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An LMI Approach to Full- and Reduced-Order Switched Luenberger Observers for Switched Affine Systems

Victor Leonardo Yoshimura[1] · **Edvaldo Assunção**[2] · **Marcelo Carvalho Minhoto Teixeira**[2]

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Abstract This work deals with reduced-order Luenberger-like observer design and control for switched affine systems. Formulation of state-dependent and dynamic output-dependent switching laws for those systems was already addressed, however, in a full-order treatment. The proposed procedure defines an estimator system only for the unmeasured part of the system state. Linear matrix inequalities for obtaining output gain matrices and for composing a reduced-order Luenberger observer-based switching law are proposed. Finally, the results are validated by design and simulation for DC–DC converters.

Keywords Switched affine systems · Linear matrix inequalities · Switched Luenberger observers · Reduced order switched Luenberger observers · DC–DC converters

1 Introduction

In the last two decades, a great effort has been dealt to understand switched systems: systems whose dynamics are both continuous and discrete in time. The interest in Switched Systems: increased significantly after the works proposed by Liberzon and Morse (1999) and DeCarlo et al. (2000), which proposed many new problems in the area.

Initially, the focus was on switched linear systems, which are still studied, and many important results were obtained. A good survey was written by Lin and Antsaklis (2009), while some books written by Liberzon (2003) and Sun and Ge (2005) cover the subject more extensively. Piecewise affine systems and switched affine systems (SASs) have drawn attention more recently, the latter, specifically for its usage on modeling DC–DC converters.

SASs have some unique features: there is no adding control input, commonly denoted **u**; hence, the control can only be achieved by choosing the system matrices among a finite set of choices, including the affine vector added to compose the state-derivative equation. Piecewise affine systems, on the other hand, have their state-space partitioned *a priori*: their control strategy uses the adding input, **u**, with, for example, an output feedback (Rodrigues and How 2003).

Back on SASs, sufficient stability conditions using an approach by average state space which demanded the average input matrix to be null were determined by Bolzern and Spinelli (2004). The treatment to the non existence of a common-mode equilibrium point in buck–boost converters was pointed out by Corona et al. (2007). Xu et al. (2008) proposed a more elaborate and general approach by defining practical stability. Design of variable-structure control of SASs applied to DC–DC converters was proposed by Cardim et al. (2009). Recently, sampled-data control (Hauroigne et al. 2011; Hetel and Fridman 2013), local stabilization (Hetel and Bernuau 2015) and $\mathcal{H}_\infty$ performance (Deaecto 2015) of those systems were addressed.

An approach for establishing SASs stability using (LMIs) was proposed by Deaecto et al. (2010). In that work, some

✉ Victor Leonardo Yoshimura
victor@facom.ufms.br

Edvaldo Assunção
edvaldo@dee.feis.unesp.br

Marcelo Carvalho Minhoto Teixeira
marcelo@dee.feis.unesp.br

1 Faculdade de Computação, Laboratório de Sistemas Computacionais de Alto Desempenho (LSCAD), Universidade Federal de Mato Grosso do Sul (UFMS), Cidade Universitária s/n, Caixa Postal 549, Campo Grande, MS CEP 79070-900, Brazil

2 Control Research Laboratory, Department of Electrical Engineering, UNESP - Univ Estadual Paulista, Carlos Rossi Avenue n 1370, Ilha Solteira, São Paulo 15.385-000, Brazil

optimal criteria were also proposed and applied to DC–DC converters. An important feature is that the average state space was not required, sufficing to use the SAS model and LMIs to elaborate a stabilizing state-dependent switching law. Recently, another proposal using a max-type state-dependent switching law was made by Scharlau et al. (2014).

LMIs are widely used in control theory (Boyd et al. 1994) and an LMI-proposal of a dynamic output-dependent switching law for SASs was addressed by Yoshimura et al. (2013): using the sensed output and imposing the same switching law to the plant and to the estimator, the estimated state could be used to substitute the real state.

Despite of the amount of research on SASs, there is no result on reducing the state estimator order, as far as the authors could search, which is an important feature in establishing observer-based control theory as already shown by Luenberger in his original works on Linear Systems (Luenberger 1964, 1971): it simplifies the observer implementation. Some helpful related insight can also be found in model reduction theory (Assunção et al. 2007) and static output feedback (Mainardi Júnior et al. 2015).

One gap in SASs theory is, then, the absence of a result that presents a way of reducing the number of state variables to be estimated by an auxiliary dynamic system. The remaining variables are to be completed by the output matrix (commonly denoted $\mathbf{y}$) with possibly an algebraic (i.e., non-dynamic) manipulation on its variables.

This paper is intended to fill that gap and is organized as follows: Sect. 2 states the problem to be dealt and important previous results; Sect. 3 presents the new results; Sect. 4 shows the application of the new results in examples involving DC–DC converters; finally, Sect. 5 presents the conclusions.

Notation

Apostrophes denote matrix transpositions. $\mathfrak{S}$ is the symmetry by addition operator, i.e., $\mathfrak{S}(\mathbf{X}) = \mathbf{X} + \mathbf{X}'$; $\mathrm{Tr}(\mathbf{X})$ is the trace of $\mathbf{X}$; $\mathrm{diag}\{\mathbf{X}_1, \mathbf{X}_2, \ldots, \mathbf{X}_N\}$ is a block-diagonal matrix with the mentioned blocks along the main diagonal, in that order; $\Phi(N)$ is the N-dimensional unit simplex set. Let $\mathbf{A}_1, \mathbf{A}_2, \ldots, \mathbf{A}_N$ be matrices of the same dimension and $\boldsymbol{\varphi} \in \Phi(N)$, a convex combination is denoted $\mathbf{A}_{\boldsymbol{\varphi}} = \sum_{i=1}^{N} \varphi_i \mathbf{A}_i$. $\mathbb{R}^{n\times n}_{\succ 0}$ denotes the set of n-order positive-definite square real matrices and $\mathbb{N}_N = \{1, 2, \ldots, N\}$.

In proofs, equation numbers will be placed above equalities/inequalities in order to facilitate the reading: they present the equation combination to obtain the respective result.

2 Problem Statement and Previous Results

SASs have the state-space model

$$\dot{\mathbf{x}} = \mathbf{A}_\sigma \mathbf{x} + \mathbf{b}_\sigma \tag{1a}$$

$$\mathbf{y} = \mathbf{C}_\sigma \mathbf{x} \tag{1b}$$

$$\mathbf{z} = \mathbf{E}_\sigma \mathbf{x} \tag{1c}$$

where $\mathbf{y}$ is the measured output, $\mathbf{z}$ is the controlled output and σ is the switching law (function), which can be state-dependent, time-dependent, output-dependent and so on. The co-domain of σ is $\mathbb{N}_N$, so it chooses the active mode $(\mathbf{A}_i, \mathbf{b}_i, \mathbf{C}_i, \mathbf{E}_i) \in (\mathbb{R}^{n\times n}, \mathbb{R}^n, \mathbb{R}^{m\times n}, \mathbb{R}^{p\times n})$, among N possibilities. It must be noted that the only possible control action is to properly design the switching law, as there is no feedback input apart from that which determines the active mode.

The initial problem in SASs, addressed by Bolzern and Spinelli (2004), was to determine which subset of the state space could provide an operation point (therein called "switched equilibrium", i.e., a point to which the state can be driven to), $\tilde{\mathbf{x}} \in \mathbb{R}^n$. A proposed solution to that problem was to pick any $\boldsymbol{\varphi} \in \Phi(N)$ and make

$$\underbrace{\left(\sum_{i=1}^{N} \varphi_i \mathbf{A}_i\right)}_{\mathbf{A}_{\boldsymbol{\varphi}}} \tilde{\mathbf{x}} + \underbrace{\sum_{i=1}^{N} \varphi_i \mathbf{b}_i}_{\mathbf{b}_{\boldsymbol{\varphi}}} = \mathbf{0} \tag{2}$$

with an adequate switching law design.

The above procedure was enhanced by Deaecto et al. (2010), who proposed LMI-designed state-dependent switching laws. The simplest is

$$\sigma(\mathbf{x}) = \arg\min_{i\in\mathbb{N}_N} (\mathbf{x} - \tilde{\mathbf{x}})'\mathbf{P}(\mathbf{A}_i\tilde{\mathbf{x}} + \mathbf{b}_i) \tag{3}$$

which requires that the operation point,[1] $\tilde{\mathbf{x}}$, be determined as in (2) and $\mathbf{P} \in \mathbb{R}^{n\times n}_{\succ 0}$ comes from the solution of the LMI

$$\mathbf{A}_i'\mathbf{P} + \mathbf{P}\mathbf{A}_i + \mathbf{C}_i'\mathbf{C}_i \prec 0 \tag{4}$$

As (3) makes it clear, the full state is necessary for a correct switching. However, it is desirable to obtain a switching law such that some state variables could be disregarded.

