



# Periodic orbits for a class of reversible quadratic vector field on $\mathbb{R}^3$ <sup>☆</sup>

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## Abstract

For a class of reversible quadratic vector fields on  $\mathbb{R}^3$  we study the periodic orbits that bifurcate from a heteroclinic loop having two singular points at infinity connected by an invariant straight line in the finite part and another straight line at infinity in the local chart  $U_2$ . More specifically, we prove that for all  $n \in \mathbb{N}$ , there exists  $\varepsilon_n > 0$  such that the reversible quadratic polynomial differential system

$$\begin{aligned}\dot{x} &= a_0 + a_1y + a_3y^2 + a_4y^2 + \varepsilon(a_2x^2 + a_3xz), \\ \dot{y} &= b_1z + b_3yz + \varepsilon b_2xy, \\ \dot{z} &= c_1y + c_4z^2 + \varepsilon c_2xz\end{aligned}$$

in  $\mathbb{R}^3$ , with  $a_0 < 0$ ,  $b_1c_1 < 0$ ,  $a_2 < 0$ ,  $b_2 < a_2$ ,  $a_4 > 0$ ,  $c_2 < a_2$  and  $b_3 \notin \{c_4, 4c_4\}$ , for  $\varepsilon \in (0, \varepsilon_n)$  has at least  $n$  periodic orbits near the heteroclinic loop.

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## 1. Introduction

A vector field  $X: \mathbb{R}^3 \rightarrow \mathbb{R}^3$  of the form  $X = (P^1, P^2, P^3)$  is called a *quadratic vector field* if  $P^1$ ,  $P^2$  and  $P^3$  are polynomials of degree less or equal to two and at least one of them has degree two.

The dynamics of linear vector fields in  $\mathbb{R}^3$  is simple and very well understand. In particular, these vector fields only present periodic orbits when they have some invariant plane with a global center; i.e. a plane with a unique singular point surrounded by periodic orbits and no other orbits. The quadratic vector fields are the easiest nonlinear vector fields which can present a rich dynamics of periodic orbits. Here we shall show the existence of an easy mechanism around two straight line solutions, already present in the quadratic vector fields, which generates as many periodic orbits as we want. Of course, this mechanism is different from a center.

A diffeomorphism  $\varphi: \mathbb{R}^3 \rightarrow \mathbb{R}^3$  is called an *involution* if  $\varphi \circ \varphi = \text{Id}$ . Given an involution  $\varphi: \mathbb{R}^3 \rightarrow \mathbb{R}^3$  we say that a vector field  $X$  over  $\mathbb{R}^3$  is  $\varphi$ -reversible, if  $\varphi_*X = -X \circ \varphi$ , i.e.,  $d\varphi_p(X(p)) = -X(\varphi(p))$ . Let  $S$  be the fixed point set of  $\varphi$ . An orbit  $\gamma$  is said *symmetric* if  $\varphi(\gamma) = \gamma$ . Hence, every singular point of  $X$  in  $S$  is *symmetric* of  $X$ . Some classical properties of reversible systems are:

- (i) The phase portrait of  $X$  is symmetric with respect to  $S$ .
- (ii) A symmetric singular point or a symmetric periodic orbit cannot be attractor or repeller.
- (iii) If  $X(p) = 0$  and  $p \notin S$ , then  $X(\varphi(p)) = 0$ .
- (iv) If a regular orbit  $\gamma$  intersects  $S$  in two distinct points, then  $\gamma$  is a periodic orbit.
- (v) If  $X(p) \neq 0$  and  $p \in S$ , then  $X(p) \notin T_p S$ .
- (vi) Any periodic orbit  $\gamma$  of  $X$  not crossing  $S$ , has a symmetric one given by  $\varphi(\gamma)$ .

For more details about these properties see [3,11,13].

In this work we deal with  $\varphi$ -reversible quadratic vector fields  $X_\varepsilon$  on  $\mathbb{R}^3$ , depending on one-parameter  $\varepsilon$ , where we assume that the dimension of the fixed point set  $S$  of the linear involution  $\varphi$  is equal to 1. Our main goal is to establish sufficient conditions for the existence of arbitrary number of periodic orbits which are not in a center.

A polynomial vector field  $X$  in  $\mathbb{R}^3$  can be extended to an analytic vector field on the closed ball of radius one, the interior of this ball is diffeomorphic to  $\mathbb{R}^3$  and its boundary (a 2-dimensional sphere  $\mathbb{S}^2$ ) plays the role of infinity. The technique for making such an extension is called Poincaré compactification, see Section 2.

We consider the *heteroclinic loop*  $\mathcal{L}$  formed by two singular points  $p$  and  $q$  at infinity connected by two orbits  $\gamma_1$  in  $\mathbb{R}^3$  and  $\gamma_2$  in  $\mathbb{S}^2$  such that  $\mathcal{L}$  satisfies: (1)  $\gamma_1$  intersects  $S$  in a point  $s$ ; (2) one endpoint of  $S$  on  $\mathbb{S}^2$  is the point  $r$  of  $\gamma_2$ , see Fig. 1.

Let  $\mathbb{N}$  be the set of positive integers. For all  $n \in \mathbb{N}$  we shall prove that there exists  $\varepsilon_n > 0$  sufficiently small such that the vector field  $X_\varepsilon$  has at least  $n$  periodic orbits near the loop  $\mathcal{L}$  for all  $\varepsilon \in (0, \varepsilon_n)$ .

