Master's Dissertation IFT–D.015/2022

Inferring new physics characteristics from the muon anomalous magnetic moment discrepancy

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November, 2022

C394i Cemin, Marcello Vinícius Hallas
Inferring new physics characteristics from the muon anomalous magnetic moment discrepancy/ Marcello Vinícius Hallas Cemin. – São Paulo, 2022
67 f.

Dissertação (mestrado) – Universidade Estadual Paulista (Unesp), Instituto de Física Teórica (IFT), São Paulo
Orientador: Juan Carlos Montero Garcia

1. Modelo padrão (Física nuclear). 2. Partículas (Física nuclear). 3. Muons. I. Título

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This thesis is dedicated to
Natália.

Acknowledgements

First of all, I would like to thank my parents and my girlfriend for all their love, patience, and understanding over the years, without whom none of this would be possible.

I thank my master's supervisor, Juan Carlos Montero, for all the support, encouragement, and weekly debate throughout those years of research.

I also thank the IFT structure, teachers, and friends for the workplace, which certainly made the job easier.

Last but not least, I would like to thank CNPq for the financial support.

Nobody ever figures out what life is all about, and it doesn't matter. Explore the world. Nearly everything is really interesting if you go into it deeply enough.

Richard P. Feynman

Resumo

Após uma rápida revisão do Modelo Padrão da Física de Partículas, nós definimos o momento de dipolo magnético anômalo do múon em termos dos fatores de forma. Desse modo, utilizando teoria de perturbação, calculamos a contribuição a 1-loop da eletrodinâmica quântica e da teoria eletrofraca. Citamos os valores encontrados na literatura para ordens superiores e com isso temos então a previsão teórica do Modelo Padrão para o momento de dipolo magnético anômalo do múon.

Discutimos a física por trás dos últimos experimentos dedicados a medir o momento magnético anômalo do múon, e vemos que o resultado obtido difere do teórico por 4.2σ, o que pode indicar a presença de nova física.

Como apenas a adição de novos bósons ou novos férmions sozinhos não é capaz de explicar tal discrepância, olhamos para a possibilidade de ao adicionarmos novos bósons, juntamente com novos férmions, obtermos contribuições que possam explicar a discrepância entre teoria e experimento. Em específico olhamos para o modelo 3-3-1 com um lepton pesado e carregado e concluímos que a contribuição da nova física só pode vir através do setor escalar visto que a adição dos novos bósons e férmions com tais interações trariam contribuições ainda menores que as do Modelo Padrão para o momento de dipolo magnético do múon.

Finalmente, consideramos interações gerais para partículas escalares neutras e carregadas e concluímos que, em princípio, elas podem contribuir para o momento magnético anômalo do múon.

Palavras Chaves: Momento de dipolo magnético do muon; Teoria eletrofraca; Invariancia de gauge; Modelo 3-3-1;

Áreas do conhecimento: Física de Partículas.

Abstract

After a quick review of the Standard Model of Particle Physics, we define the anomalous magnetic dipole moment of the muon in terms of the form factors. In this way, using perturbation theory, we calculate the 1-loop contribution from quantum electrodynamics and also from electroweak theory. We quote the values found in the literature for higher orders and with that, we have the theoretical prediction of the Standard Model for the anomalous magnetic dipole moment of the muon.

We discussed the physics behind the last experiments dedicated to measuring the anomalous magnetic moment of the muon and see that the result obtained differs from the theoretical one by 4.2σ, which may indicate the presence of new physics.

Since the addition of new bosons or new fermions alone is not able to explain such a discrepancy, we look at the possibility that by adding new bosons, together with new fermions, we can obtain contributions that can explain the discrepancy between theory and experiment. Specifically, we look at the 3-3-1 model with a heavy charged lepton and conclude that the contribution of the new physics can only come through the scalar sector since the addition of new bosons and fermions with such interactions would bring even smaller contributions than those of the Standard Model to the magnetic dipole moment of the muon.

Finally, we consider general interactions for neutral and charged scalar particles and conclude that, in principle, they can contribute to the muon anomalous magnetic moment.

Keywords: Magnetic dipole moment of the muon; Electroweak theory; Gauge invariance; 3-3-1 model;

Areas of knowledge: Particle Physics.

Contents

Chapter 1

Introduction

We start with a review of the Standard Model (SM) of particle physics, mainly focusing on the electroweak part. We explain each sector separately and also comment on some problems that the model presents. One of them is the anomalous magnetic dipole moment of the muon, and that is what we discuss in this thesis.

The magnetic dipole moment of a particle is characterized by a parameter g, the gyromagnetic factor, which tells how strong is the coupling between the particle's spin and an external magnetic field. This can be described by the following Hamiltonian:

$$H = -\vec{\mu} \cdot \vec{B}, \qquad \vec{\mu} = g\frac{e}{2m}\vec{S}. \tag{1.1}$$

From the Dirac equation, one can show that $g = 2$. It turns out that g is quite close but not equal to two. It happens because we have to add quantum corrections that slightly change that value. For that matter, we write $g = 2(1 + a)$, where a is called the anomalous magnetic moment and is the part we want to calculate. In terms of the form factors, we find: $F_2(0) = a$.

The first correction was calculated in 1948 by Julian Schwinger. We reproduced the calculation and also found the famous $\alpha/2\pi$. After that, we calculate the contributions coming from the electroweak sector and found that:

$$a_\mu^{EW}[\text{1-loop}] \simeq \frac{G_F m^2}{8\sqrt{2}\pi^2} \times \left[\frac{5}{3} + \frac{1}{3}\left(1 - 4\sin^2\theta_W\right)^2\right], \tag{1.2}$$

which is in agreement with the value found in the literature.

The anomalous magnetic dipole moment of the muon was measured with great precision. We discussed how the experiment works and the physics behind it. The theoretical and experimental values are:

$$\begin{aligned} a_\mu^{SM} &= 0.00116591810(43), \\ a_\mu^{exp} &= 0.00116592061(41), \\ \Delta a_\mu &= 0.00000000251(59). \end{aligned} \tag{1.3}$$

This means that they differ by 4.2σ, which strongly suggests the presence of new physics.

Next, we consider one of the simplest extensions of the Standard Model, the 3-3-1 model with a heavy charged lepton. We do not explore the whole model. We only look at the gauge boson sector contribution instead. In the best-case scenario, we find that:

$$a_\mu^{331}[\text{1-loop}] \simeq \frac{G_F m^2}{8\sqrt{2}\pi^2} \times 10 \left[\left(\frac{M_W}{M_U}\right)^2 - \left(\frac{M_Z}{M_{Z'}}\right)^2 \right], \tag{1.4}$$

where the number 10 comes from the order of magnitude of the integral in the Feynman parameters, and U and Z' are the heavy doubly charged gauge boson and the heavy neutral gauge boson, respectively, that the theory predicts. With the masses of U and Z' at the TeV scale, those contributions are too small to account for the discrepancy, which means that the contributions must come from the scalar sector.

Finally, we considered general interactions for the scalar sector and found that models with those characteristics have the potential to explain the theoretical-experimental discrepancy. In particular, we show that there are sets of parameters, related to the new physics, that can make the theoretical values compatible with the observed ones.

Chapter 2

The Standard Model of Particle Physics

There are 4 known fundamental forces in the universe, the strong force which binds protons and neutrons inside the nucleus, the weak force that is responsible for radioactive decay, the electromagnetic force which is present in most phenomena in our everyday life, and the gravitational force which may be the most familiar but in some sense the least known.

The Standard Model of Particle Physics describes the first three, and it is based on the symmetry group:

$$G_{SM} \equiv SU(3)_C \otimes SU(2)_L \otimes U(1)_Y, \tag{2.1}$$

where $SU(3)_C$ describes the strong sector and is an exact symmetry of nature, while $SU(2)_L \otimes U(1)_Y$ describes the electroweak sector, a unified description of electromagnetism and weak interaction but is a broken symmetry at current energies. This idea of a broken symmetry is used in subsection 2.1.3 and it is a fundamental piece of the theory since it makes it possible to generate mass for the particles. In this master thesis, we will mainly focus on the electroweak sector of the Standard Model.

2.1 Standard Model Lagrangian

The most general Lagrangian we can write that is Lorentz invariant, renormalizable, and gauge invariant under G_{SM} is:

$$\mathcal{L}_{SM} = \mathcal{L}_{Matter} + \mathcal{L}_{Scalar} + \mathcal{L}_{Gauge} + \mathcal{L}_{Yukawa}, \tag{2.2}$$

and we will be explaining each sector separately.

2.1.1 Matter Sector

From low-energy experiments, we have gathered a vast amount of information about flavor-changing processes. After a detailed analysis of β decay, such as $\mu^- \to e^- + \nu_e + \nu_\mu$, it became clear that only the left-handed fermion chiralities participate in those transitions [1]. One important consequence of this is reflected in the masses of the matter fields. A fermionic mass term is of the form:

$$m\overline{\psi}\psi = m(\overline{\psi}_L\psi_R + \overline{\psi}_R\psi_L), \tag{2.3}$$

and from those experiments, we can see that the SM should treat the left and right components of the matter fields differently somehow. This implies that a term of the form $\overline{\psi}\psi$ is forbidden, i.e., all fermions in the SM should be massless. We will see in the following subsections how to handle this problem.

Quantum Numbers:

The assignment of quantum numbers to the various components of the matter fields follows the Gell-Mann-Nishijima formula:

$$Q = I_3 + \frac{Y}{2}, \tag{2.4}$$

where Q is the electric charge, I is the weak isospin, I_3 is the third component of weak isospin, and Y is the weak hypercharge.

Leptons	I	I_3	Y	Q
ν_e	1/2	1/2	-1	0
e_L	1/2	-1/2	-1	-1
e_R	0	0	-2	-1

Table 2.1: Leptonic quantum numbers

Quarks	I	I_3	Y	Q
u_L	1/2	1/2	1/3	2/3
d_L	1/2	-1/2	1/3	-1/3
u_R	0	0	4/3	2/3
d_R	0	0	-2/3	-1/3

Table 2.2: Quarkyonic quantum numbers

Building the Lagrangian:

Taking all this into consideration, we write the left-handed fermions as doublets and the right-handed ones as singlets of $SU(2)_L$, and, since we observe three families, we will have three copies of the particles in Table 2.1 and three copies of particles in Table 2.2. This means that we can build the leptonic Lagrangian from the following objects:

$$\begin{pmatrix} \nu_e \\ e \end{pmatrix}_L, \begin{pmatrix} \nu_\mu \\ \mu \end{pmatrix}_L, \begin{pmatrix} \nu_\tau \\ \tau \end{pmatrix}_L, e_R,\ \mu_R,\ \tau_R, \tag{2.5}$$

and we can build the quarkyonic Lagrangian from the following objects:

$$\begin{pmatrix} u \\ d \end{pmatrix}_L, \begin{pmatrix} c \\ s \end{pmatrix}_L, \begin{pmatrix} t \\ b \end{pmatrix}_L, u_R,\ d_R,\ c_R, s_R,\ t_R,\ b_R, \tag{2.6}$$

where we are considering the matter Lagrangian as a sum of the leptonic and quarkyonic Lagrangian:

$$\mathcal{L}_{Matter} = \mathcal{L}_{Leptons} + \mathcal{L}_{Quarks}. \tag{2.7}$$

From this, we can already see that there is no right-handed neutrino in the SM, consequently, they will remain massless.

The matter Lagrangian is given by:

$$\mathcal{L}_{Matter} = \sum_{\substack{all \\ doublets}} \overline{\psi}_L i\gamma^\mu D^L_\mu \psi_L + \sum_{\substack{all \\ singlets}} \overline{\psi}_R i\gamma^\mu D^R_\mu \psi_R, \tag{2.8}$$

where $\psi_L(\psi_R)$ represents a fermionic doublet (singlet) and $D^{L,R}_\mu$ are covariant derivatives that are shown explicitly below. Note that in principle we could have written a more general Lagrangian considering non-diagonal terms (for leptons and quarks separately), however, we can do a rotation in flavor space to make them diagonal without loss of generality.

$$\begin{aligned} \mathcal{L}_{Matter} = & \sum_{\substack{all \\ doublets}} \overline{\psi}_L i\gamma^\mu (\partial_\mu \mathbb{1}_{2x2} + \frac{ig'}{2} Y^L B_\mu \mathbb{1}_{2x2} + \frac{ig}{2} \vec{\sigma} \cdot \vec{W}_\mu) \psi_L \\ & + \sum_{\substack{all \\ singlets}} \overline{\psi}_R i\gamma^\mu (\partial_\mu + \frac{ig'}{2} Y^R B_\mu) \psi_R, \end{aligned} \tag{2.9}$$

where σ_i are the Pauli matrices, g and g' are the coupling constants of $SU(2)_L$ and $U(1)_Y$, respectively, and the possible values of Y^L and Y^R can be seen in Table 2.1 and Table 2.2.

As we will see in the subsection 2.1.3, we can express W^a_μ and B_μ as linear combinations of the physical fields $W^{\pm}_{\mu}$, Z_{μ} and A_{μ}, so here, we will simply quote the results of writing the matter Lagrangian (2.9) in terms of these new fields:

$$\mathcal{L}_{Matter} = \sum_{\substack{i=all\\fermions}} \overline{\psi}_i i\gamma^{\mu}\partial_{\mu}\psi_i + \mathcal{L}^{(0)}_{Matter} + \mathcal{L}^{(\pm)}_{Matter}, \tag{2.10}$$

with

$$\mathcal{L}^{(0)}_{Matter} = \sum_{\substack{i=all\\fermions}} eQ_i\overline{\psi}_i\gamma^{\mu}\psi_i A_{\mu} - \frac{e}{\sin(2\theta_W)}\overline{\psi}_i\gamma^{\mu}(g^i_V\mathbb{1} - g^i_A\gamma_5)\psi_i Z_{\mu}, \tag{2.11}$$

and

$$\mathcal{L}^{(\pm)}_{Matter} = -\sqrt{\frac{M_W^2 G_F}{\sqrt{2}}} \sum_{all\,families} \overline{\nu}_e\gamma^{\mu}(\mathbb{1} - \gamma_5)eW^{+}_{\mu} + \overline{e}\gamma^{\mu}(\mathbb{1} - \gamma_5)\nu_e W^{-}_{\mu}, \tag{2.12}$$

where $\mathcal{L}^{(0)}_{Matter}$ is the neutral Lagrangian, i.e., contains the neutral bosons, and $\mathcal{L}^{(\pm)}_{Matter}$ is the charged Lagrangian, i.e., contains the charged bosons, and we have identified $e = g\sin(\theta_W)$ as the electric charge. ψ_i represents a fermion, $g^i_V \equiv I^i_3 - 2Q_i\sin^2(\theta_W)$ and $g^i_A \equiv I^i_3$ can be computed from Table 2.1 and Table 2.2, where $Q_i = \pm 1$ depending on the sign of the electric charge. $G_F \equiv \frac{g^2\sqrt{2}}{8M_W^2}$ is the Fermi constant, that we can identify by comparison with low energy phenomenology. Note also that $Q_i = 0$ for the neutrinos, so they do not couple with the photon.

2.1.2 Gauge Sector

As we have seen, we can not add a mass term for the matter fields. What about bosons? Consider a $SU(N)$ symmetry group, a bosonic mass term is of the form:

$$\frac{m^2}{2}A_{\mu}A^{\mu}, \tag{2.13}$$

where A^a_{μ} transforms as follows:

$$A^a_{\mu} \longrightarrow A'^a_{\mu} = A^a_{\mu} - \frac{1}{g}\partial_{\mu}\alpha^a - f^{abc}\alpha^b A^c_{\mu}, \tag{2.14}$$

where g is the coupling constant, α^a are the gauge transformation parameters, f^{abc} the structure constant and $a = 1, 2, \ldots, N^2 - 1$. This implies that the mass term in

(2.13) is not gauge invariant, so they are also forbidden, hence, the bosons should also be massless. Note that even if we had considered an abelian group, the mass term (2.13) would still violates gauge invariance.

This means that the gauge Lagrangian will only include the kinetic terms of the gauge bosons. In total it contains twelve bosons, one for each generator of the Standard Model group. From $SU(3)_C$ we have eight bosons $G^a_\mu (a = 1, ..., 8)$, they are the gluons, from $SU(2)_L$ and $U(1)_Y$ we have three bosons $W^a_\mu (a = 1, 2, 3)$ and one B_μ boson respectively, that combines to form the photon, the $W^\pm_\mu$, and Z_μ bosons.

The gauge Lagrangian is:

$$\mathcal{L}_{Gauge} = -\frac{1}{4} G^a_{\mu\nu} G^{a\mu\nu} - \frac{1}{4} W^a_{\mu\nu} W^{a\mu\nu} - \frac{1}{4} B_{\mu\nu} B^{\mu\nu}, \tag{2.15}$$

where the field strength tensors are given by

$$\begin{aligned} G^a_{\mu\nu} &= \partial_\mu G^a_\nu - \partial_\nu G^a_\mu + g_s f^{abc} G^b G^c, \\ W^a_{\mu\nu} &= \partial_\mu W^a_\nu - \partial_\nu W^a_\mu + g\epsilon^{abc} W^b W^c, \\ B_{\mu\nu} &= \partial_\mu B_\nu - \partial_\nu B_\mu, \end{aligned} \tag{2.16}$$

with g_s being the coupling constant and f^{abc} the structure constant, both from $SU(3)_C$. The ϵ^{abc} is the Levi-Civita totally antisymmetric symbol and also the structure constant of $SU(2)_L$.

As we will see in the next subsection, we can express W^a_μ and B_μ as linear combinations of the physical fields $W^\pm_\mu$, Z_μ, and A_μ. Here, we simply quote the results of writing the gauge Lagrangian (2.15) in terms of these new fields:

$$\mathcal{L}_{Gauge} = -\frac{1}{4} G^a_{\mu\nu} G^{a\mu\nu} - \frac{1}{2} W^+_{\mu\nu} W^{-\mu\nu} - \frac{1}{4} A_{\mu\nu} A^{\mu\nu} - \frac{1}{4} Z_{\mu\nu} Z^{\mu\nu} + \mathcal{L}^{(3)}_{Gauge} + \mathcal{L}^{(4)}_{Gauge}, \tag{2.17}$$

where

$$\begin{aligned} \mathcal{L}^{(3)}_{Gauge} = &-ie(W^-_{\mu\nu} W^{+\mu} A^\nu - W^+_{\mu\nu} W^{-\mu} A^\nu - W^-_\mu W^+_\nu A^{\mu\nu}) \\ &-ie\cot(\theta_W)(W^-_{\mu\nu} W^{+\mu} Z^\nu - W^+_{\mu\nu} W^{-\mu} Z^\nu - W^-_\mu W^+_\nu Z^{\mu\nu}), \end{aligned} \tag{2.18}$$

and

$$\begin{aligned}\mathcal{L}^{(4)}_{Gauge} = \frac{e^2}{2\sin^2(\theta_W)}&(W^+_\mu W^{-\mu} W^+_\nu W^{-\nu} - W^+_\mu W^{+\mu} W^-_\nu W^{-\nu}) \\ &-e^2(W^+_\mu W^{-\mu} A_\nu A^\nu - W^+_\mu A^\mu W^-_\nu A^\nu) \\ &-e^2\cot^2(\theta_W)(W^+_\mu W^{-\mu} Z_\nu Z^\nu - W^+_\mu Z^\mu W^-_\nu Z^\nu) \\ &-e^2\cot(\theta_W)(2W^+_\mu W^{-\mu} A_\nu Z^\nu - W^+_\mu A^\mu W^-_\nu Z^\nu - W^+_\mu Z^\mu W^-_\nu A^\nu),\end{aligned} \tag{2.19}$$

with

$$\begin{aligned} W^\pm_{\mu\nu} &= \partial_\mu W^\pm_\nu - \partial_\nu W^\pm_\mu, \\ A_{\mu\nu} &= \partial_\mu A_\nu - \partial_\nu A_\mu, \\ Z_{\mu\nu} &= \partial_\mu Z_\nu - \partial_\nu Z_\mu. \end{aligned} \tag{2.20}$$

Note that $\mathcal{L}^{(3)}_{Gauge}$ and $\mathcal{L}^{(4)}_{Gauge}$ contains all cubic and quartic bosonic electroweak interaction terms, respectively.

2.1.3 Scalar Sector

From the previous subsections, we saw that all particles in the standard model should be massless. That is, of course, not what we observe in nature, and to fix it we will use the Higgs mechanism, i.e., we will add a new doublet of $SU(2)_L$ and use the spontaneous symmetry breaking (SSB) of the vacuum to give mass to almost all particles in the SM.

This new doublet will follow the Gell-Man-Nishijima formula (2.4):

Components	I	I_3	Y	Q
ϕ^+	1/2	1/2	1	1
ϕ^0	1/2	-1/2	1	0

Table 2.3: Higgs-doublet quantum numbers

From Table 2.3, the new doublet is

$$\Phi = \begin{pmatrix} \phi^+ \\ \phi^0 \end{pmatrix}, \tag{2.21}$$

and the most general scalar Lagrangian, renormalizable, that we can write is:

$$\mathcal{L}_{Scalar} = \|D_\mu \Phi\|^2 - V(\Phi^\dagger \Phi), \tag{2.22}$$

with the covariant derivative given by:

$$D_\mu \Phi = \left(\partial_\mu \mathbb{1}_{2x2} + \frac{ig'}{2} Y^L B_\mu \mathbb{1}_{2x2} + \frac{ig}{2} \vec{\sigma} \cdot \vec{W}_\mu \right) \Phi, \tag{2.23}$$

where $Y^L = 1$, according to Table 2.3, and

$$V(\Phi^\dagger \Phi) = \mu^2 (\Phi^\dagger \Phi) + \lambda (\Phi^\dagger \Phi)^2, \tag{2.24}$$

where μ^2 and λ are arbitrary constants. Note that $\lambda > 0$, for the stability of the potential, i.e., that it is bounded from below.

If $\mu^2 > 0$, the minimum of the potential is at the origin, however, if $\mu^2 < 0$ the vacuum has a non-trivial minimum that we can parametrize as:

$$\langle \Phi \rangle_0 \ = \frac{1}{\sqrt{2}} \begin{pmatrix} 0 \\ v \end{pmatrix}, \text{ with } v \equiv \sqrt{-\frac{\mu^2}{\lambda}}. \tag{2.25}$$

where v is the vacuum expectation value (VeV). If that is the case, we have to do a shift in the Higgs-doublet so that we can expand the fields around this minimum. This means that the vacuum does not respect the original symmetry, in other words, we say that the vacuum spontaneously breaks the symmetry:

$$SU(3)_C \otimes SU(2)_L \otimes U(1)_Y \xrightarrow[\text{SSB}]{} SU(3)_C \otimes U(1)_{EM}. \tag{2.26}$$

Now, we parameterize the Higgs-doublet in terms of χ_i and H :

$$\Phi = \exp\left\{ \frac{i\vec{\sigma} \cdot \vec{\chi}}{2v} \right\} \begin{pmatrix} 0 \\ (v+H)/\sqrt{2} \end{pmatrix} \simeq \frac{1}{\sqrt{2}} \begin{pmatrix} i\sqrt{2}w^+ \\ v + H - iw_0 \end{pmatrix}, \tag{2.27}$$

where σ_i are the Pauli matrices, $w^\pm = \frac{1}{\sqrt{2}}(\chi_1 \mp i\chi_2)$, $w_0 = \frac{\chi_3}{2}$, and H is the Higgs boson. Note that (w^+, w^-, w^0) are the Goldstone bosons.

