IFT

Instituto de Física Teórica
Universidade Estadual Paulista

Doctoral Thesis

IFT-T.005/17

Radiation reaction for spinning bodies in the effective field theory approach

Natália Tenório Maia

Supervisor

Rafael A. Porto

2017, August

"Bias in Science and Philosophy, like any other kind of prejudice, is indicative of immaturity and always to the detriment of truth at last."

Ibn Sina, 980 to 1037, Persia.

To Emmy Noether, Maria Sklodowska, Katherine Johnson and all other awesome people who had to face the hostility of human society besides the very challenge of making Science.

Agradecimentos

...e a grande jornada de estudante finalmente chega ao fim. São 10 anos dedicados ao estudo dessa magnífica ciência chamada Física. Desses anos, os bons momentos guardo com afeto e me revigoro ao trazê-los à memória. Os momentos ruins, por sua vez, foram superados - alguns com muito custo - e, por ironia, acabaram por fazer minha relação com a Física se estabelecer ainda mais madura e verdadeira.

Chegar até aqui, em sentido geral, é uma grande conquista alcançada não só por mim, mas alcançada de uma forma conjunta. Seguindo a seta do tempo, as primeiras pessoas a que devo agradecer são a minha família. Meu pai Fernando Maia, minha mãe Linda Tenório e meu irmão Fernando Júnior são os alicerces do que sou, pois me ajudaram a construir o que há de melhor em mim. Serão sempre meu porto seguro e meus maiores exemplos de dignidade, honestidade, persistência e - não menos importante! - bom humor.

A superação dos obstáculos que enfrentei se deve em grande parte à compreensão e empatia de pessoas boas que encontrei pelo caminho. Sou grata à prof. Nádia Köche, prof. José W. Maluf, prof. José G. Pereira e prof. Roberto Kraenkel por terem me ajudado de boa vontade em momentos decisivos. Agradeço também ao prof. Carlos A. Galvão pela amizade e, apesar dos nossos diferentes pontos de vista, por compartilhar comigo da mesma paixão pela Ciência e pela Arte (e por compartilhar whisky também agradeço!).

Meus amigos também tiveram - e têm - grande importância nessa jornada. Em especial, agradeço à Rayssa Bruzaca por estar comigo na alegria e na tristeza, na saúde e na doença, na riqueza (sim!) e na pobreza. Meu amigo Diego Restrepo por ter me ensinado a usar o Xact com tanta boa vontade e precisão, e me fazer dar muitas risadas com suas imitações (não vou falar de quem). Meus amigos Fernando César, Carlisson Miller, Jogean Matheus, Andresa Campos, Pierre Niau, Prieslei Goulart e Pablo Carlesso por fazerem meus dias no IFT mais felizes. Não posso deixar de expressar minha gratidão aos amigos que não posso ver, mas sempre estão comigo.

Agradeço o prof. Nathan Berkovits pelo ICTP-SAIFR, que possibilita a vinda de excelentes pesquisadores a São Paulo, incluindo meu orientador Rafael Porto. Iniciativas como a do prof. Nathan, que trazem um impacto tão positivo na Ciência no Brasil e na vida dos jovens físicos da América Latina, ganham meus sinceros agradecimentos e admiração. Aproveito para agradecer a CAPES por financiar a minha pesquisa.

Finalmente agradezco a mi orientador Rafael Porto, por haber cometido la locura de aceptarme como su estudiante en una situación tan poco común. Fue esa hazaña la que me trajo de vuelta al camino de la Ciencia. Todas las palabras que escriba aquí jamas conseguirán expresar la verdadera gratitud que siento por todo lo que él ha hecho por mi. Rafael, deseo que continues haciendo la diferencia en la vida de los estudiantes. ¡Nunca cambies tu personalidad auténtica y jovial! Te admiro y te respeto.

Resumo

Nesta tese, nós investigamos os efeitos de reação de radiação devido ao spin na dinâmica de um sistema binário de corpos compactos usando uma abordagem de teoria efetiva de campos. Focamos no estágio de espiral da evolução do sistema binário que, por sua vez, provê uma hierarquia de escalas propícia à implementação de uma abordagem perturbativa, tal como a expansão pós-newtoniana. Fazemos uso de um formalismo próprio para investigar efeitos dissipativos. Provemos uma extensão desse formalismo para incluir graus de liberdade de spin. Com isso, em uma abordagem de teoria efetiva de campos, calculamos as acelerações de reação de radiação devido a efeitos de spin-órbita e spin-spin, em primeira ordem. Apresentamos, pela primeira vez, a contribuição de spin na reação de radiação devido ao tamanho finito dos corpos compactos. Também investigamos como os spins de tais corpos são afetados pela reação de radiação, na ordem pós-newtoniana de interesse. Por fim, realizamos um teste de consistência - relacionando a potência total radiada com a perda de energia induzida pelas forças dissipativas - assegurando, assim, a validade dos nossos resultados.

Palavras-chave: Teoria Efetiva de Campos, Ondas Gravitacionais, Spin, Reação de Radiação.

Abstract

In this thesis, we investigate the radiation reaction effects due to spin on the dynamics of binary compact bodies, using an effective field theory framework. We focus on the inspiral phase of the binary's evolution, which provides a hierarchy of scales that invites us to implement a perturbative approach such as the Post-Newtonian expansion. We use a formalism suitable to incorporate dissipative effects, providing an extension to include spin degrees of freedom. We use this extension of the effective field theory framework to compute the radiation reaction accelerations due to spin-orbit and spin-spin effects at leading order. We present, for the first time, the spin contribution to radiation reaction due to finite size effects. We also investigate how the spin evolution of the compact bodies is affected by the radiation reaction, at the order of interest. Finally, we perform a consistency test - relating the total radiated power to energy loss induced by the non-conservative forces - ensuring the validity of our results.

Keywords: Effective Field Theory, Gravitational Waves, Spin, Radiation Reaction.

Contents

Introduction

The successful detections of gravitational waves have encouraged precise descriptions of the dynamics of binary compact objects in order to probe the nature of the sources and to test gravity deep in the non-linear regime. The gravitational waves detected by LIGO [1] were generated by binary systems of merging black holes in the distant universe. To develop accurate theoretical models and predictions for such systems, it is necessary to know their evolution under dissipative effects to high accuracy. The work presented in this thesis entails the study of radiation reaction effects in the evolution of spinning binary systems using effective theory methods.

The heroic experimental achievement by the LIGO collaboration gives us confidence in the validity of Einstein's theory of gravity and, at the same time, it opens a new window onto the cosmos, coined "gravitational wave astronomy". Prior to the detections, the existence of gravitational waves was strongly suggested by Taylor and Hulse in the 70's and 80's, who observed for the first time a binary system composed of a pulsar and a neutron star orbiting a common centre of mass. The pulsar provided a clock which allows us to infer the relevant orbital parameters and masses from observations. Later on, Taylor and Weisberg [2] found that the orbit of the pulsar was slowly shrinking over time. This observation was then associated with the release of energy in the form of gravitational waves. The discovery of the so called Hulse-Taylor binary was awarded the nobel prize in 1993.

Gravitational waves bring a new era in Astronomy, revealing information about the Universe that cannot be observed through other ways. Since gravitational waves interact very weakly with matter, they can penetrate dust clouds and other barriers that would absorb or scatter light. Therefore, it is possible to see into the centre of dense systems, such as the cores of supernovae or the Galactic Centre, as well as further back in time, since the early universe was opaque to light prior to recombination. Furthermore, being produced by coherent mass movements, gravitational waves can also provide information about the internal structure of massive objects. This is in sharp contrast to electromagnetic radiation, which usually carries incoherent surface information. In particular, gravitational waves are extremely important to the study of black holes, since it is the only way we can directly observe them, all other information about them is indirect coming from their effects on the surrounding environment. To obtain all the information gravitational waves can provide, large investiments have been made on ground-based detectors such as LIGO/Advanced LIGO [3] in USA and VIRGO/Advanced VIRGO [4] in Italy. The planned KAGRA [5] in Japan, IndIGO [6] in India, ET [7] and the space-based eLISA [8] will be operational in the future in order to observe the universe in a more precise and broader way.

In this thesis we will focus on binary compact objects in view of their primordial importance to observational astronomy. The majority of stars is found in binary - or in multiple - systems [9]. The lifecycle of binary systems consists of three main stages: inspiral, merger and ringdown. We will concentrate here on the inspiral phase, where velocities are up to a small fraction of the speed of light. This allows us to introduce a perturbative expansion for physical observables in terms of the relative velocity. The two body problem has been extensively covered in the literature over many decades. Since the development of General Relativity, the non-linearites of the field equations make finding a close analytic solution an almost impossible task. Therefore, one approach is to solve the equations numerically [10–13]. However, due to the computational limitations, this is only suitable for the last stages of the dynamics, including merger and ringdown. In turn, this only covers a small fraction of all the cycles that we expect to observe with ground- and space-based detectors. For the inspiral phase, on the other hand, we rely on analytic approximations such as the post-Newtonian (PN) expansion [14–18]. This is the regime we focus on in here, albeit using a novel formalism borrowed from the realms of Particle Physics.

Here we will work within the Effective Field Theory (EFT) framework originally proposed by Goldberger and Rothstein in the context of non-spinning gravitationally bound extended objects. Due to its similitudes with EFTs for heavy quarks in Quantum Chromodynamics (QCD) [19–21], the EFT approach was coined Non-Relativistic General Relativity (NRGR). Even though Feynman diagrams and other tools that originated in Quantum Field Theory (QFT) will appear throughout this thesis, our work will be confined to the classical level. QFT techniques have been applied to gravity [22–30] prior to NRGR, however the latter formalism provides a systematic treatment of gravitational systems when a separation of scales is present [31–37], such is the case for a binary system during the inspiral phase. NRGR has been successful in reproducing many of the results achieved with other analytical approaches, particularly those in the conservative sector for non-spinning bodies [38–44]. An extension of NRGR was developed in [45, 46] in order to describe spinning bodies, generalizing to General Relativity the work of Hanson and Regge [47] in Minkowski space.

The EFT approach for spinning bodies has been instrumental to extend our knowledge of the binary's dynamics to high level of accuracy. Only spin-orbit effects were known, to 2.5PN order, when NRGR was being developed. The first results for spin-spin gravitational potentials at next-to-leading-order (NLO), corresponding to 3PN order, were obtained in [48–52]. This was corroborated in [53–55] through the Arnowitt-Deser-Misner (ADM) formalism [56] and posteriorly in the harmonic gauge [57]. Higher order effects have been more recently studied in the conservative sector within the EFT approach in [58–61] for the spin-orbit potential at 3.5PN order and spin-spin potential and 4PN order. These results were also calculated in [62–65] except for the finite-size contributions, which are handled by the EFT approach in an straightforward way. Moreover, the effects due to absorption were also studied in the EFT framework in [31, 32].

The subject of this thesis is radiation reaction effects due to spin, where new effects will be presented. Back reaction within an EFT framework was explored for the first time in [37, 66] by implementing the classical limit of the "in-in" approach [67, 68]. Originally computed in [69–72], the radiation reaction force at 3.5PN order was re-derived within NRGR in [73] employing the formalism for non-conservative systems developed in [74, 75]. Here we extend the formalism to incorporate spin and compute the spin-orbit and spin-spin effects, including finite size contributions for the first time. We provide a non-trivial consistency check by showing the equivalence between the energy loss induced by the resulting radiation reaction acceleration and the total emitted power in the far zone, which follows from the well-known multipole expansion. The novel work presented in this thesis is based on our papers [76] and [77].

This thesis is organized in four chapters. In the first chapter we summarize fundamental concepts on gravitational waves physics. We also briefly review the introduction of spin in General Relativity and for the two body problem in gravity. In the second chapter, we introduce NRGR in a pedagogical manner addressing only the conservative sector of the theory. We show how to construct the effective action using the weak field and PN (low velocity) approximations. The incorporation of spin effects in NRGR is discussed at the end of the chapter. In the third chapter, the radiation sector NRGR is presented. We show how to compute the energy flux in gravitational radiation to all orders, through a multipole expansion at the level of the action. We also cover the formalism to study radiation reaction effects for non-spinning bodies, which we will extend next to introduce spin effects. In the last chapter, we present the results of our work: the construction of a formalism to compute radiation reaction effects due to spin and the computation of the leading order spin-orbit and spin-spin effects for binary compact objects. Our calculations include the effects due to the finite size of the spinning bodies, which is presented for the first time. We conclude our thesis with a discussion of the results and perspectives.

Notations and Conventions

Throughout this thesis, we use geometric units, $c = G = \hbar = 1$. Newton's constant can appear conveniently, for instance to express a known result. We define the Planck mass by $m_{Pl} \equiv \frac{\hbar c}{\sqrt{32\pi G}} = \frac{1}{\sqrt{32\pi}}$. The signature of the metric is $(1, -1, -1, -1)$. Whenever repeated Latin indices in different tensors appear at the same heigh, for example $A^i B^i$, it means a contraction performed with the Euclidean metric, i.e. $A^i B^i = A^1 B^1 + A^2 B^2 + A^3 B^3$.

1. General Relativity: a brief review

The theory of General Relativity formulated by Einstein at the beginning of the last century was the first geometrical theory of gravity. In this theory, there is no absolute reference frame and dynamics is governed by the geometry of spacetime itself, where quadrupole accelerations of mass distributions generate ripples that travel at the speed of light and are named gravitational waves. This kind of radiation was first discussed by Laplace in 1805, when he explored the hypothesis of finite-speed gravitational influence. He predicted that the angular momentum of a binary star system would decrease with time, which now is interpreted as it being carried away by gravitational waves.

After the presentation of the modern form of gravitational waves by Einstein, in his 1915 publication, as solutions to a linearized form of the gravitational field equations, they were not widely studied until the 1950s, when it was proved by Hermann Bondi that gravitational waves are physically observable and in fact carry energy [78]. Subsequently, the awarded discovery of the shrinking orbit of the binary pulsar system by Hulse and Taylor triggered the scientific community to seek out further evidence of gravitational waves and ways to make possible their detection. After LIGO's achievements in 2015, we start to think of gravitational waves as promissing tools for unravelling misteries about the Universe and getting deeper in the understanding of gravitational phenomena.

In this chapter we present a brief review on general concepts of General Relativity. In the first part, we focus on gravitational waves physics. We use the weak field approximation to find plane wave solution for Einstein's equations and discuss about polarization and sources of gravitational radiation. In the second part, we show how to include a description for spinning bodies in General Relativity, which will be regarded later in the EFT framework for binary systems throughout the thesis.

1.1. Gravitational Waves

1.1.1. Einstein's equations

Einstein's field equations can be derived from a minimal action principle in a very succinct way. In order to formulate a generally covariant and locally Lorentz invariant theory of gravity, we must have an action which is a scalar under both Lorentz and coordinate transformations:

$$S = \int \sqrt{g}\left(\mathcal{L}_g + \mathcal{L}_m\right) d^4x, \tag{1.1}$$

where

$$\mathcal{L}_g = -\frac{1}{16\pi} g_{\mu\nu} R^{\mu\nu} \tag{1.2}$$

is the Lagrangian related to the gravitational field, while $\mathcal{L}_m$ contains the matter fields contributions for the dynamics. The Ricci tensor $R^{\mu\nu}$ is the contraction ${R_{\alpha\nu\mu}}^{\alpha}$ of the Riemann tensor

$${R_{\alpha\nu\mu}}^{\beta} = \partial_\alpha {\Gamma_{\nu\mu}}^{\beta} - \partial_\nu {\Gamma_{\alpha\mu}}^{\beta} + {\Gamma_{\alpha\rho}}^{\beta} {\Gamma_{\nu\mu}}^{\rho} - {\Gamma_{\nu\rho}}^{\beta} {\Gamma_{\alpha\mu}}^{\rho} \tag{1.3}$$

with the metric tensor $g_{\mu\nu}(x)$. The tensor above is defined in terms of the connection

$$\Gamma^{\lambda}_{\mu\nu} = \frac{1}{2} g^{\lambda\beta}\left(\partial_\nu g_{\beta\mu} + \partial_\mu g_{\beta\nu} - \partial_\beta g_{\mu\nu}\right). \tag{1.4}$$

We note that the infinitesimal four-volume d^4x is a Lorentz scalar, but under a general coordinate transformation it transforms like $d^4x' = \left|\frac{\partial x'}{\partial x}\right| d^4x$. The presence of $\sqrt{g}$ in (1.1) is to compensate the Jacobian at d^4x', since $\sqrt{g'} = \left|\frac{\partial x'}{\partial x}\right|^{-1} \sqrt{g}$, ensuring an invariant action under coordinate transformations.

To derive the equations of motion, the variation has to be performed with respect to the dynamical variable - the gravitational field represented by the metric of the spacetime. Defining

$$V^{\alpha} \equiv g^{\mu\nu} \delta {\Gamma_{\mu\nu}}^{\alpha} - g^{\nu\alpha} \delta {\Gamma_{\nu\beta}}^{\beta}, \tag{1.5}$$

and taking into account the metricity-postulate $D_\alpha g_{\mu\nu} = 0$, the variation of the gravitational action reads as

$$\int \delta\left(\sqrt{g} g_{\mu\nu} R^{\mu\nu}\right) d^4x = \int \partial_\alpha\left(\sqrt{g} V^{\alpha}\right) d^4x, \tag{1.6}$$

which is zero in consequence of the boundary conditions at infinity. The variation of the matter action, in turn, is given by

$$\delta S_m = \int \frac{\delta\left(\sqrt{g}\mathcal{L}_m\right)}{\delta g_{\mu\nu}} \delta g_{\mu\nu} d^4x = \frac{1}{2}\int \sqrt{g} T_{\mu\nu} \delta g_{\mu\nu} d^4x, \tag{1.7}$$

in view of the definition for the symmetric energy-momentum tensor

$$T_{\mu\nu} \equiv \frac{2}{\sqrt{g}} \frac{\delta\left(\sqrt{g}\mathcal{L}_m\right)}{\delta g_{\mu\nu}}. \tag{1.8}$$

This tensor is the source of the gravitational field and represents the energy density of the system with the exception of the gravitational one.