We will take a small interval  $G \subset S$  having  $r$  as an endpoint (see Fig. 1), and we will follow its image under the flow of  $X_\varepsilon$  until its intersection with a cross section  $\Sigma$  of the orbit  $\gamma_1$  which contains a neighborhood of  $S$  near the point  $s \in \mathcal{L}$ , see Fig. 1. We denote by  $\pi: G \rightarrow \Sigma$  the Poincaré map. Then, we shall prove that  $\pi(G)$  is a spiral near the point  $s$  giving finitely many turns for every  $\varepsilon > 0$  sufficiently small. This number of turns tends to infinity as  $\varepsilon \rightarrow 0$ . The

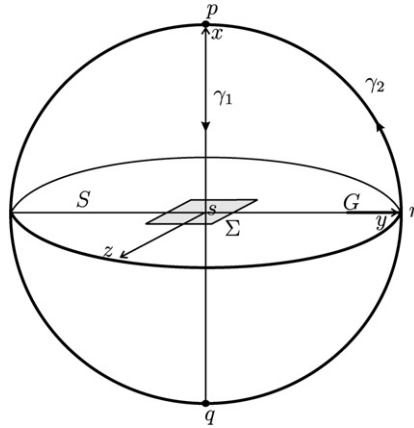


Fig. 1. The vector field  $X_\varepsilon$  on the Poincaré ball.

orbits through the points of  $\pi(G) \cap S$  are periodic because, by construction, they have two points on  $S$ . Using these ideas in Section 3 we shall prove our main results stated in the following three theorems.

**Theorem 1.** *Let  $X_\varepsilon$  be the vector field associated to the quadratic polynomial differential system:*

$$\begin{aligned} \dot{x} &= a_0 + a_1y + a_4y^2 + a_5z^2 + \varepsilon(a_2x^2 + a_3xz), \\ \dot{y} &= b_1z + b_3yz + \varepsilon b_2xy, \\ \dot{z} &= c_1y + c_4z^2 + \varepsilon c_2xz, \end{aligned} \tag{1}$$

where  $a_0 < 0$ ,  $b_1c_1 < 0$ ,  $a_2 < 0$ ,  $b_2 < a_2$ ,  $a_4 > 0$ ,  $c_2 < a_2$  and  $b_3 \notin \{c_4, 4c_4\}$ . Then, the vector field  $X_\varepsilon$  is  $\varphi$ -reversible, with  $\varphi(x, y, z) = (-x, y, -z)$ . Moreover,

- (a) For  $\varepsilon > 0$  the vector field  $X_\varepsilon$  satisfies:
  - (a<sub>1</sub>) the straight line  $\{(x, 0, 0) : x \in \mathbb{R}\}$  is an invariant line of the vector field without singular points and the flow goes in the decreasing direction of the  $x$ -axis;
  - (a<sub>2</sub>) the set  $\{(z_1, 0, 0)\}$  on the chart  $U_2$  on the Poincaré ball is an invariant straight line without singular points and the flow goes in the increasing direction of  $z_1$ -axis;
  - (a<sub>3</sub>) in the chart  $U_1$  the point  $p = (0, 0, 0)$  is a hyperbolic singular point, the plane  $\{(z_1, z_2, 0)\}$  is its local stable manifold and the line  $\{(0, 0, z_3)\}$  is its unstable manifold;
  - (a<sub>4</sub>)  $X_\varepsilon$  has the heteroclinic loop  $\mathcal{L}$  formed by the straight line  $\{(x, 0, 0)\}$  in  $\mathbb{R}^3$ , two singular points  $\{(0, 0, 0)\}$  of the Poincaré ball in the chart  $U_1$  and  $V_1$ , and the straight line  $\{(z_1, 0, 0)\}$  on the chart  $U_2$  of the Poincaré ball.
- (b) For  $\varepsilon = 0$  the vector field  $X_0$  has invariant cylinders surrounding the  $x$ -axis and the restriction of  $X_0$  to  $x = 0$  has a nonisochronous center at the origin.
- (c) For all  $n \in \mathbb{N}$  there exists  $\varepsilon_n > 0$  sufficiently small such that the vector field  $X_\varepsilon$  has at least  $n$  periodic orbits near the heteroclinic loop  $\mathcal{L}$ .

**Theorem 2.** Let  $X_\varepsilon$  be a quadratic vector field such that

- (a)  $X_\varepsilon$  is  $\varphi$ -reversible and has the heteroclinic loop  $\mathcal{L}$  on the Poincaré ball as in Theorem 1 for  $\varepsilon > 0$ ;
- (b)  $X_0$  has invariant cylinders surrounding the  $x$ -axis and the restriction of  $X_0$  to  $x = 0$  has a nonisochronous center at the origin.

Then the most general  $X_\varepsilon$  satisfying (a) and (b) is the vector field (1).

The next theorem illustrate that we may have a similar result even if the heteroclinic loop  $\mathcal{L}$  is not formed by an invariant straight line in the chart  $U_2$ .

**Theorem 3.** Let  $X_\varepsilon$  be the vector field associated to the quadratic polynomial differential system:

$$\begin{aligned} \dot{x} &= a_0 + a_1y + a_4y^2 + a_5z^2 + \varepsilon(a_2x^2 + a_3xz), \\ \dot{y} &= b_1z + b_3yz + \varepsilon b_2xy, \\ \dot{z} &= c_1y + c_3y^2 + c_4z^2 + \varepsilon c_2xz, \end{aligned} \tag{2}$$

with  $a_0 < 0$ ,  $b_1c_1 < 0$ ,  $a_2 < 0$ ,  $b_2 < a_2$ ,  $(a_3 - c_4)^2 - 4a_5(a_2 - c_2) < 0$ ,  $c_2 < a_2$ ,  $a_5(a_2 - b_2) - (a_3 - b_3)^2/4 > 0$  and  $a_4(a_5 + a_2 - b_2) > 0$  (or  $a_4 = 0$ ). Then, statement (c) of Theorem 1 holds for system (2).

