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Painlevé Integrability and mixed P_{III}-P_V system solutions

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Resumo

O presente trabalho trata de um abordagem de aplicações em física dos métodos matemáticos de integrabilidade de Painlevé, por outro lado também aborda o formalismo de hierarquias integráveis e o modelo de 2M-bosons onde são usados métodos de equações diferenciais bem como um método para soluções usando aproximantes de Padé.

Palavras Chaves: Integrabilidade clássica; Equações de Painlevé; Modelo de 2M-Bosons

Áreas do conhecimento: Ciências Exatas e da Terra; Física Teórica; Física Matemática

Abstract

The current work aims at applications of mathematical methods of Painlevé integrability in physics, on the other side it also approaches the integrable hierarchies formalism and the 2M-bose model where differential equations methods are used as well as a method for solutions using Padé approximants.

Key words: Classical Integrability; Painlevé equations; 2M-Bose Model

Areas: Natural Sciences; Theoretical Physics; Mathematical Physics

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Chapter 1

Introduction

Differential equations play a central role in theoretical physics as well as in the development of pure mathematics. By the beginning of the XX century, some group of mathematicians started to pay attention on the study of singularities of equations to be able to define new functions. This idea came primarily due to the Jacobi elliptic functions that rang a bell on this type of structure to be a good way to work.

With that, Painlevé and his colleagues discovered 6 new functions of second order and first degree, future work by many other mathematicians unveiled many more for higher degrees an orders, but this is not the most remarkable feature of this work.

These equations and their singularity structure have showed themselves to be of extreme utility on solutions for differential equations that appear in physics due to singlevaluedness, which were latter called "Painlevé Integrability". Such feature makes Painlevé equations very common to appear. Even when we are dealing with partial differential equations, we are able to find, by some reduction, the Painlevé property, this is called the "ARS-Conjecture".

In this work we were able to find this property in a very devious situation. By starting with a model that arose from the truncation of one of the most general integrable hierarchies known, the "KP-hierarchy", we could build up a very general structure, such general structure is called "2M-Bosons" model (also called sometimes, "2M-Bose" model) and is able to describe a generalized Volterra Lattice, which in turn is able to be reduced to the Volterra lattice, as well as the Toda lattice.

In the study of the 2-boson system, the P_{IV} equation was found to describe it [1], for the 4-boson system, both P_{III} and P_V equations were found and also a new equation which is a mixture of both [2].

In this work, we present all methods for the complete construction of the 2M-boson model and also treat in more detail the 4-boson model; as it was a pioneer on this mixed P_{III}-P_V equation we went for its solution. To be able to achieve it we used techniques all presented in the second chapter of this work.

The Painlevé integrability and its tools were of indispensable use to treat this new equation as well as its solutions. The study of the Painlevé integrability, as well as the Painlevé equations in general, is very advantageous due to its ubiquitous presence in physics, as a quick list: the Ising and antiferromagnet models [3], the transport of particles across boundaries (Nernst-Planck equations) [4], dilute Bose-Einstein condensates in an external one dimensional field (Gross-Pitaevskii equation) [5], Hele-Shaw problems in viscous fluids [6] as well as various approaches to quantum field theory and topological field theory, supersymmetric gauge theories, random matrix theory, statistical mechanics, plasma physics, superconductivity, nonlinear optics and fiber optics, resonant oscillations in shallow water, and polymers. The $A_3^{(1)}$ Painlevé system, corresponding to the P_V equation, appears in the context of the reduced density matrix of the impenetrable Bose gas model [7] and in connection of a unitary matrix model formulation of chiral Yang-Mills theory on the sphere [8]. The third order Chazy equation appears in the self-dual Yang-Mills equations [9], the Mixmaster universe in general relativity [10], and Prandtl boundary-layer equations [11].

This work does not pretend to be a manual or a catalog for its covered topics, rather being a general covering, elucidating introduction to non-experts and also a source to more deep topics related to its subjects. This is due to the huge amount of things that coud be written about the subjects here, but it would be unworthy if they are too advanced. By the end of each section there will be an icon, ☞, with its respective references, listed in the bibliography.

Chapter 2

Painlevé Integrability

In the end of the XIX century mathematicians were concerned about how the field of differential equations would evolve, the old way of dealing with differential equations were starting to be not as strong as it appeared to be in the beginning, then a common thought grown at the time, instead of dealing with differential equations always looking for their absolute general solution as a combination of special functions, it would be much more useful (and general) to study their solutions as just the behavior in the complex (or real) plane, it was known as the qualitative theory of differential equations.

2.1 Nonlinear differential equations

This thought is even stronger when we want to deal with nonlinear differential equations; such are of much bigger difficulty than the linear ones as there are few algorithms applicable and the existence theorems of the linear case must be replaced by those of special application very often.

Despite of its inherent difficulties, nature has shown itself very likely to veil their phenomena under nonlinear problems, where they are encountered in physics, chemistry and biology quite often.

First, some definitions:

- By *order* of a differential equation we mean the highest derivative present in the equation.
- By *linear operator* we mean the operator that obeys:

  $$L\left(k\psi + h\phi\right) = kL(\psi) + hL(\phi)$$

  where k and h are scalars and ψ and ϕ are arbitrary functions.
- An *nonlinear operator* is just an operator which is not linear.
- The *nonlinear equation* is a nonlinear operator equaled to zero or a function.

- If there is some function $u(x)$ that satisfies $L(u) = f(x)$, then we say it is a *particular solution* of the equation.
- If there is some solution that is divergent or fails the uniqueness to the initial value problem at some point it is called *singular solution*.
- If some equation has more than one particular solution, their whole set plus the singular solutions is the *general solution*.

Linear differential equations are of the form:

$$\frac{d^n y}{dz^n} + a_{n-1}(z)\frac{d^{n-1}y}{dz^n - 1} + \ldots + a_1(z)\frac{dy}{dz} + a_0(z)y = 0,$$

with the functions $a_i(z)$ being arbitrary, it is known that its general solution can only have fixed singularities [14] where the functions $a_i(z)$ are singular; however for nonlinear differential equations we can have some other feature: a *movable singularity*, which is a singularity that depends on the initial conditions. As an example:

$$\frac{dy}{dz} + y^2 = 0 \qquad \Longrightarrow \qquad y(z) = \frac{1}{z - z_0}$$

Where z_0 is arbitrary, then we see here that the pole is not fixed, but is located at z_0.

For an algebraic first order and first degree equation like $y' = \frac{\mathcal{P}(y,z)}{\mathcal{Q}(y,z)}$, where $\mathcal{P}(y,z)$ and $\mathcal{Q}(y,z)$ are polynomials in y and analytical functions of z (then the statement "algebraic"), Painlevé has proved that it has only movable singularities like poles and algebraic branch cuts.

For second order ODEs, *movable essential singularities* may appear, as well as *movable logarithmic branches*.

For third order ODEs, *natural barriers* may appear also, making the analysis of where we are able to study the problem more difficult.

When we deal with these singularities is useful to introduce the concept of a *critical singularity*, which is just a multivalued singularity like movable algebraic and logarithmic branch points and some essential singularities. Thus simple poles are *noncritical singularities* (singlevalued) as well as some essential singularities and fixed branch points.

The persistent concerning about movable and fixed singularities is justifiable by the fact that noncritical singularities can be made singular, for example, by branch cuts and Riemann surfaces, but these procedures fail for movable singularities (except for poles that are noncritical), as we should choose a cut for each initial condition. Examples are listed below the table below.

☞ [12][13][14][15]

TABLE 2.1: Possible singularities

Equation	General solution	Singularity type
$y' + y^2 = 0$	$y = \frac{1}{z-z_0}$	Movable pole
$2yy' = 1$	$y = \sqrt{z - z_0}$	Movable branch point
$y'' + y'^2 = 0$	$y = ln(z - z_0) + k$	Logarithmic branch point
$y'^4 = y\,(2y'^2y'' - 4y'^3 - yy''^2)$	$y = ke^{\frac{1}{z-z_0}}$	Isolated essential singularity
$(1 + y^2)y'' = (2y - 1)y'^2$	$y = tan(ln(k(z - z_0)))$	Nonisolated essential sing.
$y''' = yy'' - 3y'^2$		Movable natural barrier

2.2 Painlevé property

Beginning the concept of making a problem solvable, we go back to Sophie Kowalevski's work with the equations of motion of a rigid body, where she first noticed that when the dependent variables of the problem where *meromorphic functions* (complex singlevalued function, analytical in the whole domain except for isolated noncritical singularities) of time in the complex plane she could solve it explicitly. This originated what we know nowadays as the "Kowalevski top" in classical mechanics.

The problem of solving exactly a differential equation remains, according to Poincaré, to express its general solution as a finite, possibly multivalued, combination of functions. For the linear case, one can always define a function from an ODE and as a remarkable case, we have the elliptic equations being the unique first order algebraic ODE defining a new singlevalued function from a nonlinear ODE.

Guided by these ideas of finding all differential equations whose general solution has no movable critical points, Paul Painlevé and his collaborators made their main work. We can then state the broadly known:

Definition. *An ODE which general solution can be made singlevalued, has the* ***Painlevé property (PP)****.*

This statement sometimes is altered to: an ODE has the Painlevé property if all movable singularities of all solutions are poles.

In fact this is not quite correct, because when is said "all solutions" one could argue about a possible critical particular solution (the correct definition states just about the general solution), and also about "poles", for 1st and 2nd order the movable singularities are just poles, but as said above we can have a different singularity structure for third order equations.

Here we may also finally define what is meant by "Painlevé integrability", it is just a function that has the PP, that is, is singlevalued at its general solution.

By definition, all linear differential equations have the PP. Painlevé and his school have (through really hard work and brute force) found 50 first-degree second order differential equations that have the PP, by Möbius transformations any other differential equation of first degree and second order also having the PP reduces to one of those (a more detailed discussion about these transformations and equations are on Appendix A); between those 50 equations, 6 could not be reduced to known equations, thus defining new transcendents, they were named as P_I-P_{VI} (P stands for Painlevé, also):

$$y'' = 6y^2 + z \tag{2.1}$$

$$y'' = 2y^3 + zy + \alpha \tag{2.2}$$

$$y'' = \frac{(y')^2}{y} - \frac{y'}{z} + \frac{1}{z}(\alpha y^2 + \beta) + \gamma y^3 + \frac{\delta}{y} \tag{2.3}$$

$$y'' = \frac{(y')^2}{2y} + \frac{3}{2}y^3 + 4zy^2 + 2(z^2 - \alpha)y + \frac{\beta}{y} \tag{2.4}$$

$$y'' = \frac{3y-1}{2y(y-1)}(y')^2 - \frac{y'}{z} + \frac{1}{z^2}(y-1)^2\left(\alpha y + \frac{\beta}{y}\right) + \frac{\gamma y}{z} + \frac{\delta y(y+1)}{y-1} \tag{2.5}$$

$$\begin{aligned} y'' = {} & \frac{1}{2}\left(\frac{1}{y} + \frac{1}{y-1} + \frac{1}{y-z}\right)(y')^2 - \left(\frac{1}{z} + \frac{1}{z-1} + \frac{1}{y-z}\right)y' \\ & + \frac{y(y-1)(y-z)}{y^2(z-1)^2}\left(\alpha + \frac{\beta z}{y^2} + \frac{\gamma(z-1)}{(y-1)^2} + \frac{\delta y(y-1)}{(y-z)^2}\right) \end{aligned} \tag{2.6}$$

☞ [13][16][17]

2.3 Painlevé test

As was said in the beginning of this chapter, the insane search for precise exact solutions of differential equations is not so worthy, and since by any method of integration the singularity structure is maintained we may focus our attention on the the study of these singularities.

As we now know that we can attribute the quality of the PP to an equation, we may develop a test to know if such equation do or do not have the PP, this is called Painlevé test and there are some methods commonly used, here we will extend ourselves on just one (by its simplicity), but is worthy list the most common ones:

- **Painlevé α-method.** The most powerful one, also the one that was used by the pioneers on the field to discover and list all the 50 equations with PP at second order, but it's too demanding once it requires solving differential equations.

- **Kowalevski-Gambier method.** Revived by Ablowitz, Ramani and Segur, consists in requiring the existence of a Laurent series near the singularities. It is just a necessary condition (not sufficient) but it demands just algebraic calculations.

☞ [13]

2.4 Kowalevski-Gambier method

Considering an ODE for $y(z)$, which may be a scalar or a vector quantity, it has the PP if its solutions can be expanded as a Laurent series around movable singularities, this test detects if there is no branch point, although fails for essential singularities. It consists of 3 steps:

2.4.1 Identify the dominant behavior

We start by expanding like $y(z) \backsim c_0(z-z_0)^p$, where c_0 is a constant and $p < 0$. This is done because we are expanding the equations around the supposed singularity, so near z_0 the higher terms (leading terms) dominate de expression, as it is a singularity, p must be negative, if we require it to be a pole, it must also be an integer. Replacing it in the original equation, we will be able to balance the factors of p and c_0.

Example 1

$y''' + yy'' - 2y^3 + ay^2 + by = 0$

We can do this balance in 2 ways:

I) $p = -1$, $c_0 = 3$, the leading terms are y''' and yy''.

II) $p = -2$, $c_0 = 3$, the leading terms are yy'' and $-2y^3$.

Example 2

$y'' = \frac{2y(y')^2}{y^2-1}$

It becomes explicitly:

$p(p-1)c_0(z-z_0)^{p-2} = \frac{2c_0^3p^2(z-z_0)^{3p-2}}{c_0^2(z-z_0)^{2p}} \quad \longrightarrow (p-1) = 2p \longrightarrow p = -1$

and c_0 is undetermined

Actually it has the solution $y(z) = tanh(az+b)$, having no movable critical points.

If we faced a rational p, it indicates an algebraic branch cut:

Example 3

$yy'' = \frac{5}{2}(y')^2$

We may choose $p = -2/3$, then we define $y = v^{-2/3}$ and replace it back:

$v'' = 0$

Here we noticed that the original equation has an algebraic branch point, but, by a simple transformation it became linear, then having the PP.

Now obtaining a negative and integer p, we think of it as the first term in the Laurent series (like in Frobenius method):

$$y(z) = (z-z_0)^p \sum_{i=0}^{\infty} c_i(z-z_0)^i, \qquad 0 < |z-z_0| < R. \tag{2.7}$$

As z_0 is an arbitrary constant, and our equation is of $n - th$ order, there will last the other $n-1$ arbitrary constants of integration, to find at where powers they are (the resonances) we move to the next step:

2.4.2 Finding the resonances

- Now we do a small perturbation on the leading term like:

$$y(z) \backsim (z - z_0)^p (1 + \epsilon (z - z_0)^r). \tag{2.8}$$

- Then we replace it in the original equation, retain only the terms of first order in ϵ and see what values of r we get (like an indicial equation in Frobenius method).
- The value -1 is always present (it represents the arbitrariness in the choice of z_0)
- If c_0 were arbitrary in the last step, $r = 0$ will appear.
- Neglect negative roots ($r < 0$), they contradict the hypothesis of $(z - z_0)^p$ being the leading term (they also do not indicate an essential singularity)
- If there is some non-integer r, it indicates immediately an algebraic branch point, the equation does not have the PP.
- If it is not found the $n - 1$ positive integer roots, it indicates that none of the local solutions are general
- If it is found $n - 1$ nonnegative distinct integer roots, there are no branch cuts and we move on to check it back in the equation.

2.4.3 Finding the constants of integration

We replace the expression (2.7) in the original equation, if we can determine all non-arbitrary coefficients without adding any logarithm, then it indicates we were successful and our expression does not have any algebraic or logarithmic branch point.

Example 4

With the equation:
$y'' = 6y^2 - \frac{k_1}{2}$
Has the leading balance: $p = -2$ and $c_0 = 1$
As $y(z) \backsim (z - z_0)^{-2} (1 + \epsilon (z - z_0)^r)$ it implies:
$y'' - 6y^2 \backsim \epsilon \left((r-2)(r-3) - 12 \right) (z - z_0)^{r-4} = 0$

Having then $r^2 - 5r - 6 = 0$, so $r = -1$ and $r = 6$. Replacing it:

$$y = \frac{1}{(z-z_0)^2} + \frac{k_1}{20}(z-z_0)^2 + \frac{k_2}{28}(z-z_0)^4 + \dots$$

In fact we could have solved it exactly and getting $y = \wp(z - z_0; k_1, k_2)$ where $\wp$ is the Weiestrass elliptic function, and it really has movable double poles, as we have found.

☞ [14][15][18][20]

2.5 Painlevé equations and Briot-Bouquet systems

Our expression (2.7) is only justifiable if we guarantee it converges.

To do that we will use the theory of Briot-Bouquet systems, in which has a very convenient theorem to our situation. First, a system of differential equations with a noncritical singular point at $t = 0$, that is:

$$tu_j' = f_j(t, u_1, \dots, u_n), \qquad j = 1, \dots, n, \tag{2.9}$$

with analytic functions f_j around the neighborhood of the point $t = u_1 = \dots = u_n = 0$ satisfying the conditions $f_j(0, ..., 0) = 0$, forms a Briot-Bouquet system.

Rewriting the above system in a vectorial form:

$$tU' = F(t, U), \tag{2.10}$$

where $U = (u_1, u_2, ...u_n)^T$ and $F(t, U) = (f_1(t, U), ..., f_n(t, U))^T$. $F(t, U)$ may be expanded into a convergent power series in some domain around the origin:

$$F(t, U) = \sum_{j+|K|\geq 1} A_{jK} t^j U^K \tag{2.11}$$

where k is a vector of n positive integers (for the sum at each function u_i), and the matrix A is:

$$A = \begin{pmatrix} \frac{f_1}{u_1}(0,\vec{0}) & \dots & \frac{f_1}{u_n}(0,\vec{0}) \\ \vdots & & \vdots \\ \frac{f_n}{u_1}(0,\vec{0}) & \dots & \frac{f_n}{u_n}(0,\vec{0}) \end{pmatrix} \tag{2.12}$$

Theorem. *Supposing that* (2.10) *and* (2.11) *admit a solution of the form*:

$$u_j(t) = \sum_{k=1}^{\infty} c_{k,j} t^k, \qquad c_{k,j} \in \mathbb{C} \tag{2.13}$$

then our Briot-Bouquet has a solution of the form that is convergent in the domain $0 < |t| < R$ *for* $R > 0$.

As an example, we will build up the convergence for a Briot-Bouquet system correspondent to the P_{II} equation:

From the second step of the Kowalevski-Gambier method to the P_{II}, $(c_0, p) = (\sqrt{1}, 1)$, and adopting $t = z - z_0$, we have:

$$y'' = 2y^3 + zy + \alpha$$
$$y = (c_0 + u_1(t))t^{-1}$$
$$y' = (-c_0 + u_2(t))t^{-2}$$

with the corresponding Briot-Bouquet system:

$$tu_1' = u_1 + u_2 \tag{2.14}$$
$$tu_2' = 6u_1 + 2u_2 + c_0 z_0 t^2 + 6c_0 {u_1}^2 + (c_0 + \alpha)t^3 + z_0 u_1 t^2 + 2{u_1}^3 + u_1 t^3 \tag{2.15}$$

Its corresponding A-matrix has eigenvalues $\lambda = -1$ and $\lambda = 4$. The one-parameter family of solutions of P_{II} is given by:

$$y = \frac{c_0}{t} - \frac{c_0 z_0}{6}t - \frac{c_0 + \alpha}{4}t^2 + ht^3 + O(t^4) \tag{2.16}$$

with the arbitrary constant, h and z_0, appearing right in the place the Fuchs indexes indicated.

With such powerful tool in our hands, we may use some theorems (not proved here, but at [21])

Theorem. *If some rational differential equation passes in the Painlevé test then, in a neighborhood of a singular point (where the test was supposed to take place), it can be reduced to a Briot-Bouquet system, and has a convergent expression in the form* (2.7).

Thus we see that there is some correspondence between both formulations.

☞ [16][21]

2.6 PDEs and ARS conjecture

Considering a PDE with several independent variables

$$P(u, x, t, ...) = 0$$

Such has no isolated singularities in a space of independent variables $(x, t, ..)$, but rather they lie on a codimension one manifold

$$\psi(x, t, ...) - \psi_0 = 0$$

where ψ is the singular manifold variable, i.e., a function, and ψ_0 is a point in space being the movable constant.

From the Couchy-Kowalevski theorem, that states local existence and uniqueness for an analytic PDE associated to Cauchy initial value problem, we define a movable manifold that is *characteristic* when it makes inapplicable the aforementioned existence theorem.

The main difficulty with PDE is to define a general solution, but there are 2 main scenarios to define a concept of integrability.

The first is when we may linearize explicitly a PDE. The other is when a PDE admits an *inverse spectrum transform* (or *inverse scattering transform*) (IST), which is a way of representing the PDE as an integral equation (the Gelfand-Levitan-Marchenko) that can be solved.

There are classes of solutions of PDEs there are also solutions of an ODE, these are called *reductions*. They may be arranged in two groups, characteristics and non-characteristics. A reduction of a PDE to another is called non-characteristic (or exact) if it preserves the differential order and reduces 1 independent variable.

With that in mind, Ablowitz, Ramani and Segur (ARS) have conjectured that given a non-linear PDE, a non-characteristic reduction of it to an ODE via IST, yields the PP to the ODE.

As the name says, it is just a conjecture, but it has already been rigorously proved in some special cases, for example, exact reductions of nonlinear PDEs of IST class (nontrivial solutions can be found by solving the integral equation) to ODEs which solutions may be found by a linear integral equation with nonsingular kernel necessarily has the PP.

As a simple example of the ARS conjecture appearing is at the sine-Gordon equation, which is well known to be integrable:

$$u_{xt} = sin(u) \tag{2.17}$$

which under the reduction:

$$u(x,t) = w(z), \qquad z = xt \tag{2.18}$$

we arrive at

$$zw'' + w' = sin(w) \tag{2.19}$$

To one be able to apply the Painlevé test, one should first make it rational in w, in order to do that, we use $y = exp(iw)$ at the last equation, getting:

$$z(yy'' - y'^2) + yy' = \frac{y}{2}(y^2 - 1) \tag{2.20}$$

which is a special case of the P_{III}, and therefore have the PP.

☞ [13][18][20]

2.6.1 Lax Pair

Originally proposed and used as a tool when dealing whit PDEs, the Lax Pair throughout history achieved the status of a criterion of integrability. If some system possess an associated Lax Pair, then it means that all the necessary conserved quantities required for integrability can be found.

A more careful deduction of the Lax Pair can be found in the Appendix D.

It is called a Lax Pair a set of 2 linear operators L_1, L_2, whose commutativity condition is equivalent to the PDE condition $E(u) = 0$, that is, both are zero:

$$\{[L_1, L_2] = 0\} \Longleftrightarrow \{E(u) = 0\} \tag{2.21}$$

These operators can be represented as matrices or scalars. Such representation of the Lax is not unique, it can also be done in the *Lax representation*:

$$L_1 = L - \lambda, \qquad L_2 = \partial_t - P, \qquad L_1\psi = 0, \qquad L_2\psi = 0, \qquad \lambda_t = 0 \tag{2.22}$$

where the above scalar λ is called the spectral parameter, which appears in 1+1 dimensional cases; after eliminating due to the above isospectral condition ($\lambda_t = 0$) it we get:

$$L_t = [P, L] \tag{2.23}$$

Another popular equivalent representation is the zero-curvature representation:

$$\begin{gathered} L_1 = \partial_x - L, \quad L_2 = \partial_t - M, \quad L_1\psi = 0, \quad L_2\psi = 0, \\ [\partial_x - L, \partial_t - M] = L_t - M_x + LM - ML = 0 \end{gathered} \tag{2.24}$$

☞ [13][22]

2.7 Hamiltonian of P_{II}

Newton's classical treatment of mechanics demanded us to explore the world of differential equations, specially the ones of second order. From his second law, $F = m\ddot{x}$ we were merged into the urge of master second order equations once at each dealt system we arrived at some type of such differential equation.

Nature has showed itself requiring second order equations to work, and for that there is nothing we can do about it, the trouble with that relies on the difficulty on dealing with hard solutions of such order, fortunately, William Rowan Hamilton reformulated Lagrange's work in 1833 and we then had Hamiltonian mechanics, where we must just deal with first order equations.

Hamiltonian systems demand an equation that contains all information about the problem, the Hamiltonian H, and pairs of dependent variables (q, p) in the form:

$$\frac{dq}{dt} = \frac{\partial H}{\partial p}, \qquad \frac{dp}{dt} = -\frac{\partial H}{\partial q}. \tag{2.25}$$

We may also have a dependence of H on the time parameter, so generally $H = H(q(t), p(t), t)$.

Once we have a second order differential equation, we may look for its Hamiltonian representation, taking the example of a P_{II}:

$$y'' = 2y^3 + ty + b - \frac{1}{2} \tag{2.26}$$

with $y = y(t)$ and $b \in \mathbb{C}$. We may write:

$$q = y, \qquad p = y' + y^2 + \frac{t}{2} \tag{2.27}$$

This is immediately $q' = p - q^2 - \frac{t}{2}$ and $p' = 2qp + b$ and then $q'' = p' - 2qq' - \frac{1}{2}$, eliminating p and p', we recover (2.26).

