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Entanglement in the Momentum Space of Quantum Field Theories

Matheus Henrique Martins Costa

Orientador

Gastão Inácio Krein

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“The most important words a man can say are “I will do better” (...) the most important step a person can take is always the next one.”

Dalinar Kholin, *Oathbringer*

Resumo

O emaranhamento entre graus de liberdade de sistemas quânticos é uma propriedade cujo estudo, tanto teórico quanto experimental, tem expandido enormemente nossa compreensão de diversos ramos da física nos últimos anos: do desenvolvimento de novos protocolos de computação à exploração dos elos entre teorias de campo e gravidade, o uso do ponto de vista da informação quântica é sempre revelador. Com isso em mente, o objetivo desta Dissertação é investigar o emaranhamento entre modos de momento acima e abaixo de uma escala arbitrária em teorias quânticas de campos e mostrar como esse conceito é naturalmente ligado à ideia de Wilson de integrar variáveis no processo de renormalização. Para esse fim, uma nova técnica para calcular entropias de Rényi no espaço dos momentos é desenvolvida e é aplicada, primeiramente em um sistema simples de osciladores acoplados e após em teorias de campos escalares reais autointeragentes. Os resultados obtidos aqui indicam que, pelo menos no regime perturbativo explorado, o emaranhamento entre escalas aumenta à medida que a massa das excitações da teoria é levada a zero. Esta Dissertação abre espaço para trabalhos futuros nos quais o método desenvolvido é aplicado para investigar esse comportamento mais detalhadamente e em teorias de campos mais complexas.

Palavras Chaves: Emaranhamento; Entropia; Renormalização; Espaço dos Momentos; Teoria de Campos.

Áreas do conhecimento: Física; Informação Quântica; Teoria Quântica de Campos.

Abstract

Entanglement between degrees of freedom in quantum systems is a property whose study, both theoretical and experimental, has greatly expanded our understanding of several branches of physics in the last few years: from the development of new computing protocols to the exploration of the links between field theories and gravity, the use of a quantum information point of view is always illuminating. With this in mind, the objective in this Master's Thesis is to investigate entanglement between momentum modes above and below an arbitrary scale in quantum field theories and show how this concept is naturally connected to Wilson's idea of integrating out variables during the renormalization procedure. To this end, a new technique to calculate Rényi entropies in momentum-space is developed and is applied, first in a simple system of coupled harmonic oscillators and then in theories of self-interacting real scalar fields. The results obtained here indicate that, at least in the perturbative regime explored, the entanglement between scales increases as the mass of the theory's excitations is taken to zero. This Thesis opens the way for future works where the method developed is applied to investigate this behavior in more detail and in more complex theories.

Keywords: Entanglement; Entropy; Renormalization; Momentum Space; Field Theory.

Areas of Knowledge: Physics; Quantum Information; Quantum Field Theory.

Contents

1	Introduction	1
2	Foundations of Quantum Information	3
2.1	States in Quantum Mechanics	4
2.2	Composite Systems and Entanglement	9
2.3	Entanglement Measures and their Properties	16
2.4	Real-space Entanglement in Free QFTs	22
2.4.1	Replica trick in non-interacting Quantum Field Theories . .	26
2.4.2	The ultraviolet divergence	30
2.4.3	Heuristic origin of the Area Law	32
2.5	Comments on the Algebraic Formalism	33
3	Renormalization Group and Entanglement in Momentum Space	36
3.1	Wilsonian Renormalization	37
3.1.1	Steps of the RG	39
3.1.2	The Renormalization Group flow	44
3.2	Entanglement in Momentum Space	47
3.2.1	Degrees of freedom at different momentum scales	47
3.2.2	Formulation through finite temperature field theory	52
3.2.3	Correlations in momentum-space for interacting theories . .	53
3.3	A Quantum Information view of the RG	56
3.3.1	Algebra of low-momentum observables	57
3.3.2	The rescaling transformation	59
3.3.3	Entanglement between scales at a fixed point	63
4	The Momentum-Space Entropies	66
4.1	Development of the method	67

4.2	Coupled Harmonic Oscillators	71
4.3	Perturbation theory and the replica trick	76
4.4	Perturbative calculation in ϕ^3 theory	77
4.5	Results for ϕ^4 theory	83
5	Conclusions	86
A	Full entropy calculations	88
A.1	Entropy of coupled harmonic oscillators	88
A.2	Momentum-Space entropy in ϕ^3 theory	92
A.3	Momentum-Space entropy in ϕ^4 theory	96

Chapter 1

Introduction

Entanglement is one of the intrinsically quantum properties which cannot be replicated by (local) classical systems, having first appeared meaningfully in the famous Einstein-Podolsky-Rosen paper of 1935. Despite being often described with exaggerated statements, it is best characterized as a type of statistical correlation special to quantum mechanics and its potential, both experimental and theoretical, comes from the fact that it plays a crucial role in establishing such correlations between subsets of a global system.

In this Master's Thesis, entanglement will be studied in the specific context of Quantum Field Theories (QFTs) partitioned between subsystems of long wavelength and short wavelength degrees of freedom. This particular choice of partition is motivated by the fact that it naturally appears in another powerful concept: the Renormalization Group, a set of ideas and procedures which lead to great insights about field theories and phase transitions. By establishing and analysing the links between entanglement and renormalization throughout this work, the hope is that new tools to understand phases of QFTs will be developed, so they can then be applied to phenomena we still have much to learn about, such as quark confinement in Quantum Chromodynamics.

The structure of the Thesis is, then, as follows: in Chapter 2 an introduction to the basic concepts and techniques of quantum information theory is given, with a focus on the most relevant points for the remainder of the work. The most general type of state in quantum mechanics is presented, followed by a discussion on composite systems and the definition of entanglement and a derivation of the main properties of the entanglement measures used. The Chapter then closes

with an application of the techniques developed up to this point to Quantum Field Theory in the context of real-space entanglement and a brief mention of some ideas from the algebraic perspective, which will serve as inspiration for other developments later.

Chapter 3 begins with a description of the Wilsonian Renormalization Group steps and a discussion of the physical meaning of each. The presentation is directed so that the Functional (or Exact) Renormalization Group is also quickly touched on and it eventually leads to the appearance of the notion of entanglement between momentum scales in a field theory. At this point different ways of representing the state of the low-momentum subsystem are provided and an argument is given to show that this type of entanglement is non-trivial in interacting QFTs. Finally, we come full circle by formulating Wilsonian renormalization in the language of quantum information and deriving a peculiar property of the Renormalization Group fixed points, both being original results of this Thesis.

With the concepts coming from quantum information and renormalization well established, Chapter 4 is dedicated to constructing a method suitable for the calculation of entanglement measures in momentum-space (also an original contribution from this work), which is applied to a system of coupled oscillators as a test and then to the scalar field theories with cubic and quartic coupling in the perturbative regime, where some key differences from the results obtained here to other in the literature are pointed.

Finally, Chapter 5 summarizes the main lessons learned from this work and sketches lines of research moving forward.

Chapter 2

Foundations of Quantum Information

The use of tools and techniques from quantum information is proving to be a powerful way of studying the properties of quantum systems in general and field theories in particular, and the reason for this is two-fold: Mathematically, the concept of entanglement deals with fundamental structures in the Hilbert spaces of our systems, the Tensor Product Structures (TPS), and understanding them better allows us to derive new results about quantum states (which are vectors or operators in the Hilbert space) of interest. Also, from the physical point of view, a Tensor Product Structure implies the ways we can partition a system of interest into subsystems and look at the statistical correlations between them, furthermore, it is associated with the idea of restricting the observables/measurements to a subsystem while "throwing away the rest", a concept physicists are already familiar with from Thermodynamics and which also leads to new insights in this context.

With this in mind, this Chapter is dedicated to introducing the basic concepts of quantum information theory: first by discussing the nature of pure and mixed states in quantum mechanics, then introducing entanglement and entanglement measures (also proving key results about these measures which are used later in the Thesis) and concluding with the application of these ideas to the study of real-space entanglement in a Quantum Field Theory (QFT) as an example and with a brief mention of the algebraic formalism of quantum mechanics/quantum field theory, which will serve as source of many powerful ideas for the rest of the Thesis.

2.1 States in Quantum Mechanics

Given a quantum system and the associated Hilbert space $\mathcal{H}$, the most basic way to represent states of the system is via a normalized vector $|\psi\rangle \in \mathcal{H}$ (as always, any other vector of the form $e^{i\theta} |\psi\rangle$ is physically equivalent - these states are mathematically rays in the vector space). From now on, whenever a concrete example is needed for clarity, I'll take the simplest quantum system possible: the qubit, also called two-level system, with $\mathcal{H} = \mathbb{C}^2$ and:

$$|\psi\rangle = a |0\rangle + b |1\rangle \quad (2.1)$$

Where $a, b \in \mathbb{C}$; $|a|^2 + |b|^2 = 1$ and $|0\rangle, |1\rangle$ are the eigenstates of σ_z .

This description is capable of displaying the intrinsically quantum type of randomness, that is, the fact that no quantum state can ever display a definite value for non-compatible (non-commuting) observables at the same time and measurements of a subset of these observables will always display fluctuations (this is, of course, the uncertainty principle). Going back to the example of the qubit, we have an illustration in the fact that a vector state $|\psi\rangle$ can never be an eigenstate of all Pauli matrices $\sigma_x, \sigma_y, \sigma_z$ at once.

However, even though state vectors can describe the quantum fluctuations, they do not cover all possible states of a system. Consider, for example, a box with an equal amount of atoms with spin states $|1\rangle$ and $|0\rangle$ (say, collected from a Stern-Gerlach experiment). So, if we take one of the atoms at random and measure its spin in the direction of $\vec{z}$, the expectation value of the measurement must be an average of the expectation in each of the possible states we can draw (that is, $|0\rangle$ or $|1\rangle$), meaning that it is given by:

$$\langle \sigma_z \rangle = \frac{1}{2} \langle 0 | \sigma_z | 0 \rangle + \frac{1}{2} \langle 1 | \sigma_z | 1 \rangle = 0 \quad (2.2)$$

See that even though each individual atom has some definite spin projection (+1 or -1), they cancel on average. This is an impressive result, because as $|0\rangle, |1\rangle$

are the eigenstates of σ_z , we also find $\langle \sigma_x \rangle = \langle \sigma_y \rangle = 0$ and so we have a physical state in which the (average) spin projection in any axis is zero.

Now, I will show that this configuration cannot be described by any non-zero state vector $|\psi\rangle \in \mathbb{C}^2$, which will naturally lead to the more general description of quantum states:

Take $|\psi\rangle = a|0\rangle + b|1\rangle$, imposing $\langle \psi | \sigma_z | \psi \rangle = 0$ implies $|a|^2 - |b|^2 = 0 \rightarrow |a| = |b|$; now, the condition $\langle \psi | \sigma_x | \psi \rangle = 0$ leads to $ab^* + ba^* = 0 \rightarrow \Re(ab^*) = 0$ and $\langle \psi | \sigma_y | \psi \rangle = 0$ to $ab^* - ba^* = 0 \rightarrow \Im(ab^*) = 0$.

Combining all constraints it is easy to see that $ab^* = 0$, so either $a = 0$ or $b = 0$, but as $|a| = |b|$, the requirements imply $|\psi\rangle = 0$. The fact that there's no non-zero vector which gives the correct expectation values for this case proves that the physical configuration of the box (which must be encompassed by any reasonable definition of a "state", as it's analogous to what is seen in some experimental setups) cannot be described by state vectors alone.

The lesson from the study of this example is the following: it is possible to construct states of quantum systems which are not describable by vectors in a Hilbert space as long as we introduce **classical** probabilities in the configuration, as those were the ones responsible for ensuring that $\langle \sigma_z \rangle = 0$ even though each individual atom in the box had spin projection $+1$ or -1 . The word "classical" here refers to a lack of knowledge about the definite state measured (in the example of the box, each atom was certainly in a spin state $|0\rangle$ or $|1\rangle$, however there was no way of knowing which was chosen for measurement, thus the probability introduced was representative of "uncertainty" or "ignorance" in the bayesian sense). Compare and contrast with the use of probabilities in quantum mechanics, in which the uncertainty principle for non-commuting operators grants them a more fundamental meaning, the **impossibility** of having a definite value for non-compatible observables at the same time.

Thus, to represent these more general states of quantum systems, we need

to incorporate configurations where there's a statistical ensemble (the "classical probability distribution") of possible quantum states and, at the same time, introduce a formalism capable of deriving the correct expectation values for physical states. It is well known, see [1] for one introduction among many, that the way to fulfil both conditions is via the use of density matrices (or density operators for infinite-dimensional systems):

Given a quantum system with Hilbert space $\mathcal{H}$, a density matrix ρ is a positive semidefinite (meaning $\langle \psi | \rho | \psi \rangle \geq 0$ for any state vector $|\psi\rangle$), Hermitian operator $\rho : \mathcal{H} \rightarrow \mathcal{H}$ satisfying $\text{Tr } \rho = 1$. Furthermore, we say that a quantum system is in state ρ when it is described by ρ and the expectation value of any observable $\mathcal{O}$ is given by $\langle \mathcal{O} \rangle = \text{Tr}(\rho \mathcal{O})$.

From the defining properties of the density matrix, it's easy to see that any ρ can be diagonalized (since it's Hermitian) as:

$$\rho = \sum_k p_k |\psi_k\rangle \langle \psi_k| \quad (2.3)$$

Where the $|\psi_k\rangle$ are normalized state vectors and the p_k are such that $0 \leq p_k \leq 1$ (from the positive semidefinite condition) and $\sum_k p_k = 1$ (from the normalization of the trace, $\text{Tr } \rho = 1$).

The diagonalization of ρ allows us to connect the use of density matrices to the requirements listed above, as the eigenvalues p_k satisfy all properties of a (discrete) classical probability distribution and the formula for the expectation value of observables may be written as:

$$\langle \mathcal{O} \rangle = \text{Tr}(\rho \mathcal{O}) = \sum_k p_k \langle \psi_k | \mathcal{O} | \psi_k \rangle \quad (2.4)$$

That is, the expectation value is the average (weighted by the probabilities p_k) of the expectations at each state $|\psi_k\rangle$. With this, it's possible to interpret physically a given state ρ as a statistical ensemble of possible quantum states $|\psi_k\rangle$, each with probability p_k .

With a interpretation of density matrices established, we can also see the relevance of the special case in which the state is a projector $\rho = |\psi\rangle\langle\psi|$ (that is, a single probability equal to one and the rest being zero). Here we find the class of states which can be described by a definite single vector $|\psi\rangle \in \mathcal{H}$, and for this reason these are called **pure states**, as there's no classical uncertainty.

The pure states can equivalently be characterized by the condition $\rho^2 = \rho$, since this guarantees ρ is a projector and since its trace is 1 it must be a **rank-1** projector, that is $\rho = |\psi\rangle\langle\psi|$ for some normalized vector $|\psi\rangle$, automatically defining a unique state vector up to global phase (likewise, a state vector naturally defines via the exterior product with itself a rank-1 projector which is also a density matrix). As for the states for which $\rho^2 \neq \rho$ (which implies $\text{Tr} \rho^2 < 1$), they called **mixed states**, a reference to the statistical "mixture"/ensemble the system is found in. These definitions and their physical meaning will be crucial throughout this Thesis.

As the previous discussion might be somewhat abstract, we can see a practical use of the density matrix formalism by returning to the motivating example of the box with equal amounts of spins $|0\rangle$ and $|1\rangle$, where the state we constructed is described by the density matrix:

$$\rho = \frac{1}{2} |0\rangle\langle 0| + \frac{1}{2} |1\rangle\langle 1| \quad (2.5)$$

Here we can already see the representation of the statistical ensemble defined originally, which now we can characterize as an example of a mixed state. Also, if we use the formalism to calculate the expectation value of some Pauli matrix we find:

$$\langle\sigma_i\rangle = \text{Tr}(\rho\sigma_i) = \frac{1}{2} \langle 0| \sigma_i |0\rangle + \frac{1}{2} \langle 1| \sigma_i |1\rangle \quad (2.6)$$

For any $i = x, y, z$. Thus, using the density matrix we recover the expectation values we arrived at intuitively, demonstrating the power of the formalism.

Besides our motivating case of a statistical ensemble composed of 50% of atoms with spin $|0\rangle$ and 50% of atoms with spin $|1\rangle$, there's another example of a

mixed state which can be helpful to understand density matrices (and whose importance in physics cannot be overstated), which is the thermal state of a system at inverse temperature β , given by:

$$\rho_\beta := \frac{1}{Z} e^{-\beta H} \quad (2.7)$$

Where H is the Hamiltonian of the system in question and $Z = \text{Tr } e^{-\beta H}$ is the partition function.

Mathematically, ρ_β is clearly mixed, as can be seen by diagonalizing the Hamiltonian as $H = \sum_n E_n |n\rangle \langle n|$ and calculating:

$$\rho_\beta^2 = \frac{1}{(\sum_n e^{-\beta E_n})^2} \sum e^{-2\beta E_n} |n\rangle \langle n| \neq \rho_\beta \quad (2.8)$$

The inequality being valid for finite β .

It is also possible to understand physically the reason why ρ_β is mixed: a system is in a thermal state when it's in equilibrium with a heat bath (a much larger system, also called "environment"). The system equilibrates via some interaction between the system and the bath (weak enough so its only effect is creating the equilibrium) and because of this interaction the system fluctuates between its energy eigenstates and is described by a statistical ensemble characterized by the Boltzmann weights $e^{-\beta E_n}$. In summation, the mixture of the state is a consequence of the interaction with the bath and has to be described by a classical probability distribution over the energy eigenstates.

It's possible to go into much more detail on each of the topics mentioned so far, for example, the thermalization of quantum systems is an active area of research and the very notion of the a system approaching a state ρ_β must be defined carefully. However, for the purposes of this Thesis, this is all the background needed on the nature of states in quantum mechanics and, as we'll see, the use of density matrices is fundamental to understand and quantify entanglement. We'll return to this subject on the last section of this Chapter in order to comment quickly on a few subtleties that appear in QFTs and the so-called algebraic formalism, which will be a source of ideas later on.

2.2 Composite Systems and Entanglement

Heuristically, a composite system is one that can be divided into two (or more) subsystems A and B which, in turn, can be **manipulated and measured** independently of each other (in the sense that an operator can be applied to a subsystem without affecting the other).

Given a system S with Hilbert space $\mathcal{H}$, in order to implement the concept of composite systems into the formalism of quantum mechanics, there are certain requirements that the partition of S into subsystems A and B must satisfy:

1. Given all observables of A and B (the quantities which may be measured on each subsystem individually), we must be able to construct any observable of S by their linear combinations and products (the operators in A and B **generate** those in S).
2. Any pair of observables acting on the subsystem must be compatible, that is, for any $\mathcal{O}_A$ and $\mathcal{O}_B$, we must have $[\mathcal{O}_A, \mathcal{O}_B] = 0$.
3. There must exist states in $\mathcal{H}$ without any correlations between A and B , that is, such that $\langle \mathcal{O}_A \mathcal{O}_B \rangle = \langle \mathcal{O}_A \rangle \langle \mathcal{O}_B \rangle$. This condition is called **Statistical Independence** and is motivated by the same reasoning as the previous requirement, but should not be confused with the existence of independent/commuting **algebras of observables** for each subsystem.

These conditions imply, as shown by von Neumann [2], that the Hilbert space of a composite quantum system can be partitioned into a tensor product of the form:

$$\mathcal{H} = \mathcal{H}_A \otimes \mathcal{H}_B \quad (2.9)$$

Where $\mathcal{H}_A$ and $\mathcal{H}_B$ are Hilbert spaces associated with the subsystems A and B . This property is also called a Tensor Product Structure (TPS) and it's at the heart of the definition of entanglement.

Based on the result mentioned above, from now on the concepts of partitioning a system S into subsystems and partitioning a Hilbert space $\mathcal{H}$ into a Tensor Product Structure will be used interchangeably, as they are equivalent.

An important point: when talking about the complete quantum systems of interest, all states will be taken as pure states (vectors in $\mathcal{H}$), the reason for this being that the main focus of this Thesis is the study of entanglement in ground states of Quantum Field Theories, examples of pure states. Likewise, we will only consider partitions of the system into **two** subsystems A and B , leading to the so-called bipartite states.

With these comments in mind, given a Hilbert space with partition into subsystems $\mathcal{H} = \mathcal{H}_A \otimes \mathcal{H}_B$, a state $|\psi\rangle \in \mathcal{H}$ is said to be **separable** if it can be written as:

$$|\psi\rangle = |\psi_A\rangle \otimes |\psi_B\rangle \quad (2.10)$$

Where $|\psi_A\rangle \in \mathcal{H}_A$ and $|\psi_B\rangle \in \mathcal{H}_B$.

Furthermore, if $|\psi\rangle \in \mathcal{H}$ is not separable, we say it is **entangled**.

To give a concrete example of these ideas, consider the simplest possible composite quantum system (much like the two-level system was used to study states): a system of two qubits, with Hilbert space $\mathcal{H} = \mathbb{C}^2 \otimes \mathbb{C}^2$. Introducing the notation $|00\rangle := |0\rangle|0\rangle$ (likewise for the other products), a basis for $\mathcal{H}$ is given by $\{|00\rangle, |10\rangle, |01\rangle, |11\rangle\}$, this is sometimes called the computational basis for $\mathcal{H} = \mathbb{C}^2 \otimes \mathbb{C}^2$.

The elements of the computational basis are, by definition, separable with respect to the TPS of the system, to see an example of entangled state, take the vector (one of the so-called Bell states):

$$|+\rangle := \frac{|00\rangle + |11\rangle}{\sqrt{2}} \quad (2.11)$$

To prove that this is indeed entangled, assume there are complex coefficients a ,

b, c, d such that $|+\rangle = (a|0\rangle + b|1\rangle) \otimes (c|0\rangle + d|1\rangle)$. Then, comparing with the definition of $|+\rangle$, we find the conditions $bc = 0, ad = 0, ac = \frac{1}{\sqrt{2}}, bd = \frac{1}{\sqrt{2}}$. However, as the first two imply $b = 0$ or $c = 0$ and $a = 0$ or $d = 0$, no matter which pair of coefficients we choose to be zero, it becomes impossible to satisfy at least one of the latter two conditions.

Thus, there is a contradiction on the assumption that $|+\rangle$ is a separable state, meaning that it's entangled.

The Schmidt Decomposition: Having seen a concrete example of entanglement, we now move to the study of the Schmidt decomposition, a powerful result which allows us to derive many properties of bipartite entanglement in pure states and understand more deeply what is the physical meaning of this concept.

Given a bipartite quantum system $\mathcal{H} = \mathcal{H}_A \otimes \mathcal{H}_B$, where $\dim \mathcal{H}_A = n$ and $\dim \mathcal{H}_B = m$ (n or m are allowed to be infinite) and any (normalized) pure state $|\psi\rangle \in \mathcal{H}$, assuming without loss of generality that $n \geq m$ (merely a choice of the labels A and B), the Schmidt decomposition states that there are orthonormal sets $\{|u_1\rangle, \dots, |u_m\rangle\}$ in $\mathcal{H}_A$ and $\{|v_1\rangle, \dots, |v_m\rangle\}$ in $\mathcal{H}_B$ and real coefficients λ_i with $0 \leq \lambda_i \leq 1$ for all i ($i = 1, \dots, m$) and $\sum_{i=1}^m \lambda_i^2 = 1$ such that:

$$|\psi\rangle = \sum_{i=1}^m \lambda_i |u_i\rangle |v_i\rangle \quad (2.12)$$

I will not give a detailed proof of this result, but it is based on the fact that one can always write an element of $\mathcal{H}_A \otimes \mathcal{H}_B$ as an $n \times m$ matrix combined with the Singular Value Decomposition theorem, see [3]. The numbers λ_i are called Schmidt coefficients and the conditions $0 \leq \lambda_i \leq 1, \sum_{i=1}^m \lambda_i^2 = 1$ come from the normalization of $|\psi\rangle$ (they are made to be real by absorbing any phases on the vectors $|u_i\rangle$ or $|v_i\rangle$).

With the Schmidt decomposition of a vector it's easy to identify whether or not the state is entangled: either one Schmidt coefficient is 1 (and the others are zero) and the state is already given as a tensor product, thus being separable, or

at least two coefficients are non-zero and we can reproduce the argument used for the Bell state $|+\rangle$ and show that it must be entangled.

Now we have the tools to understand more what does it mean, physically, to have a state which is entangled with respect to a given partition. Historically, states of the form of $|+\rangle = \frac{|00\rangle + |11\rangle}{\sqrt{2}}$ were shown to violate Bell's inequalities and so display correlations which are not reproducible in classical theories [1], from this comes the idea of entanglement as an intrinsically quantum type of correlation, and it's possible to go even further with this concept. Given generic observables $\mathbb{O}_A$ and $\mathbb{O}_B$ acting on the subsystems, define the correlation function $C(\mathbb{O}_A, \mathbb{O}_B)$ by:

$$C(\mathbb{O}_A, \mathbb{O}_B) := \langle \mathbb{O}_A \otimes \mathbb{O}_B \rangle - \langle \mathbb{O}_A \rangle \langle \mathbb{O}_B \rangle \quad (2.13)$$

This quantity, much like in classical physics, tells how much the results of the measurements of $\mathbb{O}_A$ and $\mathbb{O}_B$ depend on each other, thus being a measure of their statistical correlation.

With this definition in mind, we can state the main result of this section: **a pure state $|\psi\rangle$ is separable with respect to a partition $\mathcal{H} = \mathcal{H}_A \otimes \mathcal{H}_B$ if, and only if, the all correlation functions $C(\mathbb{O}_A, \mathbb{O}_B)$ are zero [4].**

The proof is split into two parts:

If $|\psi\rangle = |\psi_A\rangle \otimes |\psi_B\rangle$ ($|\psi\rangle$ is separable), all $C(\mathbb{O}_A, \mathbb{O}_B)$ are zero

This is easy to see, as $|\psi\rangle = |\psi_A\rangle \otimes |\psi_B\rangle$ implies:

$$\langle \mathbb{O}_A \otimes \mathbb{O}_B \rangle = \langle \psi_A | \otimes \langle \psi_B | \mathbb{O}_A \otimes \mathbb{O}_B | \psi_A \rangle \otimes | \psi_B \rangle \quad (2.14)$$

$$= \langle \psi_A | \mathbb{O}_A | \psi_A \rangle \langle \psi_B | \mathbb{O}_B | \psi_B \rangle \quad (2.15)$$

$$= \langle \mathbb{O}_A \rangle \langle \mathbb{O}_B \rangle \quad (2.16)$$

Thus $C(\mathbb{O}_A, \mathbb{O}_B) = 0$ no matter which observables are chosen.

If all $C(\mathbb{O}_A, \mathbb{O}_B)$ are zero, $|\psi\rangle$ is separable

Via the Schmidt decomposition, for any state vector, $|\psi\rangle = \sum_{i=1}^m \lambda_i |u_i\rangle |v_i\rangle$, meaning the correlation functions are given by:

$$\sum_{i,j} \lambda_i \lambda_j \langle u_i | \mathcal{O}_A | u_j \rangle \langle v_i | \mathcal{O}_B | v_j \rangle - \sum_i \lambda_i^2 \langle u_i | \mathcal{O}_A | u_i \rangle \sum_j \lambda_j^2 \langle v_j | \mathcal{O}_B | v_j \rangle \quad (2.17)$$

Due to the cross terms $\lambda_i \lambda_j$ in the expression for $\langle \mathcal{O}_A \otimes \mathcal{O}_B \rangle$, it will be possible to construct observables $\mathcal{O}'_A = \mathcal{O}_A + |u_i\rangle \langle u_j|$ and $\mathcal{O}'_B = \mathcal{O}_B + |v_i\rangle \langle v_j|$ such that $C(\mathcal{O}'_A, \mathcal{O}'_B) \neq 0$ unless some $\lambda_i = 1$ and all others are zero. That is, we can only have $C(\mathcal{O}_A, \mathcal{O}_B) = 0$ for all $\mathcal{O}_A$ and $\mathcal{O}_B$ (all choices of observables acting in the subsystems) if $|\psi\rangle$ is separable.

This result shows not only that entanglement is indeed a quantum form of correlation in a very precise sense, but also that it is the **only** source of correlations between subsystems in composite quantum systems, as long as the global state is pure (this qualification is crucial, as the proof relies on the Schmidt decomposition, which is only valid for pure states). It's important to keep this idea in mind throughout the Thesis, at it'll be key to understanding our results, in particular during the next section, when it'll be given a quantitative form.

Also, compare with classical probability theory, where a similar result is valid (of course it's not completely analogous as classical probabilities don't have a notion of entanglement), see for example [4].

The Reduced Density Matrix: Until now, we have defined and studied entanglement from the point of view of having the state vector for the complete quantum system and then looking at its tensor product structure with respect to the partition $\mathcal{H} = \mathcal{H}_A \otimes \mathcal{H}_B$, as exemplified by the Schmidt decomposition. However, as discussed in the first section of this Chapter, the most general states of a quantum system are described by density matrices, so given a ρ for the full system, it must be possible to associate a respective ρ_A to the subsystem of Hilbert space $\mathcal{H}_A$ and from this state we should be able to extract information about the correlations between subsystems. This ρ_A is called the reduced density matrix of

subsystem A and, as we'll see shortly, it's crucial to define quantitative measures of entanglement.

To properly define ρ_A , it's necessary to consider the notion of restricting the observables to a subsystem:

Given $\mathcal{H} = \mathcal{H}_A \otimes \mathcal{H}_B$, imagine that for some reason an experiment is only able to measure operators which act on $\mathcal{H}_A$, that is to say those of the form $\mathbb{O}_A \otimes \mathbb{I}_B$ (as will be shown in later Chapters, this idea is at the heart of the concept of renormalization in QFT, where the experiments can only access degrees of freedom up to some finite energy or momentum scale), then, using the formalism of density matrices, if the global state is ρ , all expectation values measured by this experiment will be given by:

$$\langle \mathbb{O}_A \rangle = \text{Tr}(\rho \mathbb{O}_A \otimes \mathbb{I}_B) = \sum_{i,j} \langle u_i | \otimes \langle v_j | (\rho \mathbb{O}_A \otimes \mathbb{I}_B) | u_i \rangle \otimes | v_j \rangle \quad (2.18)$$

Since the trace is invariant under the choice of basis used to calculate it, there's no loss of generality in taking a basis of separable elements.