The paper is organized as follows. In Section 2 we describe the Poincaré compactification for a polynomial differential system in  $\mathbb{R}^3$ . In Section 3 we deal with the perturbed vector field  $X_\varepsilon$  for  $\varepsilon > 0$  and in Section 4 we deal with the unperturbed vector field  $X_0$ . Finally, in Sections 5–7 we prove Theorems 2, 1 and 3, respectively.

The idea of using symmetries for finding periodic orbits is very old. Thus Poincaré used it for finding periodic orbits in the restricted three-body problem, see [12]. After this technique became popular for looking for periodic orbits in different problems of Celestial Mechanics see for instance the book of Meyer [9] and the references quoted there. But this tool has been used for studying periodic orbits in problems very far from Celestial Mechanics, see for instance the paper of Devaney [3], the books of Poénaru [11] and of Sevryuk [13], the survey of Lamb and Roberts [5], and again the references quoted there.

The idea that heteroclinic loops having some component at infinity can create large amplitude periodic orbits appeared already in [6,10] for different kinds of heteroclinic loops; or again in problems related with Celestial Mechanics as in [1]; or in bifurcations of periodic orbits from infinity in planar polynomial vector fields [4] or in planar piecewise linear vector fields [7], etc.

## 2. The Poincaré compactification in $\mathbb{R}^3$

In  $\mathbb{R}^3$  we consider the polynomial differential system

$$\begin{aligned} \dot{x} &= P^1(x, y, z), \\ \dot{y} &= P^2(x, y, z), \\ \dot{z} &= P^3(x, y, z), \end{aligned}$$

or equivalently its associated polynomial vector field  $X = (P^1, P^2, P^3)$ . The degree  $n$  of  $X$  is defined as  $n = \max\{\deg(P^i): i = 1, 2, 3\}$ .

Let  $\mathbb{S}^3 = \{y = (y_1, y_2, y_3, y_4) \in \mathbb{R}^4: \|y\| = 1\}$  be the unit sphere in  $\mathbb{R}^4$ , and

$$\mathbb{S}_+ = \{y \in \mathbb{S}^3: y_4 > 0\} \quad \text{and} \quad \mathbb{S}_- = \{y \in \mathbb{S}^3: y_4 < 0\}$$

be the northern and southern hemispheres, respectively. The tangent space to  $\mathbb{S}^3$  at the point  $y$  is denoted by  $T_y\mathbb{S}^3$ . Then, the tangent hyperplane

$$T_{(0,0,0,1)}\mathbb{S}^3 = \{(x_1, x_2, x_3, 1) \in \mathbb{R}^4: (x_1, x_2, x_3) \in \mathbb{R}^3\}$$

is identified with  $\mathbb{R}^3$ .

We consider the central projections

$$f_+ : \mathbb{R}^3 = T_{(0,0,0,1)}\mathbb{S}^3 \rightarrow \mathbb{S}_+ \quad \text{and} \quad f_- : \mathbb{R}^3 = T_{(0,0,0,1)}\mathbb{S}^3 \rightarrow \mathbb{S}_-,$$

defined by

$$f_+(x) = \frac{1}{\Delta x}(x_1, x_2, x_3, 1) \quad \text{and} \quad f_-(x) = -\frac{1}{\Delta x}(x_1, x_2, x_3, 1),$$

where  $\Delta x = (1 + \sum_{i=1}^3 x_i^2)^{1/2}$ . Through these central projections,  $\mathbb{R}^3$  can be identified with the northern and the southern hemispheres, respectively. The equator of  $\mathbb{S}^3$  is  $\mathbb{S}^2 = \{y \in \mathbb{S}^3: y_4 = 0\}$ . Clearly,  $\mathbb{S}^2$  can be identified with the *infinity* of  $\mathbb{R}^3$ .

The maps  $f_+$  and  $f_-$  define two copies of  $X$ , one  $Df_+ \circ X$  in the northern hemisphere and the other  $Df_- \circ X$  in the southern one. Denote by  $\bar{X}$  the vector field on  $\mathbb{S}^3 \setminus \mathbb{S}^2 = \mathbb{S}_+ \cup \mathbb{S}_-$  which restricted to  $\mathbb{S}_+$  coincides with  $Df_+ \circ X$  and restricted to  $\mathbb{S}_-$  coincides with  $Df_- \circ X$ .

In what follows we shall work with the orthogonal projection of the closed northern hemisphere to  $y_4 = 0$ . Note that this projection is a closed ball  $B$  of radius one, whose interior is diffeomorphic to  $\mathbb{R}^3$  and whose boundary  $\mathbb{S}^2$  corresponds to the infinity of  $\mathbb{R}^3$ . We shall extend analytically the polynomial vector field  $\bar{X}$  to the boundary, in such a way that the flow on the boundary is invariant. This new vector field on  $B$  will be called the *Poincaré compactification* of  $X$ , and  $B$  will be called the *Poincaré ball*. Poincaré introduced this compactification for polynomial vector fields in  $\mathbb{R}^2$ , and its extension to  $\mathbb{R}^m$  can be found in [2].