Using (2.25) we may integrate in q and p the above equations for q' and p' and then join them to form the Hamiltonian:

$$H_{II} = \frac{p^2}{2} - \left(q^2 + \frac{t}{2}\right)p - bq = \frac{1}{2}p(p - 2q^2 - t) - bq. \tag{2.28}$$

An important feature of Hamiltonian systems is the Poisson brackets structure:

$$\{A, B\} = \frac{\partial A}{\partial p}\frac{\partial B}{\partial q} - \frac{\partial A}{\partial q}\frac{\partial B}{\partial p} \tag{2.29}$$

From that we have the well known relation for a function $A(q, p; t)$:

$$\frac{dA}{dt} = \{H, A\} + \frac{\partial A}{\partial t} \tag{2.30}$$

In such systems, it's of high importance the definition of a *canonical transformation*, which is a transformation that maps the former coordinate system (q, p) to $(\tilde{q}, \tilde{p})$, with $\tilde{q} = \tilde{q}(q, p; t)$ and $\tilde{p} = \tilde{p}(q, p; t)$ satisfying $\{\tilde{p}, \tilde{q}\} = 1$; this condition makes the pair $(\tilde{q}, \tilde{p})$ a *canonical coordinate system* and it guarantees that the new system obtained from that map is also a Hamiltonian system.

☞ [23]

2.8 Bäcklund transformations

$$\text{Recovering:} \qquad q' = p - q^2 - \frac{t}{2}, \qquad p' = 2qp + b \tag{2.31}$$

$$\text{We set:} \qquad \tilde{q} = q + \frac{b}{p}, \qquad \tilde{p} = p \tag{2.32}$$

We see here in advance that $\{\tilde{p}, \tilde{q}\} = 1$, thus it's a canonical transformation. Applying them, (2.31) becomes:

$$\tilde{q}' = \tilde{p} - \tilde{q}^2 - \frac{t}{2}, \qquad \tilde{p}' = 2\tilde{q}\tilde{p} - b \qquad . \tag{2.33}$$

It's obvious that there was a change in sign in b, so if we have a solution (q, p), the functions $(\tilde{q}, \tilde{p})$ give the solution with $(-b)$. We may define the complete transformation as $(q, p, b) \rightarrow (\tilde{q}, \tilde{p}, \tilde{b})$, where $\tilde{b} = -b$.

In this narrow sense, we may define a *Bäcklund transformation* as a change of variables that leaves a differential equation's form invariant.

Another example of complete transformation for H_{II} is:

$$\tilde{q} = -q, \qquad \tilde{p} = -p + 2q^2 + t, \qquad \tilde{b} = 1 - b \tag{2.34}$$

or, in another language:

$$r(q) = -q, \qquad r(p) = -p + 2q^2 + t, \qquad r(b) = 1 - b, \qquad r(t) = t \tag{2.35}$$

Also for the first transformation:

$$s(q) = q + \frac{b}{p}, \qquad s(p) = p, \qquad s(b) = -b, \qquad s(t) = t \tag{2.36}$$

Basically for a general function $\phi(q, p, b; t)$, $s(\phi) = \phi(\tilde{q}, \tilde{p}, \tilde{b}; t)$.

For these operations above defined s and r, we conclude that they preserve algebraic operations as well as differentiation:

$$s(\phi \pm \psi) = s(\phi) \pm s(\psi), \qquad s(\phi\psi) = s(\phi)s(\psi), \qquad s\left(\frac{\phi}{\psi}\right) = \frac{s(\phi)}{s(\psi)}, \qquad s(\phi)' = s(\phi'). \tag{2.37}$$

We may define now a *Bäcklund transformation* as an operation on the variables that leaves a differential equation's form invariant and commutes with the differentiation.

Mathematically speaking, the transformation s can be defined as the automorphism (a bijection to its own group) of the field $K = \mathbb{C}(q, p, b, t)$ that maps the generators of K as in

(2.36); as it is endowed with a derivation, s is an automorphism of the differential field K that represents the equation of the system, thus it commutes with derivation.

☞ [23]

2.9 Particular solutions of P_{II}

As we are manipulating the free parameter b, we may use $b = 0$, that is $H_{II}(b) \to H_{II}(0)$, leading us to $q' = p - q^2 - \frac{t}{2}$ and $p' = 2qp$. Letting $p = 0$:

$$q' = -q^2 - \frac{t}{2} \tag{2.38}$$

which is the Riccati equation that can be solved by linearization. Changing the variables with $q = \frac{u'}{u}$ we arrive at:

$$u'' + \frac{tu}{2} = 0 \tag{2.39}$$

which is the Airy's equation, broadly known in the theory of ordinary differential equations.

We may also put $b = 1/2$, which lead us to $y'' = 2y^3 + ty$, thus $y = q = 0$ is a solution with $p = t/2$.

From the above we apply s to the solution $(q, p) = (0, \frac{t}{2}) \to \left(\frac{1}{t}, \frac{t}{2}\right)$, generating a new solution (with a different value of b, of course).

We now move on to the thought of the composition of Bäcklund transformations is also a Bäcklund transformation. We notice that the composition $s \circ r(q, p, b) \equiv sr(q, p, b) = (\tilde{q}, \tilde{p}, \tilde{b})$ always give new solutions with different b, once $sr(b) = b + 1$. We also notice that $sr(q, p, b) \neq rs(q, p, b)$.

This forms the *affine Weyl group of type* $A_1^{(1)}$: $\tilde{W} = \langle s, r\rangle$, which indeed forms a group once we also have $s^2 = 1$ and $r^2 = 1$.

☞ [23]

2.10 The symmetric form of P_{II}

Another way of expressing the P_{II} (and also many other equations) is the symmetric form, which, as the name suggests, turns more explicit the symmetries of an equation.

The symmetric form works as a next step of the hamiltonian representation, looking back to (2.31) we may set:

$$\begin{aligned} \alpha_1 &= b, & \alpha_0 &= 1 - b & (\alpha_0 + \alpha_1 &= 1) \\ f_1 &= p, & f_0 &= r(p) = -p + 2q^2 + t \end{aligned}$$

With that we get:

$$\begin{cases} f_0' &= -2qf_0 + \alpha_0, \\ f_1' &= 2qf_1 + \alpha_1, \\ q' &= \frac{1}{2}(f_1 - f_0) \end{cases} \tag{2.40}$$

This is the symmetric form of P_{II}. We have for it:

$$(f_0 + f_1)' = 2q(f_1 - f_0) + 1 = 4qq' + 1 = (2q^2 + t)'$$

which implies:

$$f_0 + f_1 - 2q^2 = t + c$$

where c is an integration constant which, when set to zero, turns ours system the P_{II}.

As the action of r here just makes the parameters to change indexes, we relate it to a reflection, and as is customary in the literature we will call it π instead of r for the symmetric form of P_{II}:

$$\pi(\alpha_0) = (\alpha_1), \qquad \pi(\alpha_1) = \alpha_0, \qquad \pi(f_0) = f_1, \qquad \pi(f_1) = f_0; \qquad \pi(q) = -q \tag{2.41}$$

If we set $s_1 = s$ and $s_0 = rsr$, then these are the Bäcklund transformations corresponding to the reflections to $\alpha_1 = 0$ and $\alpha_0 = 0$, respectively.

In this sense we now may completely justify the extended affine Weyl group $A_1^{(1)}$; the affine Weyl group $A_1^{(1)}$ is defined just by $\langle s_0, s_1 \rangle$, here we extend it by the π element.

TABLE 2.2: Backlund transformations for the symmetric form of P_{II}

	α_0	α_1	f_0	f_1	q
s_0	$-\alpha_0$	$\alpha_1 + 2\alpha_0$	f_0	$f_1 - \frac{4\alpha_0 q}{f_0} + \frac{2\alpha_0^2}{f_0^2}$	$q - \frac{\alpha_0}{f_0}$
s_1	$\alpha_0 + 2\alpha_1$	$-\alpha_1$	$f_0 + \frac{\alpha_1 q}{f_1} + \frac{2\alpha_1^2}{f_1^2}$	f_1	$q + \frac{\alpha_1}{f_1}$
π	α_1	α_0	f_1	f_0	$-q$

☞ [23]

2.11 Arnold-Liouville integrability

As we saw previously, for some differential equations we can associate to them a Hamiltonian system, and in fact this construction also provides us with a sense of integrability which is in concept is the same as the Painlevé one, to assure that one is able to achieve a singlevalued solution to the system.

First, let's define some terms.

Some functions F_1 and F_2 on a symplectic manifold (is a smooth manifold, M, equipped with a closed nondegenerate differential 2-form) are said to be in *involtion* if their Poisson Bracket is zero:

$$\{F_1, F_2\} = 0 \tag{2.42}$$

It is said that a function F is the *first integral* of a system with Hamiltonian H when it is in involution with H:

$$\{F, H\} = 0 \tag{2.43}$$

It was proved by Liouville that if a system with n degrees of freedom (with a $2n$-dimensional phase space) has n independent first integrals in involution with themselves, then the system is integrable by quadratures.

That means to our simple cases of 2 degrees of freedom, that one must find a first integral (i.e., a second constant of motion, once the Hamiltonian is already one), nevertheless such statement is insufficient to perform a global integration (singlevaluedness).

In extension to that, the so called *Arnold-Liouville integrability* states that once you have the n first integrals, $F_1, ..., F_n$, you may consider a level set of them:

$$M_f = \{x : F_i(x) = f_i, \quad i = 1, ..., n\}. \tag{2.44}$$

Assuming the n functions F_i are independent on M_f, then:

- M_f is a smooth manifold, invariant under the phase flow with hamiltonian $H = F_1$
- If M_f is compact and connected, then it is diffeomorphic to the n-torus:

$$T^n = \{(\phi_1, ...\phi_n) \bmod 2\pi\}.$$

- The phase flow with h determines a conditionally periodic motion on M_f, that is, in angular coordinates $\vec{\phi} = (\phi_1, ...\phi_n)$ we will have:

$$\tfrac{d\vec{\phi}}{dt} = \vec{w}, \qquad \vec{w} = \vec{w}(\vec{f})$$

- The canonical equations can be integrated by quadratures.

That is, if one is able to express the system in this privileged coordinate system (called Darboux coordinates), then it can be separated and then assume a very simple form which can be globally integrated.

One can also make use of our previous way of dealing with integrability, searching for an explicit singluvalued closed form for the general solution (PP).

☞ [13][24]

Chapter 3

2M-Boson Model

3.1 Algebra of Pseudodifferential operators

Now we introduce the concept of pseudodifferential algebra and operators, first used in the context of hierarchies by Gelfand and Dikii.

The differential algebra is related to $\mathbb{C}$-valued functions and the action of differential operators like in usual calculus; the functions of one variable defined, for example, like $a(x)$ and have an action by the differential operator ∂ following the Leibniz rule $\partial(a(x)) \equiv \partial(a) = (\partial a) + a.\partial$; an element of such algebra is defined as a finite sum of such elements in many orders: $L = \sum_{i=0}^{N} a_i \partial^i$. The higher order derivative action on a function is:

$$\partial^n f = \sum_{\alpha=0}^{\infty} \binom{n}{\alpha} (\partial^\alpha f) \partial^{-n-\alpha} \tag{3.1}$$

Its extension, the so called ring of pseudodifferential operators, is done by the new element ∂^{-1} whose action is given by:

$$\begin{aligned} &\partial^{-1}\partial = \partial\partial^{-1} = 1 \\ &\partial^{-1} a = \sum_{i=0}^{\infty} (-1)^i (\partial^i a) \partial^{-i-1} \end{aligned} \tag{3.2}$$

The rule above come from infinite integrations by parts that allows the moving od ∂^{-1}; the elements of this extension are given by semi-infinite sums like $L = \sum_{i=-\infty}^{N} a_i \partial^i$.

A general set of such elements can be given by:

$$R = \left\{ A = \sum_{i=-\infty}^{N} a_i \partial^i \right\} \tag{3.3}$$

And then decomposed in $R = R_- \oplus R_+$:

$$R_+ = \left\{ A = \sum_{l=0}^{N} a_l \partial^l \right\}, \qquad R_- = \left\{ A = \sum_{l=-\infty}^{-1} a_l \partial^l \right\} \tag{3.4}$$

where R_+ is the subalgebra of differential operators and R_- is the subalgebra of integral (Volterra) operators.

We will also define an invariant non-degenerate bilinear form called "Adler trace":

$$\langle A|B\rangle = Tr\,(AB) = \int Res(AB)dx \tag{3.5}$$

where $ResA = Res\sum a_i\partial^i \equiv a_{-1}$, which we will call residue of the operator. Such definition guarantee to us that it works like a traditional trace $\langle A|B\rangle = \langle B|A\rangle$.

For illustrative proposes of the aforementioned properties of the pseudodifferential algebra, we will prove the last paragraph:

Knowing the uniqueness of writing a pseudodifferential operator with all derivatives on the right ($A = \sum_{-\infty}^{N} a_i\partial^i$), we may use them in such form like $A = a\partial^k$ and $B = b\partial^j$. If both k and j are strictly positive or negative, it's impossible to obtain a term like ∂^{-1}, then $\langle A|B\rangle = \langle B|A\rangle = 0$, so we just take $A = a\partial^k$ and $B = b\partial^{-j-1}$ with $k, j \geq 0$. Using:

$$\partial^{-j-1}a = \sum_{v=0}^{\infty}(-1)^v\binom{j+v}{v}(\partial^v a)\partial^{-j-1-v} \tag{3.6}$$

which is just a combination of (3.1) and (3.2), and also the identity for binomial coefficients

$$\binom{j+1+\mu}{\mu} = \sum_{\nu=0}^{\mu}\binom{j+\nu}{\nu}$$

we have:

$$\langle BA\rangle = (-1)^{k-j}\binom{k}{j}\int dx\left(b\partial^{k-j}a\right) = \binom{k}{j}\int dx\left(a\partial^{k-j}b\right)$$

By the same procedure, we arrive for $AB = \sum_{v=0}^{\infty}\binom{k}{v}(a\partial^v b)\partial^{k-j-v-1}$ that:

$$\langle AB\rangle = \binom{k}{j}\int dx\left(a\partial^{k-j}b\right) = \langle BA\rangle$$

☞ [25][26][27]

3.2 KP hierarchy

By using the Adler-Kostant-Symes construction to obtain hierarchies we may consider the pseudo differential operator:

$$L = \partial + \sum_{j=1}^{\infty} a_j\partial^{-j} \tag{3.7}$$

and a corresponding eigenvalue problem, $L\omega = k\omega$, we consider a formal solution:

$$\omega = e^{\xi(x,k)}\left(1 + \frac{\omega_1}{k} + \frac{\omega_2}{k^2} + \dots\right), \quad \text{where } \xi(x,k) = \sum_{j=1}^{\infty} x_j k^j \tag{3.8}$$

We can also notice that:

$$\frac{\partial}{\partial x_j} e^{\xi(x,k)} = k^j e^{\xi(x,k)}.$$

where k is a number and ω a function of x as above, we consider the linear system of equations:

$$\frac{\partial \omega}{\partial x_j} = B_j \omega, \qquad \text{where} \quad B_J = (L^j)_+ \tag{3.9}$$

From the compatibility condition between the above relations:

$$\frac{\partial L}{\partial t_n} = [(L^n)_+, L] \tag{3.10}$$

Defining each x_j as a time, we obtain infinitely many nonlinear evolution equations, and for that case, it is the *KP hierarchy*, named after the third equation from such hierarchy that was classically known, the KP (Kadomtsev-Petviashvili) equation.

The equation (3.10) is the Lax Pair related to the KP hierarchy.

If we had imposed the condition

$$(L^2)_- = 0 \tag{3.11}$$

Then we would have obtained the famous *KdV hierarchy*, which also implies that $L^2 = \partial^2 + u$, where all the functions of the hierarchy are determined by the function u. Beyond that, for even j we have $[B_j, L] = 0$ and also $\frac{\partial u}{\partial x_j} = 0$, meaning that the Kdv Hierarchy only have odd times.

TABLE 3.1: KP vs. KdV

	KP hierarchy	KdV hierarchy
time variables	$x_1,\ x_2,\ x_3, \dots$	$x_1,\ x_3,\ x_5, \dots$
pseudodiff. operator	$L = \partial + f_1\partial^{-1} + f_2\partial^{-2} + \dots$	$L = (\partial^2 + u)^{1/2}$
nonlinear equation	$\frac{3}{4}\frac{\partial^2 u}{\partial x_2^2} = \frac{\partial}{\partial x}\left(\frac{\partial u}{\partial x_3} - \frac{3u}{2}\frac{\partial u}{\partial x} - \frac{1}{4}\frac{\partial^3 u}{\partial x_3}\right)$	$\frac{\partial u}{\partial x_3} = \frac{3u}{2}\frac{\partial u}{\partial x} + \frac{1}{4}\frac{\partial^3 u}{\partial x_3}$

☞ [28]

3.3 Hamiltonian structure for the KP hierarchy

As demonstrated in [25], the KP hierarchy possess the associated Poisson bracket structure:

$$\{\langle L|X\rangle, \langle L|Y\rangle\}_1 = -\langle L|\,[X,Y]\rangle \tag{3.12}$$

$$\{\langle L|X\rangle, \langle L|Y\rangle\}_2 = Tr_A\left((LX)_+LY - (XL)_+YL\right) \tag{3.13}$$

$$+\int dxRes\left([L,X]\right)\partial^{-1}Res\left([L,Y]\right) \tag{3.14}$$

For the functions a_m from (3.7), now relabeled to u_m (to adequate to the literature) we have from the above:

$$\{u_n(x), u_m(y)\}_1 = \Omega^{(0)}_{nm}(u(x))\delta(x-y) \tag{3.15}$$

$$\{u_{-1}(x), u_{-1}(y)\}_1 = \{u_{-1}(x), u_m(y)\}_1 = 0 \tag{3.16}$$

where

$$\Omega^{(l)}_{nm}(u(x)) \equiv -\sum_{k=0}^{n+l}(-1)^k\binom{n+l}{k}u_{n+m+l-k}(x)\partial_x^k + \sum_{k=0}^{m+l}(-1)^k\binom{m+l}{k}u_{n+m+l-k}(x) \tag{3.17}$$

And also a second structure, given in function of Drinfeld-Sokolov brackets:

$$\{u_n(x), u_m(x)\}_2^D = \{u_n(x), u_m(x)\}_2 - \sum_{i=0}^{n-1}\sum_{j=0}^{m-1}(-1)^{n-i}\binom{n}{i}\binom{m}{j}u_i(x)\partial_x^{m+n-i-j-1}u_j(x)\delta(x-y) \tag{3.18}$$

where

$$\{u_n(x), u_{-1}(y)\}_2 = -\sum_{k=1}^{n}(-1)^n\binom{n}{k}u_{n-k}(x)\partial_x^k\delta(x-y) \tag{3.19}$$

$$\{u_{-1}(x), u_m(y)\}_2 = \sum_{k=1}^{m}\binom{m}{k}\partial_x^k u_{m-k}(x)\delta(x-y) \tag{3.20}$$

$$\{u_{-1}(x), u_{-1}(y)\}_2 = -\delta'(x-y) \tag{3.21}$$

From those, we define the Hamiltonians of L as:

$$H_m = \frac{1}{m}Tr(L^m) = \frac{1}{m}\int Res(L^m) \tag{3.22}$$

Satisfying

$$\frac{\partial L}{\partial t_n} = [(L^n)_+, L] = \{L, H_{m+1}\}_1 = \{L, H_m\}_2^D \tag{3.23}$$

Equivalentely:

$$\frac{\partial u_n}{\partial t_m} = \{u_n, H_{m+1}\}_1 = \{u_n, H_m\}_2^D \tag{3.24}$$

These can be indeed considered Hamiltonians because they are first integrals of (3.10):

$$\partial_m H_k \backsim \int Res\left[B_m, L^k\right] dx = 0 \tag{3.25}$$

We then readily obtain:

$$H_1 = \int u_0(x)dx, \qquad H_2 = \int u_1(x)dx, \tag{3.26}$$

$$H_3 = \int \left(u_2 + u_0^2\right) dx, \qquad H_4 = \int \left(u_3 + 3u_0u_1\right) dx \tag{3.27}$$

$$H_4 = \int \left(u_4 + 4u_0u_2 + 2u_1^2 + 2u_0^3 + u_0u_0'' - 2u_1u_0'\right) dx \tag{3.28}$$

Using then the results written in this section we are able to obtain:

$$\frac{\partial u_n}{\partial x} \equiv \frac{\partial u_n}{\partial t_1} = \{u_n, H_2\}_1 = \{u_n, H_1\}_2^D \tag{3.29}$$

$$\frac{\partial u_n}{\partial y} \equiv \frac{\partial u_n}{\partial t_2} = \{u_n, H_3\}_1 = \{u_n, H_2\}_2^D = \frac{\partial^2 u_n}{\partial x^2} + 2\frac{\partial u_{n+1}}{x} - 2\sum_{k=1}^{n}(-1)^k \binom{n}{k} u_{n-k}\frac{\partial^k u_0}{\partial x^k} \tag{3.30}$$

$$\frac{\partial u_0}{\partial t} \equiv \frac{\partial u_0}{\partial t_3} = \{u_0, H_4\}_1 = \{u_0, H_3\}_2^D = \frac{\partial^3 u_0}{\partial x^3} + 6u_0\frac{\partial u_0}{\partial x} + 3\left(\frac{\partial^2 u_1}{\partial x^2} + \frac{\partial u_2}{\partial x}\right) \tag{3.31}$$

Combining the equations above we get the KP equation (which is in Table 3.1).

3.3.1 Non-standard KP hierarchies

When on takes back the sets (3.4), one could wonder about the limits of index l used there could be changed, in fact if we change it to 1, instead of zero, we get:

$$L = L^{(1)} + L_- \equiv \partial + u_{-1} + u_0\partial^{-1} + \sum_{i\geq 1} u_i\partial^{-i-1} \tag{3.32}$$

and the Lax operator will then be:

$$\frac{\partial L}{\partial t_n} = [P_{l\geq 1}(L^n), L] \tag{3.33}$$

This new hierarchy is, at first sight, different from our original (with $l = 0$ and $L = \partial + \sum_{j=1}^{\infty} a_j \partial^{-j}$, that is no constant term), but as noticed by Sato, there is a gauge transformation:

$$K \equiv G^{-1} L G = \partial \sum_{j=1}^{\infty} v_j \partial^{-j}; \qquad G \equiv exp\left(-\int^x u_{-1} dx\right) \tag{3.34}$$

that erases the constant u_{-1} term, and we get again the Lax operator of the original form (3.7).

☞ [25]

3.4 Two-Boson KP Hierarchy

From our previous definition for the Lax (3.10), we may define it in a different way:

$$L = \partial + \bar{J}\frac{1}{\partial - J} = \partial + \bar{J}\partial^{-1} + \bar{J}\partial^{-1}J\partial^{-1} + ... \tag{3.35}$$

where we rearranged the notation briefly just to adequate it to the standard literature's [25] One may define a truncation of the non-standard Lax operator (described above):

$$L_J^{(1)} = \partial + u_0 + u_1\partial^{-1} = \partial - J + \bar{J}\partial^{-1} \tag{3.36}$$

Such restriction may appear very injunctive, but it is consistent with the $KP|_{l=1}$ hierarchy. We can define the Poisson-bracket structures for the 2 fields:

$$\begin{aligned}
\left\{\bar{J}(x), J(y)\right\}_1 &= -\delta'(x-y) \\
\left\{J(x), J(y)\right\}_1 &= \left\{\bar{J}(x), \bar{J}(y)\right\}_1 = 0 \\
\left\{\bar{J}(x), J(y)\right\}_2 &= J(x)\delta'(x-y) - \delta''(x-y) \\
\left\{\bar{J}(x), \bar{J}(y)\right\}_2 &= 2\bar{J}(x)\delta'(x-y) + \bar{J}'(x)\delta(x-y) \\
\left\{J(x), J(y)\right\}_2 &= 2\delta'(x-y)
\end{aligned}$$

The above 2-boson hierarchy is gauge-equivalent to the model:

$$L_{ce} = (\partial - e)(\partial - c)(\partial - e - c)^{-1} = \partial + (e' + ec)(\partial - e - c)^{-1} \tag{3.37}$$

Under the identification of $\bar{J} = e' + ec$ and $J = e + c$.

☞ [25] [29]

3.5 Multi-Boson KP Hierarchy

To generalize L_{ce} above to multiple bosons, one can represent the corresponding Lax as:

$$L_M = \prod_{j=1}^{M+1} \left(\frac{\partial}{\partial x} + v_{M+2-j} \right) \prod_{l=1}^{M} \left(\frac{\partial}{\partial x} + \tilde{v}_l \right)^{-1} \tag{3.38}$$

When such Lax hierarchy is subject to the constraint:

$$\sum_{j=1}^{M+1} v_j - \sum_{l=1}^{M} \tilde{v}_l = 0 \tag{3.39}$$

and the second bracket structure, subject to this constraint, has a form of a graded $SL(M+1, M)$ Kac-Moody algebra in a diagonal gauge, we are able to diagonalize it by a change of variables to a pair set of $2M$ (conjugated) variables (c_i, e_i) as:

$$\tilde{v}_l = -e_l - \sum_{p=l}^{M} c_p, \qquad v_j = -e_{j-1} - \sum_{p=j}^{M} c_p. \tag{3.40}$$

With that we are then able to obtain a canonical relation between the fundamental functions of our system:

$$\{e_i(x), c_j(y)\}_2 = -\delta_{ij}\delta_x(x-y), \qquad i, j = 1, 2, \ldots, M \tag{3.41}$$

Replacing it in (3.38) we arrive at the Lax operators:

$$\begin{aligned} L_M =& (\partial - e_M) \prod_{k=1}^{M-1} \left(\partial - e_k - \sum_{l=k+1}^{M} c_l \right) \left(\partial - \sum_{l=1}^{M} c_l \right) \\ & \times \prod_{k=1}^{M} \left(\partial - e_k - \sum_{l=k}^{M} c_l \right)^{-1} \end{aligned} \tag{3.42}$$

These pairs are the "Darboux-Poisson" canonical pairs for the second KP bracket, consequently we now have a sequence of them. Such construction may be seen as a generalized Miura transformation for multi-boson KP hierarchies.

