Companion orthogonal polynomials

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Abstract
We give some properties relating the recurrence relations of orthogonal polynomials associated with any two symmetric distributions \( d_{\phi_1}(x) \) and \( d_{\phi_2}(x) \) such that \( d_{\phi_2}(x) = (1 + kx^2) d_{\phi_1}(x) \). Applications of these properties, recurrence relations for many interesting systems of orthogonal polynomials are obtained.

Keywords: Symmetric orthogonal polynomials; Three-term recurrence relations; Strong distributions

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1. Introduction

A distribution (i.e. positive measure) \( d_\phi(x) \), defined on \( (-\infty, \infty) \), is known as a symmetric distribution if \( d_\phi(x) = -d_\phi(-x) \). The monic orthogonal polynomials \( Q_\phi^n(x) \), \( n \geq 0 \), associated with \( d_\phi(x) \) satisfy (see, for example, [1])

\[
Q_{n+1}^\phi(x) = xQ_n^\phi(x) - \alpha_n^\phi Q_{n-1}^\phi(x), \quad n \geq 1,
\]

with \( Q_0^\phi(x) = 1 \) and \( Q_1^\phi(x) = x \), where the unique coefficients \( \alpha_n^\phi \) are all positive.

Let \( d_{\phi_1}(x) \) and \( d_{\phi_2}(x) \) be two symmetric distributions such that

\[
d_{\phi_2}(x) = (1 + kx^2) d_{\phi_1}(x), \quad (1.1)
\]

where \( k \) is real and positive. In this article we give some results about how the two sequences of coefficients \( \{\alpha_n^{\phi_1}\} \) and \( \{\alpha_n^{\phi_2}\} \) are related. In particular, we prove the following result.

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Theorem 1. Associated with $d\phi_1(x)$ and $d\phi_2(x)$ there exists a sequence of positive numbers $\{\ell_n\}$, with $\ell_0 = 1$ and $\ell_n > 1$ for $n \geq 1$, such that

$$(\ell_n - 1)(\ell_{n-1} + 1) = 4k \varphi_{n+1}$$

and

$$(\ell_n - 1)(\ell_{n+1} + 1) = 4k \varphi_{n+1}$$

$$(n \geq 1).$$

We establish this result using certain properties of the monic polynomials $\tilde{B}_n^\psi(t)$, $n \geq 0$, uniquely defined by

$$\int_0^\infty t^{-n+s} \tilde{B}_n^\psi(t) d\psi(t) = 0, \quad 0 \leq s \leq n - 1,$$

where $d\psi(t)$ is a strong distribution in $(0, \infty)$. A distribution $d\psi(t)$, with support inside any interval $(a, b)$ and such that the moments $\int_a^b t^m d\psi(t)$ exist for $m = 0, \pm 1, \pm 2, \ldots$, is called a strong distribution in $(a, b)$.

2. Strong distributions

For any strong distribution $d\psi(t)$ on $(0, \infty)$, the monic polynomials $\tilde{B}_n^\psi(t)$, $n \geq 0$, satisfy the recurrence relation

$$\tilde{B}_{n+1}^\psi(t) = (t - \tilde{B}_{n+1}^\psi) \tilde{B}_n^\psi(t) - \tilde{\varphi}_{n+1} t \tilde{B}_{n-1}^\psi(t), \quad n \geq 1,$$

with $\tilde{B}_0^\psi(t) = 1$ and $\tilde{B}_1^\psi(t) = t - \tilde{\beta}_1^\psi$, where the unique coefficients $\tilde{\varphi}_{n+1}$, $n \geq 0$ and $\tilde{\varphi}_{n+1}$, $n \geq 1$ are all positive. For more information on these and other related results see [4, 7, 9]. For some results on polynomials defined by a variation of (1.3) see [6, 11].