The idea behind this is that we can now perform a $SU(2)_L$ gauge transformation with the function $\alpha_i = \chi_i/v$ (2.14) and the fields will transform as follow:

$$\begin{aligned} W^a_\mu &\longrightarrow W'^a_\mu, \\ B_\mu &\longrightarrow B'_\mu, \\ \psi_L &\longrightarrow \psi'_L, \\ \psi_R &\longrightarrow \psi'_R, \\ \Phi \longrightarrow \Phi' = \exp(-\frac{i\vec{\sigma}\cdot\vec{\chi}}{2v})\Phi &= \frac{1}{\sqrt{2}}\begin{pmatrix} 0 \\ v+H \end{pmatrix}. \end{aligned} \tag{2.28}$$

As we can see, in this gauge, the Higgs-doublet assumes a simple form that is easy to work with. This is the unitary gauge.

We can now drop the prime and look at the scalar Lagrangian (2.22) in the unitary gauge, i.e., with the parameterization (2.28):

$$\begin{aligned} \mathcal{L}_{Scalar} = \frac{1}{2}(\partial_\mu H)(\partial^\mu H) - \frac{\mu^2}{2}(v+H)^2 - \frac{\lambda}{4}(v+H)^4 \\ + \frac{g^2}{4}W^-_\mu W^{+\mu}(v+H)^2 + \frac{1}{8}(gW^3_\mu - g'B_\mu)^2(v+H)^2, \end{aligned} \tag{2.29}$$

with $W^\pm_\mu \equiv \frac{1}{\sqrt{2}}(W^1_\mu \mp iW^2_\mu)$. Since we started with symmetry eigenvectors, we have to rotate them to find the mass eigenvectors, in doing so, we find:

$$\begin{aligned} Z_\mu &= \cos(\theta_W)W^3_\mu - \sin(\theta_W)B_\mu \\ A_\mu &= \sin(\theta_W)W^3_\mu + \cos(\theta_W)B_\mu \end{aligned} \Leftrightarrow \begin{aligned} W^3_\mu &= +\cos(\theta_W)Z_\mu + \sin(\theta_W)A_\mu \\ B_\mu &= -\sin(\theta_W)Z_\mu + \cos(\theta_W)A_\mu, \end{aligned} \tag{2.30}$$

with $\cos(\theta_W) = \frac{g}{\sqrt{g^2+g'^2}}$ and $\sin(\theta_W) = \frac{g'}{\sqrt{g^2+g'^2}}$. Using (2.30) and the minimum condition (2.25) into (2.29) we obtain:

$$\begin{aligned} \mathcal{L}_{Scalar} = \frac{1}{2}[(\partial_\mu H)(\partial^\mu H) + M_H^2 H^2] - \lambda v H^3 - \frac{\lambda}{4}H^4 \\ + \frac{g^2}{4}W^-_\mu W^{+\mu}(v^2 + 2vH + H^2) \\ + \frac{g^2}{8\cos^2(\theta_W)}Z_\mu Z^\mu(v^2 + 2vH + H^2), \end{aligned} \tag{2.31}$$

where $M_H \equiv \sqrt{-2\mu^2} > 0$ is the Higgs mass, $M_{W^\pm} = \frac{gv}{2}$ is the $W^\pm$ boson mass, and $M_Z = \frac{gv}{2\cos(\theta_W)}$ is the Z boson mass.

From this, we can already see that $M_Z > M_{W^\pm}$ and that the photon does not couple with the Higgs boson and remains massless. We can also see cubic and quartic interactions between the vector bosons and the Higgs boson.

2.1.4 Yukawa Sector

For this part, let's write the Lagrangian as:

$$\mathcal{L}_{Yukawa} = \mathcal{L}^{(L)}_{Yukawa} + \mathcal{L}^{(Q)}_{Yukawa}, \tag{2.32}$$

where $\mathcal{L}^{(L)}_{Yukawa}$ represents the piece that comes from leptons, and $\mathcal{L}^{(Q)}_{Yukawa}$ represents the piece that comes from quarks.

Leptonic part of the Yukawa sector:

Since we add a new doublet (2.21) of $SU(2)_L$ in the theory, we find that we could also add a new term to the Lagrangian:

$$-G(\overline{\psi}_L \Phi \psi_R + \overline{\psi}_R \Phi^\dagger \psi_L), \tag{2.33}$$

where ψ_L represents a leptonic doublet, ψ_R represents a leptonic singlet and G is some arbitrary constant. This term is of course gauge invariant under $SU(2)_L \otimes U(1)_Y$:

$$\begin{aligned}
\psi_L &\longrightarrow \psi'_L = \exp\left\{(\frac{i\vec{\sigma}\cdot\vec{\chi}}{2})\right\}\exp\left\{(i\frac{Y^L}{2}\alpha)\right\}\psi_L, \\
\psi_R &\longrightarrow \psi'_R = \exp\left\{(i\frac{Y^R}{2}\alpha)\right\}\psi_R, \\
\Phi &\longrightarrow \Phi' = \exp\left\{(\frac{i\vec{\sigma}\cdot\vec{\chi}}{2})\right\}\exp\left\{(i\frac{Y^\Phi}{2}\alpha)\right\}\Phi,
\end{aligned} \tag{2.34}$$

as we can see by explicitly writing the transformation:

$$\begin{aligned}
\overline{\psi}'_L \Phi' \psi'_R &= \exp\left\{(\frac{i}{2}[-Y^L + Y^\Phi + Y^R]\alpha)\right\}\overline{\psi}_L \\
&\quad\times \exp\left\{(\frac{i\vec{\sigma}\cdot\vec{\chi}}{2})\right\}\exp\left\{(\frac{-i\vec{\sigma}\cdot\vec{\chi}}{2})\right\}\Phi\psi_R \\
&= \overline{\psi}_L \Phi \psi_R,
\end{aligned} \tag{2.35}$$

because $(-Y^L + Y^\Phi + Y^R) = (+1 + 1 - 2) = 0$.

The most general Yukawa Lagrangian we can write for the leptons considering three families is:

$$\begin{aligned}\mathcal{L}^{(L)}_{Yukawa} &= \sum_{i,j=1}^{3} -G_{ij}(\overline{\psi}_{iL}\Phi\psi_{jR}) + h.c \\ &= -(\overline{\mathrm{e}}_L, \overline{\mu}_L, \overline{\tau}_L)\begin{pmatrix} m_{\mathrm{ee}} & m_{\mathrm{e}\mu} & m_{\mathrm{e}\tau} \\ m_{\mu\mathrm{e}} & m_{\mu\mu} & m_{\mu\tau} \\ m_{\tau\mathrm{e}} & m_{\tau\mu} & m_{\tau\tau}\end{pmatrix}\begin{pmatrix}\mathrm{e}_R \\ \mu_R \\ \tau_R\end{pmatrix}(1+\frac{H}{v}) + h.c,\end{aligned} \tag{2.36}$$

where $m_{ij} \equiv \frac{G_{ij}}{\sqrt{2}}v$, with $i,j = 1,2,3$ corresponds to electron (e), muon (μ), and tau (τ), respectively. We can now rotate the symmetry eigenstates into the mass eigenstates using a bi-unitary transformation [2]:

$$\mathcal{L}^{(L)}_{Yukawa} = (\overline{\mathrm{e}}'_L, \overline{\mu}'_L, \overline{\tau}'_L)\begin{pmatrix} m_{\mathrm{e}}e^{i\phi_{\mathrm{e}}} & 0 & 0 \\ 0 & m_\mu e^{i\phi_\mu} & 0 \\ 0 & 0 & m_\tau e^{i\phi_\tau}\end{pmatrix}\begin{pmatrix}\mathrm{e}'_R \\ \mu'_R \\ \tau'_R\end{pmatrix}(1+\frac{H}{v})+h.c, \tag{2.37}$$

where the prime represents the rotated fields. Absorbing the phases into the right component fields

$$\begin{aligned}\psi'_{iL} &\longrightarrow \psi''_{iL} = \psi'_{iL}, \\ \psi'_{iR} &\longrightarrow \psi''_{iR} = e^{i\phi_{\psi_i}}\psi'_{iR},\end{aligned} \tag{2.38}$$

we find:

$$\mathcal{L}^{(L)}_{Yukawa} = (m_e\overline{\mathrm{e}}''\mathrm{e}'' + m_\mu\overline{\mu}''\mu'' + m_\tau\overline{\tau}''\tau'')(1+\frac{H}{v}). \tag{2.39}$$

We can see that all the charged leptons interact with the Higgs and that all became massive.

The symmetry eigenstates can now be written in terms of the mass eigenstates:

$$\begin{aligned} L_{\mathrm{e}} &= \begin{pmatrix}\nu_{\mathrm{e}} \\ \mathrm{e}\end{pmatrix}_L = \begin{pmatrix}\nu_{\mathrm{e}} \\ \alpha_{11}\mathrm{e}''_L + \alpha_{12}\mu''_L + \alpha_{13}\tau''_L\end{pmatrix}, \\ L_\mu &= \begin{pmatrix}\nu_\mu \\ \mu\end{pmatrix}_L = \begin{pmatrix}\nu_\mu \\ \alpha_{21}\mathrm{e}''_L + \alpha_{22}\mu''_L + \alpha_{23}\tau''_L\end{pmatrix}, \\ L_\tau &= \begin{pmatrix}\nu_\tau \\ \tau\end{pmatrix}_L = \begin{pmatrix}\nu_\tau \\ \alpha_{31}\mathrm{e}''_L + \alpha_{32}\mu''_L + \alpha_{33}\tau''_L\end{pmatrix},\end{aligned} \tag{2.40}$$

where α_{ij} with $i,j = (1,2,3)$ are the coefficients of one of the matrices used in (2.37) to do the transformation. We have a similar relation for the right-handed components, but with different coefficients.

Finally, we can explore the symmetry in the kinetic terms (2.8) to do a rotation, so that all the mixing terms in (2.40) go to the neutrinos, and, since they are massless, we can simply redefine the combination as the actual neutrino.

$$\begin{aligned} L''_e &= \begin{pmatrix} \widetilde{\alpha}_{11}\nu'_e + \widetilde{\alpha}_{12}\nu'_\mu + \widetilde{\alpha}_{13}\nu'_\tau \\ e''_L \end{pmatrix} \equiv \begin{pmatrix} \nu''_e \\ e'' \end{pmatrix}_L, \\ L''_\mu &= \begin{pmatrix} \widetilde{\alpha}_{21}\nu'_e + \widetilde{\alpha}_{22}\nu'_\mu + \widetilde{\alpha}_{23}\nu'_\tau \\ \mu''_L \end{pmatrix} \equiv \begin{pmatrix} \nu''_\mu \\ \mu'' \end{pmatrix}_L, \\ L''_\tau &= \begin{pmatrix} \widetilde{\alpha}_{31}\nu'_e + \widetilde{\alpha}_{32}\nu'_\mu + \widetilde{\alpha}_{33}\nu'_\tau \\ \tau''_L \end{pmatrix} \equiv \begin{pmatrix} \nu''_\tau \\ \tau'' \end{pmatrix}_L, \end{aligned} \tag{2.41}$$

where $\widetilde{\alpha}_{ij}$ with $i, j = (1, 2, 3)$ are the coefficients of the matrix that does the opposite rotation to (2.40), so $\widetilde{\alpha}_{ij} = \alpha^*_{ji}$. This implies that we could have started with a diagonal Yukawa Lagrangian for leptons (2.36) and also that both the neutral current in (2.11) and the charged current in (2.12) remain diagonal.

Quarkyonic part of the Yukawa sector:

If we follow the same procedure as in the leptonic case, we would end up with only half of the quarks being massive, the ones that are on the down component of the doublet. It turns out that we can use a special property of the SU(2) group to give mass to the remaining quarks, the ones that are on the top of the doublets.

So far we have been working with fields in the fundamental representation, but we know that, for the SU(2) group, the fundamental representation is equal to the anti-fundamental representation, so we will use this property to write another gauge invariant term:

$$-\widetilde{G}\overline{\psi}_L\widetilde{\Phi}\psi_R^{(u)}, \tag{2.42}$$

with

$$\widetilde{\Phi} \equiv i\sigma_2\Phi^* = \begin{pmatrix} (\phi^0)^* \\ -\phi^- \end{pmatrix}, \tag{2.43}$$

where $\widetilde{G}$ is some arbitrary constant, ψ_L represents a quark doublet and $\psi_R^{(u)}$ represents u_R, c_R or t_R. From this, we can see that $\widetilde{\Phi}$ have hypercharge $Y^{\widetilde{\Phi}} = -1$ and it transforms as a $SU(2)_L$ doublet. Note that we could not do this with the leptonic part because there are no right-handed neutrinos.

Thus, the Lagrangian is:

$$\mathcal{L}^{(Q)}_{Yukawa} = -\sum_{i,j=1}^{3} G_{ij}\overline{\psi}_{iL}\Phi\psi^{(d)}_{jR} + \widetilde{G_{ij}}\overline{\psi}_{iL}\widetilde{\Phi}\psi^{(u)}_{jR} + h.c\,. \tag{2.44}$$

The only difference with the leptonic part is that instead of one rotation, we have to do two rotations. Following the same procedure we find:

$$\begin{aligned}\mathcal{L}^{(Q)}_{Yukawa} = (&m_u\overline{u}''u'' + m_c\overline{c}''c'' + m_t\overline{t}''t'' \\ &+ m_d\overline{d}''d'' + m_s\overline{s}''s'' + m_b\overline{b}''b'')(1+\frac{H}{v}).\end{aligned} \tag{2.45}$$

Finally, we can write the symmetry eigenstates in terms of the mass eigenstates as in (2.40), and substitute them into the neutral current (2.11), and the charged current (2.12). We find that the neutral current remains diagonal but the charged one mixes flavors. This mixing of quark flavor is the content of the famous CKM matrix [3].

2.2 Standard Model Problems

Although successful, there are several problems that the SM does not explain, so we know that it can't be the answer to a final theory of fundamental interactions. Here we simply quote some of them, as well as some attempts to solve them.

• **Dark Matter:** From cosmological observations, about 27% of the energy in the present universe comes from dark matter, i.e., a different form of matter that only interacts weakly, and with gravity. In this sense, the SM only explains about 5% of the energy in the universe, and on top of that, it does not supply any fundamental particle that is a good candidate for dark matter. There are several candidates for dark matter such as WIMPs (Weakly Interacting Massive Particles) [4], QCD axions [5], and many others [6].

• **Dark Energy:** The remaining 68% of the energy in the universe comes from dark energy, a constant energy density that makes the universe expand. Attempts to explain dark energy in terms of vacuum energy of the SM lead to a mismatch of 120 orders of magnitude [7].

• **Matter-antimatter asymmetry:** The universe is predominantly made out of matter. In 1967 Andrei Sakharov proposed [8] a set of three necessary conditions that a model must satisfy in order to produce matter and antimatter at different rates: violation of baryon number; violation of C, and CP symmetry; and interactions out of thermal equilibrium. It turns out that the SM is in agreement with this condition, but not with a sufficient amount [9]. Beyond SM models such as Grand Unification Theories (GUTs) [10], and theories with supersymmetry (SUSY) can accommodate these conditions with a sufficient amount.

• **Neutrino Mass:** As we saw, the SM neutrinos are massless, however, neutrino oscillations have been measured with great precision [11], which implies that the neutrinos must have mass. We could, in principle, just add a right-handed neutrino, and proceed in the same way as we did for the other fermions, nonetheless, this would cause theoretical, and experimental problems such as the great difference in the order of magnitude of the masses produced by the same mechanism, and the fact that right-handed neutrinos have not yet been observed.

• **Anomalous magnetic moment of the muon:** There is no experimental result that contradicts the SM at the 5σ level, widely considered to be the threshold of a discovery in particle physics, however, in 2021 the Fermilab Muon g-2 experiment [12] give a standard deviation σ of 4.2, which can be seen as evidence of new physics. In the following sections, we will see how to define and calculate (at a one-loop level) the anomalous magnetic moment of the muon, as well as how the experiment is made.

Chapter 3

Magnetic Dipole Moment

3.1 Quantum Mechanics

Classically we can treat an electron as a charged sphere and with that, we find that the expected magnetic dipole moment is equal to g=1 [13]. The proper way to describe an electron is through quantum mechanics, i.e., the Dirac equation:

$$(i\partial\!\!\!/ - m)\Psi = 0, \tag{3.1}$$

where m is the fermion mass and Ψ is a Dirac spinor.

In the presence of an electromagnetic field, the Dirac equation (3.1) becomes:

$$(iD\!\!\!\!/ - m)\Psi = (i\partial\!\!\!/ - eA\!\!\!/ - m)\Psi = 0, \tag{3.2}$$

where $D_\mu = \partial_\mu + ieA_\mu$ is the covariant derivative, and A_μ is the 4-vector potential.

Multiplying both sides of (3.2) by $(i\partial\!\!\!/ - eA\!\!\!/ + m)$, we find:

$$\begin{aligned} 0 &= (i\partial\!\!\!/ - eA\!\!\!/ + m)(i\partial\!\!\!/ - eA\!\!\!/ - m)\Psi \\ &= \left((i\partial_\mu - eA_\mu)(i\partial_\nu - eA_\nu)\gamma^\mu\gamma^\nu - m^2\right)\Psi \\ &= \left(\frac{1}{4}\left\{(i\partial_\mu - eA_\mu),(i\partial_\nu - eA_\nu)\right\}\left\{\gamma^\mu,\gamma^\nu\right\}\right. \\ &\quad \left. +\frac{1}{4}\left[(i\partial_\mu - eA_\mu),(i\partial_\nu - eA_\nu)\right]\left[\gamma^\mu,\gamma^\nu\right] - m^2\right)\Psi. \end{aligned} \tag{3.3}$$

Using the Clifford algebra:

$$\{\gamma^\mu, \gamma^\nu\} = 2\eta^{\mu\nu}, \tag{3.4}$$

we find that

$$\left(D_\mu^2 + \frac{e}{2}F_{\mu\nu}\sigma^{\mu\nu} + m^2\right)\Psi = 0, \tag{3.5}$$

where $\sigma^{\mu\nu} \equiv \frac{i}{2}[\gamma^\mu, \gamma^\nu]$ and $F_{\mu\nu} = \partial_\mu A_\nu - \partial_\nu A_\mu$ is the electromagnetic field strength tensor.

In the Weyl representation, we find

$$\frac{e}{2} F_{\mu\nu}\sigma^{\mu\nu} = -e \begin{pmatrix} (\vec{B} + i\vec{E}) \cdot \vec{\sigma} & \\ & (\vec{B} - i\vec{E}) \cdot \vec{\sigma} \end{pmatrix}, \tag{3.6}$$

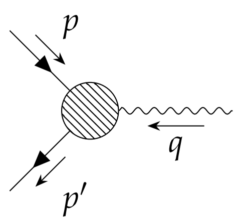
and with $\vec{S} = \frac{\vec{\sigma}}{2}$ we can see, after going to momentum space, that the predicted magnetic dipole moment is actually $g = 2$.

Another common way to show that the Dirac equation (3.1) predicts $g = 2$ is to take the non-relativistic limit of it and identify the magnetic moment interaction term. One can accomplish such a task in an elegant way through the so-called Foldy-Wouthuysen (F-W) transformation [14]. This is also a formal way to derive the Pauli-Schrodinger equation, as well as to find other correction terms such as the relativistic correction, the spin-orbit coupling, and the Darwin term. In summary, the F-W transformation provides a systematic way to collect all terms up to the desired order of expansion.

3.2 Quantum Corrections

We want to use quantum field theory to calculate quantum corrections to the magnetic dipole moment. For a parity invariant theory, such as quantum electrodynamics (QED), at all orders in perturbation theory we have [15]:

$$i\mathcal{M}^\mu =$$



$$= (-ie)\bar{u}(p') \left[F_1(q^2)\gamma^\mu + F_2(q^2) \left(\frac{i\sigma^{\mu\nu} q_\nu}{2m} \right) \right] u(p), \tag{3.7}$$

where $F_i(q^2)$ with $i = 1, 2$ are the form factors, and $q = p' - p$ is the transfer momentum. The form factors F_i are actually a function of q^2 and also of all the parameters of the theory, but for simplicity, we will only write $F_i(q^2)$.

To extract the magnetic dipole moment of (3.7), we will first take the non-relativistic limit of the scattering amplitude:

$$i\mathcal{M}^\mu A_\mu = (-ie)\bar{u}(p')\left[F_1(q^2)\gamma^\mu + F_2(q^2)\left(\frac{i\sigma^{\mu\nu}q_\nu}{2m}\right)\right]u(p)A_\mu. \tag{3.8}$$

Working in the Weyl representation:

$$u(p) = \frac{\not{p}+m}{\sqrt{E+m}}u(0) = \frac{\not{p}+m}{\sqrt{2(E+m)}}\begin{pmatrix}\chi\\ \chi\end{pmatrix} \simeq \sqrt{m}\begin{pmatrix}\left(1-\frac{\vec{p}\cdot\vec{\sigma}}{2m}\right)\chi\\ \left(1+\frac{\vec{p}\cdot\vec{\sigma}}{2m}\right)\chi\end{pmatrix}, \tag{3.9}$$

where $u(0)$ is the solution to the Dirac equation (3.1) in the rest frame, and χ is a Weyl spinor.

The first term of (3.8), considering $A^{cl}_\mu(x) = (0, A^{cl}_i(x)) = (0, -A^i)$, gives:

$$\begin{aligned}\bar{u}(p')\gamma^i u(p) &\simeq m\begin{pmatrix}\left(1-\frac{\vec{p}'\cdot\vec{\sigma}}{2m}\right)\chi'\\ \left(1+\frac{\vec{p}'\cdot\vec{\sigma}}{2m}\right)\chi'\end{pmatrix}^\dagger \gamma^0\gamma^i \begin{pmatrix}\left(1-\frac{\vec{p}\cdot\vec{\sigma}}{2m}\right)\chi\\ \left(1+\frac{\vec{p}\cdot\vec{\sigma}}{2m}\right)\chi\end{pmatrix}\\ &\simeq m\,\chi'^\dagger\left\{-\left(1-\frac{\vec{p}'\cdot\vec{\sigma}}{2m}\right)\sigma^i\left(1-\frac{\vec{p}\cdot\vec{\sigma}}{2m}\right)\right.\\ &\qquad\left.+\left(1+\frac{\vec{p}'\cdot\vec{\sigma}}{2m}\right)\sigma^i\left(1+\frac{\vec{p}\cdot\vec{\sigma}}{2m}\right)\right\}\chi\\ &\simeq \chi'^\dagger\left\{(p'+p)^i - i\epsilon^{ijk}q^j\sigma^k\right\}\chi,\end{aligned} \tag{3.10}$$

where we have used $\sigma^i\sigma^j = \delta^{ij} + i\epsilon^{ijk}\sigma^k$.