Now, as $\mathbb{O}_A$ doesn't act on $\mathcal{H}_B$, this expectation value can also be written as:

$$\langle \mathbb{O}_A \rangle = \sum_i \langle u_i | \left(\sum_j \langle v_j | \rho | v_j \rangle \right) \mathbb{O}_A | u_i \rangle \equiv \sum_i \langle u_i | \rho_A \mathbb{O}_A | u_i \rangle = \text{Tr}_A(\rho_A \mathbb{O}_A) \quad (2.19)$$

And so, it is possible to obtain all measurements of this class of observables by considering the subsystem A alone in state $\rho_A = \sum_j \langle v_j | \rho | v_j \rangle$ (and unlike the Schmidt decomposition, it doesn't matter whether ρ is pure or mixed). The process of obtaining ρ_A from ρ is called the partial trace, denoted $\rho_A = \text{Tr}_B \rho$, and it's invariant under the basis of $\mathcal{H}_B$ chosen.

This result can be understood more generally in the following way: for measurements/operators restricted to a subsystem, all expectation values must be the same, whether the description chosen considers A on its own or as a part of a larger system (after all, the results of the experiment made are independent

of the theoretical formalism). Therefore, the consistency required between these different descriptions implies that, for any $\mathcal{O}_A$, we find:

$$\text{Tr}_A \rho_A \mathcal{O}_A = \text{Tr} \rho \mathcal{O}_A \quad (2.20)$$

This condition is sufficient to define the reduce density matrix ρ_A for any system and guarantees that the partial trace is independent of any choice of basis.

Returning to our study of entanglement, take a pure state $\rho = |\psi\rangle \langle\psi|$, with the Schmidt decomposition $|\psi\rangle = \sum_{i=1}^m \lambda_i |u_i\rangle |v_i\rangle$. By the definition of the partial trace, the reduced density matrix associated with subsystem A can be calculated by:

$$\rho_A = \sum_{k=1}^m \langle v_k | \left(\sum_{i=1}^m \lambda_i |u_i\rangle \langle v_i| \sum_{j=1}^m \lambda_j \langle u_j | \langle v_j | \right) |v_k\rangle \quad (2.21)$$

$$= \sum_{k=1}^m \sum_{i=1}^m \sum_{j=1}^m \lambda_i \lambda_j |u_i\rangle \langle u_j| \delta_{ik} \delta_{jk} \quad (2.22)$$

$$= \sum_{k=1}^m \lambda_k^2 |u_k\rangle \langle u_k| \quad (2.23)$$

Redefining $p_i \equiv \lambda_i^2$:

$$\rho_A = \sum_i p_i |u_i\rangle \langle u_i| \quad (2.24)$$

Notice that, as expected, ρ_A is a properly normalized density matrix, as $\sum_i p_i = 1$ from the corresponding condition of the Schmidt decomposition.

Now, recall that I proved that a state $|\psi\rangle$ is entangled with respect to the partition $\mathcal{H} = \mathcal{H}_A \otimes \mathcal{H}_B$ if, and only if, more than one Schmidt coefficient is non-zero. As the probabilities in the diagonalization of ρ_A are simply the Schmidt coefficients squared, if we have an entangled state, then necessarily $p_i^2 \neq p_i$ for some i and:

$$\rho_A^2 = \sum_i p_i^2 |u_i\rangle \langle u_i| \neq \rho_A \quad (2.25)$$

The hallmark of a mixed state.

Therefore, the subsystems A and B are entangled if, and only if, the reduced density matrix ρ_A is a mixed state as long as the state of the complete system

is globally pure (by the symmetry of the Schmidt decomposition the previous results also apply if we focus on subsystem B instead of A). With this, we can study entanglement with only the reduced density matrix, no need for the full Schmidt decomposition (which is incredibly difficult to construct in practice), indeed the objective of the next section is to use ρ_A to define **quantitative** measures of entanglement which will be applied in the rest of the Thesis.

Finally, the fact that ρ_A being a mixed state is equivalent to having entanglement can be interpreted physically the following way: as we proved earlier, when two subsystems are entangled, measurements acting on each individually are correlated and this means that by restricting the observables to A we're effectively taking an average of the possible influences B can have (this average is what the partial trace is), by doing so we deal with a statistical ensemble of the possible quantum states of A we measure (as we can see in the bell state $|+\rangle = \frac{|00\rangle + |11\rangle}{\sqrt{2}}$, with entanglement the result of a measurement in a subsystem influences the probability distribution in its complement), thus leading to a mixed state ρ_A .

2.3 Entanglement Measures and their Properties

It is finally time to introduce some quantitative measures of entanglement of interest and prove key identities which will be crucial throughout this Thesis. When studying quantum information, these entanglement measures are incredibly useful not only because they obviously allow for a comparison of which state is more entangled than the other (with respect to a certain partition), but also, in certain conditions, they tell us a lot about the structure of the state with the advantage of being easier to work with a single number than with a state vector/density matrix in many dimensions (one of the main results in this Thesis, given at the end of Chapter 3, is an example of the power of this approach).

Once again, I note that in this Thesis, I will work only with pure states, the reason being that the Schmidt decomposition is only valid in these cases and the

fact that our measures of interest are not useful by themselves when the global state is mixed (intuitively, their intrinsic non-linearity prevents a clean separation into "entanglement" and "classical probability" terms).

Now, as shown in the end of the previous section, in a system partitioned as $\mathcal{H} = \mathcal{H}_A \otimes \mathcal{H}_B$, the pure state $|\psi\rangle \in \mathcal{H}$ is entangled if, and only if, the reduced density matrix ρ_A is mixed $\rho_A^2 \neq \rho_A$. An equivalent statement is saying that ρ_A has more than one non-zero eigenvalue, which implies, from the constraint $\text{Tr } \rho_A = 1$, that the state is entangled if, and only if, $\text{Tr } \rho_A^2 < 1$, as for the non-zero probabilities in the diagonalization of ρ_A we have $p_i^2 < p_i$.

This result naturally suggests the introduction of the entanglement measure called **purity**, defined by:

$$\gamma := \text{Tr } \rho_A^2 \quad (2.26)$$

With smaller γ implying more entanglement and the limit $\gamma = 1$ leading to a separable state. In quantum systems of finite dimension d , there's also a lower bound $\gamma \geq \frac{1}{d}$, with $\gamma = \frac{1}{d}$ meaning the system is "maximally entangled".

Of course, the same argument that lead to the introduction of γ as a good quantitative measure of entanglement can be applied to any $\text{Tr } \rho_A^n$ with $n > 1$ (n doesn't have to be integer, but in a few moments there will be comments on this topic), this means that any monotonic function of $\text{Tr } \rho_A^n$ will also be a good entanglement measure and, for reasons which will be explained shortly, we define the **Rényi entropy of order n** , denoted $H_n(\rho_A)$ by:

$$H_n(\rho_A) = \frac{1}{1-n} \log \text{Tr } \rho_A^n \quad (2.27)$$

Furthermore, with the diagonalization of ρ_A we have $\text{Tr } \rho_A^n = \sum_i p_i^n$ and, in the limit $n \rightarrow 1$:

$$\lim_{n \rightarrow 1} H_n(\rho_A) = \lim_{n \rightarrow 1} \frac{1}{1-n} \log \left(\sum_i p_i^n \right) = - \sum_i p_i \log p_i = - \text{Tr } (\rho_A \log \rho_A) \quad (2.28)$$

Thus, we arrive at our main entanglement measure of interest, the **entanglement entropy** $S_{EE}(\rho_A)$ (also called von Neumann entropy in a more general context):

$$S_{EE}(\rho_A) := -\text{Tr}(\rho_A \log \rho_A) = \lim_{n \rightarrow 1} H_n(\rho_A) \quad (2.29)$$

Note that $S_{EE}(\rho_A)$, the Rényi entropies and the purity all depend solely on the eigenvalues of ρ_A , therefore, if the global state is pure the Schmidt decomposition implies that the reduced density matrix ρ_B will have the same eigenvalues and so all entanglement measures will return the same result for ρ_A and ρ_B . This idea can be summarized by the equality $S_{EE}(\rho_A) = S_{EE}(\rho_B)$, whenever ρ is pure.

The entanglement entropy, just like the Rényi entropies, will be **zero if, and only if, the state is separable** and takes positive values for entangled states. Also, in finite-dimensional systems $S_{EE}(\rho_A)$ attains the maximum value of $\log d$ exactly when the purity is at its minimum $\gamma = \frac{1}{d}$, which shows that for such systems there is a notion of **maximally entangled states** independent of choice of measure.

Another way of arriving at the definition of the entanglement entropy, much more insightful from the physical and information point of view, is by considering Shannon's contribution to the understanding of entropy [5]: as discussed previously, when the reduced density matrix ρ_A is diagonalized as $\rho_A = \sum p_i |i\rangle \langle i|$ we find a classical probability distribution of possible quantum states $|i\rangle$ with weights p_i , Shannon showed [5] that the information content/uncertainty of such a distribution is given by minus $\sum p_i \log p_i$ and von Neumann was able to relate the same quantity (already in the context of density operators) to the classical notion of entropy of a gas of particles [2], thus, writing in the basis-independent form $\text{Tr}(\rho_A \log \rho_A)$ we arrive directly at the entanglement entropy $S_{EE}(\rho_A)$ and conclude that it allows us to apply some intuitions from thermodynamics/statistical mechanics to the study of the intrinsically quantum correlation that is entanglement.

Before moving forward to the next topic, note that the limit $\lim_{n \rightarrow 1} H_n(\rho_A)$ must in principle be taken with n real, however, in many practical calculations

it's only possible to calculate quantities with n positive integer (like when using the path integral formalism, to be introduced in the next section). In such cases, the following identity is used:

$$\log A = \lim_{n \rightarrow 0} \frac{A^n - 1}{n} = \lim_{n \rightarrow 0} \frac{dA^n}{dn} \quad (2.30)$$

With the underlying **assumption** that applying this limit to expressions derived for n integer will lead to a sensible result, equal to the calculation with n real.

This technique, together with its postulates, is called the **replica trick** and the equation above will be sometimes referred to as "the replica trick formula". The use of the replica trick is common in the study of disordered systems in statistical mechanics and recently has been used to calculate entanglement entropy in quantum field theories (see [6]), in both cases there are examples allowing for a rigorous proof of the validity of the method or a direct comparison with the answers derived some other way.

It is possible, in principle, to go beyond the replica trick: the so-called Distributional Zeta Function method is a mathematically rigorous alternative which avoids the need for an analytical continuation, see [7] for the construction of the technique and [8] for an application. It is not yet clear how to adapt this method to density matrices and their entropy, so in this Thesis the replica trick will be the main way of calculating entanglement entropy and there won't be any concern with checking it rigorously, suffice to say that its successes in many different examples will be taken as reason enough to apply it without major concerns.

Other entropy-based measures: Part of the usefulness of the entanglement entropy stems from the fact that it serves as building block for many other entanglement measures (even applicable to mixed states or multipartite systems, which, again, will not be studied here). With the help of one such measure derived from $S_{EE}(\rho_A)$, it'll be possible to prove an inequality which a realization of powerful the idea that "in pure states, all correlations between subsystems are generated by entanglement".

But first, given two generic states ρ, σ of a quantum system with Hilbert space $\mathcal{H}$, the **quantum relative entropy of ρ with respect to σ** is defined as:

$$S(\rho||\sigma) := \text{Tr} \rho (\log \rho - \log \sigma) \quad (2.31)$$

This quantity is sometimes introduced as comparing a state ρ with an approximation σ , but our main interest lies in a special case. Consider that $\mathcal{H}$ has Tensor Product Structure $\mathcal{H} = \mathcal{H}_A \otimes \mathcal{H}_B \otimes \mathcal{H}_C$, where $\mathcal{H}_C$ may or may not be equal to $\mathbb{C}$ (which means that we're taking a partition into two or more subsystems), then given a global state ρ and the reduced density matrices $\rho_{AB} = \text{Tr}_C \rho$, $\rho_A = \text{Tr}_{B \cup C} \rho$ and $\rho_B = \text{Tr}_{A \cup C} \rho$, the **quantum mutual information between subsystems A and B** is:

$$I(A : B) := S(\rho_{AB}||\rho_A \otimes \rho_B) = S(\rho_A) + S(\rho_B) - S(\rho_{AB}) \quad (2.32)$$

With the last equality coming from the properties of the logarithm of a tensor product of matrices.

Taking from now on ρ as a pure state, if A and B are the only subsystems (the system is bipartite), we have $S(\rho_A) = S(\rho_B)$ and $S(\rho_{AB}) = 0$, so the mutual information will be:

$$I(A : B) = 2S_{EE}(\rho_A) \quad (2.33)$$

The identity above will be key to the interpretation of the main result of this section, which I'll now state before proving and discussing:

Given any two observables $\mathcal{O}_A$ acting only on $\mathcal{H}_A$ and $\mathcal{O}_B$ acting only on $\mathcal{H}_B$ and defining the operator norm $|\mathcal{O}|^2 := \text{Tr}(\mathcal{O}^\dagger \mathcal{O})$, the mutual information between A and B always obeys the inequality (see [9], [10]):

$$I(A : B) \geq \frac{C(\mathcal{O}_A, \mathcal{O}_B)^2}{2|\mathcal{O}_A|^2|\mathcal{O}_B|^2} \quad (2.34)$$

With $C(\mathcal{O}_A, \mathcal{O}_B) := \langle \mathcal{O}_A \otimes \mathcal{O}_B \rangle - \langle \mathcal{O}_A \rangle \langle \mathcal{O}_B \rangle$ the (truncated) correlation function introduced previously in this Chapter.

Proof:

From its definition as a relative entropy, $I(A : B) = S(\rho_{AB} || \rho_A \otimes \rho_B)$, so with the spectral decomposition $\rho_{AB} = \sum \lambda_i |i\rangle \langle i|$ and $\rho_A \otimes \rho_B = \sum v_{i'} |i'\rangle \langle i'|$, the mutual information is given as:

$$I(A : B) = \text{Tr} \left(\sum_i \lambda_i \log \lambda_i |i\rangle \langle i| - \sum_{i,i'} \lambda_i \log v_{i'} |i\rangle \langle i| i'\rangle \langle i'| \right) \quad (2.35)$$

$$= \sum_{i,i'} (\lambda_i \log \lambda_i - \lambda_i \log v_{i'}) \text{Tr} (|i\rangle \langle i| i'\rangle \langle i'|) \quad (2.36)$$

Where the identities $\langle i|j\rangle = \delta_{ij}$ and $\sum_{i'} |i'\rangle \langle i'| = \mathbb{I}$ where used

Now, consider the function $s(x) = -x \log x$ (basically the Shannon entropy), then $s'(x) = -\log x - 1$, the second derivative obeys $s''(x) = -\frac{1}{x} \leq -1$ for any $x \in (0, 1)$ and $-\lambda_i \log v_{i'} = s(v_{i'}) + (\lambda_i - v_{i'})s'(v_{i'}) + (v_{i'} - \lambda_i)$. Therefore, using the completeness of $|i\rangle$ and $|i'\rangle$, as well as the fact that the eigenvalues of the density operators sum to one, the mutual information may be written as:

$$I(A : B) = \sum_{i,i'} (-s(\lambda_i) + s(v_{i'}) + (\lambda_i - v_{i'})s'(v_{i'})) \text{Tr} (|i\rangle \langle i| i'\rangle \langle i'|) \quad (2.37)$$

The value of this manipulation is the following: as is well known from real analysis, there will always exist a number $\alpha_{i,i'}$ between λ_i and $v_{i'}$ such that:

$$-s(\lambda_i) + s(v_{i'}) + (\lambda_i - v_{i'})s'(v_{i'}) = -\frac{1}{2}(\lambda_i - v_{i'})^2 s''(\alpha_{i,i'}) \geq \frac{1}{2}(\lambda_i - v_{i'})^2 \quad (2.38)$$

With the inequality on the right being a consequence of $s''(x) \leq -1$.

Then, using this result on the expression for the mutual information for each i and i' and summing over them all:

$$I(A : B) \geq \frac{1}{2} \sum_{i,i'} (\lambda_i - v_{i'})^2 \text{Tr} (|i\rangle \langle i| i'\rangle \langle i'|) = \frac{1}{2} \text{Tr} (\rho_{AB} - \rho_A \otimes \rho_B)^2 \quad (2.39)$$

As can easily be checked.

Finally, since the density matrices are Hermitian, the right-hand side of the equation above is basically the norm $|\mathbb{A}|^2$ from the operator inner product $\langle \mathbb{A}, \mathbb{B} \rangle := \text{Tr}(\mathbb{A}^\dagger \mathbb{B})$ for the particular case $\mathbb{A} = \rho_{AB} - \rho_A \otimes \rho_B$. Therefore, taking $\mathbb{A} =$

$\rho_{AB} - \rho_A \otimes \rho_B$ and $\mathbb{B} = \mathbb{O}_A \otimes \mathbb{O}_B$ (any two operators acting on their respective subsystems), the Cauchy-Schwarz inequality implies:

$$\text{Tr}(\rho_{AB} - \rho_A \otimes \rho_B)^2 \geq \frac{\text{Tr}[(\rho_{AB} - \rho_A \otimes \rho_B)\mathbb{O}_A \otimes \mathbb{O}_B]^2}{|\mathbb{O}_A|^2 |\mathbb{O}_B|^2} = \frac{C(\mathbb{O}_A, \mathbb{O}_B)^2}{|\mathbb{O}_A|^2 |\mathbb{O}_B|^2} \quad (2.40)$$

And so, combining the last two equations:

$$I(A : B) \geq \frac{1}{2} \text{Tr}(\rho_{AB} - \rho_A \otimes \rho_B)^2 \geq \frac{C(\mathbb{O}_A, \mathbb{O}_B)^2}{2|\mathbb{O}_A|^2 |\mathbb{O}_B|^2} \quad (2.41)$$

As was to be proved.

In particular, taking complementary subsystems (which will now be called A and $\bar{A}$), the mutual information is simply twice the entanglement entropy and we conclude:

$$S_{EE}(\rho_A) \geq \frac{C(\mathbb{O}_A, \mathbb{O}_{\bar{A}})^2}{4|\mathbb{O}_A|^2 |\mathbb{O}_{\bar{A}}|^2} \quad (2.42)$$

From the intuitive point of view, this result is incredibly powerful: as shown in the previous section, for pure, bipartite states, correlations between subsystems exist if and only if they are entangled, what the equation above does is give a **quantitative** form of this statement (only if the entanglement entropy is non-zero, can the truncated correlation functions be non-trivial). Further in the Thesis, one of the main checks for entanglement will be a calculation of a truncated correlation function, with a result other than zero indicating that there will be an interesting study to be done from the quantum information point of view.

2.4 Real-space Entanglement in Free QFTs

The goal of this section is to apply the concepts and techniques we've been studying to Quantum Field Theories as a concrete example, namely I'll show how to calculate the entanglement entropy between the degrees of freedom in two halves of space separated by a plane in a free field theory, and we'll see how the famous area law of entanglement appears (with a heuristic argument explaining why should we expect it in the first place).

This calculation is also useful because it'll allow for the introduction of the path integral representation of wavefunctions (wavefunctionals in QFT) and the replica trick, both important for the main objective of this Thesis (explored in later Chapters): the calculation of entanglement between momentum modes of QFTs at different energy scales.

Before proceeding, I note that in this Thesis the motivation to discuss entanglement in real space is purely practical while historically the study of quantum information in the context of QFT began in order to explore the entropy of black holes and ideas of quantum gravity in general. Some of the initial papers were [11] and [12], which already explored the role of entanglement in black hole entropy, [13] with a discussion from the gravity point of view and the work in [14] strengthened the connection between topics by calculating real-space entanglement entropy through the use of the famous AdS/CFT correspondence. In more recent years, [15] explains the techniques which will be presented in this Section in detail (and goes further with more advanced applications) and [16] gives a modern review of the current state of this line of research. The list of references given here is far from complete, but it may serve as a starting point for interested readers.

Now, we move forward with a well-known derivation of the path integral replica trick, for compactness I'll not go in full detail at every step, in particular, I'll use non-relativistic quantum systems as motivation before jumping to the field theory scenario. The references [16] and [6] (on which the following discussion is partially based) present a more careful treatment of the technique.

Now, as shown by Feynman, in a one-particle quantum system with Hamiltonian H , the transition amplitude between the position eigenstate $|x_i\rangle$ at time $t = 0$ and $|x_f\rangle$ at $t = T$ can be written as the Wick-rotated path integral:

$$\langle x_i, 0 | x_f, T \rangle = \int_{x(0)=x_i}^{x(T)=x_f} \mathcal{D}x(\tau) e^{-\int_0^T L[x(\tau)] d\tau} \quad (2.43)$$

With $t = -i\tau$ and $L[x(\tau)]$ is the Lagrangian obtained by the Legendre transform of H .

Now, taking the limit $T \rightarrow -\infty$, if we write the Hamiltonian in diagonal form $H = \sum_n E_n |n\rangle \langle n|$ (where the ground state is such that $E_0 = 0$) and define the wavefunctions $\psi_n(x) = \langle x|n\rangle$, we have:

$$\langle x_i, 0 | x_f, T \rangle = \langle x_i | e^{HT} | x_f \rangle = \sum_n e^{E_n T} \psi_n(x_i) \psi_n^*(x_f) \quad (2.44)$$

And, taking the limit:

$$\lim_{T \rightarrow -\infty} \langle x_i, 0 | x_f, T \rangle = \psi_0(x_i) \psi_0^*(x_f) \quad (2.45)$$

The next step is to arrive at an expression for the ground state wavefunction of the system directly from the equation above. To do this we need to take into account the following identity:

$$\int dx_f \langle x_i, 0 | x_f, T \rangle \psi_0(x_f) = \langle x_i, 0 | \left(\int dx_f |x_f, T\rangle \langle x_f, T| \right) |0\rangle = \psi_0(x_i) \quad (2.46)$$

Where we used the completeness relation $\int dx |x\rangle \langle x| = \mathbb{I}$, the fact that the vacuum doesn't evolve with time and the definition of the wavefunction.

Therefore, multiplying both sides of $\lim_{T \rightarrow -\infty} \langle x_i, 0 | x_f, T \rangle = \psi_0(x_i) \psi_0^*(x_f)$ by $\psi_0(x_f)$, integrating over x_f and using the normalization of the wavefunction, we finally arrive at:

$$\psi_0(x) = \int_{-\infty}^{x(0)=x} \mathcal{D}x(\tau) e^{-\int_{-\infty}^0 L[x(\tau)] d\tau} \quad (2.47)$$

For simplicity of notation, the lower limit in the path integral implies that all possible $x(\tau \rightarrow -\infty) = x'$ are integrated over, while the upper limit means that for $x(\tau \rightarrow 0^-)$ only the value x is taken.

Following a similar argument, we can also find a path integral expression for its complex conjugate:

$$\psi_0^*(x) = \int_{x(0)=x}^{\infty} \mathcal{D}x(\tau) e^{-\int_0^{\infty} L[x(\tau)] d\tau} \quad (2.48)$$

Thus, it is also possible to write the ground state density operator in position basis (in a continuous basis it becomes an integral operator instead of a matrix, but all its main properties remain the same):

$$\rho(x, x') = \psi_0(x) \psi_0^*(x') = \int_{-\infty}^{x(0^-)=x} \int_{x(0^+)=x'}^{\infty} \mathcal{D}x(\tau) e^{-\int_{-\infty}^{\infty} L[x(\tau)] d\tau} \quad (2.49)$$

From now, I will begin to make generalizations quickly, again referring to [16] and [6] for a more detailed approach: this construction of the wavefunction for a one-particle quantum system can be easily generalized to composite systems by adding new variables in the path integral and Lagrangian, say, x_A and x_B with $L = L[x_A(\tau), x_B(\tau)]$, with this we may take partial traces (which in the path integral means **integrating over a subset of variables**, a crucial observation for later) and write, for example, $\rho_A(x, x')$ as:

$$\rho_A(x, x') = \int_{-\infty}^{x_A(0^-)=x} \int_{x_A(0^+)=x'}^{\infty} \mathcal{D}x_A(\tau) \mathcal{D}x_B(\tau) e^{-\int L[x_A(\tau), x_B(\tau)] d\tau} \quad (2.50)$$

The fact that this is indeed a reduced density matrix/operator is implicit in the notation: the degree of freedom x_B (which was traced-over) is integrated without any restrictions at imaginary time $\tau = 0$, while the remaining variable x_A (physically, the one we're still making measurements of) has this jump from x as $\tau \rightarrow 0^-$ to x' in $\tau \rightarrow 0^+$.

With this representation, we have all the tools to apply the replica trick, that is, to find $\rho_A^n(x, x')$ via the path integral and then calculate $S_A = -\lim_{n \rightarrow 1} \partial_n \ln \text{Tr} \rho_A^n$.

The next step is generalizing this to field theories. The very idea of a QFT involves having degrees of freedom at each point in space, so naively we expect to associate to each $\vec{x}$ the Hilbert space $\mathcal{H}_{\vec{x}}$ of the possible field configurations, which can be characterized by the diagonalization of the field operator $\phi(\vec{x})$ by eigenstates $|\phi(\vec{x})\rangle$ (this is the equivalent of the coordinate representation in ordinary quantum mechanics):

$$\phi(\vec{x}) |\phi(\vec{x})\rangle = \phi(\vec{x}) |\phi(\vec{x})\rangle \quad (2.51)$$

Of course, the field operator is only Hermitian for a real scalar field, for the other cases we diagonalize the Hermitian linear combinations of $\phi^\dagger(\vec{x})$ and $\phi(\vec{x})$.

From these local Hilbert spaces, the full space of the theory is then given by $\mathcal{H} = \bigotimes_{\vec{x}} \mathcal{H}_{\vec{x}}$ and we arrive at a tensor product structure we're now free to decompose as $\mathcal{H} = \mathcal{H}_A \otimes \mathcal{H}_{\bar{A}}$ for some choice of region A in order to calculate the entanglement between the degrees of freedom in real space.

Mathematically speaking this heuristic construction of $\mathcal{H}$ is not correct (see the next section for more details, this is connected to the infinite degrees of freedom), but it's possible to make it sound as long as **ultraviolet** (UV) and **infrared** (IR) regularizations are introduced, which together lead to a finite number of degrees of freedom and, as we'll see, allow for the calculation of finite entanglement measures.

With the previous qualification in mind, we generalize the path integral representation of the wavefunction to write the ground state wavefunctional of a field $\phi(\vec{x})$ with euclidean action $S[\phi(\vec{x})]$ as (note that the use of euclidean time is a means to project the state of the system unto the ground state):

$$\psi_0[\phi(\vec{x})] = \langle \phi(\vec{x}) | \psi_0 \rangle = \frac{1}{\sqrt{Z}} \int_{\tau=-\infty}^{\phi(\vec{x}, \tau=0^-)=\phi(\vec{x})} \mathcal{D}\phi(\tau, \vec{x}) e^{-S[\phi(\vec{x})]} \quad (2.52)$$

The main difference to the representation in quantum mechanics is the need for a normalization term $\frac{1}{\sqrt{Z}}$, where Z is the partition function of the theory. It appears because of the way the integration measure $\mathcal{D}\phi(\tau, \vec{x})$ is defined and as we'll see shortly, it becomes very useful when calculating entanglement.

2.4.1 Replica trick in non-interacting Quantum Field Theories

With the wavefunctional, it is possible, as shown by [6], to write $\text{Tr} \rho_A^n$ (for some reduced density matrix associated with the degrees of freedom in region A) in terms of a path integral, meaning we can apply the replica trick identity and calculate directly the entanglement entropy.

This will be yet another moment in which I will skip details, but the key idea is to write the matrix elements of a reduced density operator for a region of space as:

$$\langle \phi_a | \rho_A | \phi'_a \rangle = \frac{1}{Z} \int \mathcal{D}\phi e^{-S[\phi(\vec{x})]} \prod_{\vec{x} \in A} \delta(\phi(0^+, \vec{x}) - \phi'_a) \delta(\phi(0^-, \vec{x}) - \phi_a) \quad (2.53)$$

Then, taking the trace of any integer power of ρ_A , there will be a Z^n factor in the denominator and one can show [6], [16], that the complicated path integral in the numerator may be rearranged as:

$$Z_n := \int_{\mathcal{M}_n} \mathcal{D}\varphi e^{-S[\varphi(\vec{x})]} \quad (2.54)$$

Where $\int_{\mathcal{M}_n}$ means the field variable now occupies a specific euclidean manifold determined by the region A which is not traced over.

For a bit of clarification in this last step, when taking the n th power of ρ_A , for each $i \in \{1, 2, \dots, n\}$ we introduce a field $\varphi_i(\vec{x}, \tau)$ in the path integral and impose the boundary conditions at $\tau = 0$:

$$\varphi_i(\vec{x}, \tau \rightarrow 0^+) = \varphi_{i+1}(\vec{x}, \tau \rightarrow 0^-) \quad (2.55)$$

If $\vec{x} \in A$ and $\varphi_i(\vec{x}, \tau)$ continuous at $\tau = 0$ if $x \notin A$. It is due to these boundary conditions that the specific geometry of $\mathcal{M}_n$ is generated.

This manipulation allows us to write the trace as the ratio between Z_n and Z^n , from which the replica trick formula $S_A = -\lim_{n \rightarrow 1} \partial_n \log \text{Tr} \rho_A^n$ leads to:

$$S_A = \lim_{n \rightarrow 1} \frac{d}{dn} (n \log Z - \log Z_n) \quad (2.56)$$

Now we have all tools necessary to perform one of the few exact calculations of an entanglement observable in QFT (with total dimension of spacetime denoted by $d + 1$): the entanglement entropy S_A , in a free theory, between the degrees of freedom in the region $A = \{(x_1, \dots, x_d) \in \mathbb{R}^d : x_1 \leq 0\}$ and those in the rest of space. For this choice of A , the manifold $\mathcal{M}_n$ is given by $\mathcal{M}_n = \mathcal{C}_n \times \mathbb{R}^{d-1}$, with $\mathcal{C}_n$ the $2d$ cone of angle $2\pi n$ whose singularity will be crucial to obtaining the entropy.