The action is invariant under infinitesimal local transformations, $\delta x^\mu = \xi^\mu$, furthermore it is possible to show that translations imply the metric to vary in the following way

$$\delta g_{\mu\nu} = D^\nu \xi^\mu + D^\mu \xi^\nu, \tag{1.9}$$

such that

$$\delta S_m = \int \sqrt{g} T_{\mu\nu} D^\nu \xi^\mu d^4x. \tag{1.10}$$

Getting rid off the surface terms, it results

$$\delta S_m = -\int \sqrt{g} D^\nu T_{\mu\nu} \xi^\mu d^4x, \tag{1.11}$$

but since the parameter of the transformation is arbitrary and the action is invariant under infinitesimal translations, it implies the conservation of the energy-momentum tensor

$$D^\nu T_{\mu\nu} = 0 \tag{1.12}$$

From considering the variations of both the actions related to gravity and matter together,

$$\delta S = -\int \frac{1}{16\pi} \sqrt{g} \left[\left(R_{\mu\nu} - \frac{1}{2} g_{\mu\nu} R\right) + T_{\mu\nu}\right] \delta g^{\mu\nu}, \tag{1.13}$$

we can readly write Einstein's equations

$$R_{\mu\nu} - \frac{1}{2} g_{\mu\nu} R = 8\pi T_{\mu\nu}. \tag{1.14}$$

The expression above represents ten partial differential equations of second order in the gravitational field $g_{\mu\nu}$, but only six of them are independent, as a consequence of Bianchi's identity.

The fact that the conservation of energy-momentum can be directly deduced from the field equations means that they contain also the equations of motion for the matter fields. In the electromagnetic theory, for instance, the situation is different, since from the Maxwell field equations one can deduce the conservation of the electromagnetic current, but not the equations of motion of charges, i.e. the Lorentz equations. In this case, we assign the charge and current distributions and then obtain the fields solving the field equations. In the gravitational case, on the other hand, the matter distribution and its motions cannot be prescribed independently from the gravitational field produced by them. This characteristic aspect of gravitation is to be ascribed to the fact that the energy plays the double role of gravitational source and inertial mass.

1.1.2. Weak Field Approximation

General Relativity is a non-linear theory and in general there is no sharp distinction between the part of the metric that represents the gravitational radiation and the other part which describes the geometry of spacetime. However, in the weak field limit we clearly define gravitational radiation. In this case, we can show that Einstein's equations have wave-like solutions when they are written and solved in a nearly flat space-time.

In the weak field approximation, the spacetime metric is considered almost flat up to a small perturbation:

$$g_{\mu\nu}(x) = \eta_{\mu\nu} + h_{\mu\nu}(x), \tag{1.15}$$

where $|h_{\mu\nu}| \ll 1$, what means that second or higher orders of the field $h_{\mu\nu}$ can be neglected. The indices of this perturbation are raised and lowered using the Minkowski metric $\eta_{\mu\nu}$. There is considerable coordinate freedom in the components $h_{\mu\nu}$, since we can wiggle and stretch the coordinate system with a comparable amplitude and change the components. This coordinate freedom is called gauge freedom, by analogy with electromagnetism. We use this freedom to enforce the Lorenz condition:

$$\partial^{\nu}\bar{h}_{\mu\nu} = 0, \tag{1.16}$$

where we regard the definition bellow

$$\bar{h}_{\mu\nu} \equiv h_{\mu\nu} - \frac{1}{2}\eta_{\mu\nu}h, \tag{1.17}$$

where $h = \eta_{\mu\nu}h^{\mu\nu}$. In this way, the Einstein's field equation can be expressed by

$$\Box\bar{h}_{\mu\nu} = -16\pi T_{\mu\nu}, \tag{1.18}$$

where $\Box \equiv \eta^{\sigma\lambda}\partial_{\sigma}\partial_{\lambda}$. It is clear that this last expression is a set of decoupled linear wave equations. The simplest case is when there is absence of matter $\Box\bar{h}_{\mu\nu} = 0$, so that the solution for Einstein's equations are plane waves:

$$\bar{h}_{\mu\nu}(x) = A_{\mu\nu}e^{i\mathbf{k}\cdot\mathbf{x}}. \tag{1.19}$$

It is important to have in mind that we can always find a gauge in which (1.16) is satisfied. A consequence of (1.16) is

$$A^{\mu\nu}k_\nu = 0, \tag{1.20}$$

which restricts $A^{\mu\nu}$ to be orthogonal to the direction of propagation.

Taking the example of the plane wave solution of Einstein's equations, we use the transverse-traceless gauge to talk about polarization of gravitational waves. The gauge redefinition

$$h_{\mu\nu} \rightarrow h_{\mu\nu} - \partial_\mu \xi_\nu - \partial_\nu \xi_\mu \tag{1.21}$$

together with (1.16), implies, in vacuum,

$$\Box \xi_\alpha = 0, \tag{1.22}$$

whose solution reads

$$\xi_\alpha = B_\alpha e^{i\mathbf{k}\cdot\mathbf{x}}. \tag{1.23}$$

The vector B_α can be chosen to impose two further restrictions on $A_{\mu\nu}$:

$$A_\mu^{\ \mu} = 0 \tag{1.24}$$

and

$$A_{\mu 0} = 0. \tag{1.25}$$

Together, (1.20), (1.24) and (1.25) constitute the so called *transverse-traceless* (TT) gauge conditions. The trace condition implies, for instance, $\overline{h}^{TT}_{\mu\nu} = h^{TT}_{\mu\nu}$. We can use the TT gauge to show that gravitational waves have only two possible polarizations. We choose the spatial coordinate axes so that the wave travels in the z direction, i.e. $k = (\omega, 0, 0, \omega)$, so that $A_{\mu z} = 0$. This means that only A_{xx}, A_{xy}, A_{yx} and A_{yy} are nonzero. Furthermore, the trace condition puts the restriction $A_{xx} = -A_{yy}$, and due to the symmetry of the metric tensor $A_{xy} = A_{yx}$. Then, in this particular frame, we see that there are only two independent components of the wave: A_{xx} and A_{xy}. We can represent these the two linear polarizations as in Fig.(1.1).

To illustrate the physics behind the interaction of gravitational waves with matter, consider two nearby particles, one at the origin and another at $x = \epsilon, y = z = 0$, both at rest. The proper distance between them when they encounter a gravitational wave is given by

$$\Delta l = \int |g_{\mu\nu} dx^\mu dx^\nu|^{\frac{1}{2}} = \int_0^\epsilon |g_{xx}|\, dx^{\frac{1}{2}} \approx \left[1 + \frac{1}{2} h^{TT}_{xx}\,(x=0)\right]\epsilon, \tag{1.26}$$

what means that the proper distance changes with time making possible to detect gravitational wave signals.

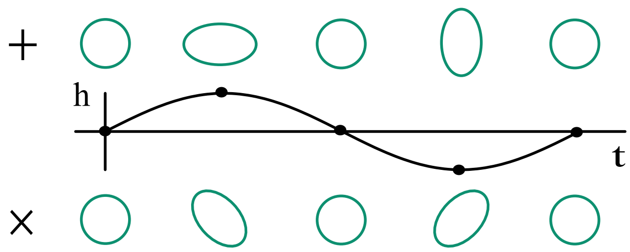


Figure 1.1.: In the upper case, $h_{xx}^{TT} \neq 0$ and $h_{xy}^{TT} = 0$, representing the + polarization. The circles under the graphic represent the × polarization, where $h_{xx}^{TT} = 0$ and $h_{xy}^{TT} \neq 0$.

1.1.3. Sources

The sources of gravitational waves can be roughly classified into two main categories. One is called cosmological origin, the other is relativistic astrophysical origin. In the cosmological case, gravitational radiation can be produced in the early stages of the Universe, for example, during the inflation and reheating epochs. In this case, the radiation is called primordial gravitational waves, and they will leave unique imprint on the cosmic microwave background, the so-called B-mode. Moreover, during the evolution of the Universe, it is expected that various phase transitions had happened as the temperature of the Universe decreases, for example, the symmetric breaking of the grand unification theory or the electroweak and quantum chromodynamics phase transitions. Gravitational waves are also expected to be produced with different features in each phase transition. Therefore, the detection of gravitational radiation due to those cosmological origins can reveal physics associated with the evolution of the Universe.

In the astrophysics side, gravitational waves can be produced in processes like rotation of non- symmetric neutron star, explosion of supernovae, inspiral, merger and ringdown of compact binaries including white dwarf, neutron star or black hole. A spinning non-symmetric neutron stars is an example of long-lived source which generates continuous waves whereas the coalescence of a compact binary is a short-lived source of gravitational waves.

Although binary star systems are common in our galaxy, only a tiny fraction experience an evolution that arrives at two compact objects in an orbit tight enough to lead to compact binary coalescence in a Hubble time [79]. The coalescence of two compact massive objects (neutron stars and black holes) into a single final black hole can be divided into three reasonably distinct stages: inspiral, merger and ringdown. During the inspiral stage, analytic expressions (perturbative post-Newtonian approximations) for gravitational waveforms are expected to be accurate. In the merger stage, strongly relativistic effects require numerical relativity calculations. The merger will result in a highly deformed single black hole which rids itself of its deformity by emitting gravitational radiation that is characteristic of the mass and spin of the final black hole. This is called the quasi-normal mode or the ringdown signal, which describes the final stage of the binary's evolution.

The inspiral stage lends itself to a natural perturbative approach: in such stage, the difference among the scales of binary system - the separation of the bodies in comparison to the wavelenght of the gravitational waves emmited by them as well as the velocity of the bodies in face of the speed of light - invites us to the usage of an effective framework, as we shall see throught out this thesis.

1.2. Spin in General Relativity

Spin in General Relativity has been explored in various ways [80–82]. In particular, spin effects for binary system have been calculated using different techniques within the PN approximation [83–87]. As it is known, the motion of a spinning particle is described by the Mathisson-Papapetrou-Dixon equations in General Relativity:

$$\frac{Dp^\mu}{d\tau} = -\frac{1}{2} R^\mu{}_{\nu\alpha\beta} u^\nu S^{\alpha\beta}, \tag{1.27}$$

$$\frac{DS^{\mu\nu}}{d\tau} = p^\mu u^\nu - p^\nu u^\mu, \tag{1.28}$$

where p^μ represents the momentum of the particle, $S^{\alpha\beta}$ is the antisymmetric spin tensor and $\frac{D}{d\tau}$ simply denotes the covariant derivative with respect to the particle's proper time.

In order to have an EFT description for spinning objects, which will be presented in the next chapter, we need to write an action that provides these equations of motion above.

1.2.1. An action for spinning bodies

To accomodate rotations, the first step is to introduce new degrees of freedom e_I^μ - elements of SO(3,1) - to the worldline, such that

$$\eta^{IJ} = e^I_\mu e^J_\nu g^{\mu\nu}, \tag{1.29}$$

$$g_{\mu\nu} = e^I_\mu e^J_\nu \eta_{IJ}. \tag{1.30}$$

The capital Latin indices are lowered and raised with the Minkowski metric $\eta^{IJ} \equiv (1, -1, -1, -1)$ and transform under a residual Lorentz invariance. We define the transport of a tetrad through the particle's worldline by

$$\dot{e}^I_\mu \equiv \frac{De^\mu_I}{d\tau} = u^\alpha \nabla_\alpha e^\mu_I = -\Omega^{\mu\nu} e_{I\nu}, \tag{1.31}$$

where $\Omega^{\mu\nu} = -\Omega^{\nu\mu}$ represents the angular velocity tensor and ∇_α is simply a covariant derivative compatible with the metric $g^{\mu\nu}$. We invert the equation above to write the angular velocity tensor in terms of the tetrads as well:

$$\Omega_{\mu\nu} = \eta^{IJ} e_{I\mu} \frac{De_{\nu J}}{d\tau}. \tag{1.32}$$

Having characterized the new degrees of freedom, we may now construct a Lagrangian for the spinning particle by demanding general covariance, local Lorentz and reparametrization invariances. There are four different scalar quantities which are given bellow [47]:

$$a_1 = u_\mu u^\mu, \tag{1.33}$$

$$a_2 = \Omega_{\mu\nu}\Omega^{\mu\nu}, \tag{1.34}$$

$$a_3 = u_\mu \Omega^{\mu\nu}\Omega_{\nu\alpha} u^\alpha, \tag{1.35}$$

$$a_4 = \frac{1}{16}\Omega_{\mu\nu}\Omega^{*\mu\nu}\Omega_{\alpha\beta}\Omega^{*\alpha\beta}, \tag{1.36}$$

where

$$\Omega^{*\mu\nu} \equiv \frac{1}{2}\epsilon_{\mu\nu\alpha\beta}\Omega^{\alpha\beta}. \tag{1.37}$$

From (1.33 - 1.36), we write

$$\delta L\left(a_1, a_2, a_3, a_4\right) = \frac{\partial L}{\partial a_1}\delta a_1 + ... + \frac{\partial L}{\partial a_4}\delta a_4. \tag{1.38}$$

On the other side, we also have [45]

$$\delta L = -p^\mu \delta u_\mu - \frac{1}{2} S^{\mu\nu}\delta\Omega_{\mu\nu}, \tag{1.39}$$

and by comparing with (1.38), considering the quantities (1.33-1.36), we find

$$p^\alpha = -2u^\alpha \frac{\partial L}{\partial a_1} - 2\Omega^{\alpha\nu}\Omega_{\nu\rho}u^\rho \frac{\partial L}{\partial a_3}, \tag{1.40}$$

$$S^{\mu\nu} = -4\Omega^{\mu\nu}\frac{\partial L}{\partial a_2} - 2\left(u^\mu \Omega^{\nu\lambda}u_\lambda - u^\nu \Omega^{\mu\lambda}u_\lambda\right)\frac{\partial L}{\partial a_3} - 8\Omega^{\nu\beta}\Omega_{\beta\alpha}\Omega^{\alpha\mu}\frac{\partial L}{\partial a_4}. \tag{1.41}$$

From (1.32), we have

$$\delta\Omega_{\mu\nu} = \delta e^J_\mu \frac{De_{\nu J}}{d\tau} + e^J_\mu \frac{D\delta e_{\nu J}}{d\tau}, \tag{1.42}$$

while the second term of (1.39) inside the action gives

$$-\frac{1}{2}\int d\tau S^{\mu\nu}\left(\delta e^J_\mu \frac{De_{\nu J}}{d\tau} + e^J_\mu \frac{D\delta e_{\nu J}}{d\tau}\right) = -\frac{1}{2}\int d\tau \left(S^{\mu\nu}\frac{De_{\nu J}}{d\tau}\delta e^J_\mu - S^{\mu\nu}\frac{De^J_\mu}{d\tau}\delta e_{\nu J} - e^J_\mu \frac{DS^{\mu\nu}}{d\tau}\delta e_{\nu J}\right), \tag{1.43}$$

which means

$$\frac{DS^{\mu\nu}}{d\tau} = S^{\mu\nu}\Omega_{\lambda}{}^{\nu} - \Omega^{\mu}{}_{\lambda}S^{\lambda\nu} = p^{\mu}u^{\nu} - u^{\mu}p^{\nu}. \tag{1.44}$$

The last equality follows from (1.40) and (1.41) using a set of identities present in the appendix A of the reference [47]. Since $S^{IJ} = S^{\mu\nu}e^{I}_{\mu}e^{J}_{\nu}$, it is straightfoward to see that the equation above implies

$$\frac{DS^{IJ}}{d\tau} = 0, \tag{1.45}$$

meaning that the spin projected on the tetrads is constant. Notice that the scalar $S^2 \equiv \frac{1}{2}S^{\mu\nu}S_{\mu\nu}$ is conserved.

To obtain the equation of motion for momentum, we have to write (1.39) in the terms of δx. To accomplish this, it is convenient to go to a locally flat coordinate system where the connection terms are zero. After computing the equations of motion we can promote the derivatives to covariant ones and find

$$\frac{Dp_{\mu}}{d\tau} = -\frac{1}{2}R_{\mu\lambda\alpha\beta}S^{\alpha\beta}u^{\lambda}. \tag{1.46}$$

Therefore, we have shown that the Mathisson-Papapetrou-Dixon equations can be derived from the Lagrangian $L\left(u^{\mu}, \Omega^{\mu\nu}\right)$:

$$L = -p^{\mu}u_{\mu} - \frac{1}{2}S^{\mu\nu}\Omega_{\mu\nu}, \tag{1.47}$$

which follows from reparametrization and diffeomorphism invariance. It is convenient to rewrite the action in the locally-flat frame introducing

$$\Omega_L^{ab} = \eta_{IJ}\Lambda^{I}_{a}\frac{D\Lambda^{J}_{b}}{d\tau} \tag{1.48}$$

and

$$e^{a}_{\mu}e^{b}_{\nu}\Omega^{\mu\nu} = \Omega_L^{ab} + u^{\mu}\omega_{\mu}^{ab}, \tag{1.49}$$

where $\omega_{\mu}^{ab} = e^{b}_{\nu}\nabla_{\mu}e^{a\nu}$ are the Ricci rotation coefficients. Finally, we write an action where the kinetic terms and the spin-gravity coupling can be easily identified:

$$S = \int d\tau \left[-p^{\mu}u_{\mu} - \frac{1}{2}S_{ab}\Omega_L^{ab} - \frac{1}{2}\omega_{\mu}^{ab}S_{ab}u^{\mu}\right]. \tag{1.50}$$

1.2.2. Routhian for spinning bodies

Whenever we treat spinning bodies, we choose to work with a Routhian instead of a Lagrangian for convenience, since in this particular physical setting the latter has no explicit dependence on the angular variables. Hence, we treat position and velocity as generalized coordinates, while angular velocity and spin are taken as a pair of canonical variables in view of the Legendre transformation

$$\mathcal{R} = \frac{1}{2} S_{ab} \Omega_L^{ab} + L, \tag{1.51}$$

yielding

$$\mathcal{R} = -p^\mu u_\mu - \frac{1}{2} \omega_\mu^{ab} S_{ab} u^\mu. \tag{1.52}$$

The equations of motion for spin is obtained through

$$\frac{dS^{ab}}{d\tau} = \left\{ S^{ab}, \mathcal{R} \right\}, \tag{1.53}$$

considering the spin algebra

$$\left\{ S^{ab}, S^{cd} \right\} = \eta^{ac} S^{bd} + \eta^{bd} S^{ac} - \eta^{ad} S^{bc} - \eta^{bc} S^{ad}. \tag{1.54}$$

Mathisson-Papapetrou-Dixon equations also follows from (1.53).