For the second term in (3.8), collecting only linear terms in q^j, we have:

$$\begin{aligned}\bar{u}(p')\left(\frac{i\sigma^{i\nu}q_\nu}{2m}\right)u(p) &\simeq \begin{pmatrix}\left(1-\frac{\vec{p}'\cdot\vec{\sigma}}{2m}\right)\chi'\\ \left(1+\frac{\vec{p}'\cdot\vec{\sigma}}{2m}\right)\chi'\end{pmatrix}^\dagger \gamma^0\left[\sigma^i,\sigma^j\right]\left(\frac{q_j}{4}\right)\begin{pmatrix}\left(1-\frac{\vec{p}\cdot\vec{\sigma}}{2m}\right)\chi\\ \left(1+\frac{\vec{p}\cdot\vec{\sigma}}{2m}\right)\chi\end{pmatrix}\\ &\simeq \chi'^\dagger\left(\frac{i\epsilon^{ijk}q_j}{2}\right)\left\{+\left(1-\frac{\vec{p}'\cdot\vec{\sigma}}{2m}\right)\sigma^k\left(1+\frac{\vec{p}\cdot\vec{\sigma}}{2m}\right)\right.\\ &\qquad\left.+\left(1+\frac{\vec{p}'\cdot\vec{\sigma}}{2m}\right)\sigma^k\left(1-\frac{\vec{p}\cdot\vec{\sigma}}{2m}\right)\right\}\chi\\ &\simeq \chi'^\dagger\left\{-i\epsilon^{ijk}q^j\sigma^k\right\}\chi.\end{aligned} \tag{3.11}$$

Combining (3.10) and (3.11) into (3.8), and only looking at terms linear in q^i (the term proportional to $(p'+p)^i$ is related to non-relativistic kinetic energy [16]),

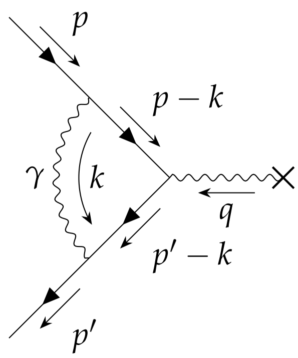


Figure 3.1: Diagram of the photon contribution at a one-loop level. All fermionic lines correspond to the muon and all momenta are indicated in the figure.

we find:

$$i\mathcal{M}^{\mu}A_{\mu} \simeq i(-2m)\left\{-2\left[F_1(0)+F_2(0)\right]\left(\frac{e}{2m}\right)\langle\vec{S}\rangle\cdot\vec{B}\right\}, \tag{3.12}$$

where we have used $B^i = i\epsilon^{ijk}q^jA^k$ and $\langle S^k\rangle = \chi'^{\dagger}\left(\frac{\sigma^k}{2}\right)\chi$. We also considered $q^2 = 0$ for the form factors, i.e., the static limit, because the actual magnetic moment is measured at non-relativistic energies ($|\vec{q}| \ll m$).

We can interpret the last result (3.12) in the context of non-relativistic quantum mechanics as the first Born approximation to the scattering of an electron from a potential well. The amplitude is given by the Fourier transform of the potential with respect to the transfer momentum [17], and the potential is simply that of a magnetic moment interaction,

$$V(q) = -\langle\vec{\mu}\rangle\cdot\vec{B}(q) = -g\left(\frac{e}{2m}\right)\langle\vec{S}\rangle\cdot\vec{B}, \tag{3.13}$$

where the coefficient g is called the Landé g-factor. Thus, we find:

$$g = 2\left[F_1(0)+F_2(0)\right], \tag{3.14}$$

and since $F_1(0) = 1$ to all orders in perturbation theory, due to charge renormalization [15], all quantum corrections must come from $F_2(0)$:

$$F_2(0) = \frac{g-2}{2} \equiv a, \tag{3.15}$$

where a is known as the anomalous magnetic moment.

3.2.1 QED Contribution

The first correction comes from quantum electrodynamics (QED). Using the Feynman rules [18], and working in the Feynman-'t Hooft gauge ($\xi = 1$), the diagram of Figure 3.1 reads:

$$\begin{aligned} i\mathcal{M}^{\mu}_{(1)} &= \int \frac{\mathrm{d}^4k}{(2\pi)^4}\bar{u}(p')(-ie\gamma^{\alpha})\left[\frac{i(\not{p}' - \not{k} + m)}{(p'-k)^2 - m^2 + i\epsilon}\right](-ie\gamma^{\mu}) \\ &\quad\times \left[\frac{i(\not{p} - \not{k} + m)}{(p-k)^2 - m^2 + i\epsilon}\right]\left[\frac{-i\eta_{\alpha\beta}}{k^2 + i\epsilon}\right](-ie\gamma^{\beta})u(p) \\ &= (-ie)(-ie^2)\int \frac{\mathrm{d}^4k}{(2\pi)^4}\bar{u}(p') \\ &\quad\times \frac{\gamma^{\alpha}(\not{p}' - \not{k} + m)\gamma^{\mu}(\not{p} - \not{k} + m)\gamma_{\alpha}}{[k^2 + i\epsilon][(p-k)^2 - m^2 + i\epsilon][(p'-k)^2 - m^2 + i\epsilon]}u(p). \end{aligned} \tag{3.16}$$

We can use the Feynman parametrization:

$$\frac{1}{A_1 \cdots A_n} = \int_0^1 \mathrm{d}x_1 \cdots \mathrm{d}x_n \frac{(n-1)!\,\delta\left(x_1 + \cdots + x_n - 1\right)}{\left[x_1 A_1 + \cdots + x_n A_n\right]^n}, \tag{3.17}$$

to write the denominator as:

$$\frac{[(p'-k)^2 - m^2 + i\epsilon]^{-1}}{[k^2 + i\epsilon][(p-k)^2 - m^2 + i\epsilon]} = \int_0^1 \mathrm{d}x\mathrm{d}y\mathrm{d}z \frac{2!\,\delta(x+y+z-1)}{D^3}, \tag{3.18}$$

where D is:

$$\begin{aligned} D &= z[k^2 + i\epsilon] + x[(p-k)^2 - m^2 + i\epsilon] + y[(p'-k)^2 - m^2 + i\epsilon] \\ &= k^2(x+y+z) - 2x(k\cdot p) - 2y(k\cdot p') + (x+y+z)i\epsilon \\ &= (k - xp - yp')^2 - (xp + yp')^2 + i\epsilon, \end{aligned} \tag{3.19}$$

and we use $p^2 = p'^2 = m^2$. Note that because of the Dirac delta in (3.18), we will freely use the relation $x + y + z = 1$ throughout the manipulations.

We can now do a shift in the variable of integration $l \equiv k - xp - yp'$, and use momentum conservation to write:

$$\begin{aligned} D &= l^2 + xyq^2 - (1-z)^2 m^2 + i\epsilon \\ &= l^2 - \Delta + i\epsilon, \end{aligned} \tag{3.20}$$

with $\Delta \equiv -xyq^2 + (1-z)^2 m^2$.

Doing the shift in the numerator, we find:

$$\begin{aligned} N^\mu &= \gamma^\alpha[\not{p}' - \not{k} + m]\gamma^\mu[\not{p} - \not{k} + m]\gamma_\alpha \\ &= \gamma^\alpha[-\not{l} - x\not{p} + (1-y)\not{p}' + m]\gamma^\mu[-\not{l} + (1-x)\not{p} - y\not{p}' + m]\gamma_\alpha \\ &= \gamma^\alpha \not{l}\gamma^\mu \not{l}\gamma_\alpha - \gamma^\alpha \not{l}\gamma^\mu[(1-x)\not{p} - y\not{p}' + m]\gamma_\alpha \\ &\quad - \gamma^\alpha[-x\not{p} + (1-y)\not{p}' + m]\gamma^\mu \not{l}\gamma_\alpha \\ &\quad + \gamma^\alpha[-x\not{p} + (1-y)\not{p}' + m]\gamma^\mu[(1-x)\not{p} - y\not{p}' + m]\gamma_\alpha. \end{aligned} \tag{3.21}$$

We can use the following identities:

$$\begin{aligned} \int \frac{d^4 l}{(2\pi)^4}\frac{l^\mu}{D^3} &= 0, \\ \int \frac{d^4 l}{(2\pi)^4}\frac{l^\mu l^\nu}{D^3} &= \int \frac{d^4 l}{(2\pi)^4}\frac{\frac{l^2}{4}\eta^{\mu\nu}}{D^3}, \end{aligned} \tag{3.22}$$

to safely ignore all odd terms in l (since the denominator is an even function).

The numerator becomes:

$$\begin{aligned} N^\mu &= \gamma^\alpha[-x\not{p} + (1-y)\not{p}' + m]\gamma^\mu[(1-x)\not{p} - y\not{p}' + m]\gamma_\alpha \\ &= 2mz(1-z)(p'+p)^\mu + 2m(2-z)(x-y)(p'-p)^\mu, \end{aligned} \tag{3.23}$$

where we have used (3.1), (3.4), and have also ignored all the terms that are proportional to γ^μ, since they will only contribute to $F_1(q^2)$, to obtain a simplified expression. We can also see that the last term in the numerator is not symmetrical under the exchange $x \leftrightarrow y$, but the rest of the integrand and the integral measure are, this means that the last term when integrated, yields zero.

Using the Gordon identity:

$$\bar{u}(p')(p'+p)^\mu u(p) = \bar{u}(p')[2m\gamma^\mu - i\sigma^{\mu\nu}q_\nu]u(p), \tag{3.24}$$

and ignoring the part proportional to γ^μ, we can write the numerator as:

$$N^\mu = -4m^2 z(1-z)\left(\frac{i\sigma^{\mu\nu}q_\nu}{2m}\right), \tag{3.25}$$

and using the following integral:

$$\int \frac{d^4 l}{(2\pi)^4}\frac{1}{(l^2-\Delta)^n} = \frac{i(-1)^n}{(4\pi)^2}\frac{1}{(n-1)(n-2)}\frac{1}{\Delta^{n-2}}, \tag{3.26}$$

with $n = 3$, we finally obtain:

$$\begin{aligned} i\mathcal{M}^{\mu}_{(1)} &= (-ie)(-ie^2)\,\bar{u}(p')\int_0^1 \mathrm{d}x\mathrm{d}y\mathrm{d}z\, 2\,\delta(x+y+z-1) \\ &\quad\times \int \frac{\mathrm{d}^4 l}{(2\pi)^4}\frac{1}{D^3}(-4m^2)z(1-z)\left(\frac{i\sigma^{\mu\nu}q_\nu}{2m}\right)u(p) \\ &= (-ie)\,\bar{u}(p')\int_0^1 \mathrm{d}z\int_0^{1-z}\mathrm{d}y(8m^2)(ie^2)\left(\frac{-i}{32\pi^2\Delta}\right) \\ &\quad\times z(1-z)\left(\frac{i\sigma^{\mu\nu}q_\nu}{2m}\right)u(p). \end{aligned} \tag{3.27}$$

Comparing (3.27) with (3.7), we find that:

$$F_2(q^2) = \frac{\alpha}{\pi}\int_0^1 \mathrm{d}z\int_0^{1-z}\mathrm{d}y\frac{z(1-z)m^2}{(1-z)^2m^2+(y+z-1)yq^2}, \tag{3.28}$$

thus, at $q^2 = 0$:

$$\begin{aligned} F_2(0) &= \frac{\alpha}{\pi}\int_0^1 \mathrm{d}z\int_0^{1-z}\mathrm{d}y\,\frac{z}{(1-z)} \\ &= \frac{\alpha}{2\pi} \simeq 0.0011614, \end{aligned} \tag{3.29}$$

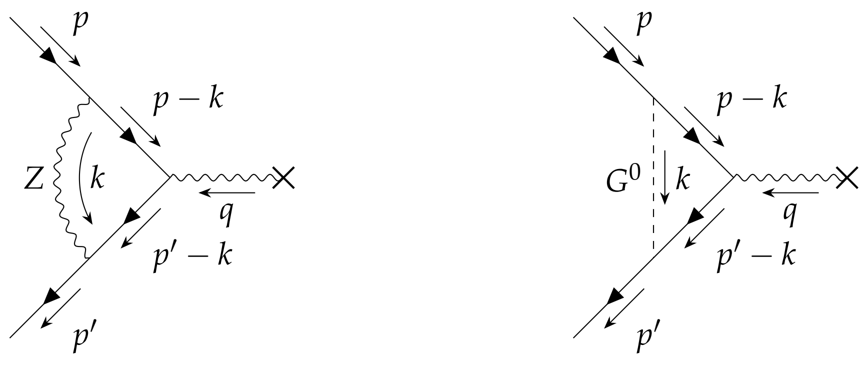
where $\alpha \equiv e^2/4\pi$ is the fine-structure constant. This calculation was first done, although not in this way, by Julian Schwinger in 1948 [19].

3.2.2 Electroweak Contribution

Similar to (3.7), we can write that matrix element for a theory that is not invariant under parity at all orders in perturbation theory as [20]:

$$\begin{aligned} i\mathcal{M}^{\mu} = (-ie)\bar{u}(p')\Big[F_1(q^2)\gamma^\mu + F_2(q^2)\left(\frac{i\sigma^{\mu\nu}q_\nu}{2m}\right) - F_3(q^2)\sigma^{\mu\nu}q_\nu\gamma^5 \\ + F_4(q^2)\left(\gamma^\mu q^2 - 2mq^\mu\right)\gamma^5\Big]\,u(p), \end{aligned} \tag{3.30}$$

where $F_3(q^2)$ and $F_4(q^2)$ are also form factors that, when evaluated at $q^2 = 0$, give the electric dipole moment and the anapole moment, respectively.



(a) (b)

Figure 3.2: Diagrams of the Z-boson contribution at a one-loop level. Since we are in the Feynman-'t Hooft gauge ($\xi = 1$), we have to consider the corresponding Goldstone-boson diagram as well. All fermionic lines correspond to the muon and all momenta are indicated in the figure.

Z-boson Contribution:

From Figure 3.2, the diagram (a) reads:

$$\begin{aligned} i\mathcal{M}^{\mu}_{(2.a)} &= \int \frac{d^4k}{(2\pi)^4} \bar{u}(p') \frac{(-ie\gamma^{\nu})}{2\sin(2\theta_W)} ([1-2\cos(2\theta_W)]+\gamma^5) \left[\frac{-i\eta_{\nu\alpha}}{k^2-M_z^2+i\epsilon}\right] \\ &\quad \times \left[\frac{i(\not{p}'-\not{k}+m)}{(p'-k)^2-m^2+i\epsilon}\right] (-ie\gamma^{\mu}) \left[\frac{i(\not{p}-\not{k}+m)}{(p-k)^2-m^2+i\epsilon}\right] \\ &\quad \times \frac{(-ie\gamma^{\alpha})}{2\sin(2\theta_W)} ([1-2\cos(2\theta_W)]+\gamma^5) u(p) \\ &= (-ie)\frac{(-ie^2)}{[2\sin(2\theta_W)]^2} \int \frac{d^4k}{(2\pi)^4} \bar{u}(p')\gamma^{\alpha}([1-2\cos(2\theta_W)]+\gamma^5) \\ &\quad \times \frac{(\not{p}'-\not{k}+m)\gamma^{\mu}(\not{p}-\not{k}+m)\gamma_{\alpha}([1-2\cos(2\theta_W)]+\gamma^5)}{[k^2-M_z^2+i\epsilon][(p-k)^2-m^2+i\epsilon][(p'-k)^2-m^2+i\epsilon]} u(p). \end{aligned} \tag{3.31}$$

Using (3.17), we can write the denominator as:

$$\begin{aligned} D &= z[k^2-M_z^2+i\epsilon] + x[(p-k)^2-m^2+i\epsilon] + y[(p'-k)^2-m^2+i\epsilon] \\ &= l^2 + xyq^2 - (1-z)^2m^2 - zM_z^2 + i\epsilon \\ &= l^2 - \Delta + i\epsilon, \end{aligned} \tag{3.32}$$

with $l \equiv k - xp - yp'$, and $\Delta \equiv -xyq^2 + (1-z)^2m^2 + zM_z^2$.

Now, for the numerator, with $V \equiv 1 - 2\cos(2\theta_W) = 3 - 4\cos^2(\theta_W)$, we have:

$$\begin{aligned} N^\mu &= \gamma^\alpha(V+\gamma^5)(\not p' - \not k + m)\gamma^\mu(\not p - \not k + m)\gamma_\alpha(V+\gamma^5) \\ &= V^2\,\gamma^\alpha(\not p' - \not k + m)\gamma^\mu(\not p - \not k + m)\gamma_\alpha \\ &\quad + \gamma^\alpha(\not p' - \not k - m)\gamma^\mu(\not p - \not k - m)\gamma_\alpha, \end{aligned} \tag{3.33}$$

where we have ignored the terms proportional to γ^5 since they will not contribute to F_2. Using (3.4) and (3.1) we can write the numerator as:

$$\begin{aligned} N^\mu &= 2m[V^2 z(1-z) - z(3+z)](p'+p)^\mu \\ &\quad + 2m[V^2(2-z) - (2+z)](x-y)(p'-p)^\mu. \end{aligned} \tag{3.34}$$

Again, we can see that the last term yields zero when integrated, and using the Gordon identity (3.24), the numerator becomes:

$$N^\mu = -4m^2[V^2 z(1-z) - z(3+z)]\left(\frac{i\sigma^{\mu\nu}q_\nu}{2m}\right). \tag{3.35}$$

With this, we have:

$$\begin{aligned} i\mathcal{M}^\mu_{(2.a)} &= (-ie)\,\bar u(p')\,\frac{G_F m^2 M_Z^2}{8\sqrt{2}\pi^2}\int_0^1 \mathrm{d}x\mathrm{d}y\mathrm{d}z\,\delta(x+y+z-1) \\ &\quad\times \frac{[V^2 z(1-z) - z(3+z)]}{-xyq^2 + (1-z)^2 m^2 + zM_z^2}\left(\frac{i\sigma^{\mu\nu}q_\nu}{2m}\right)u(p), \end{aligned} \tag{3.36}$$

where we have used (3.26) to do the l integration.

For the diagram (b) of Figure 3.2, we have:

$$\begin{aligned} i\mathcal{M}^\mu_{(2.b)} &= \int\frac{\mathrm{d}^4k}{(2\pi)^4}\bar u(p')\left(-gI_3\frac{m}{M_W}\gamma^5\right)\left[\frac{i}{k^2 - M_z^2 + i\epsilon}\right] \\ &\quad\times\left[\frac{i(\not p' - \not k + m)}{(p'-k)^2 - m^2 + i\epsilon}\right](-ie\gamma^\mu)\left[\frac{i(\not p - \not k + m)}{(p-k)^2 - m^2 + i\epsilon}\right] \\ &\quad\times\left(-gI_3\frac{m}{M_W}\gamma^5\right)u(p) \\ &= (-ie)\frac{(-2i)G_F m^2}{\sqrt{2}}\int\frac{\mathrm{d}^4k}{(2\pi)^4}\bar u(p') \\ &\quad\times\frac{\gamma^5(\not p' - \not k + m)\gamma^\mu(\not p - \not k + m)\gamma^5}{[k^2 - M_z^2 + i\epsilon][(p-k)^2 - m^2 + i\epsilon][(p'-k)^2 - m^2 + i\epsilon]}u(p), \end{aligned} \tag{3.37}$$

where g is the coupling constant for the $SU(2)_L$ group and $I_3 = 1/2$ is the third component of weak isospin.

The denominator is the same as in (3.32), so we will only look at the numerator:

$$\begin{aligned} N^\mu &= \gamma^5(\not{p}' - \not{k} + m)\gamma^\mu(\not{p} - \not{k} + m)\gamma^5 \\ &= -\not{k}\gamma^\mu\not{k} \\ &= -m(1-z)^2(p'+p)^\mu + m(1-z)(x-y)(p'-p)^\mu, \end{aligned} \tag{3.38}$$

where we have used (3.1) and (3.4) to simplify. With the Gordon identity (3.24), we write the numerator as:

$$N^\mu = (2m^2)(1-z)^2\left(\frac{i\sigma^{\mu\nu}q_\nu}{2m}\right), \tag{3.39}$$

and using (3.26) to do the l integration we arrived at:

$$\begin{aligned} i\mathcal{M}^\mu_{(2.b)} = (-ie)\,\bar{u}(p')\,\frac{G_F m^2 M_Z^2}{8\sqrt{2}\pi^2}\int_0^1 \mathrm{d}x\mathrm{d}y\mathrm{d}z\,\delta(x+y+z-1) \\ \times\left\{\frac{-2\left(\frac{m^2}{M_Z^2}\right)(1-z)^2}{-xyq^2+(1-z)^2m^2+zM_Z^2}\right\}\left(\frac{i\sigma^{\mu\nu}q_\nu}{2m}\right)u(p). \end{aligned} \tag{3.40}$$

Combining (3.36) with (3.40) and comparing to (3.30), we find that:

$$\begin{aligned} \left[F_2(q^2)\right]_Z = \frac{G_F m^2 M_Z^2}{8\sqrt{2}\pi^2}\int_0^1 \mathrm{d}x\mathrm{d}y\mathrm{d}z\,\delta(x+y+z-1) \\ \times\left\{\frac{V^2 z(1-z) - z(3+z) - 2\left(\frac{m^2}{M_Z^2}\right)(1-z)^2}{-xyq^2+(1-z)^2m^2+zM_Z^2}\right\}, \end{aligned} \tag{3.41}$$

so the Z-boson contribution at a one-loop level is given by:

$$\begin{aligned} [F_2(0)]_Z &= \frac{G_F m^2}{8\sqrt{2}\pi^2}\int_0^1 \mathrm{d}z\left\{\frac{V^2 z(1-z)^2 - z(3+z)(1-z) - 2\left(\frac{m^2}{M_Z^2}\right)(1-z)^3}{z+(1-z)^2\left(\frac{m^2}{M_Z^2}\right)}\right\} \\ &\simeq \frac{G_F m^2}{8\sqrt{2}\pi^2}\left\{\frac{1}{3}(3-4\cos^2(\theta_W))^2 - \frac{5}{3} + \mathcal{O}\left(\frac{m^2}{M_Z^2}\right) + \mathcal{O}\left(\frac{m^2}{M_Z^2}\log\left(\frac{m}{M_Z}\right)\right)\right\}, \end{aligned} \tag{3.42}$$

which agrees with Jackiw and Weinberg [21].

