\right] + \partial_x D^{(0)} - \partial_{t_N} A_0 &= 0. \end{aligned} \tag{2.8}$$

The unknown $D^{(j)}$'s can be solved starting from the highest to the lowest grade projections as functionals of A_0 and its $x-$ derivatives. Notice that, in particular the highest grade equation, namely $[E^{(1)}, D^{(N)}] = 0$ implies $D^{(N)} \in \mathcal{K}$ and henceforth $N = 2n + 1$. If we consider the fields of the theory to parametrize $A_0 = v(x, t_N)h^{(0)} \in \mathcal{M}_0$, the equations of motion are obtained from the zero grade component (2.8). Examples are,

$$N = 3 \qquad 4\partial_{t_3} v = \partial_x\left(\partial_x^2 v - 2v^3\right) \qquad \text{mKdV} \tag{2.9}$$

$$N = 5 \qquad 16\partial_{t_5} v = \partial_x\Big(\partial_x^4 v - 10v^2\big(\partial_x^2 v\big) - 10v\big(\partial_x v\big)^2 + 6v^5\Big), \tag{2.10}$$

$$\begin{aligned} N = 7 \qquad 64\partial_{t_7} v = {} & \partial_x\Big(\partial_x^6 v - 70\big(\partial_x v\big)^2\big(\partial_x^2 v\big) - 42v\big(\partial_x^2 v\big)^2 - 56v\big(\partial_x v\big)\big(\partial_x^3 v\big)\Big) \\ & - \partial_x\Big(14v^2\partial_x^4 v - 140v^3\big(\partial_x v\big)^2 - 70v^4\big(\partial_x^2 v\big) + 20v^7\Big) \\ & \cdots \text{etc.} \end{aligned} \tag{2.11}$$

For the *negative mKdV sub-hierarchy* let us propose the following form for the zero curvature representation

$$\Big[\partial_x + E^{(1)} + A_0,\, \partial_{t_{-N}} + D^{(-N)} + D^{(-N+1)} + \cdots + D^{(-1)}\Big] = 0. \tag{2.12}$$

Differently from the positive sub-hierarchy case, the lowest grade projection now yields,

$$\partial_x D^{(-N)} + \Big[A_0,\, D^{(-N)}\Big] = 0,$$

a non-local equation for $D^{(-N)}$. The process follows recursively until we reach the zero grade projection

$$\partial_{t_{-N}} A_0 - \Big[E^{(1)},\, D^{(-1)}\Big] = 0 \tag{2.13}$$

which yields the evolution equation for field A_0 according to time $t = t_{-N}$ The simplest example is to take $N = 1$ when the zero curvature decomposes into

$$\begin{aligned} &\partial_x D^{(-1)} + \Big[A_0,\, D^{(-1)}\Big] = 0, \\ &\partial_{t_{-1}} A_0 - \Big[E^{(1)},\, D^{(-1)}\Big] = 0. \end{aligned} \tag{2.14}$$

In order to solve the first equation, we define the zero grade group element $B = \exp\left(\mathcal{G}_0\right)$ and define

$$D^{(-1)} = BE^{(-1)}B^{-1}, \qquad A_0 = -\partial_x BB^{-1}, \tag{2.15}$$

where $E^{(-1)} = E_\alpha^{(-1)} + E_{-\alpha}^{(0)}$. Under such parametrization the first equation (2.14) is automatically satisfied whilst the second becomes the well known (relativistic) Leznov–Saveliev equation,

$$\partial_{t_{-1}}\Big(\partial_x BB^{-1}\Big) + \Big[E^{(1)},\, BE^{(-1)}B^{-1}\Big] = 0 \tag{2.16}$$

which for $\hat{sl}(2)$ with principal gradation $Q = 2\lambda\frac{\mathrm{d}}{\mathrm{d}\lambda} + \frac{1}{2}h$, yields the SG equation

$$\partial_{t_{-1}}\partial_x\phi = \mathrm{e}^{2\phi} - \mathrm{e}^{-2\phi}, \qquad B = \mathrm{e}^{-\phi h}, \tag{2.17}$$

where $t_{-1} = z$, $x = \bar{z}$ are the light cone coordinates and $A_0 = -\partial_x BB^{-1} = vh \equiv \partial_x\phi h$.

For higher values of $N = 3, 5, \ldots$[1] we find

$$\partial_{t_{-3}}\partial_x\phi = 4\mathrm{e}^{-2\phi}d^{-1}\Big(\mathrm{e}^{2\phi}d^{-1}(\sinh 2\phi)\Big) + 4\mathrm{e}^{2\phi}d^{-1}\Big(\mathrm{e}^{-2\phi}d^{-1}(\sinh 2\phi)\Big) \tag{2.18}$$

[1] For the negative sub-hierarchy there is no restriction upon the values of N to be odd. In particular integrable models for N even present non-trivial vacuum solution as it was explicitly discussed in [7].

$$\begin{aligned}\partial_{t_{-5}}\partial_x\phi &= 8\mathrm{e}^{-2\phi}d^{-1}\\ &\quad\times\Big(\mathrm{e}^{2\phi}d^{-1}\big(\mathrm{e}^{-2\phi}d^{-1}\big(\mathrm{e}^{2\phi}d^{-1}(\sinh 2\phi)\big)+\mathrm{e}^{2\phi}d^{-1}\big(\mathrm{e}^{-2\phi}d^{-1}(\sinh 2\phi)\big)\big)\Big)\\ &\quad+8\mathrm{e}^{2\phi}d^{-1}\\ &\quad\times\Big(\mathrm{e}^{-2\phi}d^{-1}\big(\mathrm{e}^{-2\phi}d^{-1}\big(\mathrm{e}^{2\phi}d^{-1}(\sinh 2\phi)\big)+\mathrm{e}^{2\phi}d^{-1}\big(\mathrm{e}^{-2\phi}d^{-1}(\sinh 2\phi)\big)\big)\Big),\end{aligned} \tag{2.19}$$

where $d^{-1}f = \int^x f(y)\mathrm{d}y$.

3. On the sinh-Gordon type II BT

The Lax pair $L = \partial_x + A_x$ in (2.1) for the SG model where $v(x, t) = \partial_x\phi(x, t)$ is specified by

$$A_x = \partial_x\phi h^{(0)} + E_\alpha^{(0)} + E_{-\alpha}^{(1)} = \begin{bmatrix}\partial_x\phi & 1\\ \lambda & -\partial_x\phi\end{bmatrix} \tag{3.1}$$

and for $t = t_{-1}$, we find from (2.15)

$$A_{t_{-1}} = BE^{(-1)}B^{-1} = \mathrm{e}^{-2\phi}E_\alpha^{(-1)} + \mathrm{e}^{2\phi}E_{-\alpha}^{(0)} = \begin{bmatrix}0 & \frac{1}{\lambda}\mathrm{e}^{-2\phi}\\ \mathrm{e}^{2\phi} & 0\end{bmatrix}. \tag{3.2}$$