In [9] it was given that $P_\lambda(t)$ is a real monic polynomial of degree $n \geq 2$ that satisfies

$$\int_0^\infty t^{-n+s} P_n^\psi(t) d\psi(t) = 0, \quad 1 \leq s \leq n - 1,$$

if and only if $P_n(t) = \tilde{B}_n^\psi(\lambda, t)$ for $\lambda \in \mathbb{R}$, where

$$\tilde{B}_n^\psi(\lambda, t) = \tilde{B}_n^\psi(t) - \lambda \tilde{B}_n^\psi(t).$$

(2.1)

Furthermore, it was shown that if $d\psi(t)$ has its support inside $(a, b) \subseteq (0, \infty)$ then all the zeros of $\tilde{B}_n^\psi(t)$ lie inside $(a, b)$ and at least $n - 1$ of the zeros of $\tilde{B}_n^\psi(\lambda, t)$ lie inside $(a, b)$.

A special type of distribution, first considered in [7], is the $\text{ScS}(a, b)$ distribution. A strong distribution $d\psi(t)$ with its support inside $(a, b)$ is called a $\text{ScS}(a, b)$ distribution, if

$$\frac{d\psi(t)}{\sqrt{t}} = \frac{d\psi(c/t)}{\sqrt{c/t}}, \quad t \in (a, b),$$

(2.2)

where $a, b$ and $c$ are such that $a = c/b$ and $0 < \sqrt{c} < b \leq \infty$. Taking $c = \beta^2$, we write this distribution here as a $\text{ScS}(\beta^2/b, b)$ distribution. From [7] it follows that for any $\text{ScS}(\beta^2/b, b)$ distribution
\( d\psi(t), \beta_{n+1}^\psi = \beta \) for \( n \geq 0 \) and
\[
\tilde{B}_n^\psi(t) = t^n \tilde{B}_n^\psi(\beta^2/t)/(-\beta)^n, \quad n \geq 0.
\] (2.3)

Now for \( \alpha > 0 \) and \( \beta > 0 \), if
\[
x(t) = \frac{1}{2\sqrt{\alpha}}(\sqrt{t} - \beta/\sqrt{t}), \quad t \in (0, \infty),
\] (2.4)

then from [8] the following results are obtained. For any \( A > 0 \), let
\[
d\psi(t) = A \frac{t}{t + \beta} d\phi(x(t)),
\] (2.5)

and let \( b > 0 \) and \( d > 0 \) be such that \( \sqrt{b} = \sqrt{ad^2 + \beta + \sqrt{d}} \). Then \( d\phi(x) \) is a symmetric distribution with support inside \((-d, d)\) if and only if \( d\psi(t) \) is a \( \text{SeS}(\beta^2/b, b) \) distribution. Further, in this case,
\[
\tilde{B}_n^\psi(t) = (2\sqrt{\alpha})^n Q_n^\psi(x), \quad n \geq 0
\] (2.6)

and
\[
\tilde{\alpha}_{n+1}^\psi = 4\alpha \tilde{\alpha}_{n+1}^\psi, \quad n \geq 1.
\]

In [8], these results are shown only when \( d\phi(x) = w(x)dx \) and \( d\psi(t) = v(t)dt \). When these hold then (2.5) can be written as
\[
v(t) = A^* t^{-1/2} w(x(t)),
\]

where \( A^* > 0 \). However, the extension given here is straightforward to obtain from [8].

3. The \( \text{SeS}(a, b) \) distributions

Another special strong distribution, which was first studied in [9], is the \( \text{SeS}(a, b) \) distribution. We say that a strong distribution \( d\psi(t) \) with support inside \((\beta^2/b, b)\) is a \( \text{SeS}(\beta^2/b, b) \) distribution if
\[
d\psi(t) = -d\psi(\beta^2/b), \quad t \in (\beta^2/b, b).
\]

As with the \( \text{SeS}(\beta^2/b, b) \) distribution we require \( 0 < \beta < b \leq \infty \). From [9] we have that for any \( \text{SeS}(\beta^2/b, b) \) distribution \( d\psi(t) \), the following results hold:

If \( \tilde{\gamma}_n^\psi = \tilde{\beta}_n^\psi + \tilde{\alpha}_n^\psi \), then
\[
\frac{\tilde{\gamma}_{n+1}^\psi}{\tilde{\gamma}_n^\psi} = \frac{\beta^2}{\tilde{\beta}_n^\psi \tilde{\beta}_{n+1}^\psi}, \quad n \geq 0,
\]

with \( \tilde{\gamma}_0^\psi = \tilde{\beta}_0^\psi = 1 \). For any \( n \geq 1 \), if \( \lambda \in \mathbb{R} \) and \( \eta \in \mathbb{R} \) are such that
\[
(\lambda + \tilde{\beta}_n^\psi) (\eta + \tilde{\beta}_n^\psi) = \tilde{\beta}_n^\psi \tilde{\gamma}_n^\psi,
\]
then
\[
\frac{t^n \tilde{B}_n^\psi(\lambda, \beta^2/t)}{\tilde{B}_n^\psi(\lambda, 0)} = \tilde{B}_n^\psi(\eta, t),
\]
where \( \tilde{B}_n^\psi (\lambda, t) \) are the polynomials given by (2.1). Only when \( \lambda \) is equal to

\[
\lambda_{n,1}^\psi = \sqrt{\gamma_n^\psi} \left( + \sqrt{\gamma_n^\psi} - \sqrt{\beta_n^\psi} \right) \quad \text{or} \quad \lambda_{n,2}^\psi = \sqrt{\gamma_n^\psi} \left( - \sqrt{\gamma_n^\psi} - \sqrt{\beta_n^\psi} \right),
\]

then

\[
\frac{t^n \tilde{B}_n^\psi (\lambda, \beta^2 / t)}{\tilde{B}_n^\psi (\lambda, 0)} = \tilde{B}_n^\psi (\lambda, t). \tag{3.1}
\]

All the zeros of \( \tilde{B}_n^\psi (\lambda_{n,1}^\psi, t) \) lie inside the interval \( (\beta^2 / b, b) \). However, only \( n - 1 \) of the zeros of \( \tilde{B}_n^\psi (\lambda_{n,2}^\psi, t) \) lie within \( (\beta^2 / b, b) \) and its remaining zero is equal to \(-\beta\).

4. Related ScS\((a, b)\) and ScS\((a, b)\) distributions

Let the three strong distributions \( d\psi_0(t), d\psi_1(t) \) and \( d\psi_2(t) \), all having their support inside \( (\beta^2 / b, b) \), be such that

\[
t + \beta \quad d\psi_1(t) = d\psi_0(t) = \frac{1}{t + \beta} d\psi_2(t). \tag{4.1}
\]

Then the following three statements are equivalent:

- \( d\psi_1(t) \) is a ScS\((\beta^2 / b, b)\) distribution,
- \( d\psi_0(t) \) is a ScS\((\beta^2 / b, b)\) distribution,
- \( d\psi_2(t) \) is a ScS\((\beta^2 / b, b)\) distribution.

The proof of this follows from the definitions of the ScS\((\beta^2 / b, b)\) and ScS\((\beta^2 / b, b)\) distributions.

A study involving the right-hand equality of (4.1) has already been considered in [12] and there it was also shown that

\[
\tilde{B}_{n+1}^\psi (\epsilon_{n+1}, t) = (t + \beta) \tilde{B}_n^\psi (t), \quad n \geq 0.
\]

Furthermore, if \( \epsilon_n = \sqrt{\frac{\gamma_0^\psi}{\beta_0^\psi}} \) then

\[
\tilde{\epsilon}_{n+1}^\psi = \beta (\epsilon_n - 1)(\epsilon_{n+1} + 1), \quad n \geq 1. \tag{4.2}
\]

Now we consider the polynomials \( \tilde{B}_n^\psi (t), n \geq 1 \). In (1.3) for \( d\psi_1(t) \) substituting \( t \) by \( \beta^2 / u \) and then using (2.2) and (2.3), we obtain

\[
\int_0^\infty t^{-n+s} \tilde{B}_n^\psi (u)(\beta / u) d\psi_1(u) = 0, \quad 1 \leq s \leq n.
\]