The expression for  $\bar{X}(y)$  on  $\mathbb{S}_+ \cup \mathbb{S}_-$  is

$$\bar{X}(y) = y_4 \begin{pmatrix} 1 - y_1^2 & -y_2 y_1 & -y_3 y_1 \\ -y_1 y_2 & 1 - y_2^2 & -y_3 y_2 \\ -y_1 y_3 & -y_2 y_3 & 1 - y_3^2 \\ -y_1 y_4 & -y_2 y_4 & -y_3 y_4 \end{pmatrix} \begin{pmatrix} P^1 \\ P^2 \\ P^3 \end{pmatrix},$$

where  $P^i = P^i(y_1/|y_4|, y_2/|y_4|, y_3/|y_4|)$ . Written in this way  $\bar{X}(y)$  is a vector field in  $\mathbb{R}^4$  tangent to the sphere  $\mathbb{S}^3$ .

Now we can extend analytically the vector field  $\bar{X}(y)$  to the whole sphere  $\mathbb{S}^3$  by

$$p(X)(y) = y_4^{n-1} \bar{X}(y);$$

this extended vector field  $p(X)$  is called the *Poincaré compactification* of  $X$ .

As  $\mathbb{S}^3$  is a differentiable manifold, to compute the expression for  $p(X)$  we can consider the eight local charts  $(U_i, F_i)$ ,  $(V_i, G_i)$  where  $U_i = \{y \in \mathbb{S}^3: y_i > 0\}$ , and  $V_i = \{y \in \mathbb{S}^3: y_i < 0\}$  for  $i = 1, 2, 3, 4$ ; the diffeomorphisms  $F_i : U_i \rightarrow \mathbb{R}^3$  and  $G_i : V_i \rightarrow \mathbb{R}^3$  for  $i = 1, 2, 3, 4$ , are the inverses of the central projections from the origin to the tangent planes at the points  $(\pm 1, 0, 0, 0)$ ,  $(0, \pm 1, 0, 0)$ ,  $(0, 0, \pm 1, 0)$  and  $(0, 0, 0, \pm 1)$ , respectively. We now do the computations on  $U_1$ .

Suppose that the origin  $(0, 0, 0, 0)$ , the point  $(y_1, y_2, y_3, y_4) \in \mathbb{S}^3$  and the point  $(1, z_1, z_2, z_3)$  in the tangent plane to  $\mathbb{S}^3$  at  $(1, 0, 0, 0)$  are collinear, then we have

$$\frac{1}{y_1} = \frac{z_1}{y_2} = \frac{z_2}{y_3} = \frac{z_3}{y_4},$$

and consequently

$$F_1(y) = \left( \frac{y_2}{y_1}, \frac{y_3}{y_1}, \frac{y_4}{y_1} \right) = (z_1, z_2, z_3)$$

defines the coordinates on  $U_1$ .

As

$$DF_1(y) = \begin{pmatrix} -\frac{y_2}{y_1^2} & \frac{1}{y_1} & 0 & 0 \\ -\frac{y_3}{y_1^2} & 0 & \frac{1}{y_1} & 0 \\ -\frac{y_4}{y_1^2} & 0 & 0 & \frac{1}{y_1} \end{pmatrix}$$

and  $y_4^{n-1} = (\frac{z_3}{\Delta z})^{n-1}$ , the analytical field  $p(X)$  becomes

$$\frac{z_3^n}{(\Delta z)^{n-1}} (-z_1 P^1 + P^2, -z_2 P^1 + P^3, -z_3 P^1), \tag{3}$$

where  $P^i = P^i(1/z_3, z_1/z_3, z_2/z_3)$ .

In a similar way we can deduce the expressions of  $p(X)$  in  $U_2$  and  $U_3$ . These are

$$\frac{z_3^n}{(\Delta z)^{n-1}} (-z_1 P^2 + P^1, -z_2 P^2 + P^3, -z_3 P^2), \tag{4}$$

where  $P^i = P^i(z_1/z_3, 1/z_3, z_2/z_3)$  in  $U_2$ , and

$$\frac{z_3^n}{(\Delta z)^{n-1}} (-z_1 P^3 + P^1, -z_2 P^3 + P^2, -z_3 P^3), \tag{5}$$

where  $P^i = P^i(z_1/z_3, z_2/z_3, 1/z_3)$  in  $U_3$ .

The expression for  $p(X)$  in  $U_4$  is  $z_3^{n+1}(P^1, P^2, P^3)$  where the component  $P^i = P^i(z_1, z_2, z_3)$ . The expression for  $p(X)$  in the local chart  $V_i$  is the same as in  $U_i$  multiplied by  $(-1)^{n-1}$ .

When we shall work with the expression of the compactified vector field  $p(X)$  in the local charts we shall omit the factor  $1/(\Delta z)^{n-1}$ . We can do that through a rescaling of the time.

We remark that all the points on the sphere at infinity in the coordinates of any local chart have  $z_3 = 0$ .

