Letter Section

Chawla–Numerov method revisited *

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Received 26 September 1986
Revised 1 September 1990

Abstract

Meneguette, M., Chawla–Numerov method revisited, Journal of Computational and Applied Mathematics 36 (1991) 247–250.

The well-known two-step fourth-order Numerov method was shown to have better interval of periodicity when made explicit, see Chawla (1984). It is readily verifiable that the improved method still has phase-lag of order 4. We suggest a slight modification from which linear problems could benefit. Phase-lag of any order can be achieved, but only order 6 is derived.

Keywords: Chawla–Numerov method, higher derivatives and phase-lag, periodic second-order initial-value problems.

The Numerov method

$$y_{n+2} - 2y_{n+1} + y_n = \tfrac{1}{12}h^2(f_{n+2} + 10f_{n+1} + f_n) \tag{1}$$

is the optimal two-step method of the Störmer–Cowell family for solving the initial problem

$$y'' = f(x, y), \quad y_0 = y(x_0), \quad y_0' = y'(x_0).$$

This method is fourth order and has (0, 6) as interval of periodic stability [4, Chapter 9]. Its local truncation error LTE is given by

$$\text{LTE} = \tfrac{-1}{240}h^6 y_{n+1}^{(6)} + O(h^7).$$

The stability and phase-lag can be derived by applying the method to the test equation

$$y'' = -\lambda^2 y, \quad \lambda \in \mathbb{R}_+. \tag{2}$$

* Work partially supported by CNPq (Brazil).

Solving (2) by (1) we obtain, for $H^2 = (\lambda h)^2$,

$$y_{n+2} - 2\frac{A(H^2)}{B(H^2)} y_{n+1} + y_n = 0, \tag{3}$$

where

$$A(H^2) = 1 - \tfrac{5}{12}H^2 \quad \text{and} \quad B(H^2) = 1 + \tfrac{1}{12}H^2.$$

Periodicity holds as long as the characteristic polynomial associated with (3) has zeros on the unit circle. That is so for $H^2 \in (0, 6)$.

The phase-lag analysis, for two-step methods, can be shown only to require the comparison of

$$\frac{A(H^2)}{B(H^2)} = 1 - \tfrac{1}{2}H^2 + \tfrac{1}{24}H^4 - \tfrac{1}{288}H^6 + \cdots,$$

with

$$\cos H = 1 - \frac{H^2}{2} + \frac{H^4}{4!} - \frac{H^6}{6!} + \cdots. \tag{4}$$

Clearly the order of phase-lag is 4, the actual phase-lag being $H^4/480$. It is surprising that phase analysis has never been done in this way before; for example, in [2,5] a complicated tangent expansion is used.

Chawla [1] improved (1) by making it explicit, that is, he replaced f_{n+2} by $\bar{f}_{n+2} = f(x_{n+2}, \bar{y}_{n+2})$, where

$$\bar{y}_{n+2} = 2y_{n+1} - y_n + h^2 f_{n+1}, \tag{5}$$

whose

$$\text{LTE} = \tfrac{1}{12}h^4 y_{n+1}^{(4)} + O(h^5).$$

Hence the LTE of the new formula is

$$-\frac{h^6}{240} y_{n+1}^{(6)} + \frac{\partial f}{\partial y}(x_{n+2}, y_{n+2}) \frac{h^6}{144} y_{n+1}^{(4)} + O(h^7).$$

The explicit method applied to (2) leads to (3) with

$$B(H^2) = 1 \quad \text{and} \quad A(H^2) = 1 - \frac{H^2}{2} + \frac{H^4}{4!},$$

which again imposes order 4, but the phase-lag now is $H^4/6!$. However, the interval of periodicity is enlarged to $H^2 \in (0, 12)$ [1].

Although less attractive to nonlinear problems, incorporation of higher derivatives in (5) provides phase-lag of any order; as an illustration order 6 is derived. No major difficulty is, however, imposed by linear problems as for instance from semidiscretisation of some hyperbolic problems.

Let y_{n+2} be calculated from

$$\begin{aligned} \bar{y}_{n+2} &= 2y_{n+1} - y_n + h^2 f_{n+1} + \tfrac{1}{30}h^4 f_{n+1}^{(2)}, \\ y_{n+2} - 2y_{n+1} + y_n &= \tfrac{1}{12}h^2(\bar{f}_{n+2} + 10f_{n+1} + f_n), \end{aligned} \tag{6}$$

with $\bar{f}_{n+2}$ as before and

$$f_{n+1}^{(2)} := \frac{d^2}{dx^2} f(x_{n+1}, y_{n+1}).$$

The error constant in our y_{n+2} is 1/20, therefore slightly smaller than that of Chawla's. This is also reflected in the LTE of (6), which becomes

$$-\frac{h^6}{240}y_{n+1}^{(6)} + \frac{\partial f}{\partial y}(x_{n+2},\ y_{n+2})\frac{h^6}{240}y_{n+1}^{(4)} + O(h^7). \tag{7}$$

Applying (6) to (2) we obtain (3) with

$$B(H^2) = 1 \quad \text{and} \quad A(H^2) = 1 - \frac{H^2}{2} + \frac{H^4}{4!} - \frac{H^6}{6!},$$

which gives rise, by comparison with (4), to phase-lag $H^6/8!$. Higher orders can be achieved by