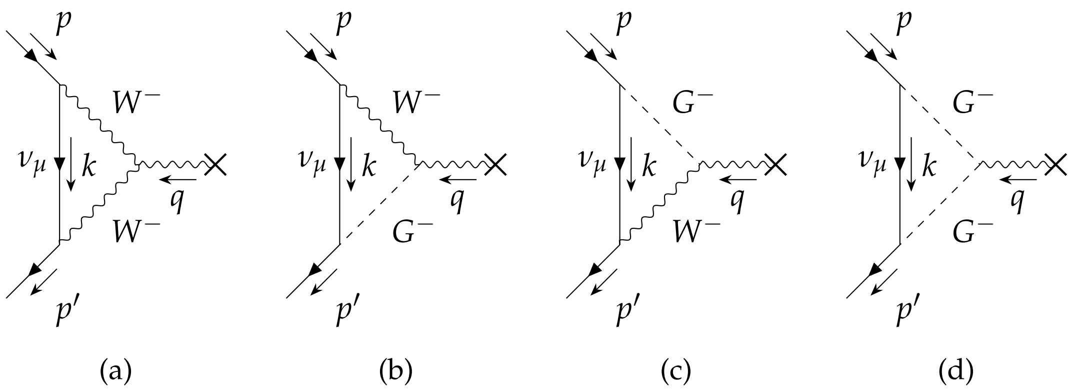


Figure 3.3: Diagrams of the W-boson contribution at a one-loop level. Since we are in the Feynman-'t Hooft gauge ($\xi = 1$), we have to consider the corresponding Goldstone-boson diagrams as well.

W-boson Contribution:

Using Feynman rules, the diagram (a) from Figure 3.3 reads:

$$\begin{aligned} i\mathcal{M}^{\mu}_{(3.a)} &= \int \frac{d^4k}{(2\pi)^4} \bar{u}(p') \left[\left(\frac{-ig}{2\sqrt{2}} \right) \gamma^{\alpha}(\mathbb{1} - \gamma^5) \right] \left[\frac{-i\eta_{\alpha\nu}}{(p'-k)^2 - M_W^2 + i\epsilon} \right] \\ &\quad \times (ie) \left[\eta^{\nu\lambda}(p+p'-2k)^{\mu} + \eta^{\lambda\mu}(k+p'-2p)^{\nu} + \eta^{\mu\nu}(k+p-2p')^{\lambda} \right] \\ &\quad \times \left[\frac{-i\eta_{\lambda\beta}}{(p-k)^2 - M_W^2 + i\epsilon} \right] \left[\frac{i\not{k}}{k^2 + i\epsilon} \right] \left[\left(\frac{-ig}{2\sqrt{2}} \right) \gamma^{\beta}(\mathbb{1} - \gamma^5) \right] u(p) \\ &= (-ie) \left(\frac{-ig^2}{8} \right) \int \frac{d^4k}{(2\pi)^4} \bar{u}(p')(\mathbb{1} + \gamma^5)\gamma_{\nu}\not{k}\gamma_{\lambda}(\mathbb{1} - \gamma^5) \\ &\quad \times \frac{\left[\eta^{\nu\lambda}(p+p'-2k)^{\mu} + \eta^{\lambda\mu}(k+p'-2p)^{\nu} + \eta^{\mu\nu}(k+p-2p')^{\lambda} \right]}{[k^2 + i\epsilon][(p-k)^2 - M_W^2 + i\epsilon][(p'-k)^2 - M_W^2 + i\epsilon]} u(p), \end{aligned} \tag{3.43}$$

and with (3.17), the denominator can be written as:

$$\begin{aligned} D &= z[k^2 + i\epsilon] + x[(p-k)^2 - M_W^2 + i\epsilon] + y[(p'-k)^2 - M_W^2 + i\epsilon] \\ &= l^2 + xyq^2 - (1-z)(M_W^2 - zm^2) + i\epsilon \\ &= l^2 - \Delta + i\epsilon, \end{aligned} \tag{3.44}$$

where $l \equiv k - xp - yp'$, and $\Delta \equiv -xyq^2 + (1-z)(M_W^2 - zm^2)$.

As for the numerator, we have:

$$\begin{aligned} N^\mu &= (\mathbb{1}+\gamma^5)\gamma_\nu \not{k} \gamma_\lambda(\mathbb{1}-\gamma^5) \\ &\quad \times [\eta^{\nu\lambda}(p+p'-2k)^\mu + \eta^{\lambda\mu}(k+p'-2p)^\nu + \eta^{\mu\nu}(k+p-2p')^\lambda] \\ &= 2[\gamma^\lambda \not{k} \gamma_\lambda (p+p'-2k)^\mu + (\not{k}+\not{p}'-2\not{p})\not{k}\gamma^\mu + \gamma^\mu \not{k}(\not{k}+\not{p}-2\not{p}')], \end{aligned} \tag{3.45}$$

where we have ignored terms proportional to γ^5 and $\gamma^\mu\gamma^5$, since they will not contribute to $F_2(q^2)$.

Now, using (3.1), (3.4), and a fair amount of algebra, the numerator becomes:

$$N^\mu = 2m(1-z)(3-2z)(p'+p)^\mu + 2m(2z-1)(x-y)(p'-p)^\mu, \tag{3.46}$$

that with (3.24), and again ignoring the part proportional to γ^μ, reads:

$$N^\mu = -4m^2(1-z)(3-2z)\left(\frac{i\sigma^{\mu\nu}q_\nu}{2m}\right). \tag{3.47}$$

Using (3.26) to do the l integration, we find:

$$\begin{aligned} i\mathcal{M}^\mu_{(3.a)} &= (-ie)\,\bar{u}(p')\,\frac{G_F m^2}{8\sqrt{2}\pi^2}\int_0^1 \mathrm{d}x\mathrm{d}y\mathrm{d}z\,\delta(x+y+z-1) \\ &\quad \times \left\{\frac{2M_W^2(1-z)(3-2z)}{-xyq^2+(1-z)(M_W^2-zm^2)}\right\}\left(\frac{i\sigma^{\mu\nu}q_\nu}{2m}\right)u(p). \end{aligned} \tag{3.48}$$

For the diagram (b) of Figure 3.3 we have:

$$\begin{aligned} i\mathcal{M}^\mu_{(3.b)} &= \int \frac{\mathrm{d}^4k}{(2\pi)^4}\bar{u}(p')\left[\left(\frac{-ig}{2\sqrt{2}}\right)\frac{m}{M_W}(\mathbb{1}-\gamma^5)\right]\left[\frac{i}{(p'-k)^2-M_W^2+i\epsilon}\right] \\ &\quad \times \left(-ieM_W\eta^{\mu\lambda}\right)\left[\frac{-i\eta_{\lambda\beta}}{(p-k)^2-M_W^2+i\epsilon}\right] \\ &\quad \times \left[\frac{i\not{k}}{k^2+i\epsilon}\right]\left[\left(\frac{-ig}{2\sqrt{2}}\right)\gamma^\beta(\mathbb{1}-\gamma^5)\right]u(p) \\ &= (-ie)\left(\frac{-ig^2}{8}\right)\int\frac{\mathrm{d}^4k}{(2\pi)^4}\bar{u}(p') \\ &\quad \times \frac{m(\mathbb{1}-\gamma^5)\not{k}\gamma^\mu(\mathbb{1}-\gamma^5)}{[k^2+i\epsilon][(p-k)^2-M_W^2+i\epsilon][(p'-k)^2-M_W^2+i\epsilon]}u(p), \end{aligned} \tag{3.49}$$

and for the diagram (c) of Figure 3.3:

$$\begin{aligned} i\mathcal{M}^{\mu}_{(3.c)} &= \int \frac{\mathrm{d}^4k}{(2\pi)^4}\bar{u}(p')\left[\left(\frac{-ig}{2\sqrt{2}}\right)\gamma^{\beta}(\mathbb{1}-\gamma^5)\right]\left[\frac{-i\eta_{\lambda\beta}}{(p'-k)^2-M_W^2+i\epsilon}\right] \\ &\times\left(-ieM_W\eta^{\mu\lambda}\right)\left[\frac{i}{(p-k)^2-M_W^2+i\epsilon}\right] \\ &\times\left[\frac{i\not{k}}{k^2+i\epsilon}\right]\left[\left(\frac{-ig}{2\sqrt{2}}\right)\frac{m}{M_W}(\mathbb{1}+\gamma^5)\right]u(p) \\ &= (-ie)\left(\frac{-ig^2}{8}\right)\int\frac{\mathrm{d}^4k}{(2\pi)^4}\bar{u}(p') \\ &\times\frac{m(\mathbb{1}+\gamma^5)\gamma^{\mu}\not{k}(\mathbb{1}+\gamma^5)}{[k^2+i\epsilon][(p-k)^2-M_W^2+i\epsilon][(p'-k)^2-M_W^2+i\epsilon]}u(p). \end{aligned} \tag{3.50}$$

Adding up the last two equations and ignoring terms that do not contribute to $F_2(q^2)$, we find:

$$\begin{aligned} i\left(\mathcal{M}^{\mu}_{(3.b)}+\mathcal{M}^{\mu}_{(3.c)}\right) &= (-ie)\left(\frac{-ig^2}{8}\right)\int\frac{\mathrm{d}^4k}{(2\pi)^4}\bar{u}(p') \\ &\times\frac{2m(\gamma^{\mu}\not{k}+\not{k}\gamma^{\mu})}{[k^2+i\epsilon][(p-k)^2-M_W^2+i\epsilon][(p'-k)^2-M_W^2+i\epsilon]}u(p). \end{aligned} \tag{3.51}$$

The denominator is the same as (3.44), and the numerator, using (3.4), is:

$$\begin{aligned} N^{\mu} &= 4mk^{\mu} = 4m(l+px+yp')^{\mu} \\ &= 2m(1-z)(p'+p)^{\mu} + 2m(x-y)(p'-p)^{\mu}. \end{aligned} \tag{3.52}$$

Again, with (3.24), the numerator becomes:

$$N^{\mu} = -4m^2(1-z)\left(\frac{i\sigma^{\mu\nu}q_{\nu}}{2m}\right), \tag{3.53}$$

and after using (3.26) to do the l integration, we have:

$$\begin{aligned} i\left(\mathcal{M}^{\mu}_{(3.b)}+\mathcal{M}^{\mu}_{(3.c)}\right) &= (-ie)\,\bar{u}(p')\,\frac{G_F m^2}{8\sqrt{2}\pi^2}\int_0^1 \mathrm{d}x\mathrm{d}y\mathrm{d}z\,\delta(x+y+z-1) \\ &\times\left\{\frac{2M_W^2(1-z)}{-xyq^2+(1-z)(M_W^2-zm^2)}\right\}\left(\frac{i\sigma^{\mu\nu}q_{\nu}}{2m}\right)u(p). \end{aligned} \tag{3.54}$$

For the diagram (d) of Figure 3.3:

$$\begin{aligned} i\mathcal{M}^{\mu}_{(3.d)} &= \int \frac{d^4k}{(2\pi)^4}\bar{u}(p')\left[\left(\frac{-ig}{2\sqrt{2}}\right)\frac{m}{M_W}(\mathbb{1}-\gamma^5)\right]\left[\frac{i}{(p'-k)^2-M_W^2+i\epsilon}\right] \\ &\quad\times\left[(-ie)(p+p'-2k)^{\mu}\right]\left[\frac{i}{(p-k)^2-M_W^2+i\epsilon}\right] \\ &\quad\times\left[\frac{i\not{k}}{k^2+i\epsilon}\right]\left[\left(\frac{-ig}{2\sqrt{2}}\right)\frac{m}{M_W}(\mathbb{1}+\gamma^5)\right]u(p) \\ &= (-ie)\left(\frac{ig^2m^2}{8M_W^2}\right)\int\frac{d^4k}{(2\pi)^4}\bar{u}(p') \\ &\quad\times\frac{2(\mathbb{1}-\gamma^5)\not{k}(p+p'-2k)^{\mu}}{[k^2+i\epsilon][(p-k)^2-M_W^2+i\epsilon][(p'-k)^2-M_W^2+i\epsilon]}u(p), \end{aligned} \tag{3.55}$$

the denominator is also the same as (3.44) and the numerator is:

$$\begin{aligned} N^{\mu} &= 2(\mathbb{1}-\gamma^5)\not{k}(p+p'-2k)^{\mu} \\ &= 2mz(1-z)(p'+p)^{\mu}+2mz(x-y)(p'-p^{\mu}). \end{aligned} \tag{3.56}$$

Using (3.24) and ignoring the terms proportional to γ^{μ}, the numerator becomes:

$$N^{\mu} = -4m^2z(1-z)\left(\frac{i\sigma^{\mu\nu}q_{\nu}}{2m}\right), \tag{3.57}$$

and after the l integration, we find:

$$\begin{aligned} i\mathcal{M}^{\mu}_{(3.d)} &= (-ie)\,\bar{u}(p')\,\frac{G_F m^2}{8\sqrt{2}\pi^2}\int_0^1 dxdydz\,\delta(x+y+z-1) \\ &\quad\times\left\{\frac{-2m^2z(1-z)}{-xyq^2+(1-z)(M_W^2-zm^2)}\right\}\left(\frac{i\sigma^{\mu\nu}q_{\nu}}{2m}\right)u(p). \end{aligned} \tag{3.58}$$

Combining (3.48), (3.54) and (3.58), and comparing to (3.30), we find that:

$$\begin{aligned} \left[F_2(q^2)\right]_W &= \frac{G_F m^2}{8\sqrt{2}\pi^2}\int_0^1 dxdydz\,\delta(x+y+z-1) \\ &\quad\times\left\{\frac{4M_W^2(2-z)(1-z)-2m^2z(1-z)}{-xyq^2+(1-z)(M_W^2-zm^2)}\right\}, \end{aligned} \tag{3.59}$$

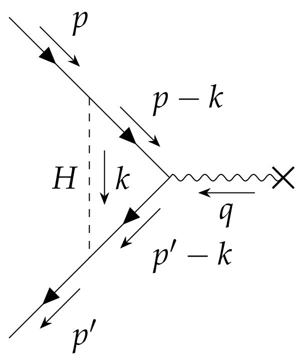


Figure 3.4: Diagram of the Higgs-boson contribution at a one-loop level. All momenta are indicated in the figure.

so the W-boson contribution, at a one-loop level, is given by:

$$\begin{aligned} [F_2(0)]_W &= \frac{G_F m^2}{8\sqrt{2}\pi^2} \int_0^1 \mathrm{d}z \left\{ \frac{4(2-z)(1-z) - 2\left(\frac{m^2}{M_W^2}\right) z(1-z)}{1 - z\left(\frac{m^2}{M_W^2}\right)} \right\} \\ &\simeq \frac{G_F m^2}{8\sqrt{2}\pi^2} \left\{ \frac{10}{3} + \mathcal{O}\left(\frac{m^2}{M_W^2}\right) \right\}. \end{aligned} \tag{3.60}$$

Higgs-boson Contribution:

Using the Feynman rules, the diagram from Figure 3.4 reads:

$$\begin{aligned} i\mathcal{M}^\mu_{(4)} &= \int \frac{\mathrm{d}^4k}{(2\pi)^4} \bar{u}(p') \left(-i\frac{g}{2}\frac{m}{M_W}\right) \left[\frac{i}{k^2 - M_H^2 + i\epsilon}\right] \left[\frac{i(\not{p}' - \not{k} + m)}{(p'-k)^2 - m^2 + i\epsilon}\right] \\ &\quad \times (-ie\gamma^\mu) \left[\frac{i(\not{p} - \not{k} + m)}{(p-k)^2 - m^2 + i\epsilon}\right] \left(-i\frac{g}{2}\frac{m}{M_W}\right) u(p) \\ &= (-ie)\left(\frac{ig^2m^2}{4M_W^2}\right) \int \frac{\mathrm{d}^4k}{(2\pi)^4} \bar{u}(p') \\ &\quad \times \frac{(\not{k} - 2m)\gamma^\mu(\not{k} - 2m)}{[k^2 - M_H^2 + i\epsilon][(p-k)^2 - m^2 + i\epsilon][(p'-k)^2 - m^2 + i\epsilon]} u(p), \end{aligned} \tag{3.61}$$

where we have used (3.1). The denominator is the same as (3.32), after substituting M_W for M_H, and the numerator is:

$$\begin{aligned} N^\mu &= (\not{k} - 2m)\gamma^\mu(\not{k} - 2m) \\ &= -m(1+z)(1-z)(p'+p)^\mu + m(1+z)(x-y)(p'-p)^\mu, \end{aligned} \tag{3.62}$$

where we have ignored all terms proportional to γ^μ, and used the (3.1), and (3.4) to simplify. With the Gordon identity (3.24), the numerator becomes (also ignoring terms proportional to γ^μ):

$$N^\mu = 2m^2(1+z)(1-z)\left(\frac{i\sigma^{\mu\nu}q_\nu}{2m}\right), \tag{3.63}$$

and after the l integration, we arrived at:

$$\begin{aligned} i\mathcal{M}^\mu_{(4)} = (-ie)\,\bar{u}(p')\,\frac{G_F m^2}{8\sqrt{2}\pi^2}\int_0^1 \mathrm{d}x\mathrm{d}y\mathrm{d}z\,\delta(x+y+z-1) \\ \times \left\{\frac{2m^2(1+z)(1-z)}{-xyq^2+(1-z)^2m^2+zM_H^2}\right\}\left(\frac{i\sigma^{\mu\nu}q_\nu}{2m}\right)u(p), \end{aligned} \tag{3.64}$$

which means that the Higgs-boson contribution, at a one-loop level, is:

$$\begin{aligned} [F_2(0)]_H &= \frac{G_F m^2}{8\sqrt{2}\pi^2}\int_0^1 \mathrm{d}z\left\{\frac{2\left(\frac{m^2}{M_H^2}\right)(1+z)(1-z)^2}{z+(1-z)^2\left(\frac{m^2}{M_H^2}\right)}\right\} \\ &\simeq \frac{G_F m^2}{8\sqrt{2}\pi^2}\left\{\mathcal{O}\left(\frac{m^2}{M_H^2}\right)+\mathcal{O}\left(\frac{m^2}{M_H^2}\log\left(\frac{m}{M_H}\right)\right)\right\}. \end{aligned} \tag{3.65}$$

Combining (3.42), (3.60), and (3.65), we obtain the electroweak contribution at a one-loop level:

$$\begin{aligned} a_\mu^{EW}[\text{1-loop}] &= [F_2(0)]_Z + [F_2(0)]_W + [F_2(0)]_H = [F_2(0)]_{EW} \\ &\simeq \frac{G_F m^2}{8\sqrt{2}\pi^2}\left\{\frac{5}{3}+\frac{1}{3}(3-4\cos^2(\theta_W))^2+\mathcal{O}\left(\frac{m^2}{M_Z^2}\right)+\mathcal{O}\left(\frac{m^2}{M_H^2}\right)\right. \\ &\quad \left. +\mathcal{O}\left(\frac{m^2}{M_Z^2}\log\left(\frac{m}{M_Z}\right)\right)+\mathcal{O}\left(\frac{m^2}{M_H^2}\log\left(\frac{m}{M_H}\right)\right)+\mathcal{O}\left(\frac{m^2}{M_W^2}\right)\right\} \\ &\simeq 194.8\times 10^{-11}. \end{aligned} \tag{3.66}$$

For completeness, we simply quote below the results for QED, electroweak and hadronic contribution at higher order calculations [22].

The QED contribution has been computed through 5-loops [23]:

$$\begin{aligned} a_\mu^{QED} &= 0.5\left(\frac{\alpha}{\pi}\right) + 0.765857420(13)\left(\frac{\alpha}{\pi}\right)^2 + 24.05050985(23)\left(\frac{\alpha}{\pi}\right)^3 \\ &\qquad + 130.8782(60)\left(\frac{\alpha}{\pi}\right)^4 + 751.0(9)\left(\frac{\alpha}{\pi}\right)^5 \\ &= 116584718.93(0.10) \times 10^{-11}, \end{aligned} \tag{3.67}$$

where the uncertainty in the second and third loops comes from the uncertainty in the muon and tau masses, and the uncertainty in the fourth and fifth loops comes from numerical integration. The 6-loop leading terms are negligible, $\mathcal{O}(10^{-12})$. The value $\alpha^{-1} = 137.035999046(27)$ [24] was used.

The electroweak 2-loop calculation gives a relatively large value [25]:

$$a_\mu^{EW}[\text{2-loop}] = -41.2(1.0) \times 10^{-11}, \tag{3.68}$$

where the uncertainty comes from the hadronic part that enters the 2-loop calculation. The 3-loop leading terms are negligible, $\mathcal{O}(10^{-12})$ [26].

Hence, after adding up (3.66) with (3.68), we obtain the electroweak contribution of the SM:

$$a_\mu^{EW} = 153.6(1.0) \times 10^{-11}. \tag{3.69}$$

The hadronic contribution gives [23]:

$$a_\mu^{Had} = 6937(40)(18) \times 10^{-11}, \tag{3.70}$$

whose errors are due to the lowest-order hadronic, and higher-order hadronic contributions, respectively.

Finally, after adding up all contributions, we find the SM prediction:

$$a_\mu^{SM} = 116591810(43) \times 10^{-11}. \tag{3.71}$$

Chapter 4

Physical Principles Behind Experiments

In 1956, Lee and Yang [27] questioned whether parity was conserved in weak transitions. This idea came as a possible explanation for the famous $\tau - \theta$ puzzle [28]. They also proposed several experiments on beta decay and meson decay which could provide evidence for parity conservation or non-conservation.

The parity conservation had already been confirmed for electromagnetic process and strong interactions, so it was a surprise for the physics community when, in 1957, Madame Wu, in collaboration with the Low Temperature Group of the US National Bureau of Standards, published their results [29], showing the non-conservation of parity in weak decays.

The experiment basically consists of analyzing the electron (e^-) distribution in the ${}^{60}_{27}\mathrm{Co} \to {}^{60}_{28}\mathrm{Ni} + e^- + \bar{\nu}_e + 2\gamma$ beta decay. The cobalt-60 (^{60}Co) atoms were aligned by a uniform magnetic field and cooled to near absolute zero. The photons came from an electromagnetic process, i.e., from the nickel-60 (^{60}Ni), that was initially in an excited state, and therefore their distribution could be used as a control for the distribution of the emitted electrons (e^-) since we know that for this process parity is conserved (they are emitted roughly equally in all direction). The decay also conserves angular momentum, and since $J_{Co} = 5$ and $J_{Ni} = 4$, we have that $J_{e^-} = J_{\nu_e} = 1/2$. With the concentration of electrons (e^-) in the opposite direction of the applied magnetic field, it became clear that only left-handed electrons were emitted, implying that only right (left) handed antineutrinos (neutrinos) participate in those transitions. This experiment became known as the Wu Experiment.