1.2.3. Spin supplementary conditions

The antisymmetric spin tensor $S^{\mu\nu}$ has six independent degrees of freedom, but only three are necessary to describe rotations. In order to get rid of these three extra degrees of freedom and fix the centre of mass, we have to impose some constraints called spin supplementary conditions (SSC) of which the most used are

$$V^\mu_{cov} = S^{\mu\nu} p_\nu, \tag{1.55}$$

$$V^\mu_{NW} = S^{\mu\nu} p_\nu - m S^{\mu 0}. \tag{1.56}$$

The first one, which is manifestly covariant, is called covariant SSC. This condition has the advantage that the relation between p_μ and u_μ is manifestly covariant. The covariant SSC implies

$$S^{i0} = S^{ij} \mathbf{v}^j. \tag{1.57}$$

The other choice (1.56) is called Newton-Wigner and, even though it is not manifestly covariant, it preserves the canonical algebra of Poisson Brackets up to effects quadratic in the spin, while the covariant SSC demands the construction of the Dirac brackets. Choosing the Newton-Wigner condition means

$$S^{i0} = \frac{1}{2} S^{ij} \mathbf{v}^j. \tag{1.58}$$

One can calculate the equations of motion in either SSC and translate between the two using a transformation of the centre of mass [85]:

$$\mathbf{r}_{cov} \to \mathbf{r}_{NW} - \frac{\mathbf{v} \times \xi}{2m}, \tag{1.59}$$

which will be discussed in the next chapter.

2. NRGR I: conservative sector

EFTs have been an indispensable tool of modern physics to study systems with multiple separate scales in a systematic fashion. The general idea of an EFT is to provide the simplest framework that captures the essential physics at each scale. To describe a physical system using an EFT, we need to determine the relevant degrees of freedom, the symmetries and the expansion parameters. In every EFT, *power counting* are rules for determining which operators are needed at a given order in the expansion parameter. In general, the coefficients that appear in the effective action are determined through a *matching* procedure by comparison with the full theory.

We may say that the key principle in this framework is that it is not necessary to know the dynamics out of the scale of interest in much detail. For instance, to describe the hydrogen atom in low energy scale we do not need to include the bottom quark. However, short distance degrees of freedom affect the dynamics in terms of coefficients in the local term in the action. This subtlety is related to the fact that when high energy degrees of freedom are integrated out in order to write an effective action that describes low energy physics, the "information" about these high energy particles becomes encoded in the coefficients of the effective theory. Even though EFTs were mostly developed in Particle Physics, they are still useful in classical settings, as we will see through this and the next chapters.

There are basically two approaches concerning EFTs. One of them is called "top-down", in which we know the high energy theory and want to build up or investigate the low energy theory. In this case, the heavier particles are integrated out and then we perform the matching onto a low energy theory, finding what are the new operators and the new low energy constants. The high and the low energy theories agree in the infrared limit, but differ in the ultraviolet regime, as expected. We can cite some EFTs that behave in this way: heavy quark EFT (HQEFT), non-relativistic QED (NRQED), non-relativistic QCD (NRQCD) and the framework used in this thesis, NRGR. The other way to proceed when we work on EFT is named "bottom-up" and what characterizes this approach is: the high energy theory is unknown. In this case, we construct the

Lagragian by writing down the most general possible operators (interactions) consistent with all symmetries and determine the coupling constants by experiment. This is the case, for instance, of the Standard Model.

Although we do not work on renormalization or explore its aspects in this thesis, it is important to mention its place in EFTs. In the traditional sense, the theory is renormalizable if at any order of perturbation the UV divergences from loop integrals can be absorbed into a finite number of parameters. In the EFT sense, a renormalizable theory must be renormalizable order by order in its expansion parameters. In order words, EFT allows for an infinite number of parameters, but only a finite number of them at any given order in the expansion, what makes possible to use renormalization group methods to understand in a simple way logarithms of scale ratios that may arise in the perturbative series for a given observable. Another substantial advantage which an EFT framework offers is having a manifest power counting in the expansion parameter of the theory[1].

Motivation

The experimental program in gravitational wave detection triggered the development of an EFT framework - NRGR - which is capable of describing the two body physics with high accuracy [38,88]. For binary systems that can be detected by LIGO , which have binary constituents with masses in the range between one or dozens of solar masses, the inspiral phase spans roughly $0.1 < v < 0.4$, corresponding to orbital separations from about 100 to a few times $m/m_{\odot}$km. A ground-based detector is sensitive to the gravitational wave phase $\Delta\phi(t)$ during the evolution of the binary system. Consider that each body of a binary system, in circular orbit, has an agular velocity $\omega(t)$. Then, through the virial theorem we have $\omega(t) = \frac{v^3}{m}$, what allows us to write the gravitational wave phase seen by the detector as

$$\Delta\phi(t) = 2\int dt\omega(t) = \frac{2}{m}\int dv v^3 \frac{1}{P}\frac{dE}{dv}, \tag{2.1}$$

since the conservation of energy assures $\frac{dE}{dt} = -P$. The expression above tells us that once the gravitational wave signal is detected, we may have information about the power loss of the system where the wave came from, for instance. The observable $\Delta\phi(t)$ has to be computed to extremely high order in the velocity expansion in order to compare the predictions of General Relativity and to probe the internal structure of those strongly bounded systems with the experimental

[1] This means that it is possible to calculate the order in the expansion parameter which a given term in the perturbation series first contributes to a given physical observable by using simple scaling arguments.

data. It is important, moreover, to have a computational approach that separates the effects that depend upon short distance physics from those that come from gravitational dynamics only. Not only does NRGR provide this, but it also deals naturally with the problems that arise in PN techniques regarding singularities by the means of a renormalization program.

Scales

To construct an EFT which describes the physics of binary systems, the first question we need to ask ourselves is: do those systems have a clear separation of scales? The answer is surely positive. Consider a binary system composed of compact bodies - it can be neutron stars or black holes - in a bound state moving non-relativistically. We can state the difference between the scales of this system in the following way:

- *Internal zone*: It is the scale corresponding to the Schwarzschild radius of the compact bodies r_s, where finite size effects take place.
- *Near zone*: Also called potential zone, it corresponds to the orbital scale represented by the separation distance r between the two compact bodies.
- *Far zone*: It is the scale corresponding to the typical wavelenght of the emitted gravitational waves $\lambda_{rad} \sim \frac{r}{v}$.

Because we consider post-Newtonian sources (slowly moving bodies), the hierarchy of scales can be represented by

$$r_s \ll r \ll \lambda_{rad}. \tag{2.2}$$

The degrees of freedom at the scale r_s are responsible for finite effects, such as tidal forces, but in general they are neglectable in comparison with purely gravitational wave damping effects [57]. At the orbital scale, the gravitational field presents itself in the potential modes, which are off-shell fields that mediate the gravitational interaction between the two compact bodies and are probes for their internal structure. While the potential modes govern the conservative dynamics, the radiation modes are responsable for dissipative effects. At the scale of radiation, the binary can be treated as a point-like source endowed with a series of multipole moments.

In what follows, we introduce the NRGR framework to describe the dynamics of compact binary systems. The first effective description used to describe the physics at distances between the finite size scale and the orbital scale is constructed by replacing the compact objects with point particle worldlines, where their finite size effects can be included with higher order terms in the point particle action. The second stage of the EFT concerns scales between the orbital and the radiation scales, where the potential modes which are responsible for physical effects at the orbital scale are integrated out from the theory. The classical limit of integrating out a field corresponds to performing the saddle point approximation, which effectively means solving for the field value and plugging it back into the action. At the end, the resulting theory describes the conservative sector, as will be presented in this chapter, as well as the radiation sector of the physics of a binary system in the inspiral phase, which will be discussed in the next chapter.

2.1. Non-spinning bodies

Here we introduce the effective action formalism for non-rotating bodies, we include spin in subsequent sections.

2.1.1. Action for binary systems

The simplest action for relativistic point particles coupled to gravity, constituting a binary system, is given by a piece describing gravity - the Einstien Hilbert action - and another part accounting for the bodies that generate the gravitational field:

$$S = S_{EH} + S_{pp}. \tag{2.3}$$

We are interested in having an EFT which provides a way to calculate the power emitted in gravitational radiation from the binary system, since this quantity can be indirectly measured by detectors such as LIGO. This is accomplished, in principle, by expanding the action using the weak field approximation,

$$g_{\mu\nu} = \eta_{\mu\nu} + \frac{h_{\mu\nu}}{m_{Pl}},, \tag{2.4}$$

where we included the Planck mass in this expansion just for convenience. Integrating out the graviton field to obtain an effective action depending only on the particles coordinates x_a with $a = 1, 2$ gives

$$e^{iS_{eff}[x_a]} = \int \mathcal{D}h_{\mu\nu} e^{iS_{EH}+iS_{pp}}. \tag{2.5}$$

Firstly, let us consider the Einstein-Hilbert action

$$S_{EH} = -2m_{Pl}^2 \int d^4x \sqrt{g} R\left(x\right). \tag{2.6}$$

In the weak field limit,

$$g^{\mu\nu} \simeq \eta^{\mu\nu} - \frac{h^{\mu\nu}}{m_{Pl}}, \tag{2.7}$$

$$\sqrt{g} \simeq 1 + \frac{h}{2m_{Pl}}, \tag{2.8}$$

the connection becomes

$$\begin{aligned}\Gamma^{\lambda}_{\mu\nu} &= \frac{1}{2}\left(\eta^{\lambda\beta} - \frac{h^{\lambda\beta}}{m_{Pl}}\right)\frac{1}{m_{Pl}}\left(\partial_\nu h_{\beta\mu} + \partial_\mu h_{\beta\nu} - \partial_\beta h_{\mu\nu}\right) \\ &\simeq \frac{1}{2m_{Pl}}\left(\partial_\nu h^{\lambda}{}_{\mu} + \partial_\mu h^{\lambda}{}_{\nu} - \partial^{\lambda} h_{\mu\nu}\right),\end{aligned} \tag{2.9}$$

what implies

$$R_{\mu\nu} = -\frac{1}{2m_{Pl}}\left(\partial^2 h_{\mu\nu} + \partial_\mu\partial_\nu h - \partial_\mu\partial_\alpha h_{\nu}{}^{\alpha} - \partial_\nu\partial_\alpha h_{\mu}{}^{\alpha}\right) + \mathcal{O}\left(\Gamma^2\right). \tag{2.10}$$

With these considerations, the Einstein-Hilbert action is re-written as

$$\begin{aligned}S_{EH} = \int d^4x &\left(\eta^{\mu\nu} + \frac{h\eta^{\mu\nu}}{2m_{Pl}} - \frac{h^{\mu\nu}}{m_{Pl}} - \frac{hh^{\mu\nu}}{2m_{Pl}^2}\right) \\ &\left[m_{Pl}\left(\partial^2 h_{\mu\nu} + \partial_\mu\partial_\nu h - \partial_\mu\partial_\alpha h_{\nu}{}^{\alpha} - \partial_\nu\partial_\alpha h_{\mu}{}^{\alpha}\right) - 2m_{Pl}^2\mathcal{O}\left(\Gamma^2\right)\right].\end{aligned} \tag{2.11}$$

The terms $\eta\partial^2 h$ (schematically) will lead to zero integration in view of the boundary conditions. Taking into account that

$$\begin{aligned}-2m_{Pl}^2\mathcal{O}\left(\Gamma^2\right) &= -2m_{Pl}^2\int \eta^{\mu\nu}\left[\Gamma^{\alpha}_{\mu\nu}\Gamma^{\sigma}_{\sigma\alpha} - \Gamma^{\alpha}_{\mu\sigma}\Gamma^{\sigma}_{\nu\alpha}\right]d^4x \\ &= \frac{1}{2}\int [2h\partial^\mu\partial^\nu h_{\mu\nu} - h\partial^2 h - 2h^{\mu\nu}\partial_\nu\partial_\lambda h^{\lambda}{}_{\mu} + h^{\mu\nu}\partial^2 h_{\nu\mu}]d^4x,\end{aligned} \tag{2.12}$$

we have

$$S_{EH} = \int\left[\frac{1}{2}h\partial^2 h - \frac{1}{2}h^{\mu\nu}\partial^2 h_{\mu\nu} - h\partial_\mu\partial_\nu h^{\mu\nu} + h^{\mu\nu}\partial_\mu\partial_\alpha h_{\nu}{}^{\alpha} + \mathcal{O}\left(h^3\right) + ...\right]d^4x. \tag{2.13}$$

Considering

$$\Gamma^{\lambda}{}_{\mu}{}^{\mu} = \Gamma^{\lambda} = \frac{1}{m_{Pl}}\left(\partial^\mu h^{\lambda}{}_{\mu} - \frac{1}{2}\partial^{\lambda} h\right), \tag{2.14}$$

we can add the gauge fixing term

$$S_{GF} = m_{Pl}^2\int \sqrt{g}\Gamma_\mu\Gamma^\mu d^4x = -\int\left(h_{\mu}{}^{\alpha}\partial_\alpha\partial_\beta h^{\mu\beta} - h_{\mu}{}^{\alpha}\partial_\alpha\partial^\mu h + \frac{1}{4}h\partial^2 h\right)d^4x, \tag{2.15}$$

to end up with the Einstein-Hilbert action with the kinetic term written in a much simpler form:

$$S_{EH} + S_{GF} = \int\frac{1}{2}\left[\frac{1}{2}h\partial^2 h - h^{\mu\nu}\partial^2 h_{\mu\nu} + ...\right]d^4x, \tag{2.16}$$

where the ellipsis represents terms which depend on higher powers of the gravitational field.

The motion of the two-body system, neglecting finite size effects, is determined by the point-particle action

$$S_{pp} = -\sum_a m_a \int d\tau_a + ... \tag{2.17}$$

where $d\tau_a = \sqrt{g_{\mu\nu} dx_a^\mu dx_a^\nu}$ is the proper time along the a-th particle worldline. For now we ignore additional degrees of freedom - such as spin, which will be introduced later in this chapter - as well as the finite size corrections for the point particle description included in the elipsis.

In order to have interaction vertices with homogenous velocity scaling, we expand the point-particle action in powers of the particles three-velocities. To accomplish this, we write the proper time using the post-Newtonian expansion:

$$\begin{aligned} d\tau &= \sqrt{\eta_{\mu\nu} dx^\mu dx^\nu + \frac{h_{\mu\nu}}{m_{Pl}} dx^\mu dx^\nu} \qquad (2.18) \\ &= \left[1 - \mathbf{v}^2 + \frac{1}{m_{pl}} \left(h_{00} + 2h_{0i}\mathbf{v}^i + h_{ij}\mathbf{v}^i\mathbf{v}^j\right)\right]^{\frac{1}{2}} dt \\ &= \left[1 - \frac{\mathbf{v}^2}{2} + \frac{1}{2m_{pl}} \left(h_{00} + 2h_{0i}\mathbf{v}^i + h_{ij}\mathbf{v}^i\mathbf{v}^j\right) - \frac{\mathbf{v}^4}{8} + \frac{\mathbf{v}^2}{4m_{pl}} \left(h_{00} + 2h_{0i}\mathbf{v}^i + h_{ij}\mathbf{v}^i\mathbf{v}^j\right) + ...\right] dt, \end{aligned}$$

where we have used $(1+x)^n = 1 + nx + \frac{n(n-1)}{2!}x^2 + ...$, for $x \ll 1$, which is a good approximation since $|\mathbf{v}| \ll 1$ as well as $|h_{\mu\nu}| \ll 1$. With this, the expanded point-particle action is given by

$$S_{pp} = \int dt \sum_{a=1,2} m_a \left[\frac{\mathbf{v}_a^2}{2} + \frac{1}{2m_{pl}} \left(h_{00} + 2h_{0i}\mathbf{v}_a^i + h_{ij}\mathbf{v}_a^i\mathbf{v}_a^j\right) - \frac{\mathbf{v}_a^4}{8} + \frac{\mathbf{v}_a^2}{4m_{pl}} \left(h_{00} + 2h_{0i}\mathbf{v}_a^i + h_{ij}\mathbf{v}_a^i\mathbf{v}_a^j\right) + ...\right]. \tag{2.19}$$

The constant term $\sum_{a=1,2} -m_a$ was ignored because it has no contribuition for the equations of motion. The fields h_{00}, h_{0i} and h_{ij} are evaluated on the particles worldlines $(x_0, \mathbf{x}_a(x_0))$.