The zero curvature condition

$$\left[\partial_t + A_t, \partial_x + A_x\right] = 0 \tag{3.3}$$

for $t = t_{-1}$ leads to the SG equation,

$$\partial_{t_{-1}}\partial_x\phi - 2\sinh(2\phi) = 0 \tag{3.4}$$

in the light cone coordinates (x, t_{-1}). In order to determine the type-II BT for the SG equation (3.4) proposed in [4] a gauge matrix $K(\phi_1, \phi_2)$ such that

$$\partial_\mu K = KA_\mu(\phi_1) - A_\mu(\phi_2)K, \qquad x_\mu = t_{-1}, x \tag{3.5}$$

was constructed in [8] (see appendix B). Let K be given by

$$K = \begin{bmatrix}1 - \dfrac{\mathrm{e}^q}{\lambda\sigma^2} & \dfrac{\mathrm{e}^{\Lambda-p}}{2\lambda\sigma}\left(\mathrm{e}^q + \mathrm{e}^{-q} + \eta\right)\\ -\dfrac{2}{\sigma}\mathrm{e}^{p-\Lambda} & 1 - \dfrac{\mathrm{e}^{-q}}{\lambda\sigma^2}\end{bmatrix}, \tag{3.6}$$

where

$$p = \phi_1 + \phi_2, \qquad q = \phi_1 - \phi_2, \tag{3.7}$$

$\Lambda(x, t_N)$ is an auxiliary field and σ the Bäcklund parameter. Equation (3.5) with $A_\mu = A_x$ and K given by (3.1) and (3.6) respectively leads to

$$\partial_x q = -\frac{1}{2\sigma}\mathrm{e}^{\Lambda-p}\left(\mathrm{e}^q + \mathrm{e}^{-q} + \eta\right) - \frac{2}{\sigma}\mathrm{e}^{p-\Lambda}, \tag{3.8}$$

$$\partial_x \Lambda = \frac{1}{2\sigma} e^{\Lambda - p}\left(e^q - e^{-q}\right). \tag{3.9}$$

For $\mu = t_{-1}$ in (3.5) we find for $A_{t_{-1}}$ given by (3.2),

$$\partial_{t_{-1}} q = -2\sigma e^{-\Lambda} - \frac{\sigma}{2} e^{\Lambda}\left(e^q + e^{-q} + \eta\right), \tag{3.10}$$

$$\partial_{t_{-1}}(p - \Lambda) = -\frac{\sigma}{2} e^{\Lambda}\left(e^q - e^{-q}\right). \tag{3.11}$$

Equations (3.8)–(3.11) are the type II BT for the SG introduced in [4] in the sense that if we act with $\partial_{t_{-1}}$ in equations (3.8) and (3.9) we obtain, using (3.10) and (3.11) relations consistent with the SG equation for ϕ_1 and ϕ_2. Conversely, acting with ∂_x in equations (3.10) and (3.11) and using (3.8) and (3.9) we obtain similar result.

In particular, from equations (3.8) and (3.10) we find the following expression for Λ,

$$\begin{aligned} e^{-\Lambda} &= -\frac{1}{2\sigma} \frac{\sigma^2 \partial_x q - e^{-p} \partial_{t_{-1}} q}{e^p - e^{-p}}, \\ e^{\Lambda} &= \frac{2}{\sigma} \frac{\sigma^2 \partial_x q - e^{p} \partial_{t_{-1}} q}{\left(e^p - e^{-p}\right)\left(e^q + e^{-q} + \eta\right)}. \end{aligned} \tag{3.12}$$

These implies constraints upon the two field configurations ϕ_1 and ϕ_2. In particular their compatibility will fix the Bäcklund parameter and provide information about the scattering data (as we shall see in section 6).

4. Type II BT for the positive mKdV sub-hierarchy

- $t_N = t_3$ Based upon the fact that the Lax operator L in (2.1) is common to the entire hierarchy we propose the *conjecture* that the gauge matrix K is also common and provide the type II BT to all members of the hierarchy. In particular this fact implies that the space component of the BT is the same for all members of the hierarchy. In order to support our conjecture we now consider the mKdV equation

$$4\partial_{t_3} v = \partial_x^3 v - 6v^2 \partial_x v, \qquad v = \partial_x \phi. \tag{4.1}$$

We shall consider A_x given by (2.1) and (3.1) together with $A_t = A_{t_3}$,

$$\begin{aligned} A_{t_3} &= D^{(3)} + D^{(2)} + D^{(1)} + D^{(0)}, \\ &= E_\alpha^{(1)} + E_{-\alpha}^{(2)} + v h^{(1)} + \frac{1}{2}\left(\partial_x v - v^2\right) E_\alpha^{(0)} - \frac{1}{2}\left(\partial_x v + v^2\right) E_{-\alpha}^{(1)} \\ &\quad + \frac{1}{4}\left(\partial_x^2 v - 2v^3\right) h^{(0)} \\ &= \begin{bmatrix} \lambda v + \frac{1}{4}\partial_x^2 v - \frac{1}{2} v^3 & \lambda - \frac{1}{2} v^2 + \frac{1}{2}\partial_x v \\ \lambda^2 - \frac{\lambda}{2} v^2 - \frac{\lambda}{2}\partial_x v & -\lambda v - \frac{1}{4}\partial_x^2 v + \frac{1}{2} v^3 \end{bmatrix} \end{aligned} \tag{4.2}$$

to generate the mKdV equation (4.1) from the zero curvature representation (3.3).

In order to derive the time component of the type II BT for the mKdV equation we employ equation (3.5) with $x_\mu = t_3$, i.e.,

$$\partial_{t_3} K = K A_{t_3}\left(\phi_1\right) - A_{t_3}\left(\phi_2\right) K, \tag{4.3}$$

where the gauge matrix K is given by (3.6).

The matrix element 11 of the above equation leads to:

$$\lambda^{-1}: 4\partial_{t_3}(\phi_1 - \phi_2) = \partial_x^3(\phi_1 - \phi_2) - 2\left((\partial_x\phi_1)^3 - (\partial_x\phi_2)^3\right), \tag{4.4}$$

$$\lambda^0: \sigma^2\partial_x^2(v_2 - v_1) - 2\sigma^2\left(v_2^3 - v_1^3\right) = 4(v_2 - v_1)e^q - 4\sigma\left(v_2^2 - \partial_x v_2\right)e^{p-\Lambda} - \sigma\left(v_1^2 + \partial_x v_1\right)\left(e^q + e^{-q} + \eta\right)e^{\Lambda-p}, \tag{4.5}$$

$$\lambda^1: \partial_x q = -\frac{1}{2\sigma}e^{\Lambda-p}\left(e^q + e^{-q} + \eta\right) - \frac{2}{\sigma}e^{p-\Lambda}. \tag{4.6}$$

Equation (4.4) is trivially satisfied for $v_1 = \partial_x\phi_1$ and $v_2 = \partial_x\phi_2$ satisfying the mkdV equation (4.1). Equation (4.6) coincides with equation (3.8) and represents the space component of the BT for the mKdV. Acting twice with ∂_x in (4.6) and using (3.9), after some tedious calculation it can be shown that (4.5) is identically satisfied.