The theoretical physicists Tsung-Dao Lee and Chen-Ning Yang were honored with the 1957 Nobel Prize in Physics, and Chien-Shiung Wu was awarded the first Wolf Prize in 1978. The question of whether Wu should have received a participation in the 1957 Nobel Prize is investigated in [30].

As also pointed out by Lee and Yang, the parity violation provides a way, through the $\pi \to \mu \to e$ decay chain, to experimentally measure the magnetic dipole moment of the muon.

Figure 4.1: Pion decay, $\pi^\pm \to \mu^\pm + \nu_\mu\ (\bar{\nu}_\mu)$, in the pion rest frame. The long arrow represents the momentum of the particle and the short one, which is above the long arrow, represents the spin of the particle.

The pion is a pseudoscalar particle, i.e., it does not have spin, so if we consider the pion decay $\pi^\pm \to \mu^\pm + \nu_\mu\ (\bar{\nu}_\mu)$, in the pion rest frame, and that all neutrinos (antineutrinos) are left (right) handed, we can use the angular momentum conservation to see that all muons are born 100% polarized (Figure 4.1). Moreover, the parity violation in the muon decay, $\mu^\pm \to e^\pm + \bar{\nu}_\mu(\nu_\mu) + \nu_e(\bar{\nu}_e)$, gives a strong correlation between the muon spin direction and the direction of emission of positrons (or electrons).

In the muon rest frame the differential decay rate for the muon is (Appendix B):

$$\mathrm{d}\Gamma^\pm = N(x_e)\,(1 \pm A(x_e)\cos\theta)\,\mathrm{d}x_e\,\mathrm{d}\Omega, \tag{4.1}$$

where $N(x_e)$ is a normalization factor, θ is the angle between the muon spin direction and the electron momentum in the muon rest frame, and $x_e = E_e/W_e$ is the reduced energy. E_e is the $e^\pm$ energy and $W_e = (m_\mu^2 + m_e^2)/2m_\mu \simeq m_\mu/2$, i.e., the maximum value for the $e^\pm$ energy. $A(x_e) = (2x_e - 1)/(3 - 2x_e)$, is the asymmetry factor and is the reason the spectrum has a strong peak for small θ.

The principles of the Fermilab E989 experiment [12] are the same as the Brookhaven E821 one [31], and of the last European Organization for Nuclear Research (CERN) muon $g-2$ experiment [32], where the method was first applied. It works as illustrated in Figure 4.2. Protons from a booster hit a target and produce pions. The electric positive (negative) pions are unstable and decay into muons plus neutrinos (antineutrinos), as described above. The polarized muons are then injected into a storage ring. Inside the storage ring, a uniform magnetic field is applied, as well as an electric quadrupole field, to provide vertical focusing (this kind of arrangement is known as a Penning Trap). Thereafter, the muons circulate hundreds of times before decaying ($\mu^\pm \to e^\pm + \bar{\nu}_\mu(\nu_\mu) + \nu_e(\bar{\nu}_e)$). The positrons are emitted along the muon spin direction and detected by twenty-four lead-scintillating fiber calorimeters that are distributed uniformly on the inside of the storage ring (Figure 4.3).

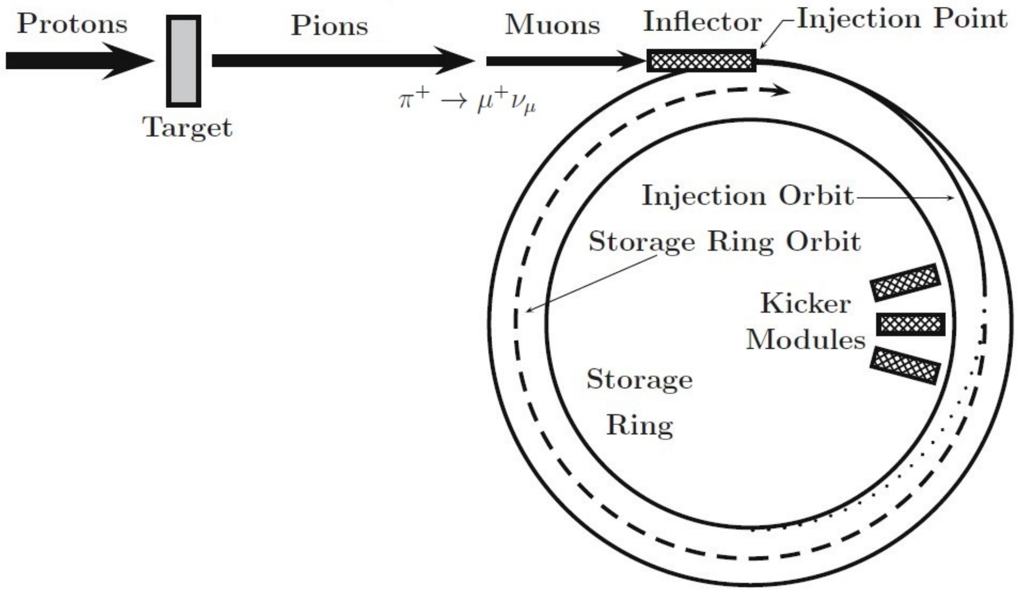
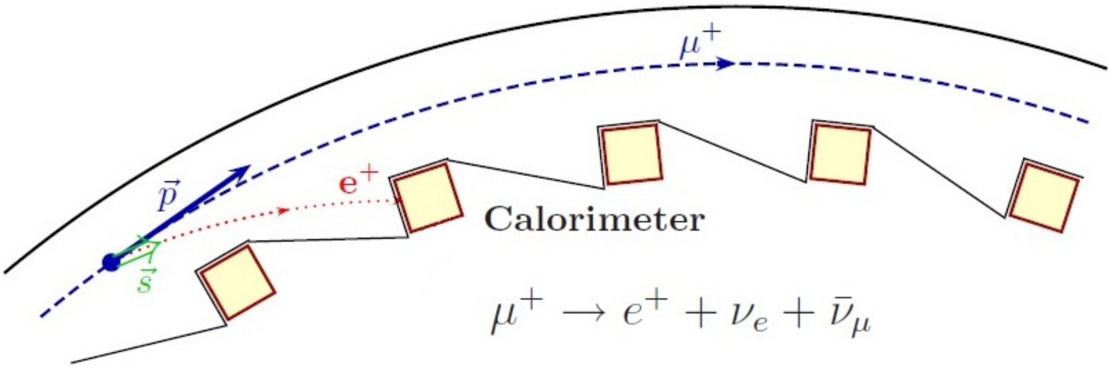
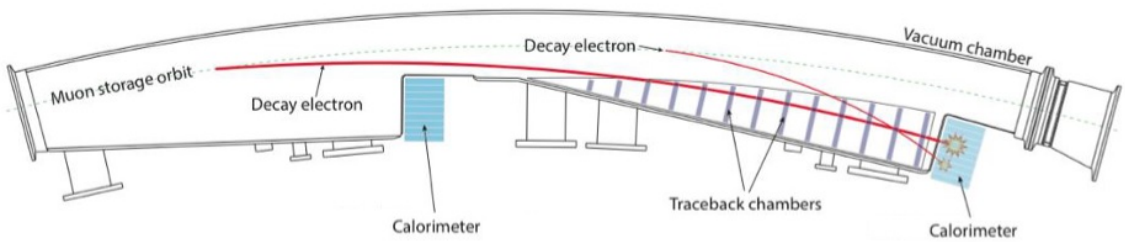


Figure 4.2: A schematic representation of the muon injection into the storage ring.



(a) Emitted positron detected by the calorimeter. The blue dashed line is the muon trajectory, $\vec{p}$ is the momentum, and $\vec{s}$ is the spin. The red dashed line is the positron trajectory.



(b) Emitted electron detected by the calorimeter. The red lines represent the electron trajectory.

Figure 4.3: Muon decay, $\mu^\pm \to e^\pm + \bar{\nu}_\mu(\nu_\mu) + \nu_e(\bar{\nu}_e)$, and the detection of the corresponded emitted particle ($e^\pm$).

The dynamics work as follows: In the presence of both electric ($\vec{E}$) and magnetic ($\vec{B}$) fields, the charged particle will feel a force:

$$\vec{F} = \frac{\mathrm{d}\vec{p}}{\mathrm{d}t} = \frac{\mathrm{d}(m\gamma\vec{v})}{\mathrm{d}t} = e\left(\vec{E} + \vec{v} \times \vec{B}\right), \tag{4.2}$$

where $\gamma^{-1} = \sqrt{(1 - v^2)}$ is the Lorentz factor, and e, m, and $\vec{v}$ are the electric charge, mass, and velocity of the particle, respectively. Now, using the vector identity:

$$(\vec{E} \times \vec{v}) \times \vec{v} = (\vec{v} \cdot \vec{E})\vec{v} - v^2\vec{E}, \tag{4.3}$$

and considering that the electric field $\vec{E}$ is perpendicular to the velocity $\vec{v}$, we can substitute (4.3) into (4.2), to find:

$$\frac{\mathrm{d}\vec{v}}{\mathrm{d}t} = \vec{\omega}_c \times \vec{v}, \quad \vec{\omega}_c = -\frac{e}{\gamma m}\left(\vec{B} + \frac{\gamma^2}{\gamma^2 - 1}\vec{E} \times \vec{v}\right), \tag{4.4}$$

where $\vec{\omega}_c$ is the cyclotron frequency.

The spin will precess with a rotation frequency that is given by [33]:

$$\frac{\mathrm{d}\vec{P}}{\mathrm{d}t} = \vec{\omega}_s \times \vec{P}, \tag{4.5}$$

with (Appendix C)

$$\vec{\omega}_s = -\frac{e}{\gamma m}\left\{(1 + \gamma a_\mu)\vec{B} + a_\mu\frac{(1-\gamma)}{v^2}(\vec{v} \cdot \vec{B})\vec{v} + \gamma\left(a_\mu + \frac{1}{\gamma + 1}\right)\vec{E} \times \vec{v}\right\}, \tag{4.6}$$

where a_μ is the anomalous magnetic dipole moment of the muon, defined in (3.15), and $\vec{P}$ is the particle polarization vector. This formula was first derived by L. H. Thomas in 1927 [34]. Thus, the spin difference frequency is:

$$\begin{aligned} \vec{\omega}_a &= \vec{\omega}_s - \vec{\omega}_c \\ &= -\frac{e}{m}\left\{a_\mu\vec{B} - a_\mu\left(\frac{\gamma}{\gamma + 1}\right)(\vec{v} \cdot \vec{B})\vec{v} + \left(a_\mu - \frac{1}{\gamma^2 - 1}\right)\vec{E} \times \vec{v}\right\}. \end{aligned} \tag{4.7}$$

To minimize the effects of the transverse electric quadrupole field ($\vec{E}$), muons with a special velocity are used, so that the coefficient of the last term in (4.7) is approximately zero. This cancellation occurs when $\gamma_m = 29.3$, where the m stands for "magic".

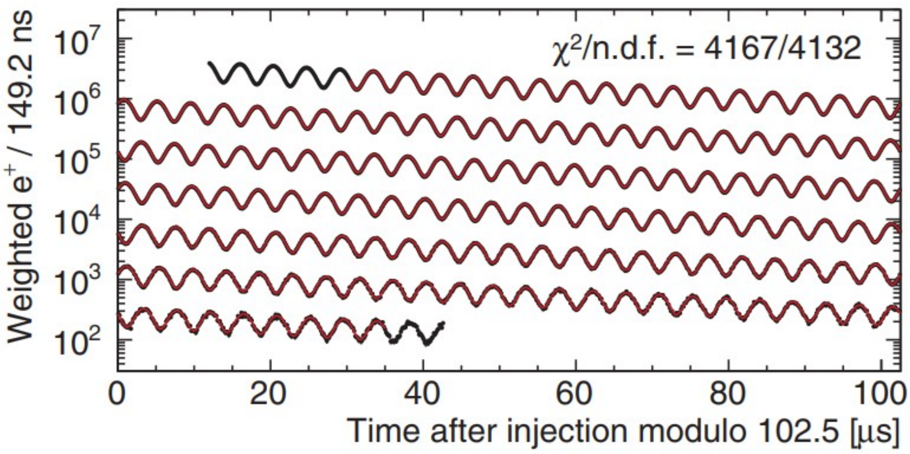


Figure 4.4: The number of high-energy positrons vs time after injection. The red line is a fit and the black points are data. This image was taken from Fermilab Muon $g-2$ Collaboration [35].

The second term in (4.7), that appears to be zero because of the transverse condition, $(\vec{v} \cdot \vec{B})$, it is not. This happens because the electric field induces a vertical pitching motion. To account for that, some corrections are needed, i.e., the Vertical Pitch Correction [36].

Another small but important correction, that also arises from the electric field, is that not all muons are at the central radius, and for that reason not at the magic momentum. This is the Radial Electric Field Correction [37].

Now, with the above considerations, one may write:

$$\vec{\omega}_a \simeq -\frac{e}{m} a_\mu \vec{B}, \tag{4.8}$$

and with two measures, ω_a and B, determine a_μ.

The magnetic field B is measured by nuclear magnetic resonance [38], and the angular frequency ω_a is determined from the time distribution of the decay positrons that are observed in the electromagnetic calorimeters (Figure 4.4). These detectors measured both the energy and the time of arrival of the particles. The time distribution gives the number of decay positrons with energies greater than E emitted at time t after the muons are injected into the storage ring [39]:

$$N(t) \simeq N_0\, e^{(-t/\gamma\tau_\mu)} \left[1 + A\cos\left(\omega_a t + \phi\right)\right], \tag{4.9}$$

where N_0 is a normalization factor, τ_μ is the muon lifetime in the muon rest frame, A is the asymmetry factor, and ϕ is the phase angle at injection.

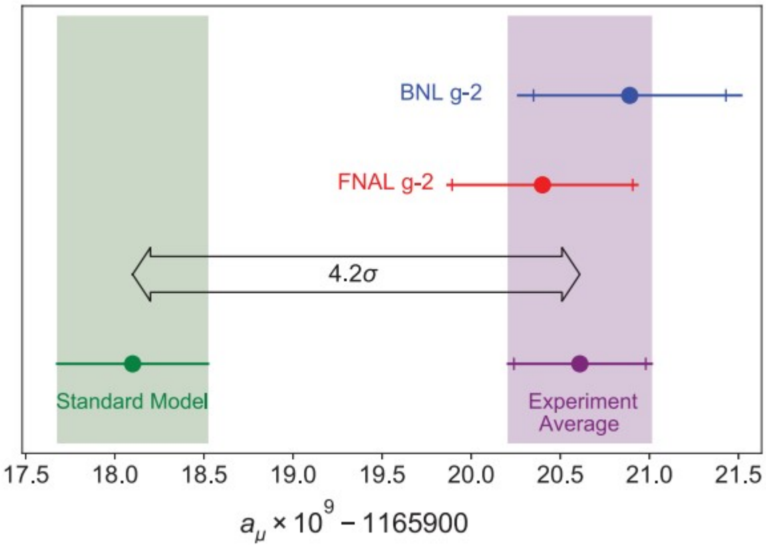


Figure 4.5: From top to bottom: experimental values of a_μ from BNL E821, FNAL E989, and the combined average. The inner tick marks indicate the statistical contribution to the total uncertainties. The SM prediction is also shown. This image was taken from Fermilab Muon $g-2$ Collaboration [35].

The predicted theoretical value and the experimental average between Fermilab (E989) and Brookhaven (E821) value of the anomalous magnetic moment of the muon are [35]:

$$\begin{aligned} a_\mu^{SM} &= 0.00116591810(43), \\ a_\mu^{exp} &= 0.00116592061(41). \end{aligned} \tag{4.10}$$

The combined results from Fermilab and Brookhaven show a difference with theory at a significance of 4.2σ (Figure 4.5). This means that the chance that the results are a statistical fluctuation is about 1 in 40,000. It is, however, slightly under the 5σ, that scientists require to claim a discovery but still strongly suggest the presence of new physics.

Chapter 5

Hints of New Physics Contributions

The discrepancy between theory and experiment indicates the possibility of new physics. It is possible to see, through the expressions calculated previously, that only the addition of new gauge bosons would not be able to explain such discrepancy since heavier bosons would bring smaller contributions than those of the SM to the anomalous magnetic moment of the muon.

Taking this into account, we investigated the possibility of adding new bosons together with new fermions. Below we will try to implement this strategy by considering an extension of the standard model as an example. We're going to work with the 3-3-1 model with a heavy charged lepton [40].

The first difference is that we now arrange the particles in the fundamental representation of the $SU(3)_L$ group instead of the $SU(2)_L$ group, so we need to work with a left triplet and not a doublet anymore. In addition, there are some variations of the 3-3-1 model, with different phenomenology. The 3-3-1 model with a heavy charged lepton means that we will add a new heavy charged lepton in the third component of the left triplet:

$$\begin{pmatrix} \nu_e \\ e^- \end{pmatrix}_L \longrightarrow \begin{pmatrix} \nu_e \\ e^- \\ E^+ \end{pmatrix}_L . \tag{5.1}$$

We will mainly focus on the gauge sector and therefore we are not going to look at the scalar part of the theory. For a $SU(N)$ group we have $N^2 - 1$ generators, so for $SU(3)_L \otimes U(1)_N$ we have $3^2 - 1 + 1 = 9$ generators, which are identified as 9 gauge bosons. In particular, the 3-3-1 model with a heavy charged lepton, has 3 neutral bosons (γ, Z, Z'), 4 charged ($W^{\pm}$, $V^{\pm}$), and 2 doubly charged ($U^{\pm\pm}$) [41].

We must then calculate the contribution of these new bosons, since the others are exactly those of the SM and, therefore, have already been calculated, but as we will see below, only two of them can contribute to the anomalous magnetic moment of the muon.

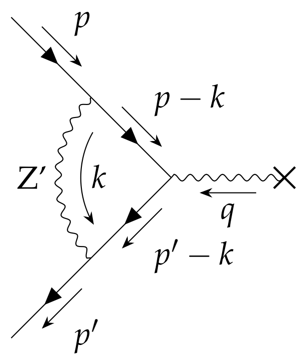


Figure 5.1: Z′ diagram at a one-loop level. All fermionic lines correspond to the muon and all momenta are indicated in the figure.

We will not perform the complete calculation at 1-loop nor prove the gauge invariance as we did for the SM. We will work in the Feynman-'t Hooft gauge ($\xi = 1$) but we will not add all the contributions, in specific we will not calculate the contributions coming from the Goldstone bosons. We justify this by arguing that those diagrams are, at best, of the same order as the ones we computed. We will first look at the neutral contribution and then look at the charged one.

The neutral Lagrangian is [40]:

$$\mathcal{L}_{NC} = -\frac{g}{2c_W}\sum_i \overline{\psi}_i \gamma^\mu [(g_V^i \mathbb{1} - g_A^i \gamma_5) Z_\mu + (f_V^i \mathbb{1} - f_A^i \gamma_5) Z'_\mu] \psi_i, \tag{5.2}$$

where ψ_i is any lepton, and the first term is simply the Z-boson of the SM (2.11). The constants f_V^i and f_A^i for neutrinos (ν), light fermions (e), and heavy fermions (E) are:

$$\begin{aligned} f_V^\nu &= \frac{1}{2}\left(\frac{1-4x}{3}\right)^{\frac{1}{2}} = f_A^\nu, \\ f_V^e &= \frac{1}{2}\left(\frac{1-4x}{3}\right)^{\frac{1}{2}} - x\left(\frac{3}{1-4x}\right)^{\frac{1}{2}}; f_A^e = \frac{1}{2}\left(\frac{1-4x}{3}\right)^{\frac{1}{2}} + x\left(\frac{3}{1-4x}\right)^{\frac{1}{2}}, \\ f_V^E &= -\left(\frac{1-4x}{3}\right)^{\frac{1}{2}} + x\left(\frac{3}{1-4x}\right)^{\frac{1}{2}}; f_A^E = -\left(\frac{1-4x}{3}\right)^{\frac{1}{2}} - x\left(\frac{3}{1-4x}\right)^{\frac{1}{2}}, \end{aligned} \tag{5.3}$$

with $x \equiv \sin^2\theta_W$.

This means that we have a similar contribution to the anomalous magnetic dipole moment of the muon coming from this new heavy neutral boson Z′.

From Figure 5.1, the diagram reads:

$$\begin{aligned} i\mathcal{M}^{\mu}_{(Z')} &= \int \frac{\mathrm{d}^4k}{(2\pi)^4}\bar{u}(p')\frac{(-ie\gamma^{\nu})}{\sin(2\theta_W)}(V'+A'\gamma^5)\left[\frac{-i\eta_{\nu\alpha}}{k^2-M_{Z'}^2+i\epsilon}\right] \\ &\quad\times\left[\frac{i(\not{p}'-\not{k}+m)}{(p'-k)^2-m^2+i\epsilon}\right](-ie\gamma^{\mu})\left[\frac{i(\not{p}-\not{k}+m)}{(p-k)^2-m^2+i\epsilon}\right] \\ &\quad\times\frac{(-ie\gamma^{\alpha})}{\sin(2\theta_W)}(V'+A'\gamma^5)u(p) \\ &= (-ie)\frac{(-ie^2)}{[\sin(2\theta_W)]^2}\int\frac{\mathrm{d}^4k}{(2\pi)^4}\bar{u}(p')\gamma^{\alpha}(V'+A'\gamma^5) \\ &\quad\times\frac{(\not{p}'-\not{k}+m)\gamma^{\mu}(\not{p}-\not{k}+m)\gamma_{\alpha}(V'+A'\gamma^5)}{[k^2-M_{Z'}^2+i\epsilon][(p-k)^2-m^2+i\epsilon][(p'-k)^2-m^2+i\epsilon]}u(p), \end{aligned} \tag{5.4}$$

where $V'\equiv f_V^e$ and $A'\equiv f_A^e$. Note that we recover the same expression as (3.31) if we set $2V'=V=1-2\cos(2\theta_W)$, $2A'=A=1$, and of course, $M_{Z'}=M_Z$. Proceeding in the same way as before, we find:

$$\begin{aligned} [F_2(0)]_{Z'} &= \frac{G_F m^2}{8\sqrt{2}\pi^2}\left(\frac{M_Z}{M_{Z'}}\right)^2\int_0^1 \mathrm{d}z\left\{\frac{4V'^2 z(1-z)^2-4A'^2 z(3+z)(1-z)}{z+(1-z)^2\left(\frac{m^2}{M_{Z'}^2}\right)}\right\} \\ &\simeq \frac{G_F m^2}{8\sqrt{2}\pi^2}\left(\frac{M_Z}{M_{Z'}}\right)^2\left\{\frac{4}{3}(V'^2-5A'^2)+\mathcal{O}\left(\frac{m^2}{M_{Z'}^2}\right)+\mathcal{O}\left(\frac{m^2}{M_{Z'}^2}\log\left(\frac{m}{M_{Z'}}\right)\right)\right\}. \end{aligned} \tag{5.5}$$

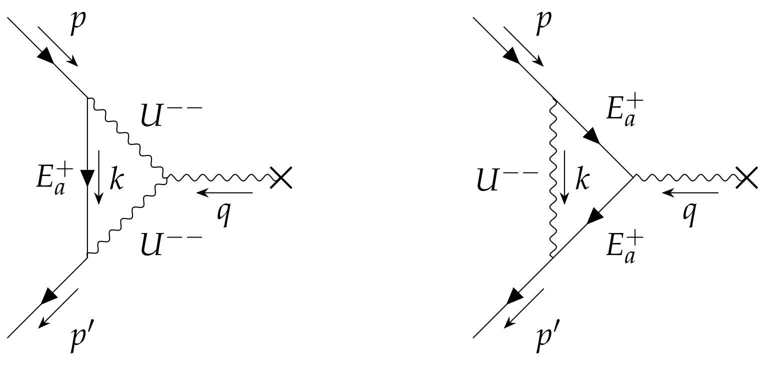
Using $sin^2(\theta_W)=0.223$ [22], we can write:

$$[F_2(0)]_{Z'} \simeq -\frac{G_F m^2}{8\sqrt{2}\pi^2}\times 10\left(\frac{M_Z}{M_{Z'}}\right)^2. \tag{5.6}$$

Considering $M_{Z'}=4$ Tev [42], we can see that Z'-boson contribution is about 0.5% of the Z-boson contribution,

$$[F_2(0)]_{Z'} \sim \mathcal{O}(10^{-11}), \tag{5.7}$$

which is negligible. Note that this contribution corresponds to the addition of only one new gauge boson, which we commented earlier would not produce a significant contribution.