2.1.2. Propagator

Let us now introduce the propagator of the gravitational field. In our worldline description we take the limit $r_s \to 0$, thus in this EFT framework the compact body is described by a localized point-like gravitational source whose stress-energy tensor is

$$T_{pp}^{\mu\nu} = m \int d\sigma \frac{u^\mu u^\nu}{\sqrt{u^2}} \frac{\delta^4 (x - x(\sigma))}{\sqrt{g}} + ..., \tag{2.20}$$

where the ellipsis account finite size effects. To accomodate the non-linearities of the gravitational field $h_{\mu\nu}$ in the Einstein-Hilbert as well as in the point-particle action, it is convenient to introduce the pseudo stress-energy tensor $\mathcal{T}^{\mu\nu}$ which incorporates those non-linearities, so that

$$S_{EH} + S_{GF} + S_{pp} = \int \frac{1}{2}\left[\left(\frac{1}{2}h\partial^2 h - h^{\mu\nu}\partial^2 h_{\mu\nu}\right) - \frac{1}{m_{Pl}}\mathcal{T}^{\mu\nu}h_{\mu\nu}\right] d^4x. \tag{2.21}$$

In order to write the propagator of the gravitational field, we compute the Euler-Lagrange equation

$$\frac{\partial L}{\partial h_{\alpha\beta}(y)} - \partial_\theta^{(y)}\frac{\partial L}{\partial\partial_\theta^{(y)}h_{\alpha\beta}(y)} = 0, \tag{2.22}$$

what leads to

$$\int d^4x\left[\left(\frac{1}{2}\eta^{\alpha\beta}\partial^2 h - \partial^2 h^{\alpha\beta} - \frac{1}{m_{Pl}}T^{\alpha\beta}_{pp}\right) - \partial_\theta^{(y)}\left(\frac{1}{2}h\partial^\theta\eta^{\alpha\beta} - h^{\alpha\beta}\partial^\theta\right)\right]\delta\left(x-y\right) = 0, \tag{2.23}$$

since $\eta_{\mu\nu}$, $h_{\mu\nu}$ and $T_{\mu\nu}$ are symmetric quantities. This implies

$$\left(\eta^{\alpha\beta}\eta^{\mu\nu} - \eta^{\alpha\mu}\eta^{\beta\nu} - \eta^{\alpha\nu}\eta^{\beta\mu}\right)\partial^2 h_{\mu\nu} = \frac{1}{m_{Pl}}\mathcal{T}^{\alpha\beta}, \tag{2.24}$$

and after inverting it turns out to be

$$h_{\mu\nu} = -\frac{i}{m_{Pl}}\int \Delta_{F\alpha\beta\mu\nu}\left(x-y\right)\mathcal{T}^{\alpha\beta}(y)dy, \tag{2.25}$$

where we use these two definitions[2]:

$$\Delta_{F\alpha\beta\mu\nu}\left(x-y\right) \equiv P_{\alpha\beta\mu\nu}\Delta_F\left(x-y\right), \tag{2.26}$$

$$P_{\alpha\beta\mu\nu} \equiv \frac{1}{2}\left(\eta_{\alpha\beta}\eta_{\mu\nu} - \eta_{\alpha\mu}\eta_{\beta\nu} - \eta_{\alpha\nu}\eta_{\beta\mu}\right). \tag{2.27}$$

At this point, we can show how to derive the Newtonian potential from this field theory approach. Regarding the linear order in $\frac{1}{m_{Pl}}$, for a static source (2.25) leads to

$$\begin{aligned}\frac{h_{\alpha\beta}}{m_{Pl}} &= -\frac{i}{2m_{Pl}^2}\int P_{\alpha\beta00}\Delta_F\left(x-x'\right)T^{00}_{pp}\left(x'\right)d^4x' \\ &= -\frac{im}{4m_{Pl}^2}\int\left(2\eta_{\alpha0}\eta_{0\beta} - \eta_{\alpha\beta}\right)\Delta_F\left(t-t',\mathbf{x}\right)dt',\end{aligned} \tag{2.28}$$

but as

$$\Delta_F\left(t-t',\mathbf{x}\right) = \int d^4p\frac{ie^{-ip_0(t-t')}e^{i\mathbf{p}\cdot\mathbf{x}}}{p_0^2 - \mathbf{p}^2 + i\epsilon}, \tag{2.29}$$

we find

$$\begin{aligned}\frac{h_{\alpha\beta}}{m_{Pl}} &= \frac{m}{4m_{Pl}^2}\left(2\eta_{\alpha0}\eta_{0\beta} - \eta_{\alpha\beta}\right)\int d^4p\frac{\delta\left(p_0\right)e^{-ip_0t}e^{i\mathbf{p}\cdot\mathbf{x}}}{p_0^2 - \mathbf{p}^2 + i\epsilon} \\ &= \frac{m}{4m_{Pl}^2}\left(2\eta_{\alpha0}\eta_{0\beta} - \eta_{\alpha\beta}\right)\int d^3\mathbf{p}\frac{e^{i\mathbf{p}\cdot\mathbf{x}}}{-\mathbf{p}^2 + i\epsilon}.\end{aligned} \tag{2.30}$$

[2] The Feynman's prescription is the right choice to compute the total radiated power using the optical theorem.

Using $m_{Pl}^2 = \frac{\hbar c}{32\pi G_N}$ and

$$\int d^3\mathbf{p}\frac{e^{i\mathbf{p}\cdot(\mathbf{x}-\mathbf{y})}}{\mathbf{p}^2} = \frac{1}{4\pi|\mathbf{x}-\mathbf{y}|}, \tag{2.31}$$

it gives us

$$\frac{h_{\alpha\beta}}{m_{Pl}} = \frac{2Gm}{r}\left(2\eta_{\alpha 0}\eta_{0\beta} - \eta_{\alpha\beta}\right), \tag{2.32}$$

which yields the Newtonian potential for the temporal component.

At this point it is necessary to make a comment. Although we have expanded the action using the weak field and the PN approximations, calculated the propagator and showed how to recover, for instance, the Newtonian potential, it is still not enough to our purporses. We need an EFT action which is capable of describing conservative but also dissipative dynamics. To accomplish this, the gravitational field has to be separated into two parts: one describing the gravitational interaction between the two particles and another describing the gravitational waves. In other words, to formulate an EFT that is tailored to the limit $v \ll 1$ we must integrate out all degrees of freedom with wavelenghts shorter than the orbital distance scale r. We discuss this next.

2.1.3. Modes of the field

Although we have performed expansion for small velocity, the propagator (2.26) for the field $h_{\mu\nu}$ still encodes fully relativistic dynamics, since it does not distinguish between potential gravitons (which must be integrated out in order to write an action for NRGR) and radiation gravitons. To make the Feynman rules scale homogeneously with v, it is convenient to decompose the graviton as

$$h_{\mu\nu} = \bar{h}_{\mu\nu} + H_{\mu\nu}, \tag{2.33}$$

where $\bar{h}_{\mu\nu}$ describes the on-shell gravitons, related to the long wavelenght radiation modes, while $H_{\mu\nu}$ represents off-shell gravitons, associated to the potential modes that mediates the gravitational interaction. We can infer the scaling of the derivatives of these fields using physical arguments. Since $\bar{h}_{\mu\nu}$ is a propagating field, the condition on the wave frequency implies

$$\partial_\alpha \bar{h}_{\mu\nu} \sim \frac{v}{r}\bar{h}_{\mu\nu}, \tag{2.34}$$

whereas the potential gravitons do not go on shell and scale as

$$\partial_0 H_{\mu\nu} \sim \frac{v}{r}H_{\mu\nu}, \tag{2.35}$$

$$\partial_i H_{\mu\nu} \sim \frac{1}{r}H_{\mu\nu}. \tag{2.36}$$

Once we regard (2.33), the NRGR action is determined through the functional integral

$$e^{iS_{NRGR}[x_a,\overline{h}]} = \int \mathcal{D}H_{\mu\nu} e^{iS[\overline{h}+H,x_a]+iS_{GF}}. \tag{2.37}$$

Now the gauge fixing action appearing in the expression above must be suitable for the potential modes, in order to preserve the gauge invariance of the NRGR action. As far as the potential modes are concerned, the radiation plays the role of a slowly varying background field, so that the background metric can be written as $\bar{g}_{\mu\nu} = \eta_{\mu\nu} + \overline{h}_{\mu\nu}$ and the gauge fixing term takes the form

$$S_{GF} = m_{Pl}^2 \int d^4x \sqrt{\bar{g}} \Gamma_\mu \Gamma^\mu, \tag{2.38}$$

with $\Gamma_\mu = D_\alpha H^\alpha_{\ \mu} - \frac{1}{2} D_\mu H^\alpha_{\ \alpha}$, being D_μ the covariant derivative associated to the background metric $\bar{g}_{\mu\nu}$. In view of the different behaviour of the potential fields regarding its scaling, it is convenient to work with its Fourier transform

$$H_{\mu\nu}(x) = \int \frac{d^3\mathbf{k}}{(2\pi)^3} e^{i\mathbf{k}\cdot\mathbf{x}} H_{\mathbf{k}\mu\nu}\left(x^0\right). \tag{2.39}$$

Taking all this into account, the action (2.21) is now written as

$$\begin{aligned} S\left[\overline{h}+H, x_a\right] + S_{GF} \;\; = \;\; & \int d^4x \left[\frac{1}{2}\left(\frac{1}{2}\overline{h}\partial^2\overline{h} - \overline{h}^{\mu\nu}\partial^2\overline{h}_{\mu\nu} + \frac{1}{2}H\partial^2 H - H^{\mu\nu}\partial^2 H_{\mu\nu}\right)\right. \\ & \left. -\frac{1}{2m_{Pl}}\mathcal{T}_{\mu\nu}\overline{h}^{\mu\nu} - \frac{1}{2m_{Pl}}\mathcal{T}_{\mu\nu}H^{\mu\nu}\right]. \end{aligned} \tag{2.40}$$

The crossed terms are zero because of the non-overlapping momenta. We can obtain the propagator for the potential gravitons regarding the kinetic and source terms related to them. Since

$$\begin{aligned} H\partial^2 H &= H\left(\partial_0^2 - \partial_i^2\right) H \\ &= \int \frac{d^3\mathbf{k}}{(2\pi)^3}\left[\mathbf{k}^2 H(\mathbf{k},x_0) H(-\mathbf{k},x_0) - \partial_0 H(\mathbf{k},x_0)\partial_0 H(-\mathbf{k},x_0)\right], \end{aligned} \tag{2.41}$$

we have

$$\begin{aligned} \frac{1}{2}H\partial^2 H - H^{\mu\nu}\partial^2 H_{\mu\nu} &= \int \frac{d^3\mathbf{k}}{(2\pi)^3}\left[\mathbf{k}^2\left[\frac{1}{2}H(\mathbf{k},x_0)H(-\mathbf{k},x_0) - H_{\mu\nu}(\mathbf{k},x_0)H^{\mu\nu}(-\mathbf{k},x_0)\right] - \right. \\ & \left. -\frac{1}{2}\partial_0 H(\mathbf{k},x_0)\partial_0 H(-\mathbf{k},x_0) + \partial_0 H_{\mu\nu}(\mathbf{k},x_0)\partial_0 H^{\mu\nu}(-\mathbf{k},x_0)\right] \\ &= \int \frac{d^3\mathbf{k}}{(2\pi)^3}\frac{1}{2}T^{\mu\nu\alpha\beta}\left[\mathbf{k}^2 H_{\mu\nu}(\mathbf{k},x_0) H_{\alpha\beta}(-\mathbf{k},x_0)\right. \\ & \left. -\partial_0 H_{\mu\nu}(\mathbf{k},x_0)\partial_0 H_{\alpha\beta}(-\mathbf{k},x_0)\right], \end{aligned} \tag{2.42}$$

where we used the definition

$$T^{\mu\nu\alpha\beta} \equiv \left(\eta^{\mu\alpha}\eta^{\nu\beta} + \eta^{\mu\beta}\eta^{\nu\alpha} - \eta^{\mu\nu}\eta^{\alpha\beta}\right). \tag{2.43}$$

The terms containing time derivatives are suppresed with respect to the first one by a power of v^2, what allows us to treat them perturbatively as operator insertions in correlations functions. Hence, taking the Lagrangian

$$L = -\frac{1}{2}\int \frac{d^3\mathbf{k}}{(2\pi)^3}\left[\mathbf{k}^2\frac{1}{2}T^{\mu\nu\alpha\beta}H_{\mu\nu}(\mathbf{k},x_0) + I^{\mu\nu\alpha\beta}\frac{\mathcal{T}_{\mu\nu}(\mathbf{k},x_0)}{m_{Pl}}\right]H_{\alpha\beta}(-\mathbf{k},x_0), \tag{2.44}$$

where

$$I^{\mu\nu\alpha\beta} \equiv \frac{1}{2}\left(\eta^{\mu\alpha}\eta^{\nu\beta} + \eta^{\mu\beta}\eta^{\nu\alpha}\right), \tag{2.45}$$

we can obtain the propagator for the potential fields, similarly to the previous section:

$$\langle H_{\mu\nu}(\mathbf{k},x_0)\, H_{\alpha\beta}(\mathbf{k}',x_0')\rangle = -i\,(2\pi)^3\, P_{\mu\nu\alpha\beta}\delta(x_0 - x_0')\,\delta^3(\mathbf{k}+\mathbf{k}')\,\frac{1}{\mathbf{k}^2}. \tag{2.46}$$

2.1.4. Power counting

Now, we are ready to see how Feynman diagrams scale in this EFT framework. Concerning that $\delta^3(\mathbf{k}+\mathbf{k}') \sim r^3$ and $\delta(x_0 - x_0') \sim \frac{v}{r}$, we may infer from (2.46)

$$H_{\mu\nu}(\mathbf{k},x_0) \sim v^{\frac{1}{2}}r^2. \tag{2.47}$$

Analogously, through (2.26) and (2.29), we have

$$\bar{h}_{\mu\nu}(x) \sim \frac{v}{r}. \tag{2.48}$$

To obtain the scale for the Planck mass, remind the equivalence in order between the kinetic energy and the Newtonian potential:

$$mv^2 \sim \frac{Gmm}{r} \quad \Rightarrow \quad G \sim \frac{v^2 r}{m}, \tag{2.49}$$

what implies $m_{Pl} \sim \sqrt{\frac{v^2 r}{m}}$ and for practical purposes it is convenient to write

$$\frac{m}{m_{Pl}} \sim \sqrt{Lv}, \tag{2.50}$$

where L is the angular momentum. Now, with this information, we can determine the order in the velocity at which any diagram in the calculation of the functional integral (2.37) as well as

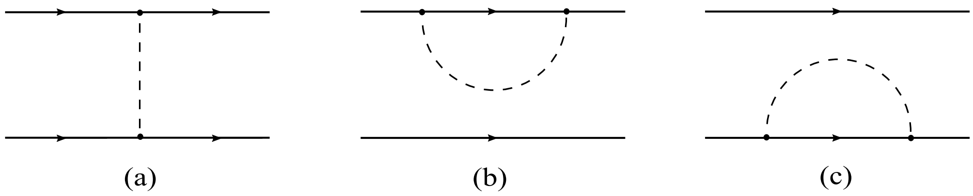


Figure 2.1.

in Feynman diagrams which comes from the NRGR action itself. As a simple illustration of the power counting rules, take the part of action corresponding to the kinetic term:

$$\int dx^0 mv^2 \sim \frac{r}{v} mv^2 \sim L, \tag{2.51}$$

and in the same way behaves the Newtonian potential part that comes from the exchange of a single potential graviton H_{00} between the two particles as shown in Fig.2.1:

$$i\frac{m_1 m_2}{8m_{Pl}^2}\int dt_1 dt_2 \delta\left(t_1 - t_2\right)\int \frac{d^3\mathbf{k}}{\left(2\pi\right)^3}\frac{e^{-i\mathbf{k}\cdot(\mathbf{x}_1-\mathbf{x}_2)}}{\mathbf{k}^2} = \frac{i}{32\pi m_{Pl}^2}\int dt \frac{m_1 m_2}{r} \sim L. \tag{2.52}$$

Without calculating the diagram[3], we could obtain its scaling by noticing that, schematically, it is determined by $\left[(dt)\left(d^3\mathbf{k}\right)\left(m/m_{Pl}\right)H\right]^2 \sim L$. The angular momentum of a binary system of compact bodies, i.e. neutron stars or black holes, is a huge quantity. All other contributions to the effective action, in general, are down by powers of v relative to the kinetic and Newtonian potential terms. We must treat the operators that scale as Lv^0 non-perturbatively while operators which scale as Lv^n must be treated perturbatively. It can also be shown that diagrams containing graviton loops are supressed by powers of $\frac{1}{L}$ relatively to tree diagrams, what enforces the fact that effective theory is perfectly suitable to describe classical physics.

2.1.5. Feynman rules

In this formalism, all observables are derivable from either the effective action for the point-particle $S_{eff}\left[x_a\right]$ defined in (2.5) or from the NRGR action for the radiation graviton mode $S_{NRGR}\left[x_a, \overline{h}\right]$ in (2.37). We can summarize some immediate rules in the following way:

- $iS_{eff}\left[x_a\right]$ is the sum of Feynman diagrams that remain connected when all particle world-lines are removed.

[3] Only the first diagram in Fig.2.1 contributes to the potential, since the other two vanish in dimensional regularization

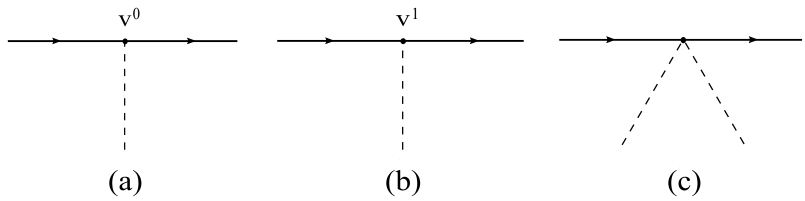


Figure 2.2.

- There is no propagators for the "field" $x_a^\mu(\tau)$ because they are treated as background whenever gravitons are considered.
- A term with n power of the radiation field $\bar{h}_{\mu\nu}$ in $iS_{NRGR}\left[x_a, \bar{h}\right]$ is equal to the *sum* of Feynman diagrams with n external radiation graviton lines.
- A internal potential graviton line with label momentum $\mathbf{k}$ and connecting points at time x_0 and x_0' corresponds to an insertion of (2.46). We can also choose to work with the propagator in coordinate space as well:

$$\langle H_{\mu\nu}(x) H_{\alpha\beta}(x')\rangle = \int \frac{d^3\mathbf{k}}{(2\pi)^3} e^{i\mathbf{k}\cdot\mathbf{x}} \int \frac{d^3\mathbf{k}'}{(2\pi)^3} e^{i\mathbf{k}'\cdot\mathbf{x}} \langle H_{\mu\nu}(\mathbf{k}, x_0) H_{\alpha\beta}(\mathbf{k}', x_0')\rangle . \qquad (2.53)$$

- An *internal* radiation graviton line connecting two points x and y denotes an insertion of (2.26).
- For each external radiation line starting at a vertex located at a spacetime point x, one includes a factor of the radiation field $\bar{h}_{\mu\nu}(x)$ and an integral $\int d^4x$.

Because of the expansion in the weak field, the action has an infinite number of interaction vertices. In what follows, we write the rules regarding the potential gravitons[4].