The matrix elements 12 and 21 of (5.4) yields respectively

$$\lambda^{-1}: \sigma\partial_{t_3}\Lambda\left(e^q + e^{-q} + \eta\right) + \sigma\partial_{t_3}q\left(e^q - e^{-q}\right) = \left(\left(v_1^2 - \partial_x v_1\right)e^{p+q} - \left(v_2^2 - \partial_x v_2\right)e^{p-q}\right)e^{-\Lambda}, \tag{4.7}$$

$$\lambda^0: \sigma^2\partial_x(v_1 - v_2) - \sigma^2\left(v_1^2 - v_2^2\right) = 2\left(e^q - e^{-q}\right) + \sigma(v_1 + v_2)\left(e^q + e^{-q} + \eta\right)e^{\Lambda-p}, \tag{4.8}$$

$$\lambda^1: 0 = 0 \tag{4.9}$$

and

$$\lambda^0: 4\sigma\partial_{t_3}\Lambda = \left(v_1^2 + \partial_x v_1\right)e^{\Lambda-q-p} - \left(v_2^2 + \partial_x v_2\right)e^{\Lambda+q-p}, \tag{4.10}$$

$$\lambda^1: \sigma^2\partial_x(v_1 - v_2) + \sigma^2\left(v_1^2 - v_2^2\right) = 2\left(e^q - e^{-q}\right) - 4\sigma(v_1 + v_2)e^{p-\Lambda}, \tag{4.11}$$

$$\lambda^2: 0 = 0. \tag{4.12}$$

Equations (4.7) and (4.11) are direct consequences of equations (3.8) and (3.9). The matrix element 22 now yields,

$$\lambda^{-1}: 4\partial_{t_3}q = \partial_x^2(v_1 - v_2) - 2\left(v_1^3 - v_2^3\right), \tag{4.13}$$

$$\lambda^0: \sigma^2\partial_x^2(v_1 - v_2) - 2\sigma^2\left(v_1^3 - v_2^3\right) = 4\sigma\left(v_1^2 - \partial_x v_1\right)e^{p-\Lambda} + \sigma\left(v_2^2 + \partial_x v_2\right)\left(e^q + e^{-q} + \eta\right)e^{-p+\Lambda} + 4(v_1 - v_2)e^{-q}. \tag{4.14}$$

Equation (4.13) is the mKdV equation for $v_1 = \partial_x\phi_1$ and $v_2 = \partial_x\phi_2$. Acting with ∂_x^2 in equation (3.8) and employing (3.9) it can be shown that (4.14) is trivially satisfied. It then follows from (9.10) in (9.7) that the following pair of equations,

$$16\sigma^3\partial_{t_3}q = e^{\Lambda-p}\left(e^q + e^{-q} + \eta\right)\left[2\sigma^2\left(\partial_{xp}^2 + \partial_x^2 q\right) + \sigma^2\left(\partial_x p + \partial_x q\right)^2 - 8e^q\right] + 4e^{p-\Lambda}\left[-2\sigma^2\left(\partial_{xp}^2 + \partial_x^2 q\right) + \sigma^2\left(\partial_x p + \partial_x q\right)^2 - 8e^{-q}\right] + 16\sigma\partial_x p\left(e^q + e^{-q} + \eta\right), \tag{4.15}$$

$$4\sigma\partial_{t_3}\Lambda = \left(v_1^2 + \partial_x v_1\right)e^{\Lambda-q-p} - \left(v_2^2 + \partial_x v_2\right)e^{\Lambda+q-p} \tag{4.16}$$

together with (3.8) and (3.9) correspond to the type-II BT for the mKdV equation. Acting with ∂_x in equation (4.15) and ∂_{t_3} in (3.8) we find in both cases consistency with the mKdV equation (4.1).

Further, from (3.8) and (4.15) we find

$$\frac{4}{\sigma}e^{-\Lambda} = -\frac{2\sigma^2\partial_{t_3}q + \partial_x q\left(\sigma^2\left(\partial_x\phi_1\right)^2 + 2\left(e^q+\eta\right) + \sigma^2\partial_x^2\phi_1\right) - 4\partial_x\phi_1\left(e^q + e^{-q}+\eta\right)}{e^p\left(\sigma^2\partial_x^2\phi_1 - \left(e^q - e^{-q}\right)\right)}, \tag{4.17}$$

$$\frac{4}{\sigma}e^{\Lambda} = \frac{8\sigma^2\partial_{t_3}q + \partial_x q\left(4\sigma^2\left(\partial_x\phi_1\right)^2 + 8\left(e^q+\eta\right) - 4\sigma^2\partial_x^2\phi_1\right) - 16\partial_x\phi_1\left(e^q + e^{-q}+\eta\right)}{e^{-p}\left(e^q + e^{-q}+\eta\right)\left(\sigma^2\partial_x^2\phi_1 - \left(e^q - e^{-q}\right)\right)} \tag{4.18}$$

- $t_N = t_5$ For $t = t_5$ we find

$$16\sigma^5\partial_{t_5}q = 4Ae^{p-\Lambda} + B\left(e^q + e^{-q} + \eta\right)e^{\Lambda-p} + C \tag{4.19}$$

$$\begin{aligned}16\sigma\partial_{t_5}\Lambda = &\Big(\partial_x^4\phi_1 - \left(\partial_x^2\phi_1\right)^2 + 2\left(\partial_x\phi_1\right)\left(\partial_x^3\phi_1\right)\\ &- 6\left(\partial_x\phi_1\right)^2\left(\partial_x^2\phi_1\right) - 3\left(\partial_x\phi_1\right)^4\Big)e^{\Lambda-q-p}\\ &- \Big(\partial_x^4\phi_2 - \left(\partial_x^2\phi_2\right)^2 + 2\left(\partial_x\phi_2\right)\left(\partial_x^3\phi_2\right)\\ &- 6\left(\partial_x\phi_2\right)^2\left(\partial_x^2\phi_2\right) - 3\left(\partial_x\phi_2\right)^4\Big)e^{\Lambda+q-p}\end{aligned} \tag{4.20}$$

with

$$\begin{aligned}A = &8\left(1 - \eta e^q - \eta^2\right) + 4\sigma^2\left(e^q+\eta\right)\left(\partial_x^2\phi_1 - \left(\partial_x\phi_1\right)^2\right)\\ &- \sigma^4\Big[\left(\partial_x^2\phi_1\right)^2 + \partial_x^4\phi_1 - 2\left(\partial_x\phi_1\right)\left(\partial_x^3\phi_1\right)\\ &- 6\left(\partial_x\phi_1\right)^2\left(\partial_x^2\phi_1\right) + 3\left(\partial_x\phi_1\right)^4\Big],\end{aligned} \tag{4.21}$$

$$\begin{aligned}B = &8\left(1 - \eta e^{-q} - \eta^2\right) - 4\sigma^2\left(e^{-q}+\eta\right)\left(\partial_x^2\phi_1 + \left(\partial_x\phi_1\right)^2\right)\\ &+ \sigma^4\Big[-\left(\partial_x^2\phi_1\right)^2 + \partial_x^4\phi_1 + 2\left(\partial_x\phi_1\right)\left(\partial_x^3\phi_1\right)\\ &- 6\left(\partial_x\phi_1\right)^2\left(\partial_x^2\phi_1\right) - 3\left(\partial_x\phi_1\right)^4\Big],\end{aligned} \tag{4.22}$$

and

$$C = -32\sigma\eta\left(\partial_x\phi_1\right)\left(e^{q} + e^{-q} + \eta\right) + 8\sigma^3\left(\partial_x^3\phi_1 - 2\left(\partial_x\phi_1\right)^3\right)\left(e^{q} + e^{-q} + \eta\right). \quad (4.23)$$

If we act with ∂_x in the equation (4.19) and use (3.8) and (3.9) we recover the equation of motion (2.11). Conversely, acting ∂_{t_5} in the equation (3.8) and using (4.19) and (4.20) we obtain the same result, (2.11).