Luenberger proposed techniques for obtaining state observers for Linear Systems: with full order (Luenberger 1964) and reduced order (Luenberger 1971). The idea is to build an auxiliary system, whose state, $\hat{\mathbf{x}}$, approaches the real state, $\mathbf{x}$, as $t \to \infty$. Consider, then, the Linear System

$$\dot{\mathbf{x}} = \mathbf{A}\mathbf{x} + \mathbf{B}\mathbf{u} \tag{5a}$$

$$\mathbf{y} = \mathbf{C}\mathbf{x} \tag{5b}$$

[1] In practical situations, the designer would like to impose $\tilde{\mathbf{x}}$ and, in order to apply (3), it would be necessary to verify if there is a $\boldsymbol{\varphi} \in \Phi(N)$, such that (2) holds.

where $\mathbf{u}$ is the control input and $\mathbf{B}$ is the control input gain matrix. For the full-order case, the proposed estimator was

$$\dot{\hat{\mathbf{x}}} = \mathbf{A}\hat{\mathbf{x}} + \mathbf{B}\mathbf{u} + \mathbf{L}(\mathbf{C}\hat{\mathbf{x}} - \mathbf{y}) \tag{6}$$

while, for the reduced-order case, the estimator was

$$\begin{aligned}\dot{\mathbf{q}} &= (\mathbf{A}_{ww} - \mathbf{L}\mathbf{A}_{yw})\mathbf{q} + (\mathbf{A}_{ww} - \mathbf{L}\mathbf{A}_{yw})\mathbf{L}\mathbf{y} \\ &\quad + (\mathbf{A}_{wy} - \mathbf{L}\mathbf{A}_{yy})\mathbf{y} + (\mathbf{B}_w - \mathbf{L}\mathbf{B}_y)\mathbf{u}\end{aligned} \tag{7}$$

with partitions $\mathbf{A} = \begin{bmatrix}\mathbf{A}_{yy} & \mathbf{A}_{yw} \\ \mathbf{A}_{wy} & \mathbf{A}_{ww}\end{bmatrix}$ and $\mathbf{B} = \begin{bmatrix}\mathbf{B}_y \\ \mathbf{B}_w\end{bmatrix}$ and a change of variable, $\mathbf{q}$, to be clarified later on. Both in (6) and (7), $\mathbf{L} \in \mathbb{R}^{n\times m}$ is the output gain matrix.

Yoshimura et al. (2013) noted that, by modifying Luenberger's idea, an observer for SASs could be devised. The proposed estimator system was

$$\dot{\hat{\mathbf{x}}} = \mathbf{A}_\sigma\hat{\mathbf{x}} + \mathbf{b}_\sigma + \mathbf{L}_\sigma(\mathbf{C}_\sigma\hat{\mathbf{x}} - \mathbf{y}) \tag{8}$$

which provided the estimation error system

$$\dot{\hat{\mathbf{x}}} - \dot{\mathbf{x}} = (\mathbf{A}_\sigma + \mathbf{L}_\sigma\mathbf{C}_\sigma)(\hat{\mathbf{x}} - \mathbf{x}) \tag{9}$$

Such observer was named switched Luenberger observer and its design is possible by searching a common quadratic Lyapunov function for the system,[2] as the estimation error system is a Switched Linear System (Lin and Antsaklis 2009).

Considering (9) and the above result on Switched Linear Systems, Lemma 1 was proposed.

Lemma 1 (Yoshimura et al. 2013) *Let (1) be a SAS and $\tilde{\mathbf{x}} \in \mathbb{R}^n$. If $\exists\boldsymbol{\varphi} \in \Phi(N)$, such that (2) holds and $\exists\mathbf{W}_1, \mathbf{W}_2, \ldots, \mathbf{W}_N \in \mathbb{R}^{n\times m}$ and $\exists\mathbf{P} \in \mathbb{R}^{n\times n}_{\succ 0}$, such that (10) holds $\forall i \in \mathbb{N}_N$, then:*

i. *There is a Switched Luenberger Observer in which the origin in (9) is global and asymptotically stable under arbitrary switching;*
ii. *(11) is a set of modal output gain matrices;*
iii. *The dynamic output switching law (12) makes $\tilde{\mathbf{x}}$ global and asymptotically stable.*

$$\begin{bmatrix}\mathfrak{S}(\mathbf{P}\mathbf{A}_i + \mathbf{W}_i\mathbf{C}_i) & \star \\ \mathbf{W}_i\mathbf{C}_i & \mathfrak{S}(\mathbf{P}\mathbf{A}_i)\end{bmatrix} \prec 0 \tag{10}$$

$$\mathbf{L}_i = \mathbf{P}^{-1}\mathbf{W}_i, \quad i = 1, 2, \ldots, N \tag{11}$$

$$\sigma(\hat{\mathbf{x}}) = \arg\min_{i\in\mathbb{N}_N} (\hat{\mathbf{x}} - \tilde{\mathbf{x}})'\mathbf{P}(\mathbf{A}_i\tilde{\mathbf{x}} + \mathbf{b}_i) \tag{12}$$

[2] Note that this approach is also possible for linear parameter-varying systems, with arbitrary rates of variation.

Considering those previous studies, the following section will aim, through LMI usage, the new results: cost-guaranteed dynamic output switching law, a solution for some cases involving non-Hurwitz modal $\mathbf{A}$ matrices and state estimation only for the unmeasured part of the state.

3 Main Results

3.1 Full-Order Observer-Based Control of Switched Affine Systems

In order to achieve the best suited switching law for the estimation problem, i.e., the one that provides faster estimation error convergence to zero, it would be necessary to minimize the functional

$$J(\sigma) = \int_0^\infty ||\mathbf{z} - \mathbf{E}_\sigma\hat{\mathbf{x}}||^2 \mathrm{d}t \tag{13}$$

However, due to σ, J is not differentiable and, therefore, very difficult to minimize (Deaecto et al. 2010). However, an upper bound for the functional can be established, using ideas from Deaecto et al. (2010) and Yoshimura et al. (2013). Firstly, define the error vector

$$\boldsymbol{\varepsilon} = \begin{bmatrix}\hat{\mathbf{x}} - \mathbf{x} \\ \hat{\mathbf{x}} - \tilde{\mathbf{x}}\end{bmatrix} \tag{14}$$

and now, a cost-guaranteed form of Lemma 1 is proposed.

Theorem 1 *Let (1) be a SAS and $\tilde{\mathbf{x}} \in \mathbb{R}^n$. If $\exists\boldsymbol{\varphi} \in \Phi(N)$, such that (2) holds and $\exists\mathbf{W}_1, \mathbf{W}_2, \ldots, \mathbf{W}_N \in \mathbb{R}^{n\times m}$ and $\exists\mathbf{P} \in \mathbb{R}^{n\times n}_{\succ 0}$, such that (15) holds $\forall i \in \mathbb{N}_N$, then:*

i. *There is a Switched Luenberger Observer in which the origin in (9) is global and asymptotically stable under arbitrary switching;*
ii. *(11) is a set of modal output gain matrices;*
iii. *$\tilde{\mathbf{x}}$ is global and asymptotically stable under the dynamic output switching law (12);*
iv. *The guaranteed cost (16) holds.*

$$\begin{bmatrix}\mathfrak{S}(\mathbf{P}\mathbf{A}_i + \mathbf{W}_i\mathbf{C}_i) + \mathbf{E}_i'\mathbf{E}_i & \star \\ \mathbf{W}_i\mathbf{C}_i & \mathfrak{S}(\mathbf{P}\mathbf{A}_i)\end{bmatrix} \prec 0 \tag{15}$$

$$\int_0^\infty ||\mathbf{z} - \mathbf{E}_\sigma\hat{\mathbf{x}}||^2 \mathrm{d}t < \boldsymbol{\varepsilon}(0)' \operatorname{diag}(\mathbf{P}, \mathbf{P})\boldsymbol{\varepsilon}(0) \tag{16}$$

Proof Items i and ii come from the direct application of Sylvester's criterion on the upper left block of (15). For item iii, take the Lyapunov candidate function (17).

$$v(\boldsymbol{\varepsilon}) = \boldsymbol{\varepsilon}' \operatorname{diag}(\mathbf{P}, \mathbf{P})\boldsymbol{\varepsilon} \tag{17}$$