(a) (b)

Figure 5.2: Diagrams of the U-boson contribution at a one-loop level.

The charged current interactions are [40]:

$$\mathcal{L}_{CC} = -\frac{g}{2\sqrt{2}}\sum_a [\bar{e}_a\gamma^\mu(1-\gamma^5)\nu_a W^-_\mu + \bar{\nu}_a\gamma^\mu(1-\gamma^5)D_{ab}E_b V^-_\mu + \bar{e}_a\gamma^\mu(1-\gamma^5)K_{ab}E_b U^{--}_\mu] + h.c., \tag{5.8}$$

where K_{ab} and D_{ab} are the matrices elements that rotated the symmetry eigenstates into the mass eigenstates, and $a = 1, 2, 3$ is a sum over the 3 families. Note that the first term is the W-boson of the SM and that the vector boson V does not couple with the light fermions, so the only contribution that we need to calculate is the one coming from U^{--}.

The diagram (a) of Figure 5.2 reads:

$$\begin{aligned}
i\mathcal{M}^\mu_{(a)} &= \int \frac{d^4k}{(2\pi)^4}\bar{u}(p')\left[\left(\frac{-igK_{ab}}{2\sqrt{2}}\right)\gamma^\alpha(\mathbb{1}-\gamma^5)\right]\left[\frac{-i\eta_{\alpha\nu}}{(p'-k)^2 - M_U^2 + i\epsilon}\right] \\
&\quad \times (2ie)\left[\eta^{\nu\lambda}(p+p'-2k)^\mu + \eta^{\lambda\mu}(k+p'-2p)^\nu + \eta^{\mu\nu}(k+p-2p')^\lambda\right] \\
&\quad \times \left[\frac{-i\eta_{\lambda\beta}}{(p-k)^2 - M_U^2 + i\epsilon}\right]\left[\frac{i(\not{k}+M_E)}{k^2 - M_E^2 + i\epsilon}\right]\left[\left(\frac{-igK_{ab}}{2\sqrt{2}}\right)\gamma^\beta(\mathbb{1}-\gamma^5)\right]u(p) \\
&= (-ie)\left(\frac{-ig^2}{8}\right)(2K^2_{ab})\int \frac{d^4k}{(2\pi)^4}\bar{u}(p')(\mathbb{1}+\gamma^5)\gamma_\nu\not{k}\gamma_\lambda(\mathbb{1}-\gamma^5) \\
&\quad \times \frac{[\eta^{\nu\lambda}(p+p'-2k)^\mu + \eta^{\lambda\mu}(k+p'-2p)^\nu + \eta^{\mu\nu}(k+p-2p')^\lambda]}{[k^2 - M_E^2 + i\epsilon][(p-k)^2 - M_U^2 + i\epsilon][(p'-k)^2 - M_U^2 + i\epsilon]}u(p).
\end{aligned} \tag{5.9}$$

The numerator is the same as (3.47), so we find:

$$
\begin{aligned}
i\mathcal{M}^{\mu}_{(a)} = (-ie)\ \bar{u}(p')\ \frac{G_F m^2}{8\sqrt{2}\pi^2} M_W^2 K_{ab}^2 \int_0^1 \mathrm{d}x\mathrm{d}y\mathrm{d}z\ \delta(x+y+z-1) \\
\times \left\{ \frac{4(1-z)(3-2z)}{-xyq^2 + M_U^2(1-z) + M_E^2 z + m^2 z(z-1)} \right\} \left(\frac{i\sigma^{\mu\nu} q_\nu}{2m} \right) u(p).
\end{aligned}
\tag{5.10}
$$

For the diagram (b) of Figure 5.2 we have:

$$
\begin{aligned}
i\mathcal{M}^{\mu}_{(b)} &= \int \frac{\mathrm{d}^4k}{(2\pi)^4} \bar{u}(p') \left[\left(\frac{-igK_{ab}}{2\sqrt{2}} \right) \gamma^\alpha (\mathbb{1}-\gamma^5) \right] \left[\frac{i(\not{p}'-\not{k}+M_E)}{(p'-k)^2 - M_E^2 + i\epsilon} \right] (ie\gamma^\mu) \\
&\times \left[\frac{i(\not{p}-\not{k}+M_E)}{(p-k)^2 - M_E^2 + i\epsilon} \right] \left[\frac{-i\eta_{\alpha\beta}}{k^2 - M_U^2 + i\epsilon} \right] \left[\left(\frac{-igK_{ab}}{2\sqrt{2}} \right) \gamma^\beta (\mathbb{1}-\gamma^5) \right] u(p) \\
&= (-ie) \left(\frac{ig^2}{8} \right) (K_{ab}^2) \int \frac{\mathrm{d}^4k}{(2\pi)^4} \bar{u}(p') \\
&\times \frac{(\mathbb{1}+\gamma^5)\gamma^\alpha(\not{p}'-\not{k}+M_E)\gamma^\mu(\not{p}-\not{k}+M_E)\gamma_\alpha(\mathbb{1}-\gamma^5)}{[k^2 - M_U^2 + i\epsilon][(p-k)^2 - M_E^2 + i\epsilon][(p'-k)^2 - M_E^2 + i\epsilon]} u(p),
\end{aligned}
\tag{5.11}
$$

which can be simplified to

$$
\begin{aligned}
i\mathcal{M}^{\mu}_{(b)} = (-ie)\ \bar{u}(p')\ \frac{G_F m^2}{8\sqrt{2}\pi^2} M_W^2 K_{ab}^2 \int_0^1 \mathrm{d}x\mathrm{d}y\mathrm{d}z\ \delta(x+y+z-1) \\
\times \left\{ \frac{4z(z+1)}{-xyq^2 + M_U^2 z + M_E^2(1-z) + m^2 z(z-1)} \right\} \left(\frac{i\sigma^{\mu\nu} q_\nu}{2m} \right) u(p).
\end{aligned}
\tag{5.12}
$$

Combining (5.10) and (5.12), we find:

$$
\begin{aligned}
[F_2(0)]_U = \frac{G_F m^2}{8\sqrt{2}\pi^2} \left(\frac{M_W}{M_U} \right)^2 K_{ab}^2 \int_0^1 \mathrm{d}z \left\{ \frac{4(1-z)^2(3-2z)}{(1-z) + \kappa^2 z + \epsilon^2 z(z-1)} \right. \\
\left. + \frac{4z(1+z)(1-z)}{z + \kappa^2(1-z) + \epsilon^2 z(z-1)} \right\},
\end{aligned}
\tag{5.13}
$$

where $\kappa \equiv (M_E/M_U)$ and $\epsilon \equiv (m/M_U)$. After expanding around $\epsilon = 0$, we find that the maximum value for the integral is of order 10 and since K_{ab} is, at best, of order 1, we have that:

$$
[F_2(0)]_U \simeq \frac{G_F m^2}{8\sqrt{2}\pi^2} \times 10 \left(\frac{M_W}{M_U} \right)^2 .
\tag{5.14}
$$

Considering $M_U = 1.5$ TeV [42], we can see that the U-boson contribution is

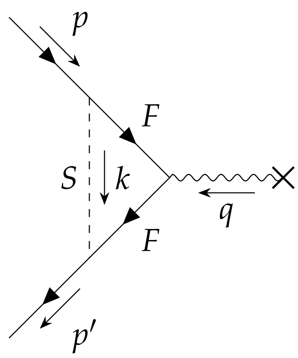


Figure 5.3: Diagram of a neutral scalar particle (S) contribution at a one-loop level.

about 1% of the W-boson contribution,

$$[F_2(0)]_U \sim 10^{-11}, \tag{5.15}$$

which is also negligible.

Therefore, the bosonic part does not contribute significantly to explain such a discrepancy between experiment and theory, and consequently, this discrepancy must be explained either by contributions coming only from the scalar sector of the theory, which is somewhat surprising when we compare it with the SM where the scalar sector contribution is of a lower order, or it does not, and we must, for that reason, consider other types of interaction.

A neutral scalar (S) contribution would only contribute with one diagram, the one on Figure 5.3:

$$\begin{aligned} i\mathcal{M}^{\mu}_{(S)} &= \int \frac{d^4k}{(2\pi)^4}\bar{u}(p')\left[i\left(C_s+C_p\gamma^5\right)\right]\left[\frac{i}{k^2-M_S^2+i\epsilon}\right]\left[\frac{i(\not{p}'-\not{k}+M_F)}{(p'-k)^2-M_F^2+i\epsilon}\right] \\ &\quad\times(-ie\gamma^{\mu})\left[\frac{i(\not{p}-\not{k}+M_F)}{(p-k)^2-M_F^2+i\epsilon}\right]\left[i\left(C_s-C_p\gamma^5\right)\right]u(p) \\ &= (-ie)\int\frac{d^4k}{(2\pi)^4}\bar{u}(p') \\ &\quad\times\frac{i(C_s+C_p\gamma^5)(\not{p}'-\not{k}+M_F)\gamma^{\mu}(\not{p}-\not{k}+M_F)(C_s-C_p\gamma^5)}{[k^2-M_S^2+i\epsilon][(p-k)^2-M_F^2+i\epsilon][(p'-k)^2-M_F^2+i\epsilon]}u(p), \end{aligned} \tag{5.16}$$

where M_S is the mass of the neutral scalar particle, M_F is the mass of a heavy fermion and we have considered the general interaction:

$$\mathcal{L}_Y = \bar{\mu}(C_s+C_p\gamma^5)F\cdot S, \tag{5.17}$$

with C_S and C_P being model dependent.

Proceeding as usual we find:

$$[F_2(0)]_S = \left(\frac{m^2}{8\pi^2}\right)\int_0^1 \mathrm{d}z \left\{(C_s^2-C_p^2)(1-z)^2\left(\frac{M_F}{m}\right)+(C_s^2+C_p^2)z(1-z)^2\right\} \times \left[M_S^2 z + M_F^2(1-z) + m^2 z(z-1)\right]^{-1}, \tag{5.18}$$

which can be rewritten in terms of the ratios of the masses $\epsilon = (m/M_S)$ and $\kappa = (M_F/M_S)$:

$$[F_2(0)]_S = \frac{1}{8\pi^2}\int_0^1 \mathrm{d}z(1-z)^2 \left\{\frac{\left(C_s^2-C_p^2\right)\epsilon\kappa + \left(C_s^2+C_p^2\right)\epsilon^2 z}{z+(1-z)(\kappa^2-\epsilon^2 z)}\right\}. \tag{5.19}$$

Since we know that ϵ is small, one can Taylor expand the integrand to find:

$$[F_2(0)]_S = \frac{1}{8\pi^2}\int_0^1 \mathrm{d}z(1-z)^2 \left\{\frac{\left(C_s^2-C_p^2\right)\epsilon\kappa + \left(C_s^2+C_p^2\right)\epsilon^2 z}{z+\kappa^2(1-z)} + \mathcal{O}(\epsilon^3)\right\}, \tag{5.20}$$

that after the last integral in the Feynman parameters yields:

$$[F_2(0)]_S = \frac{1}{16\pi^2}\left(C_s^2-C_p^2\right)\frac{\epsilon\kappa[(\kappa^2-1)(\kappa^2-3)+4\log(\kappa)]}{(\kappa^2-1)^3} + \mathcal{O}(\epsilon^2). \tag{5.21}$$

One can now expand around $\kappa = 0$, $\kappa = 1$ and $\kappa = \infty$ to find, respectively:

$$[F_2(0)]_S = \frac{\left(C_s^2-C_p^2\right)}{16\pi^2}\begin{cases} \left(\frac{m}{M_S}\right)\left(\frac{M_F}{M_S}\right)\left[4\log\left(\frac{M_S}{m}\right)-3\right] & , m \ll M_F \ll M_S \\ \frac{2}{3}\left(\frac{m}{M_F}\right) & , m \ll M_F \simeq M_S \\ \left(\frac{m}{M_F}\right) & , m \ll M_S \ll M_F. \end{cases} \tag{5.22}$$

Note that, if one wishes to recover the Higgs contribution of the SM, one also has to consider terms proportional ϵ^2. That happens because in the SM, $M_F = m$ and $M_S = M_H$, which implies that $\epsilon = \kappa$. As one can see, from (5.21), we kept only linear terms in ϵ.

Let's consider now a singly charged scalar contribution. The corresponding

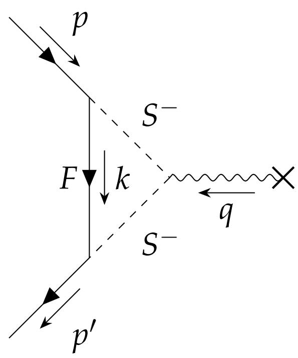


Figure 5.4: Diagram of a charged scalar particle (S^-) contribution at a one-loop level.

diagram is given in Figure 5.4, and also using (5.17), it reads:

$$\begin{aligned} i\mathcal{M}^{\mu}_{(S^-)} &= \int \frac{d^4k}{(2\pi)^4}\bar{u}(p')\left[i\left(C_s+C_p\gamma^5\right)\right]\left[\frac{i}{(p'-k)^2-M_S^2+i\epsilon}\right] \\ &\quad\times\left[(-ie)(p+p'-2k)^{\mu}\right]\left[\frac{i}{(p-k)^2-M_S^2+i\epsilon}\right] \\ &\quad\times\left[\frac{i(\not{k}+M_F)}{k^2-M_F+i\epsilon}\right]\left[i\left(C_s-C_p\gamma^5\right)\right]u(p) \\ &= (-ie)\int\frac{d^4k}{(2\pi)^4} \\ &\quad\times\frac{i\left(C_s+C_p\gamma^5\right)(\not{k}+M_F)(p+p'-2k)^{\mu}\left(C_s-C_p\gamma^5\right)}{[k^2-M_F^2+i\epsilon][(p-k)^2-M_S^2+i\epsilon][(p'-k)^2-M_S^2+i\epsilon]}u(p), \end{aligned} \tag{5.23}$$

which gives:

$$\begin{aligned} [F_2(0)]_{S^-} = \left(\frac{m^2}{8\pi^2}\right)\int_0^1 dz\left\{(C_s^2-C_p^2)z(z-1)\left(\frac{M_F}{m}\right)-(C_s^2+C_p^2)z(1-z)^2\right\} \\ \times\left[M_F^2z+M_S^2(1-z)+m^2z(z-1)\right]^{-1}. \end{aligned} \tag{5.24}$$

Rewriting in terms of $\epsilon = (m/M_S)$ and $\kappa = (M_F/M_S)$, we find:

$$[F_2(0)]_{S^-} = \frac{1}{8\pi^2}\int_0^1 dz(z-1)z\left\{\frac{\left(C_s^2-C_p^2\right)\epsilon\kappa+\left(C_s^2+C_p^2\right)\epsilon^2(1-z)}{z\kappa^2+(1-z)(1-\epsilon^2z)}\right\}. \tag{5.25}$$

Once more, we can expand around $\epsilon = 0$:

$$[F_2(0)]_{S^-} = \frac{1}{8\pi^2}\int_0^1 \mathrm{d}z(z-1)z\left\{\frac{\left(C_s^2 - C_p^2\right)\epsilon\kappa + \left(C_s^2 + C_p^2\right)\epsilon^2(1-z)}{1+z(\kappa^2-1)} + \mathcal{O}(\epsilon^3)\right\}, \quad (5.26)$$

and perform the last integral in the Feynman parameters to find:

$$[F_2(0)]_{S^-} = \frac{1}{16\pi^2}\left(C_s^2 - C_p^2\right)\frac{\epsilon\kappa[(1+\kappa^2)(1-\kappa^2)+4\kappa^2\log(\kappa)]}{(\kappa^2-1)^3} + \mathcal{O}(\epsilon^2). \quad (5.27)$$

Finally, one can expand around $\kappa = 0$, $\kappa = 1$ and $\kappa = \infty$ to find, respectively:

$$[F_2(0)]_{S^-} = -\frac{1}{16\pi^2}\left(C_s^2 - C_p^2\right)\begin{cases} \left(\frac{m}{M_S}\right)\left(\frac{M_F}{M_S}\right) & , m \ll M_F \ll M_S \\ \frac{1}{3}\left(\frac{m}{M_F}\right) & , m \ll M_F \simeq M_S \\ \left(\frac{m}{M_F}\right) & , m \ll M_S \ll M_F. \end{cases} \quad (5.28)$$

In general, C_s and C_p depend on the masses in the following way:

$$\left(C_s^2 - C_p^2\right) \simeq -4\rho\frac{m}{M_H}\frac{M_F}{M_S}, \quad (5.29)$$

where ρ is a constant that comes from the strength of the interaction and the rotation of the symmetry eigenstates into the mass eigenstates. Substituting (5.29) into (5.27), we find:

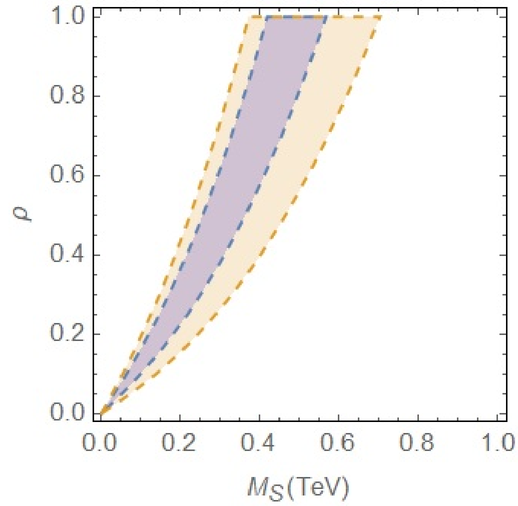
$$[F_2(0)]_{S^-} = \frac{\rho}{4\pi^2}\frac{m^2M_F^2\left[M_S^4 - M_F^4 + 4M_F^2M_S^2\log\left(\frac{M_F}{M_S}\right)\right]}{M_HM_S(M_S^2 - M_F^2)^3} + \mathcal{O}(\epsilon^2), \quad (5.30)$$

which gives a positive contribution to the anomalous magnetic moment of the muon. Note that, since (5.29) implies that the singly scalar charged particle gives a positive contribution (5.30), it also implies that the neutral scalar particle gives a negative contribution (5.21).

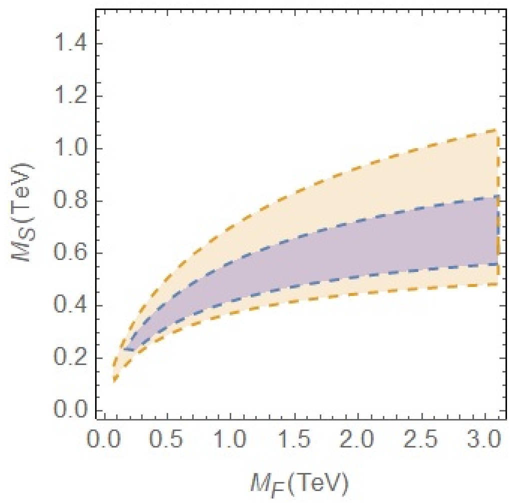
As an illustration, we have only considered the contribution of a singly charged scalar particle, thus we only have to account for the S^- contribution given in (5.30).

If that is the case, then we should expect:

$$[F_2(0)]_{S^-} = a_\mu^{exp} - a_\mu^{SM} = (251 \pm 59) \times 10^{-11}. \tag{5.31}$$



(a) Contour plot for $M_F = 1$TeV.



(b) Contour plot for $\rho = 1$.

Figure 5.5: The blue region corresponds to values up to 1σ and the orange one corresponds to values up to 2σ.

In Figure 5.5, we have used (5.30) together with (5.31) to find possible values for the masses M_S and M_F, and the constant ρ.

As a final remark, we emphasize that the formulas found in this chapter are in agreement with the work of Leveille [43] and Moore [44]. This ensures that the assumption made earlier about the order of magnitude of diagrams involving gauge bosons and the corresponding Goldstone bosons, which were not computed, was correct. The fact that the gauge bosons did not give a significant contribution comes from the fact that for the 3-3-1 model with a heavy lepton, the neutral gauge boson Z' interaction is diagonal and that the doubly charged one U^{--} has, as one can see from (5.8), $C_s^2 = C_p^2 = 1$.

Chapter 6

Conclusion

In this work, we discuss the possibility of solving the problem of the anomalous magnetic dipole moment of the muon with the addition of new particles. The simple addition of new bosons is not able to solve the problem since these additional contributions are even smaller than those found in the SM. Due to this fact, we propose that the addition of new heavy bosons must be accompanied by the addition of new heavy fermions.

To be able to perform this calculation, we choose one possible extension of the SM, the 3-3-1 model with a heavy lepton. This model has 5 new types of gauge bosons in addition to the SM ones, one neutral $Z^{'}$, two singly charged $V^{\pm}$, and two that are doubly charged $U^{\pm\pm}$. The V-boson does not couple with the SM charged leptons (electron, muon, and tau), so it can not contribute. The $Z^{'}$-boson is diagonal, which ultimately falls back to our original assumption, i.e., that this kind of contribution gives the same result as if we had only added one new heavy boson and nothing else. For the U-boson we find a possible contribution but due to its interactions, those contributions only appear in the second order term of a Taylor series expansion in ϵ. With the masses of those bosons at the TeV scale [42], we conclude that, for this specific model, the gauge sector does not yield a significant contribution.