This result, with \( u \) replaced by \( t \), added to (1.3) for \( d\psi_1(t) \) gives

\[
\int_0^\infty t^{-n+s} \tilde{B}_n^\psi (t) d\psi_0(t) = 0, \quad 1 \leq s \leq n - 1.
\]
Hence, from (2.3) and (3.1), using the result preceding (2.1), gives

\[
\tilde{B}_n^{\psi_0}(\tilde{\lambda}_n^{\psi_0}, t) = \check{B}_n^{\psi_1}(t), \quad n \geq 1.
\]

This result has already been observed in [10] when

\[
d\psi_1(t) = \frac{1}{\sqrt{b - t} \sqrt{t - a}} dt,
\]

where \( a = \beta^2/b \). Since (2.1) can also be written as

\[
\check{B}_n^{\psi_1}(t) = \frac{\lambda}{\check{\alpha}_{n+1}^{\psi}} \check{B}_n^{\psi}(t) + \left\{ \left( 1 - \frac{\lambda}{\check{\alpha}_{n+1}^{\psi}} \right) + \frac{\lambda \check{B}_{n+1}^{\psi}}{\check{\alpha}_{n+1}^{\psi} t} \right\} \check{B}_n^{\psi}(t), \quad n \geq 1,
\]

by appropriate substitutions of (2.1) and the above in the recurrence relation for \( \tilde{B}_n^{\psi_1}(t) \), we obtain (see also [10])

\[
\check{\alpha}_{n+1}^{\psi} = \beta (\ell'_n - 1)(\ell'_{n-1} + 1), \quad n \geq 1,
\]

(4.3)

where \( \ell'_n = \sqrt{\gamma'_n / \beta^{\psi_0} \check{\alpha}_n^{\psi}} \) as before. Since \( \gamma'_n = \beta_n^{\psi_0} + \check{\alpha}_n^{\psi} \), we have \( \ell'_n > 1 \) for \( n \geq 1 \).

5. The main results

First we prove Theorem 1. Choosing \( k = \alpha/\beta \), we let

\[
d\psi_1(t) = A_1 \frac{t}{t + \beta} d\phi_1(x(t))
\]

and

\[
d\psi_2(t) = A_2 \frac{t}{t + \beta} d\phi_2(x(t)),
\]

where \( A_1 \) and \( A_2 \) are positive numbers and \( x(t) \) is given by (2.4). Therefore, as in (2.5), \( d\psi_1(t) \) is a ScS(\( \beta^2/b, b \)) distribution if and only if \( d\phi_1(x) \) is a symmetric distribution with support inside \((-d, d)\). Similarly, \( d\psi_2(t) \) is a ScS(\( \beta^2/b, b \)) distribution if and only if \( d\phi_2(x) \) is a symmetric distribution with support inside \((-d, d)\). We recall that the relation between \( b \) and \( d \) is \( \sqrt{b} = \sqrt{\alpha d^2 + \beta} + \sqrt{\alpha d} \).

Furthermore, for a suitable choice of \( A_1 \) and \( A_2 \), we also obtain that the ScS(\( \beta^2/b, b \)) distributions \( d\psi_1(t) \) and \( d\psi_2(t) \) satisfy (4.1) if and only if the symmetric distributions \( d\phi_1(x) \) and \( d\phi_2(x) \) satisfy (1.1). Hence, with the use of (2.6) we obtain from (4.2) and (4.3) the result (1.2). This completes the proof of Theorem 1. \( \square \)

Now we see how (1.2) can be used to obtain information on the sequence \( \{\gamma'_n^{\psi_0}\} \) given \( \{\gamma'_n^{\psi_1}\} \), and vice versa. First we have from (1.2)

\[
\frac{\ell'_{n-1} + 1}{\ell'_{n+1} + 1} = \frac{\gamma'_n^{\psi_0}}{\gamma'_n^{\psi_1}}, \quad n \geq 1.
\]