From (8), (9) and (14), observe that

$$\dot{\boldsymbol{\varepsilon}} = \begin{bmatrix} \mathbf{A}_\sigma + \mathbf{L}_\sigma \mathbf{C}_\sigma & \mathbf{0} \\ \mathbf{L}_\sigma \mathbf{C}_\sigma & \mathbf{A}_\sigma \end{bmatrix} \boldsymbol{\varepsilon} + \begin{bmatrix} \mathbf{0} \\ \mathbf{A}_\sigma \tilde{\mathbf{x}} + \mathbf{b}_\sigma \end{bmatrix} \tag{18}$$

Differentiating (17) and using (18):

$$\begin{aligned} \dot{v}(\boldsymbol{\varepsilon}) &= \boldsymbol{\varepsilon}' \begin{bmatrix} \mathfrak{S}(\mathbf{P}\mathbf{A}_\sigma + \mathbf{P}\mathbf{L}_\sigma \mathbf{C}_\sigma) & \star \\ \mathbf{P}\mathbf{L}_\sigma \mathbf{C}_\sigma & \mathfrak{S}(\mathbf{P}\mathbf{A}_\sigma) \end{bmatrix} \boldsymbol{\varepsilon} \\ &\quad + 2(\hat{\mathbf{x}} - \tilde{\mathbf{x}})'\mathbf{P}(\mathbf{A}_\sigma \tilde{\mathbf{x}} + \mathbf{b}_\sigma) \\ &\overset{(12)}{=} \boldsymbol{\varepsilon}' \begin{bmatrix} \mathfrak{S}(\mathbf{P}\mathbf{A}_i + \mathbf{P}\mathbf{L}_i \mathbf{C}_i) & \star \\ \mathbf{P}\mathbf{L}_i \mathbf{C}_i & \mathfrak{S}(\mathbf{P}\mathbf{A}_i) \end{bmatrix} \boldsymbol{\varepsilon} \\ &\quad + 2 \min_{i \in \mathbb{N}_N} (\hat{\mathbf{x}} - \tilde{\mathbf{x}})'\mathbf{P}(\mathbf{A}_i \tilde{\mathbf{x}} + \mathbf{b}_i) \\ &\overset{(2)}{\leq} \boldsymbol{\varepsilon}' \begin{bmatrix} \mathfrak{S}(\mathbf{P}\mathbf{A}_i + \mathbf{P}\mathbf{L}_i \mathbf{C}_i) & \star \\ \mathbf{P}\mathbf{L}_i \mathbf{C}_i & \mathfrak{S}(\mathbf{P}\mathbf{A}_i) \end{bmatrix} \boldsymbol{\varepsilon} \\ &\overset{(15)}{<} -(\hat{\mathbf{x}} - \mathbf{x})'\mathbf{E}_i'\mathbf{E}_i(\hat{\mathbf{x}} - \mathbf{x}) \leq 0 \end{aligned}$$

Item iv is obtained by integrating the above Lyapunov function derivative:

$$\begin{aligned} \int_0^\infty ||\mathbf{z} - \mathbf{E}_\sigma \hat{\mathbf{x}}||^2 \mathrm{d}t &= \int_0^\infty (\hat{\mathbf{x}} - \mathbf{x})'\mathbf{E}_i'\mathbf{E}_i(\hat{\mathbf{x}} - \mathbf{x}) \mathrm{d}t \\ &< -\int_0^\infty \dot{v}(\boldsymbol{\varepsilon}) \mathrm{d}t = v(\boldsymbol{\varepsilon}(0)) \end{aligned}$$

Observe that the system stability is used in the last equality, completing the proof. □

Remark 1 The Lyapunov matrix may be substituted by $\mathrm{diag}(\mathbf{P}, \alpha\mathbf{P})$ in (17), $\alpha > 0$, for LMI relaxation. However, a one-dimensional search must be executed for an optimal solution (i.e., minimization of $\mathrm{Tr}(\mathbf{P})$).

Observe that (15) can only handle SASs with Hurwitz modes. In order to complete the theory, a modification in that LMI will be proposed, leading to the result in Theorem 2.

Theorem 2 *Let (1) be a SAS with* $\mathbf{C}_i = \mathbf{C}, i = 1, 2, \ldots, N$ *and* $\tilde{\mathbf{x}} \in \mathbb{R}^n$. *If* $\exists \boldsymbol{\varphi} \in \Phi(N)$, *such that (2) holds and* $\exists \mathbf{W}_1, \mathbf{W}_2, \ldots, \mathbf{W}_N \in \mathbb{R}^{n \times m}$ *and* $\exists \mathbf{P} \in \mathbb{R}^{n \times n}_{\succ 0}$, *such that (19) holds* $\forall i \in \mathbb{N}_N$, *then:*

i. *There is a Switched Luenberger Observer in which the origin in (9) is global and asymptotically stable under arbitrary switching;*
ii. *(11) is a set of modal output gain matrices;*
iii. *The dynamic output switching law (20) makes* $\tilde{\mathbf{x}}$ *global and asymptotically stable;*
iv. *The guaranteed cost (16) holds.*

$$\begin{bmatrix} \mathfrak{S}(\mathbf{P}\mathbf{A}_i + \mathbf{W}_i \mathbf{C}) + \mathbf{E}_i'\mathbf{E}_i & \star \\ \mathbf{W}_\varphi \mathbf{C} & \mathfrak{S}(\mathbf{P}\mathbf{A}_\varphi) \end{bmatrix} \prec 0 \tag{19}$$

$$\sigma(\hat{\mathbf{x}}) = \arg \min_{i \in \mathbb{N}_N} (\hat{\mathbf{x}} - \tilde{\mathbf{x}})'\mathbf{P}[\mathbf{A}_i \hat{\mathbf{x}} + \mathbf{b}_i + \mathbf{L}_i(\mathbf{C}\hat{\mathbf{x}} - \mathbf{y})] \tag{20}$$

Proof Items i and ii are immediate, sufficing to apply Sylvester's criterion to the upper left block of (19) and then, invoke the common quadratic Lyapunov function Lemma (Lin and Antsaklis 2009). For item iii, it is claimed that (17) is a Lyapunov function. Indeed, changing (18) into:

$$\dot{\boldsymbol{\varepsilon}} = \begin{bmatrix} \mathbf{A}_\sigma + \mathbf{L}_\sigma \mathbf{C} & \mathbf{0} \\ \mathbf{0} & \mathbf{0} \end{bmatrix} \boldsymbol{\varepsilon} + \begin{bmatrix} \mathbf{0} \\ \mathbf{A}_\sigma \hat{\mathbf{x}} + \mathbf{b}_\sigma + \mathbf{L}_\sigma(\mathbf{C}\hat{\mathbf{x}} - \mathbf{y}) \end{bmatrix}$$

and differentiating the Lyapunov candidate function:

$$\begin{aligned} \dot{v}(\boldsymbol{\varepsilon}) &= \boldsymbol{\varepsilon}' \begin{bmatrix} \mathfrak{S}(\mathbf{P}(\mathbf{A}_i + \mathbf{L}_i \mathbf{C})) & \mathbf{0} \\ \mathbf{0} & \mathbf{0} \end{bmatrix} \boldsymbol{\varepsilon} \\ &\quad + 2(\hat{\mathbf{x}} - \tilde{\mathbf{x}})'\mathbf{P}(\mathbf{A}_\sigma \hat{\mathbf{x}} + \mathbf{b}_\sigma + \mathbf{L}_\sigma(\mathbf{C}\hat{\mathbf{x}} - \mathbf{y})) \\ &\overset{(20)}{=} \boldsymbol{\varepsilon}' \begin{bmatrix} \mathfrak{S}(\mathbf{P}(\mathbf{A}_i + \mathbf{L}_i \mathbf{C})) & \mathbf{0} \\ \mathbf{0} & \mathbf{0} \end{bmatrix} \boldsymbol{\varepsilon} \\ &\quad + 2 \min_{i \in \mathbb{N}_N} (\hat{\mathbf{x}} - \tilde{\mathbf{x}})'\mathbf{P}(\mathbf{A}_i \hat{\mathbf{x}} + \mathbf{b}_i + \mathbf{L}_i \mathbf{C}(\hat{\mathbf{x}} - \mathbf{x})) \\ &\leq \boldsymbol{\varepsilon}' \begin{bmatrix} \mathfrak{S}(\mathbf{P}(\mathbf{A}_i + \mathbf{L}_i \mathbf{C})) & \mathbf{0} \\ \mathbf{0} & \mathfrak{S}(\mathbf{P}\mathbf{A}_\varphi) \end{bmatrix} \boldsymbol{\varepsilon} \\ &\quad + 2(\hat{\mathbf{x}} - \tilde{\mathbf{x}})'\mathbf{P} \left(\mathbf{A}_\varphi \tilde{\mathbf{x}} + \mathbf{b}_\varphi + \sum_{j=1}^N \varphi_j \mathbf{L}_j \mathbf{C}(\hat{\mathbf{x}} - \mathbf{x}) \right) \\ &\overset{(2)}{=} \boldsymbol{\varepsilon}' \begin{bmatrix} \mathfrak{S}(\mathbf{P}(\mathbf{A}_i + \mathbf{L}_i \mathbf{C})) & \mathbf{0} \\ \mathbf{0} & \mathfrak{S}(\mathbf{P}\mathbf{A}_\varphi) \end{bmatrix} \boldsymbol{\varepsilon} \\ &\quad + 2(\hat{\mathbf{x}} - \tilde{\mathbf{x}})'\mathbf{P}\mathbf{L}_\varphi \mathbf{C}(\hat{\mathbf{x}} - \mathbf{x}) \\ &= \boldsymbol{\varepsilon}' \begin{bmatrix} \mathfrak{S}(\mathbf{P}(\mathbf{A}_i + \mathbf{L}_i \mathbf{C})) & \star \\ \mathbf{P}\mathbf{L}_\varphi \mathbf{C} & \mathfrak{S}(\mathbf{P}\mathbf{A}_\varphi) \end{bmatrix} \boldsymbol{\varepsilon} \\ &\overset{(19)}{<} -(\hat{\mathbf{x}} - \mathbf{x})'\mathbf{E}_i'\mathbf{E}_i(\hat{\mathbf{x}} - \mathbf{x}) \leq 0 \end{aligned}$$