Let us first discuss the Feynman rules arising from the insertions of the point-particle interactions. Consider the term below belonging to the point-particle action:

$$S_{pp(H_{00})} = -\frac{m}{2m_{Pl}} \int dt H_{00}(x) . \qquad (2.54)$$

Re-writing $H_{00}(x) = \eta_{\mu 0}\eta_{\nu 0} H^{\mu\nu}(x)$, we see that the Feynman rule for this term, represented in Fig.2.2(a) is simply

$$-\frac{im}{2m_{Pl}} \int dt \eta_{\mu 0}\eta_{\nu 0}. \qquad (2.55)$$

[4]The rules presented here will be used in appendix A to perform the calculation of Einstein-Infeld-Hoffman Lagragian. It is an illustration on how potential gravitons can be integrated out to get the first PN correction to the Newtonian theory.

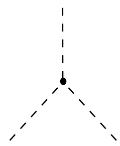


Figure 2.3.

Similarly, the term represented inFig.2.2(b)

$$S_{pp(H_{0i}v_a^i)} = -\frac{m}{m_{Pl}} \int dt H_{0i}(x)\,\mathbf{v}^i(x_0) \tag{2.56}$$

leads to the rule

$$-\frac{im}{m_{Pl}} \int dt \mathbf{v}^i(x_0)\,\delta_{i\mu}\eta_{\nu 0}. \tag{2.57}$$

There are also non-linear terms such as

$$S_{pp(H_{00}^2)} = \frac{m}{8m_{Pl}} \int dt H_{00}(x)\, H_{00}(x)\,. \tag{2.58}$$

The Feynman rule in this case, as depicted by Fig.2.2(c) has two identical terms which arise from the two independent ways of contracting the two fields in the interaction with the external graviton lines:

$$\frac{im}{4m_{Pl}} \int dt \eta_{\mu 0}\eta_{\nu 0}\eta_{0\alpha}\eta_{0\beta}. \tag{2.59}$$

These are examples of vertices arising from the interactions involving the point-particle sources, but we need also to consider the n-graviton vertices which come from the expansion of the EH action in terms of the weak field. Consider now the cubic terms in the graviton field which give rie to the three-graviton vertex (Fig.2.3). They come from many terms in the action. For instance, consider the second order terms coming from the metric and the determinant multiplying first order terms coming from the curvature.

Concerning the metric, we have to notice that if we consider $g_{\mu\nu} = \eta_{\mu\nu} + \frac{h_{\mu\nu}}{m_{Pl}}$ and impose the condition

$$g_{\mu\nu} g^{\nu\alpha} = \delta_\mu^{\ \alpha}, \tag{2.60}$$

we can see that it implies

$$g^{\mu\nu} \simeq \eta^{\mu\nu} - \frac{h^{\mu\nu}}{m_{Pl}} + \frac{h^{\mu\alpha} h_\alpha^{\ \nu}}{m_{Pl}^2} - \frac{h^\mu_{\ \alpha} h^{\alpha\beta} h_\beta^{\ \nu}}{m_{Pl}^3} + ... \tag{2.61}$$

As we know,

$$\sqrt{g} \simeq 1 + \frac{h}{2m_{Pl}} + \frac{h^2}{8m_{Pl}^2} - \frac{h^{\mu\nu} h_{\mu\nu}}{4m_{Pl}^2} + ..., \tag{2.62}$$

and putting together, it ends up as:

$$\sqrt{g}g^{\mu\nu} = \eta^{\mu\nu} + \frac{\eta^{\mu\nu}h}{2m_{Pl}} + \frac{\eta^{\mu\nu}h^2}{8m_{Pl}^2} - \frac{\eta^{\mu\nu}h^{\alpha\beta}h_{\alpha\beta}}{4m_{Pl}^2} + ... - \frac{h^{\mu\nu}}{m_{Pl}} - \frac{h^{\mu\nu}h}{2m_{Plpl}^2} - \frac{h^{\mu\nu}}{4m_{Pl}^3}\left(\frac{h^2}{2} - h^{\alpha\beta}h_{\alpha\beta}\right) + ... \tag{2.63}$$

Schematically, we have

$$(\sqrt{g}g^{\mu\nu})_{(h^0)} = \eta^{\mu\nu}, \tag{2.64}$$

$$(\sqrt{g}g^{\mu\nu})_{(h^1)} = \frac{1}{m_{Pl}}\left(\frac{\eta^{\mu\nu}h}{2} - h^{\mu\nu}\right), \tag{2.65}$$

$$(\sqrt{g}g^{\mu\nu})_{(h^2)} = \frac{1}{2m_{Pl}^2}\left(\frac{\eta^{\mu\nu}h^2}{4} - \frac{\eta^{\mu\nu}h^{\alpha\beta}h_{\alpha\beta}}{2} - h^{\mu\nu}h\right). \tag{2.66}$$

Similarly, we can also organize the terms in the connection by orders of the weak field:

$$\Gamma^{\lambda}_{\mu\nu}\left(h^1\right) = \frac{1}{2m_{Pl}}\eta^{\lambda\beta}\left(\partial_\nu h_{\beta\mu} + \partial_\mu h_{\beta\nu} - \partial_\beta h_{\mu\nu}\right), \tag{2.67}$$

$$\Gamma^{\lambda}_{\mu\nu}\left(h^2\right) = -\frac{h^{\lambda\beta}}{2m_{Pl}^2}\left(\partial_\nu h_{\beta\mu} + \partial_\mu h_{\beta\nu} - \partial_\beta h_{\mu\nu}\right), \tag{2.68}$$

$$\Gamma^{\lambda}_{\mu\nu}\left(h^3\right) = \frac{h^{\lambda\alpha}h_{\alpha}{}^{\beta}}{2m_{Pl}^3}\left(\partial_\nu h_{\beta\mu} + \partial_\mu h_{\beta\nu} - \partial_\beta h_{\mu\nu}\right). \tag{2.69}$$

- Contribution $\left(\sqrt{g}g\right)_{(h^2)} R(h^1)$

Taking the curvature term linear in the graviton field

$$R^{(1)}_{\mu\nu} = -\frac{1}{2m_{Pl}}\left(\partial^2 h_{\mu\nu} + \partial_\mu\partial_\nu h - \partial_\mu\partial_\alpha h_\nu{}^\alpha - \partial_\nu\partial_\alpha h_\mu{}^\alpha\right) \tag{2.70}$$

we can write, for instance,

$$S_{EH[g(h^2)R(h^1)]} = -\int\left(\frac{\eta^{\mu\nu}h^2}{4} - \frac{\eta^{\mu\nu}h^{\alpha\beta}h_{\alpha\beta}}{2} - h^{\mu\nu}h\right)R^{(1)}_{\mu\nu}d^4x. \tag{2.71}$$

- Contribution $\left(\sqrt{g}g\right)_{(h^1)} R(h^2)$

Consider the terms coming from the second order terms in the curvature multiplying first order terms of the metric or determinant. The connection has contributions linear and quadratic in the field, we may write the curvature tensor as

$$R_{\mu\nu}\left(h^2\right) = \partial_\lambda\Gamma^{\lambda}_{\mu\nu}\left(h^2\right) - \partial_\nu\Gamma^{\lambda}_{\mu\lambda}\left(h^2\right) + \Gamma^{\alpha}_{\mu\nu}\left(h^1\right)\Gamma^{\sigma}_{\sigma\alpha}\left(h^1\right) - \Gamma^{\alpha}_{\mu\sigma}\left(h^1\right)\Gamma^{\sigma}_{\nu\alpha}\left(h^1\right), \tag{2.72}$$

yielding

$$S_{EH[g(h^1)R(h^2)]} = -m_{Pl}\int\left(\frac{1}{2}h\eta^{\mu\nu} - h^{\mu\nu}\right)R_{\mu\nu}\left(h^2\right)d^4x. \tag{2.73}$$

- Contribution $\left(\sqrt{g}g\right)_{(h^0)} R(h^3)$

For this contribution we take the terms cubic in the field in the curvature tensor,

$$\begin{aligned} R_{\mu\nu}\left(h^3\right) &= \partial_\lambda \Gamma^\lambda_{\mu\nu}\left(h^3\right) - \partial_\nu \Gamma^\lambda_{\mu\lambda}\left(h^3\right) + \Gamma^\alpha_{\mu\nu}\left(h^2\right)\Gamma^\sigma_{\sigma\alpha}\left(h^1\right) + \Gamma^\alpha_{\mu\nu}\left(h^1\right)\Gamma^\sigma_{\sigma\alpha}\left(h^2\right) \\ & -\Gamma^\alpha_{\mu\sigma}\left(h^2\right)\Gamma^\sigma_{\nu\alpha}\left(h^1\right) - \Gamma^\alpha_{\mu\sigma}\left(h^1\right)\Gamma^\sigma_{\nu\alpha}\left(h^2\right), \end{aligned} \tag{2.74}$$

what leads to

$$S_{EH[g(h^0)R(h^3)]} = -2m^2_{Pl}\int \eta^{\mu\nu} R_{\mu\nu}\left(h^3\right) d^4x. \tag{2.75}$$

Taking into account the specifications above, the total contribution to the three-gravition vertex comes schematically from:

$$S_{EH}\left(h^3\right) = -2m^2_{Pl}\int d^4x \left[(\sqrt{g}g^{\mu\nu})_{(h^2)}(R_{\mu\nu})_{(h^1)} + (\sqrt{g}g^{\mu\nu})_{(h^1)}(R_{\mu\nu})_{(h^2)} + (\sqrt{g}g^{\mu\nu})_{(h^0)}(R_{\mu\nu})_{(h^3)}\right]. \tag{2.76}$$

As an exauting algebra would have to be performed in order to write this rule, it is convenient to use computational tools to make this evaluation. We wrote a code in Mathematica using the Feyncalc package to get

$$\left\langle T\left\{H^{00}_{\mathbf{k}_1}(x_1)\,H^{00}_{\mathbf{k}_2}(x_2)\,H^{00}_{\mathbf{k}_3}(x_3)\right\}\right\rangle = -i\frac{(2\pi)^3}{4m_{Pl}}\delta(t_1-t_2)\,\delta(t_1-t_3)\,\delta\left(\sum_i \mathbf{k}_i\right)\prod_j \frac{i}{\mathbf{k}^2_j}\sum_k \mathbf{k}^2_k. \tag{2.77}$$

We recommend the reader to go through appendix A for an illustration of the EFT usage.

2.2. Spinning bodies

At the end of the previous chapter, we introduced spin in General Relativity by including tetrads to describe the particle's rotation. Now we treat spinning bodies from the EFT perspective, presenting the power counting and Feynmann rules associated to them. The calculation of the leading order spin-orbit potential and the acceleration derived from it are presented. Moreover, we also discuss how to handle the spin supplementary conditions in the computations of the equations of motion.

2.2.1. Scaling

In order to treat spinning bodies via an EFT framework, we need to know precisely how the spin degrees of freedom scale. Let us concern the spin condition:

$$S^{i0} = \kappa S^{ij}\mathbf{v}^j, \tag{2.78}$$

where $\kappa = 1, \frac{1}{2}$ for covariant and Newton-Wigner choice, respectively. The condition above states the temporal components of the spin tensor is a half-PN order higher that the spatial components, given that the latter are expressed by

$$\mathbf{S}^i = \frac{1}{2}\epsilon^{ijk}S^{jk}. \tag{2.79}$$

As we are interested in compact objects, the moment of inertia is proportional to the Schwarzchild radius

$$I_S \sim mr_s^2 \tag{2.80}$$

implying

$$S = I_S\Omega \sim mv_{rot}r_s. \tag{2.81}$$

For maximal rotating bodies $v_{rot} \lesssim 1$, what makes spin scale as

$$S \sim Lv, \tag{2.82}$$

where $L \sim mrv$, $\frac{r_s}{r} \sim v^2$ and v is the translational velocity of the compact bodies. We can see, thus, that the spatial components of the spin are a half-PN order higher than the angular momentum. This subleading character of the spin allows it to be treated perturbatively.

2.2.2. Feynman rules

To obtain the Feynman rules we expand the metric around a Minkowski background, $g_{\mu\nu} = \eta_{\mu\nu} + \frac{h_{\mu\nu}}{m_p}$, what leads to

$$e^a_\mu = \delta^a_\mu + \frac{1}{2}\delta^a_\nu\left(h^\nu{}_\mu - \frac{1}{4}h^\nu{}_\alpha h^\alpha{}_\mu\right) + ... \tag{2.83}$$

Plugging this expasion into (1.52) and collecting only spin dependent terms, we write a Routhian representing the interacting spin-sector of the binary systems of compact bodies,

$$\begin{aligned}\mathcal{R}_{spin} =& \frac{1}{2m_{Pl}}\delta^\alpha_a\delta^\beta_b\partial_\beta h_{\alpha\gamma}u^\gamma S^{ab} + \frac{1}{4m_{Pl}^2}\delta^\beta_a\delta^\gamma_b h^\lambda_\gamma\left(\frac{1}{2}\partial_\mu h_{\beta\lambda} + \partial_\beta h_{\mu\lambda} - \partial_\lambda h_{\mu\beta}\right)u^\mu S^{ab} \\ &+ \frac{1}{2m_{Pl}}R_{\nu\alpha\rho\sigma}S^{\rho\sigma}u^\nu S^{\alpha\beta}u_\beta + ...,\end{aligned} \tag{2.84}$$

from which the Feynman rules can be read. Using the PN expansion present in the previous chapter, regarding the gravitational field split into the radiation and potential modes, we can read for instance the spin insertion up to 2PN order [45]:

$$\mathcal{R}^{NRGR}_{1PN} = \frac{1}{2m_p}\partial_k H_{i0} S^{ik}, \tag{2.85}$$

$$\mathcal{R}^{NRGR}_{1.5PN} = \frac{1}{2m_p}\left(\partial_k H_{ij} S^{ik}\mathbf{u}^j + \partial_k H_{00} S^{0k}\right), \tag{2.86}$$

$$\mathcal{R}^{NRGR}_{2PN} = \frac{1}{2m_p}\left(\partial_k H_{0j} S^{0k}\mathbf{u}^j + \partial_0 H_{i0} S^{i0}\right) + \frac{1}{4m_p^2} S^{ij}\left(H_j{}^{\lambda}\partial_i H_{0\lambda} - H_j{}^{k}\partial_k H_{0i}\right). \tag{2.87}$$

2.2.3. Finite size effects

In this EFT framework, finite size effects are included to correct the divergences that appear due to the point-particle approximation. This is done by writing the most general action respecting the symmetries of the theory - diffeomorphism and reparametrization invariance. For instance, we can add terms in the point-particle action (2.17) such as

$$S_{Q_E} = \frac{1}{2}\int d\tau Q_E^{ij} E_{ij}, \tag{2.88}$$

where E_{ij} is the electric component of the Weyl tensor $E_{\mu\nu} = C_{\mu\alpha\nu\beta}u^\alpha u^\beta$ and Q_E^{ij} is the associated electric quadrupole moment. We can also consider a similar term for the action regarding the magnetic components of the Weyl tensor[5], just by changing $E_{ij} \to B_{ij}$. The quadrupoles $Q_{ij}^{E(B)}$, which depend on time, can be split into two different components. There is a background piece which accounts for the short-distance contributions in the absence of external perturbations, and there is a response to long-wavelenght probes:

$$Q_{ij}^{E(B)} = \left\langle Q_{ij}^{E(B)}\right\rangle_S + \left(Q_{ij}^{E(B)}\right)_R. \tag{2.89}$$

The first term is the background piece and $\langle ...\rangle$ represents the expectation value computed in the background of the short modes. The response encodes tidal effects [45, 89, 90], which we will not discuss here. The background piece is proportional to the spin squared of the compact objects. To the order we are interest in this thesis, we consider only the electric part of the quadrupole moment. The body's own rotation produces a quadrupole moment given by [91]

$$\left\langle Q_{ij}^{E}\right\rangle_S = \frac{C_{ES^2} S^{ik} S_k{}^j}{2m}. \tag{2.90}$$

Regarding (2.90) in (2.88), we can write the potential - presented in the next section - which address the leading order finite size effect.

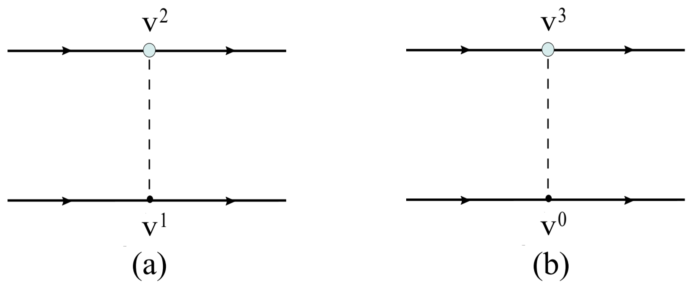


Figure 2.4.: Leading order spin-orbit interaction. (a) In this diagram we take into account the v^1 mass insertion and v^2 spin-graviton vertex. (b) In this case we take the v^3 spin-graviton vertex.

2.2.4. Potentials at leading order

As an illustration, we perform the derivation of the leading order spin-orbit contribution using (2.85) and (2.86), which comes in two pieces as it is shown in Fig.2.4, such that we can write

$$Fig2.4 = \frac{-im_2}{4m_{Pl}^2}\int dtdt'\frac{d^3\mathbf{p}}{(2\pi)^3}\partial_\beta\left(\frac{e^{-i\mathbf{p}(\mathbf{x}-\mathbf{y})}}{\mathbf{p}^2}\delta\left(t-t'\right)P_{0\epsilon\alpha\gamma}\right)S_1^{\beta\alpha}\mathbf{u}_1^\gamma\left(t\right)\mathbf{u}_2^\epsilon\left(t\right)\left(2-\delta_0^\epsilon\right). \quad (2.91)$$

Using

$$\int\frac{d^3\mathbf{p}}{(2\pi)^3}\frac{e^{-i\mathbf{p}(\mathbf{x}-\mathbf{y})}}{\mathbf{p}^2}=\frac{1}{4\pi r}, \quad (2.92)$$

and ignoring the time derivative which will give a higher order term, we have

$$\begin{aligned}Fig2.4 &= -i2m_2\partial_j\int dt\frac{1}{r}P_{0\epsilon\alpha\gamma}S_1^{\beta\alpha}\mathbf{u}_1^\gamma\left(t\right)\mathbf{u}_2^\epsilon\left(t\right)\left(2-\delta_0^\epsilon\right) \\ &= i2m_2\int dt\frac{r_j}{r^3}P_{0\epsilon\alpha\gamma}S_1^{\beta\alpha}\mathbf{u}_1^\gamma\left(t\right)\mathbf{u}_2^\epsilon\left(t\right)\left(2-\delta_0^\epsilon\right)\end{aligned} \quad (2.93)$$

For $\epsilon=0$, $\alpha=l$, $\gamma=k$,

$$i2m_2\int dt\frac{\mathbf{r}_j}{r^3}\left(\frac{-\eta_{lk}}{2}\right)S_1^{jl}\mathbf{u}_1^k\left(t\right)=-im_2\int dt\frac{\mathbf{r}_j}{r^3}S_1^{jk}\mathbf{u}_{1k}\left(t\right). \quad (2.94)$$

For $\epsilon=l$, $\alpha=k$, $\gamma=0$,

$$i2m_2\int dt\frac{\mathbf{r}_j}{r^3}\left(\frac{\eta_{lk}}{2}\right)S_1^{jk}\mathbf{u}_2^l\left(t\right)=i2m_2\int dt\frac{\mathbf{r}_j}{r^3}S_1^{jk}\mathbf{u}_2^k\left(t\right). \quad (2.95)$$

For $\epsilon=\alpha=\gamma=0$,

$$i2m_2\int dt\frac{\mathbf{r}_j}{r^3}\left(\frac{1}{2}\right)S_1^{j0}=im_2\int dt\frac{\mathbf{r}_j}{r^3}S_1^{j0}. \quad (2.96)$$

[5] We will discuss in the next chapter why it is convenient to use the electric, as well as the magnetic, component of the Weyl tensor.