(5.1)
Given $\alpha_{n+1}^{\phi_1}$, $n \geq 1$, we also obtain from (1.2) the continued fraction
\[
\ell_n - 1 = \frac{4k \alpha_{n+1}^{\phi_1}}{2 + (\ell_{n-1} - 1)},
\]
\[
= -\frac{4k \alpha_{n+1}^{\phi_1}}{2} + \frac{4k \alpha_{n}^{\phi_1}}{2} + \cdots + \frac{4k \alpha_{3}^{\phi_1}}{2} + \frac{4k \alpha_{2}^{\phi_1}}{2}, \quad n \geq 1.
\]
This can also be written as
\[
\frac{\ell_n + 1}{2} = 1 + \frac{k \alpha_{n+1}^{\phi_1}}{1} + \frac{k \alpha_{n}^{\phi_1}}{1} + \cdots + \frac{k \alpha_{3}^{\phi_1}}{1} + \frac{k \alpha_{2}^{\phi_1}}{1}, \quad n \geq 1.
\]
Hence, from the theory of continued fractions, see for example [5, p. 71], that
\[
\frac{\ell_n + 1}{2} = b_{n+1}^{(1)} b_n^{(1)}, \quad n \geq 0,
\]
where
\[
b_{n+1}^{(1)} = b_n^{(1)} + k \alpha_{n+1}^{\phi_1} b_{n-1}^{(1)}, \quad n \geq 1,
\]
with $b_0^{(1)} = 1$ and $b_1^{(1)} = 1$. From (5.1) then the following result is obtained.

**Theorem 2.** When (1.1) holds then
\[
\alpha_{n+1}^{\phi_2} = \frac{b_{n}^{(1)} b_{n+2}^{(1)}}{b_{n+1}^{(1)}} \alpha_{n+1}^{\phi_1}, \quad n \geq 1,
\]
where $b_n^{(1)}, n \geq 0$ are given by (5.2).

Now, given $\alpha_{n+1}^{\phi_2}$, $n \geq 1$, we obtain from (1.2)
\[
\frac{\ell_{n+1} + 1}{2} = \frac{k \alpha_{n+1}^{\phi_2}}{-1 + (\ell_n + 1)/2}
\]
\[
= \frac{k \alpha_{n+1}^{\phi_2}}{-1} + \frac{k \alpha_{n}^{\phi_2}}{-1} + \cdots + \frac{k \alpha_{3}^{\phi_2}}{-1} + \frac{\ell_1 + 1}{2}, \quad n \geq 1.
\]
However, since $\ell_1 - 1 = 2k \alpha_{1}^{\phi_1}$ we find $\ell_1 + 1 = 2\mu_0^{\phi_2}/\mu_0^{\phi_1}$, where $\mu_0^{\phi} = \int_{-\infty}^{\infty} d\phi(x)$. Hence,
\[
\frac{\ell_{n+1} + 1}{2} = \frac{k \alpha_{n+1}^{\phi_2}}{-1} + \frac{k \alpha_{n}^{\phi_2}}{-1} + \cdots + \frac{k \alpha_{3}^{\phi_2}}{-1} + \frac{\mu_0^{\phi_2}/\mu_0^{\phi_1}}{1}, \quad n \geq 1.
\]
It is easily verified that
\[
\frac{\ell_{n+1} + 1}{2} = \frac{b_{n}^{(2)}}{b_{n+1}^{(2)}}, \quad n \geq 1,
\]
where
\[
b_n^{(2)} = b_{n-1}^{(2)} + k \phi_2 \phi_1^{(2)} b_{n+1}^{(2)}, \quad n \geq 1,
\]  
(5.3)

with \(b_0^{(2)} = 1\) and \(b_1^{(2)} = \mu_{0}^{\phi_1}/\mu_0^{\phi_1}\). Hence, the following result is established.

**Theorem 3.** When (1.1) holds then
\[
\phi_2^{(2)} = \frac{b_2^{(2)}}{b_1^{(2)}} \phi_2^{(2)}
\]

and
\[
\phi_{n+2}^{(2)} = \frac{b_{n-1}^{(2)} b_{n+2}^{(2)}}{b_n^{(2)} b_{n+1}^{(2)}} \phi_{n+2}^{(2)}, \quad n \geq 1,
\]

where the \(b_n^{(2)}, n \geq 0\), are given by (5.3).