☞ [29][30]

3.6 The 2M-constrained KP-hierarchy

The above found $L_n^{(M)}$ is actually our previous constrained multi-boson KP-hierarchy, build for M pairs of functions $(c_k, e_k)_{k-1}^M$:

$$L_M = (\partial - e_M)\prod_{k=1}^{M-1}\left(\partial - e_k - \sum_{l=k+1}^{M} c_l\right)\left(\partial - \sum_{l=1}^{M} c_l\right) \times \prod_{k=1}^{M}\left(\partial - e_k - \sum_{l=k}^{M} c_l\right)^{-1}.$$

Now that we have realized the recurrence relation from (E.9), we are able to write the above equation as:

$$L_M = e^{\int c_M}(\partial + c_M - e_M)L_{M-1}(\partial - e_M)^{-1}e^{-\int c_M}, \qquad L_0 \equiv \partial \tag{3.43}$$

Casting now the new Lax in conventional form,

$$L_M = \partial + u_0(M)\partial^{-1} + u_1(M)\partial^{-2} + \dots \tag{3.44}$$

We obtain "equations of motion" from the so called "second flow" equation from the traditional KP-hierarchy:

$$\frac{\partial u_n}{\partial y} \equiv \frac{\partial u_n}{\partial t_2} = \{u_n, H_3\}_1 = \{u_n, H_2\}_2^D$$

noticing here that the Lax was can be written in different forms, it is just unique when we place all ∂ to the left or to the right; also the Hamiltonian H_2 is just $\int dx u_1(M)$.

As the Lax was written in so many forms we may write the correct terms in the useful languages, then we obtain:

$$u_0(M) = u_0(M-1) + (e'_M + e_M c_M) \tag{3.45}$$

$$u_1(M) = u_1(M-1) + u'_0(M-1) + 2u_0(M-1)c_M + (e'_m + e_m c_m)(e_M + c_M) \tag{3.46}$$

which can be worked out and become:

$$u_0(M) = \sum_{i=1}^{M}(e'_i + e_i c_i) \tag{3.47}$$

$$u_1(M) = \sum_{i=1}^{M-1}(M-i)(e'_i + e_i c_i)' + 2\sum_{i=1}^{M-1} u_0(i)c_{i+1} + \sum_{i=1}^{M}(e'_i + e_i c_i)(e_i + c_i) \tag{3.48}$$

Therefore the next immediate step is to in fact find the second-flows, they are given, with arbitrary value of M, by the general expressions:

$$\begin{aligned}
\frac{\partial c_j}{\partial t_2} =& \frac{d}{dx}\left(c_j' - c_j^2 - 2e_jc_j + 2\sum_{i=j+1}^{M} c_i' - 2\sum_{i=j}^{M-1} c_ic_{i+1}\right), \quad j = 1, \dots, M \\
\frac{\partial e_j}{\partial t_2} =& \frac{d}{dx}\left(-e_j' - e_j^2 - 2e_jc_j - 2u_0(j-1) - 2e_j\sum_{i=j+1}^{M} c_i\right), \quad j = 1, \dots, M
\end{aligned} \tag{3.49}$$

The system also has Darboux-Bäcklund transformations that are performed by:

$$L_M \to (\partial - e_M)^{-1} L_M (\partial - e_M) \tag{3.50}$$

It can be seem that it corresponds to the transformation on the Volterra lattice form integer to half-integer modes, similar to what was done in last section.

To summarize this transformation to each function (boson), we cast the transformations for high indexes:

$$g(e_M) = e_{M-1} + c_M, \qquad g(c_M) = -e_{M-1} + e_M - \frac{c_M'}{c_M}, \tag{3.51}$$

$$g(e_{M-1}) = e_{M-2} + e_{M-1} - e_M + c_M + c_{m-1} + \frac{c_M'}{c_M}, \tag{3.52}$$

$$g(c_{M-1}) = -e_{M-2} + e_{M-1} - \frac{\left(-e_{M-1} + e_M - c_M - c_{M-1} - \frac{c_M'}{c_M}\right)'}{\left(-e_{M-1} + e_M - c_M - c_{M-1} - \frac{c_M'}{c_M}\right)} \tag{3.53}$$

and also for the other orders:

$$g\left(e_k + \sum_{l=k+1}^{M} c_l\right) = \sum_{l=k}^{M} c_l, \quad 2 \le k \le M, \quad g\left(e_1 + \sum_{l=2}^{M} c_l\right) \sum_{l=1}^{M} c_l. \tag{3.54}$$

☞ [29][30]

3.7 4-Bosons model

Now that we have presented all the general construction of the 2M-boson model, let's focus one the current case of interest, the 4-boson model (M=2).

In that case the Lax operator (3.10) becomes:

$$
\begin{aligned}
L = &\left(\frac{\partial}{\partial x} - e_2\right)\left(\frac{\partial}{\partial x} - e_1 - c_2\right)\left(\frac{\partial}{\partial x} - c_1 - c_2\right) \\
&\times \left(\frac{\partial}{\partial x} - e_1 - c_1 - c_2\right)^{-1}\left(\frac{\partial}{\partial x} - e_2 - c_2\right)^{-1}
\end{aligned}
\tag{3.55}
$$

And by the construction made in the last section, the equations of motion for the second t_2 flow of the 4 scalars (bosons) is:

$$
\begin{aligned}
\frac{\partial c_1}{\partial t_2} &= \frac{\partial}{\partial x}\left(c_1' - c_1^2 - 2e_1c_1 + 2c_2' - 2c_1c_2\right) \\
\frac{\partial e_1}{\partial t_2} &= \frac{\partial}{\partial x}\left(-e_1' - e_1^2 - 2e_1c_1 - 2e_1c_2\right) \\
\frac{\partial c_2}{\partial t_2} &= \frac{\partial}{\partial x}\left(c_2' - c_2^2 - 2e_2c_2\right) \\
\frac{\partial e_2}{\partial t_2} &= \frac{\partial}{\partial x}\left(-e_2' - e_2^2 - 2e_2c_2 - 2(e_1' + c_1e_1)\right)
\end{aligned}
\tag{3.56}
$$

These equations have a dependence on 2 parameters (x, t_2), to eliminate one of them, we make the self-similarity limit of these equations. As conjectured by ARS this should imply equations with the Painlevé property.

To make such reduction we adopt a new variable $\xi = x/\sqrt{t_2}$ and define new fields $\tilde{e}_j(\xi)$ and $\tilde{c}_j(\xi)$:

$$
\tilde{e}_j(\xi) = e(x,t)\sqrt{t_2}, \qquad \tilde{c}_j(\xi) = c(x,t)\sqrt{t_2}, \qquad j = 1, 2. \tag{3.57}
$$

The set of equations above simplify in the way of both sides becoming total derivatives and their left-hand-sides changed by a factor:

$$
\begin{aligned}
\frac{-\xi c_1}{2} &= c_1' - c_1^2 - 2e_1c_1 + 2c_2' - 2c_1c_2 + k_1 \\
\frac{-\xi e_1}{2} &= -e_1' - e_1^2 - 2e_1c_1 - 2e_1c_2 + \bar{k}_1 \\
\frac{-\xi c_2}{2} &= c_2' - c_2^2 - 2e_2c_2 + k_2 \\
\frac{-\xi e_2}{2} &= -e_2' - e_2^2 - 2e_2c_2 - 2(e_1' + c_1e_1) + \bar{k}_2
\end{aligned}
\tag{3.58}
$$

We can simplify it even more, adopting new pairs of canonical variables:

$$
Y_1 = \tilde{e}_1 + \tilde{c}_1 + 2\tilde{c}_2, \qquad Y_2 = \tilde{e}_2 + \tilde{c}_2, \quad \tilde{e}_1(\xi) \quad \text{and} \quad \tilde{e}_2(\xi) \tag{3.59}
$$

To make the notation henceforth simpler, we will drop the tilde from the $\tilde{e}_i(\xi)$. This allows us to make the Hamiltonian:

$$\begin{aligned}\mathcal{H}_{A_4^{(1)}}(e_1, Y_1, e_2, Y_2) = & -\sum_{j=1}^{2} e_j\left(Y_j - \frac{\xi}{2}\right)(Y_j - e_j) + 2e_1\left(Y_1 - \frac{\xi}{2}\right)(Y_2 - e_2) \\ & + \bar{k}_1 Y_1 + \bar{k}_2 Y_2 - k_1 e_1 - k_2 e_2.\end{aligned} \tag{3.60}$$

where the canonical variable obey

$$\{e_i, Y_j\} = \delta_{ij}, \qquad i, j = 1, 2 \tag{3.61}$$

with relations:

$$e_{j,\xi} = \frac{\partial \mathcal{H}_{A_4^{(1)}}}{\partial Y_j}, \qquad Y_{j,\xi} = -\frac{\partial \mathcal{H}_{A_4^{(1)}}}{\partial e_j}, \qquad j = 1, 2 \tag{3.62}$$

The above Hamiltonian can be rewritten again in terms of new canonical variables in the intention of make evident the Painlevé equation contained there. To do that, we employ:

$$\begin{aligned} e_1 =& -q_1, & e_2 =& p_2 + q_2 + \xi/2, \\ Y_1 =& -q_2 - p_2 - p_1, & Y_2 =& p_2 + q_1 + \xi/2 \end{aligned}$$

$$\alpha_1 = \bar{k}_1, \quad \alpha_2 = -k_1 - \bar{k}_2, \quad \alpha_3 = \bar{k}_2 - k_2, \quad \alpha_4 = k_2 + \bar{k}_1$$

with these variables, we obtain the old Hamiltonian as:

$$\mathcal{H}_{A_4^{(1)}} = \sum_{j=1}^{2} p_j q_j (p_j + q_j + \xi/2) + 2p_1 q_1 p_2 - \sum_{j=1}^{2} \alpha_{2j} q_j + \sum_{j=1}^{2} p_j \left(\sum_{k=1}^{j} \alpha_{2k-1}\right) \tag{3.63}$$

By this Hamiltonian, the Painlevé equation becomes clear when one uses:

$$p_i = f_{2i}, \qquad q_i = \sum_{k=1}^{i} f_{2k-1}, \qquad i = 0, 1, 2, 3, 4 \tag{3.64}$$

Which implies the equations in the self-similarity limit to be:

$$f_{i,\xi} = f_i(f_{i+1} - f_{i+2} + f_{i+3} - f_{i+4}) + \alpha_i, \qquad i = 0, ..., 4 \tag{3.65}$$

that are precisely the $A_4^{(1)}$ Painlevé equations if the conditions $f_0 + \ldots + f_4 = -\xi/2$ and $\alpha_0 + \ldots + \alpha_4 = -1/2$ are satisfied.

The above Painlevé equations are invariant under the Bäcklund transformations that obeys the extended affine Weyl group $A_4^{(1)}$:

$$
\begin{aligned}
&s_i(\alpha_i) = -\alpha_i, && s_i(\alpha_j) = \alpha_j + \alpha_i \, (j = i \pm 1), && s_i(\alpha_j) = \alpha_j (j \neq i, i \pm 1),\\
&s_i(f_i) = f_i, && s_i(f_j) = f_j \pm \frac{\alpha_i}{f_i} \, (j = i \pm 1), && s_i(f_j) = f_j (j \neq i, i \pm 1),\\
&\pi(\alpha_j) = \alpha_{j+1}, && \pi(f_j) = f_{j+1}. &&
\end{aligned}
\tag{3.66}
$$

☞ [30] [31]

3.8 Dirac reduction

Now that we have already made explicit exposure of the construction of the model we are dealing with, let's start by looking for their reductions, which is precisely what is general in this model, it has many possible models for reductions.

The first Dirac reduction to be performed starts by the constraint:

$$c_2 = -e_2 = c \tag{3.67}$$

making the system of fields on the 4-bosons model to transform like

$$(e_1, c_1, e_2, c_2) \to (\tilde{e}_1 = e_1, \tilde{c}_1 = c_1, \tilde{e}_2 = (e_2 - c_2)/2, \tilde{c}_2 = (c_2 - e_2)/2) \tag{3.68}$$

with Dirac bracket:

$$\{c(x), c(y)\} = \frac{1}{2}\delta_x(x - y) \tag{3.69}$$

After the self similarity reduction, the same done in (3.57), which basically turns $\partial f/\partial t_2 \to -(xf)_x/2$, we arrive at

$$-c_1 x/2 = c_1' + 2c' - c_1^2 - 2e_1c_1 - 2c_1c + k_1 \tag{3.70}$$

$$-e_1 x/2 = -e_1' - e_1^2 - 2e_1c_1 - 2e_1c + \bar{k}_1 \tag{3.71}$$

$$-cx/2 = e_1' + e_1c_1 + k \tag{3.72}$$

Eliminating c and c_1 from the equations above, and making the replacements $y = 2e_1/x$ then followed by $z = \frac{x^2}{4}$, yields:

$$
\begin{aligned}
y_{zz} = &-\frac{y_z}{z} + \left(\frac{1}{2y} + \frac{1}{2(y-1)}\right) y_z^2 - \frac{\alpha y}{z^2(y-1)}\\
&- \frac{\beta(y-1)}{z^2 y} - \frac{\gamma}{z} y(y-1) - \delta y(y-1)(2y-1)
\end{aligned}
$$

with constants

$$\alpha = \frac{1}{8}\left(1+4k+2\bar{k}_1\right)^2, \qquad \beta = -\frac{\bar{k}_1^2}{2}, \tag{3.73}$$

$$\gamma = \frac{1}{2}(1+4k+4k_1), \qquad \delta = -\frac{1}{2} \tag{3.74}$$

which becomes the standard P_V by the transformation $w = y/(y-1)$.

Another possible transformation is done by using the results of the next section. There is not a smooth way to approach this so the reader would have to accept for awhile the pure existence of a system of equations that it will be shown that has a connection with our system.

In the next section we will show how we obtain such set of equations, but for awhile they are just:

$$zq_z = q(q - r_1 z^{-C})(2p + r_0 z^{C+1}) - (\alpha_1 + \alpha_3 + C)q + \alpha_1 r_1 z^{-C} + \epsilon_0 r_0 z^{C+1} \tag{3.75}$$

$$zp_z = p(p + r_0 z^{C+1})(r_1 z^{-C} - 2q) + (\alpha_1 + \alpha_3 + C)p - \alpha_2 r_0 z^{1+C} - \epsilon_1 r_1 z^{-C} \tag{3.76}$$

and

$$zq_z = q(q - r_1)(2p + r_0 z) - (\alpha_1 + \alpha_3)q + \alpha_1 r_1 + \epsilon_0 r_0 z^{1+2C} \tag{3.77}$$

$$zp_z = p(p + r_0 z)(r_1 - 2q) + (\alpha_1 + \alpha_3)p - \alpha_2 r_0 z - \epsilon_1 r_1 z^{-2C} \tag{3.78}$$

These two sets come from the same mixed system (shown in next section) and in fact the 4-boson model reduces to each of the sets above, for the first when it has the limits:($C = 0, r_0 = 1, \epsilon_1 = 0$). For the second, when it has the limits ($C = -1/2, r_0 = 1, \epsilon_1 = -1$).

The full reduction is a little bit tedious and is left to the reader in the Appendix F, but basically follows an idea that we have found the constraints that bring the 4-boson model to the mixed set.

☞ [30][31]

3.9 The mixed P_{III}-P_V system

As was said in the last section, and demonstrated in the Appendix F, we could reduce the integrable 4-boson model to a system of equations. Here we show that these equations are special cases of a general mixed system of equations, namely:

$$zf_{i,z} = f_i f_{i+2}(f_{i+1} - f_{i+3}) + (-1)^i f_i(\alpha_1 + \alpha_3 + C) + \alpha_i(f_i + f_{i+2}) - (-1)^{[i/2]}\epsilon_{i+1}(f_{i+1} + f_{i+3}) \tag{3.79}$$

where $i \in \mathbb{Z}_4, \epsilon \in \mathbb{Z}_2$, and $[i/2]$ is $i/2$, if i is even or $(i+1)/2$ if i is odd.

Explicitly:

$$zf_0' = (C + \alpha_1 + \alpha_3) f_0 + \alpha_0 (f_0 + f_2) + f_0 f_2 (f_1 - f_3) - \epsilon_1 (f_1 + f_3) \tag{3.80}$$
$$zf_1' = -(C + \alpha_1 + \alpha_3) f_1 + \alpha_1 (f_1 + f_3) + f_1 f_3 (-f_0 + f_2) + \epsilon_0 (f_0 + f_2) \tag{3.81}$$
$$zf_2' = (C + \alpha_1 + \alpha_3) f_2 + \alpha_2 (f_0 + f_2) + f_0 f_2 (-f_1 + f_3) + \epsilon_1 (f_1 + f_3) \tag{3.82}$$
$$zf_3' = -(C + \alpha_1 + \alpha_3) f_3 + \alpha_3 (f_1 + f_3) + f_1 f_3 (f_0 - f_2) - \epsilon_0 (f_0 + f_2) \tag{3.83}$$

The above system is actually the one that embed the $A_3^{(1)}\, P_V$ equation with the $B_2^{(1)}\, P_{III}$ equation. These equations differ from the standard $A_3^{(1)}\, P_V$ equations by a deformation by the cyclic parameters $\epsilon_i = \epsilon_{i+2}$ which are ϵ_0 and ϵ_1, and also by the usual parameters α_i, $i = 0, 1, 2, 3$ of the affine Weyl structure that also depend on C.

We can sum (3.81) and (3.83), and, separately, also sum (3.80) and (3.82) to obtain:

$$zf_{1,z} + zf_{3,z} = C(f_1 + f_3) \tag{3.84}$$
$$zf_{0,z} + zf_{2,z} = (C + \alpha_0 + \alpha_1 + \alpha_2 + \alpha_3)(f_2 + f_0) \tag{3.85}$$

Adopting $(\alpha_0 + \alpha_1 + \alpha_2 + \alpha_3) = \Omega$, we are able to integrate both terms simultaneously, giving us 2 constants of integration (r_0 and r_1), respectively:

$$f_1 + f_3 = r_1 z^{-C}, \qquad f_0 + f_2 = r_0 z^{(C+\Omega)} \tag{3.86}$$

Without any loss of generality, we will deal with $\Omega = 1$.

It's noticeable then they have the same behavior for $C = -1/2$. Further, for $C = 0$, $f_1 + f_3$ becomes a constant, the happens for $f_0 + f_2$ when $C = -1$.

We are able now to perform a change of variables defining the following canonical variables:

$$f_1 = qz^{-C}, \qquad f_2 = -pz^{C} \tag{3.87}$$

and then logically:

$$f_3 = r_1 z^{-C} - qz^{-C}, \qquad f_4 = r_0 z^{C+\Omega} + pz^{C} \tag{3.88}$$

With that we reduce our 4 equations ((3.80),(3.81),(3.82),(3.83)) to just 2:

$$zq_z = q(q - r_1)(2p + r_0 z) - (\alpha_1 + \alpha_3)q + \alpha_1 r_1 + \epsilon_0 r_0 z^{1+2C} \tag{3.89}$$
$$zp_z = p(p + r_0 z)(r_1 - 2q) + (\alpha_1 + \alpha_3)p - \alpha_2 r_0 z - \epsilon_1 r_1 z^{-2C} \tag{3.90}$$

We can eliminate p and q from these expressions, and then get 2 second-order differential equations, for q and p, which are not mixed anymore:

$$\begin{aligned} q_{zz} = &-\frac{q_z}{z} + \left(\frac{1}{2q} + \frac{1}{2(q-r_1)}\right)\left(q_z^2 - \epsilon_0^2 r_0^2 z^{4C}\right) - (2\alpha_2 + \alpha_1 + \alpha_3 - 1)\frac{(q-r_1)\,qr_0}{z} \\ &+ \frac{qr_0^2}{2}(q-r_1)(2q-r_1) + \frac{\alpha_1^2 r_1(q-r_1)}{2z^2 q} - \frac{\alpha_3^2 r_1 q}{2z(q-r_1)} \\ &+ \frac{\epsilon_0 r_0 z^{-j-2}}{q(q-r_1)}\left((\alpha_1+\alpha_3-j)q^2 + qr_1(j-2\alpha_1) + \alpha_1 r_1^2\right) - 2\epsilon_1 r_1 z^{j-1} q(q-r_1), \end{aligned} \tag{3.91}$$

$$\begin{aligned} p_{zz} = &-\frac{pp'}{z\,(p+zr_0)} + \left(\frac{1}{p} + \frac{1}{(p+zr_0)}\right)\left(\frac{p'^2}{2} - \frac{z^{2j} r_1^2 \epsilon_1^2}{2}\right) - 2z^{-1+2C} pr_0\,(p+zr_0)\,\epsilon_0 \\ &+ \frac{z^{j-1}\left(p^2\,(2C+\alpha_1+\alpha_3) - zp\,(j+2\alpha_2)\,r_0 - z^2\alpha_2 r_0^2\right) r_1\epsilon_1}{p\,(p+zr_0)} \\ &- \frac{r_0\left(-p^2\left(\alpha_1^2 + 2\alpha_2\,(-1+\alpha_3) + (-2+\alpha_3)\,\alpha_3 + 2\alpha_1\,(-1+\alpha_2+\alpha_3)\right) + 2zp\alpha_2^2 r_0 + z^2\alpha_2^2 r_0^2\right)}{2zp\,(p+zr_0)} \\ &- \frac{p\,(\alpha_1-\alpha_3)\,(p+zr_0)\,r_1}{z^2} + \frac{p\,(p+zr_0)\,(2p+zr_0)\,r_1^2}{2z^2}. \end{aligned} \tag{3.92}$$

where $j \equiv -(1+2C)$.

3.9.1 Reductions for q_{zz}

Our q_{zz} equation (3.91) in the limit $\epsilon_0 = \epsilon_1 = 0$ and $r_0 = 1$, but keeping r_1 arbitrary, then followed by the Ricatti substitution:

$$q(z) = \frac{1}{1-y(z)} \tag{3.93}$$

becomes the standard P_V equation with coefficients:

$$\begin{aligned} \alpha =& \frac{{\alpha_1}^2}{2}, & \beta =& -\frac{{\alpha_3}^2}{2}, \\ \delta =& -\frac{{r_0}^2}{2}, & \gamma =& -r_0\,(\alpha_1 + 2\alpha_2 + \alpha_3 - 1) \end{aligned} \tag{3.94}$$

The standard P_V is given in the first chapter and reproduced here for convenience:

$$y'' = \frac{3y-1}{2y(y-1)}(y')^2 - \frac{y'}{z} + \frac{1}{z^2}(y-1)^2\left(\alpha y + \frac{\beta}{y}\right) + \frac{\gamma y}{z} + \frac{\delta y(y+1)}{y-1}$$

Another way of getting a P_V out of (3.91) is by taking the limits $C = -1/2$, $r_0 = 0$, $r_1 = 1$ and then performing the transformation $q(z) = \frac{1}{1-y(z)}$ we obtain the P_V equation with:

$$\alpha = \frac{\alpha_1^2}{2}, \qquad \beta = -\frac{\alpha_3^2}{2}, \qquad \delta = 0, \qquad \gamma = -2\epsilon_1 \tag{3.95}$$

We are also able to reduce (3.91) in the limits of $r_1 = 0$ and $C = 0$ to the P_{III} equation with coefficients:

$$\alpha = r_0(1 - \alpha_1 - 2\alpha_2 - \alpha_3), \qquad \beta = r_0^2, \qquad \delta = -r_0^2\epsilon_0^2, \qquad \gamma = r_0^2 \tag{3.96}$$

The standard P_{III} equation is reproduced here also for convenience:

$$y'' = \frac{(y')^2}{y} - \frac{y'}{z} + \frac{1}{z}(\alpha y^2 + \beta) + \gamma y^3 + \frac{\delta}{y}$$

☞ [31]

Chapter 4

Solutions of the mixed P_{III}-P_V

4.1 Painlevé tests for q_{zz} and p_{zz}

Now we may look if we are able to find the existence of branch points on our equations.

As we are dealing with non-autonomous systems, we shall perform 2 analysis, for $z_0 = 0$ and $z_0 \neq 0$, in the plan of work described in Appendix C.

4.1.1 q_{zz}, $z_0 = 0$

Taking $q(z) \backsim c_0 z^p$ and simplifying, we get:

$$c_0 p(p-1)z^{p-2} = -c_0 p z^{p-2} + c_0 p^2 z^{p-2} - \frac{\epsilon^2 r_0^2}{c_0} z^{4C-p} - r_0 c_0^2 z^{2p-1}(2\alpha_2 + \alpha_1 + \alpha_3 - 1) + r_0 c_0^3 z^{3p} + \frac{\alpha_1^2 r_1}{2} z^{-2} - \frac{\alpha_3^2 r_1}{2} z^{-1} + \epsilon_0 r_0 z^{-j-2}(\alpha_1 + \alpha_3 - j) - 2\epsilon_1 r_1 z^{j-1} \quad (4.1)$$

C=0, j=-1

In this case the choice for leading powers are $(p-2,\ 2p-1,\ 3p,\ -3)$, because they are the ones we can achieve the lowest negative values for p, being $p = -1$. Picking only the above terms with such powers we get, after some cancellations:

$$c_0^3 r_0 - c_0^2(2\alpha_2 + \alpha_1 + \alpha_3 - 1) = 2\epsilon_0 \epsilon_1 r_1 \quad (4.2)$$

Immediately we notice that when we keep ϵ_1 or ϵ_0 or $r_1 \neq 0$ the system should have 3 different possible values at power z^{-1}, but if we plug such idea in the equation, we find out that such is an impossible task, failing the third step in the Kowalevski-Gambier method, thus it's not integrable.