Proceeding with the calculation of S_A for free scalar and fermionic fields, we will see how this leads to the famous area law for the entanglement entropy of spatial regions (the gauge field case will be quickly discussed next section, during some comments about the algebraic formalism). Beginning with a free massive scalar field, the euclidean action in $(d + 1)$ -dimensional space is:

$$S = \frac{1}{2} \int d^{d+1}x \left[(\partial_\mu \phi)^2 + m^2 \phi^2 \right] \quad (2.57)$$

Formally, both $\ln Z$ and $\ln Z_n$ are easily calculated from the Gaussian integral:

$$Z' = \int \mathcal{D}\phi e^{-\frac{1}{2} \int d^{d+1}x [\phi(x)(-\nabla^2 + m^2)\phi(x)]} = \det^{\frac{1}{2}} \left[\frac{1}{-\nabla^2 + m^2} \right] \quad (2.58)$$

$$\ln Z' = \frac{1}{2} \text{Tr} \ln \frac{1}{-\nabla^2 + m^2} = -\frac{1}{2} \text{Tr} \ln (-\nabla^2 + m^2) \quad (2.59)$$

For $Z' = Z$, I leave implicit in the notation the fact that the traces must be taken over a basis of states of the theory in $\mathbb{R}^{d+1}$ and, for $Z' = Z_n$, over a basis of the theory in the Riemannian surface $\mathcal{M}_n$. For clarity, I will use the notation Tr , ∇^2 whenever referring to $\mathbb{R}^{d+1}$ and $\text{Tr}_{\mathcal{M}_n}$, $\nabla_{\mathcal{M}_n}^2$ when talking about $\mathcal{M}_n$.

Now, in order to keep track of all cancellations and guarantee that the expressions are well-defined, consider the theory in a cubic box of side L (IR regulator) and introduce an UV cutoff in a minimal distance a . Obviously, L and a^{-1} are taken as large enough so that we may replace sums by integrals (and vice-versa) through the usual rules without changing the leading order terms.

Notice that, from the regularization, the Hilbert space of the theory is finite-dimensional, meaning the identity $\ln \det A = \text{Tr} \ln A$ can be used freely. Furthermore, a simple calculation of the leading terms of $\text{Tr}_{\mathcal{M}_n} \mathbb{I}$ and $\text{Tr} \mathbb{I}$ proves the useful relation $\text{Tr}_{\mathcal{M}_n} \mathbb{I} = n \text{Tr} \mathbb{I}$.

This results from the simple fact that the manifold $\mathcal{M}_n$ is obtained from gluing $\mathbb{R}^{d+1}$ spaces n times [6], thus multiplying the dimension of the Hilbert space by the same factor. This equality justifies the fact that none of the multiplicative constants in the partition function have been taken into account (in particular the common factors of 2π from the Gaussian integrals): such terms will always cancel.

Beginning the calculation of the entanglement entropy in earnest, the expression for S_A becomes:

$$S_A = \frac{1}{2} \lim_{n \rightarrow 1} \partial_n \left[-n \text{Tr} \ln (-\nabla^2 + m^2) + \text{Tr}_{\mathcal{M}_n} \ln (-\nabla_{\mathcal{M}_n}^2 + m^2) \right] \quad (2.60)$$

Next, make use of the identity (the Schwinger proper time parametrization):

$$\ln A = \int_0^\infty \frac{ds}{s} (e^{-s} - e^{-sA}). \quad (2.61)$$

And write:

$$S_A = \frac{1}{2} \lim_{n \rightarrow 1} \partial_n \int_0^\infty \frac{ds}{s} \left[n \operatorname{Tr} \left(e^{-s(-\nabla^2 + m^2)} - e^{-s} \right) + \operatorname{Tr}_{\mathcal{M}_n} (e^{-s} - e^{-s(-\nabla_{\mathcal{M}_n}^2 + m^2)}) \right] \quad (2.62)$$

Which becomes, using eq. (5) and the linearity of the trace:

$$S_A = \frac{1}{2} \lim_{n \rightarrow 1} \partial_n \int_0^\infty \frac{ds}{s} \left[n \operatorname{Tr} e^{-s(-\nabla^2 + m^2)} - \operatorname{Tr}_{\mathcal{M}_n} e^{-s(-\nabla_{\mathcal{M}_n}^2 + m^2)} \right] \quad (2.63)$$

Before moving forward, it is important to emphasize that the Schwinger parameter s has dimensions of length squared, thus, we must incorporate the IR and UV regularizations by replacing the upper and lower limits of integration with L^2 and a^2 , respectively. For simplicity, I will only make such changes whenever they become important for the calculation, in particular, there's no IR divergence in this expression.

The Laplacian operator in $\mathbb{R}^{d+1}$ is trivially diagonalized by plane waves, so:

$$\operatorname{Tr} e^{-s(-\nabla^2 + m^2)} = e^{-sm^2} \frac{L^{d+1}}{(2\pi)^{d+1}} \int d^{d+1}k e^{-s\vec{k}^2} = L^{d+1} \frac{e^{-sm^2}}{(4\pi s)^{\frac{d+1}{2}}} \quad (2.64)$$

As for $\operatorname{Tr}_{\mathcal{M}_n} e^{-s(-\nabla_{\mathcal{M}_n}^2 + m^2)}$, it can be quite complicated to find its expression properly but this has been done in [17] and the general structure of this trace, a consequence of the conical singularity introduced by the replica trick, is commented upon in [16]. In this Thesis' notation (with spacetime dimension $d+1$ and volume of $\mathcal{M}_n$ being nL^{d+1}), the leading contributions as $L \rightarrow \infty$ are given by:

$$\operatorname{Tr}_{\mathcal{M}_n} e^{-s(-\nabla_{\mathcal{M}_n}^2 + m^2)} = \frac{nL^{d+1}}{(4\pi s)^{\frac{d+1}{2}}} + \frac{\pi L^{d-1}}{3} \left(\frac{1}{n} - n \right) \frac{s}{(4\pi s)^{\frac{d+1}{2}}} \quad (2.65)$$

With this, it is easy to see that the terms proportional to L^{d+1} cancel perfectly and, incorporating the UV regularization and performing the last few calculations, we arrive at the expression for the entanglement entropy:

$$S_A = \frac{\pi L^{d-1}}{3} \int_{a^2}^\infty ds \frac{e^{-sm^2}}{(4\pi s)^{\frac{d+1}{2}}} = \frac{L^{d-1}}{6(4\pi)^d} \left[\frac{1}{(d-1)a^{d-1}} - \frac{m^2}{(d-3)a^{d-3}} + \dots \right] \quad (2.66)$$

This result can be found in [16].

The free fermion case is not very different. Due to the properties of the Berezin integral, the expression for the entanglement entropy becomes:

$$S_A = \frac{1}{2} \lim_{n \rightarrow 1} \partial_n \left[n \operatorname{Tr} \ln (-\nabla^2 + m^2) - \operatorname{Tr}_{\mathcal{M}_n} \ln (-\nabla_{\mathcal{M}_n}^2 + m^2) \right] \quad (2.67)$$

And so:

$$S_A = \frac{1}{2} \lim_{n \rightarrow 1} \partial_n \int_0^\infty \frac{ds}{s} \left[n \operatorname{Tr} \left(-e^{-s(-\nabla^2 + m^2)} + e^{-s} \right) - \operatorname{Tr}_{\mathcal{M}_n} (e^{-s} - e^{-s(-\nabla_{\mathcal{M}_n}^2 + m^2)}) \right] \quad (2.68)$$

The entropy will still be positive, despite the change of sign with respect to the scalar case, because as shown in [18], the heat kernel for anticommuting fields in the conic surface is given by:

$$\operatorname{Tr}_{\mathcal{M}_n} e^{-s(-\nabla_{\mathcal{M}_n}^2)} = \frac{nL^{d+1}}{(4\pi s)^{\frac{d+1}{2}}} + \frac{2\pi L^{d-1}}{3} \left(-\frac{1}{n} + n \right) \frac{s}{(4\pi s)^{\frac{d+1}{2}}} \quad (2.69)$$

So in the end:

$$S_A^{\text{fermion}} = \frac{L^{d-1}}{3(4\pi)^d} \left[\frac{1}{(d-1)a^{d-1}} - \frac{m^2}{(d-3)a^{d-3}} + \dots \right] \quad (2.70)$$

The same result for the entropy as before, but multiplied by a factor of 2 (which intuitively makes sense, because now there are twice as many degrees of freedom due to spin).

2.4.2 The ultraviolet divergence

Looking at equations (2.66) and (2.70) the first noteworthy property of the entropies is that they diverge with the ultraviolet cutoff a as a power law (for spatial dimension $d=1$, the divergence is logarithmic).

Divergences are commonplace in the study of Quantum Field Theories and they can usually be removed by renormalizing the parameters of the QFT in question. For this calculation, however, the divergence appears already at the free field theory, in which there is no renormalization.

One can absorb the divergent part of the entanglement entropy of spatial regions by including gravity in the system, in which case the renormalization of

Newton's constant G_N makes all quantities finite (see [19]), but, of course, we then leave the realm of Quantum Field Theory in flat space.

By staying purely in the context of QFTs without recourse to gravity (which will be the case for the entirety of this Thesis), the divergence of the entropy can be explained as follows: when using the path integral representation of the density matrix it was assumed that the Hilbert space of a field theory has a Tensor Product Structure $\mathcal{H} = \bigotimes_{\vec{x}} \mathcal{H}_{\vec{x}}$, which is true when the lattice regularization is introduced through a (the lattice implicitly, of course, as the cutoff appears in the Schwinger proper time representation) but no longer valid in the continuum limit $a \rightarrow 0$, as explained in [20].

Physically, the assumption of a TPS in configuration space fails because in the continuum limit the system gains degrees of freedom which are infinitely near to each other, so a strict partition between the inside and outside of a region A becomes ill-defined as an observable with such infinite resolution would require, by the uncertainty principle, the use of an infinite amount of energy. Furthermore, when calculating a measure of the entanglement between arbitrarily close points as was done above for A and its complement, the counting of such degrees of freedom naturally returns a divergent entropy and prevents it from being a well defined quantity in QFTs.

Finally, it is possible to work fully in the continuum if we exclude a safety corridor between regions A and B in order to avoid the infinite proximity between points when studying entanglement between them. Recalling a concept from earlier in this Chapter, this is a perfect situation to use the mutual information $I(A : B)$ (a good measure of entanglement between two subsystems even when a third one is excluded), which does lead to completely finite quantities, as shown in [21].

2.4.3 Heuristic origin of the Area Law

Another key characteristic of the entanglement entropies for both scalar and fermionic fields is that they are proportional to L^{d-1} . So, as the region of space A has volume L^d , we see that the entanglement entropy found is proportional to the **area of the boundary** of A , instead of the volume as would be expected for a thermodynamic notion of entropy.

Furthermore, even though the calculation made is only for free theories and a region consisting of the half-space, this area law has been shown to be valid for a variety of systems and shapes of A (including bounded regions, which have an entropy without the infrared divergence of our example). It would be impossible to include a complete list of references here, but two excellent reviews are [16] and [22].

In order to understand why the entanglement for a spatial region is proportional to the area of its boundary, return to the powerful identity for the entanglement entropy in a bipartite pure state:

$$S_A \geq \left(\frac{\langle \mathbf{O}_A \otimes \mathbf{O}_{\bar{A}} \rangle - \langle \mathbf{O}_A \rangle \langle \mathbf{O}_{\bar{A}} \rangle}{2|\mathbf{O}_A||\mathbf{O}_{\bar{A}}|} \right)^2 \quad (2.71)$$

Now, in a QFT where the energy spectrum is gapped (the so-called mass gap, which will be denoted by m), the truncated correlation functions obey the Cluster Decomposition Property, [23], [24], [25], which takes the form:

$$\lim_{\vec{r} \rightarrow \infty} (\langle \mathbf{O}(\vec{x}) \mathbf{O}'(\vec{x} + \vec{r}) \rangle - \langle \mathbf{O}(\vec{x}) \rangle \langle \mathbf{O}'(\vec{x} + \vec{r}) \rangle) \propto e^{-m|\vec{r}|} \quad (2.72)$$

That is, the correlations decay exponentially with distance.

Therefore, the intuition gained from combining the two previous equations is that, as source of all correlations in a pure state, the entanglement between spatial regions must decay quickly with distance and so the dominant contributions for the entropy S_A (which measures the entanglement **between** A and $\bar{A}$) must come

from the boundary of A , its bulk being too distant for sizes greater than the length scale m^{-1} , thus leading to S_A being proportional to the area of ∂A , the area law.

This intuition (decaying correlations imply area law) hasn't yet been proven to be correct for general QFTs, however, there are other strong indications of its validity:

- A complete proof for systems in one spatial dimension was given in [26].
- Known violations of the area law, where the entropy is proportional to $l^{d-1} \log l$ (l a characteristic length of the region), appear exactly when the system is gapless, such as in critical points or in the presence of a Fermi surface [22]. In these gapless systems, the decay of correlations is **algebraic** instead of exponential [25] and, following our argument, this allows the bulk of A to contribute to the entropy in a non-negligible way, leading to the logarithmic correction to the area law.
- Finally, the authors in [21] find an almost-exponential decay of entanglement (measured by the mutual information of regions A and B which are not complementary) via the use the algebraic formalism and provide approximate bounds as to how this property would lead to the area law.

With this, we conclude our exploration of entanglement in real space for QFTs and of the techniques involved, many of which will be used later. As the last item above mentions yet again the algebraic framework of field theories, this will be the topic (quickly) discussed in the next section.

2.5 Comments on the Algebraic Formalism

Up to this point, the algebraic point of view of quantum physics was already mentioned several times, mainly in reference to ideas it inspires and its use for mathematically rigorous calculations in QFT. The goal of this section is to explain

what it consists of and present a result which will be conceptually key to the investigation of entanglement in momentum space later in this Thesis.

In practice, no time will be spent on the mathematical details of the formalism and they won't be used at all in the Thesis, interested readers are referred to Haag's seminal book [23] for the physical motivation and meaning of this approach and [27], [28], for a proper mathematical treatment.

When investigating entanglement in a quantum system, which from now on will be taken as the ground state $|0\rangle$ of a QFT, one of the key objects to be studied is the reduced density operator of the subsystem in question $\rho_A \equiv \text{Tr}_B |0\rangle \langle 0|$. The definition of the partial trace Tr_B assumes the Hilbert space $\mathcal{H}$ of the system has a tensor product structure $\mathcal{H} = \mathcal{H}_A \otimes \mathcal{H}_B$, however, as discussed earlier, the essential characteristic of a reduced density operator (which does not depend on a specific structure on $\mathcal{H}$) is that for any operator $\mathcal{O}_A$ acting non-trivially only on A , the equality $\text{Tr}_A \rho_A \mathcal{O}_A = \langle 0 | \mathcal{O}_A | 0 \rangle$ is valid.

This fact implies that a subsystem may be defined in terms of the observables which act on it and, as explained in [23], the **algebra** of observables contains all the necessary information. Then, in the algebraic framework, a subsystem is equivalent to a subalgebra of the full algebra of observables and the relation between the two can be used to define both ρ_A and its respective entanglement entropy, see [29] where the authors show this in detail (and explain how this allows us to talk about entanglement of fermions while avoiding unphysical conclusions due to the Pauli exclusion principle), as well as [9]. The paper [30] also discusses this approach in the case of gauge fields, which were avoided here for their subtleties.

The idea that subsystems and subalgebras are intimately connected was already implicit earlier in the Chapter, when composite quantum systems were introduced, but from the generalized point of view it can also be applied to systems where there is no proper TPS: it's known that Hilbert spaces of QFTs cannot be written as a tensor product of local spaces associated with spatial regions [23],

this is a consequence of the infinite degrees of freedom infinitely close to each other in a QFT and in practice can only be dealt with a lattice regularization which will present UV divergences (see the real-space entropy previously calculated).

A final advantage of using these ideas (even though the mathematical details will be ignored) is that in the path integral formalism is much simpler to find what constitutes a subalgebra of observables then to prove that a Tensor Product Structure exists in some regularized version of the theory. In the next Chapter, momentum-space entanglement of QFTs will be introduced (more specifically, the entanglement between modes with high and low momentum, separated by an arbitrary scale μ) and in this context the relation $\text{Tr}_A \rho_A \mathcal{O}_A = \langle 0 | \mathcal{O}_A | 0 \rangle$ can be proven almost trivially with the path integral, allowing for a precise definition of ρ_A .

Chapter 3

Renormalization Group and Entanglement in Momentum Space

The Renormalization Group (RG) is a set of ideas which came to fruition in the second half of the 20th century, having its origins in both high-energy physics and condensed matter theory. At the heart of the concept is a simple observation: when studying the **long distance** or **low momentum** behavior of a system (which, by the Fourier transform, are the same thing), all physical observables - the expectation values - must be the same regardless of the description used for the short distance/high momentum sector. By implementing this reasoning in the formalism of Quantum Field Theory and Statistical Mechanics, fundamental insights were obtained as to the nature of phase transitions and QFTs themselves.

Now, the reason to discuss the RG in this Thesis is that, by its very definition, renormalization involves restricting our attention only to the observables of a subsystem characterized as the long distance degrees of freedom, therefore, in a quantum mechanical system this naturally leads to the appearance of a reduced density matrix and (in the case of pure states) to an entanglement entropy between the variables considered and those ignored. Concretely speaking, the Hilbert space of a QFT can be written as a tensor product of spaces for low and high momentum modes, the scale of separation μ being arbitrary (much like the shape of region A is arbitrary in the study of real-space entanglement).

Such Tensor Product Structure can be related to Kenneth Wilson's definition of renormalization and as the RG is such a powerful way of understanding field

theories, this leads to the hope that by exploring its information-theoretical aspects, the entropy in particular, we may arrive at new insights about the non-trivial/interacting QFTs.

Thus, the structure of this Chapter is as follows: first, Wilsonian renormalization will be described in detail and the concept of the RG flow and its (formally) complete implementation in the Functional Renormalization Group will be introduced. Then, entanglement in momentum-space for field theories will be properly defined (the path integral allows for the characterization of a low-momentum reduced density matrix ρ_μ) alongside an argument as to why most interacting theories must display non-zero entanglement entropy when the Hilbert space partitioned into modes at different momentum scales. Finally, each step of the RG will be fully translated to the language of quantum information, this will allow us to understand any Renormalization Group transformation as a map between density matrices and will lead to the first major result of this Thesis: an argument showing that fixed points of the RG have no entanglement between different momentum scales (and thus are separable with respect to this partition). Both the understanding of Wilson's RG as a map of density matrices and the associated result for fixed points are original contributions from this work.

A final comment before moving forward: everything done here is in the context of quantum systems and QFTs; in particular, the Renormalization Group's immense value in statistical mechanics will not be explored in this Thesis. Interested readers are referred to the excellent books [31] and [32].

3.1 Wilsonian Renormalization

As mentioned above, Wilson's RG is born naturally from the idea that in many contexts we're only interested in looking at the behavior of a physical system at long distances or low momenta.

A seminal example is the study of the phase diagram and thermodynamics

of some theory, as macroscopic phases are defined by observables accessible at scales large compared to those of the microscopic dynamics (these observables can be order parameters in the Landau paradigm of symmetry breaking or more subtle, but still global, quantities in topological phases); likewise in critical phenomena, where fluctuations extend from the short to the large distances and in the latter experiments find scaling and universal behaviors. Even when starting from a high-energy physics point of view this focus on distances above some cut-off (which is inversely proportional to the maximum momentum scale of interest) is physically relevant: after all, experiments in some particle collider or equivalent only have a finite resolution associated to the momentum imparted to the particles and the detectors are transparent to excitations which are too energetic.

Therefore, following this reasoning we may take as starting point a Lagrangian or Hamiltonian which is known to describe the microscopic theory of interest and look for the **effective dynamics** of the long-distance variables, under the condition that they **reproduce the same expectation values as the full theory** for the observables at these scales (there's already a glimpse of the quantum information aspects of this procedure, as we're restricting observables to a subsystem, later in this Chapter more details will be given).

There are many ways to implement this idea practically, the condition of full and effective theories returning the same results for the long distance observables being often referred to as "insensitivity to the high-energies" (see [33] and [34] for an introduction to some **renormalization schemes**, as the different implementations are called). Wilson's realization [35] in **momentum space** (from now on we'll speak of low and high momenta instead of long and short distances) is based on the fact that all expectation values in a field theory may be obtained from the partition function:

$$Z = \int \prod_k \mathcal{D}\phi_k e^{-S_E[\phi_k]} \quad (3.1)$$

Where the path integral in Wick-rotated time was taken and $S_E[\phi_k]$ is the euclidean action describing the interaction between field modes of all momenta (ϕ_k

here is meant to be very generic and can in principle represent any microscopic variables of the theory).

Then, to study the physics at the sector with momentum lower than some scale μ , the modes with $|k| > \mu$ are only of interest in the way they affect the remaining ones and, as the action for a set of fields is sufficient (alongside the path integral) to describe their behavior, the effective dynamics of the low-momentum variables is determined by **integrating out completely** the field modes with $|k| > \mu$ while maintaining those with $|k| < \mu$ untouched. With this, the **Wilsonian effective action** $S_\mu[\phi_{|k| \leq \mu}]$ (not to be confused with the generating function of the one-particle irreducible correlators, constructed by a Legendre transform of $\log Z$) is defined by the equation:

$$e^{-S_\mu[\phi_{|k| \leq \mu}]} := \int \prod_{|k| > \mu} \mathcal{D}\phi_k e^{-S_E[\phi_k]} \quad (3.2)$$

Thanks to the interactions between different momenta, $S_\mu[\phi_{|k| \leq \mu}]$ contains new terms when compared to the form of $S_E[\phi_k]$ and so it captures the influence of the high-momentum modes. Furthermore, if the remaining variables are also integrated out:

$$\int \prod_{|k| \leq \mu} \mathcal{D}\phi_k e^{-S_\mu[\phi_{|k| \leq \mu}]} = \int \prod_{|k| \leq \mu} \mathcal{D}\phi_k \int \prod_{|k| > \mu} \mathcal{D}\phi_k e^{-S_E[\phi_k]} = Z \quad (3.3)$$

Thus the original partition function of the theory is reproduced and this is sufficient to guarantee that the low-momentum observables are given the same expectation values in both approaches.

3.1.1 Steps of the RG

In the first part of this Chapter, the physical intuition behind the Renormalization Group was given as well as Wilson's insight as to how to implement it (historically, he was inspired by Kadanoff's blocking procedure in statistical mechanics, see [31], [35]). However, the full RG involves more than what was shown

above, as it also allows for the **comparison between theories defined at the same scale**. The goal now is to explain all steps and discuss their implications.

Step 1: Integrating out the high-momentum degrees of freedom

This part of the procedure was already explained above when connecting the basic idea of the RG to the partition function of a QFT: by performing first the full path integral over certain variables, the interaction between field modes in the original action leads to the appearance of a Wilsonian effective action for the low-momentum variables and thanks to this effective action the long distance observables/expectation values may be calculated without reference to the degrees of freedom which are not being measured. In the specific case of integrating part of the momenta, this is also called **coarse-graining**.

Now, assume that the QFT being studied has an overall momentum cutoff (a maximum momentum scale for the field modes) a^{-1} . In the context of condensed matter physics a is naturally given by (or proportional to) the minimal distance between lattice sites, in high-energy physics the idea is to eventually take $a^{-1} \rightarrow \infty$, but either way it is present when first defining the theory.

With this cutoff, we write the original euclidean action as $S[\phi_k]$ and the integration of the high-momentum modes up to scale μ is given by:

$$e^{-S_\mu[\phi_{|k|\leq\mu}]} = \int \prod_{a^{-1} \geq |k| > \mu} \mathcal{D}\phi_k e^{-S[\phi_k]} \quad (3.4)$$

The overall cutoff serves to guarantee that the expressions are properly convergent (the path integral becomes finite-dimensional), but it also makes clear that, formally, the integration of part of the modes takes us from a QFT defined at scale a^{-1} to another one defined at μ . Another way of putting this fact: in principle a QFT for degrees of freedom up to momentum μ could be defined simply by choosing the action to be $S_\mu[\phi_{|k|\leq\mu}]$.

To make the last comment above clear, notice that, even for the simplest of cases, integrating out a part of the degrees of freedom in a path integral leads to

effective actions which contain terms non-local in space and time (for concrete examples, see the next Chapter). Physically, the reason for this is precisely the fact that short distances are being ignored: thanks to the coarse-graining, any interactions or fluctuations happening at distances smaller than μ^{-1} are averaged and become effectively instant.

However, for many practical purposes (and conceptual clarity) it's useful to make the derivative expansion of any non-local kernels (writing them as a Taylor series of differentiation operators). With this and a Fourier transform of the remaining field modes back to real space, the Lagrangian density associated with the Wilsonian effective action can be written generically as:

$$\mathcal{L}_\mu = \frac{1}{2}z(\mu)(\nabla\phi)^2 + \sum_{i=1}^{\infty} a_i(\mu)\phi^i + \sum_{i=3}^{\infty} b_i(\mu)(\nabla^2)\phi^i + \dots \quad (3.5)$$

That is, all terms involving powers of the field and its derivatives at any order are generated (the term proportional to $(\nabla\phi)^2$ was isolated for reasons which will be explained soon and $z(\mu)$ is not to be confused with the partition function Z). As a side comment, note that any symmetries of the original action will be present in the effective Lagrangian for the low-momentum field modes, after all, any violation would contradict the original symmetry and the Wilsonian effective action must give the same results as the original interaction for the calculation of any physical quantity.

With the equation above, we can see indeed that the Wilsonian effective action defines a new theory at scale μ . Based on this observation, the next steps of the Renormalization Group are introduced as a way to obtain from $\mathcal{L}_\mu$ a Lagrangian which defines a theory back at the original momentum cutoff a^{-1} . This theory will, in general, be different from the original model and so the RG will define a flow in the space of possible Lagrangians, but before entering this discussion, the next steps must be introduced.

Step 2: Rescaling the momentum and fields

In the previous step, an effective theory at momentum scale μ was obtained from the original QFT at scale a^{-1} , so in order to find a third action functional that can be properly compared to $S[\phi_k]$, the momenta in the effective action (equivalently, the derivatives in the Fourier-transformed Lagrangian) must be **rescaled** as $k \rightarrow \sigma k$, with $\sigma = (a\mu)^{-1}$, this rescaling also including the integration measure in the action. The original cutoff of the QFT must be recovered by a procedure of this form because the operation of integrating out the degrees of freedom with high momentum is not invertible (more comments will be made on this topic at the last section of the Chapter).

Besides the rescaling of momenta, the remaining field modes themselves must also be transformed:

- Mathematically, the reason for this is the fact that inside the path integral, ϕ_k is a dummy variable (as its being integrated over) and its multiplication by a constant doesn't change any physical quantity when properly handled, so in order to have a meaningful way of comparing theories at the same scale by comparing their Lagrangians/actions some convention must be imposed and this is what will be discussed further.
- From the physical point of view (that is, thinking of ϕ as an actual operator $\hat{\phi}$ which can be measured instead of a label being integrated over), the field ϕ has some unit or dimension which describes its spread over space (much like a density or probability amplitude) and since the transformation $k \rightarrow \sigma k$ implies lengths are being shortened as $l \rightarrow \sigma^{-1}l$, so must ϕ be scaled by some factor depending on σ and its dimension thanks to this "zooming out".

Understanding the need of rescaling correctly the field alongside the momentum, the convention imposed (and the one most commonly chosen) is that in the

Lagrangian the coefficient of the term $(\nabla\phi)^2$ is always $\frac{1}{2}$ and this is the reason this exact term was isolated previously.

To make the statement above more clear, note that prior to integrating out modes, the coefficient in the original Lagrangian is already set to $\frac{1}{2}$ and the dimension of ϕ can be determined by simple dimensional analysis to be some d_ϕ^0 (by taking the reduced Planck's constant as one, the action becomes adimensional and this is sufficient to determine d_ϕ^0 in terms of the length dimension) and in principle this leads to the transformation $\phi \rightarrow s\sigma d_\phi^0 \phi$, however, after integrating the fields with high momentum the coefficient of $(\nabla\phi)^2$ is changed and to impose the convention ϕ **must scale differently than dimensional analysis suggests** and instead it transforms as $\phi \rightarrow \sigma^{d_\phi} \phi$, with d_ϕ obtained by making $\phi \rightarrow z(\mu)^{-\frac{1}{2}} \phi$ and following with the scaling $\phi \rightarrow \sigma^{d_\phi^0} \phi$. This feature leads to the definition of the **anomalous dimension** η by the equation $d_\phi = d_\phi^0 + \frac{\eta}{2}$, so it's easy to see that η contains the effect of the change in dimensionality of ϕ due to the integration of modes.

Before moving on, it's important to make clear that despite the many mentions of the convention regarding the coefficient of $(\nabla\phi)^2$ in the Lagrangian, the appearance and value of the anomalous dimension is **not** a mere mathematical artefact: it is a real deviation of the scaling behaviors expected from dimensional analysis and it can be concretely measured in experiments, both in condensed matter and high-energy settings. Physically, a non-zero η is a manifestation of the fact that the interactions in a QFT make the fields behave in a qualitatively different manner than in the free theory case, where the dimension d_ϕ^0 is indeed correct.

Finally, with a full understanding of how the scaling step of the RG works, the new Lagrangian obtained is:

$$\mathcal{L}'_{(\sigma)} = \frac{1}{2}(\nabla\phi)^2 + \sum_{i=1}^{\infty} a'_i(\mu)\phi^i + \sum_{i=3}^{\infty} b'_i(\mu)(\nabla^2)\phi^i + \dots \quad (3.6)$$

Here the subscript (σ) indicates that the momenta had to be rescaled by $\sigma = (a\mu)^{-1}$. Furthermore, the associated action is denoted $S_{(\sigma)}$ and its partition function is given by:

$$Z_{(\sigma)} = \int \mathcal{D}\phi e^{-S_{(\sigma)}[\phi]} \quad (3.7)$$

When discussing the Functional Renormalization Group, the form of the action $S_{(\sigma)}$ in momentum space (the setting actually used in this Thesis) will be shown.