6. Some special cases

In this section we consider some examples of pairs of symmetric distributions that satisfy the relation (1.1).

**Example 1.** Let \(d = 1\) and let
\[
\text{d}\phi_1(x) = \frac{1}{\sqrt{1-x^2}} \text{d}x, \quad \text{d}\phi_2(x) = \frac{1+kx^2}{\sqrt{1-x^2}} \text{d}x,
\]

where \(k > 0\). Since \(\phi_2^{(1)} = \frac{1}{2}\) and \(\phi_{n+1}^{(1)} = \frac{1}{4}, n \geq 2\), using the theory of difference equations we obtain from (5.2) that
\[
b_0^{(1)} = 1, \quad b_n^{(1)} = \left(\frac{1+\sqrt{1+k}}{2}\right)^n + \left(\frac{1-\sqrt{1+k}}{2}\right)^n, \quad n \geq 1.
\]

Hence, we can write
\[
\phi_1^{(2)} = \frac{1}{4} K_{n-1} K_{n+2}, \quad n \geq 1,
\]

where \(K_n = \left\{1+\sqrt{1+k}\right\}^n + \left\{1-\sqrt{1+k}\right\}^n, n \geq 0\). This result has also been obtained in [8].

Though we have proved the above result for \(k > 0\), we believe that it holds for all \(k \geq -1\). It certainly holds for \(k = -1\) and \(k = 0\). When \(k = 0\), we must take \(K_0 = 2\).

**Example 2.** Let \(d = 1\) and suppose that
\[
\text{d}\phi_1(x) = \sqrt{1-x^2} \text{d}x, \quad \text{d}\phi_2(x) = (1+kx^2)\sqrt{1-x^2} \text{d}x,
\]
where $k > 0$. It is known that $\phi^{n+1}_{n+1} = \frac{1}{4}$, $n \geq 1$. Hence, from (5.2) it follows that

$$k^{(1)}_n = \frac{1}{\sqrt{1 + k}} \left\{ \frac{1 + \sqrt{1 + k}}{2} \right\}^{n+1} - \frac{1}{\sqrt{1 + k}} \left\{ \frac{1 - \sqrt{1 + k}}{2} \right\}^{n+1}, \quad n \geq 0.$$ 

Thus, we can write

$$\phi^{n+1}_{n+1} = \frac{1}{4} \frac{L_n - L_{n+2}}{L_n L_{n+1}}, \quad n \geq 1,$$

where $L_n = \{1 + \sqrt{1 + k}\}^{n+1} - \{1 - \sqrt{1 + k}\}^{n+1}, \quad n \geq 0$. We believe that this result also holds for any $k \geq -1$.

**Example 3.** For $d = 1$ and $k > 0$, let

$$d\phi_1(x) = \frac{1}{(1 + kx^2)\sqrt{1 - x^2}} \, dx, \quad d\phi_2(x) = \frac{1}{\sqrt{1 - x^2}} \, dx.$$

Here, instead of using Theorem 3 we shall consider a different approach. Since $\phi^{n+1}_{n+1} = \frac{1}{4}$, $n \geq 2$, we obtain from (1.2)

$$\ell_n - 1 = \frac{4k\phi^{n+1}_{n+1}}{2 + (\ell_{n+1} - 1)} = \frac{k}{2 + (\ell_{n+1} - 1)}, \quad n \geq 2.$$

This leads to the convergent periodic continued fraction

$$\ell_n - 1 = \frac{k}{2 + \frac{k}{2 + \frac{k}{2 + \ldots}}},$$

the limit of which is $-1 + \sqrt{1 + k}$. Hence, $\ell_n + 1 = 1 + \sqrt{1 + k}, \quad n \geq 2$.