Using (4.19) together with (3.8) we find the following expression for Λ.

$$e^{\Lambda} = \frac{16\sigma^5\partial_{t_5}q + 2\sigma A\partial_x q - C}{e^{-p}(B - A)\left(e^{q} + e^{-q} + \eta\right)},$$
$$e^{-\Lambda} = \frac{16\sigma^5\partial_{t_5}q + 2\sigma B\partial_x q - C}{4e^{p}(A - B)}. \quad (4.24)$$

As in equations (3.12), (4.17) and (4.18) equation (4.24) provide constraints upon the Bäcklund solutions. These will be specified later in section 6. Note that the auxiliary field Λ depends upon the time evolution and therefore the expressions for $e^{\pm\Lambda}$ varies from model to model.

5. Type-II BT for negative grade mKdV sub-hierarchy

- $t_N = t_{-3}$

  We now derive the type-II BT for the $t = t_{-3}$ equation,

$$\partial_{t_{-3}}\partial_x\phi = 4e^{-2\phi}d^{-1}\left(e^{2\phi}d^{-1}(\sinh 2\phi)\right) + 4e^{2\phi}d^{-1}\left(e^{-2\phi}d^{-1}(\sinh 2\phi)\right) \quad (5.1)$$

obtained from the zero curvature representation (3.3) with A_x given by (3.1) and

$$A_{t_{-3}}(\phi) = e^{-2\phi}E_{\alpha}^{(-2)} + e^{2\phi}E_{-\alpha}^{(-1)} - 2I(\phi)h^{(-1)} - 4e^{-2\phi}\int^x e^{2\phi}I(\phi)E_{\alpha}^{(-1)} + 4e^{2\phi}\int^x e^{-2\phi}I(\phi)E_{-\alpha}^{(0)}, \quad (5.2)$$

where

$$I_i\left(\phi_i\right) = \int^x \sinh\left(2\phi_i(y)\right)dy. \quad (5.3)$$

The Bäcklund is generated by

$$\partial_{t_{-3}}K = KA_{t_{-3}}\left(\phi_1\right) - A_{t_{-3}}\left(\phi_2\right)K, \quad (5.4)$$

where the gauge matrix K is the same given by (3.6). It therefore follows that the matrix element 11 of the above equation leads to:

$$\lambda^{-2}: I_1 - I_2 = -\frac{\sigma e^{\Lambda}}{4}\left(e^{q} + e^{-q} + \eta\right) - \sigma e^{-\Lambda} \quad (5.5)$$

$$\lambda^{-1}: \partial_{t_{-3}}q = 2\sigma^2\left(I_1 - I_2\right)e^{-q} - 2\sigma e^{\Lambda}\left(e^{q} + e^{-q} + \eta\right)\int^x I_1 e^{-2\phi_1} + 8\sigma e^{-\Lambda}\int^x I_2 e^{2\phi_2} \quad (5.6)$$

the matrix element 22 to

$$\lambda^{-2}: \partial_{t_{-3}} q = 2\sigma^2 \left(I_1 - I_2\right) e^{q} - 2\sigma e^{\Lambda} \left(e^{q} + e^{-q} + \eta\right) \int^x I_2 e^{-2\phi_2} + 8\sigma e^{-\Lambda} \int^x I_1 e^{2\phi_1}. \tag{5.7}$$

The element 12 leads to

$$\lambda^{-3}: \left(I_1 + I_2\right)\left(e^{q} + e^{-q} + \eta\right) = \sigma e^{-\Lambda}\left(e^{q} - e^{-q}\right) - \frac{4}{\sigma} e^{-\Lambda} \int^x \left(I_1 e^{2\phi_1} - I_2 e^{2\phi_2}\right), \tag{5.8}$$

$$\lambda^{-1}: \partial_{t_{-3}}(\Lambda - p)\left(e^{q} + e^{-q} + \eta\right) + \partial_{t_{-3}} q \left(e^{q} - e^{-q}\right) = -8\sigma e^{-\Lambda}\left(e^{-q} \int^x I_1 e^{2\phi_1} - e^{q} \int I_2 e^{2\phi_2}\right). \tag{5.9}$$

Element 21 yields,

$$\lambda^{-1}: I_1 + I_2 = -\frac{\sigma}{4} e^{\Lambda}\left(e^{q} - e^{-q}\right) + \frac{e^{\Lambda}}{\sigma} \int^x \left(I_1 e^{-2\phi_1} - I_2 e^{-2\phi_2}\right), \tag{5.10}$$

$$\lambda^{-1}: \partial_{t_{-3}}(p - \Lambda) = -2\sigma e^{\Lambda + q} \int^x I_1 e^{-2\phi_1} + 2\sigma e^{\Lambda - q} \int^x I_2 e^{-2\phi_2}. \tag{5.11}$$

Acting with ∂_x in equations (5.5), (5.8) and (5.10) and using (3.8) and (3.9) with we find that they are identically satisfied. It therefore follows that the type-II BT for the equation of motion (5.1) is given by

$$\partial_{t_{-3}}(p - \Lambda) = -2\sigma e^{\Lambda + q} \int^x I_1 e^{-2\phi_1} + 2\sigma e^{\Lambda - q} \int^x I_2 e^{-2\phi_2}, \tag{5.12}$$

$$\partial_{t_{-3}} q = 2\sigma^2 \left(I_1 - I_2\right) e^{-q} - 2\sigma e^{\Lambda}\left(e^{q} + e^{-q} + \eta\right) \int^x I_1 e^{-2\phi_1} + 8\sigma e^{-\Lambda} \int^x I_2 e^{2\phi_2}, \tag{5.13}$$

$$\partial_{t_{-3}} q = 2\sigma^2 \left(I_1 - I_2\right) e^{q} - 2\sigma e^{\Lambda}\left(e^{q} + e^{-q} + \eta\right) \int^x I_2 e^{-2\phi_2} + 8\sigma e^{-\Lambda} \int^x I_1 e^{2\phi_1}, \tag{5.14}$$

$$\partial_{t_{-3}}(\Lambda - p)\left(e^{q} + e^{-q} + \eta\right) + \partial_{t_{-3}} q \left(e^{q} - e^{-q}\right) = -8\sigma e^{-\Lambda}\left(e^{-q} \int^x I_1 e^{2\phi_1} - e^{q} \int I_2 e^{2\phi_2}\right) \tag{5.15}$$

together with

$$\partial_x \Lambda = \frac{1}{2\sigma} e^{\Lambda - p}\left(e^{q} - e^{-q}\right), \tag{5.16}$$

$$\partial_x q = -\frac{1}{2\sigma} e^{\Lambda - p}\left(e^{q} + e^{-q} + \eta\right) - \frac{2}{\sigma} e^{p - \Lambda}. \tag{5.17}$$