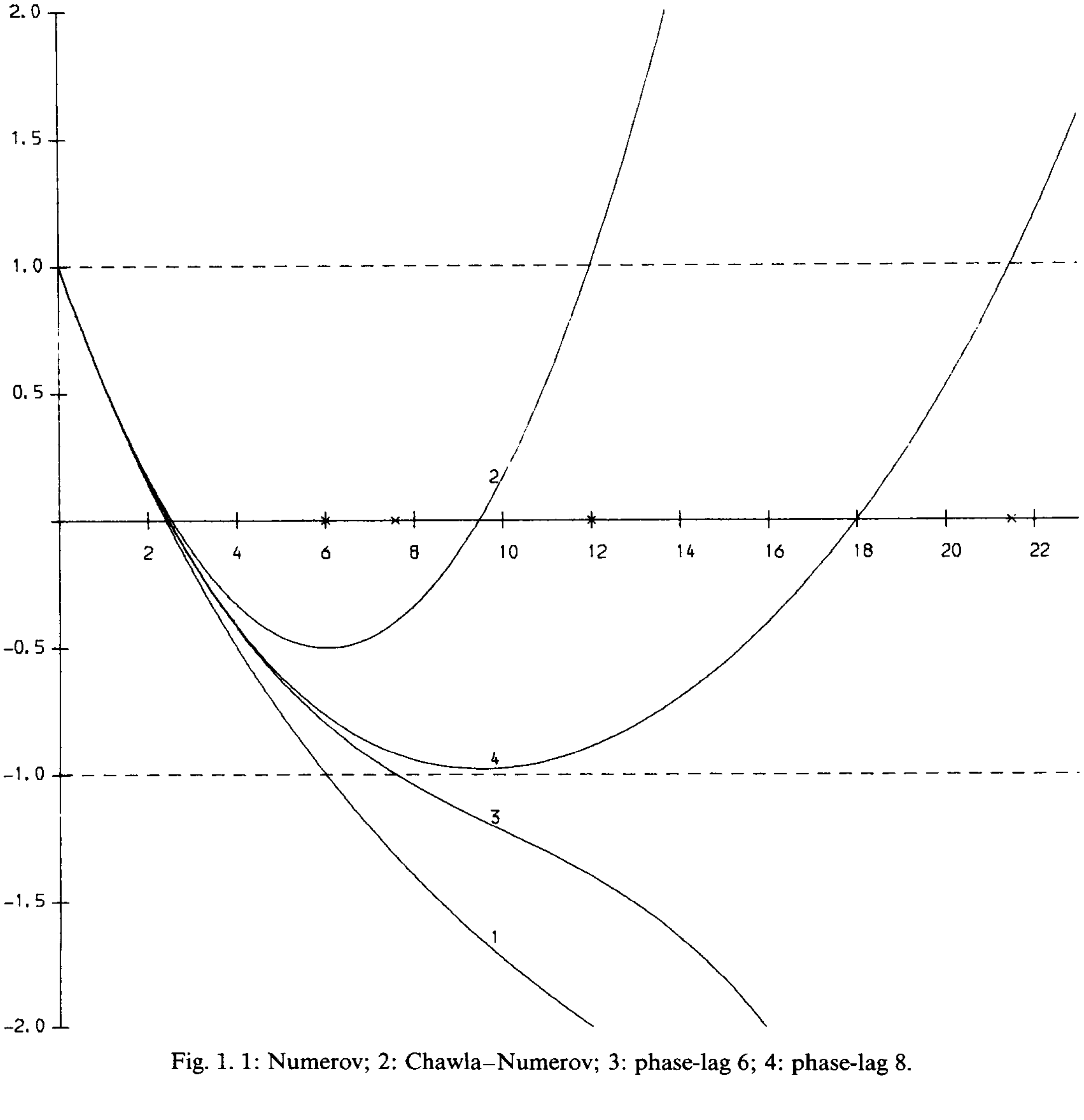

Fig. 1. 1: Numerov; 2: Chawla–Numerov; 3: phase-lag 6; 4: phase-lag 8.

using higher derivatives with coefficients to match the cos H expansion, that is, the predicted value y_{n+2} is given by

$$\bar{y}_{n+2} = 2y_{n+1} - y_n + h^2 f_{n+1} + \sum_{j=1}^{L} h^{2j+2}\gamma_j \frac{d^{2j}}{dx^{2j}} f(x_{n+1}, y_{n+1}),$$

where $\gamma_j = 4!/(2j+4)!$ and L is a positive integer.

Remark. A fourth-order explicit method with phase-lag of order 6 and similar periodicity characteristics as in (6) was presented in [3].

The interval of periodicity is calculated from $|A(H^2)/B(H^2)| < 1$, therefore a straightforward search followed by a bisection computation provides (0, 7.57) as the interval for (6). Although this is a smaller interval, it is still reasonably large. However, if the fourth total derivative of f is also incorporated in (6), the new method has the same LTE (7); the phase-lag becomes $H^8/10!$ and the interval of periodicity is (0, 21.48121). Figure 1 gives the behaviour of $A(H^2)/B(H^2)$ against H^2 for the various methods.

Acknowledgements

The author is grateful to Professor S. McKee and the referee for improvements in this note.

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