3.1.2 The Renormalization Group flow

In the previous section, the Wilsonian view of the RG was established as a transformation between Lagrangians $\mathcal{L} \rightarrow \mathcal{L}'(\mu)$ where $\mathcal{L}$ gives the dynamics of the full theory including all momenta up to the cutoff a^{-1} and $\mathcal{L}'(\mu)$ has the same cutoff (meaning it can be compared to $\mathcal{L}$) but is able to describe the original theory in the sector with momentum below μ thanks to the integration of field modes with high momentum used to generate it. Through the derivative expansion of any non-local terms generated by the integration at the first step (the rescaling preserves locality), any RG transformation from a^{-1} to μ translates to a map of the infinite-dimensional vector composed of the coefficients of all terms in the Lagrangians, this map depending on the initial coefficients in a very non-linear manner, of course.

Therefore, we have a **flow** in the space of possible Lagrangians (a dynamical system in the mathematical sense), parameterized by μ and such that at any point the physics at momentum scales below μ is preserved (the variable $t := -\log(a\mu)$ is frequently introduced as parameter instead of μ).

This property is the source of the power of the Renormalization Group: by studying the flow at small or large μ , the UV and IR limits of the original theory can be studied and these regimes can be connected, and as the asymptotics of dynamical systems are well studied (see [36] for an introduction) they can be used

to describe these extremes in a simpler manner while also explaining phenomena like universality in critical systems [31], [32].

Dynamical systems can have many different behaviors, but the most widespread in the RG evolutions are the **fixed points**, which can be **attractors** or **repellers** (so-called limit cycles can also appear in some contexts [37], though they will not be discussed here). Fixed points of the RG are theories whose action remain invariant under a combination of integrating out degrees of freedom and rescaling the remaining ones and by linearizing the flow in the space of possible Lagrangians near these points, the appearance of universal behavior in many different contexts is explained.

Explaining properly the physics of theories near a fixed point and how the RG transformation can behave near them would deviate too much from the objectives of this Thesis, readers are referred instead to Wilson's and Kogut's classical review [35] for a deep discussion while a very brief summary will be given here:

Theories lying at a fixed point (rather abstract mathematical objects) possess **exact scale symmetry**, meaning that the physics are the same at all distances and, despite interactions, the degrees of freedom of all wavelengths follow the same behavior. Though these theories are never realized exactly in nature, the fact that the RG flow stops at them implies that physical Lagrangians (which don't have exact scale symmetry) near these points will share many characteristics in common in some momentum scale (ultraviolet if the theory flows **away** under RG transformations, infrared if it flows **towards** the fixed point).

In particular, this is what explains the physics of the critical point of a phase transition, since at criticality the Lagrangian is fine-tuned to lie in a region flowing towards a fixed point (its **basin of attraction** named the **critical manifold** associated to some phase transition) and so all systems in this regime will share similar infrared behavior as long as they approach the same point.

In this Thesis' objective of studying entanglement between momentum degrees of freedom, fixed points of the RG will be briefly analysed at the end of the Chapter and an interesting property will be derived, with an exploration of physical theories at their critical points left as one of the directions this work can be taken going forward.

Functional Renormalization Group:

As we'll see in Chapter 4, the exact form (or at least a justified approximation) of the effective action obtained from the RG steps is crucial to derive the entanglement between momentum scales. With this in mind, it's useful to spend a short time discussing the Functional Renormalization Group (also called Exact RG).

Like renormalization itself, the Functional RG has many different formulations and applications (see the recent review [38]), here the focus will be on the **sharp cutoff** version of [39], also motivated in [35], as it will be the one to be directly related to a partial trace in momentum space.

In a nutshell, the FRG is focused on deriving **exact** equations for the variations in the action at each step of the Renormalization Group. The way this is done is by making the steps involve only an infinitesimal shell of momenta between scales μ and $\mu - d\mu$ and arriving at a functional differential equation whose initial condition can be taken as the original action of the theory.

Denoting by $\frac{dS_\mu}{d\sigma}$ the variation due to infinitesimal rescaling and $\frac{dS_\mu}{d\mu}$ the variation of the Wilsonian effective action by integrating out the infinitesimal shell, the total change will be $\dot{S}_\mu = \frac{dS_\mu}{d\sigma} + \frac{dS_\mu}{d\mu}$, with [39]:

$$\frac{dS_\mu}{d\sigma} = - \left(\int \frac{d^{d-1}k}{(2\pi)^{d-1}} \varphi_{\vec{k}} \vec{k} \cdot \nabla_{\vec{k}} \frac{\delta}{\delta \varphi_{\vec{k}}} + d_\phi \int \frac{d^{d-1}k}{(2\pi)^{d-1}} \varphi_{\vec{k}} \frac{\delta}{\delta \varphi_{\vec{k}}} \right) S_\mu \quad (3.8)$$

$$\frac{dS_\mu}{d\mu} = \frac{1}{2} \int_{|\vec{k}|=\mu} \log \left(\frac{\delta^2 S_\mu}{\delta \varphi_{\vec{k}} \delta \varphi_{-\vec{k}}} \right) - \frac{1}{2} \int_{|\vec{k}|=\mu} \frac{\delta S_\mu}{\delta \varphi_{\vec{k}}} \left(\frac{\delta^2 S_\mu}{\delta \varphi_{\vec{k}} \delta \varphi_{-\vec{k}}} \right)^{-1} \frac{\delta S_\mu}{\delta \varphi_{-\vec{k}}} \quad (3.9)$$

Where $\int_{|\vec{k}|=\mu}$ denotes the integration only over momenta $\vec{k}$ with fixed modulus μ and the spatial dimension considered is $(d - 1)$.

If the Wilsonian effective action $S_\mu[\phi_{\vec{k}}]$ is written formally as a sum of powers of $\phi_{\vec{k}}$, its expression becomes:

$$S_\mu[\phi_k] = \sum_{i=0}^{\infty} \int_{k_1} \dots \int_{k_i} \delta(k_1 + \dots + k_i) g_i(k_1, \dots, k_i) \phi_{k_1} \dots \phi_{k_i} \quad (3.10)$$

The $g_i(k_1, \dots, k_i)$ are called the **coupling functions** and, unlike the derivative expansion in real-space, they can in principle have any dependence on the momenta generated by the RG flow.

These equations will not be used in practice in this Thesis, however, it's important to keep in mind their existence, as they might be conceptually useful and, thinking of further work, they are good tools to study some non-perturbative aspects of QFTs, I refer again to the review [38].

3.2 Entanglement in Momentum Space

It's time to introduce in detail the main object of study for the remainder of this Thesis: the entanglement between the degrees of freedom at low and high momenta in a QFT, separated by a (chosen) scale μ . This partition is intimately related to the Wilsonian RG as first pointed out in [40] and throughout the rest of this Thesis this insight will be developed further, beyond what is in the literature so far.

3.2.1 Degrees of freedom at different momentum scales

When introducing the wavefunctional for the ground state of a QFT (see section 2.4 for more details), the field operator $\hat{\phi}(x)$ was formally diagonalized by the equation $\hat{\phi}(x) |\phi(\vec{x})\rangle = \phi(\vec{x}) |\phi(\vec{x})\rangle$, where the eigenvectors $|\phi(\vec{x})\rangle$ are understood as states in which the field operator takes a **definite value at each point**

in space and with the resulting configuration being described by the function $\phi(\vec{x})$. Furthermore, this is a basis of **separable** elements as there are no correlations between the field values at different points and thus it is ideal to define and calculate entanglement between the degrees of freedom (heuristically, the field variable at each point $\vec{x}$) in different regions of space, as was done in the end of the previous Chapter.

The idea, then, is to now do the same procedure with the Fourier transform of the field, the operator $\hat{\phi}_{\vec{k}}$: it is diagonalized by vectors of the form $|\phi_{\vec{k}}\rangle$ which, much like in the real-space case, represent states in which **every Fourier component of the field configuration takes on a definite value**, the values for different momenta being independent of each other.

With this, the Hilbert space of the full QFT is spanned by the set of all $|\phi_{\vec{k}}\rangle$ and, taking the ground state $|0\rangle$, its wavefunctional in this momentum-space basis $\psi_0[\phi_{\vec{k}}]$ is given by the path integral expression:

$$\psi_0[\phi_{\vec{k}}] := \langle \phi_{\vec{k}} | 0 \rangle = \frac{1}{\sqrt{Z}} \int_{\tau=-\infty}^{\varphi_{\vec{k}}(\tau=0^-)=\phi_{\vec{k}}} \mathcal{D}\varphi_{\vec{k}}(\tau) e^{-S[\varphi_{\vec{k}}(\tau)]} \quad (3.11)$$

Where $S[\varphi_{\vec{k}}(\tau)]$ is the action of the QFT written in terms of the Fourier-transformed field.

Before moving forward, some comments are in order:

- First, the path integral expression written above is perfectly valid and does lead to the wavefunctional desired, this is because Feynman's construction of the transition amplitude $\langle \phi_{\vec{k}} | 0 \rangle$ can be reproduced step by step using the canonically-conjugate variables $\phi_{\vec{k}}$ and $i\partial_{\tau}\phi_{\vec{k}}$ and evolving them in (imaginary) time with the system's Hamiltonian (which leads to the appearance of exactly the action $S[\varphi_{\vec{k}}(\tau)]$).
- Secondly, note that throughout this discussion, the Fourier-transformed field considered is labeled by the **spatial momentum** $\vec{k}$ without mention of the

component associated with the time variable. This is because the actual degrees of freedom in a QFT are spread in space with time indicating their dynamics instead of introducing new variables (this can also be seen in the canonical formalism, in which the phase space does not include fields labeled by time). Another way of seeing this is through the expression for the wavefunctional, where the imaginary time is present merely as a trick to projecting the basis vector $|\phi_{\vec{k}}\rangle$ into the ground state. The main consequence of this fact for our purposes is the fact that when making a connection to Wilsonian renormalization, a slight modification will be made in the procedure.

- Finally, one may wonder about how well defined is the diagonalization of $\hat{\phi}_{\vec{k}}$ as it is complex even in the case of $\hat{\phi}(x)$ the real scalar field and may not commute with its Hermitian conjugate. There's no reason for concern because the same question appears in the calculation of the wavefunctional of a complex scalar field in real space and the solution in both cases is to actually diagonalize some operator which acts on the same Hilbert space as $\hat{\phi}_{\vec{k}}$, like the (Hermitian) linear combinations $\frac{1}{2}(\hat{\phi}_{\vec{k}}^\dagger + \hat{\phi}_{\vec{k}})$ and $\frac{1}{2i}(\hat{\phi}_{\vec{k}}^\dagger - \hat{\phi}_{\vec{k}})$. This avoids any problems and the restriction to operators which act in the same space as the field guarantees that the degrees of freedom considered are still the same (the notion of diagonalizing $\hat{\phi}_{\vec{k}}$ is still maintained in the notation and at the heuristic level for convenience).

As mentioned earlier, the state vectors $|\phi_{\vec{k}}\rangle$ are separable according to the partition of the Hilbert space $\mathcal{H}$ of the QFT as $\mathcal{H} = \bigotimes_{\vec{k}} \mathcal{H}_{\vec{k}}$ and this can be seen from the fact that field amplitudes of different momenta are uncorrelated. The Tensor Product Structure $\bigotimes_{\vec{k}} \mathcal{H}_{\vec{k}}$ may be thought of as the occupation number representation of the Fock space constructed by the QFT's single-particle states (it's well known that a Fock space of identical particles is isomorphic to an ordinary tensor product of the mode occupations, [41] has a discussion on this fact and the associated consequences for entanglement), the introductory parts of [40] explore

this point of view, though the TPS introduced via the path integral can be more general.

Now, as the Hilbert space is partitioned into a tensor product of components labeled by momentum, the entanglement between subsystems made of these degrees of freedom may be discussed, in particular, this study for the **ground states** of QFTs is specially interesting, both for these states' intrinsic physical interest and for the fact that a path integral expression for their wavefunctionals is already available. Based on the Renormalization Group ideas introduced in the beginning of this Chapter, it becomes almost natural to partition the momenta by **scale**, that is, take $\mathcal{H} = \mathcal{H}_{|\vec{k}| \leq \mu} \otimes \mathcal{H}_{|\vec{k}| > \mu}$ by considering the subsystems to be the degrees of freedom whose momenta have modulus below and above the arbitrarily chosen scale μ .

From this, drawing once again from the path integral representation of a reduced density matrix in real-space (see section 2.4), the partial trace over the high-momentum modes (defined here as those such that $|\vec{k}| > \mu$) leads to matrix elements given by:

$$\langle \phi_{\vec{k}} | \rho_{\mu} | \tilde{\phi}_{\vec{k}} \rangle = \frac{1}{Z} \int_{-\infty}^{\varphi_{\vec{k}}(0^-)=\tilde{\phi}_{\vec{k}}}^{\varphi_{\vec{k}}(0^-)=\tilde{\phi}_{\vec{k}}} \int_{\varphi_{\vec{k}}(0^+)=\phi_{\vec{k}}}^{\infty} \prod_{|\vec{k}| \leq \mu} \mathcal{D}\varphi_{\vec{k}}(\tau) \prod_{|\vec{k}| > \mu} \mathcal{D}\varphi'_{\vec{k}}(\tau) e^{-S[\varphi_{\vec{k}}, \varphi'_{\vec{k}}]} \quad (3.12)$$

Where ρ_{μ} is the reduced density matrix of the low-momentum degrees of freedom obtained from the ground state $|0\rangle$ and the boundary conditions of the path integral should be read as containing the discontinuity at $\tau = 0$ only for the $\varphi_{\vec{k}}$ variables (those with momentum below μ) while the $\varphi'_{\vec{k}}$ are completely integrated over as when calculating the partition function Z .

The equation above initially seems as intractable as in the real-space case (perhaps even more, since the action in momentum-space is non-local due to translation symmetry), but here there's a fundamental insight not available when calculating entanglement measures between regions of space: the total integration of field modes with momentum above μ is **exactly** what is done in the first step

of the Renormalization Group and since this part of the procedure generates a Wilsonian effective action at scale μ , denoted from here on out by S_μ , the matrix elements of ρ_μ are also given by (leaving implicit the fact that all remaining variables have $|\vec{k}| \leq \mu$):

$$\langle \phi_{\vec{k}} | \rho_\mu | \hat{\phi}_{\vec{k}} \rangle = \frac{1}{Z} \int_{-\infty}^{\phi_{\vec{k}}(0^-)=\hat{\phi}_{\vec{k}}}^{\phi_{\vec{k}}(0^+)=\hat{\phi}_{\vec{k}}} \int_{\phi_{\vec{k}}(0^+)=\hat{\phi}_{\vec{k}}}^{\infty} \mathcal{D}\phi_{\vec{k}}(\tau) e^{-S_\mu[\phi_{\vec{k}}]} \quad (3.13)$$

Thus, if there's entanglement between different momentum scales, and so, non-zero $S_{EE}(\rho_\mu) = -\text{Tr}(\rho_\mu \log \rho_\mu)$, the operator ρ_μ will be a mixed state and this will be determined by the Wilsonian effective action S_μ alone.

In other words, the expression for S_μ alone contains all details about whether the density matrix it generates describes a pure state or a mixture and so any entanglement measure (in this partition) can, in principle, be derived from the effective action. Furthermore, the effective action's relation to the RG gives weight to the idea that quantum information techniques applied to momentum-space might be a useful way of learning more about the nature of the ground states of QFTs and their phase transitions.

As a technical side note, it's important to make clear that the momenta integrated out to generate S_μ have to obey the condition $|\vec{k}| > \mu$ which relates only to the **spatial** components of the momentum four-vector and with the euclidean k_0 being unrestricted during integration. As explained in the comments above, this is because the true degrees of freedom are identified by $\vec{k}$ and k_0 appears in the path integral merely as a consequence of the use of the euclidean time τ to project the theory into its ground state.

The act of treating $\vec{k}$ and k_0 differently does not affect the physical meaning (and consequences) of renormalization and is just a specific scheme for implementing the RG. However, this separation **does** violate Lorentz symmetry explicitly and the reason why such violation isn't problematic is because the very concept of high and low momentum is frame-dependent, so two observers in

relative motion may consider the same field configuration (say, a particle with definite velocity) to be in different sectors of the theory, thus leading to an expected discrepancy. These aspects of the momentum-space entanglement won't be relevant for this Thesis, if necessary, some specific frame of reference might be chosen (and in any applications for condensed matter, a preferred frame is given by the lattice).

3.2.2 Formulation through finite temperature field theory

Another way of finding (in principle) the matrix elements of the reduced density operator ρ_μ was given in [40] through the techniques of thermal QFT (an introduction to which is the book [42]). The reason to introduce this representation is that it is less complicated than the previous and I will use its specific properties to construct a completely new technique to calculate entanglement between momentum scales in the next chapter.

With the path integral, the matrix elements of the global thermal state of a QFT with inverse temperature β are given by:

$$\langle \phi_{\vec{k}} | \rho_\beta | \tilde{\phi}_{\vec{k}} \rangle = \frac{1}{Z(\beta)} \int_{\phi_{\vec{k}}(0)=\phi_{\vec{k}}}^{\phi_{\vec{k}}(\beta)=\tilde{\phi}_{\vec{k}}} \mathcal{D}\phi_{\vec{k}}(\tau) e^{-S^\beta[\phi_{\vec{k}}]} \quad (3.14)$$

Where all spatial momenta of the system are included, the imaginary time variable is restricted to the interval $[0, \beta]$ (leading to the action $S^\beta[\phi_{\vec{k}}] = \int_0^\beta d\tau L[\phi_{\vec{k}}]$) and the thermal partition $Z(\beta)$ is:

$$Z(\beta) = \int_{0 \leq \tau \leq \beta} \mathcal{D}\phi_{\vec{k}}(\tau) e^{-S[\phi_{\vec{k}}]} \quad (3.15)$$

The path integral being only over paths periodic under $\tau \rightarrow \tau + \beta$.

Through this formalism, renormalization can be performed at finite temperature and the partial trace over high momentum modes is calculated by identifying the boundary conditions at $\tau = 0$ and $\tau = \beta$ and summing over all periodic paths

in this interval **only** for the variables $\varphi_{\vec{k}}$ with $|\vec{k}| > \mu$, the remaining ones still having the conditions imposed by the matrix element being represented. Thus, this leads to an Wilsonian effective action at scale μ and temperature β :

$$e^{-S_\mu^\beta} = \int_{0 \leq \tau \leq \beta} \mathcal{D}\varphi_{|\vec{k}| > \mu} e^{-S^\beta[\varphi_{\vec{k}}]} \quad (3.16)$$

And, taking the zero temperature limit via $\beta \rightarrow \infty$, we get the expression:

$$\langle \phi_{\vec{k}} | \rho_\mu | \tilde{\phi}_{\vec{k}} \rangle = \lim_{\beta \rightarrow \infty} \frac{1}{Z(\beta)} \int_{\varphi_{\vec{k}}(0)=\phi_{\vec{k}}}^{\varphi_{\vec{k}}(\beta)=\tilde{\phi}_{\vec{k}}} \mathcal{D}\varphi_{\vec{k}}(\tau) e^{-S_\mu^\beta[\varphi_{\vec{k}}]} \quad (3.17)$$

Due to the periodicity of the fields and bounded interval for the imaginary time shown here, taking the system's temperature to be finite during calculations allows the path integral to be manipulated more easily and, as will be shown in Chapter 4, is key to constructing a method to calculate measures of entanglement between degrees of freedom above and below an arbitrary scale μ .

3.2.3 Correlations in momentum-space for interacting theories

The discussion has so far been very abstract, with the entanglement between momentum scales somehow being encoded in the Wilsonian effective action. As a remedy to this situation, an example will now be given to show that such entanglement is already present in the usual calculations done in QFT, with the crux of the argument being equation (2.42) in Chapter 2 - a quantitative realization of the fact that all correlations between subsystems which are in a joint pure state **must** come from entanglement.

For this end, consider the $\lambda\phi^4$ model with ϕ a real scalar field and the coupling λ small enough that perturbation theory may be applied. Then, as is well known, the euclidean action in real space is given by:

$$S[\phi] = \int d^d x \left[\frac{1}{2} \partial_\nu \phi(x) \partial^\nu \phi(x) + \frac{1}{2} m^2 \phi^2(x) + \frac{\lambda}{4!} \phi^4(x) \right] \quad (3.18)$$

With the space-time dimension d left arbitrary.

As the focus here is on the field modes in momentum space, taking the Fourier transform of $\phi(x)$, the action can be easily shown to become:

$$S[\phi] = \int \frac{d^d k}{(2\pi)^d} \left[\frac{1}{2}(k^2 + m^2)\phi_k^* \phi_k + \frac{\lambda}{4!} \int \frac{d^d p}{(2\pi)^d} \int \frac{d^d q}{(2\pi)^d} \phi_k^* \phi_p^* \phi_q \phi_{k+p-q} \right] \quad (3.19)$$

Now, when $\lambda = 0$, we have the free theory with a Fock space structure whose ground state is given by the separable vector $|0\rangle = \bigotimes_{\vec{k}} |0\rangle_{\vec{k}}$ (each $|0\rangle_{\vec{k}}$ indicates no occupation of the single-particle state with momentum $\vec{k}$), this separability can also be seen directly from the wavefunctional (a product of Gaussian functions, one for each momentum, see [43]) and from the fact that integrating out field modes with momentum above μ does not modify the action for the remaining ones, thus implying the reduced density matrix is a pure state and so $|0\rangle$ presents no entanglement with respect to this partition.

By making λ non-zero and small the ground state is no longer known, but in this setting it's possible to calculate the correlator $\langle \phi_k \phi_p \phi_q \phi_{-k-p-q} \rangle$ to first order in λ and choosing the momenta to be all different and such that $|\vec{k}|, |\vec{p}|, |\vec{q}| \leq \mu$ and $|\vec{k} + \vec{p} + \vec{q}| > \mu$. This expectation value is guaranteed to be zero in the free theory as can be seen by applying Wick's theorem or simply from the fact that without interactions there can be no scattering in the model.

Applying the standard methods of perturbation theory for the interacting case (ignoring some known numerical factors which are irrelevant for the discussion), we will calculate the correlation function at lowest order, with the starting point being the path integral:

$$\langle \phi_k \phi_p \phi_q \phi_{-k-p-q} \rangle = \frac{1}{Z} \int \mathcal{D}\phi \phi_k \phi_p \phi_q \phi_{-k-p-q} e^{-S[\phi]} \quad (3.20)$$

Separating the free action $S_0[\phi]$ in the exponential and expanding the interaction at first order in the coupling, the $\mathcal{O}(\lambda^0)$ is automatically zero as the momenta of the fields where chosen to be different and none of the delta functions generated by contractions are satisfied. Going to $\mathcal{O}(\lambda)$, the expectation value is proportional

to:

$$\frac{1}{Z} \int \mathcal{D}\phi_{\tilde{k}} \int \prod_{i=1}^4 d^d \tilde{k}_i \delta(\sum_{i=1}^4 \tilde{k}_i) \phi_{\tilde{k}_1} \phi_{\tilde{k}_2} \phi_{\tilde{k}_3} \phi_{\tilde{k}_4} \phi_k \phi_p \phi_q \phi_{-k-p-q} e^{-S_0[\phi]} \quad (3.21)$$

Contracting the fields via Wick's theorem and using the momentum integrals, the only non-zero contribution comes from contracting each of the fields with momentum $\tilde{k}_i$ with one of the field modes we are measuring, producing one propagator for each of the four momenta $k, p, q, -k - p - q$ (multiplied by a symmetry factor we are ignoring). With this, the momentum delta function in the equation above automatically becomes $\delta(0)$, which is proportional to VT when the theory is inserted in a spatial box of volume V and limited to a time interval of duration T .

At the end of this calculation, we then find that the correlator of these four field modes is, at lowest order in λ , proportional to:

$$\langle \phi_k \phi_p \phi_q \phi_{-k-p-q} \rangle \propto \lambda VT \frac{1}{k^2 + m^2} \frac{1}{p^2 + m^2} \frac{1}{q^2 + m^2} \frac{1}{(k + p + q)^2 + m^2} \quad (3.22)$$

The appearance of the space-time volume VT is physically intuitive, as the Fourier-transformed fields are delocalized and thus their correlations have contributions from each point. In fact, this property of the momentum modes will be shown in Chapter 4 to lead to a **volume law** for the entanglement between scales, as opposed to the **area law in real space** (the partitions of the degrees of freedom of a QFT in real and momentum spaces define different Tensor Product Structures and so their entropies are not guaranteed to be directly related to each other).

Also, note that this expectation value is by itself already the **truncated** correlation function of these operators (when focusing on observables at low and high momentum), because, by the condition imposed on the momenta, the two correlators only differ by a product of $\langle \phi_k \phi_p \phi_q \rangle$ with $\langle \phi_{-k-p-q} \rangle$ and both terms are zero as they are Gaussian averages of odd numbers of fields.

For the interests of this Thesis, the quantity which should be actually calculated is $\langle \phi_{\tilde{k}}(\tau) \phi_{\tilde{p}}(\tau) \phi_{\tilde{q}}(\tau) \phi_{-\tilde{k}-\tilde{p}-\tilde{q}}(\tau) \rangle$, which would introduce more complexities

in the calculation but would **not** change the main conclusion that **in the interacting theory there are non-trivial correlations between observables acting on different momentum scales**. Therefore, from equation (2.42), this proves that **this interaction leads to entanglement between low and high momentum degrees of freedom**.

In particular, as $S_{EE}(\rho_\mu) \gtrsim \langle \phi_{\vec{k}}(\tau) \phi_{\vec{p}}(\tau) \phi_{\vec{q}}(\tau) \phi_{-\vec{k}-\vec{p}-\vec{q}}(\tau) \rangle^2$ one might expect the entanglement entropy calculated to be, at lowest order, proportional λ^2 . As we'll see in the next Chapter, that's not the case due to the nature of the von Neumann entropy.

Obviously the procedure shown here is done in a perturbative setting, but the lesson is broader: most theories with interactions can be proven, in one way or another, to lead to non-zero truncated correlation functions between an observable in the low momentum and one in the high momentum sectors and this in turn guarantees that there will be entanglement with respect to this partition.

One note of warning, however, is that the partition is heavily dependent on the field variables chosen to describe the QFT, in other words, **the partition of the Hilbert space changes if some change of variables is performed** (that's exactly what happened when going from the real-space field $\phi(x)$ to the momentum modes $\phi_{\vec{k}}$ via the Fourier transform). For example, if the system's Hamiltonian can be diagonalized by introducing new fields as functions of ϕ , there will be no entanglement in the new partition as there will be no correlations (or interactions to generate them). This last comment being particularly pertinent for the next section.

3.3 A Quantum Information view of the RG

The way that the partition of the Hilbert space of a QFT as a tensor product of spaces of low and high momentum modes has been discussed until now relies on

the introduction of the state vectors $|\phi_{\vec{k}}\rangle$ and on the argument that the field amplitudes labeled by $\vec{k}$ are uncorrelated among themselves in these vectors, leading to the appearance of a Tensor Product Structure (since the $|\phi_{\vec{k}}\rangle$ are separable) which can then be divided into subsystems below and above a chosen scale μ .

There is, however, a more direct and physically relevant way of proving that the desired TPS indeed exists, as was explained by the authors in [40]. Their argument will be reproduced in this section and connected to the idea of partial trace as a restriction of the observables to a subsystem discussed in Chapter 2 (in particular, section 2.5).

Now, this will be yet another proof in this Chapter that the Wilsonian integration of short wavelength variables is equivalent to the tracing out of the associated degrees of freedom (and the practical calculation of entanglement measures of the reduced density matrix obtained by this process will be the focus of Chapter 4), but the tracing out by itself is only a part of the Renormalization Group. As explained earlier, the RG includes scaling transformations of the momenta and fields and another objective of this section is to show that this step also has a very clear interpretation as a **map between density matrices**.

Thus, we'll prove that the complete Wilsonian Renormalization Group has an equivalent interpretation as quantum information map, a so-called **quantum channel** and from this an interesting property of the RG fixed points will be derived.

3.3.1 Algebra of low-momentum observables

Throughout this Thesis we've been using the fact that the path integral allows for a representation of a system's ground state wavefunction (in the case of QFTs, wavefunctional). Thanks to this representation, the calculation of the vacuum (ground state) expectation value of any observable $\mathcal{O}$ in a QFT with euclidean

action $S[\phi]$ can be made with the equation:

$$\langle \mathbb{O} \rangle = \frac{1}{Z} \int \mathcal{D}\phi \mathbb{O} e^{-S[\phi]} \quad (3.23)$$

Which is nothing other than the identity $\langle \mathbb{O} \rangle = \langle 0 | \mathbb{O} | 0 \rangle = \text{Tr}(\rho \mathbb{O})$, with $\rho = |0\rangle \langle 0|$ the ground state density matrix.

This is implicitly the Schrödinger representation (as long as only the expectation value of time-independent observables are taken, as will be done here) introduced via the path integral. Focusing on field theories, this means that **any** observable is represented by a function of the representations ϕ and $i \frac{\delta}{\delta \phi}$ of canonically conjugate variables [43].

Therefore, following the authors in [40] and focusing on observables acting only on the sector of the theory with momentum below scale μ , they are all of the form:

$$\mathbb{O} = \mathbb{O} \left(\phi_{\vec{k}}, \frac{\delta}{\delta \phi_{\vec{k}}} \right) \quad (3.24)$$

Such representation of an operator is both valid mathematically and physically relevant, after all any detector or measuring device which can be built in a lab can resolve phenomena only up to a certain momentum scale/minimal distance, meaning its relevant aspects can be described by a function of the $\phi_{\vec{k}}$ and $\frac{\delta}{\delta \phi_{\vec{k}}}$ acting in its range of resolution.

Consequently, the expectation value of such a low-momentum observable is:

$$\langle \mathbb{O} \rangle = \frac{1}{Z} \int \mathcal{D}\phi_{\vec{k}} \mathbb{O}(\phi_{\vec{k}}, \frac{\delta}{\delta \phi_{\vec{k}}}) e^{-S[\phi_{\vec{k}}]} = \frac{1}{Z} \int \prod_{|\vec{k}| \leq \mu} \mathcal{D}\phi_{\vec{k}} \mathbb{O}(\phi_{\vec{k}}, \frac{\delta}{\delta \phi_{\vec{k}}}) e^{-S_{\mu}[\phi_{|\vec{k}| \leq \mu}]} \quad (3.25)$$

With the last equality a direct consequence of the fact that, as $\mathbb{O}$ does not act on the modes above scale μ , the path integral **automatically** traces them out.