Since $\ell_1 - 1 = 2k/(\ell_2 + 1)$, it also follows that $\ell_1 + 1 = 2\sqrt{1 + k}$. Thus, from (5.1),

$$\phi^{n+1}_2 = \frac{1}{1 + \sqrt{1 + k}}, \quad \phi^{n+1}_3 = \frac{\sqrt{1 + k}}{2(1 + \sqrt{1 + k})}$$

and

$$\phi^{n+1}_{n+1} = \frac{1}{4}, \quad n \geq 3.$$

This result has already been obtained in [3] for all $k \geq -1$.

**Example 4.** With $d = 1$ and $k > 0$ let

$$d\phi_1(x) = \frac{\sqrt{1 - x^2}}{(1 + kx^2)} \, dx, \quad d\phi_2(x) = \sqrt{1 - x^2} \, dx.$$
Here, since $\alpha_{n+1} = \frac{1}{4}$, $n \geq 1$, we obtain from (1.2) that $\ell_n + 1 = 1 + \sqrt{1 + k}$, $n \geq 1$. Hence, from (5.1)

$$\alpha_2^{\phi} = \frac{1}{2(1 + \sqrt{1 + k})}$$

and

$$\alpha_{n+1}^{\phi} = \frac{1}{4}, \quad n \geq 2.$$

This result follows from [2] for all $k \geq -1$. For information on the results of the last two examples, see also [1, p. 205].

**Example 5.** Let $d = 1$ and that for $k > 0$ and $k_1 > 0$, let

$$d\phi_1(x) = \frac{\sqrt{1 - x^2}}{(1 + kx^2)(1 + k_1x^2)} \, dx, \quad d\phi_2(x) = \frac{\sqrt{1 - x^2}}{(1 + k_1x^2)} \, dx.$$

From the previous example we note that $\alpha_2^{\phi_1} = \frac{1}{2} \left( 2(1 + \sqrt{1 + k}) \right)^{-1}$ and $\alpha_2^{\phi_2} = \frac{1}{4}$, $n \geq 2$. Thus, we obtain from (1.2) that $\ell_n + 1 = 1 + \sqrt{1 + k}$, $n \geq 2$. Furthermore, also from (1.2)

$$\ell_1 - 1 = \frac{2k}{(1 + \sqrt{1 + k})(1 + \sqrt{1 + k_1})},$$

and hence,

$$\ell_1 + 1 = \frac{2[\sqrt{1 + k} + \sqrt{1 + k_1}]}{(1 + \sqrt{1 + k_1})}.$$

Consequently, from (5.1)

$$\alpha_2^{\phi_1} = \frac{1}{(1 + \sqrt{1 + k})(1 + \sqrt{1 + k_1})}, \quad \alpha_2^{\phi_2} = \frac{(\sqrt{1 + k} + \sqrt{1 + k_1})}{2(1 + \sqrt{1 + k})(1 + \sqrt{1 + k_1})}$$

and

$$\alpha_{n+1}^{\phi_1} = \frac{1}{4}, \quad n \geq 3.$$

**Example 6.** Here, with $d = 1$, $k > 0$ and $k_1 > 0$ we take

$$d\phi_1(x) = \frac{\sqrt{1 - x^2}}{(1 + kx^2)} \, dx, \quad d\phi_2(x) = \frac{(1 + kx^2)}{(1 + k_1x^2)} \sqrt{1 - x^2} \, dx.$$

Since $\alpha_2^{\phi_1} = \frac{1}{2} \left( 2(1 + \sqrt{1 + k}) \right)^{-1}$ and $\alpha_2^{\phi_2} = \frac{1}{4}$, $n \geq 2$, we obtain from Theorem 2 that

$$\alpha_2^{\phi_2} = \frac{1}{2(1 + \sqrt{1 + k}) b_2^{(1)}} \quad \text{and} \quad \alpha_{n+1}^{\phi_2} = \frac{1}{4} b_n^{(1)} b_{n+1}^{(1)}, \quad n \geq 2.$$