Equations (5.12), (5.13) and (5.16), (5.17) are compatible in the sense that acting with $\partial_{t_{-3}}$ in (5.16), (5.17) we recover the same equation of motion (5.1). Conversely, applying ∂_x on equations (5.13) and (5.14) we obtain the equation of motion (5.1) after adding both results and using the equations (5.16) and (5.17).
From the above equations we find

$$e^{\Lambda} = -\frac{1}{2\sigma} \times \frac{(\partial_{t_{-3}} q)e^{p} + 4\sigma^2(\partial_x q)\int^x I(\phi_2)e^{2\phi_2}dy - 2\sigma^2\big(I(\phi_1) - I(\phi_2)\big)e^{-q+p}}{(e^{q} + e^{-q} + \eta)\left(e^{p}\int^x I(\phi_1)e^{-2\phi_1}dy + e^{-p}\int^x I(\phi_2)e^{2\phi_2}dy\right)}, \tag{5.18}$$

and

$$e^{-\Lambda} = \frac{1}{8\sigma} \times \frac{(\partial_{t_{-3}} q)e^{-p} - 4\sigma^2(\partial_x q)\int^x I(\phi_1)e^{-2\phi_1}dy - 2\sigma^2\big(I(\phi_1) - I(\phi_2)\big)e^{-q-p}}{\left(e^{p}\int^x I(\phi_1)e^{-2\phi_1}dy + e^{-p}\int^x I(\phi_2)e^{2\phi_2}dy\right)}. \tag{5.19}$$

- $t_N = t_{-5}$
  Similarly we have worked out the type II BT for $t = t_{-5}$ evolution equation (2.19). The general construction (2.12) leads to

$$\begin{aligned} A_{t_{-5}} = {} & e^{-2\phi}E_{\alpha}^{(-3)} + e^{2\phi}E_{-\alpha}^{(-2)} - 2I(\phi)h^{(-2)} - 4e^{-2\phi}\int^x I(\phi)e^{2\phi}E_{\alpha}^{(-2)} \\ & + 4e^{2\phi}\int^x I(\phi)e^{-2\phi}E_{-\alpha}^{(-1)} - 4W(\phi)h^{(-1)} - 8e^{-2\phi}\int^x W(\phi)e^{2\phi}E_{\alpha}^{(-1)} \\ & + 8e^{2\phi}\int^x W(\phi)e^{-2\phi}E_{-\alpha}^{(0)}, \end{aligned} \tag{5.20}$$

where

$$W(\phi) = \int^x \left(e^{2\phi}\int^y I(\phi)e^{-2\phi} + e^{-2\phi}\int^y I(\phi)e^{2\phi}\right) \tag{5.21}$$

and the gauge transformation (3.5) for $t = t_{-5}$ with K given by (3.6) yields the following BT,

$$\begin{aligned} \partial_{t_{-5}} q = {} & -4\sigma^2\big(W(\phi_2) - W(\phi_1)\big)e^{-q} \\ & - 4\sigma e^{\Lambda}(e^{q} + e^{-q} + \eta)\int^x W(\phi_1)e^{-2\phi_1} \\ & + 16\sigma e^{-\Lambda}\int^x W(\phi_2)e^{2\phi_2}, \end{aligned} \tag{5.22}$$

$$\begin{aligned} \partial_{t_{-5}}(p - \Lambda) = {} & -4\sigma e^{\Lambda+q}\int^x W(\phi_1)e^{-2\phi_1} \\ & + 4\sigma e^{\Lambda-q}\int^x W(\phi_2)e^{-2\phi_2}, \end{aligned} \tag{5.23}$$

$$\begin{aligned} \partial_{t_{-5}} q = {} & -4\sigma^2\big(W(\phi_2) - W(\phi_1)\big)e^{q} - 4\sigma e^{\Lambda}(e^{q} + e^{-q} + \eta)\int^x W(\phi_2)e^{-2\phi_2} \\ & + 16\sigma e^{-\Lambda}\int^x W(\phi_1)e^{2\phi_1}, \end{aligned} \tag{5.24}$$

$$\partial_{t_{-5}}(\Lambda - p)\left(e^{q} + e^{-q} + \eta\right) = -\partial_{t_{-5}}q\left(e^{q} - e^{-q}\right) - 16\sigma e^{-\Lambda-q}\int^{x} W\left(\phi_1\right)e^{2\phi_1} + 16\sigma e^{-\Lambda+q}\int^{x} W\left(\phi_2\right)e^{2\phi_2}. \qquad (5.25)$$

By direct calculation we have verified that the compatibility of equations (5.16), (5.17) with (5.22), (5.23), (5.24), (5.25) indeed leads to the equations of motion (2.19). It thus follows that

$$e^{\Lambda} = -\frac{1}{4\sigma} \times \frac{\partial_{t_{-5}}q e^{p} + 8\sigma^2\partial_x q\int^{x} W\left(\phi_2\right)e^{2\phi_2} + 4\sigma^2\left(W\left(\phi_2\right) - W\left(\phi_1\right)\right)e^{2\phi_2}}{\left(e^{q} + e^{-q} + \eta\right)\left(e^{p}\int^{x} W\left(\phi_1\right)e^{-2\phi_1} + e^{-p}\int^{x} W\left(\phi_2\right)e^{2\phi_2}\right)}, \qquad (5.26)$$

and

$$e^{-\Lambda} = \frac{1}{16\sigma} \times \frac{\partial_{t_{-5}}q e^{-p} - 8\sigma^2\partial_x q\int^{x} W\left(\phi_1\right)e^{-2\phi_1} + 4\sigma^2\left(W\left(\phi_2\right) - W\left(\phi_1\right)\right)e^{-2\phi_1}}{\left(e^{p}\int^{x} W\left(\phi_1\right)e^{-2\phi_1} + e^{-p}\int^{x} W\left(\phi_2\right)e^{2\phi_2}\right)}. \qquad (5.27)$$

6. Solutions

We now discuss some solutions for the type-II BT already derived in the previous section.

6.1. Vacuum–vacuum solution

Let us consider

$$\phi_1 = 0 \; \phi_2 = 0. \qquad (6.1)$$

It thus follows that, for $N = -5, -3, -1, 3, 5$

$$e^{2\Lambda} = -\frac{4}{2 + \eta}. \qquad (6.2)$$

Writing $\eta = -(e^{2\tau} + e^{-2\tau})$ equation (6.2) becomes

$$e^{2\Lambda} = \frac{1}{\sinh^2 \tau}. \qquad (6.3)$$

The vacuum–vacuum solution therefore leads to a constant Λ [5].

6.2. Vacuum to one-soliton solution

Let us consider the following field configurations[2]

$$\phi_1 = 0, \qquad \phi_2 = \ln\left(\frac{1 + \rho}{1 - \rho}\right), \qquad (6.4)$$

where $\rho(x, t_N) = \exp(2kx + 2(k)^N t_N)$.