This is nothing other than the identity $\langle \mathbb{O} \rangle = \text{Tr}(\rho_A \mathbb{O}_A) = \text{Tr}(\rho \mathbb{O}_A \otimes \mathbb{I})$ which characterizes completely the reduced density operator ρ_A of a subsystem A ! Thus, from the moment the path integral is written in terms of the Fourier-transformed fields $\phi_{\vec{k}}$ the Tensor Product Structure $\mathcal{H} = \bigotimes_{\vec{k}} \mathcal{H}_{\vec{k}}$ is established.

Note also that the TPS arises naturally in the form of restricting the algebra of observables to a subsystem, to use some concepts discussed in Chapter 2. The subsystem being, of course, the low-momentum degrees of freedom and the algebraic aspect coming from the fact that functions of $\phi_{\vec{k}}$ and its functional derivative (with $|\vec{k}| \leq \mu$) may be added, multiplied and combined as to form a great variety of operators, but they'll never act on any variable above scale μ .

3.3.2 The rescaling transformation

With the first step of the Renormalization Group firmly established as a partial trace over degrees of freedom with high momentum, it is now time to explore more the scaling transformation which follows. For the rest of this particular discussion, a^{-1} will be the overall cutoff of the QFT and the scaling parameter will be $\sigma := (a\mu)^{-1}$ (σ is the multiplication necessary to take the momenta back to range $[0, a^{-1}]$ after the modes above μ were integrated out).

So, drawing from the introduction to RG earlier in this Chapter, the rescaling of field modes is given by:

$$\vec{k} \rightarrow s\vec{k} \quad (3.26)$$

$$\phi_{\vec{k}} \rightarrow \sigma^{d_\phi} \phi_{\sigma\vec{k}} \quad (3.27)$$

Where d_ϕ is the appropriate dimension of the Fourier-transformed field.

From this, if $S[\phi_{\vec{k}}]$ is the original action of the QFT, let $S_{(\sigma)}[\phi_{\vec{k}}] := S_\mu[\sigma^{d_\phi} \phi_{\sigma\vec{k}}]$ denote the new action at scale a^{-1} obtained from the scaling transformation, with the momenta in the coupling functions of the action also rescaled. Then, by the path integral construction of matrix elements of a state operator, this action (and thus, the operation of rescaling the fields and momenta) will naturally define the density matrix $\rho_{(\sigma)}$ by:

$$\langle \phi_{\vec{k}} | \rho_{(\sigma)} | \tilde{\phi}_{\vec{k}} \rangle = \frac{1}{Z_{(\sigma)}} \int_{-\infty}^{\varphi_{\vec{k}}(0^-)=\tilde{\phi}_{\vec{k}}}^{\varphi_{\vec{k}}(0^+)=\phi_{\vec{k}}} \mathcal{D}\varphi_{\vec{k}}(\tau) e^{-S_{(\sigma)}[\varphi_{\vec{k}}]} \quad (3.28)$$

$$Z_{(\sigma)} = \int \mathcal{D}\phi_{\vec{k}}(\tau) e^{-S_{(\sigma)}[\phi_{\vec{k}}]} \quad (3.29)$$

It is important to note that this is a non-trivial transformation despite its simple definition, since it reintroduces degrees of freedom with momentum between μ and a^{-1} .

Also notice that there's no a priori reason to believe that $\rho_{(\sigma)} = |0\rangle \langle 0|$, where $|0\rangle$ is the ground state vector, as the action $S_{(\sigma)}[\phi_{\vec{k}}]$ will be generally different from the original $S[\phi_{\vec{k}}]$ and the process of obtaining an effective action by integrating part of the momentum modes can be inverted by a scaling transformation only at a fixed point of the RG (this will soon be explored further).

Now, the theory after rescaling will give the same results as the effective action S_μ for all correlation functions and observables restricted to momenta lower than μ . The way the results can be recovered is not by the usual partial trace of the first step of the RG, but rather via a more subtle analysis.

As explained brilliantly in the first sections of [44], both steps of a Renormalization Group transformation are guaranteed to preserve the **probability distributions** of the low-momentum degrees of freedom. The tracing out ensures this by definition (via the generation of effective dynamics for the remaining modes, in practice a marginal distribution), while the expectation values of any operator $\mathcal{O}(\phi_{\vec{k}}, \frac{\delta}{\delta\phi_{\vec{k}}})$ can be recovered from the average of an associated $\tilde{\mathcal{O}}(\phi_{\vec{k}}, \frac{\delta}{\delta\phi_{\vec{k}}}) := \mathcal{O}(\sigma^{d_\phi} \phi_{\sigma\vec{k}}, \sigma^{-d_\phi} \frac{\delta}{\delta\phi_{\sigma\vec{k}}})$ by the equation:

$$\langle \mathcal{O} \rangle = \frac{1}{Z} \int \prod_{|\vec{k}| \leq \mu} \mathcal{D}\phi_{\vec{k}} \mathcal{O}(\phi_{\vec{k}}, \frac{\delta}{\delta\phi_{\vec{k}}}) e^{-S[\phi_{\vec{k}}]} = \frac{1}{Z_{(\sigma)}} \int \mathcal{D}\phi_{\vec{k}} \tilde{\mathcal{O}}(\phi_{\vec{k}}, \frac{\delta}{\delta\phi_{\vec{k}}}) e^{-S_{(\sigma)}[\phi_{\vec{k}}]} \quad (3.30)$$

As can be seen from a change of variables in the integral.

Intuitively, the possibility of recovering all the expectation values at scale μ after the rescaling transformation should imply some straightforward relation between the entropy of the density matrix ρ_μ and that of $\rho_{(\sigma)}$, since "no information is being lost".

I will show that this intuition does indeed correspond to what happens, but first a short deviation is necessary in order to prove that the scaling transformation is well defined not only for ρ_μ , which is obtained via a partial trace, but also for **any density matrix acting on the low-momentum degrees of freedom**.

Any operator $\tilde{\rho}$ satisfying the defining properties of a density matrix (see Chapter 2) can be written in terms of a so-called **modular Hamiltonian**, a Hermitian operator $K_{\tilde{\rho}}$ [23], via:

$$\rho = \frac{1}{\text{Tr } e^{-K_{\tilde{\rho}}}} e^{-K_{\tilde{\rho}}} \quad (3.31)$$

Despite its name, $K_{\tilde{\rho}}$ does not have any of the restrictions of a physical Hamiltonian (locality, lower bound in the spectrum, and so on), but it can still be used in a construction of the path integral via the time-slicing procedure.

In more detail, we may write $K_{\tilde{\rho}} = \sum_{i=1}^N \frac{1}{N} K_{\tilde{\rho}_i}$, split the exponential in the equation above and (focusing on the case of the field modes below momentum scale μ) insert a basis of the eigenvalues of $\hat{\phi}_{\vec{k}}$ between each term, following the usual process until we arrive at:

$$\langle \phi_{\vec{k}} | \tilde{\rho} | \tilde{\phi}_{\vec{k}} \rangle = \frac{1}{Z_{\tilde{\rho}}} \int_{\phi_{\vec{k}}(0)=\tilde{\phi}_{\vec{k}}}^{\phi_{\vec{k}}(1)=\phi_{\vec{k}}} \mathcal{D}\pi_{\vec{k}}(\tau) \mathcal{D}\phi_{\vec{k}}(\tau) e^{-\int_0^1 d\tau \left[K_{\tilde{\rho}}(\pi_{\vec{k}}(\tau), \phi_{\vec{k}}(\tau)) - \pi_{\vec{k}}^*(\tau) \frac{d\phi_{\vec{k}}(\tau)}{d\tau} \right]} \quad (3.32)$$

Here, $\pi_{\vec{k}}$ is the momentum canonically conjugate to $\phi_{\vec{k}}$ and $Z_{\tilde{\rho}}$ is calculated by integrating over all periodic configurations of the low-momentum fields between $\tau = 0$ and $\tau = 1$ and is introduced, like the partition functions, to preserve the normalization of the operator.

With this construction the rescaling of momenta and fields can now be defined for **any** $\tilde{\rho}$ just as it was done for ρ_μ , leading to a $\tilde{\rho}_{(\sigma)}$ (in the path-integral, $\pi_{\vec{k}}$ scales exactly like $i \frac{\delta}{\delta \phi_{\vec{k}}}$, as the latter is just a representation of the former).

Furthermore, the map between density matrices defined by this transformation is an example of what is called a **quantum channel**, also known as Completely Positive Trace Preserving map (see section 8.2.4 of [45] for a definition of the term):

- It is trace-preserving by construction since the trace of the new density matrices is still 1 and linearity extends this property for the rest of the operators.
- It is positive as given any observable $\mathcal{O}$ with only positive eigenvalues, $\text{Tr}(\tilde{\rho}_{(\sigma)}\mathcal{O}) \geq 0$, since in the path-integral representation this is an average taken only over positive numbers.
- Finally, checking the **complete** positivity involves tensoring $\tilde{\rho}$ with an ancilla of arbitrary dimension and seeing if after the transformation the resulting operator is still positive, in our case this condition is satisfied trivially, as the rescaling transformation is defined to act only on the field degrees of freedom, irrespective of any ancilla.

Again I refer to [45], Chapter 8, for a more detailed discussion of the concept of quantum channel and complete positivity.

With this, it was proved that the rescaling map is well defined for any low-momentum density matrix and it is a completely positive and trace-preserving map. This is also a map which allows us to recover any original $\tilde{\rho}$ by applying a scaling of σ^{-1} , in technical terms the map **has a left inverse**. Now, the reason we went to such lengths to prove these previous statements is that the authors in [46] showed that this means $\tilde{\rho}_{(\sigma)}$ can be written as:

$$\tilde{\rho}_{(\sigma)} = U (\tilde{\rho} \otimes \omega) U^\dagger \quad (3.33)$$

Where U is an unitary transformation and ω a density matrix in the Hilbert space of the **new degrees of freedom introduced by the map**, that is, the modes between μ and a^{-1} which are restored.

Thus, using the fact that the von Neumann entropy of a density matrix is invariant under unitary transformation and that the entropy of a tensor product of density matrices is simply the sum of each individual entropy (two well known

properties [45], [9]), we conclude that:

$$S_{EE}(\tilde{\rho}_{(\sigma)}) = S_{EE}(\tilde{\rho}) + S_{EE}(\omega) \geq S_{EE}(\tilde{\rho}) \quad (3.34)$$

Thus **the scaling can at most increase the entropy of a density matrix** and this is a direct consequence of the intuitive idea that scaling transformations preserve the physics of the system: the entropy must behave monotonically under them.

3.3.3 Entanglement between scales at a fixed point

The Wilsonian Renormalization Group transformation (which, in terms of density matrices is the map $\rho \rightarrow \rho_{(\sigma)}$) has now been characterized in the language of quantum information theory as the composition of a partial trace over high-momentum variables (map $\rho \rightarrow \rho_\mu$) and a specific quantum channel induced by the rescaling operation (map $\rho_\mu \rightarrow \rho_{(\sigma)}$).

To see how this formulation might be valuable, we apply it to study the momentum-space entanglement in the ground states of RG fixed points and this will quickly lead to a novel result about the theories at those points. Drawing from the Functional Renormalization Group equations quickly presented earlier, for a fixed point the effective action $S_\mu[\phi_{\vec{k}}]$ at any scale obeys:

$$\dot{S}_\mu = \frac{dS_\mu[\phi_{\vec{k}}]}{d\mu} + \frac{dS_\mu[\phi_{\vec{k}}]}{d\sigma} = 0 \quad (3.35)$$

Where, as before, $\frac{d}{d\mu}$ is the variation due to tracing out an infinitesimal momentum shell and $\frac{d}{d\sigma}$ the variation under the scaling transformation which restores the same shell.

The same statement can be made for finite RG transformations, but the main point is that, by definition, $S_{(\sigma)}[\phi_{\vec{k}}] = S[\phi_{\vec{k}}]$ no matter how many degrees of freedom are integrated over in the first step and, consequently, $\rho_{(\sigma)} = \rho = |0\rangle\langle 0|$, meaning $\rho_{(\sigma)}$ is a pure state with zero entropy.

Now, the actual entropy of interest is $S_{EE}(\rho_\mu) = -\text{Tr}(\rho_\mu \log \rho_\mu)$ which gives the entanglement between low and high momenta, but we showed that the rescaling quantum channel can never decrease the entropy and this map is (in this context only) the inverse of the partial trace, so this implies:

$$S_{EE}(\rho_\mu) \leq S_{EE}(\rho_{(\sigma)}) = 0 \quad (3.36)$$

And since the von Neumann entropy of a density matrix is always positive or zero, necessarily $S_{EE}(\rho_\mu) = 0$ for all μ , in other words, **theories at a fixed point of the Renormalization Group have no entanglement between different momentum scales.**

One implication of this result is that the wavefunctionals of the ground state of these theories must be separable: a simple product of terms labeled by the momentum scale. Another consequence is that truncated correlation functions of observables acting on different momentum scales must be all zero (from the relation between entanglement and correlations discussed in Chapter 2), thus leading to a family of identities involving expectation values of such operators.

It is important to remember, however, that in no way the lack of entanglement between momentum scales implies that theories in a RG fixed point have trivial or completely unentangled vacuum: as mentioned before, the notion of entanglement is heavily dependent on the partition of the Hilbert space chosen and separability with respect to one Tensor Product Structure does not imply anything about other partitions.

The analysis made here seems to have plenty of potential and might serve as starting point for other investigations (perhaps of critical points of more realistic QFTs), it would be also interesting to prove this result using methods from Conformal Field Theories, as those include a large portion of the possible fixed points (also such a proof would not require the introduction of a cutoff, in the formulation presented here first we consider some finite a^{-1} and then take it to infinity while renormalizing the interactions of the theory so that it is kept in the

fixed point). Nevertheless, this is as far as the Thesis will go in this direction and the next Chapter will be focused on practical calculations of momentum-space entanglement in simple models.

Chapter 4

The Momentum-Space Entropies

Having build in the previous Chapters a conceptual basis regarding how entanglement between momentum scales is generated and why it is interesting to study it, we now turn to the practical problem of calculating a measure of this entanglement in some models.

For this end, in the first section of this Chapter is dedicated to the development of a new technique (not seen previously in the literature) which allows for the calculation of the **Rényi entropies** with integer order and, via the replica trick, the entanglement entropy itself. The key idea behind this method is that the entropies must be determined directly from the effective action, as it is all the necessary to construct the reduced density matrix.

Once the details are established, I begin to apply the method to some models: first a system of coupled harmonic oscillators whose entanglement entropy is calculated exactly in [16] as a "sanity check". This example will also be useful to demonstrate some properties of the effective action which lead to a non-zero entropy, drawing insights from the theory of open quantum systems. Then, the technique will be used in the simplest interacting QFTs, namely the ϕ^3 and ϕ^4 theories, to calculate the momentum-space entropy at first order in perturbation theory (the method seems amenable for higher order calculations as well, though they will not be done in this Thesis).

A final note before moving forward: the calculations involved in applying the replica trick in momentum space to practical models are very extensive, so this Chapter will only include the main steps and general discussion and the

full derivations will be given in the Appendix. Furthermore, the machinery of Feynman diagrams and perturbation theory in QFTs will be widely used, readers are referred to [33] for an introduction to these techniques.

4.1 Development of the method

Given the reduced density matrix of a subsystem and its associated effective action (here taken to be the operator ρ_μ of low-momentum degrees of freedom and the action S_μ), the idea is to use the representation, introduced in the previous Chapter, of ρ_μ as the $\beta \rightarrow \infty$ limit of a thermal path integral:

$$\langle \phi_{\vec{k}} | \rho_\mu | \tilde{\phi}_{\vec{k}} \rangle = \lim_{\beta \rightarrow \infty} \frac{1}{Z(\beta)} \int_{\varphi_{\vec{k}}(0)=\phi_{\vec{k}}}^{\varphi_{\vec{k}}(\beta)=\tilde{\phi}_{\vec{k}}} \mathcal{D}\varphi_{\vec{k}}(\tau) e^{-S_\mu^\beta[\varphi_{\vec{k}}]} \quad (4.1)$$

Now, the partition function $Z(\beta)$ in the expression above is still that of the full system, but it is possible to modify the equation as to make all terms relate **only to the subsystem**:

When a partial trace is taken, the effective action generated contains a **constant** term, independent of the remaining degrees of freedom (in the context of renormalization, this is the term of zeroth order in the low-momentum fields), this term can be thought of as the contribution of the ignored variables to the partition function. Now, if such term is simply thrown away, the effective action can still be used to construct a density matrix, but if the equation above is used, the operator will no longer obey $\text{Tr} \rho_\mu = 1$. Thus, to restore the normalization of ρ_μ , we are led to define:

$$Z(\mu, \beta) := \int \mathcal{D}\varphi_{\vec{k}}(\tau) e^{-S_\mu^\beta[\varphi_{\vec{k}}]} \quad (4.2)$$

With $S_\mu[\varphi_{\vec{k}}]$ understood to contain **no field-independent terms** from now on. A finite-temperature version $Z(\mu, \beta)$ is also constructed by taking the integral only over paths periodic in $[0, \beta]$.

This in turn adjusts the path integral representation of the matrix elements to:

$$\langle \phi_{\vec{k}} | \rho_\mu | \tilde{\phi}_{\vec{k}} \rangle = \lim_{\beta \rightarrow \infty} \frac{1}{Z(\mu, \beta)} \int_{\varphi_{\vec{k}}(0)=\phi_{\vec{k}}}^{\varphi_{\vec{k}}(\beta)=\tilde{\phi}_{\vec{k}}} \mathcal{D}\varphi_{\vec{k}}(\tau) e^{-S_\mu^\beta[\varphi_{\vec{k}}]} \quad (4.3)$$

Though somewhat convoluted to introduce verbally, this formulation is more practical and is often used implicit in ordinary renormalization calculations (by ignoring the free energy term generated by the RG flow).

It is important to indicate that **from this point forward all results are novel**: the authors in [40] made the initial connection between the effective action and the low-momentum density matrix, but did not develop the idea further nor construct the path integral method to calculate entanglement which will be shown.

Now, with the expression above at hand, the quantity we want to calculate is $\text{Tr } \rho_\mu^n$, where n is an integer, in order to apply the replica trick introduced in Chapter 2. So, performing the matrix multiplication and then the trace:

$$\text{Tr } \rho_\mu^n = \lim_{\beta \rightarrow \infty} \frac{\int \mathcal{D}\varphi_1 \dots \int \mathcal{D}\varphi_n}{[Z(\mu, \beta)]^n} \int_{\varphi_{\vec{k}}(0)=\varphi_1}^{\varphi_{\vec{k}}(\beta)=\varphi_2} \mathcal{D}\varphi_{\vec{k}}(\tau) e^{-S_\mu^\beta} \dots \int_{\varphi_{\vec{k}}(0)=\varphi_n}^{\varphi_{\vec{k}}(\beta)=\varphi_1} \mathcal{D}\varphi_{\vec{k}}(\tau) e^{-S_\mu^\beta} \quad (4.4)$$

Clearly this is not practical, as it still contains boundary conditions which cannot be easily dealt with. However, by writing the coarse-grained action as the integral of an effective Lagrangian, $S_\mu^\beta = \int_0^\beta d\tau L_\mu^\beta$, a trick may be performed: in the equation above, shift the the limits of integration in τ of the $u + 1$ path integral (u an integer in $\{0, \dots, n - 1\}$) from $[0, \beta]$ to $[u\beta, (u + 1)\beta]$.

This allows the integrals to be combined into a **single one** taken over paths periodic in $[0, n\beta]$ (it can be seen from the fact that, due to the trace and operator multiplication, the boundary conditions at multiples of β match perfectly and are integrated over). And so, summing the integrals in the exponent:

$$\text{Tr } \rho_\mu^n = \lim_{\beta \rightarrow \infty} \frac{1}{[Z(\mu, \beta)]^n} \int_{0 \leq \tau \leq n\beta} \mathcal{D}\varphi_{\vec{k}}(\tau) e^{-\sum_{u=0}^{n-1} \int_{u\beta}^{(u+1)\beta} d\tau L_\mu^\beta} =: \lim_{\beta \rightarrow \infty} \frac{Z_n(\mu, \beta)}{[Z(\mu, \beta)]^n} \quad (4.5)$$

Where I defined:

$$Z_n(\mu, \beta) := \int_{0 \leq \tau \leq n\beta} \mathcal{D}\varphi_{\vec{k}}(\tau) e^{-\sum_{u=0}^{n-1} \int_{u\beta}^{(u+1)\beta} d\tau L_\mu^\beta} \quad (4.6)$$

Note the similarities to the expression for real-space entanglement in a QFT ([6], [16] and section 2.4 of this Thesis), the difference is that in the momentum-space

scenario, there's an effective action which allows the partial trace to be performed and the high-momentum degrees of freedom completely ignored.

At a first glance it might be tempting to assume $\sum_{u=0}^{n-1} \int_{u\beta}^{(u+1)\beta} d\tau L_\mu^\beta = \int_0^{n\beta} L_\mu^\beta$, however this is **not** correct. As will be explained in detail in the next section, the effective action has the general form:

$$S_\mu^\beta = \int_0^\beta d\tau L_{local}(\tau) + \int_0^\beta d\tau \int_0^\beta d\tau' \tilde{L}(\tau - \tau') + \dots \quad (4.7)$$

Where $L_{local}(\tau)$ indicates that it is the part of the Lagrangian **local in time** (composed of differential operators $\frac{d}{d\tau}$), while terms like $\tilde{L}(\tau - \tau')$ involve **non-local kernels** and are essential to ensure that the effective action generates a mixed state density matrix (and thus, confirms the presence of entanglement between modes).

As mentioned, this aspect of non-locality in τ will be better explored in the next section, but know that this sort of term is ubiquitous in the study of open quantum systems [47]. Furthermore, by the energy conservation in the global system and the fact that we are dealing with ground states (stationary by definition), any terms in the effective action, even those non-local in time, can only depend on **time differences** $\tau - \tau'$ (equivalently, the subsystem of low momentum degrees of freedom is not driven).

Now, given such structure for the effective action, a practical way to perform the shift used to define $Z_n(\mu, \beta)$ is by shifting all imaginary time variables by the same amount. Since any non-local terms appear as functions of $\tau - \tau'$, this means that this operation only affects the limits of integration of these variables and the sum $\sum_{u=0}^{n-1} \int_{u\beta}^{(u+1)\beta} d\tau L_\mu^\beta$ becomes:

$$\sum_{u=0}^{n-1} \int_{u\beta}^{(u+1)\beta} d\tau L_\mu^\beta = \int_0^{n\beta} d\tau L_{local}(\tau) + \sum_{u=0}^{n-1} \int_{u\beta}^{(u+1)\beta} d\tau \int_{u\beta}^{(u+1)\beta} d\tau' \tilde{L}(\tau - \tau') + \dots \quad (4.8)$$

So the part of the action local in time **is** simply taken to an integral from 0 to $n\beta$, but the non-local ones are **not**, as can be seen by the trivial statement:

$$\sum_{u=0}^{n-1} \int_{u\beta}^{(u+1)\beta} d\tau \int_{u\beta}^{(u+1)\beta} d\tau' \tilde{L}(\tau - \tau') \neq \int_0^{n\beta} d\tau \int_0^{n\beta} d\tau' \tilde{L}(\tau - \tau') \quad (4.9)$$

Having discussed the details of the technique constructed here, the trace of the n -th power of the reduced density matrix $\text{Tr } \rho_\mu^n$ can be used to obtain the Rényi entropies by:

$$H_n(\rho_\mu) = \frac{1}{n-1} \lim_{\beta \rightarrow \infty} (n \log Z(\mu, \beta) - \log Z_n(\mu, \beta)) \quad (4.10)$$

And, of course, the entanglement entropy via the formal limit:

$$S_{EE}(\rho_\mu) = \lim_{n \rightarrow 1} H_n(\rho_\mu) \quad (4.11)$$

Now is the moment to call attention to a very important fact: consider that under some approximation scheme the effective action S_μ^β is taken to contain only the local terms in time, that is, $S_\mu^\beta = \int_0^\beta d\tau L_{local}(\tau)$. Then, the reduced density matrix generated by this approximate effective action will have **zero entropy**, in other words, it will be a pure state.

To prove this, we can take the new Lagrangian L_{local} and, by the Legendre transform, construct an associated Hamiltonian $\tilde{H}$. By the usual path integral construction of finite-temperature systems [42], it is easy to see that:

$$\text{Tr } e^{-\beta \tilde{H}} = \int_{0 \leq \tau \leq \beta} \mathcal{D}\phi_{\vec{k}} e^{-\int_0^\beta d\tau L_{local}(\tau)} \quad (4.12)$$

Therefore, the trace of ρ_μ^n will be:

$$\text{Tr } \rho_\mu^n = \lim_{\beta \rightarrow \infty} \frac{\int_{0 \leq \tau \leq n\beta} \mathcal{D}\phi_{\vec{k}} e^{-\int_0^{n\beta} d\tau L_{local}(\tau)}}{\left[\int_{0 \leq \tau \leq \beta} \mathcal{D}\phi_{\vec{k}} e^{-\int_0^\beta d\tau L_{local}(\tau)} \right]^n} = \lim_{\beta \rightarrow \infty} \frac{\text{Tr } e^{-n\beta \tilde{H}}}{(\text{Tr } e^{-\beta \tilde{H}})^n} = 1 \quad (4.13)$$

With the last equality coming from using the diagonalization of $\tilde{H}$ to calculate the traces.

This is the mathematical way of understanding the result, physically what happens is that by approximating the full effective action by some expression local in time, we construct an effective Hamiltonian whose ground state (a state vector, thus pure) replaces the complete reduced density matrix ρ_μ .

From the discussion above, we can conclude that the inclusion of the non-local terms is **necessary** to find the entanglement entropy and this is an important

guide in the choice of whatever approximation will be used when applying the method to concrete models. For practical applications, this means that when calculating the entropy at **lowest order** in perturbation theory in the next sections, any local terms generated perturbatively can be safely ignored as they do not generate entropy by themselves and, at this order, do not mix with the non-local terms (this will be shown in detail later).

To conclude this section, it is useful to point out that although the idea of having the low-momentum density be determined by the Wilsonian effective action was presented in [40], this is a brand new technique to calculate the entropic entanglement measures developed during the work on this Thesis (obviously inspired by the replica trick in real-space).

The next sections of this Chapter are, then, dedicated to analysing the usefulness of this new method by first applying it to an exactly solvable model in quantum mechanics and later using it to obtain momentum-space entropies in QFTs.

4.2 Coupled Harmonic Oscillators

In the review paper [16], one of the cases on entanglement studied is that of a quantum-mechanical system consisting of two particles (whose positions are denoted x_A and x_B) moving in one dimension inside a quadratic potential and linearly coupled to each other.

Due to their interaction, the degrees of freedom x_A and x_B are entangled in the ground state and [16] calculates the exact form of the entanglement entropy in this state by writing the wavefunction, taking the partial trace over x_B and diagonalizing exactly the resulting reduced density matrix ρ_A . It's clear that such procedure cannot be followed in the general case, but since this is a simple system with result known by different means, it serves as a perfect test of the method developed in the previous section.

Thus, the euclidean Lagrangian in [16] describing the system is of the form:

$$L_E = \frac{1}{2} \left(\frac{dx_A}{d\tau} \right)^2 + \frac{1}{2} M^2 x_A^2 + \frac{1}{2} \left(\frac{dx_B}{d\tau} \right)^2 + \frac{1}{2} M^2 x_B^2 - l x_A x_B \quad (4.14)$$

Where M^2 can be related to the actual parameters used in [16].

Applying the new technique for this system will consist of performing the Gaussian path integral over x_B to generate an effective action for x_A alone (discarding any terms independent of x_A which may appear) and from this effective action calculating at finite temperature the associated $Z_n(A, \beta)$ and $Z(A, \beta)$.

So, diagonalizing the differential operators in frequency space, the system's euclidean action is easily seen to be given by:

$$S = \frac{1}{2} \int d\omega \left[\left(\omega^2 + M^2 \right) x_A^* x_A + \left(\omega^2 + M^2 \right) x_B^* x_B - 2l x_A^* x_B \right] \quad (4.15)$$

And the path integral over x_B leads to the effective action:

$$S_{eff} = \frac{1}{2} \int \frac{d\omega}{2\pi} \left(\omega^2 + M^2 - l^2 \frac{1}{\omega^2 + M^2} \right) x_A^*(\omega) x_A(\omega) \quad (4.16)$$

Now, returning to imaginary time, the variables $x_A(\omega)$ are written as Fourier transforms of $x_A(\tau)$ and S_{eff} becomes:

$$S_{eff} = \frac{1}{2} \int d\tau d\tau' \left[\left(-\frac{d^2}{d\tau^2} + M^2 \right) \delta(\tau - \tau') - \frac{l^2}{2M} e^{-M|\tau - \tau'|} \right] x_A(\tau) x_A(\tau') \quad (4.17)$$

And indeed a term showing non-locality in time appears!

Technically all calculations must be done at finite temperature, meaning the non-local kernel is actually:

$$\frac{1}{\beta} \sum_n \frac{e^{-i\omega_n(\tau - \tau')}}{\omega_n^2 + M^2} = \frac{e^{-M|\tau - \tau'|}}{2M} + \frac{1}{e^{\beta M} - 1} \frac{\cosh(M|\tau - \tau'|)}{M} \quad (4.18)$$

However, since the second term goes to zero **exponentially** as $\beta \rightarrow \infty$, it can be safely ignored, as it won't affect the zero-temperature results (this is just like ignoring irrelevant operators after a scaling transformation).

Now, the sort of effective action of equation (4.17) is far from uncommon in the study of open quantum systems [47], the reason being that non-local kernels like $e^{-M|\tau-\tau'|}$ enter the path integral as a dissipation term responsible for making the expectation values calculated with this action be those of a mixed state. In fact, returning to the discussions in section 2.2, this sort of phenomenon is expected, as the entanglement between subsystem A and its complement creates an effective bath for A due to the fluctuations (and this makes the reduced density matrix no longer a pure state).

Regardless, it's clear that the non-local term in S_{eff} is the only which can possibly lead to a finite entropy, as the other ones merely describe the propagation of particle A in the harmonic potential.