Since the gauge sector did not yield a significant contribution, we rely on the scalar sector to find such contributions. We consider general interactions for the scalar sector and calculate possible contributions coming from a new heavy neutral scalar particle S and a new heavy singly charged scalar particle S^{-}. Finally, we analyze the S^{-} contribution, which can be found in a different model, such as the 3-3-1 model with a heavy neutral lepton.

Our results show the viability of the strategy proposed in this work, which was implemented through the extra degrees of freedom of the 3-3-1 models. Ultimately, we find possible values for the masses of the scalar S^{-} and the neutral fermion F that can explain the discrepancy between theory and experiment, they are shown in Figure 5.5.

Appendix A

Gauge Invariance Proof of Each Boson Contribution at 1-Loop Level

To prove the gauge invariance of each contribution, we will work in the R_ξ gauge and show that the final result does not depend on the gauge parameter ξ.

Photon Contribution:

The photon contribution corresponds to the diagram of the Figure 3.1. In the R_ξ gauge, the photon propagator reads [18]:

$$\Delta_{\alpha\beta}(k,\xi) = \frac{-i}{k^2+i\epsilon}\left[\eta_{\alpha\beta} - (1-\xi)\,\frac{k_\alpha k_\beta}{k^2+i\epsilon}\right]. \tag{A.1}$$

Using the above propagator, we can see that the term proportional to $\eta_{\alpha\beta}$ gives the exact same result as (3.16), but now we have another term that needs to be evaluated. This term is proportional to $k_\alpha k_\beta$ and gives:

$$\begin{aligned} i\mathcal{M}^{\mu}_{(1)} &\sim \int \frac{d^4k}{(2\pi)^4}\bar{u}(p')(-ie\gamma^\alpha)\left[\frac{i(\not{p}'-\not{k}+m)}{(p'-k)^2-m^2+i\epsilon}\right](-ie\gamma^\mu) \\ &\quad\times\left[\frac{i(\not{p}-\not{k}+m)}{(p-k)^2-m^2+i\epsilon}\right]\left[\frac{i(1-\xi)k_\alpha k_\beta}{(k^2+i\epsilon)(k^2+i\epsilon)}\right](-ie\gamma^\beta)u(p) \\ &\sim (-ie)(-ie^2)\int\frac{d^4k}{(2\pi)^4}\bar{u}(p') \\ &\quad\times\frac{-(1-\xi)\not{k}(\not{p}'-\not{k}+m)\gamma^\mu(\not{p}-\not{k}+m)\not{k}}{[k^2+i\epsilon][k^2+i\epsilon][(p-k)^2-m^2+i\epsilon][(p'-k)^2-m^2+i\epsilon]}u(p). \end{aligned} \tag{A.2}$$

Using the Feynman parametrization (3.17), we can write the denominator of (A.2) as:

$$\frac{[k^2+i\epsilon]^{-1}[(p'-k)^2-m^2+i\epsilon]^{-1}}{[k^2+i\epsilon][(p-k)^2-m^2+i\epsilon]} = \int_0^1 d^4x\frac{3!\,\delta(w+x+y+z-1)}{D^4}, \tag{A.3}$$

where $\mathrm{d}^4x = \mathrm{d}w\mathrm{d}x\mathrm{d}y\mathrm{d}z$, and D is:

$$D = l^2 - \Delta + i\epsilon, \tag{A.4}$$

with $l = k - xp - yp'$, and $\Delta = -xyq^2 + (x+y)^2m^2$.

The numerator of (A.2) is:

$$\begin{aligned} N^\mu &= \not{k}(\not{p}' - \not{k} + m)\gamma^\mu(\not{p} - \not{k} + m)\not{k} \\ &= \left[k^2 - 2(p' \cdot k)\right]\gamma^\mu\left[k^2 - 2(p \cdot k)\right] \\ &= \left\{l^2 - 2l \cdot [xp + (y-1)p'] + m^2(x+y)(x+y-2)\right\}\gamma^\mu \\ &\quad \times \left\{l^2 - 2l \cdot [(x-1)p + yp'] + m^2(x+y)(x+y-2)\right\}, \end{aligned} \tag{A.5}$$

where we have used (3.4), and $q^2 = 0$, to simplify. From (A.5), one can see that there are only terms proportional to γ^μ, and since those terms do not contribute to the form factor $F_2(0)$, we conclude that the photon contribution is gauge invariant, i.e., it does not depend on the parameter ξ.

Z-boson Contribution:

The Z-boson contribution corresponds to the diagrams (a) and (b) of the Figure 3.2. In the R_ξ gauge, the Z-boson propagator reads [18]:

$$\begin{aligned} \Delta_{\alpha\beta}(k,\xi) &= \frac{-i}{k^2 - M_Z^2 + i\epsilon}\left[\eta_{\alpha\beta} - (1-\xi)\frac{k_\alpha k_\beta}{k^2 - \xi M_Z^2 + i\epsilon}\right] \\ &= \frac{-i\eta_{\alpha\beta}}{k^2 - M_Z^2 + i\epsilon} + \frac{ik_\alpha k_\beta/M_Z^2}{k^2 - M_Z^2 + i\epsilon} - \frac{ik_\alpha k_\beta/M_Z^2}{k^2 - \xi M_Z^2 + i\epsilon}. \end{aligned} \tag{A.6}$$

For the diagram (a) we can see that the first term of (A.6) gives the same contribution as (3.31). The contribution of the second and third terms are:

$$\begin{aligned} i\mathcal{M}^\mu_{(2.a)} \sim (-ie)&\frac{(-ie^2)}{[2\sin(2\theta_W)]^2}\int\frac{\mathrm{d}^4k}{(2\pi)^4}\bar{u}(p')\not{k}(V+\gamma^5) \\ &\times \frac{(\not{p}' - \not{k} + m)\gamma^\mu(\not{p} - \not{k} + m)\not{k}(V+\gamma^5)}{[(p-k)^2 - m^2 + i\epsilon][(p'-k)^2 - m^2 + i\epsilon]} \\ &\times \left(\frac{-1}{M_Z^2}\right)\left[\frac{1}{k^2 - M_z^2 + i\epsilon} - \frac{1}{k^2 - \xi M_z^2 + i\epsilon}\right]u(p). \end{aligned} \tag{A.7}$$

where $V = 1 - 2\cos 2\theta_W$.

Using the Feynman parametrization (3.17), we can write the last expression as:

$$
\begin{aligned}
i\mathcal{M}^{\mu}_{(2.a)} \sim (-ie)\frac{(-ie^2)}{[2\sin(2\theta_W)]^2}\bar{u}(p')\int_0^1 \mathrm{d}x\mathrm{d}y\mathrm{d}z\, 2!\, \delta(x+y+z-1) \\
\times\left(-\frac{1}{M_Z^2}\right)\int\frac{\mathrm{d}^4 l}{(2\pi)^4}N^{\mu}\left(\frac{1}{(l^2-\Delta_1)^3}-\frac{1}{(l^2-\Delta_2)^3}\right)u(p),
\end{aligned}
\tag{A.8}
$$

where the shift is $l = k - xp - yp'$, and Δ_i with $i = 1, 2$ are:

$$
\begin{aligned}
\Delta_1 &= -xyq^2 + (1-z)^2 m^2 + zM_Z^2, \\
\Delta_2 &= -xyq^2 + (1-z)^2 m^2 + z\xi M_Z^2,
\end{aligned}
\tag{A.9}
$$

and N^{μ} is the numerator:

$$
\begin{aligned}
N^{\mu} &= \not{k}(V+\gamma^5)(\not{p}'-\not{k}+m)\gamma^{\mu}(\not{p}-\not{k}+m)\not{k}(V+\gamma^5) \\
&= V^2\not{k}(\not{p}'-\not{k}+m)\gamma^{\mu}(\not{p}-\not{k}+m)\not{k} \\
&\quad + \not{k}(\not{p}'-\not{k}-m)\gamma^{\mu}(\not{p}-\not{k}-m)\not{k} \\
&= \not{k}(\not{p}'-\not{k}-m)\gamma^{\mu}(\not{p}-\not{k}-m)\not{k} \\
&= (-2m^2)[(1-3z)l^2 + 2m^2(1-z)^3]\left(\frac{i\sigma^{\mu\nu}q_{\nu}}{2m}\right),
\end{aligned}
\tag{A.10}
$$

where we have ignored all terms proportional to $\gamma^{\mu}\gamma^5$, and γ^5, and also used (3.1), (3.4), and (3.24), to simplify (the term proportional to V^2 is the same one that appears in (A.5)).

Now, using (3.26) to do the l integration, we find:

$$
\begin{aligned}
i\mathcal{M}^{\mu}_{(2.a)} \sim (-ie)\,\bar{u}(p')\,\frac{G_F m^2}{8\sqrt{2}\pi^2}\int_0^1 \mathrm{d}x\mathrm{d}y\mathrm{d}z\,\delta(x+y+z-1) \\
\times\left[(1-3z)\log\left(\frac{\Delta_2}{\Delta_1}\right) + m^2(1-z)^3\left(\frac{1}{\Delta_2}-\frac{1}{\Delta_1}\right)\right] \\
\times\left(\frac{i\sigma^{\mu\nu}q_{\nu}}{2m}\right)u(p).
\end{aligned}
\tag{A.11}
$$

For the diagram (b), the propagator of the correspondent Goldstone-boson in the R_{ξ} gauge is:

$$
\Delta(k,\xi) = \frac{i}{k^2 - \xi M_Z^2 + i\epsilon},
\tag{A.12}
$$

so the contribution is the same as (3.40), after substituting Δ_1 by Δ_2:

$$i\mathcal{M}^{\mu}_{(2.b)} \sim (-ie)\,\bar{u}(p')\,\frac{G_F m^2}{8\sqrt{2}\pi^2}\int_0^1 \mathrm{d}x\mathrm{d}y\mathrm{d}z\,\delta(x+y+z-1) \times \left\{\frac{-2m^2(1-z)^2}{\Delta_2}\right\}\left(\frac{i\sigma^{\mu\nu}q_\nu}{2m}\right)u(p). \tag{A.13}$$

Adding up (A.11) and (A.13), with (3.36), and comparing with (3.30), we find that the Z-boson contribution to the form factor $F_2(0)$ in the R_ξ gauge is:

$$\begin{aligned}[F_2(0)]_Z = \frac{G_F m^2}{8\sqrt{2}\pi^2}\int_0^1 & \mathrm{d}x\mathrm{d}y\mathrm{d}z\,\delta(x+y+z-1)\\ &\times\left\{\frac{M_Z^2(V^2 z(1-z)-z(3+z))-m^2(1-z)^3}{\Delta_1}\right.\\ &\left.+\frac{m^2(x+y)^2(x+y-2)}{\Delta_2}+[3(x+y)-2]\log\left(\frac{\Delta_2}{\Delta_1}\right)\right\}.\end{aligned} \tag{A.14}$$

Taking the derivative of the last expression with respect to the gauge parameter ξ, we find:

$$\frac{\mathrm{d}}{\mathrm{d}\xi}[F_2(0)]_Z = \frac{G_F m^2}{8\sqrt{2}\pi^2}M_Z^2\int_0^1 \mathrm{d}x\int_0^{1-x}\mathrm{d}y\,(1-x-y) \times\left\{\frac{3(x+y)-2}{\Delta_2}+\frac{m^2(x+y)^2(2-x-y)}{[\Delta_2]^2}\right\}. \tag{A.15}$$

Performing the following change of variables:

$$\begin{aligned} t &= (x+y) & & x = t(1+s)/2\\ s &= (x-y)/t & \Leftrightarrow\ & y = t(1-s)/2,\end{aligned} \tag{A.16}$$

we arrive at:

$$\begin{aligned}\frac{\mathrm{d}}{\mathrm{d}\xi}[F_2(0)]_Z &= \frac{G_F m^2}{8\sqrt{2}\pi^2}M_Z^2\int_0^1 \mathrm{d}t\left\{(3t^2-2t)\cdot\frac{(1-t)}{t^2m^2+(1-t)\xi M_Z^2}\right.\\ &\qquad\left.+(t^3-t^2)\cdot\frac{m^2(t^2-2t)}{[t^2m^2+(1-t)\xi M_Z^2]^2}\right\}\\ &= \frac{G_F m^2}{8\sqrt{2}\pi^2}M_Z^2\left[(t^3-t^2)\cdot\frac{(1-t)}{t^2m^2+(1-t)\xi M_Z^2}\right]\Bigg|_0^1,\end{aligned} \tag{A.17}$$

where we have used the product rule in the last step. The above result shows that the Z-boson contribution is indeed gauge invariant.

W-Boson Contribution:

The W-boson contribution corresponds to the diagrams of the Figure 3.3. In the R_ξ gauge, the W-boson propagator is the same as (A.6), after we substitute M_Z by M_W, so the diagram (a) reads:

$$\begin{aligned} i\mathcal{M}^{\mu}_{(3.a)} = &\int \frac{\mathrm{d}^4k}{(2\pi)^4}\bar{u}(p')\left[\left(\frac{-ig}{2\sqrt{2}}\right)\gamma^{\alpha}(\mathbb{1}-\gamma^5)\right]\Delta_{\alpha\nu}(p'-k,\xi) \\ &\times (ie)\left[\eta^{\nu\lambda}(p+p'-2k)^{\mu}+\eta^{\lambda\mu}(k+p'-2p)^{\nu}+\eta^{\mu\nu}(k+p-2p')^{\lambda}\right] \\ &\times \Delta_{\lambda\beta}(p-k,\xi)\left[\frac{i\not{k}}{k^2+i\epsilon}\right]\left[\left(\frac{-ig}{2\sqrt{2}}\right)\gamma^{\beta}(\mathbb{1}-\gamma^5)\right]u(p). \end{aligned} \tag{A.18}$$

Using the Feynman parametrization (3.17), we rewrite the last expression as:

$$\begin{aligned} i\mathcal{M}^{\mu}_{(3.a)} = &(-ie)\left(\frac{-ig^2}{8}\right)\bar{u}(p')\int_0^1 \mathrm{d}x\mathrm{d}y\mathrm{d}z\, 2!\,\delta(x+y+z-1) \\ &\times\int\frac{\mathrm{d}^4l}{(2\pi)^4}\left[\eta^{\nu\lambda}(p+p'-2k)^{\mu}+\eta^{\lambda\mu}(k+p'-2p)^{\nu}+\eta^{\mu\nu}(k+p-2p')^{\lambda}\right] \\ &\times 2(\mathbb{1}+\gamma^5)\gamma^{\alpha}\not{k}\gamma^{\beta}\Bigg\{\frac{\eta_{\alpha\nu}\eta_{\lambda\beta}}{(l^2-\Delta_1)^3} \\ &-\frac{\eta_{\alpha\nu}}{M_W^2}(p-k)_{\lambda}(p-k)_{\beta}\left(\frac{1}{(l^2-\Delta_1)^3}-\frac{1}{(l^2-\Delta_2)^3}\right) \\ &-\frac{\eta_{\lambda\beta}}{M_W^2}(p'-k)_{\alpha}(p'-k)_{\nu}\left(\frac{1}{(l^2-\Delta_1)^3}-\frac{1}{(l^2-\Delta_3)^3}\right) \\ &+\frac{1}{M_W^4}(p-k)_{\lambda}(p-k)_{\beta}(p'-k)_{\alpha}(p'-k)_{\nu} \\ &\times\left(\frac{1}{(l^2-\Delta_1)^3}-\frac{1}{(l^2-\Delta_2)^3}-\frac{1}{(l^2-\Delta_3)^3}+\frac{1}{(l^2-\Delta_4)^3}\right)\Bigg\}u(p), \end{aligned} \tag{A.19}$$

where the shift is $l = k - xp - yp'$, and Δ_i with $i = 1,2,3,4$ are:

$$\begin{aligned} \Delta_1 &= -xyq^2 + M_Z^2(x+y) \;\; - m^2(x+y)(1-x-y), \\ \Delta_2 &= -xyq^2 + M_Z^2(\xi x+y) - m^2(x+y)(1-x-y), \\ \Delta_3 &= -xyq^2 + M_Z^2(x+\xi y) - m^2(x+y)(1-x-y), \\ \Delta_4 &= -xyq^2 + \xi M_Z^2(x+y) - m^2(x+y)(1-x-y). \end{aligned} \tag{A.20}$$

The term proportional to $\eta_{\alpha\nu}\eta_{\lambda\beta}$ is the same as (3.48). The term proportional to $\eta_{\alpha\nu}$ combined with the term proportional to $\eta_{\lambda\beta}$, after the l integration, gives:

$$
\begin{aligned}
i\mathcal{M}^{\mu}_{(3.a)} \sim (-ie)\,\bar{u}(p')\,\frac{G_F m^2}{8\sqrt{2}\pi^2}\int_0^1 \mathrm{d}x\mathrm{d}y\mathrm{d}z\,\delta(x+y+z-1)\Big\{2[4(x+y)^2 \\
-9(x+y)+2+6y]\log\left(\frac{\Delta_1}{\Delta_2}\right)+2m^2(x+y-1) \\
\times[2y+(x+y-1)(x+y)]\left(\frac{1}{\Delta_1}-\frac{1}{\Delta_2}\right)\Big\}\left(\frac{i\sigma^{\mu\nu}q_\nu}{2m}\right)u(p),
\end{aligned}
\tag{A.21}
$$

and the last term of (A.19) yields zero after the l integration.

Now, for the diagram (b), we have:

$$
\begin{aligned}
i\mathcal{M}^{\mu}_{(3.b)} = \int\frac{\mathrm{d}^4k}{(2\pi)^4}\bar{u}(p')\left[\left(\frac{-ig}{2\sqrt{2}}\right)\frac{m}{M_W}(\mathbb{1}-\gamma^5)\right]\left[\frac{i}{(p'-k)^2-M_W^2+i\epsilon}\right] \\
\times\left(-ieM_W\eta^{\mu\lambda}\right)\Delta_{\lambda\beta}(p-k,\xi)\left[\frac{i\not{k}}{k^2+i\epsilon}\right]\left[\left(\frac{-ig}{2\sqrt{2}}\right)\gamma^\beta(\mathbb{1}-\gamma^5)\right]u(p),
\end{aligned}
\tag{A.22}
$$

and for the diagram (c):

$$
\begin{aligned}
i\mathcal{M}^{\mu}_{(3.c)} = \int\frac{\mathrm{d}^4k}{(2\pi)^4}\bar{u}(p')\left[\left(\frac{-ig}{2\sqrt{2}}\right)\gamma^\beta(\mathbb{1}-\gamma^5)\right]\Delta_{\lambda\beta}(p'-k,\xi)\left(M_W\eta^{\mu\lambda}\right) \\
\times(-ie)\left[\frac{i}{(p-k)^2-M_W^2+i\epsilon}\right]\left[\frac{i\not{k}}{k^2+i\epsilon}\right]\left[\left(\frac{-ig}{2\sqrt{2}}\right)\frac{m}{M_W}(\mathbb{1}+\gamma^5)\right]u(p).
\end{aligned}
\tag{A.23}
$$

Combining both diagrams, using (3.17), and doing the l integration, we obtain:

$$
\begin{aligned}
i\left(\mathcal{M}^{\mu}_{(3.b)}+\mathcal{M}^{\mu}_{(3.c)}\right) \sim (-ie)\,\bar{u}(p')\,\frac{G_F m^2}{8\sqrt{2}\pi^2}\int_0^1 \mathrm{d}x\mathrm{d}y\mathrm{d}z\,\delta(x+y+z-1) \\
\times\Bigg\{\frac{4M_W^2 y}{\Delta_2}+2[3(x+y)-2]\log\left(\frac{\Delta_4}{\Delta_2}\right) \\
+2m^2(x+y-1)^2(x+y)\left(\frac{1}{\Delta_4}-\frac{1}{\Delta_2}\right)\Bigg\}\left(\frac{i\sigma^{\mu\nu}q_\nu}{2m}\right)u(p).
\end{aligned}
\tag{A.24}
$$

For the diagram (d) we have the same contribution as (3.58), after we substitute the denominator by Δ_4:

$$i\mathcal{M}^{\mu}_{(3.d)} \sim (-ie)\,\bar{u}(p')\,\frac{G_F m^2}{8\sqrt{2}\pi^2}\int_0^1 \mathrm{d}x\mathrm{d}y\mathrm{d}z\;\delta(x+y+z-1) \times \left\{\frac{-2m^2 z(1-z)}{\Delta_4}\right\}\left(\frac{i\sigma^{\mu\nu}q_\nu}{2m}\right)u(p). \tag{A.25}$$

Combining (3.48), (A.21), (A.24), and (A.25), and comparing with (3.30), we find that the W-boson contribution to the form factor $F_2(0)$ in the R_ξ gauge is:

$$\begin{aligned}[F_2(0)]_W = \frac{G_F m^2}{8\sqrt{2}\pi^2}&\int_0^1 \mathrm{d}x\mathrm{d}y\mathrm{d}z\;\delta(x+y+z-1)\\ \times&\left\{2M_W^2(x+y)[2(x+y)+1]\left(\frac{1}{\Delta_1}\right)\right.\\ &+2[4(x+y)^2-9(x+y)+2+6y]\log\left(\frac{\Delta_1}{\Delta_2}\right)\\ &+2m^2(x+y-1)[2y+(x+y-1)(x+y)]\left(\frac{1}{\Delta_1}-\frac{1}{\Delta_2}\right)\\ &+\frac{4M_W^2 y}{\Delta_2}-\frac{2m^2 z(1-z)}{\Delta_4}+2[3(x+y)-2]\log\left(\frac{\Delta_4}{\Delta_2}\right)\\ &\left.+2m^2(x+y-1)^2(x+y)\left(\frac{1}{\Delta_4}-\frac{1}{\Delta_2}\right)\right\}.\end{aligned} \tag{A.26}$$

Taking the derivative of the last expression with respect to the gauge parameter ξ, we find:

$$\begin{aligned}\frac{\mathrm{d}}{\mathrm{d}\xi}[F_2(0)]_W = \frac{G_F m^2}{8\sqrt{2}\pi^2}(2M_W^2)\int_0^1\mathrm{d}x\int_0^{1-x}\mathrm{d}y&\\ \times\left\{(x+y)\left[\frac{3(x+y)-2}{\Delta_4}+\frac{m^2(x+y)^2(1-x-y)}{[\Delta_4]^2}\right]\right.&\\ \left.-x\left[\frac{4(x+y)^2-6x}{\Delta_2}+\frac{2M_W^2 y-m^2(x+y-1)^2[2y+(x+y)^2]}{[\Delta_2]^2}\right]\right\},&\end{aligned} \tag{A.27}$$

that with the change of variables (A.16), becomes:

$$\begin{aligned}\frac{\mathrm{d}}{\mathrm{d}\xi}[F_2(0)]_W = \frac{G_F m^2}{8\sqrt{2}\pi^2}M_W^2\int_0^1\mathrm{d}t\int_{-1}^1\mathrm{d}s\left\{(3t^2-2t)\frac{t}{\Delta_4}+(t^3-t^2)\frac{(-m^2 t)}{[\Delta_4]^2}\right.&\\ \left.-\frac{t^2}{2}(1+s)\left[\frac{4t^2-3t(1+s)}{\Delta_2}+\frac{2M_W^2 t(1-s)-m^2(t-1)^2[t^2+t(1-s)]}{[\Delta_2]^2}\right]\right\}.&\end{aligned} \tag{A.28}$$