Taking these pieces together, and using Euclidean metric and considering the proper time as the worldline parameter, we have

$$W = \int dt m_2 \frac{\mathbf{r}^j}{r^2} \left[-S_1^{jk} \mathbf{v}_1^k(t) + 2 S_1^{jk} \mathbf{v}_2^k(t) - S_1^{j0} \right] + 1 \Leftrightarrow 2, \tag{2.97}$$

hence

$$V_{SO} = m_2 \frac{\mathbf{r}^j}{r^3} \left[S_1^{jk} \mathbf{v}_1^k - 2 S_1^{jk} \mathbf{v}_2^k + S_1^{j0} \right] + 1 \Leftrightarrow 2. \tag{2.98}$$

Similarly, by considering (40) twice, the spin1-spin2 leading order potential reads

$$V_{S_1 S_2} = -\frac{1}{r^3} \left[\mathbf{S}_1 \cdot \mathbf{S}_2 - 3 \left(\mathbf{S}_1 \cdot \hat{\mathbf{r}} \right) \left(\mathbf{S}_2 \cdot \hat{\mathbf{r}} \right) \right] \tag{2.99}$$

and, going further, the analysis of finite size effects gives the spin-spin leading order potential:

$$V_{S^2} = -\sum_{a,b} \frac{m_b}{2m_a} C_{ES^2}^{(a)} \left[\frac{1}{r^3} \mathbf{S}_a^2 - \frac{3}{r^5} \left(\mathbf{S}_a \cdot \mathbf{r} \right)^2 \right]. \tag{2.100}$$

The leading order spin accelerations can be readily derived from these potentials:

$$\mathbf{a}_{SO}^{cov} = \frac{1}{r^3} \left\{ 6\mathbf{n} \left[(\mathbf{n} \times \mathbf{v}) \cdot \left(2\mathbf{S} + \frac{\delta m}{m} \mathbf{\Sigma} \right) \right] - \mathbf{v} \times \left(7\mathbf{S} + 3 \frac{\delta m}{m} \mathbf{\Sigma} \right) + 3\dot{r} \left[\mathbf{n} \times \left(3\mathbf{S} + \frac{\delta m}{m} \mathbf{\Sigma} \right) \right] \right\}, \tag{2.101}$$

in the covariant SSC,

$$\mathbf{a}_{SO}^{NW} = \frac{1}{r^3} \left\{ \frac{3}{2} \mathbf{n} \left[(\mathbf{n} \times \mathbf{v}) \cdot \left(7\mathbf{S} + 3 \frac{\delta m}{m} \mathbf{\Sigma} \right) \right] - \mathbf{v} \times \left(7\mathbf{S} + 3 \frac{\delta m}{m} \mathbf{\Sigma} \right) + \frac{3}{2} \dot{r} \left[\mathbf{n} \times \left(7\mathbf{S} + 3 \frac{\delta m}{m} \mathbf{\Sigma} \right) \right] \right\}, \tag{2.102}$$

in the Newton-Wigner SSC,

$$\mathbf{a}_{S_1 S_2} = -\frac{3}{m \nu r^4} \left[\mathbf{n} \left(\mathbf{S}_1 \cdot \mathbf{S}_2 \right) - 5\mathbf{n} \left(\mathbf{S}_1 \cdot \mathbf{n} \right) \left(\mathbf{S}_2 \cdot \mathbf{n} \right) + \mathbf{S}_1 \left(\mathbf{S}_2 \cdot \mathbf{n} \right) + \mathbf{S}_2 \left(\mathbf{S}_1 \cdot \mathbf{n} \right) \right], \tag{2.103}$$

and

$$\mathbf{a}_{S_1^2, S_2^2} = \sum_{K \neq Q} \frac{3 m_Q}{2 m_K} \frac{C_{ES^2}^{(K)}}{m \nu} \frac{1}{r^4} \left[-\mathbf{n} \mathbf{S}_K^2 + \mathbf{5n} \left(\mathbf{S}_K \cdot \mathbf{n} \right)^2 - 2 \mathbf{S}_K \left(\mathbf{S}_K \cdot \mathbf{n} \right) \right]. \tag{2.104}$$

We recommend the reader to go through appendix B where we compute the spin-orbit accelerations and discuss the SSC issue. In particular, there we show how to go from one choice to another.

3. NRGR II: Radiation sector

Describing the evolution of a binary system of compact objects in NRGR is achieved by integrating out the gravitational perturbations from the action. This is accomplished by solving for the gravitational perturbations and substituting these solutions into the original action resulting in an effective action for the worldlines of the binary, which is efficiently carried out using Feynman diagrams. This procedure is well-suited for conservative interactions, such as for computing PN corrections to the binding potential of a compact binary. However, when investigating non-conservative processes, such as the inspiral of a binary due to the emission of gravitational radiation, this procedure requires modification since dissipative systems generally do not admit Lagrangian or Hamiltonian descriptions. In this chapter we show how to investigate radiation reaction using the formalism developed in [74], which generalizes the usual action principle to handle dissipative dynamics by doubling the number of degrees of freedom.

3.1. Multipole Expansion

The radiation sector of NRGR has been further developed and explored beyond the leading order in [33], where the general form of the action for the radiation sector was constructed. Its form is determined by the underlying symmetries, reparameterization invariance and diffeomorphism invariance, and is applicable to arbitrary gravitational wave sources in the long wavelength approximation. This action of the effective long wavelength radiation theory is in the form of a multipole expansion and is a derivative expansion where higher order terms are suppressed by powers of the ratio of the size of the source over the wavelength. The Wilson coefficients of the action, the multipole moments, are not determined by the symmetries and need to be fixed through a matching calculation.

Multipole expansion is a tool that has been used not only in Electromagnetism but in gravitational wave physics as well [92–94]. However, in both cases it is commonly applied at the level of the solution of the equations of motion for the fields. Here, the multipole expansion is considered at the level of the action by Taylor expanding the radiation modes around a point lying within the source, what leads to a uniform power counting [95, 96]. Multipole expansions are related to the underlying rotational symmetry of the physical laws. Even in the cases which the source terms may not be symmetrical, one can expand them in terms of irreducible representations of the rotational symmetry group. It makes calculations simpler and more transparent and it also ensures the absence of mixing of multipole moments for example in the energy flux in a linear theory. Performing the multipole expansion in the NRGR framework includes a decomposition into multipole moments which are in irreducible representations of SO(3), where we use symmetric trace free (STF) tensors. This approach, which we present here, was formally developed in [34].

3.1.1. Scalars

To illustrate how to perform the multipole expansion at the level of the action, let us first consider the simplest case of a scalar field ϕ coupled linearly to a source J, i.e. let us regard the binary system as being described by scalar gravity. As in electromagnetism, if the particles' positions and velocities do not vary too rapidly, the typical wavelenght of the emitted radiation is given by $\lambda_{rad} \sim \frac{r}{v}$ implying $\lambda_{rad} \gg r$. This allows us to treat the system as a single localized source endowed with a series of multipole moments, whose effective radiation action can be written as

$$S_{eff}^{rad} = \frac{1}{M_\phi} \int dt \left(J_{(0)} \left(t\right) \phi \left(t, \mathbf{x}_{cm}\right) + J^{i}_{(1)} \left(t\right) \partial_i \phi \left(t, \mathbf{x}_{cm}\right) + J^{ij}_{(2)} \left(t\right) \partial_i \partial_j \phi \left(t, \mathbf{x}_{cm}\right) + ... \right), \quad (3.1)$$

where $\mathbf{x}_{cm}$ is the centre-of-mass of the bound state. To understand what are these $J^{i_1 ... i_n}_{(n)}(t)$ - or Wilson coefficients in the EFT language - it is necessary a matching procedure. For this purpose, we write the multipole expansion for the scalar fields

$$\phi\left(t, \mathbf{x}\right) = \phi\left(t, \mathbf{x}_{cm}\right) + \left(\mathbf{x} - \mathbf{x}_{cm}\right)^i \partial_i \phi\left(t, \mathbf{x}_{cm}\right) + \frac{1}{2} \left(\mathbf{x} - \mathbf{x}_{cm}\right)^i \left(\mathbf{x} - \mathbf{x}_{cm}\right)^j \partial_i \partial_j \phi\left(t, \mathbf{x}_{cm}\right) + ... \quad (3.2)$$

and plug this expansion - we are free to choose the coordinate system such that $\mathbf{x}_{cm} = 0$ - into the non-effective interaction action $J\phi$:

$$\int d^4x J(x)\phi(x) = \int dt \left[\left(\int d^3x J\left(t, \mathbf{x}\right) \right) \phi\left(t, 0\right) + \left(\int d^3x J\left(t, \mathbf{x}\right) x^i \right) \partial_i \phi\left(t, 0\right) + ... \right], \quad (3.3)$$

what allows us to read off

$$J^{i_1 ... i_n}_{(n)}(t) = M_\phi \int d^3x J\left(t, \mathbf{x}\right) \mathbf{x}^{i_1} ... \mathbf{x}^{i_n}, \quad (3.4)$$

where i_n corresponds only to spatial indices. We write this same expression in a more compact way:

$$J^N_{(n)} = M_\phi \int d^3x J\left(t, \mathbf{x}\right) \mathbf{x}^N. \tag{3.5}$$

In order to bring the source action into the form of a multipole expansion we need to decompose the moments (3.5) in irreducible representations of the rotation group SO(3) for which we use symmetric trace free (STF) tensors. We can see that these moments are already symmetric in their indices. When they are written in a TF form, the effective action becomes [34]

$$S^{rad}_{eff} = \int dt \sum_{l=0}^{\infty} \frac{1}{l!} I^L \partial_L \phi, \tag{3.6}$$

with the multipole moments defined by

$$I^L = \sum_{p=0}^{\infty} \frac{(2l+1)!!}{(2p)!!\,(2l+2p+1)!!} \int d^3\mathbf{x}\, |\mathbf{x}|^{2p}\, \mathbf{x}^L_{STF} \partial_t^{2p} J\left(t, \mathbf{x}\right), \tag{3.7}$$

whose normalization is chosen such that for $p = 0$ we have

$$I^L_0 = \int d^3\mathbf{x} J\left(t, \mathbf{x}\right) \mathbf{x}^L_{STF}. \tag{3.8}$$

With the general expression (3.7), the power loss can be computed to all orders. The amplitude [97]

$$i\mathcal{A}^{(l)} = i\frac{(-i)^l}{l!} I^L k^L, \tag{3.9}$$

must be considered at the generalized formula of the energy flux by taking its modulus squared in a time average and integrating over momentum, yielding

$$\frac{dE}{dt} = \sum_{l=0}^{\infty} \frac{1}{4\pi l!\,(2l+1)!!} \left\langle \left(\frac{d^{l+1}}{dt^{l+1}} I^L \right)^2 \right\rangle. \tag{3.10}$$

3.1.2. Gravity

The same idea presented last section is used in NRGR in order to obtain the multipole moments necessary to calculate the power loss in the binary system due to the emission of gravitational waves. We can express the action in terms of manifestly gauge invariant operators by including in operators consistent with the symmetries. In General Relativity, the gauge invariant quantities are the Riemann tensor, the Ricci tensor and Ricci scalar and their covariant derivatives, but due to vaccum equations of motion only structures involving the Riemann tensor

- or rather the Weyl tensor - yield physical terms which can contribute to observables. For our purposes, it is convenient to work with the electric and magnetic components of the Weyl tensor:

$$E_{ij} = R_{0i0j}, \tag{3.11}$$
$$B_{ij} = \frac{1}{2}\epsilon_{imn}R_{0jmn}, \tag{3.12}$$

where we chose to place the centre-of-mass of the binary at the origin and at rest with respect to distant observers. With this, and regarding the binary system as a source of long wavelenght gravitational waves, the effective radiation action can be written as

$$S_{eff}^{rad} = -\int d\tau \left[M\left(t\right) - \sum_{l=2} \frac{1}{l!} I^L \nabla_{L-2} E_{i_{l-1}i_l} + \frac{2l}{(2l+1)!} J^L\left(t\right) \nabla_{L-2} B_{i_{l-1}i_l}\right], \tag{3.13}$$

where $M\left(t\right)$ represents the binding mass/energy of the binary system. Note that the multipoles I^L and J^L include contributions from the spin. We need to determine the electric and magnetic-type multipole moments by matching with source action presented in the last chapter

$$-\frac{1}{2m_{Pl}}\int d^4x \mathcal{T}^{\mu\nu}\bar{h}_{\mu\nu}, \tag{3.14}$$

which represents the coupling between the short-distance modes and the radiation field. After Taylor expanding the radiation modes

$$\bar{h}_{\mu\nu}\left(x\right) = \sum_{n=0} \frac{1}{n!} x^N \partial_N \bar{h}_{\mu\nu}\left(t, 0\right) \tag{3.15}$$

and plugging back into (3.14), we can determine the multipole moments (after an extensive use of relations which follow from the conservation of the pseudo stress-energy tensor as well as the wave equation) [34]:

$$\begin{aligned} I^L = \int d^3\mathbf{x} \sum_{p=0}^{\infty} \frac{(2l+1)!!}{(2p)!!\,(2l+2p+1)!!} \Bigg\{ \left(1 + \frac{8p\left(l+p+1\right)}{\left(l+1\right)\left(l+2\right)}\right) \left[\partial_0^{2p}\mathcal{T}^{00}\left|\mathbf{x}\right|^{2p}\mathbf{x}^L\right]_{STF} \\ + \left(1 + \frac{4p}{\left(l+1\right)\left(l+2\right)}\right) \left[\partial_0^{2p}\mathcal{T}^{kk}\left|\mathbf{x}\right|^{2p}\mathbf{x}^L\right]_{STF} - \left(\frac{4}{l+1}\right)\left(1 + \frac{2p}{l+2}\right) \left[\partial_0^{2p+1}\mathcal{T}^{0m}\left|\mathbf{x}\right|^{2p}\mathbf{x}^{mL}\right]_{STF} \\ + \left(\frac{2}{\left(l+1\right)\left(l+2\right)}\right) \left[\partial_0^{2p+2}\mathcal{T}^{mn}\left|\mathbf{x}\right|^{2p}\mathbf{x}^{mnL}\right]_{STF} \Bigg\}, \end{aligned} \tag{3.16}$$

which are the mass moments whereas the current moments are

$$\begin{aligned} J^L = \int d^3\mathbf{x} \sum_{p=0}^{\infty} \frac{(2l+1)!!}{(2p)!!\,(2l+2p+1)!!} \Bigg\{ \left(1 + \frac{2p}{l+2}\right) \left[\epsilon^{k_l mn}\partial_0^{2p}\mathcal{T}^{0m}\left|\mathbf{x}\right|^{2p}\mathbf{x}^{nL-1}\right]_{STF} \\ - \left(\frac{1}{l+2}\right) \left[\epsilon^{k_l mr}\partial_0^{2p+1}\mathcal{T}^{mn}\left|\mathbf{x}\right|^{2p}\mathbf{x}^{nrL-1}\right]_{STF} \Bigg\}. \end{aligned} \tag{3.17}$$

We can write, for instance, the leading order and next to leading order mass and current quadrupole moments, as given bellow:

$$I^{ij}_{(0)} = \int d^3\mathbf{x}\mathcal{T}^{00}\left[\mathbf{x}^i\mathbf{x}^j\right]_{TF}, \tag{3.18}$$

$$I^{ij}_{(1)} = \int d^3\mathbf{x}\left(\mathcal{T}^{ll} - \frac{4}{3}\dot{\mathcal{T}}^{0l}\mathbf{x}^l + \frac{11}{42}\ddot{\mathcal{T}}^{00}\mathbf{x}^2\right)\left[\mathbf{x}^i\mathbf{x}^j\right]_{TF}, \tag{3.19}$$

$$J^{ij}_{(0)} = \int d^3\mathbf{x}\left[\epsilon^{ilk}\mathcal{T}^{0k}\mathbf{x}^j\mathbf{x}^k\right]_{STF}, \tag{3.20}$$

$$J^{ij}_{(1)} = \int d^3\mathbf{x}\left[\epsilon^{ilk}\left(\frac{1}{28}\dot{\mathcal{T}}^{lm}\mathbf{x}^j\mathbf{x}^k\mathbf{x}^m + \frac{3}{28}\dot{\mathcal{T}}^{jk}\mathbf{x}^j\mathbf{x}^2\right)\right]_{STF}, \tag{3.21}$$

whereas the leading order octopole moments read as

$$I^{ijk}_{(0)} = \int d^3\mathbf{x}\mathcal{T}^{00}\left[\mathbf{x}^i\mathbf{x}^j\mathbf{x}^k\right]_{TF}, \tag{3.22}$$

$$J^{ijk}_{(0)} = \int d^3\mathbf{x}\left[\epsilon^{iml}\mathcal{T}^{0l}\mathbf{x}^m\mathbf{x}^j\mathbf{x}^k\right]_{STF}. \tag{3.23}$$