The goal is now to calculate the partition functions $Z(A, \beta)$ and $Z_n(A, \beta)$. Beginning with the first, the finite temperature effective action is given by:

$$S_{eff}[x_j] = \frac{1}{2} \sum_j \left(\omega_j^2 + M^2 - \frac{l^2}{\omega_j^2 + M^2} \right) x_j^* x_j \quad (4.19)$$

Where $\omega_j = \frac{2\pi j}{\beta}$ are the Matsubara frequencies [42].

The Gaussian path integral over all x_j is simple and leads to the expression:

$$\log Z(A, \beta) = -\frac{1}{2} \sum_j \log \left(\omega_j^2 + M^2 - \frac{l^2}{\omega_j^2 + M^2} \right) \quad (4.20)$$

Now, it can be easily checked that:

$$\begin{aligned} \log \left(\omega_j^2 + M^2 - \frac{l^2}{\omega_j^2 + M^2} \right) &= \log \left(\omega_j^2 + M^2 - l \right) + \log \left(\omega_j^2 + M^2 + l \right) \\ &\quad - \log \left(\omega_j^2 + M^2 \right) \end{aligned} \quad (4.21)$$

Using the following identity for Matsubara sums [42]

$$\sum_j \log \left(\omega_j^2 + C^2 \right) = 2 \log \sinh \left(\frac{\beta C}{2} \right) \quad (4.22)$$

We arrive at the final expression for $Z(A, \beta)$:

$$Z(A, \beta) = \frac{\sinh\left(\frac{\beta M}{2}\right)}{\sinh\left(\frac{\beta\sqrt{M^2+l}}{2}\right) \sinh\left(\frac{\beta\sqrt{M^2-l}}{2}\right)} \quad (4.23)$$

The next step is calculating $Z_n(A, \beta)$ and for that it is necessary to have the particular sum $\sum_{u=0}^{n-1} \int_{u\beta}^{(u+1)\beta} d\tau L_\mu^\beta$. As shown previously, the local terms simply add up to the same expression at inverse temperature $n\beta$, so the focus now is on:

$$-\frac{l^2}{2M} \sum_{u=0}^{n-1} \int_{u\beta}^{(u+1)\beta} d\tau \int_{u\beta}^{(u+1)\beta} d\tau' e^{-M|\tau-\tau'|} x_A(\tau) x_A(\tau') \quad (4.24)$$

As the position $x_A(\tau)$ is now the Fourier sum of components x_j with Matsubara frequencies $\omega_j = \frac{2\pi j}{n\beta}$, the sum becomes:

$$\frac{1}{n\beta} \sum_{u=0}^{n-1} \int_{u\beta}^{(u+1)\beta} d\tau \int_{u\beta}^{(u+1)\beta} d\tau' e^{-M|\tau-\tau'|+i\omega_j\tau+i\omega_{j'}\tau'} x_j x_{j'} \quad (4.25)$$

To perform the imaginary time integrals, a good trick is writing $e^{-M|\tau-\tau'|} = e^{-M(\tau-\tau')}\theta(\tau-\tau') + e^{-M(\tau'-\tau)}\theta(\tau'-\tau)$, where $\theta(\tau)$ is the Heaviside step function.

Skipping part of the long, but easy, calculations, the non-local term becomes:

$$\begin{aligned} & \frac{1}{n\beta} \sum_{j,j'} x_j x_{j'} \sum_{u=0}^{n-1} e^{2\pi i \frac{(j+j')}{n} u} \frac{2M}{M^2 + \omega_{j'}^2} \frac{e^{2\pi i \frac{(j+j')}{n}} - 1}{i(\omega_j + \omega_{j'})} \\ & + \frac{1}{M + i\omega_{j'}} \frac{e^{-\beta M + 2\pi i \frac{j}{n}} - 1}{M - i\omega_j} + \frac{1}{M + i\omega_j} \frac{e^{-\beta M + 2\pi i \frac{j'}{n}} - 1}{M - i\omega_{j'}} \end{aligned} \quad (4.26)$$

Through the identity $\sum_{u=0}^{n-1} e^{2\pi i \frac{(j+j')}{n} u} = n\delta_{vn}^{j+j'}$ (v being any integer) the equation above can be simplified, though for some terms the sum over j' is not fully eliminated. Once again avoiding to show all tedious calculations (see Appendix), the final result (restoring the overall multiplicative constant $\frac{l^2}{2M}$) is:

$$\sum_j \frac{l^2}{\omega_j^2 + M^2} x_j^* x_j + \frac{l^2}{2M\beta} \sum_{j,v} \left[\frac{e^{-\beta M + 2\pi i \frac{j}{n}} - 1}{(M + i\omega_{nv} - i\omega_j)(M - i\omega_j)} + h.c. \right] x_j x_{nv-j} \quad (4.27)$$

With $h.c.$ denoting the Hermitian conjugate of the adjacent term.

The first factor of the expression above plus the local part of the effective action will add up **exactly** to the action at $n\beta$ used to calculate $Z(A, \beta)$ and the

second factor clearly connects components x_j of different frequencies, thus being the non-diagonal terms of a matrix. This is as far as we can go exactly, but we can still make a perturbative expansion (and compare with the expansion of the exact result in [16]).

So, taking the lowest order term in the expansion and disregarding the vanishing contributions, we have:

$$Z_n(A, \beta) = \int \mathcal{D}x_j e^{-S_{eff}^{n\beta}} \times \left(1 - \frac{nl^2}{4M} \frac{1}{n\beta} \sum_{j,\nu} \left[\frac{1}{(M + i\omega_{n\nu} - i\omega_j)(M - i\omega_j)} + h.c. \right] x_j x_{n\nu-j} \right) \quad (4.28)$$

And so:

$$\text{Tr} \rho_A^n = \lim_{\beta \rightarrow \infty} \frac{Z(A, n\beta)}{[Z(A, \beta)]^n} \times \left(1 - n \frac{l^2}{4M} \frac{1}{n\beta} \sum_{j,\nu} \left[\frac{1}{(M + i\omega_{n\nu} - i\omega_j)(M - i\omega_j)} + h.c. \right] \langle x_j x_{n\nu-j} \rangle \right) \quad (4.29)$$

Now, $\lim_{\beta \rightarrow \infty} \frac{Z(A, n\beta)}{[Z(A, \beta)]^n} = 1$ as equation (19) shows (in the limit, only the leading exponentials matter and those cancel each other perfectly). Also, by Wick's theorem $\langle x_j x_{n\nu-j} \rangle = \delta_n^0 (\omega^2 + M^2 - \frac{l^2}{\omega^2 + M^2})^{-1}$.

Therefore, performing all necessary manipulations (Wick contraction, Matsubara sum, expansion over l , see Appendix), we arrive at the simple lowest-order result:

$$\text{Tr} \rho_A^n = 1 - n \frac{l^2}{16M^4} \quad (4.30)$$

In [16] this trace is calculated exactly and is given by $\text{Tr} \rho_A^n = \frac{(1-\xi)^n}{1-\xi^n}$, with $\xi = \left(\frac{(M^2+l)^{\frac{1}{4}} - (M^2-l)^{\frac{1}{4}}}{(M^2+l)^{\frac{1}{4}} + (M^2+l)^{\frac{1}{4}}} \right)^2$ in our notation. Expanding the exact result up to order l^2 , we get $\text{Tr} \rho_A^n = 1 - n \frac{l^2}{16M^4}$, precisely what we obtained via the effective action, which shows that we can rely on this method.

4.3 Perturbation theory and the replica trick

Before moving forward to apply the technique to QFTs, there is a subtlety involving the use of the replica trick and a perturbative expansion in the same calculation that must be explained.

Consider, then, the n th Rényi entropy derived from equation (4.30) above and expanded in l up to order l^2 :

$$H_n(\rho_A) = \frac{1}{1-n} \log \text{Tr} \rho_A^n = \frac{n}{n-1} \frac{l^2}{16M^4} + \mathcal{O}(l^4) \quad (4.31)$$

If the limit $n \rightarrow 1$ is taken naively in the expression above, we would arrive at the absurd conclusion of $S_A = \infty$. The reason for this is that the expansion in the parameter l is made **at fixed n** and terms like l^{2n} are ignored for being of higher-order than desired, this means that terms which are important for the entanglement in the limit $n \rightarrow 1$ are thrown away and cannot be recovered via the limiting procedure.

There's still a way of finding the lowest-order contribution to the entanglement entropy through this method, however. Consider that the reduced density matrix ρ_A is diagonalized exactly and its eigenvalues (as functions of l) are given by $p_i(l)$. For $l = 0$ all but one of the probabilities must be zero (as the starting point of the perturbative expansion made here is a separable state) and, as shown in [40], the eigenvalues are modified only at order l^2 and above, so they may be labeled such that $p_i(l) = l^2 a_i$ for $i \geq 1$ and $p_0 = 1 - l^2 \sum_i a_i$ (note that the condition $\sum_{i=0}^{\infty} p_i = 1$ must always be preserved).

Thus, calculating the entanglement entropy from these probabilities:

$$S_A = - \left(1 - l^2 \sum_{i=1}^{\infty} a_i \right) \log \left(1 - l^2 \sum_{i=1}^{\infty} a_i \right) - l^2 \sum_{i=1}^{\infty} a_i \log(l^2 a_i) \quad (4.32)$$

And, taking the dominant term as $l \rightarrow 0$, we find:

$$S_A = - \sum_{i=0}^{\infty} p_i \log p_i = -l^2 \log l^2 \sum_i a_i + \mathcal{O}(l^2) \quad (4.33)$$

The presence of a term $-l^2 \log l^2$ is ubiquitous, see [40] and the exact result in [16] and is particular to the entanglement/von Neumann entropy, as the $x \log x$ expression is non-analytical at $S_A = 0$.

Following the same procedure to calculate any Rényi entropy leads to:

$$H_n(\rho_A) = \frac{1}{1-n} \log \left(\sum_{i=0}^{\infty} p_i^n \right) = \frac{n}{n-1} l^2 \sum_i a_i + \mathcal{O}(l^4) \quad (4.34)$$

This means that we may find the entanglement entropy by making the substitution $\frac{n}{n-1} \rightarrow \log \frac{1}{l^2}$, keeping in mind that terms of order l^2 or higher are being discarded and that these must be calculated through more sophisticated procedures (in fact, perhaps a method based on distributional zeta function, mentioned in Chapter 2, would be more appropriate for calculating higher order contributions).

With this, for the remainder of this Thesis, the results will be given in terms of the Rényi entropies, which at lowest order are all proportional to each other and to the entanglement entropy, with the constants of proportionality known.

4.4 Perturbative calculation in ϕ^3 theory

As a reminder, the strategy in this and in the next section will be the following: to calculate entanglement measures between the degrees of freedom at different momentum scales (as always, in the ground state of the theory), we will split the field variable as a sum of high and low momentum parts and integrate the high momentum modes (introducing at first an overall UV cutoff a^{-1}) perturbatively to find the effective action $S_\mu[\phi_{|\vec{k}| \leq \mu}]$ at the lowest order in which entanglement appears and then apply the method constructed earlier to calculate the n th order Rényi entropies. Many of the tricks and calculations of the previous section will allow us to streamline the procedure, as will quickly become apparent.

Integrating out modes with **spatial** momentum obeying $|\vec{k}| > \mu$, the perturbative corrections to the effective action are obtained by the usual Feynman di-

agrams under the condition that all internal lines have momentum above the scale μ , furthermore, since the effective action is an exponent, only the **connected** diagrams must be considered and, by the construction of the momentum-space replica trick, terms independent of $\phi_{\vec{k} \leq \mu}$ are discarded (overall, this is just the usual Wilsonian procedure [35]).

For ϕ^3 theory, the euclidean bare action is, in momentum space:

$$S[\phi] = \frac{1}{2} \int \frac{d^d k}{(2\pi)^d} (k^2 + m^2) \phi_k^* \phi_k + \frac{\lambda}{3!} \int \prod_{i=1}^3 \frac{d^d k_i}{(2\pi)^d} (2\pi)^d \delta(\sum_i k_i) \phi_{k_1} \phi_{k_2} \phi_{k_3} \quad (4.35)$$

At order λ the only contribution to S_μ besides the already existing $\lambda\phi^3$ comes from the so-called Tadpole diagram, however this only shifts the expectation value of ϕ and is local in τ , so by the previous discussions on the need of a non-local term, this means $\text{Tr} \rho_\mu^n = 1 + \mathcal{O}(\lambda^2)$ and the first order generation of entanglement is zero.

In fact, it was proven in [40] that the first non-zero perturbative contribution to entanglement is **always** of order λ^2 and we'll see that path integral calculations agree with this result.

Thus, looking at the diagrams with two vertices, a ϕ^4 is generated, but it can be shown to be local. We also get the **one-loop correction to the mass** and this will be the source of entropy at this order. Writing explicitly this term in the effective action (with symmetry factor already included), we have:

$$S_\mu = S_{local} - \frac{\lambda^2}{2} \int^* \frac{dk_0}{2\pi} \frac{d^{d-1}k}{(2\pi)^{d-1}} \int^* \frac{dq_0}{2\pi} \frac{d^{d-1}q}{(2\pi)^{d-1}} \times \frac{1}{q_0^2 + \vec{q}^2 + m^2} \frac{1}{(k_0 - q_0)^2 + (\vec{k} - \vec{q})^2 + m^2} \phi_{-k_0, -\vec{k}} \phi_{k_0, \vec{k}} \quad (4.36)$$

Where S_{local} indicates all terms which are local in τ and the asterisk in the integrals indicate that k_0 and q_0 are integrated over the range $(-\infty, \infty)$ while $\vec{k}$ and $\vec{q}$ are restricted to the regions $|\vec{k}| \leq \mu$ and $\mu < |\vec{q}|, |\vec{k} - \vec{q}| \leq a^{-1}$, respectively.

This last comment needs more discussion. So, recalling the previous Chapter:

Mathematically, the spatial components of momenta are restricted while the time components go over their full range because the actual degrees of freedom of a QFT are labeled by $\vec{k}$ (so the large spatial momenta are the variables integrated over) and k_0 is unlimited in order to ensure that the path integral projects the theory in its ground state correctly.

Physically, besides the selection of the proper degrees of freedom, we can think of this distinction as coming from the fact that we can have many low-momentum excitations such that the total energy of the system gets arbitrarily high while still being kept in the low-momentum sector of the theory (as individually, each particle has momentum below scale μ), thus the time components of momenta, which indicate the energy, don't have to be restricted.

It was important to have this discussion, because we'll now use the integration over the full real line in the next calculation. Fourier transforming only the time components of the fields in the equation above, the one-loop contribution to the effective action becomes (leaving aside for the moment the integration over spatial components and giving the detailed steps in the Appendix):

$$\int d\tau \int d\tau' \frac{e^{-(\sqrt{\vec{q}^2+m^2}+\sqrt{(\vec{k}-\vec{q})^2+m^2})|\tau-\tau'|}}{4\sqrt{\vec{q}^2+m^2}\sqrt{(\vec{k}-\vec{q})^2+m^2}} \phi_{\vec{k}}(\tau) \phi_{-\vec{k}}(\tau') \quad (4.37)$$

And, again, we find a term which is non-local in time, thus leading to entropy. This non-locality came to be because an external momentum, in this case $\vec{k}$, appeared in one of the propagators, thus leading to the kernel above when the Fourier transform of the fields was taken (since the **fields** are the ones depending on τ , it's crucial that the **external** momentum was in the loop, because it's then affected by the transform).

Based on this result, it's easy to see that at the perturbative level non-local kernels are generated by **loop diagrams where the external momenta also run inside a loop**.

Note that, for fixed $\vec{k}$ and $\vec{q}$, we have an exponential of $|\tau - \tau'|$ just as in with the coupled harmonic oscillators, with the decay rate now being $M = \sqrt{\vec{q}^2 + m^2} + \sqrt{(\vec{k} - \vec{q})^2 + m^2}$, this allows for a quick adaptation of the results for the coupled harmonic oscillators, as long as the integrals over $\vec{k}$ and $\vec{q}$ are left for the end (which is possible to do in perturbation theory). Thus, the effective action at this order is:

$$S_\mu = S_{local} - \frac{\lambda^2}{2} \int^* \frac{d^{d-1}k}{(2\pi)^{d-1}} \frac{d^{d-1}q}{(2\pi)^{d-1}} \times \int d\tau \int d\tau' \frac{e^{-(\sqrt{\vec{q}^2+m^2}+\sqrt{(\vec{k}-\vec{q})^2+m^2})|\tau-\tau'|}}{4\sqrt{\vec{q}^2+m^2}\sqrt{(\vec{k}-\vec{q})^2+m^2}} \phi_{\vec{k}}(\tau) \phi_{-\vec{k}}(\tau') \quad (4.38)$$

With integration limits $|\vec{k}| \leq \mu$ and $\mu < |\vec{q}|$, $|\vec{k} - \vec{q}| \leq a^{-1}$.

Now, instead of calculating the trace $\text{Tr} \rho_\mu^n$, it is better to go directly to the Rényi entropies $H_n(\rho_\mu)$, as they take the logarithm of the trace and this means only the connected diagrams are relevant. Leaving the detailed steps to the Appendix (as they are just an adaptation of those for the coupled oscillators), the entropies at lowest order in λ are given by the expression:

$$H_n(\rho_\mu) = \frac{n}{n-1} \frac{\lambda^2}{8} \int^* \frac{d^{d-1}k}{(2\pi)^{d-1}} \int^* \frac{d^{d-1}q}{(2\pi)^{d-1}} \frac{1}{\sqrt{\vec{q}^2+m^2}\sqrt{(\vec{k}-\vec{q})^2+m^2}} \times \lim_{\beta \rightarrow \infty} \frac{1}{n\beta} \sum_{j,\nu} \frac{2M^2 - 2\omega_j^2 + 2\omega_{n\nu}\omega_j}{[M^2 + (\omega_{n\nu} - \omega_j)^2] (M^2 + \omega_j^2)} \langle \phi_{j,\vec{k}} \phi_{n\nu-j,-\vec{k}} \rangle \quad (4.39)$$

With $M = \sqrt{\vec{q}^2 + m^2} + \sqrt{(\vec{k} - \vec{q})^2 + m^2}$ and the same momentum integration regions as before (remember, they arise naturally from the Wilsonian integration of fast modes).

Now, the field average is, as usual, taken over the free action of the field ϕ , therefore we have $\langle \phi_{j,\vec{k}} \phi_{n\nu-j,-\vec{k}} \rangle = \delta_\nu^0 \frac{(2\pi)^{d-1} \delta(\vec{k}-\vec{k})}{\omega_j^2 + \vec{k}^2 + m^2}$. The delta function $\delta(\vec{k} - \vec{k})$ is not well defined, so we consider the theory in a finite volume V and have: $(2\pi)^{d-1} \delta(\vec{k} - \vec{k}) = \int d^{d-1}x e^{i\vec{x}(\vec{k}-\vec{k})} = V$. This means that the entanglement entropy is extensive as found in [40] and we can calculate an entropy density, this makes sense as the momentum degrees of freedom are uniformly spread throughout

space and so the total entropy should be proportional to the volume of the system.

Following through with the calculation and making the substitution $M = \sqrt{\vec{q}^2 + m^2} + \sqrt{(\vec{k} - \vec{q})^2 + m^2}$ at the end, the n th Rényi entropy for the entanglement between momentum modes separated by scale μ in ϕ^3 theory is:

$$\begin{aligned} \frac{H_n(\rho_\mu)}{V} &= \frac{n}{n-1} \frac{\lambda^2}{8} \int^* \frac{d^{d-1}k}{(2\pi)^{d-1}} \int^* \frac{d^{d-1}q}{(2\pi)^{d-1}} \frac{1}{\sqrt{\vec{k}^2 + m^2}} \frac{1}{\sqrt{\vec{q}^2 + m^2}} \\ &\times \frac{1}{\sqrt{(\vec{k} - \vec{q})^2 + m^2}} \frac{1}{\left(\sqrt{\vec{k}^2 + m^2} + \sqrt{\vec{q}^2 + m^2} + \sqrt{(\vec{k} - \vec{q})^2 + m^2} \right)^2} \end{aligned} \quad (4.40)$$

The same equation gives the entanglement entropy after replacing $\frac{n}{n-1}$ by $-\log \lambda^2$, as shown in the previous section.

A comment is in order: the result above reproduces almost exactly that of [40] with the integrand and overall numerical factors being the same. The difference between expressions comes from the limits of integration in $\vec{k}$ and $\vec{q}$, in this Thesis the only momentum below scale μ is $\vec{k}$ while in [40] the integration includes the possibility of two momenta being smaller than μ .

The reason as to why these entropies differ is not yet clear, but it is important to point out that the new approach developed here follows completely the Wilsonian integration of high momentum modes (giving automatically the integration regions above already at the level of the effective action), while the authors in [40] use a Hamiltonian method which is not shown to correspond exactly to the path integral.

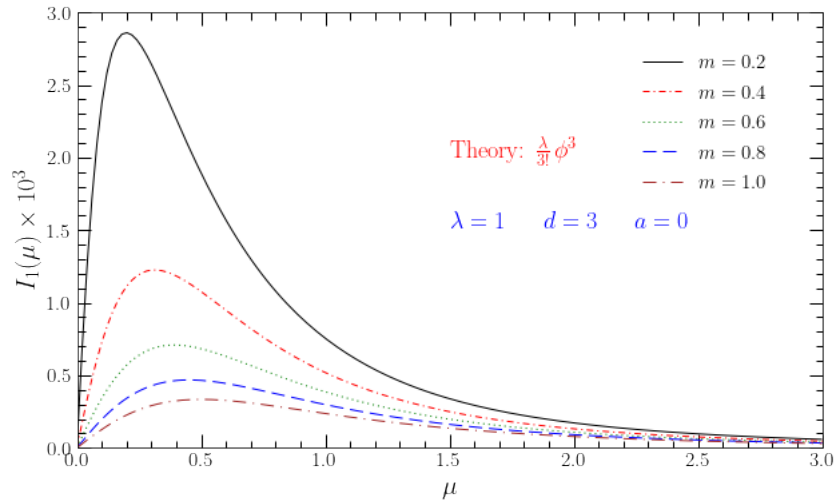
The expression for the entropy is clearly very complicated due to the integrals, so it is difficult to move forward with analytical calculations. We can still gain an understanding of how this entropy density varies with the scale of separation μ through numerical integration, this allows us to also study the effects of changing the mass parameter m of the theory.

So, we may define:

$$I_1(\mu) = \frac{\lambda^2}{8} \int_{|\vec{k}| \leq \mu} \frac{d^{d-1}k}{(2\pi)^{d-1}} \int_{\mu < |\vec{q}|, |\vec{k}-\vec{q}| \leq a^{-1}} \frac{d^{d-1}q}{(2\pi)^{d-1}} \frac{1}{\sqrt{\vec{k}^2 + m^2}} \frac{1}{\sqrt{\vec{q}^2 + m^2}} \times \frac{1}{\sqrt{(\vec{k} - \vec{q})^2 + m^2}} \frac{1}{\left(\sqrt{\vec{k}^2 + m^2} + \sqrt{\vec{q}^2 + m^2} + \sqrt{(\vec{k} - \vec{q})^2 + m^2} \right)^2} \quad (4.41)$$

Which, excepting known proportionality constants, gives all Rényi entropies, as well as the entanglement entropy.

And now, making for convenience $d = 3$, $\lambda = 1$, $a = 0$ and giving all quantities in units of an arbitrarily chosen mass m_0 , we may plot $I_1(\mu)$ for several values of m :



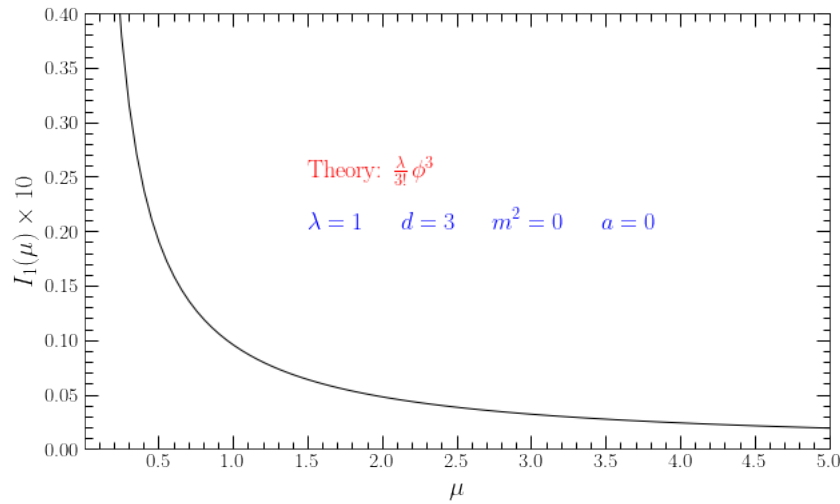
Note that the entropy density goes to zero as $\mu \rightarrow 0$ and $\mu \rightarrow \infty$, the reason for this being the fact that, as explained in section 3.3.3, at such limits the state is pure.

Furthermore, the maximum of the entropies is reached when μ is of order m , which can be understood as follows: the maximum of entanglement entropy naturally defines a characteristic scale for a theory, as it is the momentum scale across which modes are correlated the most, so this must be one of the scales of the theory and m is the only one available in such a simple model (this also relates

to the result obtained in Chapter 3 for RG fixed points: since any maximum of entanglement defines a characteristic scale for a theory, any scale-invariant QFT must have zero entropy for all μ).

Another interesting property is the increase of entanglement as the mass is reduced, meaning momentum scales are more easily correlated by the interaction in the presence of low-energy excitations. This needs to be studied further and could be interesting to investigate in theories with Goldstone bosons, for example.

Giving weight to the idea of low-lying modes being important to the entanglement of scales is the plot for the entropy of the massless ϕ^3 theory:



Where the entropy is greater than in the previous massive cases and as $\mu \rightarrow 0$ an infrared divergence (going as μ^{-1}) violates the prediction of zero entropy at this limit, leaving unclear, at the moment, how can it be regulated (or if it is just a pathology of the lowest order in perturbation theory).

4.5 Results for ϕ^4 theory

Drawing from the lessons of the previous sections, we can also calculate the momentum-space entanglement for $\lambda\phi^4$ theory and with a greater understanding

of how the steps proceed. For this model, I will present only analytical results, leaving numerical plots of the entropy as a function of μ for further work.

So given the bare action at UV cutoff a^{-1} of this theory:

$$S[\phi] = \int \frac{d^d k}{2(2\pi)^d} (k^2 + m^2) \phi_k^* \phi_k + \frac{\lambda}{4!} \int \prod_{i=1}^4 \frac{d^d k_i}{(2\pi)^d} (2\pi)^d \delta(\sum_i k_i) \phi_{k_1} \phi_{k_2} \phi_{k_3} \phi_{k_4} \quad (4.42)$$

Integrating out modes with momentum $\vec{k}$ such that $|\vec{k}| > \mu$ perturbatively, the discussion of the previous section made clear that to get entropy only diagrams which have an external momentum inside a loop contribute at lowest order.

At order λ the only Feynman diagram is the Tadpole and it gives a mass renormalization without generating a non-local term. At order λ^2 two diagrams will lead to non-locality in time: the Setting Sun and the one-loop renormalization of the coupling constant. With this, the contribution to the effective action at scale μ from each diagram is, respectively (all symmetry factors already included):

$$- \frac{\lambda^2}{6} \int^* \frac{d^d k}{(2\pi)^d} \int^* \frac{d^d q}{(2\pi)^d} \frac{d^d p}{(2\pi)^d} \frac{1}{q^2 + m^2} \frac{1}{p^2 + m^2} \frac{1}{(k - q - p)^2 + m^2} \phi_k^* \phi_k \quad (4.43)$$

$$- \frac{\lambda^2}{16} \int^* \prod_{i=1}^4 \frac{d^d k_i}{(2\pi)^d} \int^* \frac{d^d q}{(2\pi)^d} \frac{1}{q^2 + m^2} \frac{(2\pi)^d \delta(\sum_i k_i)}{(k_1 + k_2 - q)^2 + m^2} \phi_{k_1} \phi_{k_2} \phi_{k_3} \phi_{k_4} \quad (4.44)$$

The integrals with asterisks indicating the specific the momentum regions arising from the tracing out of high-momentum modes.

The next step is to find how exactly these new expressions are non-local in imaginary time. Following the Appendix, we have, respectively:

$$\int d\tau \int d\tau' \frac{e^{-(\sqrt{\vec{q}^2 + m^2} + \sqrt{\vec{p}^2 + m^2} + \sqrt{(\vec{k} - \vec{q} - \vec{p})^2 + m^2})|\tau - \tau'|}}{8\sqrt{\vec{q}^2 + m^2} \sqrt{\vec{p}^2 + m^2} \sqrt{(\vec{k} - \vec{q} - \vec{p})^2 + m^2}} \phi_{\vec{k}}(\tau) \phi_{-\vec{k}}(\tau') \quad (4.45)$$

$$\int d\tau d\tau' \frac{e^{-(\sqrt{\vec{q}^2 + m^2} + \sqrt{(\vec{k}_1 + \vec{k}_2 - \vec{q})^2 + m^2})|\tau - \tau'|}}{4\sqrt{\vec{q}^2 + m^2} \sqrt{(\vec{k}_1 + \vec{k}_2 - \vec{q})^2 + m^2}} \phi_{\vec{k}_1}(\tau) \phi_{\vec{k}_2}(\tau) \phi_{\vec{k}_3}(\tau') \phi_{\vec{k}_4}(\tau') \quad (4.46)$$

Once again exponential kernels appear, which allows for a quick adaptation of previous calculations.

As this is a lowest order calculation, it is useful to remember that each of the terms above will contribute to the entropy independently of the other and the end result will simply be their sum (see section 4.1 and the Appendix).