Here, from (5.2)

$$b_n^{(1)} = c_1 \left\{ \frac{1 + \sqrt{1 + k}}{2} \right\}^n + c_2 \left\{ \frac{1 - \sqrt{1 + k}}{2} \right\}^n, \quad n \geq 1,$$
where
\[ c_1 = \frac{\sqrt{1 + k + \sqrt{1 + k_1}}}{\sqrt{1 + k(1 + \sqrt{1 + k_1})}}, \quad c_2 = \frac{\sqrt{1 + k - \sqrt{1 + k_1}}}{\sqrt{1 + k(1 + \sqrt{1 + k_1})}}. \]

**Example 7.** For \( d = 1, \ k > 0 \) and \( k_1 > 0 \) let
\[ d \phi_1(x) = \frac{1}{(1 + kx^2)(1 + k_1x^2)\sqrt{1 - x^2}} \ dx, \quad d \phi_2(x) = \frac{1}{(1 + k_1x^2)\sqrt{1 - x^2}} \ dx. \]
Using the results of Example 3, we obtain from (1.2) that \( \ell_n + 1 = 1 + \sqrt{1 + k}, \ n \geq 3, \)
\[ \ell_2 + 1 = \frac{2(1 + \sqrt{1 + k\sqrt{1 + k_1}})}{1 + \sqrt{1 + k_1}} \]
and
\[ \ell_1 + 1 = \frac{2\sqrt{1 + k}(\sqrt{1 + k + \sqrt{1 + k_1}})}{1 + \sqrt{1 + k\sqrt{1 + k_1}}}. \]
Thus, the use of (5.1) gives
\[ \alpha_2^{\phi_1} = \frac{1}{1 + \sqrt{1 + k\sqrt{1 + k_1}}}, \]
\[ \alpha_3^{\phi_1} = \frac{\sqrt{1 + k\sqrt{1 + k_1}(\sqrt{1 + k + \sqrt{1 + k_1})}}}{(1 + \sqrt{1 + k})(1 + \sqrt{1 + k_1})(1 + \sqrt{1 + k\sqrt{1 + k_1})}}, \]
\[ \alpha_4^{\phi_1} = \frac{(1 + \sqrt{1 + k\sqrt{1 + k_1})}}{2(1 + \sqrt{1 + k})(1 + \sqrt{1 + k_1})} \]
and
\[ \alpha_n^{\phi_1} = \frac{1}{4}, \quad n \geq 4. \]

**Example 8.** Finally, for \( d = 1, \ k > 0 \) and \( k_1 > 0 \) let
\[ d \phi_1(x) = \frac{1}{(1 + kx^2)\sqrt{1 - x^2}} \ dx, \quad d \phi_2(x) = \frac{(1 + kx^2)}{(1 + k)x^2)\sqrt{1 - x^2}} \ dx. \]
Then from Theorem 2
\[ \alpha_2^{\phi_2} = \frac{1}{1 + \sqrt{1 + k_1} b_2^{(1)}}, \quad \alpha_3^{\phi_2} = \frac{\sqrt{1 + k_1}}{2(1 + \sqrt{1 + k_1}) b_2^{(1)} b_3^{(1)}} \]
and
\[ \alpha_n^{\phi_2} = \frac{1}{4} \frac{b_n^{(1)} b_{n+2}^{(1)}}{b_n^{(1)} b_{n+1}^{(1)}}, \quad n \geq 3. \]
Here, from (5.2) it follows that

\[ b_n^{(1)} = c_1 \left( \frac{1 + \sqrt{1 + k}}{2} \right)^{n-1} - c_2 \left( \frac{1 - \sqrt{1 + k}}{2} \right)^{n-1}, \quad n \geq 2, \]

where

\[ c_1 = \frac{\sqrt{1 + k} + \sqrt{1 + k_1}}{(1 + \sqrt{1 + k_1})}, \quad c_2 = \frac{\sqrt{1 + k} - \sqrt{1 + k_1}}{(1 + \sqrt{1 + k_1})}. \]

In all four of the above examples we believe that the results hold for \( k > -1 \) and \( k_1 > -1 \).

References