### 3. The perturbed vector field $X_\epsilon$

We write the quadratic polynomial differential system in  $\mathbb{R}^3$  in the form

$$\begin{aligned} \dot{x} &= \sum_{0 \leq i+j+k \leq 2} a_{ijk} x^i y^j z^k, \\ \dot{y} &= \sum_{0 \leq i+j+k \leq 2} b_{ijk} x^i y^j z^k, \\ \dot{z} &= \sum_{0 \leq i+j+k \leq 2} c_{ijk} x^i y^j z^k. \end{aligned} \tag{6}$$

We consider that its associated vector field  $X_\varepsilon$  is  $\varphi$ -reversible with respect to the linear involution

$$\varphi(x, y, z) = (-x, y, -z),$$

which has the fix point set  $S = \{(0, y, 0)\}$ . So, the system becomes

$$\begin{aligned}\dot{x} &= a_{000} + a_{010}y + a_{200}x^2 + a_{101}xz + a_{020}y^2 + a_{002}z^2, \\ \dot{y} &= b_{100}x + b_{001}z + b_{110}xy + b_{011}yz, \\ \dot{z} &= c_{000} + c_{010}y + c_{200}x^2 + c_{101}xz + c_{020}y^2 + c_{002}z^2.\end{aligned}\quad (7)$$

We assume that the straight line  $\{(x, 0, 0): x \in \mathbb{R}\}$  is invariant for system (7), and that the flow along this line goes in the decreasing direction of the  $x$ -axis. Therefore, we get that system (7), after a renaming of the parameters, has the form

$$\begin{aligned}\dot{x} &= a_0 + a_1y + a_2x^2 + a_3xz + a_4y^2 + a_5z^2, \\ \dot{y} &= b_1z + b_2xy + b_3yz, \\ \dot{z} &= c_1y + c_2xz + c_3y^2 + c_4z^2.\end{aligned}\quad (8)$$

Here, we have that  $a_0 < 0$  and  $a_2 < 0$ .

The next step is to analyze system (8) at infinity.

**Lemma 4.** *If the straight line  $\{(z_1, 0, 0)\}$  on the chart  $U_2$  is formed by a unique orbit of system (8) and the flow goes in the increasing direction of the  $z_1$ -axis, then  $c_3 = 0$ ,  $b_2 - a_2 < 0$  and  $a_4 > 0$ .*

**Proof.** According to (4), system (8) in the chart  $U_2$  has the expression

$$\begin{aligned}\dot{z}_1 &= a_4 + a_1z_3 + (a_2 - b_2)z_1^2 + (a_3 - b_3)z_1z_2 + a_5z_2^2 + a_0z_3^2 - b_1z_1z_2z_3, \\ \dot{z}_2 &= c_3 + c_1z_3 + (c_4 - b_3)z_2^2 + (c_2 - b_2)z_1z_2 - b_1z_2^2z_3, \\ \dot{z}_3 &= -b_2z_1z_3 - b_3z_2z_3 - b_1z_2z_3^2.\end{aligned}$$

In order to have the invariant straight line  $\{(z_1, 0, 0)\}$  with the flow going in the increasing direction of the  $z_1$ -axis we need the conditions  $c_3 = 0$ ,  $b_2 - a_2 < 0$  and  $a_4 > 0$ .  $\square$

Now, we analyze the vector field on the chart  $U_1$ .

**Lemma 5.** *If  $b_2 - a_2 < 0$ ,  $c_2 - a_2 < 0$  and  $a_2 < 0$ , then on the chart  $U_1$  we have that  $(0, 0, 0)$  is a hyperbolic singular point such that  $\{(z_1, z_2, 0)\}$  and  $\{(0, 0, z_3)\}$  are the local stable and the unstable manifold of  $(0, 0, 0)$ , respectively.*

**Proof.** According to (3), on the chart  $U_1$ , the system has the expression

$$\begin{aligned}\dot{z}_1 &= (b_2 - a_2)z_1 + (b_3 - a_3)z_1z_2 + b_1z_2z_3 - a_4z_1^3 - a_5z_1z_2^2 - a_1z_1^2z_3 - a_0z_1z_3^2, \\ \dot{z}_2 &= (c_2 - a_2)z_2 + c_3z_1^2 + (c_4 - a_3)z_2^2 + c_1z_1z_3 - a_4z_1^2z_2 - a_5z_2^3 - a_1z_1z_2z_3 - a_0z_2z_3^2, \\ \dot{z}_3 &= -a_2z_3 - a_3z_2z_3 - a_4z_1^2z_3 - a_5z_2^2z_3 - a_1z_1z_3^2 - a_0z_3^3.\end{aligned}$$

So, the conditions to have  $(0, 0, 0)$  as an attractor node on  $\mathbb{S}^2$  and a repeller on  $\mathbb{R}^3$  are  $b_2 - a_2 < 0$ ,  $c_2 - a_2 < 0$  and  $a_2 < 0$ .  $\square$

#### 4. The unperturbed vector field $X_0$

Until now we have that a system satisfying the hypotheses of Lemmas 4 and 5 can be written in the form

$$\begin{aligned} \dot{x} &= a_0 + a_1y + a_2x^2 + a_3xz + a_4y^2 + a_5z^2, \\ \dot{y} &= b_1z + b_2xy + b_3yz, \\ \dot{z} &= c_1y + c_2xz + c_4z^2, \end{aligned} \tag{9}$$

with  $a_0 < 0$ ,  $a_2 < 0$ ,  $a_4 > 0$ ,  $b_2 - a_2 < 0$  and  $c_2 - a_2 < 0$ .

We choose the cross section  $\Sigma$  to the orbit  $\{(x, 0, 0)\}$  as the plane  $x = 0$  in a neighborhood of  $(0, 0, 0)$ . We want that for  $\varepsilon = 0$  there exist invariant cylinders surrounding the  $x$ -axis and that the restriction of system (9) to  $\Sigma$  has a nonisochronous center at the origin. In order to obtain this we need to take  $b_2 = c_2 = 0$ .