Thus, after the calculations detailed in the Appendix, the contribution of the mass renormalization diagram, $H_n^{m^2}(\rho_\mu)$ to the entropy is:

$$\begin{aligned} \frac{H_n^{m^2}(\rho_\mu)}{V} &= \frac{n}{n-1} \frac{\lambda^2}{48} \int_{|\vec{k}|, |\vec{q}| \leq \mu} \frac{d^{d-1}k}{(2\pi)^{d-1}} \frac{d^{d-1}q}{(2\pi)^{d-1}} \int_{\mu \leq |\vec{p}|, |\vec{k}-\vec{q}-\vec{p}| \leq a^{-1}} \frac{d^{d-1}p}{(2\pi)^{d-1}} \\ &\times \frac{\left(\sqrt{\vec{k}^2 + m^2} + \sqrt{\vec{q}^2 + m^2} + \sqrt{\vec{p}^2 + m^2} + \sqrt{(\vec{k} - \vec{q} - \vec{p})^2 + m^2} \right)^{-2}}{\sqrt{\vec{k}^2 + m^2} \sqrt{\vec{q}^2 + m^2} \sqrt{\vec{p}^2 + m^2} \sqrt{(\vec{k} - \vec{q} - \vec{p})^2 + m^2}} \end{aligned} \quad (4.47)$$

This contribution is also very similar to the full entropy derived in [40], the main difference again lying in the specific region of integration and now the numerical coefficient found is one-third of that calculated in the paper.

There is also a completely new term compared to [40], with origin in the one-loop coupling renormalization. After the calculations, the contribution $H_n^\lambda(\rho_\mu)$ is given by:

$$\begin{aligned} \frac{H_n^\lambda(\rho_\mu)}{V} &= \frac{n}{n-1} \frac{\lambda^2}{64} \int_{|\vec{k}|, |\vec{p}| \leq \mu} \frac{d^{d-1}k}{(2\pi)^{d-1}} \frac{d^{d-1}p}{(2\pi)^{d-1}} \int_{\mu \leq |\vec{q}|, |\vec{k}+\vec{p}-\vec{q}| \leq a^{-1}} \frac{d^{d-1}q}{(2\pi)^{d-1}} \\ &\times \frac{I(\vec{k}, \vec{p}, \vec{q})}{\sqrt{\vec{k}^2 + m^2} \sqrt{\vec{p}^2 + m^2} \sqrt{\vec{q}^2 + m^2} \sqrt{(\vec{k} + \vec{p} - \vec{q})^2 + m^2}} \end{aligned} \quad (4.48)$$

With:

$$\begin{aligned} I(\vec{k}, \vec{p}, \vec{q}) &= \frac{1}{\left(\sqrt{\vec{k}^2 + m^2} + \sqrt{\vec{p}^2 + m^2} + \sqrt{\vec{q}^2 + m^2} + \sqrt{(\vec{k} + \vec{p} - \vec{q})^2 + m^2} \right)^2} \\ &+ \frac{1}{2 \left(\sqrt{\vec{q}^2 + m^2} + \sqrt{(\vec{k} + \vec{p} - \vec{q})^2 + m^2} \right)^2} \end{aligned} \quad (4.49)$$

Finally, the total entropy is $H_n(\rho_\mu) = H_n^{m^2}(\rho_\mu) + H_n^\lambda(\rho_\mu)$. The expressions above make clear that plotting this function, even numerically, is more laborious than in the ϕ^3 case and for this reason was left as further work beyond this Thesis.

Chapter 5

Conclusions

In this Thesis, I reviewed the fundamental concepts of quantum information in Chapter 2, those being the notions of separable and entangled states, reduced density matrices and entanglement measures, the entanglement entropy being the most important among them as it sheds much light on the nature of entanglement via equation (2.42). I concluded Chapter 2 by briefly applying these ideas in the context of Quantum Field Theories and entanglement in real space, reproducing seminal results such as the area law of entanglement entropy.

In Chapter 3, the main topic of this Thesis was discussed: entanglement in momentum space of field theories. After an introduction to renormalization group concepts, I showed how the Wilsonian RG combined with path integral techniques derived in Chapter 2 naturally defines a reduced density matrix describing the low-momentum degrees of freedom of a QFT in its ground state, meaning the renormalization group can also be used to calculate entanglement measures, as well as how each RG step can be interpreted in the language of quantum information. The Chapter was concluded by using this interpretation in order to derive a property of the momentum-space entanglement in theories lying at fixed points of the RG.

Finally, in Chapter 4 the replica trick was adapted to momentum space and used to develop a new method of calculating Rényi entropies between momentum scales based on non-local kernels generated by integrating out the high-momentum modes of a QFT. This method was first verified by reproducing in perturbation theory the known entropy of coupled harmonic oscillators and then

it was applied to the ϕ^3 and ϕ^4 real scalar fields, where in the former I was able to numerically plot the expression derived and interpret the results.

The work done in this Thesis opens the way for many applications: obviously the entropy for the ϕ^4 theory can be plotted as a function of the scale μ and by varying the mass it will be possible to study the behavior of the entanglement between momentum scales through a phase transition (in this particular case, spontaneous symmetry breaking when $m^2 = 0$).

More broadly, the new technique developed here can be used to more complex QFTs, with the connection between entanglement and the renormalization group possibly being a source of many insights. For example, in theories such as the $O(N)$ model (where the large N expansion allows for the study of non-perturbative features including symmetry breaking and the appearance of Goldstone bosons) or the Gross-Neveu model (which is fermionic and presents asymptotic freedom) it is very likely that the non-perturbative aspects will leave marks in the momentum-space entanglement. In the longer term another interesting possibility is to investigate how the method can be adapted to gauge theories, where a quantum information point of view may help to shed light on phenomena which are not completely understood, like quark confinement in the theory of strong interactions, Quantum Chromodynamics.

Appendix A

Full entropy calculations

In this Appendix I will give all details of the entanglement entropy calculations of Chapter 4, namely the derivation of the lowest-order entropies for the coupled harmonic oscillators and momentum-space entanglement in ϕ^3 and ϕ^4 through the method developed in this Thesis. In order to streamline the exposition of the calculations, some of the steps shown in the main body of the Thesis will be repeated here.

As a reminder, the goal of this version of the replica trick is to calculate the partition functions $Z(A, \beta)$ and $Z_n(A, \beta)$ associated with the subsystem A in order to obtain the Rényi entropies $H_n(\rho_A) = \frac{1}{n-1} \lim_{\beta \rightarrow \infty} (n \log Z(A, \beta) - \log Z_n(A, \beta))$.

A.1 Entropy of coupled harmonic oscillators

After tracing/integrating out one of the oscillators in section 4.2, the effective action of the remaining degree of freedom is given, as a non-local integral kernel in τ by:

$$S_{eff} = \frac{1}{2} \int d\tau d\tau' \left[\left(-\frac{d^2}{d\tau^2} + M^2 \right) \delta(\tau - \tau') - \frac{l^2}{2M} e^{-M|\tau - \tau'|} \right] x_A(\tau) x_A(\tau') \quad (\text{A.1})$$

With any terms which vanish as $\beta \rightarrow \infty$ being ignored (they go to zero at the level of the action and, given the perturbative calculations here, it can be shown that even taking them into account the zero temperature result remains unchanged).

In order to calculate $Z(A, \beta)$, the operator $\frac{1}{-\frac{d^2}{d\tau^2} + M^2}(\tau, \tau')$ must be restricted to acting on periodic functions with period β (as consequence of the finite tempera-

ture), so we have:

$$\frac{1}{-\frac{d^2}{d\tau^2} + M^2}(\tau, \tau') \rightarrow \frac{1}{\beta} \sum_j \frac{e^{i\omega_j(\tau-\tau')}}{\omega_j^2 + M^2} \quad (\text{A.2})$$

With this, $S_{eff}^\beta = \frac{1}{2} \sum_j \left(\omega_j^2 + M^2 - \frac{l^2}{\omega_j^2 + M^2} \right) x_j^* x_j$ and we can perform another Gaussian integral to arrive at:

$$\log Z(A, \beta) = -\frac{1}{2} \sum_j \log \left(\omega_j^2 + M^2 - \frac{l^2}{\omega_j^2 + M^2} \right) \quad (\text{A.3})$$

Decomposing the logarithm and using the Matsubara sum $\sum_j \log(\omega_j^2 + C^2) = 2 \log \sinh\left(\frac{\beta C}{2}\right)$:

$$\begin{aligned} \log \left(\omega_j^2 + M^2 - \frac{l^2}{\omega_j^2 + M^2} \right) &= \log \left(\omega_j^2 + M^2 - l \right) + \log \left(\omega_j^2 + M^2 + l \right) \\ &\quad - \log \left(\omega_j^2 + M^2 \right) \end{aligned} \quad (\text{A.4})$$

$$Z(A, \beta) = \frac{\sinh\left(\frac{\beta M}{2}\right)}{2 \sinh\left(\frac{\beta \sqrt{M^2 + l}}{2}\right) \sinh\left(\frac{\beta \sqrt{M^2 - l}}{2}\right)} \quad (\text{A.5})$$

The extra factor of 2 in the denominator does not change any physical expectation value, but it allows the reduced density matrix to be properly normalized.

The next step is calculating $Z_n(A, \beta)$ and for that we need the particular sum $\sum_{u=0}^{n-1} \int_{u\beta}^{(u+1)\beta} d\tau L_\mu^\beta$. As shown previously, the local terms simply add up to the same expression at inverse temperature $n\beta$, so the focus now is on the non-local part **after taking the variables as periodic on β** (this comes from the very construction of the method):

$$\frac{1}{n\beta} \sum_{u=0}^{n-1} \sum_{j,j'} \int_{u\beta}^{(u+1)\beta} d\tau \int_{u\beta}^{(u+1)\beta} d\tau' \frac{1}{\beta} \sum_{j_0} \frac{e^{i\omega_{j_0}(\tau-\tau')}}{\omega_{j_0}^2 + M^2} e^{i\omega_j \tau + i\omega_{j'} \tau'} x_j x_{j'} \quad (\text{A.6})$$

Here is important to make clear that the Fourier coefficients of the variables are normalized as: $x_A(\tau) = \frac{1}{\sqrt{n\beta}} \sum_j e^{i\omega_j \tau} x_j$ (the **same** choice will be maintained in the field theory case). Furthermore, $\omega_{j_0} = \frac{2\pi j_0}{\beta}$ and $\omega_j = \frac{2\pi j}{n\beta}$, $\omega_{j'} = \frac{2\pi j'}{n\beta}$.

Thus, using the zero temperature kernel we have:

$$\frac{1}{n\beta} \sum_{u=0}^{n-1} \sum_{j,j'} \int_{u\beta}^{(u+1)\beta} d\tau \int_{u\beta}^{(u+1)\beta} d\tau' \frac{1}{2M} e^{-M|\tau-\tau'|} e^{i\omega_j\tau+i\omega_{j'}\tau'} x_j x_{j'} \quad (\text{A.7})$$

$$e^{-M|\tau-\tau'|} = e^{-M(\tau-\tau')}\theta(\tau-\tau') + e^{-M(\tau'-\tau)}\theta(\tau'-\tau) \quad (\text{A.8})$$

The integral over τ' results in:

$$\frac{2Me^{i(\omega_j+\omega_{j'})\tau}}{M^2+\omega_{j'}^2} - \frac{e^{-M\tau+i\omega_j\tau}}{M+i\omega_{j'}} e^{u(\beta M+2\pi i \frac{j'}{n})} - \frac{e^{M\tau+i\omega_j\tau}}{M-i\omega_{j'}} e^{-(u+1)(\beta M+2\pi i \frac{j'}{n})} \quad (\text{A.9})$$

Which, after the integrating over τ , becomes:

$$\frac{1}{n\beta} \sum_{u=0}^{n-1} e^{2\pi i \frac{j+j'}{n} u} \times \left[\frac{2M}{M^2+\omega_{j'}^2} \frac{e^{2\pi i \frac{j+j'}{n}} - 1}{i(\omega_j+\omega_{j'})} + \frac{e^{-\beta M+2\pi i \frac{j}{n}} - 1}{(M+i\omega_{j'})(M-i\omega_j)} + \frac{e^{-\beta M+2\pi i \frac{j'}{n}} - e^{2\pi i \frac{j+j'}{n}}}{(M-i\omega_{j'})(M+i\omega_j)} \right] \quad (\text{A.10})$$

Using the identity $\sum_{u=0}^{n-1} e^{2\pi i \frac{j+j'}{n} u} = n\delta_{j+j', \nu}^{nv}$, $\nu \in \mathbb{Z}$ (which can be checked explicitly using the geometric progression formula) as well as the fact that $\delta_{j+j', \nu}^{nv} \frac{e^{2\pi i \frac{j+j'}{n}} - 1}{i \frac{j+j'}{n}} = \delta_{j+j', \nu}^0$ (also easily checked), the terms in the effective action become:

$$\frac{l^2}{4M} \sum_{j,j'} \frac{2M}{M^2+\omega_{j'}^2} \delta_{j+j', \nu}^0 x_j x_{j'} = \frac{1}{2} \sum_j \frac{l^2}{M^2+\omega_j^2} x_j^* x_j \quad (\text{A.11})$$

$$\begin{aligned} & n \frac{l^2}{4M} \frac{1}{n\beta} \sum_{j,j'} \delta_{j+j', \nu}^{nv} \left[\frac{e^{-\beta M+2\pi i \frac{j}{n}} - 1}{(M+i\omega_{j'})(M-i\omega_j)} + \frac{e^{-\beta M+2\pi i \frac{j'}{n}} - e^{2\pi i \frac{j+j'}{n}}}{(M-i\omega_{j'})(M+i\omega_j)} \right] x_j x_{j'} \\ & = -n \frac{l^2}{4M} \frac{1}{n\beta} \sum_{j,\nu} \left[\frac{1 - e^{-\beta M+2\pi i \frac{j}{n}}}{(M+i\omega_{nv}-i\omega_j)(M-i\omega_j)} + h.c. \right] x_j x_{nv-j} \end{aligned} \quad (\text{A.12})$$

Where "h.c" indicates the Hermitian conjugate of the adjacent term and the extra factors of $\frac{1}{2}$ come from the overall action.

Notice that the first term is **exactly** the same as in the calculation of $Z(A, \beta)$ (summing this term with the local part of the action results in $S_{eff}^{n\beta}$) but now the inverse temperature is $n\beta$, meaning this term will be responsible for canceling the

denominator in the equation for $\text{Tr } \rho_A^n$ and won't generate entropy by itself. We'll see that this happens both in this model and in perturbation theory for QFT.

The second term contains a contribution which goes exponentially to zero as $\beta \rightarrow \infty$ meaning it can be thrown away as done previously. With this, making a perturbative expansion and taking the lowest order term:

$$Z_n(A, \beta) = \int \mathcal{D}x_j e^{-S_{eff}^{n\beta}} \times \left(1 - n \frac{l^2}{4M} \frac{1}{n\beta} \sum_{j,\nu} \left[\frac{1}{(M + i\omega_{n\nu} - i\omega_j)(M - i\omega_j)} + h.c. \right] x_j x_{n\nu-j} \right) \quad (\text{A.13})$$

And so:

$$\text{Tr } \rho_A^n = 1 - n \frac{l^2}{4M} \times \lim_{\beta \rightarrow \infty} \left(\frac{1}{n\beta} \sum_{j,\nu} \left[\frac{1}{(M + i\omega_{n\nu} - i\omega_j)(M - i\omega_j)} + h.c. \right] \langle x_j x_{n\nu-j} \rangle \right) \quad (\text{A.14})$$

Having used $\lim_{\beta \rightarrow \infty} \frac{Z(A, n\beta)}{[Z(A, \beta)]^n} = 1$.

By Wick's theorem, $\langle x_j x_{n\nu-j} \rangle = \delta_\nu^0 (\omega^2 + M^2 - \frac{l^2}{\omega^2 + M^2})^{-1}$, therefore, summing the Hermitian conjugate terms inside the brackets and eliminating the Kronecker delta:

$$\frac{1}{(M + i\omega_{n\nu} - i\omega_j)(M - i\omega_j)} + h.c. = \frac{2M^2 - 2\omega_j^2 + 2\omega_{n\nu}\omega_j}{[M^2 + (\omega_{n\nu} - \omega_j)^2] (M^2 + \omega_j^2)} \quad (\text{A.15})$$

$$\text{Tr } \rho_A^n = 1 - n \frac{l^2}{4M} \lim_{\beta \rightarrow \infty} \frac{1}{n\beta} \sum_j \frac{2M^2 - 2\omega_j^2}{(M^2 + \omega_j^2)^2} \frac{1}{\omega^2 + M^2 - \frac{l^2}{\omega^2 + M^2}} \quad (\text{A.16})$$

And the term inside the sum can be written as:

$$\frac{4M^2}{(M^2 + \omega_j^2)(M^2 + \omega_j^2 + l)(M^2 + \omega_j^2 - l)} - \frac{2}{(M^2 + \omega_j^2 + l)(M^2 + \omega_j^2 - l)} \quad (\text{A.17})$$

It is necessary to now expand the denominators as functions of l in order to find the correct leading term. Also, to use the formula $\lim_{\beta \rightarrow \infty} \frac{1}{\beta} \sum_j \frac{1}{\omega_j^2 + E^2} = \frac{1}{2E}$ [42], the denominators above must be split using the following partial fraction expansions:

$$\begin{aligned} \frac{1}{(x+a)(x+b)(x+c)} &= \frac{1}{(a-b)(a-c)} \frac{1}{x+a} \\ &+ \frac{1}{(b-a)(b-c)} \frac{1}{x+b} + \frac{1}{(c-a)(c-b)} \frac{1}{x+c} \end{aligned} \quad (\text{A.18})$$

$$\frac{1}{(x+a)(x+b)} = \frac{1}{b-a} \left(\frac{1}{x+a} - \frac{1}{x+b} \right) \quad (\text{A.19})$$

Thus, the zero temperature limit of the sum is:

$$4M^2 \left(\frac{1}{2l^2} \frac{1}{2\sqrt{M^2-l}} + \frac{1}{2l^2} \frac{1}{2\sqrt{M^2+l}} - \frac{1}{l^2} \frac{1}{2M} \right) - \frac{1}{l} \left(\frac{1}{2\sqrt{M^2-l}} - \frac{1}{2\sqrt{M^2+l}} \right) \quad (\text{A.20})$$

Finally, this means that at lowest order in l :

$$\text{Tr } \rho_A^n = 1 - n \frac{l^2}{16M^4} \quad (\text{A.21})$$

Which, as shown in Chapter 4, reproduces the exact result at same order.

A.2 Momentum-Space entropy in ϕ^3 theory

As explained in section 4.4, by applying the method to momentum-space entanglement in ϕ^3 , the term in the effective action responsible for generating entropy at lowest order is given by (one-loop propagator diagram):

$$- \frac{\lambda^2}{2} \int_{-\infty}^{\infty} \frac{dk_0}{2\pi} \int_{|\vec{k}| \leq \mu} \frac{d^{d-1}k}{(2\pi)^{d-1}} \int_{-\infty}^{\infty} \frac{dq_0}{2\pi} \int_{\mu < |\vec{q}|, |\vec{k}-\vec{q}| \leq a^{-1}} \frac{d^{d-1}q}{(2\pi)^{d-1}} \times \frac{1}{q_0^2 + \vec{q}^2 + m^2} \frac{1}{(k_0 - q_0)^2 + (\vec{k} - \vec{q})^2 + m^2} \phi_{-k_0, -\vec{k}} \phi_{k_0, \vec{k}} \quad (\text{A.22})$$

Fourier transforming only the time components of the fields in the equation above, the one-loop contribution to the effective action becomes (leaving aside for the moment the integration over spatial components):

$$\begin{aligned} & \int_{-\infty}^{\infty} \frac{dk_0}{2\pi} \int_{-\infty}^{\infty} \frac{dq_0}{2\pi} \frac{1}{q_0^2 + \vec{q}^2 + m^2} \frac{e^{ik_0(\tau-\tau')}}{(k_0 - q_0)^2 + (\vec{k} - \vec{q})^2 + m^2} \phi_{-\vec{k}}(\tau') \phi_{\vec{k}}(\tau) = \\ & \int d\tau \int d\tau' \int_{-\infty}^{\infty} \frac{dk_0}{2\pi} \int_{-\infty}^{\infty} \frac{dq_0}{2\pi} \frac{e^{iq_0(\tau-\tau')}}{q_0^2 + \vec{q}^2 + m^2} \frac{e^{i(k_0-q_0)(\tau-\tau')}}{(k_0 - q_0)^2 + (\vec{k} - \vec{q})^2 + m^2} \phi_{\vec{k}}(\tau) \phi_{-\vec{k}}(\tau') \\ & = \frac{1}{4\sqrt{\vec{q}^2 + m^2} \sqrt{(\vec{k} - \vec{q})^2 + m^2}} \int d\tau \int d\tau' e^{-(\sqrt{\vec{q}^2 + m^2} + \sqrt{(\vec{k} - \vec{q})^2 + m^2})|\tau - \tau'|} \phi_{\vec{k}}(\tau) \phi_{-\vec{k}}(\tau') \end{aligned}$$

As noted in the main body of the Thesis, this is an exponential kernel with decay rate $M = \sqrt{\vec{q}^2 + m^2} + \sqrt{(\vec{k} - \vec{q})^2 + m^2}$ and, as long as the integrals over $\vec{k}$ and $\vec{q}$ are

left for the end, this allows for the use of the same calculations as in the previous case of coupled harmonic oscillators.

Equivalently, going to finite temperature and writing the kernel in terms of the Matsubara frequencies it becomes:

$$\frac{1}{\beta} \sum_{j,j'} \frac{1}{\omega_j^2 + \vec{q}^2 + m^2} \frac{1}{(\omega_j - \omega_{j'})^2 + (\vec{k} - \vec{q})^2 + m^2} \phi_{j,\vec{k}}^* \phi_{j,\vec{k}} \quad (\text{A.23})$$

Using the formula for Matsubara sums in loops [42] to eliminate j' :

$$\frac{1}{\beta} \sum_{j'} \frac{1}{\omega_j^2 + E_1^2} \frac{1}{(\omega_j - \omega_{j'})^2 + E_2^2} = \frac{E_1 + E_2}{2E_1 E_2} \frac{1}{\omega_j^2 + (E_1 + E_2)^2} + \mathcal{O}(e^{-\beta}) \quad (\text{A.24})$$

With $\mathcal{O}(e^{-\beta})$ indicating terms which go to zero exponentially as $\beta \rightarrow \infty$.

Thus we get:

$$\sum_j \frac{\sqrt{\vec{q}^2 + m^2} + \sqrt{(\vec{k} - \vec{q})^2 + m^2}}{2\sqrt{\vec{q}^2 + m^2} \sqrt{(\vec{k} - \vec{q})^2 + m^2}} \frac{1}{\omega_j^2 + (\sqrt{\vec{q}^2 + m^2} + \sqrt{(\vec{k} - \vec{q})^2 + m^2})^2} \phi_{j,\vec{k}}^* \phi_{j,\vec{k}} \quad (\text{A.25})$$

Ignoring, as before, the $\mathcal{O}(e^{-\beta})$ terms.

There is no need to find $Z(\mu, \beta)$ explicitly, it's enough to keep in mind which terms it contains in order to make all cancellations in the last steps. So, moving on to $Z_n(\mu, \beta)$, the sum of double integrals over τ and τ' can be calculated by simply using the formulas (A.7) to (A.10) with the specification that, as mentioned earlier, $M = \sqrt{\vec{q}^2 + m^2} + \sqrt{(\vec{k} - \vec{q})^2 + m^2}$ (leaving the spatial momentum integrals intact). This means the non-local term contributes (after discarding all terms vanishing at zero temperature) as:

$$\frac{\lambda^2}{2} \int_{|\vec{k}| \leq \mu} \frac{d^{d-1}k}{(2\pi)^{d-1}} \int_{\mu < |\vec{q}|, |\vec{k} - \vec{q}| \leq a^{-1}} \frac{d^{d-1}q}{(2\pi)^{d-1}} \frac{1}{4\sqrt{\vec{q}^2 + m^2} \sqrt{(\vec{k} - \vec{q})^2 + m^2}} \times$$

$$\left[\sum_j \frac{2M^2}{\omega_j^2 + M^2} \phi_{j,\vec{k}}^* \phi_{j,\vec{k}} - n \frac{1}{n\beta} \sum_{j,\nu} \frac{2M^2 - 2\omega_j^2 + 2\omega_{\nu}\omega_j}{[M^2 + (\omega_{\nu} - \omega_j)^2] (M^2 + \omega_j^2)} \phi_{j,\vec{k}} \phi_{\nu-j,-\vec{k}} \right]$$

Once again there is a term similar to that in $Z(\mu, \beta)$. To see that they still cancel each other at lowest order in perturbation theory, consider the following: when

taking the contribution of this term to the logarithm of $Z(\mu, \beta)$ and $Z_n(\mu, \beta)$ at $\mathcal{O}(\lambda^2)$, Wick's theorem replaces the product of fields by a propagator times an multiplicative constant. This means that in $\log Z_n(\mu, \beta) - n \log Z(\mu, \beta)$ we get, ignoring the constants and momentum integration (which are the same for both partition functions):

$$\sum_j \frac{1}{\omega_j^2 + M^2} \frac{1}{\omega_j^2 + \vec{k}^2 + m^2} - n \sum_{j_0} \frac{1}{\omega_{j_0}^2 + M^2} \frac{1}{\omega_{j_0}^2 + \vec{k}^2 + m^2} \quad (\text{A.26})$$

Now, the frequencies ω_{j_0} are those of inverse temperature β , while ω_j are of $n\beta$, therefore, taking the limit $\beta \rightarrow \infty$ and using the Matsubara sum previously introduced:

$$\lim_{\beta \rightarrow \infty} \left(\sum_j \frac{1}{\omega_j^2 + M^2} \frac{1}{\omega_j^2 + \vec{k}^2 + m^2} - n \sum_{j_0} \frac{1}{\omega_{j_0}^2 + M^2} \frac{1}{\omega_{j_0}^2 + \vec{k}^2 + m^2} \right) = 0 \quad (\text{A.27})$$

With this, expanding the logarithms and taking the zero temperature limit, the Rényi entropies are:

$$H_n(\rho_\mu) = \frac{n}{n-1} \frac{\lambda^2}{2} \int_{|\vec{k}| \leq \mu} \frac{d^{d-1}k}{(2\pi)^{d-1}} \int_{\mu < |\vec{q}|, |\vec{k}-\vec{q}| \leq a^{-1}} \frac{d^{d-1}q}{(2\pi)^{d-1}} \frac{1}{4\sqrt{\vec{q}^2 + m^2}} \times \quad (\text{A.28})$$

$$\frac{1}{\sqrt{(\vec{k}-\vec{q})^2 + m^2}} \lim_{\beta \rightarrow \infty} \frac{1}{n\beta} \sum_{j,v} \frac{2M^2 - 2\omega_j^2 + 2\omega_{nv}\omega_j}{[M^2 + (\omega_{nv} - \omega_j)^2] (M^2 + \omega_j^2)} \langle \phi_{j,\vec{k}} \phi_{nv-j,-\vec{k}} \rangle$$

The field average is given by Wick's theorem, so the Matsubara sum to be calculated is:

$$\lim_{\beta \rightarrow \infty} \frac{1}{n\beta} \sum_j \frac{2M^2 - 2\omega_j^2}{(M^2 + \omega_j^2)^2 (\omega_j^2 + \vec{k}^2 + m^2)} = \quad (\text{A.29})$$

$$\lim_{\beta \rightarrow \infty} \frac{1}{n\beta} \sum_j \left[\frac{4M^2}{(M^2 + \omega_j^2)^2 (\omega_j^2 + \vec{k}^2 + m^2)} - \frac{2}{(M^2 + \omega_j^2)(\omega_j^2 + \vec{k}^2 + m^2)} \right]$$

Besides the Matsubara sum identity and partial fraction identities presented in the previous section of this Appendix, it is also necessary to split the first fraction with Matsubara frequencies via:

$$\frac{1}{(x+a)^2(x+b)} = \frac{1}{(a-b)^2} \frac{1}{x+b} - \frac{1}{(a-b)^2} \frac{1}{x+a} - \frac{1}{a-b} \frac{1}{(x+a)^2} \quad (\text{A.30})$$

To deal with a sum of the form $\frac{1}{\beta} \sum_j \frac{1}{(\omega_j^2 + E^2)^2}$ at the zero temperature limit, a good trick is:

$$\lim_{\beta \rightarrow \infty} \frac{1}{\beta} \sum_j \frac{1}{(\omega_j^2 + E^2)^2} = \left(-\frac{d}{dE^2} \right) \lim_{\beta \rightarrow \infty} \frac{1}{\beta} \sum_j \frac{1}{\omega_j^2 + E^2} = -\frac{d}{dE^2} \frac{1}{2E} = \frac{1}{4E^3} \quad (\text{A.31})$$

With this, the zero temperature limit of the total sum becomes:

$$\begin{aligned} & \frac{2M^2}{\sqrt{k^2 + m^2} \left(M^2 - \vec{k}^2 - m^2 \right)^2} - \frac{2M}{\left(M^2 - \vec{k}^2 - m^2 \right)^2} - \frac{1}{M \left(M^2 - \vec{k}^2 - m^2 \right)} \\ & - \frac{1}{\sqrt{k^2 + m^2} \left(M^2 - \vec{k}^2 - m^2 \right)} + \frac{1}{M \left(M^2 - \vec{k}^2 - m^2 \right)} \end{aligned} \quad (\text{A.32})$$

Besides the trivially cancelling terms, this result can be simplified by writing all terms in the same fraction and doing some manipulations:

$$\begin{aligned} & \frac{2M^2}{\sqrt{k^2 + m^2} \left(M^2 - \vec{k}^2 - m^2 \right)^2} - \frac{2M}{\left(M^2 - \vec{k}^2 - m^2 \right)^2} - \frac{1}{\sqrt{k^2 + m^2} \left(M^2 - \vec{k}^2 - m^2 \right)} \\ & = \frac{M^2 - 2M\sqrt{\vec{k}^2 + m^2} + \vec{k}^2 + m^2}{\sqrt{\vec{k}^2 + m^2} \left(M - \sqrt{\vec{k}^2 + m^2} \right)^2 \left(M^2 + \sqrt{\vec{k}^2 + m^2} \right)^2} \\ & = \frac{1}{\sqrt{\vec{k}^2 + m^2} \left(M + \sqrt{\vec{k}^2 + m^2} \right)^2} \end{aligned}$$

Finally, making the substitution $M = \sqrt{\vec{q}^2 + m^2} + \sqrt{(\vec{k} - \vec{q})^2 + m^2}$, we can restore the multiplicative constants and integrals and arrive at the complete formula for the Rényi entropy in ϕ^3 theory (and, as shown in Chapter 4, the use of momentum-space delta functions leads to the appearance of an overall volume factor and thus to an entropy density):

$$\begin{aligned} \frac{H_n(\rho_\mu)}{V} &= \frac{n}{n-1} \frac{\lambda^2}{8} \int_{|\vec{k}| \leq \mu} \frac{d^{d-1}k}{(2\pi)^{d-1}} \int_{\mu < |\vec{q}|, |\vec{k} - \vec{q}| \leq a^{-1}} \frac{d^{d-1}q}{(2\pi)^{d-1}} \frac{1}{\sqrt{\vec{k}^2 + m^2}} \frac{1}{\sqrt{\vec{q}^2 + m^2}} \\ &\times \frac{1}{\sqrt{(\vec{k} - \vec{q})^2 + m^2}} \frac{1}{\left(\sqrt{\vec{k}^2 + m^2} + \sqrt{\vec{q}^2 + m^2} + \sqrt{(\vec{k} - \vec{q})^2 + m^2} \right)^2} \end{aligned} \quad (\text{A.33})$$

A.3 Momentum-Space entropy in ϕ^4 theory

This time the bare action at UV cutoff a^{-1} is given by:

$$S[\phi] = \int \frac{d^d k}{2(2\pi)^d} (k^2 + m^2) \phi_k^* \phi_k + \frac{\lambda}{4!} \int \prod_{i=1}^4 \frac{d^d k_i}{(2\pi)^d} (2\pi)^d \delta(\sum_i k_i) \phi_{k_1} \phi_{k_2} \phi_{k_3} \phi_{k_4} \quad (\text{A.34})$$

To find the diagrams which are relevant to the entropy (generated by integrating out high-momentum modes up to scale μ), once again we have to focus on those with external momenta running inside a loop. At order λ the diagram leads solely to a mass renormalization and does not satisfy the criterion for generating entanglement, now at order λ^2 there are two diagrams do satisfy it: the two-loop correction to the propagator and the one-loop renormalization of the coupling constant. With this, the terms in the effective action at scale μ leading to non-zero entropy are (symmetry factors included)[35]:

$$- \frac{\lambda^2}{6} \int^* \frac{d^d k}{(2\pi)^d} \int^* \frac{d^d q}{(2\pi)^d} \frac{d^d p}{(2\pi)^d} \frac{1}{q^2 + m^2} \frac{1}{p^2 + m^2} \frac{1}{(k - q - p)^2 + m^2} \phi_k^* \phi_k \quad (\text{A.35})$$

$$- \frac{\lambda^2}{16} \int^* \prod_{i=1}^4 \frac{d^d k_i}{(2\pi)^d} \int^* \frac{d^d q}{(2\pi)^d} \frac{1}{q^2 + m^2} \frac{(2\pi)^d \delta(\sum_i k_i)}{(k_1 + k_2 - q)^2 + m^2} \phi_{k_1} \phi_{k_2} \phi_{k_3} \phi_{k_4} \quad (\text{A.36})$$

The integrals with asterisks indicating the specific the momentum regions arising from the tracing out of high-momentum modes.