If ϵ_1 or ϵ_0 or $r_1 = 0$, then

$$c_0 = \frac{2\alpha_2 + \alpha_1 + \alpha_3 - 1}{r_0}$$

And our equation really passes the third step, giving a full expansion. The case of $r_1 = 0$ is precisely the P_{III} limit treated before, the other two besides passing this fast Painlevé test must be object of further investigation.

4.1.2 q_{zz}, $z_0 \neq 0$

Taking $q(z) \backsim c_0(z-z_0)^p = c_0 w^p$, when $w \to 0$ and $z \to z_0$, and simplifying, we get:

$$\begin{aligned} c_0 p(p-1)w^{p-2} = &- \frac{c_0 p}{z_0} w^{p-1} + c_0 p^2 w^{p-2} - \frac{\epsilon^2 r_0^2}{c_0} z_0^{4C} w^{-p} - (2\alpha_2 + \alpha_1 + \alpha_3 - 1)\frac{r_0 c_0^2}{z_0} w^{2p} \\ &+ r_0 c_0^3 w^{3p} + \frac{\alpha_1^2 r_1}{2 z_0^2} - \frac{\alpha_3^2 r_1}{2 z_0^1} + \epsilon_0 r_0 z_0^{-j-2}((\alpha_1 + \alpha_3 - j) - 2\epsilon_1 r_1 z_0^{j-1}) \end{aligned} \tag{4.3}$$

C=0, j=-1

For this case, the leading powers are $p-2$ and $3p$, which give us the lowest negative power of $q(z)$, $p = -1$, picking only the terms above with these powers, we may arrange them to give:

$$2c_0 = c_0 + r_0 c_0^3 \quad \longrightarrow \quad c_0^2 = \frac{1}{r_0^2}$$

4.1.3 p_{zz}, $z_0 = 0$

Taking $p(z) \backsim c_0 z^p$ and simplifying, we get:

$$\begin{aligned} c_0 p(p-1) z^{p-2} = &- c_0^2 p z^{2p-2} + c_0 p^2 z^{p-2} - z^{-2-p-4C}\frac{r_1^2 \epsilon_1^2}{c_0} - 2 r_0 \epsilon_0 c_0^2 z^{2p+2C-1} + z^{-2(1+C)}(...) \\ &+ z^{-1}\frac{r_0}{2}(...) - r_1(\alpha_1 - \alpha_3) c_0^2 z^{2p-2} + r_1^2 c_0^3 z^{3p-2} \end{aligned} \tag{4.4}$$

which does not admit an expansion with negative integer p, failing the Painlevé test.

4.2 Briot-Bouquet system

As was already done before in this presentation, the work with Briot-Bouquet Theorem for convergence in Painlevé equations have been very successful, so we will apply it on our equation also.

Using as a basis the work of [32] that has the Briot-Bouquet system for the P_V, we may adapt it to our situation.

First we really make the connection identifying the equations presented in [32]:

$$\begin{aligned} z u_z &= -h - (a+c)u + v/4 - u^2 v \\ z v_z &= \gamma z + 2\delta z^2 u + (a+c)v + u v^2 \end{aligned} \tag{4.5}$$

where $h = (a-c)/2$, $a = \sqrt{2\alpha}$ and $c = \sqrt{-2\beta}$, for the α and β from the P_V equation.

4.2.1 $\epsilon_0 = \epsilon_1 = 0$

Replacing properly the terms with $q = u + r_1/2$ and $p = -(v + r_0 z)/2$ we obtain:

$$\begin{aligned} zu_z &= \frac{r_1(\alpha_1-\alpha_3)}{2} - (\alpha_1+\alpha_3)u + \frac{vr_1^2}{4} - u^2 v \\ zv_z &= (\alpha_1+\alpha_3+2\alpha_2-1)r_0 z - r_0^2 z^2 u + (\alpha_1+\alpha_3)v + uv^2 \end{aligned} \tag{4.6}$$

Using the series $u = \sum_{i=0}^{\infty} u_i z^i$ and $v = \sum_{i=0}^{\infty} v_i z^i$, the Briot-Bouquet system holds for the coordinates $\bar{u} = u - u_0$ and $\bar{v} = v - v_0$, giving $v_0 = 0$ and automatically $u_0 = \frac{r_1(\alpha_1-\alpha_3)}{2(\alpha_1+\alpha_3)}$.

4.2.2 Negative j

Using $j < 0$ and $\epsilon_0 \neq 0$ and $\epsilon_1 = 0$, the system becomes:

$$\begin{aligned} zu_z &= \frac{r_1(\alpha_1-\alpha_3)}{2} - (\alpha_1+\alpha_3)u + \frac{vr_1^2}{4} - u^2 v + \epsilon_0 r_0 z^{|j|} \\ zv_z &= (\alpha_1+\alpha_3+2\alpha_2-1)r_0 z - r_0^2 z^2 u + (\alpha_1+\alpha_3)v + uv^2 \end{aligned} \tag{4.7}$$

Also considering the aforementioned power series, we must have the same condition for convergence once the term $\epsilon_0 r_0 z^{|j|}$ goes to zero with $z \to 0$.

4.2.3 Positive j

Using $j > 0$ and $\epsilon_0 \neq 0$ and $\epsilon_1 = 0$, the system becomes:

$$\begin{aligned} zu_z &= \frac{r_1(\alpha_1-\alpha_3)}{2} - (\alpha_1+\alpha_3)u + \frac{vr_1^2}{4} - u^2 v + \epsilon_0 r_0 z^{-j} \\ zv_z &= (\alpha_1+\alpha_3+2\alpha_2-1)r_0 z - r_0^2 z^2 u + (\alpha_1+\alpha_3)v + uv^2 \end{aligned} \tag{4.8}$$

From the relations $u = q - r_1/2$ and $v = -2p - r_0 z$, it follows that u starts at $u_j z^j$ and $v = -4\epsilon_0 r_0 z^{-j}/r_1^2 - 2(\alpha_1-\alpha_3)/r_1 + v_{2j} z^{2j} + \ldots$.

Adopting $p_> \equiv \sum_{i=1}^{\infty} p_i z^i$, $u \equiv q_> = \sum_{i=1}^{\infty} q_i z^i$ and $v = -4\epsilon_0 r_0 z^{-j}/r_1^2 - 2(\alpha_1-\alpha_3)/r_1 + v_>$, and we need to show that $q_>$ and $v_>$ satisfy the Briot-Bouquet system of equations with holomorphic and bounded (for small z) expressions.

Rewriting zu_z:

$$zu_z = -(\alpha_1+\alpha_3)u + \frac{v_> r_1^2}{4} - u^2\left(-\frac{4\epsilon_0 r_0}{r_1^2} z^{-j} - 2\frac{\alpha_1-\alpha_3}{r_1} + v_>\right) \tag{4.9}$$

Then rewriting everything in terms of $q_>$ and $p_>$:

$$
\begin{aligned}
zq_{>z} =& q_>^2(2p_0 + 2p_> + r_0 z) + q_>^2\frac{4\epsilon_0 r_0}{r_1^2}z^{-j} - \frac{r_1^2}{2}(2p_> + r_0 z) - (\alpha_1 + \alpha_3)q_> \qquad (4.10)\\
zp_{>z} =& \left(\frac{2\epsilon_0 r_0}{r_1^2}\right)^2 z^{-2j}(-2q_> + 2q_j z^j)\\
&+ \frac{4\epsilon_0 r_0}{r_1^2}z^{-j}(p_0 + p_>)(-2q_>) + \frac{2\epsilon_0 r_0}{r_1^2}z^{-j}r_0 z(-2q_>)\\
&+ (p_0 + p_> + r_0 z)(p_0 + p_>)(-2q_>) + (\alpha_1 + \alpha_3)(p_o + p_>) - \alpha_2 r_0 z. \qquad (4.11)
\end{aligned}
$$

As $q_>$ starts at z^j, the right hand side of (4.10) starts at z^j as well. Furthermore, since $q_> = q_j z^j$ starts at order z^{2j}, the right hand side of (4.11) starts at order z. Therefore both sides of (4.11) and (4.10) will vanish for $z \to 0$.

4.3 Rational solutions by Padé method

The search for rational solutions of (3.91) is of high importance once such differential equation cannot be reduced to some traditional combination of special functions, but under certain conditions on the parameters we can obtain exact rational functions as particular solutions.

Our goal in this section is to show a systematic way of finding such solutions roughly by an oriented try-and-error method which will be the 2-point Padé approximant.

The power of shrinking many terms of a truncated power series in a short rational function is well and long known for the Padé approximants. Here we make use of such good representation of powers series to actually turn it into a solution by setting all the discrepancies from such approximation to the actual solution to zero, that is, specifying the conditions for the parameters of such discrepancies for it to be a solution.

4.3.1 2-point Padé approximant rational solutions

First let us briefly present the 1-point Padé approximant.

Once you have, for example, a Taylor series expansion of a certain function (or power series solution of a differential equation) $f(x)$ around some number x_0 we can truncate it at a certain order, say τ: $a_0 + a_1 z + \ldots + a_\tau z^\tau$, (where $z = (x - x_0)$) and then compare its coefficients with the ones of the expansion of a rational function like:

$$
f(x) \backsim \frac{b_0 + b_1 x + b_2 x^2 + \ldots + b_s x^s}{1 + c_1 x + c_2 x^2 + \ldots + c_r x^r} \qquad (4.12)
$$

where the coefficients of the above rational expression are (for $x_0 = 0$):

$$
b_0 + (b_1 - b_0 c_1)\,x + \left(b_2 - b_1 c_1 + b_0 c_1^2 - b_0 c_2\right)x^2 + \ldots \qquad (4.13)
$$

If we have the condition $r+s=\tau$ satisfied, we'll be able to find all $b's$ and $c's$ in terms of $a's$.

For a 2-point Padé approximant the proceeding is quite the same, but now we'll have 2 expansions: $a_0+a_1z+\ldots+a_\tau z^\tau$ and $d_0+d_1\zeta+\ldots+d_\rho\zeta^\rho$, where $z=(x-x_0)$ and $\zeta=(x-x_1)$, and now we make 2 expansions for our rational function (here on called just Padé, for brevity), one around x_0 and other around x_1 and compare with the expansions above again, the calculus may just work for $\tau+\rho=r+s+1$.

Example

As an useful example, let us take the P_V equation with parameters $\alpha,\beta,\delta,\gamma$:

$$q''=\left(\frac{1}{2q}+\frac{1}{q-1}\right)(q')^2-\frac{q'}{z}+\frac{(q-1)^2}{z^2}\left(\alpha q+\frac{\beta}{q}\right)+\frac{q\gamma}{z}+\delta q\frac{(q+1)}{q-1}. \tag{4.14}$$

With Taylor expansion around zero (for small z):

$$-\frac{i\sqrt{\beta}}{\sqrt{\alpha}}+\frac{iz\sqrt{\beta}\gamma}{\sqrt{\alpha}\left(-1+2\alpha+4i\sqrt{\alpha}\sqrt{\beta}-2\beta\right)}+\ldots \tag{4.15}$$

And an asymptotic expansion (for large z):

$$-1+\frac{2\gamma}{z\delta}+\frac{2\left(-\gamma^2+4\alpha\delta+4\beta\delta\right)}{z^2\delta^2}-\frac{2\left(-\gamma^3-2\gamma\delta+16\alpha\gamma\delta\right)}{z^3\delta^3}+\ldots \tag{4.16}$$

We want to create a (1,1)Pade ($\frac{a_0+a_1z}{1+b_1z}$), so we will need 3 terms combined from the asympotic and Taylor expansion of P_V.

Taking $-\frac{i\sqrt{\beta}}{\sqrt{\alpha}}$ (from Taylor) and $-1+\frac{2\gamma}{z\delta}$ (from asymptotic), we are able to compare the expansions and get a first "raw" Pade:

$$a_0=-\frac{i\sqrt{\beta}}{\sqrt{\alpha}},\, a_1=-\frac{\left(\sqrt{\alpha}-i\sqrt{\beta}\right)\delta}{2\sqrt{\alpha}\gamma},\, b_1=\frac{\left(\sqrt{\alpha}-i\sqrt{\beta}\right)\delta}{2\sqrt{\alpha}\gamma}. \tag{4.17}$$

Inserting such Pade back to the P_V equation, we obtain a large expression which we can separate a whole denominator which must be zero at each power of z, a condition that simplifies most of the equations is $\beta=-\alpha$. The equations which do not reduce to zero in this limit can be taken there by applying the second condition of $\alpha=\frac{1}{8}$ and then we obtain a solution of the P_V with specific conditions:

$$q(z)=\frac{\gamma-z\delta}{\gamma+z\delta},\qquad \text{for} \qquad \alpha=\frac{1}{8} \quad \text{and} \quad \beta=-\frac{1}{8}. \tag{4.18}$$

4.3.2 C=0, Taylor series starting with constant

Now we are capable of looking for rational solutions for our equation (3.91), let's begin with the simplifying limit of $\epsilon_1 = 0$ and $c = 0$, we get for the Taylor expansion around zero:

$$q(z) = \frac{\alpha_1 r_1}{\alpha_1+\alpha_3} + \\ \frac{z r_0\left(\alpha_1^3\epsilon_0 + (\alpha_3-1)\,\alpha_3^2\epsilon_0 - \alpha_1^2\left(\alpha_3\left(r_1^2-3\epsilon_0\right)+\epsilon_0\right) + \alpha_1\alpha_3\left((1-2\alpha_2-\alpha_3)\,r_1^2 + (-2+3\alpha_3)\,\epsilon_0\right)\right)}{(\alpha_1+\alpha_3)^{\,2}\left(-1+\alpha_1^2+2\alpha_1\alpha_3+\alpha_3^2\right)} + \dots \tag{4.19}$$

And for the asymptotic (around ∞) we obtain 5 different series expansions due to a fifth-root of the constant term, so we'll choose one of those:

$$q(z) = \frac{1}{2}\left(r_1 + \sqrt{r_1^2+4\epsilon_0}\right) + \tag{4.20}$$

$$\frac{4\,(-1+\alpha_2)\,\epsilon_0 + \alpha_1 r_1\left(-r_1+\sqrt{r_1^2+4\epsilon_0}\right) - \alpha_3 r_1\left(r_1+\sqrt{r_1^2+4\epsilon_0}\right)}{2 z r_0\left(r_1^2+4\epsilon_0\right)} + \dots \tag{4.21}$$

As we have this repeating term $\sqrt{r_1^2+4\epsilon_0}$, we'll call it A and eliminate ϵ_0. As done in the example above we'll go for a (1,1)Padé, solving it we obtain it's parameters:

$a_0 = \frac{\alpha_1 r_1}{\alpha_1+\alpha_3}$

$a_1 = \frac{\left(r_0(A+r_1)\left(\alpha_1^3\left(A^2-r_1^2\right)+(-1+\alpha_3)\alpha_3^2\left(A^2-r_1^2\right)+\alpha_1^2\left(-A^2+r_1^2+\alpha_3\left(3A^2-7r_1^2\right)\right)+\alpha_1\alpha_3\left(\alpha_3\left(3A^2-7r_1^2\right)-2\left(A^2+(-3+4\alpha_2)r_1^2\right)\right)\right)\right)}{4\left(\alpha_1^3+3\alpha_1^2\alpha_3+\alpha_3\left(-1+\alpha_3^2\right)+\alpha_1\left(-1+3\alpha_3^2\right)\right)(\alpha_1(A-r_1)+\alpha_3(A+r_1))}$

$b_1 = \frac{r_0\left(\alpha_1^3\left(A^2-r_1^2\right)+(-1+\alpha_3)\alpha_3^2\left(A^2-r_1^2\right)+\alpha_1^2\left(-A^2+r_1^2+\alpha_3\left(3A^2-7r_1^2\right)\right)+\alpha_1\alpha_3\left(\alpha_3\left(3A^2-7r_1^2\right)-2\left(A^2+(-3+4\alpha_2)r_1^2\right)\right)\right)}{2\left(\alpha_1^3+3\alpha_1^2\alpha_3+\alpha_3\left(-1+\alpha_3^2\right)+\alpha_1\left(-1+3\alpha_3^2\right)\right)(\alpha_1(A-r_1)+\alpha_3(A+r_1))}$

Sending it back to the equation we get our first condition:

$$A = \frac{(-\alpha_1+\alpha_3)\,r_1}{\alpha_1+\alpha_3}. \tag{4.22}$$

Now to erase the whole equation one more condition arises: $\alpha_2 = 1-\alpha_1-\alpha_3$, rewriting everything, the Pade and the conditions reduces themselves to:

$$q(z) = \frac{\alpha_1 r_1}{\alpha_1+\alpha_3}, \quad \text{for} \quad \epsilon_0 = -\frac{\alpha_1\alpha_3 r_1^3}{(\alpha_1+\alpha_3)^2} \quad \text{and} \quad \alpha_2 = 1-\alpha_1-\alpha_3. \tag{4.23}$$

We can go even further with this method, as we have discovered the condition on A before the condition on α_2 just because it was easier computationally speaking, doesn't mean that we couldn't find it on the opposite order, in fact if it's imposed $\alpha_2 = 1-\alpha_1-\alpha_3$ and then searched a condition on A we obtain the following:

$$A = \frac{(\alpha_1-\alpha_3)\,r_1}{\alpha_1+\alpha_3}, \qquad A = -\frac{(\alpha_1-\alpha_3)\,r_1}{\alpha_1+\alpha_3}, \qquad A = \frac{(\alpha_1-\alpha_3)\,r_1}{2+\alpha_1+\alpha_3}. \tag{4.24}$$

The first and the second ones above are the same for ϵ_0 (since one is just the opposite of the other, and we must square it to obtain ϵ_0), but the third one is new, in fact we have just obtained another solution with a different condition:

$$q(z) = \frac{\frac{\alpha_1 r_1}{\alpha_1+\alpha_3} + \frac{z r_0 \left(r_1+\frac{(\alpha_1-\alpha_3)r_1}{2+\alpha_1+\alpha_3}\right)\left(\alpha_3\left(-r_1+\frac{(\alpha_1-\alpha_3)r_1}{2+\alpha_1+\alpha_3}\right)+\alpha_1\left(r_1+\frac{(\alpha_1-\alpha_3)r_1}{2+\alpha_1+\alpha_3}\right)\right)}{4\left(\alpha_1+\alpha_1^2+\alpha_3+2\alpha_1\alpha_3+\alpha_3^2\right)}}{1+\frac{z r_0\left(\alpha_3\left(-r_1+\frac{(\alpha_1-\alpha_3)r_1}{2+\alpha_1+\alpha_3}\right)+\alpha_1\left(r_1+\frac{(\alpha_1-\alpha_3)r_1}{2+\alpha_1+\alpha_3}\right)\right)}{2\left(\alpha_1+\alpha_1^2+\alpha_3+2\alpha_1\alpha_3+\alpha_3^2\right)}} \tag{4.25}$$

for $\epsilon_0 = -\frac{(1+\alpha_1)(1+\alpha_3)r_1^2}{(2+\alpha_1+\alpha_3)^2}$ and $\alpha_2 = 1-\alpha_1-\alpha_3$.

By using this imposition of $\alpha_2 = 1-\alpha_1-\alpha_3$, we are able to try a (2,2)Pade, that has the form $\frac{a_0+a_1z+a_2z^2}{1+b_1z+b_2z^2}$, and calculate it quickly. From that we obtain:

$$A = \frac{(-\alpha_1+\alpha_3)\,r_1}{\alpha_1+\alpha_3},\ A = \frac{(\alpha_1-\alpha_3)\,r_1}{\alpha_1+\alpha_3},\ A = \frac{(\alpha_1-\alpha_3)\,r_1}{2+\alpha_1+\alpha_3},\ A = \frac{(\alpha_1-\alpha_3)\,r_1}{4+\alpha_1+\alpha_3} \tag{4.26}$$

As before, the previous results are also solutions for this Pade, in fact, if we use the previous values of A at a higher order Pade, it will just set to zero the higher order terms and return our Pade to the form (and value) of the corresponding Pade for that A, that is, if we use, for instance, any of the 2 first A's above, we'll get again the equation (4.23).

Using the new value for the A, we arrive at:

$$q(z) = \frac{r_1\left(\alpha_1(1+H)(4+H)^3+2z\left(\alpha_1-\alpha_3\right)\left(4+H\right)\left(2+\alpha_3+\alpha_1(4+H)\right)r_0r_1+z^2\left(2+\alpha_1\right)\left(\alpha_1-\alpha_3\right)^2r_0^2}{\left(4+H\right)\left((H)(1+H)(4+H)^2+2z\left(\alpha_1-\alpha_3\right)\left(1+H\right)(4+H)r_0r_1+z^2\left(\alpha_1-\alpha_3\right)^2r_0^2r_1^2\right)} \tag{4.27}$$

$$q(z) = \frac{num}{den}, \tag{4.28}$$

where

$$\begin{aligned} num =& r_1\left(\alpha_1(1+H)(4+H)^3+2z\left(\alpha_1-\alpha_3\right)\left(4+H\right)\left(2+\alpha_3+\alpha_1(4+H)\right)r_0r_1+\right. \\ & \left. z^2\left(2+\alpha_1\right)\left(\alpha_1-\alpha_3\right)^2r_0^2r_1^2\right) \\ den =& (4+H)\left((H)(1+H)(4+H)^2+2z\left(\alpha_1-\alpha_3\right)(1+H)(4+H)r_0r_1+z^2\left(\alpha_1-\alpha_3\right)^2r_0^2r_1^2\right) \end{aligned}$$

where $H = \alpha_1+\alpha_3$, $A = \frac{(\alpha_1-\alpha_3)r_1}{4+\alpha_1+\alpha_3}$ or $\epsilon_0 = -\frac{(2+\alpha_1)(2+\alpha_3)r_1^2}{(4+\alpha_1+\alpha_3)^2}$.

As can be seen now, we can carry on this calculation forever and then getting new results at each calculation with the conditions $\alpha_2 = 1-\alpha_1-\alpha_3$ and $A = \frac{(\alpha_1-\alpha_3)r_1}{2n+\alpha_1+\alpha_3}$ for $n = 1, 2, 3, \ldots$.

In time, in the beginning of this section was said that we should choose one of the 5 possible asymptotic expansions, but what about the others?

Well, it seems that the one we used (the one which starts like $\frac{1}{2}\left(r_1+\sqrt{r_1^2+4\epsilon_0}\right)$) is absolutely equivalent to another one, $\frac{1}{2}\left(r_1-\sqrt{r_1^2+4\epsilon_0}\right)$, in fact, both give the same Pade in the end, and curiously, all these Pades have the asymptotic expansion behavior like the series with minus sign, even if it were calculated initially with the plus sign.

The other 3 possible asymptotic expansions does not have possible Pades.

4.3.3 C=0, Taylor series starting at 1/z

It's also possible to consider the other Taylor expansion available which begins like:

$$q(z) = \frac{\alpha_1 + 2\alpha_2 + \alpha_3 - 1}{zr_0} + \frac{\left(2\alpha_2^2 + 2\alpha_2\left(-1+\alpha_3\right) + \left(-1+\alpha_3\right)\alpha_3 + \alpha_1\left(-1+2\alpha_2+\alpha_3\right)\right)r_1}{\alpha_1^2 + 4\alpha_2^2 + 4\alpha_2\left(\alpha_3-1\right) + \left(\alpha_3-2\right)\alpha_3 + 2\alpha_1\left(2\alpha_2+\alpha_3-1\right)} + \dots \tag{4.29}$$

In this case we must use an adaptation of the usual Pade, and for that we will multiply it by $\frac{1}{z}$ so that the expansion around zero starts at order z^{-1}. By doing this trick, things start being not that simple, we can no longer think on some 2 polynomials and start calculating because the orders of expansions around 0 and ∞ may not match with the ones we have got from the differential equations. This is this case where we can just appeal to a simple 1-point Pade approximant. Doing all the proceeding as before for a (1,1)Pade, but now using 3 terms from the Taylor expansion, we cannot go further.

All of the calculations presented here were made by computer due to their size, actually one could try to make all these calculations forever, for any type of Pade, but as the degree of the polynomials in the Pade get bigger, the calculations grow much faster in complexity and it turns out that this method is not longer useful.

Still exploring this case, our biggest simplification, would be to reduce the Pade to a simple polynomial, which would let us don't use the term proportional to z from the Taylor expansion (which is immense). By doing this we get 2 possible solutions imposing conditions on ϵ_0 and α_2:

$$q(z) = \frac{-1+\alpha_1+\alpha_3}{zr_0} + \frac{\left(-1+\alpha_3\right)r_1}{-2+\alpha_1+\alpha_3} \tag{4.30}$$

for $\epsilon_0 = \frac{(-1+\alpha_1)(-1+\alpha_3)r_1^2}{(-2+\alpha_1+\alpha_3)^2}$ and $\alpha_2 = 0$.