[2] Solutions ϕ_1 and ϕ_2 with appropriate $\rho(x, t_N)$ solves all equations within the mKdV hierarchy, i.e., equations (2.9)–(2.11) and (3.4)–(2.19) , etc for $N = 3, 5, 7, \ldots$ and $-3, -5, \ldots$

It therefore follows for the SG model where $t_N = t_{-1}$ that inserting (6.4) in (3.12),

$$e^{\Lambda} = \frac{2}{\sigma}\frac{\partial_{t_{-1}}\phi_2 e^{\phi_2} - \sigma^2\partial_x\phi_2}{\left(e^{\phi_2} - e^{-\phi_2}\right)\left(e^{\phi_2} + e^{-\phi_2} + \eta\right)}$$
$$= 2(1+\rho)\frac{\left(1+\rho - k^2\sigma^2(1-\rho)\right)}{k\sigma\left((2+\eta) + \rho^2(2-\eta)\right)}, \tag{6.5}$$

$$e^{-\Lambda} = -\frac{1}{2\sigma}\frac{\partial_{t_{-1}}\phi_2 e^{-\phi_2} - \sigma^2\partial_x\phi_2}{\left(e^{\phi_2} - e^{-\phi_2}\right)}$$
$$= \frac{\left(-1 + k^2\sigma\right) + \rho\left(1 + k^2\sigma\right)}{2k\sigma(1+\rho)}. \tag{6.6}$$

The identity

$$e^{\Lambda}e^{-\Lambda} - 1 = \frac{\left(1 + \eta k^2\sigma^2 + k^4\sigma^4\right)}{2 + \eta + \rho^2(2-\eta)}\left(\frac{1-\rho^2}{k^2\sigma^2}\right) = 0 \tag{6.7}$$

implies

$$1 + \eta k^2\sigma^2 + k^4\sigma^4 = 0 \tag{6.8}$$

which leads to values of the Bäcklund parameter σ. Parametrizing $\sigma = e^{\varphi}$, $k = e^{\theta}$, $\eta = -(e^{2\tau} + e^{-2\tau})$ we find four solutions for the identity (6.8)

$$\varphi = -\theta \pm \tau, \tag{6.9}$$

$$\varphi = -\theta \pm \tau + i\pi. \tag{6.10}$$

We have also verified, using Mathematica program, that the *very same condition* (6.8) is required for $t = t_{-3}$ and $t = t_{-5}$ obtained from (5.18), (5.19) and from (5.26), (5.27) respectively.

For the mKdV equation where $t_N = t_3$, we find after substituting (6.4) into (4.17) and (4.18),

$$e^{\Lambda}e^{-\Lambda} - 1 = \frac{\left(1 + \eta k^2\sigma^2 + k^4\sigma^4\right)}{2 + \eta + (2-\eta)\rho^2}\left(A_3 + B_3\rho^2\right) = 0, \tag{6.11}$$

where

$$A_3 = 2 + \eta + k^2\sigma^2, \qquad B_3 = -2 + \eta + k^2\sigma^2. \tag{6.12}$$

Similarly for $t_N = t_5$,

$$e^{\Lambda}e^{-\Lambda} - 1 = \frac{\left(1 + \eta k^2\sigma^2 + k^4\sigma^4\right)}{2 + \eta + (2-\eta)\rho^2}\left(\frac{A_5 + B_5\rho^2}{\eta^2}\right) = 0, \tag{6.13}$$

where

$$A_5 = -\eta^3 + \eta^2\left(k^2\sigma^2 - 2\right) + \eta k^2\sigma^2\left(2 + k^2\sigma^2\right) - k^2\sigma^2\left(1 + k^4\sigma^4\right)$$
$$B_5 = \eta^3 - \eta^2\left(k^2\sigma^2 + 2\right) + \eta k^2\sigma^2\left(2 - k^2\sigma^2\right) + k^2\sigma^2\left(1 + k^4\sigma^4\right). \tag{6.14}$$

Identities (6.11) and (6.13) are both satisfied by condition (6.8) and henceforth the same result (6.9) and (6.10) holds for all cases considered, i.e., $N = -5, -3, -1, 3, 5$. It is interesting to observe that in all cases considered the result for the auxiliary field Λ is precisely the same, namely

$$e^{\Lambda} = 2e^{\tau}\frac{1+\rho}{\pm 1 \mp e^{2\tau} + \left(1+e^{2\tau}\right)\rho}, \quad \text{for} \quad \varphi = -\theta \mp \tau \tag{6.15}$$

or

$$e^{\Lambda} = -2e^{\tau}\frac{1+\rho}{\pm 1 \mp e^{2\tau} + \left(1+e^{2\tau}\right)\rho}, \quad \text{for} \quad \varphi = -\theta \mp \tau + i\pi. \tag{6.16}$$

Most probably this is a direct consequence of the universality of the Lax space component within the hierarchy.

6.3. Scattering of one-soliton solutions

Let us consider the one-soliton solutions of the form

$$\phi_1 = \ln\left(\frac{1+\rho(x, t_N)}{1-\rho(x, t_N)}\right), \qquad \phi_2 = \ln\left(\frac{1+R\rho(x, t_N)}{1-R\rho(x, t_N)}\right), \tag{6.17}$$

where $\rho(x, t_N) = \exp\left(2kx + 2(k)^N t_N\right)$. It thus follows that

$$\partial_{t_N} q = \partial_{t_N}\left(\phi_1 - \phi_2\right) = 4k^N \rho\left(\frac{1}{1-\rho^2} - \frac{R}{1-R^2\rho^2}\right) \tag{6.18}$$

and similarly for $\partial_x q$.

For the SG, substituting ϕ_1 and ϕ_2 into the equation (3.12) we find that

$$\begin{aligned}&\left(1 - e^{\Lambda}e^{-\Lambda}\right)\frac{k^2\sigma^2(1+R)^2}{\left(1-\rho^2\right)\left(1-R^2\rho^2\right)}\\ &= \frac{\left(1+k^2\eta\sigma^2+k^4\sigma^4\right)\left(1+R^2\right) - 2R\left(1-k^2(4+\eta)\sigma^2+k^4\sigma^4\right)}{2+\eta-\left(-2+8R+R^2(-2+\eta)+\eta\right)\rho^2+(2+\eta)R^2\rho^4} = 0\end{aligned} \tag{6.19}$$

and henceforth the delay R satisfies

$$\left(1+k^2\eta\sigma^2+k^4\sigma^4\right)\left(1+R^2\right) - 2R\left(1-k^2(4+\eta)\sigma^2+k^4\sigma^4\right) = 0. \tag{6.20}$$

The two solutions are

$$R_1 = \frac{1-4k^2\sigma^2-k^2\eta\sigma^2+k^4\sigma^4-2k\sigma\left(k^2\sigma^2-1\right)\sqrt{-2-\eta}}{1+k^2\eta\sigma^2+k^4\sigma^4}, \tag{6.21}$$

and

$$R_2 = \frac{1-4k^2\sigma^2-k^2\eta\sigma^2+k^4\sigma^4+2k\sigma\left(k^2\sigma^2-1\right)\sqrt{-2-\eta}}{1+k^2\eta\sigma^2+k^4\sigma^4}. \tag{6.22}$$

We can rewrite R_1 and R_2 in a more simple form if we make the following parametrization

$$\eta = -e^{2\tau} - e^{-2\tau}, \qquad k = e^{\theta}, \qquad \sigma = e^{\varphi}. \tag{6.23}$$

With these choices we find that

$$\begin{aligned}R_1 &= \frac{e^{\tau} - e^{\theta+\varphi} + e^{\theta+2\tau+\varphi} - e^{2\theta+\tau+2\varphi}}{e^{\tau} + e^{\theta+\varphi} - e^{\theta+2\tau+\varphi} - e^{2\theta+\tau+2\varphi}}\\ &= -\coth\left(\frac{-\theta-\varphi-\tau}{2}\right)\tanh\left(\frac{\theta+\varphi-\tau}{2}\right),\end{aligned} \tag{6.24}$$

and

$$R_2 = \frac{1}{R_1}. \tag{6.25}$$

Similarly from (5.18), (5.19) for $t = t_{-3}$ and (5.26), (5.27) for $t = t_{-5}$ we find the same result (6.19) and the same two solutions for R. For the mKdV the condition (6.20) remains true (see appendix A) and leads to the same two solutions for R. We therefore conclude that the delay R is the same for the $t = t_{-5}$, t_{-3}, t_{-1} and t_3. Such result might as well be true for all times.