The restriction of system (9) on  $\Sigma$  becomes

$$\begin{aligned} \dot{y} &= b_1z + b_3yz, \\ \dot{z} &= c_1y + c_4z^2. \end{aligned} \tag{10}$$

The next result proved by Loud [8] is useful for classifying the nonisochronous centers of quadratic differential systems in the plane.

**Theorem 6.** *The origin is an isochronous center of the quadratic system*

$$\begin{aligned} \dot{x} &= -y + a_{20}x^2 + a_{11}xy + a_{02}y^2, \\ \dot{y} &= x + b_{20}x^2 + b_{11}xy + b_{02}y^2, \end{aligned}$$

if and only if the system can be brought to one of the following systems:

- (a)  $\dot{x} = -y + x^2 - y^2, \quad \dot{y} = x(1 + 2y);$
- (b)  $\dot{x} = -y + x^2, \quad \dot{y} = x(1 + y);$
- (c)  $\dot{x} = -y - \frac{4}{3}x^2, \quad \dot{y} = x\left(1 - \frac{16}{3}y\right);$
- (d)  $\dot{x} = -y + \frac{16}{3}x^2 - \frac{4}{3}y^2, \quad \dot{y} = x\left(1 + \frac{8}{3}y\right)$

through a linear change of coordinates and a rescaling of time.

**Lemma 7.** *If  $a_2 = 0$ ,  $a_3 = 0$ ,  $b_2 = 0$ ,  $c_2 = 0$ ,  $b_1c_1 < 0$ , and  $b_3 \notin \{c_4, 4c_4\}$ , then the origin is a nonisochronous center of system (10) and for system (9) we have invariant cylinders surrounding the  $x$ -axis.*

**Proof.** The vector field (10) is reversible with respect to the involution  $(y, z) \mapsto (y, -z)$ .

If  $c_1b_1 < 0$ , then the origin is a center or a focus of (10). Using the fact that (10) is reversible we conclude that it is a center.

According to Theorem 6 the center (10) is isochronous if and only if there is a linear change of variables and a time rescaling such that the system (10) becomes one of the four systems given in Theorem 6.



First of all we apply a rescaling of time to system (10). By multiplying the system by the factor  $\sqrt{\frac{1}{-b_1c_1}}$ , we have

$$\begin{aligned} \dot{y} &= -\sqrt{-\frac{b_1}{c_1}}z + \frac{b_3}{\sqrt{-b_1c_1}}yz, \\ \dot{z} &= \sqrt{-\frac{c_1}{b_1}}y + \frac{c_4}{\sqrt{-b_1c_1}}z^2. \end{aligned} \tag{11}$$

We call  $\beta = \sqrt{-\frac{b_1}{c_1}}$ . We can prove that all linear transform of variables that change the linear part of (11) in

$$\begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}$$

are of the form

$$B = \begin{pmatrix} \frac{d}{\beta} & -b\beta \\ b & d \end{pmatrix},$$

with  $d^2 + b^2\beta^2 \neq 0$ .

The nonlinear part of (11) is given by  $f(y, z) = (\frac{b_3}{\sqrt{-b_1c_1}}yz, \frac{c_4}{\sqrt{-b_1c_1}}z^2)$ . We impose that  $Bf(B^{-1}(y, z))$  is equal to one of the four for nonlinear parts of the systems given in Theorem 6. Then, we obtain that the vector field can be brought to vector field (b) or (c), if  $b_3 = c_4$  or  $b_3 = 4c_4$ , respectively.

In short we get that if  $b_3 \notin \{c_4, 4c_4\}$ , then the origin is a nonisochronous center for (11).

Using the fact that the origin is a center for (11), for  $y$  and  $z$  small enough the periodic orbit passing through  $(y, z)$  can be given implicitly by  $h(y, z) = 0$ . Observe that the cylinder  $\{(x, y, z) \in \mathbb{R}^3 : h(y, z) = 0\}$  is invariant by the flow of (10) and surrounds the  $x$ -axis.  $\square$

### 5. Proof of Theorem 2

According to Lemmas 4 and 5 the vector field  $X_\varepsilon$  for  $\varepsilon > 0$  must be

$$\begin{aligned} \dot{x} &= a_0 + a_1y + a_4y^2 + a_5z^2 + a_2x^2 + a_3xz, \\ \dot{y} &= b_1z + b_3yz + b_2xy, \\ \dot{z} &= c_1y + c_4z^2 + c_2xz, \end{aligned} \tag{12}$$

where  $a_0 < 0, a_2 < 0, b_2 - a_2 < 0, a_4 > 0$  and  $c_2 - a_2 < 0$ .

According to Lemma 7, for  $\varepsilon = 0$  the vector field must satisfy

$$\begin{aligned} \dot{x} &= a_0 + a_1y + a_4y^2 + a_5z^2, \\ \dot{y} &= b_1z + b_3yz, \\ \dot{z} &= c_1y + c_4z^2, \end{aligned} \tag{13}$$

where  $b_1c_1 < 0$  and  $b_3 \notin \{c_4, 4c_4\}$ . So, we put in front of  $a_2, a_3, b_2$  and  $c_2$  the parameter  $\varepsilon$  and we obtain the vector field of the statement of Theorem 2.  $\square$

6. Proof of Theorem 1

The proof of statements (a) and (b) of Theorem 1 follows directly from Theorem 2. So, the only thing that remains to prove in Theorem 1 is that for all  $n \in \mathbb{N}$  there is  $\varepsilon_n > 0$  such that for  $\varepsilon \in (0, \varepsilon_n)$  the system  $X_\varepsilon$  has at least  $n$  periodic orbits near the heteroclinic loop  $\mathcal{L}$ .