The probability amplitude to emit a graviton can be written as

$$\begin{aligned} i\mathcal{A}_h \;=\; & \frac{i}{2m_{Pl}}\sum_{l=2}^{\infty}(-i)^{l-2}\left[\frac{1}{l!}I^L\left(|\mathbf{k}|\right)k^{L-2}\left|\mathbf{k}\right|^2\varepsilon^{k_{l-1}k_l*}\left(\mathbf{k},h\right)\right. \\ & \left.+\frac{2l}{(l+1)!}J^L\left(|\mathbf{k}|\right)k^{jL-2}\left|\mathbf{k}\right|\epsilon^{ijk_{l-1}}\varepsilon^{k_li*}\left(\mathbf{k},h\right)\right], \end{aligned} \tag{3.24}$$

where the polarization tensors obey [33]

$$\begin{aligned} \sum_h \varepsilon^{ij}\left(\mathbf{k},h\right)\varepsilon^{kl*}\left(\mathbf{k},h\right) = \frac{1}{2}\Big[&\delta^{ik}\delta^{jl} + \delta^{il}\delta^{jk} - \delta^{ij}\delta^{kl} - \frac{1}{\mathbf{k}^2}\left(\delta^{ij}\mathbf{k}^k\mathbf{k}^l + \delta^{kl}\mathbf{k}^i\mathbf{k}^j\right) \\ &-\frac{1}{\mathbf{k}^2}\left(\delta^{ik}\mathbf{k}^j\mathbf{k}^l + \delta^{il}\mathbf{k}^j\mathbf{k}^k + \delta^{jk}\mathbf{k}^i\mathbf{k}^l + \delta^{jl}\mathbf{k}^i\mathbf{k}^k\right) + \frac{1}{\mathbf{k}^4}\mathbf{k}^i\mathbf{k}^j\mathbf{k}^k\mathbf{k}^l\Big] \end{aligned} \tag{3.25}$$

in the transverse-traceless gauge. Upon squaring of the amplitude, summing over polarizations and momentum, we find

$$\frac{dE}{dt} = \sum_{l=2}^{\infty}\left[\frac{(l+1)(l+2)}{l(l-1)\,l!\,(2l+1)!!}\left\langle\left(\frac{d^{l+1}I^L}{dt^{l+1}}\right)^2\right\rangle + \frac{4l(l+2)}{(l-1)(l+1)!\,(2l+1)!!}\left\langle\left(\frac{d^{l+1}J^L}{dt^{l+1}}\right)^2\right\rangle\right], \tag{3.26}$$

or, likewise, it can be expressed in a more explicit form:

$$\frac{dE}{dt} = \frac{1}{5}\left\langle\left(\frac{d^3I^{ij}}{dt^3}\right)^2\right\rangle + \frac{16}{45}\left\langle\left(\frac{d^3J^{ij}}{dt^3}\right)^2\right\rangle + \frac{1}{189}\left\langle\left(\frac{d^4I^{ijk}}{dt^4}\right)^2\right\rangle + \frac{1}{84}\left\langle\left(\frac{d^4J^{ijk}}{dt^4}\right)^2\right\rangle + ..., \tag{3.27}$$

which is the energy flux to all orders in the multipole expansion. Note that the moments presented in this general expression for the power loss are the complete moments. For instance, the total mass quadrupole moment includes all PN corrections:

$$I^{ij} = I^{ij}_{(0)} + I^{ij}_{(1)} + ..., \tag{3.28}$$

so that the mass quadrupole contribution to the power loss reads

$$\frac{1}{5}\left\langle\left(\frac{d^3I^{ij}}{dt^3}\right)^2\right\rangle = \frac{1}{5}\left\langle\frac{d^3I^{ij}_{(0)}}{dt^3}\frac{d^3I^{ij}_{(0)}}{dt^3}\right\rangle + 2\left\langle\frac{d^3I^{ij}_{(0)}}{dt^3}\frac{d^3I^{ij}_{(1)}}{dt^3}\right\rangle + \left\langle\frac{d^3I^{ij}_{(1)}}{dt^3}\frac{d^3I^{ij}_{(1)}}{dt^3}\right\rangle + ..., \tag{3.29}$$

and the same follows for the other multipole moments, including spin contributions. This means that we need a power counting to determine what multipole moments are necessary to evaluate the power loss for the binary system to any given PN order.

3.1.3. Moments

Ignoring non-linear gravitational effects, at leading order we can use (2.20) instead of the pseudo tensor. As to all orders, we can read off the stress energy pseudo tensor by calculating all diagrams with one external radiation graviton leg. With this, we find, for instance,

$$I^{ij}_{(0)} = \sum_a m_a \left[\mathbf{x}^i_a \mathbf{x}^j_a\right]_{TF}, \tag{3.30}$$

$$I^{ij}_{(1)} = \sum_a m_a \left[\frac{3}{2}\mathbf{v}^2_a \mathbf{x}^i_a \mathbf{x}^j_a + \frac{11}{42}\frac{d^2}{dt^2}\left(\mathbf{x}^2_a \mathbf{x}^i_a \mathbf{x}^j_a\right) - \frac{4}{3}\frac{d}{dt}\left(\mathbf{x}_a\cdot\mathbf{v}_a \mathbf{x}^i_a \mathbf{x}^j_a\right)\right]_{TF} - \sum_{a,b}\frac{m_a m_b}{r}\left[\mathbf{x}^i_a \mathbf{x}^j_a\right]_{TF}, \tag{3.31}$$

$$J^{ij}_{(0)} = \sum_a m_a \left[\left(\mathbf{x}_a\times\mathbf{v}_a\right)^i \mathbf{x}^j_a\right]_{STF}, \tag{3.32}$$

$$J^{ij}_{(1)} = \sum_a m_a \left[\frac{\mathbf{v}^2_a}{2}\left(\mathbf{x}_a\times\mathbf{v}_a\right)^i \mathbf{x}^j_a\right]_{STF} + \sum_{a,b}\frac{m_a m_b}{r}\left[2\left(\mathbf{x}_a\times\mathbf{v}_a\right)^i \mathbf{x}^j_a - \frac{11}{4}\left(\mathbf{x}_b\times\mathbf{v}_a\right)^i \mathbf{x}^j_b \right. \tag{3.33}$$

$$\left. -\frac{3}{4}\left(\mathbf{x}_b\times\mathbf{v}_a\right)^i \mathbf{x}^j_a + \left(\mathbf{x}_a\times\mathbf{v}_a\right)^i \mathbf{x}^j_b + \frac{7}{4}\left(\mathbf{x}_a\times\mathbf{x}_b\right)^i \mathbf{v}^j_a + \frac{\mathbf{v}_a\cdot\mathbf{r}}{4r^2}\left(\mathbf{x}_a\times\mathbf{x}_b\right)^i\left(\mathbf{x}^j_a + \mathbf{x}^j_b\right)\right]_{STF}$$

$$+\frac{1}{28}\frac{d}{dt}\left[\sum_a m_a\left(\mathbf{x}_a\times\mathbf{v}_a\right)^i\left(3\mathbf{x}^2_a\mathbf{v}^j_a - \mathbf{x}_a\cdot\mathbf{v}_a\mathbf{x}^i_a\right)\right. \tag{3.34}$$

$$\left. +\sum_{a,b}\frac{m_a m_b}{2r^3}\mathbf{x}^i_a\left(\mathbf{x}_a\times\mathbf{x}_b\right)^j\left(6\mathbf{x}^2_a - 7\mathbf{x}_a\cdot\mathbf{x}_b + 7\mathbf{x}^2_b\right)\right]_{STF}, \tag{3.35}$$

$$I^{ijk}_{(0)} = \sum_a m_a\left[\mathbf{x}^i_a\mathbf{x}^j_a\mathbf{x}^k_a\right]_{TF}, \tag{3.36}$$

$$J^{ijk}_{(0)} = \sum_a m_a\left[\left(\mathbf{x}_a\times\mathbf{v}_a\right)^i\mathbf{x}^j_a\mathbf{x}^k_a\right]_{STF}. \tag{3.37}$$

To write multipoles with spin contributions, we consider the Routhian (2.84) expanded in terms of the weak field. Splitting the gravitational field into the sum of radiation and potential modes we get the Feynman rules linear in spin, some of them are shown in the second chapter in terms of the potential fields. Some of the spin induced multipole moments were computed and discussed in [98]. We write below the moments required to compute the leading order spin effects in radiation reaction of binary system which will be presented in next chapter:

$$I^{ij}_{S(0)} = \sum_a \left[\left(2\kappa + \frac{2}{3}\right)(\mathbf{v}_a \times \mathbf{S}_a)^i \mathbf{x}^j_a - \frac{4}{3}(\mathbf{x}_a \times \mathbf{S}_a)^i \mathbf{v}^j_a\right]_{STF}, \tag{3.38}$$

$$I^{ij}_{S^2(0)} = \sum_a \frac{C^{(a)}_{ES^2}}{m_a}\left[\mathbf{S}^i_a \mathbf{S}^i_a\right]_{STF}, \tag{3.39}$$

$$J^{ij}_{S(0)} = \sum_a \frac{3}{2}\left[\mathbf{S}^i_a \mathbf{x}^j_a\right]_{STF}, \tag{3.40}$$

where $\kappa = 1, \frac{1}{2}$ when we choose to work on covariant SSC or Newton-Wigner SSC, respectively. $C^{(a)}_{ES^2}$ are coefficients describing short-distance physics at the scale r_s, this finite effect arises from the coupling

$$\mathcal{R}^0_{ES} = -\frac{C_{ES^2}}{2mm_{Pl}}\partial_i\partial_j h_{00} S^{ik} S^{jk}, \tag{3.41}$$

which comes from a self-induced term in the Routhian [52]. An extensive list of multipole moments with and without spin, as well as the couplings that are necessary to computed them, is presented in [91].

It is straightfoward to perform the power counting of the multipole moments, once we know from the last chapter how each of the particles' quantities scale. Hence, for the mass moments we have

$$I^{ij}_{(0)} \sim mr^2, \qquad I^{ij}_{(1)} \sim mr^2v^2, \qquad I^{ijk}_{(0)} \sim mr^3, \qquad I^{ij}_{S(0)} \sim mr^2v^3, \qquad I^{ij}_{S^2(0)} \sim mr^2v^4, \tag{3.42}$$

whereas for the current quadrupole moments:

$$J^{ij}_{(0)} \sim mr^2v, \qquad J^{ij}_{(1)} \sim mr^2v \qquad J^{ijk}_{(0)} \sim mr^3v, \qquad J^{ij}_{S(0)} \sim mr^2v^2. \tag{3.43}$$

With this power counting we are able to tell what multipole moments are required to computed the power loss, for instance, to any desired order.

3.1.4. Leading order power loss due to spin

Spin-orbit

From (3.27) we can write the leading order spin contributions to the power loss of the binary system. Let us consider, firstly, the spin-orbit effect. In this case, we have to consider the leading order mass and current quadrupole moments with and without spin, in the following way:

$$\left(\frac{dE}{dt}\right)_{SO} = -\frac{1}{5}\left\langle I^{ij(3)}_{(0)} I^{ij(3)}_{(0)}\right\rangle - \frac{1}{5}\left\langle I^{ij(3)}_{(0)} I^{ij(3)}_{S(0)}\right\rangle - \frac{16}{45}\left\langle J^{ij(3)}_{(0)} J^{ij(3)}_{S(0)}\right\rangle, \tag{3.44}$$

where in the first term, the second time derivatives of the position with respect to time have to be reduced using the spin-orbit acceleration. The leading order spin mass quadrupole moment $I^{ij}_{S(0)}$ depends on the SSC. We wrote a code in Mathematica, using with the package "Xtensor"[1], to compute the time derivatives of the multipole moments, yielding

$$\left(\frac{dE}{dt}\right)^{cov}_{SO} = \frac{8m^2\nu}{15r^6}\left[(\mathbf{L}\cdot\mathbf{S})\left(12\frac{m}{r}+37v^2-27\dot{r}^2\right)+(\mathbf{L}\cdot\boldsymbol{\xi})\left(-4\frac{m}{r}+43v^2-51\dot{r}^2\right)\right], \quad (3.45)$$

$$\left(\frac{dE}{dt}\right)^{NW}_{SO} = \frac{8m^2\nu}{15r^6}\left[(\mathbf{L}\cdot\mathbf{S})\left(12\frac{m}{r}+37v^2-27\dot{r}^2\right)+(\mathbf{L}\cdot\boldsymbol{\xi})\left(8\frac{m}{r}+19v^2-18\dot{r}^2\right)\right], \quad (3.46)$$

where

$$\boldsymbol{S} \equiv \boldsymbol{S}_1 + \boldsymbol{S}_2, \quad (3.47)$$

$$\boldsymbol{\xi} \equiv \frac{m_2}{m_1}\boldsymbol{S}_1 + \frac{m_1}{m_2}\boldsymbol{S}_2. \quad (3.48)$$

These results agree with [85] and [84], where the power loss was computed using other methods.

Spin-spin

We can also compute the leading order energy flux due to spin-spin effects, which is given by

$$\left(\frac{dE}{dt}\right)_{SS} = -\frac{1}{5}\left\langle I^{ij(3)}_{(0)} I^{ij(3)}_{(0)}\right\rangle - \frac{1}{5}\left\langle I^{ij(3)}_{(0)} I^{ij(3)}_{S^2(0)}\right\rangle - \frac{16}{45}\left\langle J^{ij(3)}_{S(0)} J^{ij(3)}_{S(0)}\right\rangle, \quad (3.49)$$

where again the first term accounts for reduced contributions. However, in this case, we have to consider also the spin1-spin2 and spin-spin accelerations given at the end of the previous chapter. We can then separate the spin-spin contributions to the energy flux into parts:

$$\left(\frac{dE}{dt}\right)_{SS} = \left(\frac{dE}{dt}\right)_{s_1 s_2} + \left(\frac{dE}{dt}\right)_{s^2}, \quad (3.50)$$

where

$$\left(\frac{dE}{dt}\right)_{s_1 s_2} = \frac{4m^2\nu}{15r^8}\Big\{-3r^2(\boldsymbol{S}_1\cdot\boldsymbol{S}_2)\left(47v^2-55\dot{r}^2\right)+3(\boldsymbol{S}_1\cdot\boldsymbol{r})(\boldsymbol{S}_2\cdot\boldsymbol{r})\left(168v^2-269\dot{r}^2\right)$$
$$+171\dot{r}\left((\boldsymbol{S}_1\cdot\boldsymbol{r})(\boldsymbol{S}_2\cdot\boldsymbol{v})+(\boldsymbol{S}_1\cdot\boldsymbol{v})(\boldsymbol{S}_2\cdot\boldsymbol{r})\right)-71(\boldsymbol{S}_1\cdot\boldsymbol{v})(\boldsymbol{S}_2\cdot\boldsymbol{v})\Big\}, \quad (3.51)$$

[1] www.xact.es

and, in agreement with the literature [85, 99], we have

$$\left(\frac{dE}{dt}\right)_{s^2} = -\frac{2m^4\nu^2}{15r^8}\Bigg\{3r^2\left(\frac{\boldsymbol{S}_1^2}{m_1^2}+\frac{\boldsymbol{S}_2^2}{m_2^2}\right)\left(v^2+3\dot{r}^2\right)+9\dot{r}^2\left(\frac{(\boldsymbol{S}_1\cdot\boldsymbol{r})^2}{m_1^2}+\frac{(\boldsymbol{S}_2\cdot\boldsymbol{r})^2}{m_2^2}\right)$$
$$-6r\dot{r}\left(\frac{(\boldsymbol{S}_1\cdot\boldsymbol{r})(\boldsymbol{S}_1\cdot\boldsymbol{v})}{m_1^2}+\frac{(\boldsymbol{S}_2\cdot\boldsymbol{r})(\boldsymbol{S}_2\cdot\boldsymbol{v})}{m_2^2}\right)+r^2\left(\frac{(\boldsymbol{S}_1\cdot\boldsymbol{v})^2}{m_1^2}+\frac{(\boldsymbol{S}_2\cdot\boldsymbol{v})^2}{m_2^2}\right)\Bigg\}$$
$$-\frac{8}{5r^8}\sum_{a\neq b}C_{ES^2}^{(a)}m_b^2\Big\{r^2\boldsymbol{S}_a^2\left(12v^2-13\dot{r}^2\right)+2\left(\boldsymbol{S}_a\cdot\boldsymbol{r}\right)^2\left(-21v^2+34\dot{r}^2\right)$$
$$-29r\dot{r}\left(\boldsymbol{S}_a\cdot\boldsymbol{r}\right)\left(\boldsymbol{S}_a\cdot\boldsymbol{v}\right)+6r^2\left(\boldsymbol{S}_a\cdot\boldsymbol{v}\right)^2\Big\}. \tag{3.52}$$

We will use these results, both spin-orbit as well as spin-spin effects to the power loss, in the next chapter, for the consistency test. In other words, we will compare these results with the power loss computed from the results that we will obtain for the radiation reaction accelerations.

3.2. Radiation Reaction

3.2.1. Formalism

Computing radiative effects in the EFT approach presents a unique challenge since the formalism makes heavy use of an action formulation of the binary system. It is well-known that Lagrangians and Hamiltonians are not generally applicable to dissipative systems. In [37] it was indicated how this might be overcome employing a language and notation from quantum field theory. A formal extension of Hamilton's variational principle was presented in [74] that yields a Lagrangian and Hamiltonian formulations that suitably and correctly describes dissipative (and also conservative) systems [2].

The total system formed by the gravitational perturbations $h_{\mu\nu}$ and the worldlines of the compact bodies, x_a^μ with $a = 1, 2$, is closed. Only when the gravitational perturbations are integrated out is the dynamics of the worldlines open. When integrating out the long wavelength perturbations at the level of the action one have to be careful when applying Hamilton's principle of stationary action to the effective action since it is formulated by specifying boundary conditions in time, not initial conditions. If we follow the usual Hamilton's principle, the resulting effective action for the worldlines would describe conservative dynamics, the Green function for the gravitational perturbations that would appear in the effective action would be time-symmetric

[2] We recommend the reader to go through appendix C, where we present the main aspects of this formulation in the context of Classical Mechanics

(as these are the ones satisfying boundary conditions according to Sturm-Liouville theory) and hence do not account for the dissipative effects of radiation reaction.