7. Conclusions and further developments

We have verified, within the mKdV hierarchy that there exist a single gauge transformation that generates the BT for the first few integrable equations associated to both, positive and negative graded time evolution equations. In particular, we have shown that the very same gauge transformation generating the BT for the SG model was verified to generate BT for the mKdV and other higher (lower) graded evolution equations.

Moreover we have verified that the vacuum–vacuum, vacuum-one-soliton transitions and the scattering of one-soliton solutions of the BT are correlated according to the different time evolution parameters. In fact, the Bäcklund parameters are fixed (for a specific model) and remain the same within all equations of motion considered (i.e. are independent of the model in question).

We conclude this paper by establishing the conjecture that the type I as well as type II BT for a given hierarchy may be systematically derived from a fundamental principle of the universality of the space Lax operator. This fact induces the universality of a gauge transformation which maps the two field configurations related by the BT for the entire hierarchy.

So far we have restricted ourselves purely in the construction of the BT for the various integrable equations within the mKdV hierarchy. The general structure of integrable hierarchies involves an infinite number of conserved charges which play the role of Hamiltonians. Each of which, generates a particular time evolution under a Poisson bracket structure. The conservation of all these charges with respect of all times is guaranteed by the existence of a classical $r-$ matrix satisfying the fundamental Poisson bracket relation (see [11]). In particular it implies that all charges commute under the Poisson bracket.

Although this paper does not involve a direct discussion of defects, a Poisson bracket structure for systems within the mKdV hierarchy with defects described by the type II BT may be derived very similarly to the one described in [5], involving Dirac Brackets for the NLS model. Integrability may be described imposing the fundamental Poisson bracket relation structure and constructing the classical r matrix.

More general hierarchies beyond the mKdV hierarchy may be considered also. In particular the type II BT for the $N = 1$ super SG models was recently derived in [9] and may be extended to other equations of motion within the hierarchy. For the NLS equation, the BT was derived in terms of a Riccati equation [10]. It would be interesting to verify our conjecture within the NLS hierarchy.

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Appendix A. One soliton scattering for mKdV $t = t_3$

For the mKdV equation $t = t_3$ the one soliton scattering with solitions (6.17) yields

$$
\begin{aligned}
&\mathrm{e}^{\Lambda}\mathrm{e}^{-\Lambda} - 1 \\
&\quad = \frac{\left(\left(1 + R^2\right)\left(1 + \eta k^2\sigma^2 + k^4\sigma^4\right) - 2R\left(1 - (4 + \eta)k^2\sigma^2 + k^4\sigma^4\right)\right)}{d_1 d_2} \\
&\quad\quad \times \sum_{n=0}^{6} c_n \rho(x, t)^{2n} = 0
\end{aligned}
$$

with

$$
\begin{aligned}
c_0 &= -\left(2 + \eta + k^2\sigma^2\right) \\
c_1 &= \left(2 + 3\eta + 3k^2\sigma^2\right) + \left(12 + 2\eta - 2k^2\sigma^2\right)R + \left(-2 + \eta + k^2\sigma^2\right)R^2 \\
c_2 &= \left(2 - 3\eta - 3k^2\sigma^2\right) + \left(-20 - 6\eta + 6k^2\sigma^2\right)R - 4\left(4 + \eta + k^2\sigma^2\right)R^2 \\
&\quad + \left(4 - 2\eta + 2k^2\sigma^2\right)R^3 \\
c_3 &= \left(-2 + \eta + k^2\sigma^2\right) + \left(4 + 6\eta - 6k^2\sigma^2\right)R + 6\left(6 + \eta + k^2\sigma^2\right)R^2 \\
&\quad + \left(4 + 6\eta - 6k^2\sigma^2\right)R^3 + \left(-2 + \eta + k^2\sigma^2\right)R^4 \\
c_4 &= \left(2 - 3\eta - 3k^2\sigma^2\right) - 2\left(-2 + \eta - k^2\sigma^2\right)R - 4\left(4 + \eta + k^2\sigma^2\right)R^2 \\
&\quad + \left(-20 - 6\eta + 6k^2\sigma^2\right)R^3 \\
c_5 &= \left(-2 + \eta + k^2\sigma^2\right)R^2 + \left(12 + 2\eta - 2k^2\sigma^2\right)R^3 + \left(2 + 3\eta + 3k^2\sigma^2\right)R^4 \\
c_6 &= -\left(2 + \eta + k^2\sigma^2\right)R^4
\end{aligned}
$$

and

$$
\begin{aligned}
d_1 &= 2 + \eta - \left(-2 + 8R + R^2(-2 + \eta) + \eta\right)\rho^2 + (2 + \eta)R^2\rho^4 \\
d_2 &= \left[1 - R - 2k^2\sigma^2 + \left(-1 - 2k^2\sigma^2 + \left(1 + 2k^2\sigma^2\right)R^2\right)\rho^2 \right. \\
&\quad \left. + \left(R - \left(1 - 2k^2\sigma^2\right)R^2\right)\right]^2
\end{aligned}
$$

and the condition for the delay coincide with (6.20), i.e.,

$$
\left(1 + R^2\right)\left(1 + \eta k^2\sigma^2 + k^4\sigma^4\right) - 2R\left(1 - (4 + \eta)k^2\sigma^2 + k^4\sigma^4\right) = 0. \tag{8.1}
$$

Appendix B. Obtaining the K matrix

In order to find the BTs to the equations of mKdV hierarchy, we suppose that the K matrix is given by

$$K = \beta + \frac{\gamma}{\lambda} = \begin{bmatrix} \beta_{11} + \frac{\gamma_{11}}{\lambda} & \beta_{12} + \frac{\gamma_{12}}{\lambda} \\ \beta_{21} + \frac{\gamma_{21}}{\lambda} & \beta_{22} + \frac{\gamma_{22}}{\lambda} \end{bmatrix}, \tag{9.1}$$

and we can write

$$\partial_\mu K = K A_\mu(\phi_1) - A_\mu(\phi_2) K, \tag{9.2}$$

where $x_\mu = x, t_N$.