In order to prove this, we construct a Poincaré map in the following way. Consider a cross section  $\bar{\Sigma}$  at the point  $r = (0, 0, 0)$  in the chart  $U_2$  of the Poincaré ball that contains the small interval  $G$ . In a neighborhood  $U$  of  $p$  in the chart  $U_1$  we take the cross sections  $\Sigma_1$  and  $\Sigma_2$  of the orbits  $\gamma_1$  and  $\gamma_2$  such that  $q_1 \in \gamma_1 \cap \Sigma_1$  and  $q_2 \in \gamma_2 \cap \Sigma_2$ , respectively. Finally, we consider the cross section  $\Sigma$  of the orbit  $\gamma_1$  which contains a neighborhood of  $s$  near the point  $s$  of  $\mathcal{L}$ , see Fig. 2. We denote by  $\pi$  the Poincaré map going from  $\bar{\Sigma}$  to  $\Sigma$ . The loop  $\mathcal{L}$  and the local phase portrait of the hyperbolic singular point  $p$  of  $U_1$  guarantee the existence of the Poincaré map  $\pi$ .

We may consider diffeomorphisms induced by the flow of the vector field  $X_\varepsilon: \bar{\Sigma} \rightarrow \Sigma_2$  and  $\pi_0: \Sigma_1 \rightarrow \Sigma$ . Since the orbits going from  $\bar{\Sigma}$  to  $\Sigma_2$  and  $\Sigma_1$  to  $\Sigma$  spend only a bounded time,  $\pi_1$  and  $\pi_2$  are well defined.

In the neighborhood  $U$  of the hyperbolic singular point  $p = (0, 0, 0)$ , we get that the plane  $\{(z_1, z_2, 0)\}$  is the local stable manifold of  $p$  and the line  $\{(0, 0, z_3)\}$  is the unstable manifold of  $p$ . So, we may consider a diffeomorphism  $\pi_1: \Sigma_2 \rightarrow \Sigma_1$  induced by the flow of  $X_\varepsilon$  in the neighborhood  $U$ . Thus, we consider the Poincaré map given by  $\pi = \pi_0 \circ \pi_1 \circ \pi_2: \bar{\Sigma} \rightarrow \Sigma$ .

The image of the segment  $G$  with endpoint  $r$  and contained in  $\bar{\Sigma}$  by  $\pi_2$  is the arc  $\pi_2(G)$  in  $\Sigma_2$  having the point  $q_2$  as an endpoint. And consequently,  $(\pi_1 \circ \pi_2)(G)$  is an arc in  $\Sigma_1$  having  $q_1$  as an endpoint.

We claim that for  $\varepsilon > 0$  sufficiently small the arc  $(\pi_1 \circ \pi_2)(G)$  spirals with infinitely many turns around the point  $q_1$ . This is due to two facts. First, the time going from the point  $p$  to the point  $q_1$  is infinite, and for  $\varepsilon = 0$  the flow of system (1) restricted to  $\Sigma$  has a nonisochronous center.

Since  $\pi_0$  is a diffeomorphism, we have that the arc  $\pi(G) = (\pi_0 \circ \pi_1 \circ \pi_2)(G)$  is an arc in  $\Sigma$  which spirals to the point  $s$  giving infinitely many turns around  $s$ . The orbits through the intersection points of  $\pi(G)$  with the  $y$ -axis are periodic because, by construction, they have two points on the  $y$ -axis and  $X_\varepsilon$  is  $\varphi$ -reversible. This completes the proof of Theorem 1.

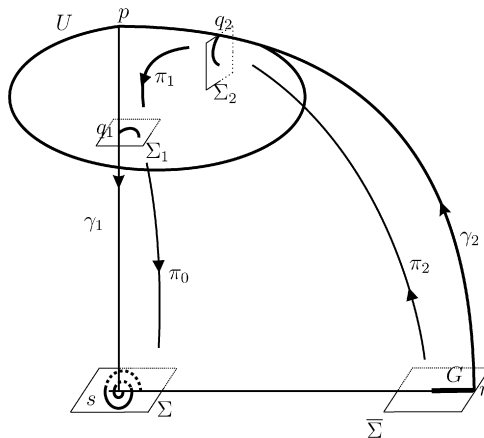


Fig. 2. The half-heteroclinic loop  $\mathcal{L}$  on  $x \geq 0$ .

### 7. Proof of Theorem 3

We recall that system (8), with  $a_0 < 0$  and  $a_2 < 0$ , is  $\varphi$ -reversible and it has  $\{(x, 0, 0)\}$  as an invariant straight line with the flow going in the decreasing direction of  $x$ -axis.

We assume that  $c_3 \neq 0$ , then by Lemma 4 we have that the straight line  $\{(z_1, 0, 0)\}$  on the chart  $U_2$  is no more an invariant set. We will find conditions for system (8) to have only the singular points  $(0, 0, 0)$  on charts  $U_1$  and  $V_1$  and no more singular points on the Poincaré sphere  $\mathbb{S}^2$ . We will also find conditions for these points to be an attractor on  $U_1$  and repellor on  $V_1$ , restrict to  $\mathbb{S}^2$ . This fact guarantees the existence of the orbit that plays the role of  $\gamma_2$  on  $\mathbb{S}^2$ , because there is exactly one orbit departing from the repellor and arriving to the attractor passing through the point  $r$ .