And item iv is obtained by integration of the Lyapunov function, as in Theorem 1. □

Remark 2 Observe that (19) uses $\mathfrak{S}(\mathbf{P}\mathbf{A}_\varphi)$, which shows that only a Hurwitz convex combination of modes is necessary. On the other hand, modal $\mathbf{C}$ matrices cannot be considered as the minimum must be tested with every choice of $i \in \mathbb{N}_N$ (but the same $\mathbf{y}$, of course) and, if multiple $\mathbf{C}$ matrices were allowed, that would possibly lead to mixed-mode $\mathbf{C}$ matrices, preventing the error signal to be obtained in the second equality involving $\dot{v}$.

3.2 Reduced-Order Observers

It must be reminded that the objective is to provide a state estimation with fewer state variables to be estimated. The strategy is to use the part of $\mathbf{y}$ that provides some of those variables. The first step is to conveniently partition the state space as the next subsection shows.

3.2.1 Partitioning the State Space

Let (1) be a SAS such that m state variables are directly measured from the output. Admitting a matrix $\mathcal{C} = \text{diag}\{c_1, c_2, \ldots, c_m\}$, $c_i \neq 0, \forall i \in \mathbb{N}_m$, the state is partitioned as

$$\mathbf{x} = [\underbrace{x_1 \quad x_2 \quad \ldots \quad x_m}_{(\mathcal{C}^{-1}\mathbf{y})'} \quad \underbrace{x_{m+1} \quad \ldots \quad x_n}_{\mathbf{w}'}]' \tag{21}$$

It is reasonable to assume that the sensors are not affected by the switching law, as their placement will not change during the system operation. Therefore, all $\mathbf{C}$-matrices can be assumed equal and (1) can be partitioned into

$$\dot{\mathbf{x}} = \begin{bmatrix} \mathcal{C}^{-1}\dot{\mathbf{y}} \\ \dot{\mathbf{w}} \end{bmatrix} = \begin{bmatrix} \mathbf{A}_{yy\sigma} & \mathbf{A}_{yw\sigma} \\ \mathbf{A}_{wy\sigma} & \mathbf{A}_{ww\sigma} \end{bmatrix} \begin{bmatrix} \mathcal{C}^{-1}\mathbf{y} \\ \mathbf{w} \end{bmatrix} + \begin{bmatrix} \mathbf{b}_{y\sigma} \\ \mathbf{b}_{w\sigma} \end{bmatrix} \tag{22a}$$

$$\mathbf{y} = \begin{bmatrix} \mathcal{C} & \mathbf{0} \end{bmatrix} \mathbf{x} = \begin{bmatrix} \mathcal{C} & \mathbf{0} \end{bmatrix} \begin{bmatrix} \mathcal{C}^{-1}\mathbf{y} \\ \mathbf{w} \end{bmatrix} \tag{22b}$$

$$\mathbf{z} = \begin{bmatrix} \mathbf{0} & \mathcal{E} \end{bmatrix} \mathbf{x} = \begin{bmatrix} \mathbf{0} & \mathcal{E} \end{bmatrix} \begin{bmatrix} \mathcal{C}^{-1}\mathbf{y} \\ \mathbf{w} \end{bmatrix} \tag{22c}$$

As $\mathbf{y}$ is directly measured from sensors, only $\mathbf{w}$ must be estimated. The estimated state is, thus, defined as

$$\hat{\mathbf{x}} = \left[(\mathcal{C}^{-1}\mathbf{y})' \ \hat{\mathbf{w}}' \right]' \tag{23}$$

3.2.2 Reduced-Order Switched Luenberger Observers

In order to compose the observer, an estimation error system, $\hat{\mathbf{w}} - \mathbf{w}$, must be defined in a way that its origin is guaranteed to be global and asymptotically stable. For that matter, let $\mathbf{L}_1, \mathbf{L}_2, \ldots, \mathbf{L}_N \in \mathbb{R}^{(n-m)\times m}$ be modal output gain matrices. The proposed modified Luenberger observer, henceforth called Reduced-Order Switched Luenberger Observer (ROSLO), is

$$\begin{aligned} \dot{\mathbf{q}} = (\mathbf{A}_{ww\sigma} - \mathbf{L}_\sigma \mathbf{A}_{yw\sigma})\mathbf{q} + (\mathbf{A}_{ww\sigma} - \mathbf{L}_\sigma \mathbf{A}_{yw\sigma})\mathbf{L}_\sigma \mathcal{C}^{-1}\mathbf{y} \\ + (\mathbf{A}_{wy\sigma} - \mathbf{L}_\sigma \mathbf{A}_{yy\sigma})\mathcal{C}^{-1}\mathbf{y} + \mathbf{b}_{w\sigma} - \mathbf{L}_\sigma \mathbf{b}_{y\sigma} \end{aligned} \tag{24}$$

in which, $\mathbf{q} \in \mathbb{R}^{n-m}$ is defined as

$$\mathbf{q} = \hat{\mathbf{w}} - \mathbf{L}_\sigma \mathcal{C}^{-1}\mathbf{y} \tag{25}$$

and its purpose is to avoid the need for sensing $\dot{\mathbf{y}}$ (Luenberger 1971). If the intention in a given design is only to estimate the state, Theorem 3 will show a sufficient LMI procedure, which will also be helpful for subsequent analysis.

Theorem 3 *Let (22) be a SAS. If* $\exists \mathbf{P} \in \mathbb{R}^{(n-m)\times(n-m)}_{\succ 0}$ *and* $\exists \mathbf{W}_1, \mathbf{W}_2, \ldots, \mathbf{W}_N \in \mathbb{R}^{(n-m)\times m}$, *such that*

$$\mathfrak{S}(\mathbf{P}\mathbf{A}_{wwi} - \mathbf{W}_i \mathbf{A}_{ywi}) \prec 0 \tag{26}$$

holds $\forall i \in \mathbb{N}_N$, *then there is a ROSLO, where* $\hat{\mathbf{w}} \to \mathbf{w}$ *as* $t \to \infty$ *under arbitrary switching. Moreover,*

$$\mathbf{L}_i = \mathbf{P}^{-1}\mathbf{W}_i, \quad i = 1, 2, \ldots, N \tag{27}$$

is a set of modal output injection matrices for such an observer.