Then, we start by the spacial part because $A_x(\phi)$ is the same for all equations of the hierarchy, having the following form

$$A_x(\phi) = \begin{bmatrix} \partial_x \phi & 1 \\ \lambda & -\partial_x \phi \end{bmatrix}. \tag{9.3}$$

Calculating the terms of equation (9.2), with $x_\mu = x$, we obtain a matrix whose 11 element is given by

$$\lambda^{-1}: \partial_x \gamma_{11} = \gamma_{11} \partial_x q - \gamma_{21}, \tag{9.4}$$

$$\lambda^0: \partial_x \beta_{11} = \beta_{11} \partial_x q + \gamma_{12} - \beta_{21}, \tag{9.5}$$

$$\lambda^1: \beta_{12} = 0; \tag{9.6}$$

the 12 matrix element in turn is

$$\lambda^{-1}: \partial_x \gamma_{12} = -\gamma_{12} \partial_x p + \gamma_{11} - \gamma_{22}, \tag{9.7}$$

$$\lambda^0: \partial_x \beta_{12} = -\beta_{12} \partial_x p + \beta_{11} - \beta_{22}. \tag{9.8}$$

We also have the 21 matrix element

$$\lambda^{-1}: \partial_x \gamma_{21} = \gamma_{21} \partial_x p, \tag{9.9}$$

$$\lambda^0: \partial_x \beta_{21} = \beta_{21} \partial_x p + \gamma_{22} - \gamma_{11}, \tag{9.10}$$

$$\lambda^1: \beta_{22} = \beta_{11}; \tag{9.11}$$

and the 22 matrix element given by

$$\lambda^{-1}: \partial_x \gamma_{22} = -\gamma_{22} \partial_x q + \gamma_{21}, \tag{9.12}$$

$$\lambda^0: \partial_x \beta_{22} = -\beta_{22} \partial_x q + \beta_{21} - \gamma_{12}, \tag{9.13}$$

$$\lambda^1: \beta_{12} = 0. \tag{9.14}$$

Using the equations (9.5) and (9.11) together with equations (9.4) and (9.13) we obtain

$$\beta_{12} = 0, \tag{9.15}$$

$$\beta_{22} = \beta_{11} = \text{constant} \equiv b_{11}, \tag{9.16}$$

$$\beta_{21} = b_{11} \partial_x q + \gamma_{12}. \tag{9.17}$$

Let us now, consider the time component of equation (9.2). Putting

$$A_{t_{-1}} = \begin{bmatrix} 0 & \frac{1}{\lambda} e^{-2\phi} \\ e^{2\phi} & 0 \end{bmatrix} \tag{9.18}$$

in the equation (9.2) we find for the λ^{-2} coefficient of the 11 matrix element to be

$$\lambda^{-2} : \gamma_{21} = 0. \tag{9.19}$$

The same result, $\gamma_{21} = 0$ follows for $A_{t_{-3}}$ and $A_{t_{-5}}$.

Taking now A_{t_3} as

$$A_{t_3}(\phi) = \begin{bmatrix} v\lambda + \frac{1}{4}\partial_x^2 v - \frac{1}{2}v^3 & \lambda - \frac{1}{2}v^2 + \frac{1}{2}\partial_x v \\ \lambda^2 - \frac{1}{2}v^2\lambda - \frac{1}{2}\partial_x v\lambda & -v\lambda - \frac{1}{4}\partial_x^2 v + \frac{1}{2}v^3 \end{bmatrix} \tag{9.20}$$

in equation (9.2) we obtain that the term proportional to λ^{-1} of the matrix element 21 to be

$$\partial_{t_3}\gamma_{21} = \left(\frac{1}{4}\partial_x^2 v_1 + \frac{1}{4}\partial_x^2 v_2 - \frac{1}{2}v_1^3 - \frac{1}{2}v_2^3\right)\gamma_{21}, \tag{9.21}$$

which can be rewritten as

$$\partial_{t_3}\gamma_{21} = \partial_{t_3}(\phi_1 + \phi_2)\gamma_{21}. \tag{9.22}$$

Again $\gamma_{21} = 0$ is a particular solution. The same type of equation (9.22) for γ_{21} follows for $N = 5$. In this work, we will assume

$$\gamma_{21} = 0, \tag{9.23}$$

but the more general assumption still to be studied.

With this choice we conclude using the equations (9.4) and (9.12) that

$$\gamma_{11} = c_{11}\mathrm{e}^{q}, \qquad \gamma_{22} = c_{22}\mathrm{e}^{-q}. \tag{9.24}$$

We have then two possible solutions:

- $\gamma_{11} = \gamma_{22}$.

  This implies that $c_{11} = c_{22} = 0$ and henceforth from (9.7) and (9.10),

$$\beta_{21} = b_{21}\mathrm{e}^{p} \qquad \gamma_{12} = c_{12}\mathrm{e}^{-p}. \tag{9.25}$$

Setting $b_{11} \equiv 1$, $b_{21} = c_{12} = -\frac{\beta}{2}$ we obtain

$$K = \begin{bmatrix} 1 & -\frac{\beta}{2\lambda}\mathrm{e}^{-p} \\ -\frac{\beta}{2}\mathrm{e}^{p} & 1 \end{bmatrix} \tag{9.26}$$

the K matrix that generates the *type-I BTs*.

- $\gamma_{11} \neq \gamma_{22}$.

  We start by the following expression

$$\partial_x(\gamma_{12}\beta_{21}) = (\partial_x\gamma_{12})\beta_{21} + \gamma_{12}(\partial_x\beta_{21}) \tag{9.27}$$

and using the equations (9.7), (9.10) and (9.24) we have

$$\partial_x(\gamma_{12}\beta_{21}) = b_{11}(c_{11}\mathrm{e}^{q} - c_{22}\mathrm{e}^{-q})\partial_x q \tag{9.28}$$

and assuming $c_{11} = c_{22}$, results in

$$\gamma_{12}\beta_{21} = b_{11}c_{11}\left(\mathrm{e}^{q} + \mathrm{e}^{-q} + \eta\right) \tag{9.29}$$

with η is an integration constant. Generalizing (9.25)

$$\beta_{21} = b_{21}\mathrm{e}^{p-\Lambda} \tag{9.30}$$

by (9.10) gives

$$\partial_x\Lambda = \frac{c_{11}}{b_{21}}\mathrm{e}^{\Lambda-p}\left(\mathrm{e}^{q} - \mathrm{e}^{-q}\right) \tag{9.31}$$

which has the form (3.9). Expression (9.29) implies:

$$\gamma_{12} = \frac{b_{11}c_{11}b_{21}}{e}^{\Lambda-p}\left(\mathrm{e}^{q} + \mathrm{e}^{-q} + \eta\right). \tag{9.32}$$

Putting $b_{11} \equiv 1$, $c_{11} = -\frac{1}{\sigma^2}$ and $b_{21} = -\frac{2}{\sigma}$, we obtain

$$K = \begin{bmatrix} 1 - \frac{1}{\sigma^2\lambda}\mathrm{e}^{q} & \frac{1}{2\sigma\lambda}\mathrm{e}^{\Lambda-p}\left(\mathrm{e}^{q} + \mathrm{e}^{-q} + \eta\right) \\ -\frac{2}{\sigma}\mathrm{e}^{p-\Lambda} & 1 - \frac{1}{\sigma^2\lambda}\mathrm{e}^{-q} \end{bmatrix} \tag{9.33}$$

which is the K matrix that generates the *type-II BTs* .

As a conclusion of this apppendix, we have shown that for $t = t_{-1}, t_{-3}$ and t_{-5} we have universality of the matrix K while for $t = t_3, t_5, \ldots$ we have it as a particular solution.

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