The integral that involves Δ_4 can be done in the same way as (A.17), so we are only left with:

$$\frac{\mathrm{d}}{\mathrm{d}\xi}\left[F_2(0)\right]_W = \frac{G_F m^2}{8\sqrt{2}\pi^2}\int_0^1 \mathrm{d}t \int_{-1}^1 \mathrm{d}s \left(\frac{-M_W^2}{2}\right) t^2(1+s)\left\{\frac{[4t^2-3t(1+s)]}{\Delta_2} + \frac{M_W^2 t(1-s) - m^2(t-1)^2[t^2+t(1-s)]}{[\Delta_2]^2}\right\}. \tag{A.29}$$

Integrating in the s variable first, we find:

$$\frac{\mathrm{d}}{\mathrm{d}\xi}\left[F_2(0)\right]_W = \frac{G_F m^2}{8\sqrt{2}\pi^2}\int_0^1 \mathrm{d}t\, \frac{t}{2(\xi-1)^3 M_W^4}\left\{M_W^4 f_1 + M_W^2 m^2 f_2 + \left[M_W^4 f_3 + M_W^2 m^2 f_4 + m^4 f_5\right]\log(f_6) + \frac{f_7\cdot f_8}{m^2\, f_6}\right\}\Bigg|_{-1}^{+1}, \tag{A.30}$$

where f_i with $i = 1, ..., 8$ are:

$$\begin{aligned}
f_1 &= s(\xi-1)(3\xi(s+2)t - 3(s+6)t - 8(\xi-1)t^2 + 4),\\
f_2 &= 4s(\xi-1)(1-t)(4t-1),\\
f_3 &= 8(2(\xi-1)t^2 + 3t - (\xi+1)),\\
f_4 &= 4(t-1)(5(\xi-1)t^2 + (\xi+15)t - 2(\xi+3)),\\
f_5 &= 8(t-1)^2(5t-2),\\
f_6 &= (\xi+1) + (\xi-1)s + 2(m/M_W)^2(t-1),\\
f_7 &= 8(M_W^2 + m^2(t-1)),\\
f_8 &= -2\xi M_W^4 + M_W^2 m^2 (t-1)((\xi-1)t^2 + (\xi+1)t - 2(\xi+1)) + 2m^4(t-1)^3.
\end{aligned} \tag{A.31}$$

After substituting the limits of integration and performing the integral in the t variable, we obtain:

$$\frac{\mathrm{d}}{\mathrm{d}\xi}\left[F_2(0)\right]_W = \frac{G_F m^2}{8\sqrt{2}\pi^2}\left[\frac{2(t-1)t^2}{(\xi-1)^3 M_W^4}\left\{M_W^4 f_9 + M_W^2 m^2 f_{10} + f_{11}\log\left(\frac{m^2(t-1)+\xi M_W^2}{m^2(t-1)+M_W^2}\right)\right\}\right]\Bigg|_0^1, \tag{A.32}$$

where f_9, f_{10}, and f_{11} are:

$$\begin{aligned} f_9 &= (\xi-1)((\xi-1)t+2), \\ f_{10} &= 2(1-\xi)(t-1), \\ f_{11} &= (M_W^2+m^2(t-1))(M_W^2(\xi+(\xi-1)t+1)+2m^2(t-1)). \end{aligned} \tag{A.33}$$

Finally, after substituting the limits of integration, we see that (A.32) yields zero, confirming that the W-boson contribution is gauge invariant.

This appendix was inspired by the great work of Fujikawa, Lee, and Sanda [45].

Appendix B

Muon Decay

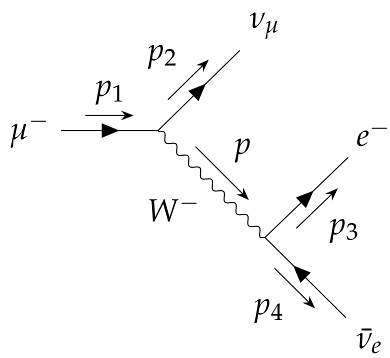


Figure B.1: Tree level diagram of the muon decay. All momenta are indicated in the figure.

The muon decay is characterized by the diagram of Figure B.1. The amplitude for this process can be read from (2.12):

$$\begin{aligned}\mathcal{M} &= \bar{u}(p_2)\left(\frac{-ig}{\sqrt{2}}\gamma^\mu\left(\mathbb{1}-\gamma^5\right)\right)u(p_1)\left(\frac{-i}{p^2-M_W^2+i\epsilon}\right)\left(\eta_{\mu\nu}-\frac{p_\mu p_\nu}{M_W^2}\right)\\ &\quad\times\bar{u}(p_3)\left(\frac{-ig}{\sqrt{2}}\gamma^\nu\left(\mathbb{1}-\gamma^5\right)\right)v(p_4)\\ &= \left(\frac{ig^2}{8\left(p^2-M_W^2\right)}\right)\Big\{\bar{u}(p_2)\left[\gamma_\mu\left(\mathbb{1}-\gamma^5\right)\right]u(p_1)\bar{u}(p_3)\left[\gamma^\mu\left(\mathbb{1}-\gamma^5\right)\right]v(p_4)\\ &\qquad -\left(\frac{m_e m}{M_W^2}\right)\bar{u}(p_2)\left(\mathbb{1}-\gamma^5\right)u(p_1)\bar{u}(p_3)\left(\mathbb{1}-\gamma^5\right)v(p_4)\Big\}\\ &\simeq \left(\frac{-ig^2}{8M_W^2}\right)\bar{u}(p_2)\left[\gamma_\mu\left(\mathbb{1}-\gamma^5\right)\right]u(p_1)\bar{u}(p_3)\left[\gamma^\mu\left(\mathbb{1}-\gamma^5\right)\right]v(p_4),\end{aligned} \tag{B.1}$$

where m is the muon mass, and m_e is the electron mass.

We also have considered $p^2 = (p_1 - p_2)^2 \sim \mathcal{O}(m^2) \ll M_W^2$ and ignored the second term because $\left(m_e m / M_W^2\right) \simeq 10^{-8}$.

Now, with those considerations, the square amplitude becomes:

$$\|\mathcal{M}\|^2 \simeq \left(\frac{g^2}{8M_W^2}\right)^2 Tr\left\{v(p_4)\bar{v}(p_4)\gamma_\nu\left(\mathbb{1}-\gamma^5\right)u(p_3)\bar{u}(p_3)\gamma_\mu\left(\mathbb{1}-\gamma^5\right)\right\} \\ \times Tr\left\{u(p_2)\bar{u}(p_2)\gamma^\mu\left(\mathbb{1}-\gamma^5\right)u(p_1)\bar{u}(p_1)\gamma^\nu\left(\mathbb{1}-\gamma^5\right)\right\}. \tag{B.2}$$

Using the following spinor identity [46]:

$$u(k)\bar{u}(k) = (\not{k} + m)\frac{\mathbb{1} + \gamma^5 \not{s}}{2}, \tag{B.3}$$

for the muon, where s^α is the spin direction, we find, after summing over the final spin states, that the square amplitude is:

$$\|\mathcal{M}\|^2 \simeq 2\left(\frac{g}{M_W}\right)^4 ((p_1 - ms_\mu)\cdot p_4)(p_2 \cdot p_3), \tag{B.4}$$

where the index μ refers to the muon, i.e., it is not a Lorentz index.

For an n-particle decay $A \longrightarrow a_1 + ... + a_n$, the differential decay rate is given by:

$$\mathrm{d}\Gamma = \frac{S}{2E}\|\mathcal{M}\|^2 \prod_{i=1}^{n} \frac{\mathrm{d}^3 p_i}{(2\pi)^3 2E_i}(2\pi)^4 \delta^4\left(P - \sum_{i=1}^{n} p_i\right), \tag{B.5}$$

where E and P are the mother particle energy and 4-momentum, respectively, and S is the symmetry factor. Specializing for our case, we have:

$$\mathrm{d}\Gamma = \frac{1}{2E_1}\|\mathcal{M}\|^2 \frac{\mathrm{d}^3 p_2}{(2\pi)^3 2E_2}\frac{\mathrm{d}^3 p_3}{(2\pi)^3 2E_3}\frac{\mathrm{d}^3 p_4}{(2\pi)^3 2E_4}(2\pi)^4\delta^4(p_1 - p_2 - p_3 - p_4), \tag{B.6}$$

and since the neutrino ν_μ and anti-neutrino $\bar{\nu}_e$ momenta are not measured, we integrate over p_2 and p_4:

$$\mathrm{d}\Gamma = \frac{\mathrm{d}^3 p_3}{2E_1 E_3 (2\pi)^5}\left(\frac{g}{M_W}\right)^4 (p_1 - ms_\mu)_\alpha (p_3)_\beta \\ \times \int \frac{\mathrm{d}^3 p_2}{2E_2}\frac{\mathrm{d}^3 p_4}{2E_4}\delta^4(p_1 - p_2 - p_3 - p_4) p_4^\alpha p_2^\beta. \tag{B.7}$$

Using the following integral in phase space:

$$\int \frac{d^3p}{2p_0}\frac{d^3q}{2q_0}\delta^4(k-p-q)p_\alpha q_\beta = \frac{\pi}{24k^6}\lambda^{1/2}(k^2,p^2,q^2)\Big\{k^2\lambda(k^2,p^2,q^2)\eta_{\alpha\beta} + 2[k^4+k^2(p^2+q^2)-2(p^2-q^2)^2]k_\alpha k_\beta\Big\}, \tag{B.8}$$

where $\lambda(x,y,z) \equiv x^2+y^2+z^2-2xy-2xz-2yz$, we find that:

$$d\Gamma = \left(\frac{2g^4 d^3p_3}{192E_1 M_W^4(2\pi)^4 E_3}\right)\Big\{(p_1-p_3)^2(p_1-ms_\mu)\cdot p_3 + 2(p_1-p_3)\cdot(p_1-ms_\mu)(p_1-p_3)\cdot p_3\Big\}. \tag{B.9}$$

Choosing a coordinate system in the rest frame of the muon, where the angle between the muon spin and the electron momentum is θ, we arrive at:

$$\begin{aligned} d\Gamma &= \left(\frac{2g^4 E_3 dE_3 d\Omega}{192mM_W^4(2\pi)^4}\right)\Big\{(m^2-2mE_3)(mE_3+mE_3\cos\theta) \\ &\qquad + 2(m^2-mE_3-mE_3\cos\theta)(mE_3)\Big\} \\ &= \left(\frac{2g^4 E_3 dE_3 d\Omega}{192mM_W^4(2\pi)^4}\right)\Big\{3m^3E_3-4m^2E_3^2+m^3E_3\cos\theta-4m^2E_3^2\cos\theta\Big\}, \end{aligned} \tag{B.10}$$

where we have ignored the electron mass ($m_e \ll m$).

Finally,

$$\frac{d^2\Gamma}{dx_e d\cos\theta} = \frac{G_F^2 m^5}{192\pi^3}x_e^2\left[(3-2x_e)-(2x_e-1)\cos\theta\right], \tag{B.11}$$

where $x_e = E_3/W_e$ is the reduced energy, $W_e \simeq m/2$ is the maximum value for the electron energy in the muon rest frame, and the coefficient in front is the total decay rate.

A more general calculation, where one also considers the mass and spin direction of the electron, can be found in [47].

Appendix C

Spin Dynamics

The relation between the rate of change of some vector quantity in an inertial frame and a rotating (non-inertial) frame is:

$$\left(\frac{\mathrm{d}\vec{S}}{\mathrm{d}t}\right) = \left(\frac{\mathrm{d}\vec{S}}{\mathrm{d}t}\right)_R + \vec{\omega}_T \times \vec{S}, \tag{C.1}$$

where the subscript R refers to a rotating frame, i.e., the muon frame, $\vec{S}$ is the muon spin, and t is the time as measured by the inertial frame, i.e., the laboratory frame. The vector $\vec{\omega}_T$ is known as the Thomas precession frequency [34]:

$$\vec{\omega}_T = \left(\frac{\gamma^2}{\gamma + 1}\right) \vec{a} \times \vec{v}, \tag{C.2}$$

where γ is the Lorentz factor, $\vec{v}$ is the muon velocity, and $\vec{a}$ is the muon acceleration, as observed in the laboratory frame.

From the Minkowski force:

$$\frac{\mathrm{d}U^\mu}{\mathrm{d}\tau} = \frac{e}{m} F^{\mu\nu} U_\nu, \tag{C.3}$$

we find that:

$$\frac{\mathrm{d}\vec{v}}{\mathrm{d}t} = \vec{a} = \frac{e}{\gamma m}\left(\vec{E} + \vec{v} \times \vec{B} - (\vec{v} \cdot \vec{E})\vec{v}\right), \tag{C.4}$$

where U^μ is the 4-velocity, $F^{\mu\nu}$ is the electromagnetic tensor, and e and m are the particle's charge and mass, respectively.

Now, in the muon rest frame, we have:

$$\left(\frac{\mathrm{d}\vec{S}}{\mathrm{d}t}\right)_R = \frac{1}{\gamma}\left(\frac{\mathrm{d}\vec{S}}{\mathrm{d}\tau}\right)_R = \frac{1}{\gamma}\, g \frac{e}{2m} \vec{S} \times \vec{B}_R = \frac{1}{\gamma}\, \vec{\mu} \times \vec{B}_R, \tag{C.5}$$

where B_R is the magnetic field, as perceived by the muon. In the laboratory, we have a magnetic field $\vec{B}$ and an electric field $\vec{E}$. These quantities are related by a Lorentz transformation:

$$\vec{B}_R = \gamma(\vec{B} + \vec{E} \times \vec{v}) - \frac{\gamma^2}{\gamma+1}(\vec{v}\cdot\vec{B})\vec{v}. \tag{C.6}$$

Combining everything into (C.1), we arrived at:

$$\begin{aligned}\left(\frac{\mathrm{d}\vec{S}}{\mathrm{d}t}\right) &= \frac{1}{\gamma}\, g\frac{e}{2m}\vec{S}\times\vec{B}_R + \vec{S}\times\left(\frac{\gamma^2}{\gamma+1}\right)\vec{v}\times\vec{a} \\ &= g\frac{e}{2m}\vec{S}\times\left\{\vec{B} + \vec{E}\times\vec{v} - \frac{\gamma}{\gamma+1}(\vec{v}\cdot\vec{B})\vec{v}\right\} \\ &\quad + \vec{S}\times\left\{\frac{\gamma}{\gamma+1}\vec{v}\times\frac{e}{m}\left(\vec{E} + \vec{v}\times\vec{B} - (\vec{v}\cdot\vec{E})\vec{v}\right)\right\}.\end{aligned} \tag{C.7}$$

Finally, using $a = \frac{g}{2} - 1$, we find:

$$\left(\frac{\mathrm{d}\vec{S}}{\mathrm{d}t}\right) = \frac{e}{m}\vec{S}\times\left\{\left(a+\frac{1}{\gamma}\right)\vec{B} - a\left(\frac{\gamma}{\gamma+1}\right)\left(\vec{v}\cdot\vec{B}\right)\vec{v} + \left(a+\frac{1}{\gamma+1}\right)\vec{E}\times\vec{v}\right\}, \tag{C.8}$$

which is the frequency used in (4.6).

Bibliography

[1] A. Pich. The standard model of electroweak interactions. *arXiv preprint arXiv:1201.0537*, 2012.

[2] T. P. Cheng and L. F. Li. *Gauge theory of elementary particle physics*. Oxford university press, 1994.

[3] M. Kobayashi and T. Maskawa. Cp-violation in the renormalizable theory of weak interaction. *Progress of theoretical physics*, 49(2):652–657, 1973.

[4] L. Roszkowski, E. M. Sessolo, and S. Trojanowski. Wimp dark matter candidates and searchescurrent status and future prospects. *Reports on Progress in Physics*, 81(6):066201, 2018.

[5] D. J. E. M. Marsh. Axions and alps: a very short introduction, 2017.

[6] J. L. Feng. Dark matter candidates from particle physics and methods of detection. *Annual Review of Astronomy and Astrophysics*, 48:495–545, 2010.

[7] S. M. Carroll. The cosmological constant. *Living reviews in relativity*, 4(1):1–56, 2001.

[8] A. D. Sakharov. Violation of CP Invariance, C asymmetry, and baryon asymmetry of the universe. *Pisma Zh. Eksp. Teor. Fiz.*, 5:32–35, 1967.

[9] A. Riotto and M. Trodden. Recent progress in baryogenesis. *Annual Review of Nuclear and Particle Science*, page 3575, Dec 1999.

[10] E. W. Kolb and M. S Turner. Grand unified theories and the origin of the baryon asymmetry. *Annual Review of Nuclear and Particle Science*, 33(1):645–696, 1983.

[11] A. Bellerive, J. R. Klein, A. B. McDonald, A. J. Noble, A. W. P. Poon, SNO Collaboration, et al. The sudbury neutrino observatory. *Nuclear Physics B*, 908:30–51, 2016.

[12] T. Albahri, A. Anastasi, et al. Measurement of the anomalous precession frequency of the muon in the fermilab muon g- 2 experiment. *Physical Review D*, 103(7):072002, 2021.

[13] D. J. Griffiths. *Introduction to quantum mechanics*. Pearson International Edition (Pearson Prentice Hall, Upper Saddle River, 2005), 1962.

[14] C. Itzykson and J. B. Zuber. *Quantum field theory*. Courier Corporation, 2012.

[15] M. D. Schwartz. *Quantum field theory and the standard model*. Cambridge University Press, 2014.

[16] M. E. Peskin. *An introduction to quantum field theory*. CRC press, 2018.

[17] J. J. Sakurai and J. Napolitano. *Modern Quantum mechanics, 2nd edition*. Cambridge University Press, 2011.

[18] J. C. Romão and J. P. Silva. A resource for signs and feynman diagrams of the standard model. *International Journal of Modern Physics A*, 27(26):1230025, 2012.

[19] J. Schwinger. On quantum-electrodynamics and the magnetic moment of the electron. *Physical Review*, 73(4):416, 1948.

[20] M. Nowakowski, E.A. Paschos, and J.M. Rodriguez. All electromagnetic form factors. *European journal of physics*, 26(4):545, 2005.

[21] R. Jackiw and S. Weinberg. Weak-interaction corrections to the muon magnetic moment and to muonic-atom energy levels. *Physical Review D*, 5(9):2396, 1972.

[22] Particle Data Group, R. L. Workman, et al. Review of particle physics. *Progress of Theoretical and Experimental Physics*, 2022(8):083C01, 2022.

[23] T. Aoyama et al. The anomalous magnetic moment of the muon in the standard model. *Physics reports*, 887:1–166, 2020.

[24] R. H. Parker, C. Yu, W. Zhong, B. Estey, and H. Müller. Measurement of the fine-structure constant as a test of the standard model. *Science*, 360(6385):191–195, 2018.

[25] C. Gnendiger, D. Stöckinger, and H. Stöckinger-Kim. The electroweak contributions to (g- 2) μ after the higgs-boson mass measurement. *Physical Review D*, 88(5):053005, 2013.

[26] G. Degrassi and G. F. Giudice. Qed logarithms in the electroweak corrections to the muon anomalous magnetic moment. *Physical Review D*, 58(5):053007, 1998.

[27] T. D. Lee and C. N. Yang. Question of parity conservation in weak interactions. *Physical Review*, 104:254, 1956.

[28] A. K. Wróblewski. The downfall of parity–the revolution that happened fifty years ago. *Acta Physica Polonica B*, 39(2), 2008.

[29] C. S. Wu, E. Ambler, R. W. Hayward, D. D. Hoppes, and R. P. Hudson. Experimental test of parity conservation in beta decay. *Physical Review*, 105(4):1413, 1957.

[30] M. Hargittai. Credit where credit's due? *Physics World*, 25(09):38, 2012.

[31] G. W. Bennett, B. Bousquet, et al. Final report of the e821 muon anomalous magnetic moment measurement at bnl. *Physical Review D*, 73(7):072003, 2006.

[32] F. Combley, F. J. M. Farley, and E. Picasso. The cern muon (g-2) experiments. *Physics Reports*, 68(2):93–119, 1981.

[33] F. Jegerlehner. *The anomalous magnetic moment of the muon*. Springer, 2008.

[34] L. H. Thomas. I. the kinematics of an electron with an axis. *The London, Edinburgh, and Dublin Philosophical Magazine and Journal of Science*, 3(13):1–22, 1927.

[35] B. Abi et al. Measurement of the positive muon anomalous magnetic moment to 0.46 ppm. *Physical Review Letters*, 126(14):141801, 2021.

[36] F. J. N. Farley. Pitch correction in (g-2) experiments. *Physics Letters B*, 42(1):66–68, 1972.

[37] J. Bailey et al. Final report on the cern muon storage ring including the anomalous magnetic moment and the electric dipole moment of the muon, and a direct test of relativistic time dilation. *Nuclear Physics B*, 150:1–75, 1979.

[38] X. Fei, V. W. Hughes, and R. Prigl. Precision measurement of the magnetic field in terms of the free-proton nmr frequency. *Nuclear Instruments and Methods in Physics Research Section A: Accelerators, Spectrometers, Detectors and Associated Equipment*, 394(3):349–356, 1997.

[39] V. W. Hughes and E. P. Sichtermann. The anomalous magnetic moment of the muon. *International Journal of Modern Physics A*, 18(supp01):215–272, 2003.

[40] V. Pleitez and M.D. Tonasse. Heavy charged leptons in an su (3) l u (1) n model. *Physical Review D*, 48(5):2353, 1993.

[41] F. Pisano and V. Pleitez. Su (3) u (1) model for electroweak interactions. *Physical Review D*, 46(1):410, 1992.

[42] A. A. Nepomuceno and B. Meirose. Limits on 331 vector bosons from lhc proton collision data. *Physical Review D*, 101(3):035017, 2020.

[43] J. P. Leveille. The second-order weak correction to (g-2) of the muon in arbitrary gauge models. *Nuclear Physics B*, 137(1-2):63–76, 1978.

[44] S. R. Moore, K. Whisnant, and Bing-Lin Y. Second-order corrections to the muon anomalous magnetic moment in alternative electroweak models. *Physical Review D*, 31(1):105, 1985.

[45] K. Fujikawa, B. W. Lee, and A. I. Sanda. Generalized renormalizable gauge formulation of spontaneously broken gauge theories. *Physical Review D*, 6(10):2923, 1972.

[46] W. Greiner et al. *Relativistic quantum mechanics*, volume 2. Springer, 2000.

[47] W. Greiner, B. Müller, et al. *Gauge theory of weak interactions*, volume 5. Springer, 1996.