Proof Combining (22) and (24), it is obtained

$$\begin{aligned} \dot{\mathbf{q}} - \dot{\mathbf{w}} = (\mathbf{A}_{ww\sigma} - \mathbf{L}_\sigma \mathbf{A}_{yw\sigma})(\mathbf{q} + \mathbf{L}_\sigma \mathcal{C}^{-1}\mathbf{y}) \\ - \mathbf{L}_\sigma \mathbf{A}_{yy\sigma}\mathcal{C}^{-1}\mathbf{y} - \mathbf{L}_\sigma \mathbf{b}_{y\sigma} - \mathbf{A}_{ww\sigma}\mathbf{w} \end{aligned}$$

which, by eliminating $\mathbf{q}$ with (25), turns into

$$\begin{aligned} \dot{\hat{\mathbf{w}}} - \dot{\mathbf{w}} - \mathbf{L}_\sigma \mathcal{C}^{-1}\dot{\mathbf{y}} = (\mathbf{A}_{ww\sigma} - \mathbf{L}_\sigma \mathbf{A}_{yw\sigma})\hat{\mathbf{w}} \\ - \mathbf{L}_\sigma \mathbf{A}_{yy\sigma}\mathcal{C}^{-1}\mathbf{y} - \mathbf{L}_\sigma \mathbf{b}_{y\sigma} - \mathbf{A}_{ww\sigma}\mathbf{w} \end{aligned}$$

Using (22) again to eliminate $\mathcal{C}^{-1}\dot{\mathbf{y}}$:

$$\begin{aligned} \dot{\hat{\mathbf{w}}} - \dot{\mathbf{w}} = (\mathbf{A}_{ww\sigma} - \mathbf{L}_\sigma \mathbf{A}_{yw\sigma})\hat{\mathbf{w}} - \mathbf{L}_\sigma \mathbf{A}_{yy\sigma}\mathcal{C}^{-1}\mathbf{y} - \mathbf{L}_\sigma \mathbf{b}_{y\sigma} \\ - \mathbf{A}_{ww\sigma}\mathbf{w} + \mathbf{L}_\sigma(\mathbf{A}_{yy\sigma}\mathcal{C}^{-1}\mathbf{y} + \mathbf{A}_{yw\sigma}\mathbf{w} + \mathbf{b}_{y\sigma}) \end{aligned}$$

Then, the estimation error system is the following Switched Linear System

$$\dot{\hat{\mathbf{w}}} - \dot{\mathbf{w}} = (\mathbf{A}_{ww\sigma} - \mathbf{L}_\sigma \mathbf{A}_{yw\sigma})(\hat{\mathbf{w}} - \mathbf{w}) \tag{28}$$

for which

$$v(\hat{\mathbf{w}} - \mathbf{w}) = (\hat{\mathbf{w}} - \mathbf{w})'\mathbf{P}(\hat{\mathbf{w}} - \mathbf{w})$$

is a common quadratic Lyapunov function as (26) is feasible from hypothesis. Invoking the result by Lin and Antsaklis (2009), the claim follows. □

Remark 3 Despite of Theorem 3 being established, it is still not enough to guarantee the usage of (24) in closed-loop control, i.e., to formulate a reduced-order observer-based switching law. In addition, such a switching law must secure, for $t \to \infty$, that:

(i) the estimation error converges to zero, and;
(ii) either the state or the estimated state converges to the desired operation point.

It will be noted later on that it is easier to use the approach of the estimated state converging to the desired operation point, where Theorem 3, precisely (28), will prove itself useful.

3.2.3 Reduced-Order Observer-Based Control

Using (21), an operation point will be established. Its representation is

$$\tilde{\mathbf{x}} = \left[(\mathcal{C}^{-1}\tilde{\mathbf{y}})' \; \tilde{\mathbf{w}}'\right]' \tag{29}$$

Remark 3 with (29) suggests the usage of

$$\boldsymbol{\varepsilon} = \begin{bmatrix} \hat{\mathbf{w}} - \mathbf{w} \\ \mathcal{C}^{-1}\mathbf{y} - \mathcal{C}^{-1}\tilde{\mathbf{y}} \\ \hat{\mathbf{w}} - \tilde{\mathbf{w}} \end{bmatrix} \tag{30}$$

as the partitioned state for the augmented system, the first partition is already defined in (28). The second is obtained with a modification in (22).

$$\begin{aligned} \mathcal{C}^{-1}\dot{\mathbf{y}} &= \underbrace{\mathbf{A}_{yy\sigma}(\mathcal{C}^{-1}\mathbf{y} - \mathcal{C}^{-1}\tilde{\mathbf{y}}) + \mathbf{A}_{yy\sigma}\mathcal{C}^{-1}\tilde{\mathbf{y}}}_{\mathbf{A}_{yy\sigma}\mathcal{C}^{-1}\mathbf{y}} + \mathbf{b}_{y\sigma} \\ &\quad - \underbrace{\mathbf{A}_{yw\sigma}(\hat{\mathbf{w}} - \mathbf{w}) + \mathbf{A}_{yw\sigma}(\hat{\mathbf{w}} - \tilde{\mathbf{w}}) + \mathbf{A}_{yw\sigma}\tilde{\mathbf{w}}}_{-\mathbf{A}_{yw\sigma}\mathbf{w}} \\ &= \begin{bmatrix} -\mathbf{A}_{yw\sigma} & \mathbf{A}_{yy\sigma} & \mathbf{A}_{yw\sigma} \end{bmatrix}\boldsymbol{\varepsilon} + \begin{bmatrix} \mathbf{A}_{yy\sigma} & \mathbf{A}_{yw\sigma} \end{bmatrix}\tilde{\mathbf{x}} + \mathbf{b}_{y\sigma} \end{aligned} \tag{31}$$

The third state partition is taken from a combination of (24) and (25).

$$\begin{aligned} \dot{\hat{\mathbf{w}}} &= (\mathbf{A}_{ww\sigma} - \mathbf{L}_\sigma\mathbf{A}_{yw\sigma})\hat{\mathbf{w}} + (\mathbf{A}_{wy\sigma} - \mathbf{L}_\sigma\mathbf{A}_{yy\sigma})\mathcal{C}^{-1}\mathbf{y} \\ &\quad + \mathbf{b}_{w\sigma} - \mathbf{L}_\sigma\mathbf{b}_{y\sigma} + \mathbf{L}_\sigma\underbrace{(\mathbf{A}_{yy\sigma}\mathcal{C}^{-1}\mathbf{y} + \mathbf{A}_{yw\sigma}\mathbf{w} + \mathbf{b}_{y\sigma})}_{\mathcal{C}^{-1}\dot{\mathbf{y}}} \\ &= \begin{bmatrix} -\mathbf{L}_\sigma\mathbf{A}_{yw\sigma} & \mathbf{A}_{wy\sigma} & \mathbf{A}_{ww\sigma} \end{bmatrix}\boldsymbol{\varepsilon} + \begin{bmatrix} \mathbf{A}_{wy\sigma} & \mathbf{A}_{ww\sigma} \end{bmatrix}\tilde{\mathbf{x}} + \mathbf{b}_{w\sigma} \end{aligned} \tag{32}$$

The augmented system is obtained from (28), (31) and (32), in which it must be noted that $\mathcal{C}^{-1}\dot{\tilde{\mathbf{y}}} = \mathbf{0}$ and $\dot{\tilde{\mathbf{w}}} = \mathbf{0}$.

$$\dot{\boldsymbol{\varepsilon}} = \begin{bmatrix} \mathbf{A}_{ww\sigma} - \mathbf{L}_\sigma\mathbf{A}_{yw\sigma} & \mathbf{0} & \mathbf{0} \\ -\mathbf{A}_{yw\sigma} & \mathbf{A}_{yy\sigma} & \mathbf{A}_{yw\sigma} \\ -\mathbf{L}_\sigma\mathbf{A}_{yw\sigma} & \mathbf{A}_{wy\sigma} & \mathbf{A}_{ww\sigma} \end{bmatrix}\boldsymbol{\varepsilon} + \begin{bmatrix} \mathbf{0} \\ \mathbf{A}_\sigma\tilde{\mathbf{x}} + \mathbf{b}_\sigma \end{bmatrix} \tag{33}$$

Now, a theorem on reduced-order observer-based control, the main result of this paper, can be stated as follows.

Theorem 4 *Let (22) be a (partitioned) SAS and $\tilde{\mathbf{x}} \in \mathbb{R}^n$. If $\exists\boldsymbol{\varphi} \in \Phi(N)$, such that (2) holds, $\exists\mathbf{P}_y \in \mathbb{R}^{m\times m}_{\succ 0}$, $\exists\mathbf{P}_w \in \mathbb{R}^{(n-m)\times(n-m)}_{\succ 0}$ and $\exists\mathbf{W} \in \mathbb{R}^{(n-m)\times m}$, such that (34) holds $\forall i \in \mathbb{N}_N$, then:*

i. *There is a ROSLO in which the origin in (28) is global and asymptotically stable under arbitrary switching;*
ii. *(35) is an output gain matrix;*
iii. *$\tilde{\mathbf{x}}$ is global and asymptotically stable under the dynamic output switching law (36);*
iv. *The guaranteed cost (37) holds.*

$$\begin{bmatrix} \mathfrak{S}(\mathbf{P}_w\mathbf{A}_{wwi} - \mathbf{W}\mathbf{A}_{ywi}) + \mathcal{E}'\mathcal{E} & \star & \star \\ -\mathbf{P}_y\mathbf{A}_{ywi} & \mathfrak{S}(\mathbf{P}_y\mathbf{A}_{yyi}) & \star \\ -\mathbf{W}\mathbf{A}_{ywi} & \mathbf{P}_w\mathbf{A}_{wyi} + \mathbf{A}_{ywi}{}'\mathbf{P}_y & \mathfrak{S}(\mathbf{P}_w\mathbf{A}_{wwi}) \end{bmatrix} \prec 0 \tag{34}$$

$$\mathbf{L} = \mathbf{P}_w^{-1}\mathbf{W} \tag{35}$$

$$\sigma(\hat{\mathbf{x}}) = \arg\min_{i\in\mathbb{N}_N} (\hat{\mathbf{x}} - \tilde{\mathbf{x}})' \operatorname{diag}(\mathbf{P}_y, \mathbf{P}_w)(\mathbf{A}_i\tilde{\mathbf{x}} + \mathbf{b}_i) \tag{36}$$

$$\int_0^\infty ||\mathbf{z} - \mathcal{E}\hat{\mathbf{w}}||^2 \mathrm{d}t < \boldsymbol{\varepsilon}(0)' \operatorname{diag}(\mathbf{P}_w, \mathbf{P}_y, \mathbf{P}_w)\boldsymbol{\varepsilon}(0) \tag{37}$$