In the sequel we establish these conditions. First, we choose the cross section  $\Sigma$  of the orbit  $\{(x, 0, 0)\}$  as the plane  $x = 0$  in a neighborhood of  $(0, 0, 0)$ . We want that for  $\varepsilon = 0$  there exist invariant cylinders surrounding the  $x$ -axis and that the restriction of system (8) on  $\Sigma$  has a nonisochronous center at the origin. In order to obtain this we need to take  $b_2 = c_2 = 0$ .

The restriction of system (8) on  $\Sigma$  becomes

$$\begin{aligned} \dot{y} &= b_1z + b_3yz, \\ \dot{z} &= c_1y + c_3y^2 + c_4z^2. \end{aligned} \tag{14}$$

In similar way of the proof of Lemma 7 we obtain that if  $b_1c_1 < 0$ , then the origin is a nonisochronous center of system (14) and for system (8) we have invariant cylinders surrounding the  $x$ -axis.

System (8) satisfying the condition  $b_1c_1 < 0$ , according to (3), on the chart  $U_1$  has the expression

$$\begin{aligned} \dot{z}_1 &= (b_2 - a_2)z_1 + (b_3 - a_3)z_1z_2 + b_1z_2z_3 - a_4z_1^3 - a_5z_1z_2^2 - a_1z_1^2z_3 - a_0z_1z_3^2, \\ \dot{z}_2 &= (c_2 - a_2)z_2 + c_3z_1^2 + (c_4 - a_3)z_2^2 + c_1z_1z_3 - a_4z_1^2z_2 - a_5z_2^3 - a_1z_1z_2z_3 - a_0z_2z_3^2, \\ \dot{z}_3 &= -a_2z_3 - a_3z_2z_3 - a_4z_1^2z_3 - a_5z_2^2z_3 - a_1z_1z_3^2 - a_0z_3^3. \end{aligned}$$

So, the conditions to have  $(0, 0, 0)$  as an attractor node on  $\mathbb{S}^2$  and repellor on  $\mathbb{R}^3$  are  $b_2 - a_2 < 0$ ,  $c_2 - a_2 < 0$  and  $a_2 < 0$ .

**Lemma 8.** *If  $a_5(a_2 - b_2) - (a_3 - b_3)^2/4 > 0$  and  $a_4 = 0$  or  $a_4(a_5 + a_2 - b_2) > 0$ , then there are no singular points on  $U_2$  for  $z_3 = 0$ .*

**Proof.** The system (8) on the chart  $U_2$  for  $z_3 = 0$  is

$$\begin{aligned} \dot{z}_1 &= a_4 + (a_2 - b_2)z_1^2 + (a_3 - b_3)z_1z_2 + a_5z_2^2, \\ \dot{z}_2 &= c_3 + (c_4 - b_3)z_2^2 + (c_2 - b_2)z_1z_2, \\ \dot{z}_3 &= 0. \end{aligned}$$

We recall that for a conic  $f(z_1, z_2) = \sum_{i+j \geq 0} a_{ij}z_1^i z_2^j = 0$ , we have that

$$D_3 = \det \begin{pmatrix} a_{20} & \frac{a_{11}}{2} & \frac{a_{10}}{2} \\ \frac{a_{11}}{2} & a_{02} & \frac{a_{01}}{2} \\ \frac{a_{10}}{2} & \frac{a_{01}}{2} & a_{00} \end{pmatrix}, \quad D_2 = \det \begin{pmatrix} a_{20} & \frac{a_{11}}{2} \\ \frac{a_{11}}{2} & a_{02} \end{pmatrix}.$$

It is well known that if  $D_2 > 0$  and  $D_3 = 0$ , then  $f(z_1, z_2) = 0$  is a pair of imaginary parallel straight lines, and if  $D_2 > 0$  and  $D_3(a_{20} + a_{02}) > 0$ , then  $f(z_1, z_2) = 0$  is an imaginary ellipse.

The equation  $\dot{z}_1 = 0$  is a conic in the plane  $z_1 z_2$  and we get that  $D_2 = a_5(a_2 - b_2) - (a_3 - b_3)^2/4 > 0$  and  $D_3 = a_4 D_2$ . In short  $\dot{z}_1 = 0$  has no real solutions.  $\square$

**Lemma 9.** *If  $(a_3 - c_4)^2 - 4a_5(a_2 - c_2) < 0$ , then there are no singular points on  $U_3$  for  $z_2 = z_3 = 0$ .*

**Proof.** System (8) on the chart  $U_2$  for  $z_3 = 0$  is

$$\begin{aligned}\dot{z}_1 &= a_5 + (a_3 - c_4)z_1 + (a_2 - c_2)z_1^2, \\ \dot{z}_2 &= 0, \\ \dot{z}_3 &= 0.\end{aligned}$$

So, by hypothesis the discriminant of the  $\dot{z}_1 = 0$  is negative and we conclude the proof.  $\square$

With Lemmas 8 and 9 we conclude that on the Poincaré sphere  $\mathbb{S}^2$  there are only the two singular points at  $(0, 0, 0)$  of  $U_1$  and  $V_1$ . So, the proof of Theorem 3 is completed.

We observe here that the conditions presented in Lemmas 8 and 9 are sufficient, but not necessary, conditions.

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