Also

$$q(z) = -\frac{-1+\alpha_1+\alpha_3}{zr_0} + \frac{\left(-1+\alpha_3\right)r_1}{-2+\alpha_1+\alpha_3} \tag{4.31}$$

for $\epsilon_0 = -\frac{(-1+\alpha_1)(-1+\alpha_3)r_1^2}{(-2+\alpha_1+\alpha_3)^2}$ and $\alpha_2 = 1-\alpha_1-\alpha_3$.

If we try to find solutions with conditions on ϵ_0 and α_1, we get:

$$q(z) = \frac{r_1}{2} + \frac{-1+2\alpha_2+2\alpha_3}{zr_0} \tag{4.32}$$

for $\epsilon_0 = \frac{(-1+2\alpha_3)r_1^2}{-4+8\alpha_2+8\alpha_3}$ and $\alpha_1 = \alpha_3$.

Yet some other possibility arises:

$$q(z) = \frac{r_1}{2} + \frac{-1 + 2\alpha_2}{z r_0} \tag{4.33}$$

for $\epsilon_0 = 0$, $\alpha_1 = -\alpha_3 = \pm 1/2$.

The first case for this expansion starting at z^{-1} that we can truly use our method, in the case of $C = 0$, is a Pade like: $\frac{a_0 + a_1 z + a_2 z^2}{z + b_1 z^2}$, but the calculation of the possible conditions for a solution are already too big for a reasonable free calculation. Knowing that limitation all we can do is to attempt some simplification guided by the previous results, that is, shot an imposition over α_2 and seek some condition for ϵ_0 (or the opposite).

By doing so, we impose $\alpha_2 = 1 - \alpha_1 - \alpha_3$ and we arrive at three possible solutions for A in this Pade:

$$A = 0, \qquad A = \frac{(\alpha_1 - \alpha_3)\, r_1}{-2 + \alpha_1 + \alpha_3}, \qquad A = \frac{(\alpha_1 - \alpha_3)\, r_1}{-4 + \alpha_1 + \alpha_3}. \tag{4.34}$$

The two first ones basically give us 0/0 due to the way of calculating our Pade (just taking the numerator of huge fractions equaled to zero, then sometimes the solution that vanishes such numerator also does for the denominator, so we are already aware of it), but the third one give us a new solution which is:

$$q(z) = \frac{-\frac{-1+\alpha_1+\alpha_3}{r_0} - \frac{z(2+\alpha_1-2\alpha_3)r_1}{-4+\alpha_1+\alpha_3} + \frac{z^2(\alpha_1-\alpha_3)(-2+\alpha_3)r_0 r_1^2}{(-4+\alpha_1+\alpha_3)^2(-2+\alpha_1+\alpha_3)}}{z\left(1 + \frac{z(\alpha_1-\alpha_3)r_0 r_1}{(-4+\alpha_1+\alpha_3)(-2+\alpha_1+\alpha_3)}\right)} \tag{4.35}$$

for $\alpha_2 = 1 - \alpha_1 - \alpha_3$ and $\epsilon_0 = -\frac{(-2+\alpha_1)(-2+\alpha_3)r_1^2}{(-4+\alpha_1+\alpha_3)^2}$.

It's interesting to notice the solution $A = \frac{(\alpha_1-\alpha_3)r_1}{-2+\alpha_1+\alpha_3}$ is actually the condition found previously ($\epsilon_0 = -\frac{(-1+\alpha_1)(-1+\alpha_3)r_1^2}{(-2+\alpha_1+\alpha_3)^2}$), but it doesn't fit here.

Keeping track of the developments above we can go even further using this condition on α_2 and then look for the condition on ϵ_0 e discover the rational solution of higher order. Bringing to work the words we have:

For a Pade like $q(z) = \frac{a_0 + a_1 z + a_2 z^2 + a_3 z^3}{z + b_1 z^2 + b_2 z^3}$ we'll get:

$$
\begin{aligned}
a_0 &= -\frac{-1+\alpha_1+\alpha_3}{r_0} \\
a_1 &= -\frac{(3+2\alpha_1-3\alpha_3)\,r_1}{-6+\alpha_1+\alpha_3} \\
a_2 &= -\frac{(\alpha_1^2+\alpha_1(6-4\alpha_3)+3(-2+\alpha_3)\,\alpha_3)\,r_0r_1^2}{(-6+\alpha_1+\alpha_3)^{\,2}(-2+\alpha_1+\alpha_3)} \\
a_3 &= \frac{(\alpha_1-\alpha_3)^{\,2}(-3+\alpha_3)\,r_0^2r_1^3}{(-6+\alpha_1+\alpha_3)^{\,3}(6+\alpha_1^2-5\alpha_3+\alpha_3^2+\alpha_1(-5+2\alpha_3))} \\
b_1 &= \frac{2(\alpha_1-\alpha_3)\,r_0r_1}{(-6+\alpha_1+\alpha_3)(-2+\alpha_1+\alpha_3)} \\
b_2 &= \frac{(\alpha_1-\alpha_3)^{\,2}r_0^2r_1^2}{(-6+\alpha_1+\alpha_3)^{\,2}(6+\alpha_1^2-5\alpha_3+\alpha_3^2+\alpha_1(-5+2\alpha_3))}
\end{aligned}
$$

for $\epsilon_0 = -\frac{(-3+\alpha_1)(-3+\alpha_3)r_1^2}{(-6+\alpha_1+\alpha_3)^2}$ or $A = \frac{r_1(\alpha_3-\alpha_1)}{-6+\alpha_1+\alpha_3}$

For the following one, $\frac{a_0+a_1z+a_2z^2+a_3z^3+a_4z^4}{z+b_1z^2+b_2z^3+b_3z^4}$, we'll have:

$$
\begin{aligned}
a_0 &= -\frac{-1+\alpha_1+\alpha_3}{r_0} \\
a_1 &= -\frac{(4+3\alpha_1-4\alpha_3)\,r_1}{-8+\alpha_1+\alpha_3} \\
a_2 &= \frac{3(\alpha_1^2+\alpha_1(4-3\alpha_3)+2(-2+\alpha_3)\,\alpha_3)\,r_0r_1^2}{(-8+\alpha_1+\alpha_3)^{\,2}(-2+\alpha_1+\alpha_3)} \\
a_3 &= \frac{(12+\alpha_1-4\alpha_3)(\alpha_1-\alpha_3)^{\,2}r_0^2r_1^3}{(-8+\alpha_1+\alpha_3)^{\,3}(6+\alpha_1^2-5\alpha_3+\alpha_3^2+\alpha_1(-5+2\alpha_3))} \\
a_4 &= \frac{(\alpha_1-\alpha_3)^{\,3}(-4+\alpha_3)\,r_0^3r_1^4}{(-8+\alpha_1+\alpha_3)^{\,4}(-24+\alpha_1^3+3\alpha_1^2(-3+\alpha_3)+26\alpha_3-9\alpha_3^2+\alpha_3^3+\alpha_1(26-18\alpha_3+3\alpha_3^2))} \\
b_1 &= \frac{3(\alpha_1-\alpha_3)\,r_0r_1}{(-8+\alpha_1+\alpha_3)(-2+\alpha_1+\alpha_3)} \\
b_2 &= \frac{3(\alpha_1-\alpha_3)^{\,2}r_0^2r_1^2}{(-8+\alpha_1+\alpha_3)^{\,2}(6+\alpha_1^2-5\alpha_3+\alpha_3^2+\alpha_1(-5+2\alpha_3))} \\
b_3 &= \frac{(\alpha_1-\alpha_3)^{\,3}r_0^3r_1^3}{(-8+\alpha_1+\alpha_3)^{\,3}(-24+\alpha_1^3+3\alpha_1^2(-3+\alpha_3)+26\alpha_3-9\alpha_3^2+\alpha_3^3+\alpha_1(26-18\alpha_3+3\alpha_3^2))}
\end{aligned}
$$

for $\epsilon_0 = -\frac{(-4+\alpha_1)(-4+\alpha_3)r_1^2}{(-8+\alpha_1+\alpha_3)^2}$ or $A = \frac{r_1(\alpha_3-\alpha_1)}{-8+\alpha_1+\alpha_3}$.

Still in the case of $C = 0$, we may wonder some other family of solutions derived from the condition now of $\epsilon_0 = 0$ and $\epsilon_1 \neq 0$. Actually the Taylor and asymptotic series for this case are way more simple than the previous ones, but unfortunately any kind of solution that we struggle at give us the same condition: $r_1 = 0$, which is nothing more than reducing our equation to the limit where it is a P_{III}, where there are many known results in the literature.

Now we move on to the case of $C = -1$ ($j = 1$). For this case we also have two types of Taylor expansion, one beginning with a constant, and the other at order z^{-1}. Starting by our most successful case with $C = 0$, the one with order z^{-1}, we arrive at an extremely complex expansion since its very first term, due to fact that such term comes from a quartic equation:

$$-q_{-1}^2 r_0 \left(1 + q_{-1} r_0 - \alpha_1 - 2\alpha_2 - \alpha_3\right) - r_0 \left(-1 + \alpha_1 + \alpha_3\right) \epsilon_0 + \frac{r_0^2 \epsilon_0^2}{q_{-1}} = 0 \tag{4.36}$$

which let all the next terms to become humongous, so we must look for some simplification; the first attempt, and also a successful one, is $a_2 = 1 - a_1 - a_3$, which turns our problem treatable.

By doing the first possible Pade as before, we start with the polynomial $(a_0 + a_1 z)/z$ and use the first 2 terms of one of the Taylor expansions:

$$q(z) = -\frac{-1 + A + \alpha_1 + \alpha_3}{2zr_0} + \frac{\left(-1 + A^2 - 3\alpha_1 - A\alpha_1 - 2\alpha_1 \left(-1 + \alpha_3\right) + \alpha_3 + A\alpha_3 + 2\alpha_1\alpha_3\right) r_1}{2\left(-1 + A^2\right)} + \dots \tag{4.37}$$

where now $A = \sqrt{\left(-1 + \alpha_1 + \alpha_3\right)^2 + 4r_0^2\epsilon_0}$. By the same procedure as before we arrive at two solutions, each with a typical condition on A:

$$q(z) = -\frac{\alpha_1}{zr_0} \tag{4.38}$$

for $\epsilon_0 = -\frac{\alpha_1(-1+\alpha_3)}{r_0^2}$ or $A = 1 + \alpha_1 - \alpha_3$. Also

$$q(z) = -\frac{\alpha_3}{zr_0} + r_1 \tag{4.39}$$

for $\epsilon_0 = -\frac{(-1+\alpha_1)\alpha_3}{r_0^2}$ or $A = 1 - \alpha_1 + \alpha_3$.

If we move to the next Pade, $\frac{a_0+a_1z+a_2z^2}{z+b_1z^2}$, we unfortunately get exactly the same result as before, that is, the same condition that sends a_2 and b_1 to zero and return us to the same $q(z)$, this also happens to all subsequent Pade orders.

Any other type of condition tried, like $\alpha_1 = \alpha_3$ (from (3.91)), were also unsuccessful with the Pade method.

4.4 Laguerre-hypergeometric solutions

For our original first order set of equations, (4.75) and (4.74), one observes that by the special values:

$$p(z) = -r_0 z, \qquad \epsilon_1 = 0, \qquad \alpha_2 = 1 - \alpha_1 - \alpha_3 \tag{4.40}$$

the equation for $p(z)$ is satisfied, and the one for $q(z)$ gets a form which can be treated classically, in fact it becomes a Riccati equation:

$$q_z = -r_0 q(q - r_1) - \frac{(\alpha_1 + \alpha_3)q}{z} + \frac{\alpha_1 r_1}{z} + \epsilon_0 r_0 z^{2C} \tag{4.41}$$

so it can be linearized by the Riccati transformation $q = (1/(\omega r_0))\omega_z$, giving us the equation:

$$\omega_{zz} = (r_0 r_1 - \frac{\alpha_1 + \alpha_3}{z})\omega_z + (\frac{\alpha_1 r_0 r_1}{z} + \epsilon_0 r_0^2 z^{2C})\omega \tag{4.42}$$

Such equation can be solved exactly by the transformation:

$$\omega(z) = z^{-\frac{\alpha_1+\alpha_3}{2}} Exp(\frac{r_0 r_1 z}{2}) f(\tau), \quad \tau = r_0 r_1 z \tag{4.43}$$

that lead us to Whittaker equations as solutions.

However, there is another way of solving (4.42), where we obtain a combination of Generalized Laguerre Polynomials and Confluent Hypergeometric Functions:

$$\begin{aligned} \omega_{C=0} =& \\ & e^{\frac{1}{2} z r_0 (r_1 - A)} \left(\kappa_1 \text{HypergeometricU}\left[\frac{1}{2}\left(\alpha_3 \left(1 - \frac{r_1}{A}\right) + \alpha_1 \left(1 + \frac{r_1}{A}\right)\right), \alpha_1 + \alpha_3, z r_0 A \right] + \right. \\ & \left. \kappa_2 \text{LaguerreL}\left[\frac{1}{2}\left(\alpha_1 \left(-1 - \frac{r_1}{A}\right) + \alpha_3 \left(-1 + \frac{r_1}{A}\right)\right), -1 + \alpha_1 + \alpha_3, z r_0 A \right] \right) \end{aligned} \tag{4.44}$$

where $A = \sqrt{r_1^2 + 4\epsilon_0}$ and κ_1 and κ_2 are constants.

$$\begin{aligned} \omega_{C=-\frac{1}{2}} = & \kappa_1 \text{HypergeometricU}\left[\alpha_1 + \frac{r_0 \epsilon_0}{r_1}, \alpha_1 + \alpha_3, z r_0 r_1 \right] + \\ & \kappa_2 \text{LaguerreL}\left[-\alpha_1 - \frac{r_0 \epsilon_0}{r_1}, -1 + \alpha_1 + \alpha_3, z r_0 r_1 \right] \end{aligned} \tag{4.45}$$

$$\begin{aligned} \omega_{C=-1} = & z^{\frac{1}{2}(1 - \alpha_1 - \alpha_3 + B)} \left(\kappa_1 \text{HypergeometricU}\left[\frac{1}{2}(1 + \alpha_1 - \alpha_3 + B), 1 + B, z r_0 r_1 \right] + \right. \\ & \left. \kappa_2 \text{LaguerreL}\left[\frac{1}{2}(-1 - \alpha_1 + \alpha_3 - B), B, z r_0 r_1 \right] \right) \end{aligned} \tag{4.46}$$

where $B = \sqrt{1 + \alpha_1^2 + 2\alpha_1(-1 + \alpha_3) - 2\alpha_3 + \alpha_3^2 + 4 r_0^2 \epsilon_0}$

Here, LaguerreL[n,a,x] is the Generalized Laguerre Polynomial $L_n^a(x)$ [33], and HypergeometricU[a,b,z] is the confluent hypergeometric function $U(a, b, z)$ [33].

The choice of these functions for our solution relies on the fact that we can choose the constant κ_2 to be zero, that is, to neglect the infinite-degree series and keep the Laguerre polynomials as solutions.

By taking this choice, we know that the values of the first (lower) index of the Laguerre functions are integers, so we can now look for the conditions for the parameters that turns it really integers. By choosing such condition we determine, for instance ϵ_0, for convenience.

4.4.1 C=0, Laguerre part

As our first case, let explore $C = 0$. The $q(z)$ solution for (0.47) is:

$$\frac{1}{2}\left(A\left(-1-\frac{2\text{LaguerreL}\left[\frac{-2A-(A+r_1)\alpha_1+(-A+r_1)\alpha_3}{2A},\alpha_1+\alpha_3,Azr_0\right]}{\text{LaguerreL}\left[\frac{-(A+r_1)\alpha_1+(-A+r_1)\alpha_3}{2A},-1+\alpha_1+\alpha_3,Azr_0\right]}\right)+r_1\right). \tag{4.47}$$

Noticing that the first index of the numerator's Laguerre is the denominator's minus 1, we first choose it to be zero, and then we get $\epsilon_0 = -\frac{r_1^2(1+\alpha_1)(1+\alpha_3)}{(2+\alpha_1+\alpha_3)^2}$ or $A = \frac{-r_1\alpha_1+r_1\alpha_3}{2+\alpha_1+\alpha_3}$ and a $q(z)$:

$$\frac{1}{2}r_1\left(1+\frac{(\alpha_1-\alpha_3)\left(2+\alpha_1+\alpha_3+\frac{zr_0r_1(\alpha_1-\alpha_3)}{2+\alpha_1+\alpha_3}\right)}{(2+\alpha_1+\alpha_3)\left(\alpha_1+\alpha_3+\frac{zr_0r_1(\alpha_1-\alpha_3)}{2+\alpha_1+\alpha_3}\right)}\right) \tag{4.48}$$

The next term we are guided to is the one with the smaller Laguerre's lower index equals 1, which give us:

$$q(z)=\frac{1}{2}r_1\left(1+\frac{(-\alpha_1+\alpha_3)\left(-1+\frac{-4(1+H)(4+H)^2-4z(\alpha_1-\alpha_3)(4+H)r_0r_1}{(H)(1+H)(4+H)^2+2z(\alpha_1-\alpha_3)(1+H)(4+H)r_0r_1+z^2(\alpha_1-\alpha_3)^2r_0^2r_1^2}\right)}{4+H}\right) \tag{4.49}$$

where $H = \alpha_1 + \alpha_3$ and we have $A = \frac{(-\alpha_1+\alpha_3)r_1}{4+\alpha_1+\alpha_3}$ or $\epsilon_0 = -\frac{(2+\alpha_1)(2+\alpha_3)r_1^2}{(4+\alpha_1+\alpha_3)^2}$.

Such $q(z)$'s are exactly the same as the ones obtained by the Pade method for these same conditions, up to algebraic manipulations.

We can do it continuously, by seeing the solution of $\frac{-2A-(A+r_1)\alpha_1+(-A+r_1)\alpha_3}{2A} = n$ which gives $A = \frac{(-\alpha_1+\alpha_3)r_1}{(2n+2)+\alpha_1+\alpha_3}$, and then with $A = \sqrt{r_1^2+4\epsilon_0}$ we arrive at $\epsilon_0 = -\frac{((n+1)+\alpha_1)((n+1)+\alpha_3)r_1^2}{(2(n+1)+\alpha_1+\alpha_3)^2}$, for $n = 0, 1, 2, \ldots$.

4.4.2 C=0, Hypergeometric part

Now, as we have exhausted the manipulations with the Laguerre part, let's move to the Hypergeometric one. We can obtain sets of special functions by specifying the hypergeometric functions to certain values, as to say, the solution with $C = 0$ without the Laguerre terms is:

$$q(z) = \frac{\mathrm{U}\left[\eta, a_1 + a_3, Azr_0\right](-A + r_1) - \mathrm{U}\left[\eta + 1, 1 + a_1 + a_3, Azr_0\right](a_3(A - r_1) + a_1(A + r_1))}{2\mathrm{U}\left[\eta, a_1 + a_3, Azr_0\right]} \tag{4.50}$$

where $\eta = \frac{a_3(A-r_1)+a_1(A+r_1)}{2A}$ and $A = \sqrt{r_1^2 + 4\epsilon_0}$.

Taking the 2 indexes of U to have the same value $\lambda \in \mathbb{C}$, we get:

$$q(z) = \frac{1}{2}\left(A\left(-1 - \frac{2\lambda\Gamma\left[-\lambda, Azr_0\right]}{\Gamma\left[1 - \lambda, Azr_0\right]}\right) + r_1\right) \tag{4.51}$$

where $\Gamma[a, z]$ is the incomplete gamma function; to do that we had to impose

$$\alpha_1 = \frac{\lambda(A + r_1)}{2r_1}, \qquad \alpha_3 = \frac{\lambda}{2} - \frac{A\lambda}{2r_1}$$

By imposing $\alpha_1 = \alpha_3$ we arrive at:

$$q(z) = \frac{1}{2}\left(-A - \frac{2Ae^{-\frac{1}{2}Azr_0}\sqrt{\pi}\mathrm{U}\left[1 + \alpha_3, 1 + 2\alpha_3, Azr_0\right]\alpha_3\left(Azr_0\right)^{-\frac{1}{2}+\alpha_3}}{\mathrm{BesselK}\left[-\frac{1}{2} + \alpha_3, \frac{1}{2}Azr_0\right]} + r_1\right) \tag{4.52}$$

From this equation we are able to get some ratios of Bessel function for some values for α_3:

$$q(z) = \frac{1}{2}\left(-\frac{A\mathrm{BesselK}\left[0, \frac{1}{2}Azr_0\right]}{\mathrm{BesselK}\left[1, \frac{1}{2}Azr_0\right]} + r_1\right) \qquad \alpha_3 = -\frac{1}{2}$$

$$q(z) = \frac{1}{2}\left(-\frac{A\mathrm{BesselK}\left[1, \frac{1}{2}Azr_0\right]}{\mathrm{BesselK}\left[0, \frac{1}{2}Azr_0\right]} + r_1\right) \qquad \alpha_3 = \frac{1}{2}$$

$$q(z) = \frac{1}{2}\left(-\frac{A\mathrm{BesselK}\left[0, \frac{1}{2}Azr_0\right]}{\mathrm{BesselK}\left[1, \frac{1}{2}Azr_0\right]} - \frac{4}{zr_0} + r_1\right) \qquad \alpha_3 = \frac{3}{2}$$

If we set the value of α_3 to any integer, we get rational solutions, for example, for $\alpha_3 = -1, 0, 1, 2, 3, 4$, respectively:

$$
\begin{aligned}
q(z) &= \frac{2r_1 + Azr_0\,(-A+r_1)}{4+2Azr_0} \\
q(z) &= 1/2(-A+r_1) \\
q(z) &= \frac{1}{2}\left(-A - \frac{2}{zr_0} + r_1\right) \\
q(z) &= \frac{1}{2}\left(-A - \frac{4\,(3+Azr_0)}{zr_0\,(2+Azr_0)} + r_1\right) \\
q(z) &= \frac{1}{2}\left(-A - \frac{6\,(20+8Azr_0+A^2z^2r_0^2)}{zr_0\,(12+6Azr_0+A^2z^2r_0^2)} + r_1\right) \\
q(z) &= \frac{1}{2}\left(-A - \frac{8\,(210+90Azr_0+15A^2z^2r_0^2+A^3z^3r_0^3)}{zr_0\,(120+60Azr_0+12A^2z^2r_0^2+A^3z^3r_0^3)} + r_1\right)
\end{aligned}
$$

If we want to choose the values of η to be negative integers, we end up imposing a condition on A, e.g., for $\eta = 0$, we get $A = \frac{(-\alpha_1+\alpha_3)r_1}{\alpha_1+\alpha_3}$, which is just the condition on A from the Laguerre polynomials, and in fact, if we calculate it for the hypergeometric function we arrive at:

$$q(z) = \frac{\alpha_1 r_1}{\alpha_1+\alpha_3} \tag{4.53}$$

which is the same result from that same imposition when applied to the Laguerre polynomial. In fact, if we keep equaling η to the next negative integers we we'll get the same conditions and the same ratios as in the Laguerre case, which are the same found in the Pade case (all solvable situations found by different approaches).

As the hypergeometric functions are, generally, more powerful than the Laguerre ones, we will try to go for the negative integers on the Laguerre index, but put the conditions on the hypergeometric functions instead. For a better notation, we abandon here the η symbol, to apply the condition $\epsilon_0 = -\frac{((n+1)+\alpha_1)((n+1)+\alpha_3)r_1^2}{(2(n+1)+\alpha_1+\alpha_3)^2}$, if we put $n = -1$ and simplify the expression we arrive at:

$$q(z) = \frac{\alpha_1 r_1}{\alpha_1+\alpha_3}, \qquad \text{with} \qquad \epsilon_0 = -\frac{\alpha_1\alpha_3 r_1^2}{(\alpha_1+\alpha_3)^2}$$

Moving to more negative n, the equations start to become harder to simplify, therefore we replace $\epsilon_0 = -\frac{((n+1)+\alpha_1)((n+1)+\alpha_3)r_1^2}{(2(n+1)+\alpha_1+\alpha_3)^2}$ on the original equation, then we solve it and neglect the Laguerre part, getting:

$$q(z) = \frac{r_1\left((-1-n)\alpha_1 U_2 - \alpha_1^2 U_2 + (1+n+\alpha_3)\,(U_1+\alpha_3 U_2)\right)}{(2+2n+\alpha_1+\alpha_3)\,U_1} \tag{4.54}$$

where $U_1 = U\left[1+n+\alpha_1+\alpha_3, \alpha_1+\alpha_3, \frac{z(\alpha_1-\alpha_3)r_0r_1}{2+2n+\alpha_1+\alpha_3}\right]$ and
$U_2 = U\left[2+n+\alpha_1+\alpha_3, 1+\alpha_1+\alpha_3, \frac{z(\alpha_1-\alpha_3)r_0r_1}{2+2n+\alpha_1+\alpha_3}\right]$.

And then, for $n = -2, -3$, respectively:

$$q(z) = -\frac{-1+\alpha_1+\alpha_3}{zr_0} + \frac{(-1+\alpha_3)\, r_1}{-2+\alpha_1+\alpha_3} \tag{4.55}$$

$$\begin{aligned} q(z) = &-\frac{-2+\alpha_1+\alpha_3}{zr_0} + \frac{(-2+\alpha_3)\, r_1}{-4+\alpha_1+\alpha_3} \\ &- \frac{(-4+\alpha_1+\alpha_3)\,(-2+\alpha_1+\alpha_3)}{zr_0\,(8+\alpha_1^2+\alpha_3\,(-6+\alpha_3-zr_0r_1)+\alpha_1\,(-6+2\alpha_3+zr_0r_1))} \end{aligned} \tag{4.56}$$

Which are exactly the singular solutions obtained from the Pade method as said above.