Proof Items i and ii are immediate, sufficing to apply Sylvester's criterion to the upper left block of (34) and then, invoke Theorem 3. For item iii, a common quadratic Lyapunov function must be exhibited (Lin and Antsaklis 2009). Using (30), it is claimed here that

$$v(\boldsymbol{\varepsilon}) = \boldsymbol{\varepsilon}' \operatorname{diag}(\mathbf{P}_w, \mathbf{P}_y, \mathbf{P}_w)\boldsymbol{\varepsilon} \tag{38}$$

is one such function. Indeed, its derivative is

$$\begin{aligned} \dot{v}(\boldsymbol{\varepsilon}) &= 2\boldsymbol{\varepsilon}' \operatorname{diag}(\mathbf{P}_w, \mathbf{P}_y, \mathbf{P}_w)\dot{\boldsymbol{\varepsilon}} \\ &\overset{(33)}{=} \boldsymbol{\varepsilon}' \begin{bmatrix} \mathfrak{S}(\mathbf{P}_w\mathbf{A}_{ww\sigma} - \mathbf{W}\mathbf{A}_{yw\sigma}) & \star & \star \\ -\mathbf{P}_y\mathbf{A}_{yw\sigma} & \mathfrak{S}(\mathbf{P}_y\mathbf{A}_{yy\sigma}) & \star \\ -\mathbf{W}\mathbf{A}_{yw\sigma} & \mathbf{P}_w\mathbf{A}_{wy\sigma} + \mathbf{A}_{yw\sigma}{}'\mathbf{P}_y & \mathfrak{S}(\mathbf{P}_w\mathbf{A}_{ww\sigma}) \end{bmatrix} \boldsymbol{\varepsilon} + 2\boldsymbol{\varepsilon}' \operatorname{diag}(\mathbf{P}_w, \mathbf{P}_y, \mathbf{P}_w) \begin{bmatrix} \mathbf{0} \\ \mathbf{A}_\sigma\tilde{\mathbf{x}} + \mathbf{b}_\sigma \end{bmatrix} \\ &= \boldsymbol{\varepsilon}' \begin{bmatrix} \mathfrak{S}(\mathbf{P}_w\mathbf{A}_{ww\sigma} - \mathbf{W}\mathbf{A}_{yw\sigma}) & \star & \star \\ -\mathbf{P}_y\mathbf{A}_{yw\sigma} & \mathfrak{S}(\mathbf{P}_y\mathbf{A}_{yy\sigma}) & \star \\ -\mathbf{W}\mathbf{A}_{yw\sigma} & \mathbf{P}_w\mathbf{A}_{wy\sigma} + \mathbf{A}_{yw\sigma}{}'\mathbf{P}_y & \mathfrak{S}(\mathbf{P}_w\mathbf{A}_{ww\sigma}) \end{bmatrix} \boldsymbol{\varepsilon} + 2(\hat{\mathbf{x}} - \tilde{\mathbf{x}})' \operatorname{diag}(\mathbf{P}_y, \mathbf{P}_w)(\mathbf{A}_\sigma\tilde{\mathbf{x}} + \mathbf{b}_\sigma) \\ &\overset{(36)}{=} \boldsymbol{\varepsilon}' \begin{bmatrix} \mathfrak{S}(\mathbf{P}_w\mathbf{A}_{wwi} - \mathbf{W}\mathbf{A}_{ywi}) & \star & \star \\ -\mathbf{P}_y\mathbf{A}_{ywi} & \mathfrak{S}(\mathbf{P}_y\mathbf{A}_{yyi}) & \star \\ -\mathbf{W}\mathbf{A}_{ywi} & \mathbf{P}_w\mathbf{A}_{wyi} + \mathbf{A}_{ywi}{}'\mathbf{P}_y & \mathfrak{S}(\mathbf{P}_w\mathbf{A}_{wwi}) \end{bmatrix} \boldsymbol{\varepsilon} + 2\min_{i\in\mathbb{N}_N} (\hat{\mathbf{x}} - \tilde{\mathbf{x}})' \operatorname{diag}(\mathbf{P}_y, \mathbf{P}_w)(\mathbf{A}_i\tilde{\mathbf{x}} + \mathbf{b}_i) \\ &\le \boldsymbol{\varepsilon}' \begin{bmatrix} \mathfrak{S}(\mathbf{P}_w\mathbf{A}_{wwi} - \mathbf{W}\mathbf{A}_{ywi}) & \star & \star \\ -\mathbf{P}_y\mathbf{A}_{ywi} & \mathfrak{S}(\mathbf{P}_y\mathbf{A}_{yyi}) & \star \\ -\mathbf{W}\mathbf{A}_{ywi} & \mathbf{P}_w\mathbf{A}_{wyi} + \mathbf{A}_{ywi}{}'\mathbf{P}_y & \mathfrak{S}(\mathbf{P}_w\mathbf{A}_{wwi}) \end{bmatrix} \boldsymbol{\varepsilon} + 2(\hat{\mathbf{x}} - \tilde{\mathbf{x}})' \operatorname{diag}(\mathbf{P}_y, \mathbf{P}_w)(\mathbf{A}_{\boldsymbol{\varphi}}\tilde{\mathbf{x}} + \mathbf{b}_{\boldsymbol{\varphi}}) \end{aligned}$$

$$\stackrel{(2)}{=} \boldsymbol{\varepsilon}' \begin{bmatrix} \mathfrak{S}(\mathbf{P}_w\mathbf{A}_{wwi} - \mathbf{W}\mathbf{A}_{ywi}) & \star & \star \\ -\mathbf{P}_y\mathbf{A}_{ywi} & \mathfrak{S}(\mathbf{P}_y\mathbf{A}_{yyi}) & \star \\ -\mathbf{W}\mathbf{A}_{ywi} & \mathbf{P}_w\mathbf{A}_{wyi} + \mathbf{A}_{ywi}{}'\mathbf{P}_y & \mathfrak{S}(\mathbf{P}_w\mathbf{A}_{wwi}) \end{bmatrix} \boldsymbol{\varepsilon}$$
$$\stackrel{(34)}{<} -(\hat{\mathbf{w}} - \mathbf{w})'\mathcal{E}'\mathcal{E}(\hat{\mathbf{w}} - \mathbf{w}) \leq 0$$

And item iv is again obtained from integration of the Lyapunov function derivative. □

Remark 4 It can be shown that the other (up to now) possible approach, pointed out in Remark 3, i.e., trying to make the (real) state converge to the desired operation point is not convenient. Indeed, it suffices to subtract the first line from the last in (33), reaching:

$$\dot{\mathbf{w}} - \dot{\tilde{\mathbf{w}}} = \begin{bmatrix} -\mathbf{A}_{ww\sigma} & \mathbf{A}_{wy\sigma} & \mathbf{A}_{ww\sigma} \end{bmatrix} \boldsymbol{\varepsilon} + \begin{bmatrix} \mathbf{A}_{wy\sigma} & \mathbf{A}_{ww\sigma} \end{bmatrix} \tilde{\mathbf{x}} + \mathbf{b}_{w\sigma} \tag{39}$$

And trying to compose a similar result to that from Theorem 4 returns a full-order state-dependent switching law, which is useless for the intent herein described.[3]

Remark 5 The usage of modal output gain matrices is not possible with ROSLOs. The reason is that the switching law would depend on $\mathbf{q}$, as $\hat{\mathbf{w}}$ is not directly available (it must be computed with (25)). That, in turn, would make the switching law dependent on $\mathbf{L}_\sigma$, i.e., σ would depend on itself. Such effect is avoided by using only a common-mode output gain matrix.