The quadratic term in the fields is the most similar to the previous examples and its Fourier transform of the temporal component is (leaving the spatial momentum integrals aside for now):

$$\begin{aligned} & \int d\tau \int d\tau' \int \frac{dk_0}{2\pi} \frac{dq_0}{2\pi} \frac{dp_0}{2\pi} \frac{1}{q_0^2 + \vec{q}^2 + m^2} \frac{1}{p_0^2 + \vec{p}^2 + m^2} \times \\ & \frac{1}{(k_0 - q_0 - p_0)^2 + (\vec{k} - \vec{q} - \vec{p})^2 + m^2} e^{ik_0(\tau - \tau')} \phi_{\vec{k}}(\tau) \phi_{-\vec{k}}(\tau') \end{aligned} \quad (\text{A.37})$$

Which can be re-written as:

$$\begin{aligned} & \int d\tau \int d\tau' \int \frac{dk_0}{2\pi} \frac{dq_0}{2\pi} \frac{dp_0}{2\pi} \frac{e^{iq_0(\tau - \tau')}}{q_0^2 + \vec{q}^2 + m^2} \frac{e^{ip_0(\tau - \tau')}}{p_0^2 + \vec{p}^2 + m^2} \\ & \times \frac{e^{i(k_0 - q_0 - p_0)(\tau - \tau')}}{(k_0 - q_0 - p_0)^2 + (\vec{k} - \vec{q} - \vec{p})^2 + m^2} \phi_{\vec{k}}(\tau) \phi_{-\vec{k}}(\tau') \end{aligned} \quad (\text{A.38})$$

And, performing the integrals over temporal components of momenta:

$$\int d\tau \int d\tau' \frac{e^{-(\sqrt{\vec{q}^2+m^2}+\sqrt{\vec{p}^2+m^2}+\sqrt{(\vec{k}-\vec{q}-\vec{p})^2+m^2})|\tau-\tau'|}}{8\sqrt{\vec{q}^2+m^2}\sqrt{\vec{p}^2+m^2}\sqrt{(\vec{k}-\vec{q}-\vec{p})^2+m^2}} \phi_{\vec{k}}(\tau) \phi_{-\vec{k}}(\tau') \quad (\text{A.39})$$

Thus, it is an exponential kernel $e^{-M|\tau-\tau'|}$ with $M = \sqrt{\vec{q}^2+m^2} + \sqrt{\vec{p}^2+m^2} + \sqrt{(\vec{k}-\vec{q}-\vec{p})^2+m^2}$.

As for the four-field term, a similar Fourier transform can be written as:

$$\begin{aligned} & \int d\tau_1 d\tau_2 d\tau_3 d\tau_4 \int \prod_{i=1}^4 \frac{d(k_i)_0}{2\pi} \int \frac{dq_0}{2\pi} \frac{2\pi\delta((k_1)_0 + (k_2)_0 + (k_3)_0 + (k_4)_0)}{q_0^2 + \vec{q}^2 + m^2} \\ & \times \frac{e^{i(k_1)_0\tau_1 + i(k_2)_0\tau_2 + i(k_3)_0\tau_3 + i(k_4)_0\tau_4}}{((k_1)_0 + (k_2)_0 - q_0)^2 + (\vec{k}_1 + \vec{k}_2 - \vec{q})^2 + m^2} \phi_{\vec{k}_1}(\tau_1) \phi_{\vec{k}_2}(\tau_2) \phi_{\vec{k}_3}(\tau_3) \phi_{\vec{k}_4}(\tau_4) \\ & = \int d\tau_1 d\tau_2 d\tau_3 d\tau_4 \int \prod_{i=1}^3 \frac{d(k_i)_0}{2\pi} \int \frac{dq_0}{2\pi} \frac{1}{q_0^2 + \vec{q}^2 + m^2} \\ & \times \frac{e^{i(k_1)_0(\tau_1-\tau_4) + i(k_2)_0(\tau_2-\tau_4) + i(k_3)_0(\tau_3-\tau_4)}}{((k_1)_0 + (k_2)_0 - q_0)^2 + (\vec{k}_1 + \vec{k}_2 - \vec{q})^2 + m^2} \phi_{\vec{k}_1}(\tau_1) \phi_{\vec{k}_2}(\tau_2) \phi_{\vec{k}_3}(\tau_3) \phi_{\vec{k}_4}(\tau_4) \\ & = \int d\tau_1 d\tau_2 d\tau_3 d\tau_4 \int \prod_{i=1}^3 \frac{d(k_i)_0}{2\pi} \int \frac{dq_0}{2\pi} \frac{e^{iq_0(\tau_1-\tau_4)}}{q_0^2 + \vec{q}^2 + m^2} \\ & \times \frac{e^{i((k_1)_0 + (k_2)_0 - q_0)(\tau_1-\tau_4) + i(k_2)_0(\tau_2-\tau_1) + i(k_3)_0(\tau_3-\tau_4)}}{((k_1)_0 + (k_2)_0 - q_0)^2 + (\vec{k}_1 + \vec{k}_2 - \vec{q})^2 + m^2} \phi_{\vec{k}_1}(\tau_1) \phi_{\vec{k}_2}(\tau_2) \phi_{\vec{k}_3}(\tau_3) \phi_{\vec{k}_4}(\tau_4) \end{aligned} \quad (\text{A.40})$$

Performing the Fourier transform of the Lorentzian functions and integrating the remaining time components of momenta to obtain delta functions, equation (A.40) becomes:

$$\begin{aligned} & \int \prod_{i=1}^4 d\tau_i \frac{\delta(\tau_2 - \tau_1) \delta(\tau_3 - \tau_4)}{4\sqrt{\vec{q}^2+m^2}\sqrt{(\vec{k}_1 + \vec{k}_2 - \vec{q})^2+m^2}} \int \prod_{i=1}^3 \frac{d(k_i)_0}{2\pi} \int \frac{dq_0}{2\pi} \frac{e^{iq_0(\tau_1-\tau_4)}}{q_0^2 + \vec{q}^2 + m^2} \\ & \times e^{-(\sqrt{\vec{q}^2+m^2}+\sqrt{(\vec{k}_1+\vec{k}_2-\vec{q})^2+m^2})|\tau_1-\tau_4|} \phi_{\vec{k}_1}(\tau_1) \phi_{\vec{k}_2}(\tau_2) \phi_{\vec{k}_3}(\tau_3) \phi_{\vec{k}_4}(\tau_4) \\ & = \int d\tau d\tau' \frac{e^{-(\sqrt{\vec{q}^2+m^2}+\sqrt{(\vec{k}_1+\vec{k}_2-\vec{q})^2+m^2})|\tau-\tau'|}}{4\sqrt{\vec{q}^2+m^2}\sqrt{(\vec{k}_1 + \vec{k}_2 - \vec{q})^2+m^2}} \phi_{\vec{k}_1}(\tau) \phi_{\vec{k}_2}(\tau) \phi_{\vec{k}_3}(\tau') \phi_{\vec{k}_4}(\tau') \end{aligned} \quad (\text{A.41})$$

Where in the last equality τ_1 and τ_4 were renamed to τ and τ' , respectively.

This new, four-field, term also has an exponential integral kernel with constant $\tilde{M} = \sqrt{\vec{q}^2+m^2} + \sqrt{(\vec{k}_1 + \vec{k}_2 - \vec{q})^2+m^2}$ and it, alongside the previous two-

field term, can be manipulated using the same strategies as in the oscillator and ϕ^3 cases.

At lowest order each of the terms above can be treated on its own and the total entropy taken as simply the sum of each contribution, as any cross terms will necessarily be of higher order.

For the two-field term, whose entropy will be denoted by $H_n^{m^2}(\rho_\mu)$ the steps of the ϕ^3 calculation can be followed verbatim, meaning that the following Matsubara sum appears:

$$\lim_{\beta \rightarrow \infty} \frac{1}{n\beta} \sum_j \left[\frac{4M^2}{(M^2 + \omega_j^2)^2(\omega_j^2 + \vec{k}^2 + m^2)} - \frac{2}{(M^2 + \omega_j^2)(\omega_j^2 + \vec{k}^2 + m^2)} \right] \quad (\text{A.42})$$

Again drawing from the ϕ^3 case, the limit of the Matsubara sum will simplify to be $\left[\sqrt{\vec{k}^2 + m^2} \left(M + \sqrt{\vec{k}^2 + m^2} \right)^2 \right]^{-1}$, so replacing M by its full expression and including all momentum integrals and numerical factors, the entropy density is given by:

$$\begin{aligned} \frac{H_n^{m^2}(\rho_\mu)}{V} &= \frac{n}{n-1} \frac{\lambda^2}{48} \int^* \frac{d^{d-1}k}{(2\pi)^{d-1}} \frac{d^{d-1}q}{(2\pi)^{d-1}} \frac{d^{d-1}p}{(2\pi)^{d-1}} \times \\ &\quad \left(\sqrt{\vec{k}^2 + m^2} + \sqrt{\vec{q}^2 + m^2} + \sqrt{\vec{p}^2 + m^2} + \sqrt{(\vec{k} - \vec{q} - \vec{p})^2 + m^2} \right)^{-2} \\ &\quad \frac{1}{\sqrt{\vec{k}^2 + m^2} \sqrt{\vec{q}^2 + m^2} \sqrt{\vec{p}^2 + m^2} \sqrt{(\vec{k} - \vec{q} - \vec{p})^2 + m^2}} \end{aligned} \quad (\text{A.43})$$

With momentum integration regions being those shown in Chapter 4.

To deal with the four-field term it's important to calculate again the sum of double integrals $\sum_{u=0}^{n-1} \int_{u\beta}^{(u+1)\beta} d\tau \int_{u\beta}^{(u+1)\beta} d\tau'$. As before, it is simpler to keep the zero-temperature kernel and take just the fields at inverse temperature $n\beta$. Because of the way the imaginary times of the fields are paired, this calculation leads to:

$$\begin{aligned} &\frac{1}{(n\beta)^2} \sum_{j_1, j_2, j_3, j_4} \sum_{u=0}^{n-1} \int_{u\beta}^{(u+1)\beta} d\tau \int_{u\beta}^{(u+1)\beta} d\tau' \times \\ &e^{-\tilde{M}|\tau - \tau'| + i(\omega_{j_1} + \omega_{j_2})\tau + i(\omega_{j_3} + \omega_{j_4})\tau'} \phi_{j_1, \vec{k}_1} \phi_{j_2, \vec{k}_2} \phi_{j_3, \vec{k}_3} \phi_{j_4, \vec{k}_4} \end{aligned} \quad (\text{A.44})$$

In order to use the calculations of the previous section to shortcut the steps here, we have to include an extra factor of $\frac{1}{n\beta}$ (whose origin is the new pair of field variables appearing in this contribution) and make the replacements $\omega_j \rightarrow \omega_{j_1} + \omega_{j_2}$ and $\omega_{j'} \rightarrow \omega_{j_3} + \omega_{j_4}$. Then, using again (A.7) to (A.10), equation (A.44) becomes:

$$\frac{1}{n\beta} \sum_{j_1, j_2, j_3, j_4} \frac{2\tilde{M}}{\tilde{M}^2 + (\omega_{j_3} + \omega_{j_4})^2} \delta_{j_1+j_2+j_3+j_4}^0 \phi_{j_1, \vec{k}_1} \phi_{j_2, \vec{k}_2} \phi_{j_3, \vec{k}_3} \phi_{j_4, \vec{k}_4} - n \frac{1}{(n\beta)^2} \times$$

$$\sum_{j_1, j_2, j_3, j_4} \delta_{j_1+j_2+j_3+j_4}^{mv} \left[\frac{1}{\tilde{M} + i(\omega_{j_3} + \omega_{j_4})} \frac{1}{\tilde{M} - i(\omega_{j_1} + \omega_{j_2})} + h.c. \right] \phi_{j_1, \vec{k}_1} \phi_{j_2, \vec{k}_2} \phi_{j_3, \vec{k}_3} \phi_{j_4, \vec{k}_4}$$

Once again, the first term will be cancelled when calculating the entropy. To see how this happens, remember that the partition function $Z(\mu, \beta)$ has the following contribution from the one-loop renormalization of the coupling at finite temperature:

$$- \frac{\lambda^2}{16\beta^2} \sum_{j_1, j_2} \frac{1}{\omega_j^2 + \vec{q}^2 + m^2} \frac{\delta_{j_1+j_2+j_3+j_4}^0 \phi_{j_1, \vec{k}_1} \phi_{j_2, \vec{k}_2} \phi_{j_3, \vec{k}_3} \phi_{j_4, \vec{k}_4}}{(\omega_j - \omega_{j_1} - \omega_{j_2})^2 + (\vec{k}_1 + \vec{k}_2 - \vec{q})^2 + m^2} \quad (\text{A.45})$$

With the inclusion of the same spatial momentum integrals as in the effective action.

Now, summing over j in the finite-temperature action and discarding the terms which go exponentially to zero as $\beta \rightarrow \infty$, we get:

$$- \frac{\lambda^2}{16} \frac{\sqrt{\vec{q}^2 + m^2} + \sqrt{(\vec{k}_1 + \vec{k}_2 - \vec{q})^2 + m^2}}{2\sqrt{\vec{q}^2 + m^2} \sqrt{(\vec{k}_1 + \vec{k}_2 - \vec{q})^2 + m^2}} \times$$

$$\frac{1}{\beta} \sum_{j_1, j_2} \frac{\delta_{j_1+j_2+j_3+j_4}^0 \phi_{j_1, \vec{k}_1} \phi_{j_2, \vec{k}_2} \phi_{j_3, \vec{k}_3} \phi_{j_4, \vec{k}_4}}{(\omega_{j_1} + \omega_{j_2})^2 + (\sqrt{\vec{q}^2 + m^2} + \sqrt{(\vec{k}_1 + \vec{k}_2 - \vec{q})^2 + m^2})^2} \quad (\text{A.46})$$

By including all missing integrations and numerical factors both in the expression above and in the first term generated by the one-loop renormalization of the coupling, it's easy to see that they're exactly the same. Thus, these terms do cancel each other.

This means that the only remaining contribution to the entropy involves the

sum:

$$\sum_{j_1, j_2, j_3, j_4} \frac{\delta_{j_1+j_2+j_3+j_4}^{nv}}{(n\beta)^2} \left[\frac{1}{\tilde{M} + i(\omega_{j_3} + \omega_{j_4})} \frac{1}{\tilde{M} - i(\omega_{j_1} + \omega_{j_2})} + h.c. \right] \langle \phi_{j_1, \vec{k}_1} \phi_{j_2, \vec{k}_2} \phi_{j_3, \vec{k}_3} \phi_{j_4, \vec{k}_4} \rangle \quad (\text{A.47})$$

The average $\langle \phi_{j_1, \vec{k}_1} \phi_{j_2, \vec{k}_2} \phi_{j_3, \vec{k}_3} \phi_{j_4, \vec{k}_4} \rangle$ is calculated via Wick's theorem and each contraction is given $\langle \phi_{j_1, \vec{k}_1} \phi_{j_2, \vec{k}_2} \rangle = \delta_{j_1+j_2}^0 \frac{(2\pi)^{d-1} \delta(\vec{k}_1 - \vec{k}_2)}{\omega_j^2 + \vec{k}_1^2 + m^2}$. With this, the sum simplifies to:

$$\begin{aligned} & \frac{1}{(n\beta)^2} \sum_{j_1, j_4} \frac{2}{\tilde{M}^2} \frac{1}{\omega_{j_1}^2 + \vec{k}_1^2 + m^2} \frac{1}{\omega_{j_4}^2 + \vec{k}_4^2 + m^2} \\ & + \frac{2}{(n\beta)^2} \sum_{j_1, j_4} \left[\frac{1}{\tilde{M} - i(\omega_{j_1} - \omega_{j_4})} \frac{1}{\tilde{M} - i(\omega_{j_1} - \omega_{j_4})} + h.c. \right] \frac{1}{\omega_{j_1}^2 + \vec{k}_1^2 + m^2} \frac{1}{\omega_{j_4}^2 + \vec{k}_4^2 + m^2} \end{aligned} \quad (\text{A.48})$$

Which in turn is:

$$\begin{aligned} & \frac{2}{\tilde{M}^2} \left(\frac{1}{n\beta} \sum_{j_1} \frac{1}{\omega_{j_1}^2 + \vec{k}_1^2 + m^2} \right) \left(\frac{1}{n\beta} \sum_{j_1} \frac{1}{\omega_{j_1}^2 + \vec{k}_4^2 + m^2} \right) + \\ & \frac{2}{(n\beta)^2} \sum_{j_1, j_4} \left[\frac{4\tilde{M}^2}{(\tilde{M}^2 + (\omega_{j_1} - \omega_{j_4})^2)^2} - \frac{2}{\tilde{M}^2 + (\omega_{j_1} - \omega_{j_4})^2} \right] \frac{1}{\omega_{j_1}^2 + \vec{k}_1^2 + m^2} \frac{1}{\omega_{j_4}^2 + \vec{k}_4^2 + m^2} \end{aligned} \quad (\text{A.49})$$

The sums in the first line of equation (A.49) are simple and result in the expression $(2\tilde{M}^2 \sqrt{\vec{k}_1^2 + m^2} \sqrt{\vec{k}_4^2 + m^2})^{-1}$. The second line can be written as:

$$\frac{4\tilde{M}^2}{(\tilde{M}^2 + (\omega_{j_1} - \omega_{j_4})^2)^2} = - \lim_{E \rightarrow \tilde{M}} \frac{d}{dE^2} \frac{4\tilde{M}^2}{E^2 + (\omega_{j_1} - \omega_{j_4})^2} \quad (\text{A.50})$$

Performing the necessary Matsubara sums and zero temperature limits, as well as a fair bit of algebra, this function can be simplified to:

$$\frac{1}{\sqrt{\vec{k}_1^2 + m^2} \sqrt{\vec{k}_4^2 + m^2} \left(\tilde{M} + \sqrt{\vec{k}_1^2 + m^2} + \sqrt{\vec{k}_4^2 + m^2} \right)^2} \quad (\text{A.51})$$

Renaming $\vec{k}_1$ by $\vec{k}$, $\vec{k}_4$ by $\vec{k}$ and $\vec{k}_2$ by $\vec{p}$ (they get the same name as one always appears without the other after using delta functions to eliminate integrals), using the explicit formula for $\tilde{M}$ and collecting the previous results, the contribution to the

Rényi entropy density coming from the one-loop renormalization of the coupling, denoted by $H_n^\lambda(\rho_\mu)$, is:

$$\begin{aligned} \frac{H_n^\lambda(\rho_\mu)}{V} &= \frac{n}{n-1} \frac{\lambda^2}{64} \int_{|\vec{k}|, |\vec{p}| \leq \mu} \frac{d^{d-1}k}{(2\pi)^{d-1}} \frac{d^{d-1}p}{(2\pi)^{d-1}} \int_{\mu \leq |\vec{q}|, |\vec{k}+\vec{p}-\vec{q}| \leq a^{-1}} \frac{d^{d-1}q}{(2\pi)^{d-1}} \\ &\times \frac{I(\vec{k}, \vec{p}, \vec{q})}{\sqrt{\vec{k}^2 + m^2} \sqrt{\vec{p}^2 + m^2} \sqrt{\vec{q}^2 + m^2} \sqrt{(\vec{k} + \vec{p} - \vec{q})^2 + m^2}} \end{aligned} \quad (\text{A.52})$$

With:

$$\begin{aligned} I(\vec{k}, \vec{p}, \vec{q}) &= \frac{1}{\left(\sqrt{\vec{k}^2 + m^2} + \sqrt{\vec{p}^2 + m^2} + \sqrt{\vec{q}^2 + m^2} + \sqrt{(\vec{k} + \vec{p} - \vec{q})^2 + m^2} \right)^2} \\ &+ \frac{1}{2 \left(\sqrt{\vec{q}^2 + m^2} + \sqrt{(\vec{k} + \vec{p} - \vec{q})^2 + m^2} \right)^2} \end{aligned} \quad (\text{A.53})$$

Finally, the total entropy is $H_n(\rho_\mu) = H_n^{m^2}(\rho_\mu) + H_n^\lambda(\rho_\mu)$, as shown previously.

Bibliography

- [1] L. E. Ballentine. *Quantum Mechanics: A Modern Development*. World Scientific, 1998.
- [2] J. von Neumann. *Mathematical Foundations of Quantum Mechanics*. Princeton University Press, 1955.
- [3] S. Axler. *Linear Algebra Done Right*. Springer-Verlag, 1997.
- [4] B. Zeng et al. *Quantum Information Meets Quantum Matter*. Springer, 2019.
- [5] C. E. Shannon. “A mathematical theory of communication”. In: *The Bell System Technical Journal* 27.3 (1948), pp. 379–423.
- [6] P. Calabrese and J. Cardy. “Entanglement entropy and conformal field theory”. In: *J. Phys. A: Math. Theor.* **42**.504005 (2009).
- [7] B. F. Svaiter and N. F. Svaiter. “The Distributional Zeta-Function in Disordered Field Theory”. In: *Int. J. Mod. Phys. A* 31.25 (2016), p. 1650144.
- [8] R. Acosta-Diaz et al. “Disordered Bose–Einstein condensate in hard walls trap”. In: *J. Phys. A: Math. Theor.* 52.44 (Sept. 2019), p. 445401.
- [9] M. Ohya and D. Petz. *Quantum Entropy and Its Use*. Springer-Verlag, 1993.
- [10] M. M. Wolf et al. “Area Laws in Quantum Systems: Mutual Information and Correlations”. In: *Phys. Rev. Lett.* **100**.070502 (2014).
- [11] L. Bombelli et al. “Quantum source of entropy for black holes”. In: *Phys. Rev. D* 34 (1986), pp. 373–383.
- [12] M. Srednicki. “Entropy and area”. In: *Phys. Rev. Lett.* 71 (1993), pp. 666–669.
- [13] G. ’t Hooft. “On the Quantum Structure of a Black Hole”. In: *Nucl. Phys. B* 256 (1985), pp. 727–745.

- [14] S. Ryu and T. Takayanagi. “Holographic derivation of entanglement entropy from AdS/CFT”. In: *Phys. Rev. Lett.* **96** (2006), p. 181602.
- [15] S. N. Solodukhin. “Entanglement entropy of black holes”. In: *Living Rev. Rel.* **14** (2011), p. 8.
- [16] T. Nishioka. “Entanglement entropy: Holography and renormalization group”. In: *Rev. Mod. Phys.* **90.3** (2018).
- [17] D. V. Fursaev. “The heat-kernel expansion on a cone and quantum fields near cosmic strings”. In: *Class. Quantum Grav.* **11.1431** (1994).
- [18] D. V. Fursaev and G. Miele. “Cones, spins and heat kernels”. In: *Nucl. Phys. B* **484** (1997), pp. 697–723.
- [19] D. V. Fursaev and S. N. Solodukhin. “On one-loop renormalization of black-hole entropy”. In: *Phys. Lett. B* **365** (1996), pp. 51–55.
- [20] D. Pontello. *Aspects of Entanglement Entropy in Algebraic Quantum Field Theory*. 2020. arXiv: [2005.13975 \[hep-th\]](https://arxiv.org/abs/2005.13975).
- [21] S. Hollands and J. Sanders. *Entanglement Measures and Their Properties in Quantum Field Theory*. Springer International Publishing, 2018.
- [22] J. Eisert, M. Cramer, and M.B. Plenio. “Colloquium: Area laws for the entanglement entropy”. In: *Rev. Mod. Phys.* **82.277** (2010).
- [23] R. Haag. *Local Quantum Physics: Fields, Particles, Algebras*. Springer-Verlag, 1992.
- [24] F. Strocchi. *An Introduction to Non-Perturbative Foundations of Quantum Field Theory*. Oxford University Press, 2013.
- [25] H. Araki, K. Hepp, and D. Ruelle. “On the asymptotic behaviour of Wightman functions in space-like directions”. In: *Helv. Phys. Acta* **35.III** (1962), pp. 164–174.
- [26] F.G.S.L. Brandão and M. Horodecki. “Exponential Decay of Correlations Implies Area Law”. In: *Commun. Math. Phys.* **333** (2015), pp. 761–798.

- [27] O. Bratteli and D.W. Robinson. *Operator Algebras and Quantum Statistical Mechanics 1*. Springer-Verlag, 1979.
- [28] O. Bratteli and D.W. Robinson. *Operator Algebras and Quantum Statistical Mechanics 2*. Springer-Verlag, 1981.
- [29] P. Zanardi; D. A. Lidar and S. Lloyd. “Quantum tensor product structures are observable-induced”. In: *Phys. Rev. Lett.* **92**.060402 (2004).
- [30] J. Lin and D. Radicevic. *Comments on Defining Entanglement Entropy*. 2018. arXiv: [1808.05939 \[hep-th\]](#).
- [31] N. Goldenfeld. *Lectures on Phase Transitions and the Renormalization Group*. Perseus Books Publishing, 1992.
- [32] J. Cardy. *Scaling and Renormalization in Statistical Physics*. Cambridge University Press, 1996.
- [33] S. Weinberg. *The Quantum Theory of Fields - Volume 1: Foundations*. Cambridge University Press, 1996.
- [34] M. E. Peskin and D. V. Schroeder. *An Introduction to Quantum Field Theory*. Addison-Wesley Publishing Company, 1996.
- [35] K. G. Wilson and J. B. Kogut. “The Renormalization group and the epsilon expansion”. In: *Phys. Rept.* **12** (1974), pp. 75–199.
- [36] S. H. Strogatz. *Nonlinear Dynamics and Chaos: with applications to physics, biology, chemistry and engineering*. Westview Press, 2015.
- [37] H.-W. Hammer and L. Platter. “Efimov States in Nuclear and Particle Physics”. In: *Ann. Rev. Nucl. Part. Sci.* **60** (2010), pp. 207–236.
- [38] N. Dupuis et al. “The nonperturbative functional renormalization group and its applications”. In: *Phys. Rept.* **910** (2021), pp. 1–114.
- [39] F. J. Wegner and A. Houghton. “Renormalization Group Equation for Critical Phenomena”. In: *Phys. Rev. A* **8**.1 (1973).

- [40] V. Balasubramanian; M. B. McDermott and M. Van Raamsdonk. "Momentum-space entanglement and renormalization in quantum field theory". In: *Phys. Rev. D* **86**.045014 (2012).
- [41] P. Zanardi. "Quantum entanglement in fermionic lattices". In: *Phys. Rev. A* **65**.042101 (2002).
- [42] M. Le Bellac. *Thermal Field Theory*. Cambridge University Press, 1996.
- [43] R. Jackiw. "Schrodinger picture analysis of boson and fermion quantum field theories". In: Sept. 1987.
- [44] S.-K. Ma. "The renormalization group and the large n limit". In: *J. Math. Phys.* **15**.1866 (1974).
- [45] M. A. Nielsen and I. L. Chuang. *Quantum Computation and Quantum Information (10th ed.)* Cambridge University Press, 2010.
- [46] A. Nayak and P. Sen. "Invertible Quantum Operations and Perfect Encryption of Quantum States". In: *Quantum Info. Comput.* 7.1 (2007).
- [47] U. Weiss. *Quantum Dissipative Systems*. World Scientific Publishing, 1993.