4.4.3 C=-1/2, Laguerre part

The Laguerre solution for $C = \frac{-1}{2}$, already neglecting the hypergeometric part is:

$$q[z] = -\frac{\text{LaguerreL}\left[-1-\alpha_1-\frac{r_0\epsilon_0}{r_1}, \alpha_1+\alpha_3, zr_0r_1\right] r_1}{\text{LaguerreL}\left[-\alpha_1-\frac{r_0\epsilon_0}{r_1}, -1+\alpha_1+\alpha_3, zr_0r_1\right]} \tag{4.57}$$

By doing the same as before, we start with the lowest possible Laguerre (n=0), which give us:

$$q(z) = -\frac{r_1}{-zr_0r_1+\alpha_1+\alpha_3} \tag{4.58}$$

with $\epsilon_0 \to -\frac{r_1(1+\alpha_1)}{r_0}$.

The next one, with $n = 1$ in the numerator's Laguerre, give us:

$$q(z) = \frac{2r_1\,(-1+zr_0r_1-\alpha_1-\alpha_3)}{z^2r_0^2r_1^2+\alpha_1+\alpha_3+(\alpha_1+\alpha_3)^{\,2}-2zr_0r_1\,(1+\alpha_1+\alpha_3)} \tag{4.59}$$

with $\epsilon_0 = -\frac{r_1(2+\alpha_1)}{r_0}$.

It is noticeable here that we are getting continuously rational solutions by seeing the solution of $-1-\alpha_1-\frac{r_0\epsilon_0}{r_1} = n$ which gives $\epsilon_0 = -\frac{r_1((n+1)+\alpha_1)}{r_0}$ for $n = 0, 1, 2, \ldots$

4.4.4 C=-1/2, Hypergeometric part

We can obtain sets of special functions by specifying the hypergeometric functions to certain values, as to say, the solution with $C = -\frac{1}{2}$ without the Laguerre terms is:

$$q(z) = -\frac{U\left[1+\alpha_1+\frac{r_0\epsilon_0}{r_1}, 1+\alpha_1+\alpha_3, zr_0r_1\right](\alpha_1r_1+r_0\epsilon_0)}{U\left[\alpha_1+\frac{r_0\epsilon_0}{r_1}, \alpha_1+\alpha_3, zr_0r_1\right]} \tag{4.60}$$

Taking the 2 indexes of U to have the same value $\lambda \in \mathbb{C}$, we get:

$$q(z) = -\frac{(-1+\lambda)\Gamma\left[1-\lambda, zr_0r_1\right] r_1}{\Gamma\left[2-\lambda, zr_0r_1\right]} \tag{4.61}$$

where $\Gamma[a,z]$ is the incomplete gamma function; to do that we had to impose

$$\alpha_1 = -1 + \lambda - \frac{r_0\epsilon_0}{r_1}, \qquad\qquad \alpha_3 = \frac{r_0\epsilon_0}{r_1}.$$

By making the imposition $\epsilon_0 = -\frac{(1+\alpha_1-\alpha_3)r_1}{2r_0}$ we arrive at:

$$q(z) = -\frac{e^{\frac{1}{2}zr_0r_1}\text{BesselK}\left[\frac{1}{2}(\alpha_1+\alpha_3), \frac{1}{2}zr_0r_1\right](-1+\alpha_1+\alpha_3)\, r_1 \,(zr_0r_1)^{\frac{1}{2}(-\alpha_1-\alpha_3)}}{2\sqrt{\pi}U\left[\frac{1}{2}(-1+\alpha_1+\alpha_3), \alpha_1+\alpha_3, zr_0r_1\right]} \tag{4.62}$$

From this equation we are able to get some ratios of Bessel function for some values of n when $\alpha_3 = -\alpha_1 + n$:

$$q(z) = \frac{\text{BesselK}\left[0, \frac{1}{2}zr_0r_1\right]}{z\left(\text{BesselK}\left[0, \frac{1}{2}zr_0r_1\right] + \text{BesselK}\left[1, \frac{1}{2}zr_0r_1\right]\right) r_0} \qquad n = 0$$

$$q(z) = -\frac{\text{BesselK}\left[1, \frac{1}{2}zr_0r_1\right]}{z\left(\text{BesselK}\left[0, \frac{1}{2}zr_0r_1\right] + \text{BesselK}\left[1, \frac{1}{2}zr_0r_1\right]\right) r_0} \qquad n = 2$$

If we set n to any odd number, we arrive at proper rational solutions, for example, with $n = -5, -3, -1, 1, 3, 5$, we get, respectively:

$$q(z) = \frac{3r_1\left(12 + zr_0r_1\left(6 + zr_0r_1\right)\right)}{60 + zr_0r_1\left(36 + zr_0r_1\left(9 + zr_0r_1\right)\right)}$$

$$q(z) = \frac{2r_1\left(2 + zr_0r_1\right)}{6 + zr_0r_1\left(4 + zr_0r_1\right)}$$

$$q(z) = \frac{r_1}{1 + zr_0r_1}$$

$$q(z) = 0$$

$$q(z) = -\frac{2 + zr_0r_1}{zr_0 + z^2r_0^2r_1}$$

$$q(z) = -\frac{2\left(12 + zr_0r_1\left(6 + zr_0r_1\right)\right)}{zr_0\left(6 + zr_0r_1\left(4 + zr_0r_1\right)\right)}$$

If we choose the values of the first index of U $(1+\alpha_1+\frac{r_0\epsilon_0}{r_1})$ to be a negative integer (as in the last section with η), we end up imposing a condition on ϵ_0, e.g., for it to be zero, $\epsilon_0 = -\frac{(1+\alpha_1)r_1}{r_0}$, which is just the condition on ϵ_0 from the Laguerre polynomials, and in fact, if we calculate it for the hypergeometric function we arrive at:

$$q(z) = -\frac{r_1}{\alpha_1 + \alpha_3 - zr_0r_1} \tag{4.63}$$

which is the same result from that same imposition when applied to the Laguerre polynomial. In fact, if we keep equaling the first indexes to the next negative integers we we'll get the same conditions and the same ratios as in the Laguerre case, for example the next one becomes:

$$q(z) = \frac{2r_1\left(-1-\alpha_1-\alpha_3+zr_0r_1\right)}{\alpha_1^2+\alpha_3^2+\alpha_3\left(1-2zr_0r_1\right)+\alpha_1\left(1+2\alpha_3-2zr_0r_1\right)+zr_0r_1\left(-2+zr_0r_1\right)} \tag{4.64}$$

and $\epsilon_0 = -\frac{(2+\alpha_1)r_1}{r_0}$.

If we choose these first index to be positive, when it is 1, we have $\epsilon_0 = -\frac{\alpha_1 r_1}{r_0}$ and $q(z) = 0$. When it is 2, we have:

$$q(z) = \frac{e^{-zr_0r_1}r_1\left(-1+e^{zr_0r_1}E_{1-\alpha_1-\alpha_3}\left(zr_0r_1\right)\left(1-\alpha_1-\alpha_3+zr_0r_1\right)\right)}{E_{2-\alpha_1-\alpha_3}\left(zr_0r_1\right)\left(-1+\alpha_1+\alpha_3\right)}$$
$$\epsilon_0 = -\frac{\left(-1+\alpha_1\right)r_1}{r_0}$$

When it is 3, we have:

$q(z) =$
$-\frac{r_1\left(-3+\alpha_1+\alpha_3-zr_0r_1+e^{zr_0r_1}E_{1-\alpha_1-\alpha_3}(zr_0r_1)\left(2+\alpha_1^2-3\alpha_3+4zr_0r_1+\alpha_1(-3+2\alpha_3-2zr_0r_1)+(\alpha_3-zr_0r_1)^2\right)\right)}{(-1+\alpha_1+\alpha_3)\left(1+e^{zr_0r_1}E_{2-\alpha_1-\alpha_3}(zr_0r_1)(-2+\alpha_1+\alpha_3-zr_0r_1)\right)}$
$\epsilon_0 = -\frac{(-2+\alpha_1)r_1}{r_0}$, where $E_n(z)$ is the integral exponential function. We can do it continuously to the next natural numbers.

4.4.5 C=-1, Laguerre Part

The Laguerre solution for $C = -1$, already neglecting the hypergeometric part is:

$$q(z) = \frac{1}{2}\left(\frac{1+B-\alpha_1-\alpha_3}{zr_0} - \frac{2\text{LaguerreL}\left[\frac{1}{2}\left(-3-B-\alpha_1+\alpha_3\right), 1+B, zr_0r_1\right]r_1}{\text{LaguerreL}\left[\frac{1}{2}\left(-1-B-\alpha_1+\alpha_3\right), B, zr_0r_1\right]}\right) \tag{4.65}$$

where $B = \sqrt{1+\alpha_1^2+2\alpha_1\left(-1+\alpha_3\right)-2\alpha_3+\alpha_3^2+4r_0^2\epsilon_0}$.

By doing the same as before, we start with the lowest possible Laguerre (n=0), which give us:

$$q(z) = \frac{1}{2}\left(-\frac{2\left(1+\alpha_1\right)}{zr_0} + \frac{2r_1}{2+zr_0r_1+\alpha_1-\alpha_3}\right) \tag{4.66}$$

with $\epsilon_0 = -\frac{(1+\alpha_1)(-2+\alpha_3)}{r_0^2}$ or $B = -3-\alpha_1+\alpha_3$.

The next one, with $n = 1$ in the numerator's Laguerre, give us:

$$q(z) = \frac{1}{2}\left(-\frac{2\left(2+\alpha_1\right)}{zr_0} + \frac{4r_1\left(3+zr_0r_1+\alpha_1-\alpha_3\right)}{z^2r_0^2r_1^2+2zr_0r_1\left(3+\alpha_1-\alpha_3\right)+\left(3+\alpha_1-\alpha_3\right)\left(4+\alpha_1-\alpha_3\right)}\right) \tag{4.67}$$

with $\epsilon_0 = -\frac{(2+\alpha_1)(-3+\alpha_3)}{r_0^2}$ or $B = -5 - \alpha_1 + \alpha_3$.

It is noticeable here that we are getting continuously rational solutions by seeing the solution of $\frac{1}{2}\left(-3 - B - \alpha_1 + \alpha_3\right) = n$ which gives $B = -(2n+3) - \alpha_1 + \alpha_3$ and then with $B = \sqrt{1+\alpha_1^2+2\alpha_1\left(-1+\alpha_3\right)-2\alpha_3+\alpha_3^2+4r_0^2\epsilon_0}$ we arrive at $\epsilon_0 = -\frac{((n+1)+\alpha_1)(-(n+2)+\alpha_3)}{r_0^2}$, for $n = 0, 1, 2, \ldots$.

4.4.6 C=-1, Hypergeometric part

We can obtain sets of special functions by specifying the hypergeometric functions to certain values, as to say, the solution with $C = -1$ without the Laguerre terms is:

$$\begin{aligned} q(z) = \Bigg(&(1+B-\alpha_3)\Bigg(U\left[\frac{1}{2}\left(1+B+\alpha_1-\alpha_3\right), 1+B, zr_0r_1\right] \\ &-zU\left[1+\frac{1}{2}\left(1+B+\alpha_1-\alpha_3\right), 2+B, zr_0r_1\right]r_0r_1\Bigg) \\ &-\alpha_1\Bigg(U\left[\frac{1}{2}\left(1+B+\alpha_1-\alpha_3\right), 1+B, zr_0r_1\right] \\ &+zU\left[1+\frac{1}{2}\left(1+B+\alpha_1-\alpha_3\right), 2+B, zr_0r_1\right]r_0r_1\Bigg)\Bigg)\Bigg/ \\ &\left(2zU\left[\frac{1}{2}\left(1+B+\alpha_1-\alpha_3\right), 1+B, zr_0r_1\right]r_0\right) \end{aligned} \tag{4.68}$$

where here, $B = \sqrt{1+\alpha_1^2+2\alpha_1\left(-1+\alpha_3\right)-2\alpha_3+\alpha_3^2+4r_0^2\epsilon_0}$.

Taking the 2 indexes of U to have the same value $\lambda \in \mathbb{C}$, we get:

$$q(z) = -\frac{\alpha_3}{zr_0} - \frac{\lambda\Gamma\left[-\lambda, zr_0r_1\right]r_1}{\Gamma\left[1-\lambda, zr_0r_1\right]} \tag{4.69}$$

where $\Gamma[a, z]$ is the incomplete gamma function; to do that we had to impose

$$B = -1 + \lambda, \qquad \alpha_1 = \lambda + \alpha_3.$$

By making the imposition $\alpha_1 = \alpha_3$ we arrive at:

$$q(z) = \frac{1+B-2\alpha_3}{2zr_0} - \frac{(1+B)e^{-\frac{1}{2}zr_0r_1}\sqrt{\pi}U\left[1+\frac{1+B}{2}, 2+B, zr_0r_1\right] r_1\left(zr_0r_1\right)^{B/2}}{2\text{BesselK}\left[-\frac{1}{2}+\frac{1+B}{2}, \frac{1}{2}zr_0r_1\right]} \tag{4.70}$$

From this equation we are able to get some ratios of Bessel function for some values of B:

$$\begin{aligned}
q(z) &= -\frac{1+2\alpha_3}{2zr_0} + \frac{\left(-\text{BesselK}\left[0, \frac{1}{2}zr_0r_1\right] + \text{BesselK}\left[1, \frac{1}{2}zr_0r_1\right]\right) r_1}{2\text{BesselK}\left[1, \frac{1}{2}zr_0r_1\right]} && B=-2 \\
q(z) &= -\frac{-1+2\alpha_3}{2zr_0} + \frac{\left(\text{BesselK}\left[0, \frac{1}{2}zr_0r_1\right] - \text{BesselK}\left[1, \frac{1}{2}zr_0r_1\right]\right) r_1}{2\text{BesselK}\left[0, \frac{1}{2}zr_0r_1\right]} && B=0 \\
q(z) &= -\frac{1+2\alpha_3}{2zr_0} + \frac{\left(-\text{BesselK}\left[0, \frac{1}{2}zr_0r_1\right] + \text{BesselK}\left[1, \frac{1}{2}zr_0r_1\right]\right) r_1}{2\text{BesselK}\left[1, \frac{1}{2}zr_0r_1\right]} && B=2
\end{aligned}$$

If we set B to any odd number, we arrive at proper rational solutions, also $q(z;B) = q(z;-B)$, that is, $q(z)$ is even for B, as an example, with $B = 1, 3, 5$, we get, respectively:

$$\begin{aligned}
q(z) &= -\frac{\alpha_3}{zr_0} \\
q(z) &= -\frac{\alpha_3}{zr_0} - \frac{2}{zr_0\left(2+zr_0r_1\right)} \\
q(z) &= -\frac{\alpha_3}{zr_0} - \frac{6\left(4+zr_0r_1\right)}{zr_0\left(12+6zr_0r_1+z^2r_0^2r_1^2\right)}
\end{aligned}$$

If we choose the values of the first index of U $\left(\frac{1}{2}\left(1+B+\alpha_1-\alpha_3\right)\right)$ to be a negative integer (as in the first section with η), we end up imposing a condition on B, e.g., for it to be zero, $B = -1 - \alpha_1 + \alpha_3$, which is just the condition on B from the Laguerre polynomials, and in fact, if we calculate it for the hypergeometric function we arrive at:

$$q(z) = -\frac{\alpha_1}{zr_0} \tag{4.71}$$

which is the same result from that same imposition when applied to the Laguerre polynomial. In fact, if we keep equaling the first indexes to the next negative integers we we'll get the same conditions and the same ratios as in the Laguerre case, for example the next one becomes:

$$q(z) = \frac{-2-\alpha_1^2+\alpha_3+\alpha_1\left(-3+\alpha_3-zr_0r_1\right)}{zr_0\left(2+\alpha_1-\alpha_3+zr_0r_1\right)} \tag{4.72}$$

and $B = -3 - \alpha_1 + \alpha_3$.

If we choose these first index to be positive, when it is 1, we have:

$$q(z) = -\frac{e^{-zr_0r_1}}{zE_{\alpha_1-\alpha_3}\left(zr_0r_1\right)r_0} + \frac{-\alpha_3 + zr_0r_1}{zr_0}$$
$$B = 1 - \alpha_1 + \alpha_3$$

When it is 2, we have:

$$q(z) = \frac{-zr_0r_1 + \alpha_3 + e^{zr_0r_1}ze_{-2+\alpha_1-\alpha_3}r_0r_1\left(2 - 2zr_0r_1 + 3\alpha_3 + \left(-zr_0r_1 + \alpha_3\right)^2 - \alpha_1\left(1 - zr_0r_1 + \alpha_3\right)\right)}{zr_0\left(-1 + e^{zr_0r_1}ze_{-2+\alpha_1-\alpha_3}r_0r_1\left(-2 + zr_0r_1 + \alpha_1 - \alpha_3\right)\right)}$$
$$B = 3 - \alpha_1 + \alpha_3 \quad (4.73)$$

where $E_n(z)$ is the integral exponential function. We can do it continuously to the next natural numbers.

4.5 Other rational solutions

There are also some other rational solutions still in research, they may be of special interest when looking for some chain of solutions by Bäcklund transformations. The specific Bäcklund transformation to be used are still unknown.

4.5.1 Polynomial solutions for $p(z)$

It's already known that we can impose a solution for $p(z)$ like $p(z) = g_0 + g_1z$ and obtain simplifications on the first order equation for $p(z)$ that lead us to good solutions for $q(z)$.

Now we wonder answer the question that if we could impose a polynomial of different order (higher or lower) and also obtain solution for $q(z)$.

First let's explore the case $\epsilon_1 = 0$ and $C = 0$, then we have for the first order equations:

$$zq_z = q(q - r_1)(2p + r_0z) - (\alpha_1 + \alpha_3)q + \alpha_1r_1 + \epsilon_0r_0z \quad (4.74)$$
$$zp_z = p(p + r_0z)(r_1 - 2q) + (\alpha_1 + \alpha_3)p - \alpha_2r_0z \quad (4.75)$$

Imposing $p(z) = g_0 + g_1z + g_2z^n$ yields in (4.75) (collecting $q(z)$):

$$q(z) = \frac{-zg_1 - nz^ng_2 + \left(g_0 + zg_1 + z^ng_2\right)\left(g_0 + z^ng_2 + z\left(g_1 + r_0\right)\right)r_1 - zr_0\alpha_2 + \left(g_0 + zg_1 + z^ng_2\right)\left(\alpha_1 + \alpha_3\right)}{2\left(g_0 + zg_1 + z^ng_2\right)\left(g_0 + z^ng_2 + z\left(g_1 + r_0\right)\right)} \quad (4.76)$$

When we replace this in (4.74) we obtain a very large equation with many undefined powers of z (that is, with some power of z^n), what we want to do is to look at those cases of $n > 1$ and $n < 0$ for integer value of n.

There will be a power of z which is dominant and doesn't combine with any other, we will look at it because it's more likely to find a more strict condition on the parameters.

n>1

In this case we look at the biggest powers of z for positive n first, they are:
$9,\ 8+n,\ 7+2n,\ 6+3n,\ 5+4n,\ 4+5n,\ 3+6n,\ 2+7n,\ 1+8n,\ 9n$
We can see that if $n > 1$, the dominating exponent is always $9n$, and its coefficient is exactly:

$$-4g_2^9 r_1^2$$

Of course, when we want to cancel all orders of z, this single term must vanish too, then we are forced to put g_2 to zero.

We may notice here, that in the case of $n = 1$ (or zero) there will be many other terms with the same order, then we will have other than trivial solutions.

n<0

In this case we look at the smallest powers of z for negative n first, they are:
$0,\ 1+2n,\ 1+3n,\ 1+4n,\ 1+5n,\ 1+6n,\ 1+7n,\ 1+8n,\ 9n$
We can see that if $n < 1$, the dominating exponent is always $9n$, and its coefficient is exactly (as above):

$$-4g_2^9 r_1^2$$

Of course, when we want to cancel all orders of z, this single term must vanish too, then we are forced to put g_2 to zero again.

Then it is proved that we cannot impose other than the $p(z) = g_0 + g_1 z$ solution.

Just for the record, all the powers of z that appear are:

$$0, 1, 2, 3, 4, 5, 6, 7, 8, 9, n, 2n, 3n, 4n, 5n, 6n, 7n, 8n, 9n, 1+n, 2+n, 3+n, 4+n, 5+n, 6+n, 7+n, 8+n,$$
$$1+2n, 2+2n, 3+2n, 4+2n, 5+2n, 6+2n, 7+2n, 1+3n, 2+3n, 3+3n, 4+3n, 5+3n, 6+3n,$$
$$1+4n, 2+4n, 3+4n, 4+4n, 5+4n, 1+5n, 2+5n, 3+5n, 4+5n, 1+6n, 2+6n, 3+6n, 1+7n, 2+7n,$$

The attempt of a solution of rational type for $p(z)$ with $\epsilon_1 = 0$ was unsuccessful for new solutions.

4.5.2 General solutions for simple polynomial in $p(z)$

Beginning with the set of two first order equations:

$$zq_z = q(q - r_1)(2p + r_0 z) - (\alpha_1 + \alpha_3)q + \alpha_1 r_1 + \epsilon_0 r_0 z^{1+2C} \tag{4.77}$$

$$zp_z = p(p + r_0 z)(r_1 - 2q) + (\alpha_1 + \alpha_3)p - \alpha_2 r_0 z - \epsilon_1 r_1 z^{-2C} \tag{4.78}$$

We go to the limits of $C = 0$ and $\epsilon_1 = 0$ and impose a solution like $p(z) = ar_0 z + p_0$. This yields in (4.75) (collecting $q(z)$):

$$q(z) = \frac{a(1+a)z^2 r_0^2 r_1 + p_0\left(p_0 r_1 + \alpha_1 + \alpha_3\right) + zr_0\left(-a + (1+2a)p_0 r_1 + a\alpha_1 - \alpha_2 + a\alpha_3\right)}{2\left(p_0 + azr_0\right)\left(p_0 + (1+a)zr_0\right)} \tag{4.79}$$

Replacing this value in (4.74) we obtain

$$\begin{aligned}
0 = {} & 2\left(p_0 + azr_0\right)^2\left(p_0 + (1+a)zr_0\right)^2 \\
& \left(-\left(zr_0\left(p_0\left(\alpha_1 + 2\left(a + \alpha_2\right) + \alpha_3\right) + z\left(a\alpha_1 + (1+2a)\left(a + \alpha_2\right) + a\alpha_3\right) r_0\right) + \right.\right. \\
& \left.\left(p_0 + azr_0\right)\left(p_0 + (1+a)zr_0\right)\left(2p_0 + (z + 2az)r_0\right) r_1\right) \times \\
& \left(-z\left(a + \alpha_2\right) r_0 + \alpha_1\left(p_0 + azr_0\right) + \left(p_0 + azr_0\right)\left(\alpha_3 + \left(p_0 + (1+a)zr_0\right) r_1\right)\right) + \\
& 2\left(2{p_0}^4\alpha_1 r_1 + 2a^2(1+a)^2 z^5 r_0^5\epsilon_0 + 2a(1+a)z^4 r_0^4\left(a(1+a)\alpha_1 r_1 + 2(1+2a)p_0\epsilon_0\right) + \right. \\
& {p_0}^2 zr_0\left(a + \alpha_2 + (1+a)\alpha_3 + \alpha_1\left(1 + a + 4(1+2a)p_0 r_1\right) + 2{p_0}^2\epsilon_0\right) + \\
& 2p_0 z^2 r_0^2\left(a(1+a)\alpha_3 + \alpha_1\left(a + a^2 + (1 + 6a(1+a))p_0 r_1\right) + 2(1+2a){p_0}^2\epsilon_0\right) + \\
& \left.\left. z^3 r_0^3\left(a(1+a)\left(-\alpha_2 + a\left(-1 + \alpha_3\right) + \alpha_1\left(a + 4(1+2a)p_0 r_1\right)\right) + 2(1 + 6a(1+a)){p_0}^2\epsilon_0\right)\right)\right)
\end{aligned} \tag{4.80}$$

In the equation above, not a single term was canceled by some ratio simplification, they are all there, and we must discover for what values of the parameters such equation is satisfied; for that we take each power in z and cancel all them.

As an example (as it's the simplest one), the coefficient with no power in z is:

$$-4{p_0}^8 r_1\left(-\alpha_1 + \alpha_3 + p_0 r_1\right) \tag{4.81}$$

we see from that we must have $r_1 = 0$, or $p_0 = 0$ or $\alpha_1 = \alpha_3 + p_0 r_1$.

We don't want $r_1 = 0$ because that is a reduction to the case of a P_V.

With $p_0 = 0$, we can just obtain few values of parameters, or $a = 0$, or $a = -1$, or $r_0 = 0$.