Remark 6 The results obtained herein are essentially theoretical and may be difficult to reproduce in a physical device (Bolzern and Spinelli 2004). In practical situations, when dealing with uncertain parameters (which may deviate the operation point, $\tilde{\mathbf{x}}$) or disturbed data (which may modify the state trajectory), exponential and even asymptotic stability must be abandoned in favor of practical stability, as addressed, for instance, by Xu et al. (2008) and Hetel and Fridman (2013) in other contexts. That would secure the state trajectory within a bounded region containing, from a given instant t_b on.

4 Design Examples in DC–DC Converters

DC–DC power converters are natural applications of SAS theory, as recent works show in their examples (Scharlau et al. 2014; Corona et al. 2007; Deaecto et al. 2010; Mainardi Júnior et al. 2012; Yoshimura et al. 2013). Modal matrices for the two-dimensional buck (step down), boost (step up) and buck–boost (step down or up, inverter) converters, taken from Deaecto et al. (2010) (also used by Yoshimura et al. 2013), are:

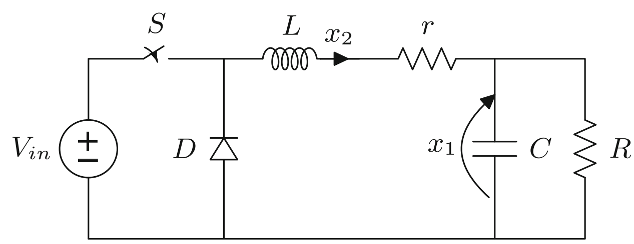

Fig. 1 Buck converter power circuit

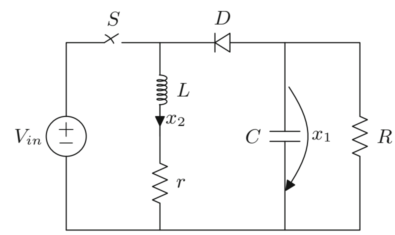

Fig. 2 Buck–boost converter power circuit

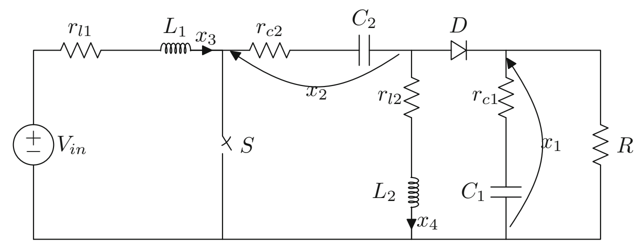

Fig. 3 SEPIC power circuit

$$\mathbf{A}_1 = \begin{bmatrix} -\dfrac{1}{RC} & 0 \\ 0 & -\dfrac{r}{L} \end{bmatrix} \quad \mathbf{A}_2 = \begin{bmatrix} -\dfrac{1}{RC} & \dfrac{1}{C} \\ -\dfrac{1}{L} & -\dfrac{r}{L} \end{bmatrix}$$
$$\mathbf{b}_1 = \begin{bmatrix} 0 & \dfrac{V_{in}}{L} \end{bmatrix}' \quad \mathbf{b}_2 = \begin{bmatrix} 0 & 0 \end{bmatrix}'$$

The buck converter in Fig. 1 has switch mode on $(\mathbf{A}_2, \mathbf{b}_1)$ and switch mode off $(\mathbf{A}_2, \mathbf{b}_2)$, while the buck–boost converter in Fig. 2 has switch mode on $(\mathbf{A}_1, \mathbf{b}_1)$ and switch mode off $(\mathbf{A}_2, \mathbf{b}_2)$.

The parameters for the buck and buck–boost converters are $V_{\text{in}} = 100\,\text{V}$, $R = 50\,\Omega$, $r = 2\,\Omega$, $L = 500\,\mu\text{H}$, $C = 470\,\mu\text{F}$, for all designs and simulations. Observe that r is a parasitic resistor, while R is the load.

Another important converter is the four-dimensional two-modal single-ended primary inductor converter (SEPIC), as in Fig. 3, with switch mode on $(\mathbf{A}_1, \mathbf{b}_1)$ and switch mode off $(\mathbf{A}_2, \mathbf{b}_2)$, where:

[3] Of course, the second line in (33) must also be altered. However, showing (39) is enough to reject the second approach.

$$\mathbf{A}_1 = \begin{bmatrix} -\frac{1}{(R+r_{c1})C_1} & 0 & 0 & 0 \\ 0 & -\frac{r_{c2}}{L_2} & 0 & \frac{1}{C_2} - \frac{r_{c2}r_{l2}}{L_2} \\ 0 & 0 & -\frac{r_{l1}}{L_1} & 0 \\ 0 & -\frac{1}{L_2} & 0 & -\frac{r_{l2}}{L_2} \end{bmatrix}$$

$$\mathbf{A}_2 = \begin{bmatrix} -\frac{L_1L_2 + Rr_{c1}C_1(L_1+L_2)}{L_1L_2C_1(R+r_{c1})} & -\frac{r_{c2}}{L_1} & -\frac{1}{L_1} & \frac{1}{L_2} \\ -\frac{Rr_{c1}}{(R+r_{c1})L_1} & -\frac{r_{c2}}{L_1} & -\frac{r_{l1}}{L_1} & 0 \\ \frac{R}{(R+r_{c1})C_1}\left(1 - \frac{r_{c1}r_{l1}C_1}{L_1}\right) & \frac{L_1 - r_{l1}r_{c2}C_2}{L_1C_2} & -\frac{r_{l1}}{L_1} & 0 \\ \frac{R}{(R+r_{c1})C_1}\left(\frac{r_{c1}r_{l2}C_1}{L_2} - 1\right) & 0 & 0 & -\frac{r_{l2}}{L_2} \end{bmatrix}'$$

$$\mathbf{b}_1 = \begin{bmatrix} 0 \; 0 \; \frac{V_{in}}{L_1} \; 0 \end{bmatrix}' \quad \mathbf{b}_2 = \begin{bmatrix} \frac{r_{c1}C_1V_{in}}{L_1} \; \frac{r_{c2}V_{in}}{L_1} \; \frac{V_{in}}{L_1} \; 0 \end{bmatrix}'$$

The parameters for all designs and simulations involving the SEPIC are $V_{\text{in}} = 100\,\text{V}$, $R = 50\,\Omega$, $L_1 = 500\,\mu\text{H}$, $L_2 = 600\,\mu\text{H}$, $C_1 = 800\,\mu\text{F}$, $C_2 = 600\,\mu\text{F}$, $r_{l1} = 2\,\Omega$, $r_{l2} = 3\,\Omega$ and $r_{c1} = r_{c2} = 100\,\text{m}\,\Omega$. Again, R is the load and all remaining resistors are parasitic.

4.1 Full-Order Design

For the buck and buck–boost converters, $\mathbf{C}_1 = \mathbf{C}_2 = \begin{bmatrix}1 \; 0\end{bmatrix}$ and $\mathbf{E}_1 = \mathbf{E}_2 = \begin{bmatrix}1 \; 1\end{bmatrix}$ will be adopted. The design is provided by MATLAB®, using LMILab (Gahinet et al. 1994) via Yalmip (Löfberg 2004), minimizing $\text{Tr}(\mathbf{P})$ in Theorem 1.

For the buck converter, the resulting matrices were:

$$\mathbf{P} = \begin{bmatrix} 0.2662 & 0.2184 \\ 0.2184 & 0.2112 \end{bmatrix} \times 10^{-3}$$

$$\mathbf{L}_1 = \mathbf{L}_2 = \begin{bmatrix} 2.9032 \\ -3.54 \end{bmatrix} \times 10^{-3}$$

while, for the buck–boost converter:

$$\mathbf{P} = \begin{bmatrix} 0.0034 & 0.0003 \\ 0.0003 & 0.0062 \end{bmatrix}$$

$$\mathbf{L}_1 = \begin{bmatrix} 11.0258 \\ -77.0672 \end{bmatrix} \quad \mathbf{L}_2 = \begin{bmatrix} 20.3073 \\ -121.5431 \end{bmatrix}$$

Finally, for the SEPIC and adopting $\mathbf{C}_i = \mathbf{E}_i = \begin{bmatrix}1 \; 1 \; 0 \; 0\end{bmatrix}$ the designed matrices are:

$$\mathbf{P} = \begin{bmatrix} 0.01 & 0.0008 & 0 & 0.0002 \\ 0.0008 & 0.0018 & 0 & 0.0004 \\ 0 & 0 & 0.0008 & -0.0006 \\ 0.0002 & 0.0004 & -0.0006 & 0.0006 \end{bmatrix}$$

$$\mathbf{L}_1 = \begin{bmatrix} -42.5 \; -1202.9 \; 3539.8 \; 4783.1 \end{bmatrix}'$$

$$\mathbf{L}_2 = \begin{bmatrix} -42.6 \; -1198.1 \; 3522.6 \; 4758.5 \end{bmatrix}'$$

4.2 Reduced-Order Design

For the reduced-order design, the same software is used, but $2\,\text{Tr}(\mathbf{P}_w) + \text{Tr}(\mathbf{P}_y)$, which is the Lyapunov matrix in (38), is minimized in Theorem 4 instead.