With $\alpha_1 = \alpha_3 + p_0 r_1$ we obtain a set of interesting solutions:

$$a \to 0, \alpha_3 \to \frac{1}{2}\left(-\alpha_2 - p_0 r_1\right), \epsilon_0 \to \frac{1}{4}\left(-\frac{\alpha_2^2}{{p_0}^2} + r_1^2\right)$$
$$a \to 0, \alpha_3 \to \frac{1}{2}\left(2 - \alpha_2 - p_0 r_1\right), \epsilon_0 \to \frac{1}{4}\left(-\frac{\alpha_2^2}{{p_0}^2} + r_1^2\right)$$
$$a \to -\alpha_2, \alpha_3 \to -\frac{p_0 r_1}{2}, \epsilon_0 \to \frac{1}{4}\left(1 - 2\alpha_2\right) r_1^2$$
$$a \to -\alpha_2, \alpha_3 \to 1 - \alpha_2 - \frac{p_0 r_1}{2}, \epsilon_0 \to \frac{1}{4}\left(1 - 2\alpha_2\right) r_1^2$$
$$a \to \alpha_2, \alpha_3 \to 1 - \frac{p_0 r_1}{2}, \epsilon_0 \to \frac{1}{4}\left(1 + 2\alpha_2\right) r_1^2$$
$$a \to \alpha_2, \alpha_3 \to -\alpha_2 - \frac{p_0 r_1}{2}, \epsilon_0 \to \frac{1}{4}\left(1 + 2\alpha_2\right) r_1^2$$

It was tried to extend these results to higher order Pade type, but it was unsuccessful, what may indicate (as we think) that there is no chain of rational solutions with Bäcklund transformations starting by these values.

4.5.3 Explorations with $\alpha_1 = -\alpha_3$

We will use the condition $\alpha_1 = -\alpha_3$ throughout the whole description below.

Using for Taylor series:

$$\frac{r_1}{2} - \frac{z r_0 \left(-a_3^3 \epsilon_0 + a_3^2\left(1 + a_3\right)\epsilon_0 + a_3^2\left(\left(-1 + 2a_2 + a_3\right) r_1^2 + \left(2 + a_3\right)\epsilon_0\right) + a_3^2\left(\epsilon_0 - a_3\left(r_1^2 + \epsilon_0\right)\right)\right)}{4a_3^2\left(-1 + 4a_3^2\right)} + \ldots \tag{4.82}$$

And for the asymptotics:

$$\frac{1}{2}\left(r_1 - \sqrt{r_1^2 + 4\epsilon_0}\right) + \frac{4\left(-1 + a_2\right)\epsilon_0 + a_3 r_1\left(-r_1 + \sqrt{r_1^2 + 4\epsilon_0}\right) + a_3 r_1\left(r_1 + \sqrt{r_1^2 + 4\epsilon_0}\right)}{2 z r_0\left(r_1^2 + 4\epsilon_0\right)} + \ldots \tag{4.83}$$

By the Pade method, looking for a rational function like $\frac{a_0 + a_1 z}{1 + b_1 z}$ we arrive at:

$$q(z) = \frac{r_1}{2}, \qquad \epsilon_0 \to \frac{1}{4}\left(-1 + 2\alpha_2\right) r_1^2 \tag{4.84}$$

Another possible solution is:

$$q(z) = \frac{r_1\left(2\alpha_3^2 + z\left(1 + \alpha_3\right) r_0 r_1\right)}{2\alpha_3\left(2\alpha_3 + z r_0 r_1\right)}, \qquad \epsilon_0 \to \frac{\left(-1 + \alpha_3^2\right) r_1^2}{4\alpha_3^2}, \qquad \alpha_2 \to -1 \tag{4.85}$$

Now we notice a symmetry specifically for the case of $C = 0$ and $\alpha_1 = -\alpha_3$:

$$\begin{aligned} q_{zz}(z) =& \frac{1}{2}\left(\frac{1}{q}+\frac{1}{q-r_1}\right)q'^2 - \frac{q'}{z} - \frac{q\left(-1+2\alpha_2\right)r_0\left(q-r_1\right)}{z} - \frac{q\alpha_3^2 r_1}{2z^2\left(q-r_1\right)} + \frac{\alpha_3^2\left(q-r_1\right)r_1}{2z^2 q} \\ &+ \frac{1}{2}qr_0^2\left(2q^2-3qr_1+r_1^2\right) + \frac{r_0\left(q^2-q\left(1-2\alpha_3\right)r_1-\alpha_3 r_1^2\right)\epsilon_0}{zq\left(q-r_1\right)} - \frac{1}{2}r_0^2\left(\frac{1}{q}+\frac{1}{q-r_1}\right)\epsilon_0^2 \end{aligned}$$

By inspection we see that this equation is invariant under $z \to -z$, $\epsilon_0 \to -\epsilon_0$ and replace our choice $\alpha_2 = -1 \to \alpha_2 = 2$ (in order to keep the term term invariant). Thus we generate a new solution from our previous one (4.85):

$$q(z) = \frac{r_1\left(2\alpha_3^2+z\left(1+\alpha_3\right)r_0 r_1\right)}{2\alpha_3\left(2\alpha_3-zr_0 r_1\right)}, \qquad \epsilon_0 \to -\frac{\left(-1+\alpha_3^2\right)r_1^2}{4\alpha_3^2}, \qquad \alpha_2 \to 2 \tag{4.86}$$

Appendix A

Topics in Hamiltonian formalism

A.1 Dirac-Lagrange reduction

As such classical treatment of constrained dynamics also features an indispensable role in our work, we will present it here briefly.

As a starting point, we are dealing with the Lagrangian $L(q, \dot{q}, t)$ spanning the configuration space of positions $q = (q_1, \ldots, q_N)$ and velocities $\dot{q} = (\dot{q}_1, \ldots, \dot{q}_N)$. We may define the conjugate momenta:

$$p_n = \frac{\partial L}{\partial \dot{q}_n}, \qquad n = 1, \ldots, N \tag{A.1}$$

One can identify that if such Lagrangian has a null Hessian matrix determinant,

$$\det\left(\frac{\partial^2 L}{\partial \dot{q}_i \partial \dot{q}_j}\right) = 0, \tag{A.2}$$

then such system has primary constraints ϕ_m such that

$$\phi_m(q, p) = 0, \qquad m = 1, \ldots, M \tag{A.3}$$

for all M physical constraints of our problem.

Given the constrained action:

$$S = \int \{L + \lambda_m \phi_m\}\, dt = \int \{p_n \dot{q}_n - H(q, p) + \lambda_m \phi_m\}\, dt \tag{A.4}$$

where L is the Lagrangian, H is its Lagrange transformation, the Hamiltonian, λ_m are the Lagrange multipliers over the constraints; the repetition of indexes is assumed to de considered a summation over them.

Using the null variation of the action, $\delta S = 0$ we arrive at:

$$\dot{q}_n = \frac{\partial H}{\partial p_n} + \lambda_m \frac{\partial \phi_m}{\partial p_n} \tag{A.5}$$

$$\dot{p}_n = -\frac{\partial H}{\partial q_n} - \lambda_m \frac{\partial \phi_m}{\partial q_n}. \tag{A.6}$$

Also for any given function $g(q, p, t)$:

$$\frac{dg}{dt} = \frac{\partial g}{\partial q}\dot{q} + \frac{\partial g}{\partial p}\dot{p} + \frac{\partial g}{\partial t} \tag{A.7}$$

Using the above definitions for $\dot{q}$ and $\dot{p}$:

$$\frac{dg}{dt} = \frac{\partial g}{\partial q}\left(\frac{\partial H}{\partial p_n} + \lambda_m \frac{\partial \phi_m}{\partial p_n}\right) - \frac{\partial g}{\partial p}\left(\frac{\partial H}{\partial q_n} + \lambda_m \frac{\partial \phi_m}{\partial q_n}\right) + \frac{\partial g}{\partial t} \tag{A.8}$$

We may write this in a more compact way using the Poisson brackets:

$$\frac{dg}{dt} = \{g, H\} + \lambda_m \{g, \phi_m\} + \frac{\partial g}{\partial t} \tag{A.9}$$

Then we may write the above as:

$$\dot{g} = \{g, H + \lambda_m \phi_m\} = \{g, H\} + \{g, \lambda_m\} \phi_m + \lambda_m \{g, \phi_m\} + \frac{\partial g}{\partial t} \tag{A.10}$$

$$= \{g, H\} + \lambda_m \{g, \phi_m\} + \frac{\partial g}{\partial t} \tag{A.11}$$

From the above we notice that appear many ϕ_m throughout the calculation, even though they being defined as zero ($\phi_m(q, p) = 0$), when operated inside the Poisson brackets they are not, nevertheless we had to use that fact to cancel the middle term in the above equation.

That is, we shall cancel the ϕ_m when they are not inside Poisson brackets, this is what is called "weak equality", which makes our calculations even more compact and is represented as

$$\phi_m \approx 0 \tag{A.12}$$

then

$$\dot{g} \approx \{g, H + \lambda_m \phi_m\} + \frac{\partial g}{\partial t} \equiv \{g, H_T\} + \frac{\partial g}{\partial t} \tag{A.13}$$

Now if we use $g = \phi_n$ in the above, we shall obtain:

$$\frac{d\phi_n}{dt} = \{\phi_n, H\} + \lambda_m \{\phi_n, \phi_m\} + \frac{\partial \phi_n}{\partial t} \approx 0 \tag{A.14}$$

Once arrived here, we may have 4 possibilities with the above relation:

- We get some absurd relation like $1 = 0$, this would mean that our original Lagrangian is ill-defined, meaning that its equations of motion already carry such error.

- We get $0 = 0$, which is fine and means that it is consistent.

- It reduces to an independent equation of the λ_m, then we define new constraints over the system, the *secondary constraints*:

$$\chi_t(q,p) = 0, \qquad t = M+1, \ldots, K \tag{A.15}$$

- It does not reduce to any of the cases above and we have some condition on the λ_m.

If we do obtain secondary constraints, we repeat the process (A.14) with them, then if we obtain again new constraints we perform (A.14) again until we run out of new constraints appearing. With the complete set of T constraints obtained:

$$\dot{\phi}_n \approx \{\phi_n, H\} + \lambda_m \{\phi_n, \phi_m\} + \frac{\partial \phi_n}{\partial t} \approx 0, \qquad n = 1, ..., T \tag{A.16}$$

If the matrix $\{\phi_n, \phi_m\}$ is invertible, that is $\det(\{\phi_n, \phi_m\}) \neq 0$, then we can isolate the Lagrange multiplier:

$$\lambda_m = -\left(\{\phi,\phi\}\right)^{-1}_{nm} \left(\{\phi_n, H\} + \frac{\partial \phi_n}{\partial t}\right) \tag{A.17}$$

Replacing this back to the relation of general time evolution:

$$\dot{g} \approx \{g, H\}^D + \frac{\partial g}{\partial t} \tag{A.18}$$

where we finally define the correct bracket, the *Dirac Bracket*, for constrained systems:

$$\{g, H\}^D \equiv \{g, H\} - \{g, \phi_m\} \left(\{\phi,\phi\}\right)^{-1}_{nm} \left(\{\phi_n, H\} + \frac{\partial \phi_n}{\partial t}\right) \tag{A.19}$$

where the summation is over the T constraints found.

This bracket reduces back to the traditional Poisson bracket for

$$\{g, H_T\}^D = \{g, H_T\}, \tag{A.20}$$

and also carries the same properties of the Poisson bracket like the Jacobi identity, antisymmetry, linearity and the Leibniz rule.

☞ [34]

A.2 Canonical transformations

For many Hamiltonian systems it may be troublesome to deal with the coordinate system that the original theory is presented in. For that we may use canonical transformations, that are basically transformations that keep the infinitesimal action null $\delta S = 0$.

If one has a Hamiltonian $H = H(q,p,t)$ and wants another like $K = K(P,Q,t)$ where $Q_i = Q_i(q,p,t)$ and $P_i = P_i(q,p,t)$ satisfying:

$$\dot{Q}_i = \frac{\partial K}{\partial P_i}, \qquad \dot{P}_i = -\frac{\partial K}{\partial Q_i} \tag{A.21}$$

One requires that:

$$\delta S = \delta \int_{t_1}^{t_2} (p_i \dot{q}_i - H(q,p,t)dt) = \delta \int_{t_1}^{t_2} \left(P_i \dot{Q}_i - K(Q,P,t)dt \right) \tag{A.22}$$

Then it is true if:

$$\lambda(p_i \dot{q}_i - H) = P_i \dot{Q}_i - K + \frac{\partial F}{dt} \tag{A.23}$$

Where λ is a scaling factor, which would specially be non zero when we change also the time variable from a t to a ξ with Jacobian different from zero; F is the generating function of the transformation, explicit examples can be found in any Analytical Mechanics textbook.

But in summary, we may respect such rules for transformation, but as a more refined criterion for a transformation to be canonical is that once we have a canonical coordinate system, namely (q,p) such their Poisson bracket is $\{q,p\} = 1$, the map to other system (Q,P) is canonical if its Jacobian determinant is 1, then it yields that the differential equation now described by Q and P can also be written in a Hamiltonian form.

☞ [35]

Appendix B

Möbius transformations and other equivalences on equations

As was said before, Painlevé and his school, over a century ago, have discovered all the first-degree second-order rational ordinary differential equations, that is, equations of the form

$$\frac{\partial^2 y}{\partial z^2} = F\left(\frac{\partial y}{\partial z}, y, z\right) \tag{B.1}$$

where F is rational in $\frac{\partial y}{\partial z}$ and y and analytic in z.

All these equations cast a list of 50 equations that can be found in [17], from those 50, 6 were irreducible (could not be reduced as a combination of simpler functions), and they are the 6 Painlevé equations. As an example, one of the reducible equations found, the $XXXIV$, has solutions on the P_{II} transcendent:

$$XXXIV: \qquad \frac{\partial^2 y}{\partial z^2} = \frac{(y')^2}{2y} + \alpha y^2 - zy - \frac{1}{2y} \tag{B.2}$$

Solution:

$$2\alpha y = v' + v^2 + \frac{z}{2} \tag{B.3}$$

where

$$v'' = 2v^3 + zv + 2\alpha - 1/2$$

But all the 50 equations clearly do not represent absolutely all possible-to-write first-degree second-order ODE with the PP, actually, one may obtain the others via a class of transformations that leave the singularity structure invariant, and therefore keep the PP. This is called the *homographic group*, or Möbius group (also denoted PSL(2,$\mathbb{C}$)):

$$(u, x) \mapsto (U, X), \quad u(x) = \frac{\alpha(x)U(X) + \beta(x)}{\gamma(x)U(X) + \delta(x)}, \quad X = \xi(x) \tag{B.4}$$

where $(\alpha(x), \beta(x), \gamma(x), \delta(x), \xi(x))$ are arbitrary functions satisfying $\alpha\delta - \beta\gamma \neq 0$.

The reason why they preserve the PP is that when one chooses a regular point x_0 and looks in its neighborhood, the above functions will seem to be constant, then driving our function replacement to approximate to a Möbius transformation:

$$x \mapsto \frac{\alpha x + \beta}{\delta x + \gamma} \tag{B.5}$$

which is largely known to be the more general bijection (one-to-one map) between two Riemann spheres.

Because of that, if our previous function had a critical (respectively non-critical) point at x_0, then X_0 will also be critical (non-critical). That's due to the one-to-one mapping that would send each branch at x_0 to a different map, preserving this part of the PP.

About the "movability" of the solution, once the functions $(\alpha(x), \beta(x), \gamma(x), \delta(x), \xi(x))$ don't depend on x_0, then one could not cancel or create a movable point, compelling the invariance of this property on the equation by this transformation and, consequently, preserving the PP as said above.

An example of such transformation is the Ricatti transformation, used in our case of the mixed P_{III}-P_V when we wanted to go to the P_V limit:

$$q(z) = \frac{1}{1 + y(z)}. \tag{B.6}$$

Basically what it says is that when someone want to identify if an ODE has the PP, one first perform the Panlevé test, if no critical points was found by the method, than on should look for a Möbius transformation that makes its equation to be listed in its classification (like second-degree second-order). Huge lists of equations in different classifications with the PP were already made, for example, for second-order [17][36][37][38][39][40], third-order [41][42][43], fourth-order [43][44] and higher orders [45].

☞ [13][17]

Appendix C

Painlevé analysis for non-autonomous systems

In the second chapter, we have described the Painlevé method, but all the examples used there were autonomous equations, that is, they do not depend explicitly on the variable z.

Here I explain what one should do in the case when the variable z is apparent. The issue of the non-autonomous equations is that when we expand $q(z) \backsim c_0(z-z_0)^p$ we are actually making the limits

$$q(z) \to \infty \qquad \text{and} \qquad z \to z_0. \tag{C.1}$$

Then when we have, for example, the ratio:

$$\frac{q_z}{z} \backsim \frac{c_0 p(z-z_0)^{p-1}}{z} \tag{C.2}$$

we must think on the situations of $z_0 = 0$ and $z_0 \neq 0$. For $z_0 = 0$, the simpler one, we are able to analyze the correct behavior like:

$$\frac{q_z}{z} \backsim \frac{c_0 p z^{p-1}}{z} = c_0 p z^{p-2} \tag{C.3}$$

Which is evidently different from the case with $z_0 \neq 0$, where we must remember that $z \to z_0$ just for the explicit z's:

$$\frac{q_z}{z} \backsim \frac{c_0 p(z-z_0)^{p-1}}{z_0} \tag{C.4}$$

The explicit z_0 is just a number, then if it have appeared in a summation we could have get rid of it, but as it is in a division (the same in a multiplication), it has the same status as c_0, for example.

Appendix D

Lax pairs and hierarchies

D.1 Pseudodifferential deduction of the KdV Lax Pair

In this work we won't do the original deduction of the Lax Pair, done by Peter Lax considering operator's eingenvalues constant under nonlinear evolution, such can be found in []. Using the power of pseudo differential operators, we will do a careful deduction of the KdV hierarchy.

Using an operator of the form:

$$L(t) = \partial^2 + u/6 \tag{D.1}$$

we may define the square root of this operator being an infinite series of ∂ and ∂^{-1}:

$$L(t)^{\frac{1}{2}} = \partial + a_0(u) + \sum_{n=1}^{\infty} a_n(u)\partial{-}n \tag{D.2}$$

where the functionals $a_n(u)$ are determined in the way on making the square of $L(t)^{\frac{1}{2}}$ being $L(t)$. We may also extend it to:

$$L(t)^{\frac{2m+1}{2}} = L(t)^m L(t)^{\frac{1}{2}} = (\partial^2 + u/6)^m L(t)^{\frac{1}{2}} \tag{D.3}$$

Letting $L(t)_+^{\frac{2m+1}{2}}$ denote the the part of the series with degree $\geq$ zero, and $L(t)_-^{\frac{2m+1}{2}}$ the part with degree $<$ zero, n the same way we have done when dealing with pure pseudodifferential operators.

Since the highest derivative in $L(t)^m$ is ∂^{2m}, to know $L(t)_+^{\frac{2m+1}{2}}$ one must know the coefficient ∂^{-2m} in $L(t)^{\frac{1}{2}}$ to cancel it. One may also note that:

$$[L(t)^{\frac{2m+1}{2}}, L(t)] = 0 \tag{D.4}$$

then

$$[L(t)_+^{\frac{2m+1}{2}}, L(t)] = -[L(t)_-^{\frac{2m+1}{2}}, L(t)] \tag{D.5}$$

For this equality to hold, once each side has different degrees of derivatives, both sides correspond to a multiplicative operator. Labeling:

$$B_m(t) = \alpha_m L(t)_+^{\frac{2m+1}{2}} \tag{D.6}$$

and recovering:

$$L(t)_+^{\frac{1}{2}} = \partial + a_0(u) \tag{D.7}$$

we may note that to determine the value of a_0 one must expand $L(t)^{\frac{1}{2}}$ up to the power ∂^{-1}:

$$L(t)^{\frac{1}{2}} = \partial + a_0(u) + a_1(u)\partial^{-1} \tag{D.8}$$

then squaring this and comparing coefficients, we obtain:

$$L(t)^{\frac{1}{2}} = \partial + \frac{u}{12}\partial^{-1} \tag{D.9}$$

then $B_0(t) = L(t)_+^{\frac{1}{2}} = \partial$.

If we make the exercise of moving to the next order,

$$B_1(t) = \alpha_1 L(t)_+^{\frac{3}{2}} = \alpha_1(\partial^3 + 1/8(\partial u + u\partial)) \longrightarrow 4\partial^3 + 1/2(\partial u + u\partial) \tag{D.10}$$

which are exactly the one that obeys:

$$L_t = [B, L] \quad \Longrightarrow u_t = uu_x + u_{xxx} \tag{D.11}$$

the KdV equation.

☞ [46]

D.2 The hierarchy for the P_{II} equation

As a common milestone in the study of the integrability of the Painlevé transcendents, the deduction of the P_{II} hierarchy shows the strict correlation between the P_{II} and KdV hierarchy, here all written.

D.2.1 KdV to P_{II}

We begin by the normally guiding transformations that lead the traditional Kdv equation to the P_{II} equation. To do that we start by the KdV equation:

$$U_t + 6UU_x + U_{xxx} = 0 \tag{D.12}$$

Which can be reduced from one variable by the transformation:

$$U(x,t) = \frac{u(z)}{(3t)^{\frac{2}{3}}}, \qquad z = \frac{x}{(3t)^{\frac{1}{3}}}. \tag{D.13}$$

resulting an non-linear third order ODE:

$$u'''(z) + 6u'(z)u(z) - zu'(z) - 2u(z) = 0 \tag{D.14}$$

This equation can still be transformed again by:

$$u(z) = y_z - y^2 \tag{D.15}$$

Which gives:

$$2y^2 - 2y' + 12y^3y' + 2yzy' - 12y\left(y'\right)^2 - 6y^2y'' - zy'' - 2yy^{(3)} + y^{(4)} = 0 \tag{D.16}$$

Where $y(z)$ satisfies the P_{II} equation:

$$y''(z) = 2y^3(z) + zy(z) + \alpha, \qquad \alpha \text{ const.} \tag{D.17}$$

D.2.2 KdV hierarchy

The usefulness of the Lax Pairs on the construction of hierarchies comes from the possibility of a recursive relation that gives the whole hierarchy.

For the KdV hierarchy, such must obey the following relation:

$$\frac{\partial U}{\partial t_{2n+1}} = \{U(x,t), H_{2n+1}\}_1 = \{U(x,t), H_{2n-1}\}_2, \qquad n = 0,1,2,... \tag{D.18}$$

where the first Poisson brackets are

$$\begin{aligned}
\{U(x,t)U(y,t)\}_1 &= -\,\delta_x(x-y) \\
\{U(x,t)U(y,t)\}_2 &= -\,\delta_{xxx}(x-y) - 4U\delta_x(x-y) - 2U_x\delta(x-y)
\end{aligned}$$

and Hamiltonian

$$H_{-1} = \int U dy$$

Resulting in the Lénard relation:

$$\left(\frac{\partial^3}{\partial x^3} + 4U\frac{\partial}{\partial x} + 2U_x\right)\frac{\delta H_{2n-1}}{\delta U} = \frac{\partial}{\partial x}\frac{\delta H_{2n+1}}{\delta U}$$

Compressing this relation we arrive at:

$$U_{t_{2n+1}} + \partial_x \mathcal{L}_{n+1}[U] = 0, \qquad n \geq 0 \tag{D.19}$$

where

$$\begin{aligned} \partial_x \mathcal{L}_{n+1}[U] &= (\partial_{xxx} + 4U\partial_x + 2U_x)\, \mathcal{L}_n[U], \\ \mathcal{L}_1[U] &= U \end{aligned} \tag{D.20}$$

we may emphasize that we will use x instead of z here.

The first members of this hierarchy are given in the table below

Table D.1: KdV Hierarchy

n	$\mathcal{L}_{n+1}$	Equation
0	U	$U_{t_1} + U_x = 0$
1	$U_{xx} + 3U^2$	$U_{t_3} + 6UU_x + U_{xxx} = 0$
2	$U_{xxxx} + 10U^3 + 5U_x^2 + 10UU_{xx}$	$U_{t_5} + U_{xxxxx} + 10UU_{xxx} + 20U_x u_{xx} + 30U^2U_x = 0$

D.2.3 MKdV hierarchy

Now we introduce the Modified KdV hierarchy, that appears by the use of the Miura map $U = W_x - W^2$ again, but now directly on the original KdV, which, after some manipulations, become:

$$W_t - 6W^2W_x + W_{xxx} = 0 \tag{D.21}$$

By doing the same replacement in the KdV hierarchy's recursion relation we immediately get:

$$\partial_{t_{2n+1}}(W_x - W^2) + \partial_x \mathcal{L}_{n+1}\left[W_x - W^2\right] = 0 \tag{D.22}$$

By manipulating the ∂_x derivative acting on $\mathcal{L}_{n+1}$ we achieve a more treatable expression:

$$\partial_{t_{2n+1}} W + \partial_x(\partial_x + 2W)\mathcal{L}_n\left[W_x - W^2\right] = 0 \tag{D.23}$$

which works like in the previous case as a recurrence relation for the MKdV hierarchy.

D.2.4 P_{II} hierarchy

And, as in the second last section, to bring the 2-variable equation to a 1-variable equation we may perform the reduction:

$$W(x, t_3) = \frac{y(z)}{(3t_3)^{\frac{1}{3}}}, \qquad z = \frac{x}{(3t_3)^{\frac{1}{3}}} \tag{D.24}$$

that, when it acts on the MKdV equation:

$$y''' = 6y^2y' + zy' + y \tag{D.25}$$

we are lead again to the P_{II} equation.

But for the hierarchy, we make the following replacements:

$$W(x, t_{2n+1}) = \frac{y(z)}{((2n+1)t_{2n+1})^{\frac{1}{2n+1}}}, \qquad z = \frac{x}{((2n+1)t_{2n+1})^{\frac{1}{2n+1}}} \tag{D.26}$$

$$\begin{aligned} W_x - W^2 &= \frac{1}{((2n+1)t_{2n+1})^{\frac{2}{2n+1}}}(y' - y^2), \\ W_{t_{2n+1}} &= -\frac{y}{((2n+1)t_{2n+1})^{\frac{2n+2}{2n+1}}} - \frac{zy'}{((2n+1)t_{2n+1})^{\frac{2n+2}{2n+1}}} \end{aligned} \tag{D.27}$$

It can be proved by extensive calculation that the above replacements connect the Lax in two different coordinates:

$$\mathcal{L}_k\left[U\right] = \frac{1}{((2n+1)t)^{\frac{2k}{2n+1}}}\mathcal{L}_k\left[y' - y^2\right] \tag{D.28}$$

now substituting the known $\mathcal{L}_k\left[U\right]$ we arrive at

$$\left(\frac{d}{dz} + 2y\right)\mathcal{L}_n\left[y' - y^2\right] = zy + \alpha_n, \qquad n \geq 1 \tag{D.29}$$

which is in fact the P_{II} hierarchy, with first members are list below

TABLE D.2: P_{II} Hierarchy

n	Equation
0	$y = -\frac{\alpha_0}{z-1}$
1	$y'' = 2y^3 + zy + \alpha_1$
2	$y'''' - 10yy'^2 - 10y^2y'' + 6y^5 = zy + \alpha_2$

☞ [47]

Appendix E

Square-root lattice

Before we move with our construction let's make a connection with other models that makes such work sensible.

Many continuum KP-type hierarchies are governed by discrete lattice-like structures like the AKNS hierarchy whose Bäcklund transformation's structure comes from the Toda lattice and lead to Hirota-type equations for the Toda chain of tau-functions.

Another lattice that provides the symmetry transformations for a AKNS hierarchy is the "square-root" (or "half-integer" or "generalized Volterra") lattice. Its foundations rely on two spectral equations:

$$\lambda^{1/2}\tilde{\Psi}_{n+\frac{1}{2}} = \Psi_{n+1} + \mathcal{A}^{(0)}_{n+1}\Psi_n + \sum_{p=1}^{M}\mathcal{A}^{(p)}_{n-p+1}\Psi_{n-p} \tag{E.1}$$

$$\lambda^{1/2}\Psi_n = \tilde{\Psi}_{n+\frac{1}{2}} + \mathcal{B}^{(0)}\tilde{\Psi}_{n-\frac{1}{2}} \tag{E.2}$$

and evolution equations:

$$\tilde{\Psi}_{n+\frac{1}{2}} = \left(\partial - \mathcal{B}^{(0)}_n - \mathcal{A}^{(0)}_n\right)\tilde{\Psi}_{n-\frac{1}{2}} \tag{E.3}$$

$$\Psi_{n+1} = \left(\partial - \mathcal{B}^{(0)}_n - \mathcal{A}^{(0)}_{n+1}\right)\Psi_n \tag{E.4}$$

here all objects are labeled by integers and half-integers. When we look at the equation (E.1) we notice that if one removes the term $\sum_{p=1}^{M}\mathcal{A}^{(p)}_{n-p+1}\Psi_{n-p}$ the above described system reduces to the traditional Volterra lattice. Also if we remove from the system above all half-integer modes it reduces to the Toda lattice equations.

Combining the equations we arrive at:

$$\lambda^{1/2}\tilde{\Psi}_{n+\frac{1}{2}} = \left(\partial - \mathcal{B}^{(0)}_n + \sum_{p=1}^{M}\mathcal{A}^{(p)}_{n-p+1}\left(\partial - \mathcal{B}^{(0)}_{n-p} - \mathcal{A}^{(0)}_{n-p+1}\right)^{-1}\dots\left(\partial - \mathcal{B}^{(0)}_{n-1} - \mathcal{A}^{(0)}_n\right)^{-1}\right)\Psi_n \tag{E.5}$$

and

$$\lambda^{1/2}\Psi_n = (\partial - \mathcal{A}_n^{(0)})\tilde{\Psi}_{n-\frac{1}{2}} \tag{E.6}$$

Eliminating the half integer modes above we arrive at the relation:

$$\lambda\Psi_n = (\partial - \mathcal{A}_n^{(0)}) \left(\partial - \mathcal{B}_{n-1}^{(0)} + \sum_{p=1}^{M} \mathcal{A}_{n-p}^{(p)} \left(\partial - \mathcal{B}_{n-p-1}^{(0)} - \mathcal{A}_{n-p}^{(0)}\right)^{-1} \ldots \left(\partial - \mathcal{B}_{n-2}^{(0)} - \mathcal{A}_{n-1}^{(0)}\right)^{-1}\right) \\ \times (\partial - \mathcal{B}_{n-1}^{(0)} - \mathcal{A}_n^{(0)})^{-1}\Psi_n \tag{E.7}$$

and then at the spectral relation:

$$\lambda\Psi_n = L_n^{M+1}\Psi_n \tag{E.8}$$

At this level, we already recognize it as a relation for a Lax operator, which can be generalized to obey the above integer relations as:

$$L_n^{M+1} = e^{\int \mathcal{B}_{n-1}^{(0)}} \left(\partial - \mathcal{A}_n^{(0)} + \mathcal{B}_{n-1}^{(0)}\right) L_n^M \left(\partial - \mathcal{A}_n^{(0)}\right)^{-1} e^{-\int \mathcal{B}_{n-1}^{(0)}} \tag{E.9}$$

where

$$L_n^{(M)} = \partial + \sum_{p=1}^{M} \mathcal{A}_{n-p}^{(0)} \left(\partial + \mathcal{B}_{n-1}^{(0)} - \mathcal{B}_{n-p-1}^{(0)} - \mathcal{A}_{n-p}^{(0)}\right)^{-1} \ldots \left(\partial + \mathcal{B}_{n-1}^{(0)} - \mathcal{B}_{n-2}^{(0)} - \mathcal{A}_{n-1}^{(0)}\right)^{-1} \tag{E.10}$$

completing a chain of recurrence relations.

☞ [30]

Appendix F

Dirac reduction of the $\mathcal{H}_{A_4^{(1)}}$

Our starting point is the Hamiltonian:

$$\begin{aligned}\mathcal{H}_{A_4^{(1)}}(e_1, Y_1, e_2, Y_2) = &-\sum_{j=1}^{2} e_j \left(Y_j - \frac{\xi}{2}\right)(Y_j - e_j) + 2e_1 \left(Y_1 - \frac{\xi}{2}\right)(Y_2 - e_2) \\ &+ \bar{k}_1 Y_1 + \bar{k}_2 Y_2 - k_1 e_1 - k_2 e_2.\end{aligned} \tag{F.1}$$

First defining the canonical variables P and Q:

$$P = -N\xi e_1, \qquad Q = \frac{Y_1 - M\xi}{N\xi} \tag{F.2}$$

Redefining then the Hamiltonian to:

$$\mathcal{H}_{A_4^{(1)}}(P, Q, e_2, Y_2) = \mathcal{H}_{A_4^{(1)}}(e_1, Y_1, e_2, Y_2) - \frac{PQ}{\xi} - \frac{MP}{N\xi} \tag{F.3}$$

To eliminate the old variables e_2 and Y_2 we impose the second class constraints:

$$\phi_1 = e_2 - \frac{1}{\xi}(D\xi^2 + EP + FPQ) \tag{F.4}$$

$$\phi_2 = Y_2 - A\xi \tag{F.5}$$

These constraints are very *ad hoc* due to the fact that we wanted to find what kind of constraints reduced our system to the mixed one. Historically, the mixed system have arisen after the reductions of the Hamiltonian to the P_{III} and P_V being discovered, then it suggested, naturally, that should be a more complex structure hidden before this double reduction; the mixed system was then proposed and here we are able to indeed reduce the Hamiltonian to it.

By the use of Lagrange multipliers:

$$\mathcal{H}_{A_4^{(1)}} \to \mathcal{H}^{\lambda}_{A_4^{(1)}} + \lambda_1 \phi_1 + \lambda_2 \phi_2 \tag{F.6}$$

For such constraints $\{\phi_1, \phi_2\}^{-1} = -1 = \{\phi_2, \phi_1\}^{-1}$, as seen in Appendix A; then we get:

$$\mathcal{H}^{\lambda}_{A_4^{(1)}} = \mathcal{H}_{A_4^{(1)}} - \sum_{n=1,m=1}^{2} \phi_n \{\phi_n, \phi_m\}^{-1} \left(\left\{ \phi_m, \mathcal{H}_{A_4^{(1)}} \right\} + \frac{\partial \phi_m}{\partial \xi} \right) \tag{F.7}$$

The equation of motion for P and Q are:

$$P_\xi = \left\{ P, \mathcal{H}_{A_4^{(1)}} \right\} - \{P, \phi_1\} \{\phi_1, \phi_2\}^{-1} \left(\left\{ \phi_2, \mathcal{H}_{A_4^{(1)}} \right\} + \frac{\partial \phi_2}{\partial \xi} \right) |_{\phi_i \approx 0}$$
$$Q_\xi = \left\{ Q, \mathcal{H}_{A_4^{(1)}} \right\} - \{Q, \phi_1\} \{\phi_1, \phi_2\}^{-1} \left(\left\{ \phi_2, \mathcal{H}_{A_4^{(1)}} \right\} + \frac{\partial \phi_2}{\partial \xi} \right) |_{\phi_i \approx 0}$$

Which give us through explicit calculation:

$$\left\{ \phi_2, \mathcal{H}_{A_4^{(1)}} \right\} = -\frac{\partial \mathcal{H}_{A_4^{(1)}}}{\partial e_2}, \qquad \{P, \phi_1\} = \frac{FP}{\xi}, \qquad \{Q, \phi_1\} = -\frac{E + FQ}{\xi} \tag{F.8}$$

Inserting these results we readily obtain:

$$\xi P_\xi = -\xi \frac{\partial \mathcal{H}_{A_4^{(1)}}}{\partial Q} - FP \left(\frac{\partial \mathcal{H}_{A_4^{(1)}}}{\partial e_2} + A \right) |_{\phi_i \approx 0} \tag{F.9}$$
$$\xi Q_\xi = \xi \frac{\partial \mathcal{H}_{A_4^{(1)}}}{\partial P} + (E + FQ) \left(\frac{\partial \mathcal{H}_{A_4^{(1)}}}{\partial e_2} + A \right) |_{\phi_i \approx 0} \tag{F.10}$$

Now as stated before the above system assumes the correct Hamilton equations for certain values of the constraints' constants. Comparing the above system with the mixed one with $C = 0, r_0 = 1, \epsilon_1 = 0$, that is:

$$zq_z = q(q - r_1)(2p + z) - (\alpha_1 + \alpha_3)q + \alpha_1 r_1 + \epsilon_0 z$$
$$zp_z = p(p + z)(r_1 - 2q) + (\alpha_1 + \alpha_3)p - \alpha_2 z,$$

we identify $P = p$ and $Q = q$ for $z = \xi^2$ and also:

$$N = 2, \qquad A = \frac{1}{2} + \frac{32\epsilon_0}{16r_1^2 - 1}, \qquad 16\epsilon_0 F^2 + (16r_1^2 - 1)(F - 1) = 0,$$
$$G = 1 - F + 4r_1^2 F^2 \qquad D = \frac{1}{F^2} \left(-F + F^2/4 + 1 \pm \sqrt{G}/2 \right),$$
$$M = \frac{1}{F^3} \left((1 - 2r_1)F^3 - 2F^2 + 2F \pm F(F - 2)\sqrt{G} \right), \qquad E = -\frac{1}{4F} \left(2r_1 F^2 + F \pm F\sqrt{G} \right),$$
$$\bar{k}_1 = \alpha_2, \qquad k_2 = \frac{1}{2F^2} \left(F^2 + 4 - 6F + 4F(\alpha_1 + \alpha_3) \right),$$
$$k_1 = \frac{1}{2F^3} \left(F^3(1 + 4r_1(\alpha_1 - \alpha_3)) + 2F - F^2(2(\alpha_1 + \alpha_3) + 1 \pm 2F(F(\alpha_1 + \alpha_3) - 1)\sqrt{G} \right)$$

We may also compare it with the mixed system with $C = -1/2, r_0 = 1, \epsilon_1 = -1$:

$$\begin{aligned} zq_z =& q(q - r_1)(2p + z) - (\alpha_1 + \alpha_3)q + \alpha_1 r_1 + \epsilon_0 \\ zp_z =& p(p + z)(r_1 - 2q) + (\alpha_1 + \alpha_3)p - \alpha_2 z + r_1 z \end{aligned}$$

we identify $P = p$ and $Q = q$ for $z = \xi^2$ and also:

$$F = 1, \qquad A = \frac{1}{2}, \qquad N = 2, \qquad D = \frac{1}{4} \pm r_1$$

$$E = \begin{cases} -1/4 \\ -(r_1 + 1/4) \end{cases}, \qquad M = \begin{cases} 1/2 - 2r_1 \\ 1/2 \end{cases},$$

$$\bar{k}_1 = \alpha_2 - r_1, \qquad 3/2 - 2(\alpha_1 + \alpha_3),$$

$$k_{1\pm} = \alpha_0 + \alpha_2 + 2r_1(2\alpha - 1 - 1 + 4\epsilon_0) + \begin{cases} 0 \\ 4r_1(\alpha_0 + \alpha_2) \end{cases}$$

☞ [31]

Appendix G

Solving the Laguerre-hypergeometric equation

After it was linearized, the equation for $q(z)$ is now in terms of $\omega(z)$, and now it must be solved. First, let us realize that the solution for the generalized associated Laguerre equation,

$$xy_{xx} = -(\nu + 1 - x)y_x - \lambda y = 0$$

is

$$y(x) = c_1 U(-\lambda, 1+\nu, x) + L^{\nu}_{\lambda}(x),$$

where $U(a, b, x)$ is the confluent hypergeometric function of the first kind and $L^{\nu}_{\lambda}(x)$ is the generalized Laguerre polynomial. [Iyanaga and Kawada 1980, p. 1481; Zwillinger 1997, p. 124].

In our case we have the equation:

$$\omega_{zz} = (r_0 r_1 - \frac{\alpha_1 + \alpha_3}{z})\omega_z + (\frac{\alpha_1 r_0 r_1}{z} + \epsilon_0 r_0^2 z^{2C})\omega \tag{G.1}$$

For the sake of simplicity here, let's use other parameters, getting:

$$\omega_{zz} = (\alpha + \frac{\beta}{z})\omega_z + (\frac{\gamma}{z} + \delta z^{2C})\omega \tag{G.2}$$

where $\alpha = r_0 r_1$, $\beta = -\alpha_1 - \alpha_3$, $\gamma = \alpha_1 r_0 r_1$, $\delta = \epsilon_0 r_0^2$.

G.0.5 C=0

For the case $C = 0$ we'll have:

$$\omega_{zz} = (\alpha + \frac{\beta}{z})\omega_z + (\frac{\gamma}{z} + \delta)\omega \tag{G.3}$$

Performing the substitution:

$$\omega(z) = e^{\frac{1}{2}z\left(\alpha-\sqrt{\alpha^2+4\delta}\right)}z^{1+\beta}f(\tau) \tag{G.4}$$

for $\tau = z\sqrt{\alpha^2+4\delta}$, we arrive at

$$f" = \frac{-2-b+\tau}{\tau}f' + \frac{2+b+\frac{ab}{\sqrt{a^2+4d}}+\frac{2c}{\sqrt{a^2+4d}}}{2\tau}f \tag{G.5}$$

which has the form of the generalized associated Laguerre equation above. Solving it for ω we get:

$$\omega(z) = e^{\frac{1}{2}z\left(\alpha-\sqrt{\alpha^2+4\delta}\right)}z^{1+\beta}\left(c_1\text{HypergeometricU}\left[\frac{\alpha\beta+2\gamma+(2+\beta)\sqrt{\alpha^2+4\delta}}{2\sqrt{\alpha^2+4\delta}}, 2+\beta, z\sqrt{\alpha^2+4\delta}\right]+\right.$$
$$\left. c_2\text{LaguerreL}\left[\frac{-\alpha\beta-2\gamma-(2+\beta)\sqrt{\alpha^2+4\delta}}{2\sqrt{\alpha^2+4\delta}}, 1+\beta, z\sqrt{\alpha^2+4\delta}\right]\right) \tag{G.6}$$

which is identical to the equation (5), up to these identities: $U(a,b,z) = z^{1-b}U(1+a-b, 2-b, z)$ and $L_n^{-m}(z) = \frac{z^m}{(-n)_m}L_{n-m}^m(z)$, where$(-n)_m$ is the Pochhammer symbol, which will go within the multiplicative constant.

G.0.6 C=-1/2

For the case $C = -\frac{1}{2}$ we'll have:

$$\omega_{zz} = (\alpha + \frac{\beta}{z})\omega_z + (\frac{\gamma+\delta}{z})\omega \tag{G.7}$$

Performing the substitution:

$$\omega(z) = z^{1+\beta}f(\tau) \tag{G.8}$$

for $\tau = az$, we arrive at

$$f" = \frac{(\alpha+\alpha\beta+\gamma+\delta)f}{\alpha\tau} + \frac{(-2-\beta+\tau)f'}{\tau} \tag{G.9}$$

which has the form of the generalized associated Laguerre equation above. Solving it for ω we get:

$$\omega(z) = z^{1+\beta}c_1\text{HypergeometricU}\left[-\frac{-\alpha-\alpha\beta-\gamma-\delta}{\alpha}, 2+\beta, z\alpha\right]+$$
$$z^{1+\beta}c_2\text{LaguerreL}\left[\frac{-\alpha-\alpha\beta-\gamma-\delta}{\alpha}, 1+\beta, z\alpha\right] \tag{G.10}$$

G.0.7 C=-1

For the case $C = -1$ we'll have:

$$\omega_{zz} = (\alpha + \frac{\beta}{z})\omega_z + (\frac{\gamma}{z} + \frac{delta}{z^2})\omega \tag{G.11}$$

Performing the substitution:

$$\omega(z) = z^{\frac{1}{2}\left(1+\beta+\sqrt{1+2\beta+\beta^2+4\delta}\right)} f(\tau) \tag{G.12}$$

for $\tau = az$, we arrive at

$$f" = \frac{\left(2\gamma + \alpha\left(1+\beta+\sqrt{(1+\beta)^2+4\delta}\right)\right)}{2\alpha\tau} f - \frac{1+\sqrt{(1+\beta)^2+4\delta}-\tau}{\tau} f' \tag{G.13}$$

which has the form of the generalized associated Laguerre equation above. Solving it for ω we get:

$$\omega(z) = z^{\frac{1}{2}(1+\beta+\rho)} \Bigg(c_1 \text{HypergeometricU}\left[\frac{2\gamma+\alpha(1+\beta+\rho)}{2\alpha}, 1+\rho, z\alpha\right] + \\ c_2 \text{LaguerreL}\left[-\frac{2\gamma+\alpha(1+\beta+\rho)}{2\alpha}, \rho, z\alpha\right]\Bigg) \tag{G.14}$$

where $\rho = \sqrt{1+2\beta+\beta^2+4\delta}$.

Bibliography

[1] H. Aratyn, J. F. Gomes, A. H. Zimerman, Nonlinear and Modern Mathematical Physics: Proc. 1st Int. Workshop (AIP Conf. Proc. vol 1212) p. 146-153, and arXiv: 0909.3532

[2] H. Aratyn, J. F. Gomes, D. V. Ruy, A. H. Zimerman. Journal of Physics A: Math. Theor. 49, (2016)

[3] B. M. McCoy, C. A. Tracy and T. T. Wu, Phys. Rev. Lett. 38 (1977) 793-796.

[4] A J Bracken, L Bass and C Rogers, J. Phys. A: Math. Theor. 45 105204 (2012), [arXiv:math-ph/1201.0673]

[5] A. Aftalion, Q. Du and Y. Pomeau, Phys. Rev. Lett. 91, 090407 (2003)

[6] A.S. Fokas and S. Tanveer, Math. Proc. Cambridge Philos. Soc., 124, no.1, 169-191 (1998)

[7] M. Jimbo, T. Miwa, Y. Mori and M. Sato, Physica D 1 (1980) 80-158.

[8] R. J. Szabo and M. Tierz, J. Phys. A: Math. Theor. 45 (2012) 085401 [arXiv:hep-th/1102.3640]

[9] S. Chakravarty , M. J. Ablowitz , and P. A. Clarkson , Reductions of self-dual Yang-Mills fields and classical systems, Phys. Rev. Lett. 65:108.1087 (1990). [Erratum: 65:2086 (1990)].

[10] A. Latifi , M. Musette, and R. Conte , The Bianchi IX (mixmaster) cosmological model is not integrable, Phys. Lett. A 194:83.92 (1994) .[Erratum: 197:459.460 (1995)].

[11] P. A. Clarkson and P. J. Olver , Symmetry and the Chazy equation, J. Differential Equations, 124:225.246 (1986)

[12] Rod Halburd, Movable Singularities of Nonlinear ODEs, Presentation on FASDEII, Bedlewo,August 2011

[13] Robert M. Conte, Micheline Musette, The Painlevé Handbook, 2008, Springer Netherlands

[14] Andrew N. W. Hone, Springer Lect. Notes Phys. 767 (2009) 245-277

[15] Martin D.Kruskal, Nalini Joshi, Rod Halburd, arXiv:solv-int/9710023v1

[16] Valerii I. Gromak, Ilpo Laine, Shun Shimomura; Painlevé Differential Equations in the Complex Plane; 2002; de Gruyter Studies in Mathematics, Volume 28

[17] E. L. Ince, Ordinary Differential Equations (1926). Reprint: Dover Publications, New York (1956)

[18] M. J. Ablowitz and A. Ramani and H. Segur, A connection between nonlinear evolution equations and ordinary differential equations of P-type I, Journal of Mathematical Physics, v.21, Issue 4, pp. 715-721 (1980)

[19] M. J. Ablowitz and A. Ramani and H. Segur, A connection between nonlinear evolution equations and ordinary differential equations of P-type II, Journal of Mathematical Physics, v.21, Issue 5, pp. 1006-1015 (1980)

[20] Decio Levi, Pavel Winternitz, Painlevé Transcendents, Nato Science Series B, v.278, Springer US (1992)

[21] Gritsuk, E.V., Gromak, V.I. Diff. Equat. (2010) 46: 1379

[22] Lax, P. (1968), Integrals of nonlinear equations of evolution and solitary waves, Comm. Pure Applied Math., 21 (5): 467-490

[23] Masatoshi Noumi, Painlevé Equations through Symmetry (2004)

[24] V. I. Arnol'd, Mathematical Methods of Classical Mechanics, Springer, 1989

[25] H. Aratyn, L.A. Ferreira, J.F. Gomes and A.H. Zimerman,“ On W_{infty} Algebras, Gauge Equivalence of KP Hierarchies, Two Boson Realizations and their KdV Reductions; VII J.A. Swieca Summer School, Particles and Fields, 10-23/01/1993. Ed. O. Eboli and V. Rivelles, World Sci, p. 630-666

[26] O. Babelon *et al.*, Introduction to Classical Integrable Systems, Cambridge Monographs on Mathematical Physics (2007)

[27] L.A. Dickey, Soliton Equations and Hamilton Systems, World Scientific

[28] T. Miwa, M. Jimbo, E. Date, Solitons: Differential equations, symmetries, and infinite-dimensional algebras, Cambridge University Press (2000)

[29] H.Aratyn, L.A. Ferreira, J.F. Gomes, A.H. Zimerman, Nucl.Phys. B402 (1993) 85-117

[30] H. Aratyn, J. F. Gomes, A. H. Zimerman, 2011 J. Phys. A: Math. Theor. 44 235202

[31] H. Aratyn, J. F. Gomes, D. V. Ruy, A. H. Zimerman, Journal of Physics A: Math. Theor. 49, (2016), 045201

[32] V. I. Gromak, G. V. Filipuk, On Algebraic Solutions of the Fifth Painlevé Equation, Differ. Uravn., 39:3 (2003), 299-307

[33] Arfken and Weber, Mathematical Methods For Physicists, 5th edition, sec. 13.2

[34] P.A.M. Dirac, Lectures on Quantum Mechanics, Dover (2001)

[35] H. Goldstein *et al.*, Classical Mechanics, 3rd ed., Pearson (2001)

[36] Gambier B 1910 Acta. Math. 33 1-55

[37] Bureau F J 1972 Ann. Mat. Pura Appl. XCI 163-281

[38] Cosgrove C M 1993 Stud. Appl. Math. 90 119-87

[39] Cosgrove C M and Scoufis G 1993 Stud. Appl. Math. 88 25-87

[40] Cosgrove C M 1997 Stud. Appl. Math. 98 355-433

[41] Chazy J 1911 Acta Math. 34 317-385

[42] Cosgrove C M, 2000 Stud. Appl. Math., 104 171-228 (http://www.maths.usyd.edu.au:8000/res/Nonlinear/Cos/1998-23.html)

[43] Bureau F J 1964 Ann. Mat. Pura Appl. LXVI 1-116

[44] Cosgrove C M, 2000 Stud. Appl. Math., 104 1-65 (http://www.maths.usyd.edu.au:8000/res/Nonlinear/Cos/1998-22.html)

[45] Exton H 1971 Rend. Mat. 4 385-448

[46] A. Das, Integrable Models, World Scientific (1989)

[47] N. Joshi, Publ. RIMS, Kyoto Univ., 40 (2004), 1039-1061