For the buck and buck–boost designs, $\mathbf{E}_1 = \mathbf{E}_2 = 1$ was chosen. The resulting matrices are, for the buck converter, $\mathbf{P}_w = 1.3917 \times 10^{-4}$, $\mathbf{P}_y = 8.7924 \times 10^{-5}$, $\mathbf{L} = 7.7057$ and, for the buck–boost converter, $\mathbf{P}_w = 1.3914 \times 10^{-4}$, $\mathbf{P}_y = 8.7904 \times 10^{-5}$, $\mathbf{L} = 7.706$.

For the SEPIC, with $\mathbf{E}_1 = \mathbf{E}_2 = \begin{bmatrix}1 \; 1\end{bmatrix}$, the resulting matrices are:

$$\mathbf{P}_w = \begin{bmatrix} 0.125 & 0.0496 \\ 0.0496 & 0.0689 \end{bmatrix} \times 10^{-3}$$

$$\mathbf{P}_y = \begin{bmatrix} 0.3586 & -0.0397 \\ -0.0397 & 0.7915 \end{bmatrix} \times 10^{-4}$$

$$\mathbf{L} = \begin{bmatrix} 2.3464 & -0.989 \\ -6.4368 & 12.0639 \end{bmatrix}$$

4.3 Simulation Results

Simulations were run with ODE45 in MATLAB®, initial state $\begin{bmatrix}\mathbf{x}(0)' \; \hat{\mathbf{x}}(0)'\end{bmatrix} = \begin{bmatrix}20 \; 0.1 \; 0 \; 0\end{bmatrix}$ (buck and buck–boost) or $\begin{bmatrix}\mathbf{x}(0)' \; \hat{\mathbf{x}}(0)'\end{bmatrix} = \begin{bmatrix}20 \; 20 \; 1 \; 1 \; 0 \; 0 \; 0 \; 0\end{bmatrix}$ (SEPIC), desired operation points $\tilde{\mathbf{x}} = \begin{bmatrix}50 \; 1\end{bmatrix}'$ (buck), $\tilde{\mathbf{x}} = \begin{bmatrix}120 \; 6\end{bmatrix}'$ (buck–boost) and $\tilde{\mathbf{x}} = \begin{bmatrix}40.53 \; 83.2 \; 12.88 \; -2.988\end{bmatrix}'$ (SEPIC).

For the reduced-order case, it must be noted that the initial state is shortened, as some variables are no longer necessary. For the buck and buck–boost converters, it is now $\begin{bmatrix}\mathbf{x}(0)' \; \hat{\mathbf{x}}(0)'\end{bmatrix} = \begin{bmatrix}20 \; 0.1 \; 0\end{bmatrix}$ while, for the SEPIC, it is now $\begin{bmatrix}\mathbf{x}(0)' \; \hat{\mathbf{x}}(0)'\end{bmatrix} = \begin{bmatrix}20 \; 20 \; 1 \; 1 \; 0 \; 0\end{bmatrix}$. The obtained results[4] are in Figs. 4, 5 and 6, respectively, to the buck, buck–boost and SEPIC.

Every simulation led the state to its desired operation point. Observe that, in all simulations, Theorem 4 showed better results for the convergence of the state variables to be estimated, i.e., the error converged faster to zero. More-

[4] The reduced order x_2 is the lower curve in Fig. 5. Dash-dots joined due to the number of plotted points.

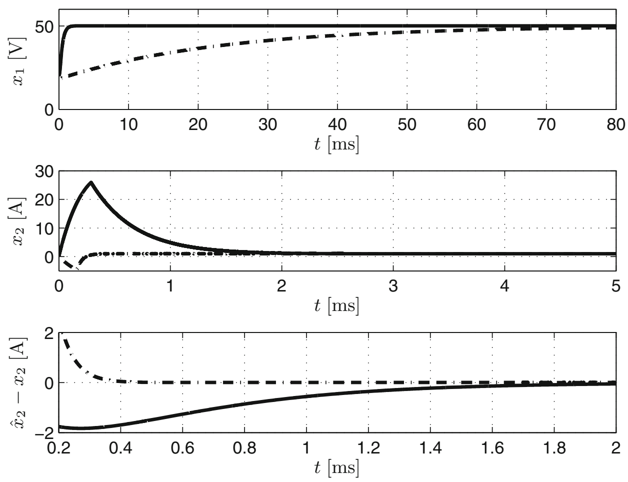

Fig. 4 Simulation results for the buck converter with Theorem 1 (*solid*) and Theorem 4 (*dash-dot*)

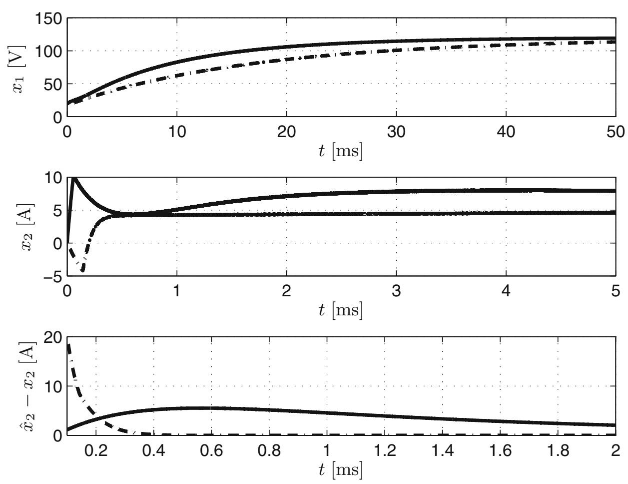

Fig. 5 Simulation results for the buck–boost converter with Theorem 1 (*solid*) and Theorem 4 (*dash-dot*)

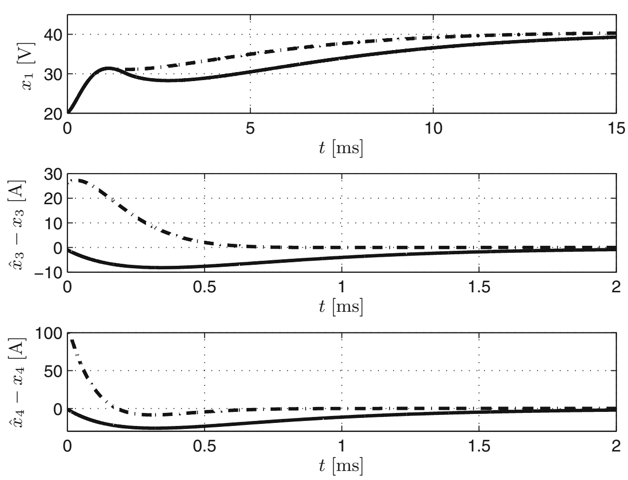

Fig. 6 Simulation results for the SEPIC converter with Theorem 1 (*solid*) and Theorem 4 (*dash-dot*)

over, for the SEPIC, it also drove faster the output, x_1, to the desired point.

The main reason for that behavior is that the initial error is smaller for the reduced-order case as some state variables are not estimated. As for the output to converge faster to the operation value, in DC–DC converters, that depends on the available inductor current, whose peaks are suppressed as a consequence of the minimization of (13). It must be noted, however, that this is a secondary effect of the design as (13) is the optimization target.

5 Conclusions

New results on full-order observer-based SASs control are presented in Theorems 1 and 2. Those theorems inserted matrix blocks for enhancing the state estimation by guaranteeing an upper bound for the error energy as in (16). As previous results given by Yoshimura et al. (2013), Deaecto et al. (2010) and Mainardi Júnior et al. (2012), Theorem 1 has an immediate application, the DC–DC converters. Although Theorem 2 presents no immediate application, that result is important as it could also be applied to DC–DC converters and completes the theory on full-order SASs state estimation.

The main result is the composition of a cost-guaranteed ROSLO (Theorem 4). That technique allowed the design of an estimator only for the unavailable part of the state, i.e., the measured variables are not estimated. The reduced-order observer presented better error convergence in all designs and, in the case of the SEPIC, better output convergence to the desired point of output operation.

The ROSLO, unlike the full-order observer, has the major drawback of imposing a common-mode $\mathbf{W}$ in (26), leading to possible feasibility problems in other applications. Finally, note that despite the only $\mathbf{L}$ matrix, the observer is still switched, as its structure is determined by (24).

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