To overcome this problem, one formally doubles the degrees of freedom so that

$$h_{\mu\nu} \to (h_{\mu\nu 1}, h_{\mu\nu 2}), \tag{3.53}$$

$$\mathbf{x}_a \to (\mathbf{x}_{a1}, \mathbf{x}_{a2}), \tag{3.54}$$

and then constructs the effective action

$$S[\mathbf{x}_{a1}, \mathbf{x}_{a2}, h_{\mu\nu 1}, h_{\mu\nu 2}] = S[\mathbf{x}_{a1}, h_{\mu\nu 1}] - S[\mathbf{x}_{a2}, h_{\mu\nu 2}], \tag{3.55}$$

where each action on the right side consists of the gauge fixed Einstein-Hilbert action together with the point particle action. Integrating out the long wavelength gravitational perturbations using Feynamn diagrams at the desired PN order gives the effective action for the open dynamic of the binary's inspiral. After all variations performed, the physical limit has to be taken, i.e., $\mathbf{x}_{a1}, \mathbf{x}_{a2} \to \mathbf{x}_a$. It is convenient, however, to work with another set of variables, which are given by

$$\mathbf{x}_{a+} = \frac{\mathbf{x}_{a1} + \mathbf{x}_{a2}}{2}, \tag{3.56}$$

$$\mathbf{x}_{a-} = \mathbf{x}_{a1} - \mathbf{x}_{a2}. \tag{3.57}$$

This change of variable is motivated by the physical limit that now becomes $\mathbf{x}_{a-} \to 0$ and $\mathbf{x}_{a+} \to \mathbf{x}_a$. Finally, by computing the following variation

$$\left[\frac{\delta S_{eff}[\mathbf{x}_{1\pm}, \mathbf{x}_{2\pm}]}{\delta \mathbf{x}_{a-}}\right]_{p.l.} = 0 \tag{3.58}$$

we obtain a set of worldline equations of motion that properly incorporates radiation reaction effects.

The relative acceleration, $\mathbf{a} = \mathbf{a}_1 - \mathbf{a}_2$, for a spinning two-body system can be expanded in terms of PN corrections:

$$\mathbf{a} = \mathbf{a}_0 + \mathbf{a}_1 + \mathbf{a}_{1.5} + \mathbf{a}_2 + \mathbf{a}_{2.5} + \mathbf{a}_3 + \mathbf{a}_{3.5} + \mathbf{a}_4 + \mathbf{a}_{4.5} + ..., \tag{3.59}$$

where the numerical labels represent the PN order of each correction. The first term $\mathbf{a}_0$ is the Newtonian acceleration,

$$\mathbf{a}_0 = -\frac{m\mathbf{r}}{r^3}, \tag{3.60}$$

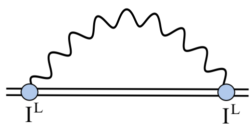
where $m = m_1 + m_2$ and $\mathbf{a}_1$ is the 1PN correction that can be derived from the Einstein-Infeld-Hoffman Lagrangian computed in chapter 2, which can be written as

$$\begin{aligned}\mathbf{a}_1 \quad &= \quad -(3+\nu)\frac{m}{r}\mathbf{a} - m\nu\frac{(\mathbf{a}\cdot\mathbf{r})\,\mathbf{r}}{r^3} - (1-3\nu)\left(\mathbf{a}\frac{v^2}{2} + \mathbf{v}\,(\mathbf{a}\cdot\mathbf{v})\right) \\ &\quad -\frac{m}{r^2}\left\{\frac{\mathbf{r}}{r}\left[-\frac{m}{r} + \frac{3}{2}(1+\nu)\,v^2 - \frac{3}{2}\eta\dot{r}^2\right] - \mathbf{v}\dot{r}\,(3+\nu)\right\}.\end{aligned} \tag{3.61}$$

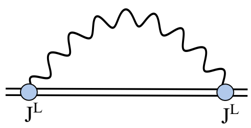
The third correction comes at 1.5PN order and represents the spin-orbit acceleration $\mathbf{a}_{1.5}$, given by (2.101) or (2.102) depending on the SSC choice. The 2PN correction $\mathbf{a}_2$, in turn, accounts for conservative effects which in part comes from the spin1-spin2 (2.103) and the spin-spin (2.104) leading order contributions, but also has terms that independ on spin. The first dissipative effect enters at 2.5PN order, which is due to radiation reaction. The 3PN correction is due to conservative effects, but the terms which follows after that have corrections due to dissipative effects. In particular, spin enters at radiation reaction at 4PN order for spin-orbit interaction and at 4.5PN order for spin1-spin2 and spin-spin interactions, as we will present in the next chapter.

In the next two sections we show how to compute the leading order and the next to leading order radiation reaction forces at 2.5PN and 3.5PN orders, respectively. To accomplish this, we use a Routhian written in terms of multipole moments that accounts for the dissipative effects. At leading order in Newton's constant, after the gravitational perturbations are integrated out, the effective action

$$iS_{\text{eff}}[\boldsymbol{x}_a^{(\pm)}] = \sum_{\ell\geq 2}$$



$+$



(3.62)

gives the Routhian that describes the non-conservative dynamics

$$\int dt R_{eff} = \sum_{l=2}^{\infty}\frac{(-1)^{l+1}(l+2)}{l-1}\int dt\left[\frac{2^l\,(l+1)}{l\,(2l+1)}I_-^L(t)\,I_+^{L(2l+1)}(t) + \frac{2^{l+3}l}{(2l+2)!}J_-^L(t)\,J_+^{L(2l+1)}(t)\right], \tag{3.63}$$

which can be also expressed by

$$R_{eff}\,[\mathbf{r}_\pm] = -\frac{1}{5}I_-^{ij}(t)\,I_+^{ij(5)} - \frac{16}{45}J_-^{ij}(t)\,J_+^{ij(5)} - \frac{1}{189}I_-^{ijk}(t)\,I_+^{ijk(7)} + ..., \tag{3.64}$$

where the superscripts (n) in the two last equations represent the number of the time derivatives.

3.2.2. Radiation reaction at 2.5PN order

Using the power counting presented previously, we can see that the lowest order for radiation reaction comes by simply taking the leading order term for the mass quadrupole moment:

$$-\frac{1}{5}\int dt I^{ij}_{(0)-}(t)\, I^{ij(5)}_{(0)+} \sim \left(\frac{r}{v}\right) mr^2 \left(\frac{v}{r}\right)^5 mr^2 = Lv^5, \tag{3.65}$$

which we can see scales as a 2.5PN order term in the effective action. Then, we take the leading order mass moment given in (3.30) and write it according to the formalism for non-conservative system, i.e., doubling its degrees of freedom:

$$I^{ij}_{(0)1} = m\nu \left[\mathbf{r}^i_1 \mathbf{r}^j_1\right]_{TF}, \tag{3.66}$$

$$I^{ij}_{(0)2} = m\nu \left[\mathbf{r}^i_2 \mathbf{r}^j_2\right]_{TF}, \tag{3.67}$$

where $\mathbf{r} = \mathbf{x}_1 - \mathbf{x}_2$ is the relative position and $\nu = \frac{m_1 m_2}{m^2}$. Changing to the $\pm$ variables, i.e using $\mathbf{r}_1 = \mathbf{r}_+ + \frac{\mathbf{r}_-}{2}$ and $\mathbf{r}_2 = \mathbf{r}_+ - \frac{\mathbf{r}_-}{2}$ regarding

$$I^{ij}_{(0)-} = I^{ij}_{(0)1} - I^{ij}_{(0)2}, \tag{3.68}$$

$$I^{ij}_{(0)+} = \frac{I^{ij}_{(0)1} + I^{ij}_{(0)2}}{2}, \tag{3.69}$$

we get

$$I^{ij}_{(0)-} = m\nu \left[\mathbf{r}^i_+ \mathbf{r}^j_- + \mathbf{r}^i_- \mathbf{r}^j_+\right]_{TF}, \tag{3.70}$$

$$I^{ij}_{(0)+} = m\nu \left[\mathbf{r}^i_+ \mathbf{r}^j_+\right]_{TF}. \tag{3.71}$$

Note that terms of quadratic or higher order in the "$-$" variables can be ignored because it will not affect the equations of motion when the physical limit is taken. Thus, we consider the Routhian

$$R_{eff}\left[\mathbf{r}_\pm\right] = -\frac{1}{5} I^{ij}_{(0)-} I^{ij(5)}_{(0)+} \tag{3.72}$$

in (3.58) to obtain the leading order radiation reaction acceleration:

$$\mathbf{a}^i_{2.5} = -\frac{2}{5}\mathbf{r}^j I^{ij(5)}_{(0)}. \tag{3.73}$$

3.2.3. Radiation reaction at 3.5PN order

Using the power counting, we can see that the next correction due to radiation reaction comes from the terms

$$\int dt \left[-\frac{1}{5} I^{ij}_{(0)-}(t)\, I^{ij(5)}_{(1)+} - \frac{1}{5} I^{ij}_{(1)-}(t)\, I^{ij(5)}_{(0)+} - \frac{16}{45} J^{ij}_{(0)-}(t)\, J^{ij(5)}_{(0)+} - \frac{1}{189} I^{ijk}_{(0)-}(t)\, I^{ijk(7)}_{(0)+}\right] \tag{3.74}$$

in the effective action. These terms scale as Lv^7 and therefore constitute a 3.5PN correction. Following the same steps presented in the previous section, we write the moments doubling the degrees of freedom

$$\begin{aligned} I^{ij}_{(1)-} = \Bigg[& m\nu\left(1-3\nu\right)\frac{3}{2}\left\{\mathbf{v}^l_+\mathbf{v}^l_+\left(\mathbf{r}^i_+\mathbf{r}^j_- + \mathbf{r}^i_-\mathbf{r}^j_+\right) + 2\mathbf{v}^l_+\mathbf{v}^l_-\mathbf{r}^i_+\mathbf{r}^j_+ +\right\} + \\ & + m\nu\left(1-3\nu\right)\frac{11}{42}\frac{d^2}{dt^2}\left\{\mathbf{r}^l_+\mathbf{r}^l_+\left(\mathbf{r}^i_+\mathbf{r}^j_- + \mathbf{r}^i_-\mathbf{r}^j_+\right) + 2\mathbf{r}^l_+\mathbf{r}^l_-\mathbf{r}^i_+\mathbf{r}^j_+\right\} + \\ & - m\nu\left(1-3\nu\right)\frac{4}{3}\frac{d}{dt}\left\{\mathbf{r}^l_+\mathbf{v}^l_+\left(\mathbf{r}^i_+\mathbf{r}^j_- + \mathbf{r}^i_-\mathbf{r}^j_+\right) + \left(\mathbf{r}^l_+\mathbf{v}^l_- + \mathbf{r}^l_-\mathbf{v}^l_+\right)\mathbf{r}^i_+\mathbf{r}^j_+\right\} + \\ & -\nu\left(m_1^2+m_2^2\right)\left\{\frac{\mathbf{r}^i_+\left(\mathbf{r}_+ + \frac{\mathbf{r}_-}{2}\right)^j + \frac{\mathbf{r}^i_-}{2}\mathbf{r}^j_+}{\left|\mathbf{r}_+ + \frac{\mathbf{r}_-}{2}\right|} - \frac{\mathbf{r}^i_+\left(\mathbf{r}_+ - \frac{\mathbf{r}_-}{2}\right)^j - \frac{\mathbf{r}^i_-}{2}\mathbf{r}^j_+}{\left|\mathbf{r}_+ - \frac{\mathbf{r}_-}{2}\right|}\right\}\Bigg]_{STF}, \end{aligned} \tag{3.75}$$

$$\begin{aligned} I^{ij}_{(1)+} = \Bigg[& m\nu\left(1-3\nu\right)\frac{3}{2}\left\{\mathbf{v}^l_+\mathbf{v}^l_+\mathbf{r}^i_+\mathbf{r}^j_+\right\} + m\nu\left(1-3\eta\right)\frac{11}{42}\frac{d^2}{dt^2}\left\{\mathbf{r}^l_+\mathbf{r}^l_+\mathbf{r}^i_+\mathbf{r}^j_+\right\} \\ & - m\nu\left(1-3\nu\right)\frac{4}{3}\frac{d}{dt}\left\{\mathbf{r}^l_+\mathbf{v}^l_+\mathbf{r}^i_+\mathbf{r}^j_+\right\} \\ & -\frac{\nu\left(m_1^2+m_2^2\right)}{2}\left\{\frac{\mathbf{r}^i_+\left(\mathbf{r}_+ + \frac{\mathbf{r}_-}{2}\right)^j + \frac{\mathbf{r}^i_-}{2}\mathbf{r}^j_+}{\left|\mathbf{r}_+ + \frac{\mathbf{r}_-}{2}\right|} + \frac{\mathbf{r}^i_+\left(\mathbf{r}_+ - \frac{\mathbf{r}_-}{2}\right)^j - \frac{\mathbf{r}^i_-}{2}\mathbf{r}^j_+}{\left|\mathbf{r}_+ - \frac{\mathbf{r}_-}{2}\right|}\right\}\Bigg]_{STF}, \end{aligned} \tag{3.76}$$

whereas for the mass octupole moment and current quadrupole moment we have, respectively,

$$I^{ijk}_{(0)-} = -\nu\delta m\left[\mathbf{r}^i_+\mathbf{r}^j_+\mathbf{r}^j_- + \mathbf{r}^i_+\mathbf{r}^j_-\mathbf{r}^j_+ + \mathbf{r}^i_-\mathbf{r}^j_+\mathbf{r}^j_+\right]_{TF}, \tag{3.77}$$

$$I^{ijk}_{(0)+} = -\nu\delta m\left[\mathbf{r}^i_+\mathbf{r}^j_+\mathbf{r}^j_+\right]_{TF}, \tag{3.78}$$

$$J^{ij}_{(0)-} = -\nu\delta m\left[\left(\mathbf{r}_+\times\mathbf{v}_+\right)^i\mathbf{r}^j_- + \left(\mathbf{r}_+\times\mathbf{v}_-\right)^i\mathbf{r}^j_+ + \left(\mathbf{r}_-\times\mathbf{v}_+\right)^i\mathbf{r}^j_+\right]_{STF}. \tag{3.79}$$

$$J^{ij}_{(0)+} = -\nu\delta m\left[\left(\mathbf{r}_+\times\mathbf{v}_+\right)^i\mathbf{r}^j_+\right]_{STF}, \tag{3.80}$$

where $\delta m = m_1 - m_2$ and $\mathbf{v} = \mathbf{v}_1 - \mathbf{v}_2$. Regarding these multipoles in (3.74) and then computing (3.58), we get

$$\mathbf{a}_{3.5} = \mathbf{a}_{mq} + \mathbf{a}_{mo} + \mathbf{a}_{cq} + \mathbf{a}_{red}, \tag{3.81}$$

where

$$\begin{aligned} \mathbf{a}^m_{mq} = & -\frac{2}{5}\delta^{mi}\mathbf{r}^j I^{(5)}_{ij(1)} - \frac{3}{5}I^{(5)}_{ij(0)}\Bigg[\frac{m}{3r}\left(1-2\nu\right)\left(\frac{\mathbf{r}^m\mathbf{r}^i\mathbf{r}^j}{r^2} - 2\delta^{mi}\mathbf{r}^j\right) \\ & - \left(1-3\nu\right)\left(\mathbf{a}^m\mathbf{r}^i\mathbf{r}^j + \mathbf{v}^m\mathbf{v}^i\mathbf{r}^j + \mathbf{v}^m\mathbf{r}^i\mathbf{v}^j - v^2\mathbf{r}^i\delta^{mj}\right)\Bigg]_{STF} \qquad (3.82) \\ & - \frac{1}{15}I^{(6)}_{ij(0)}\left[\left(1-3\nu\right)\left\{8\left(\mathbf{r}\cdot\mathbf{v}\right)\mathbf{r}^i\delta^{mj} - 9\mathbf{v}^m\mathbf{r}^i\mathbf{r}^j - 8\mathbf{r}^m\mathbf{v}^i\mathbf{r}^j\right\}\right]_{STF} \\ & - \frac{1}{5}I^{(7)}_{ij(0)}\left[\left(1-3\nu\right)\left\{\frac{11}{21}r^2\mathbf{r}^i\delta^{mj} - \frac{17}{3}\mathbf{r}^m\mathbf{r}^i\mathbf{r}^j\right\}\right]_{STF} \qquad (3.83) \end{aligned}$$

for the mass quadrupole contribution,

$$\mathbf{a}^m_{mo} = -\frac{\delta m}{63m} I^{(7)}_{ijk(0)} \left[\mathbf{r}^i \mathbf{r}^j \delta^{mk}\right]_{STF} \tag{3.84}$$

from the mass octopole part,

$$\mathbf{a}^m_{cq} = -\frac{16\delta m}{45m} \left[J^{(5)}_{ij(0)} \left[\epsilon^{ilk}\mathbf{r}^l\mathbf{v}^k\delta^{mj} + 2\epsilon^{imk}\mathbf{r}^j\mathbf{v}^k - \epsilon^{ilm}\mathbf{r}^l\mathbf{v}^j\right]_{STF} - J^{(6)}_{ij(0)} \left[\epsilon^{ilm}\mathbf{r}^l\mathbf{r}^j\right]_{STF}\right], \tag{3.85}$$

for the current quadrupole, and there is an additional contribution

$$\mathbf{a}^i_{red} = -\frac{2}{5}\mathbf{r}^j I^{ij(5)}_{(0)} + \frac{2}{5}\left[(3+\nu)\frac{m}{r}\mathbf{r}^j\delta^{ik} + \frac{m\nu}{r^3}\mathbf{r}^j\mathbf{r}^k\mathbf{r}^i + (1-3\nu)\left(\mathbf{r}^j\delta^{ik}\frac{v^2}{2} + \mathbf{v}^i\mathbf{r}^j\mathbf{v}^k\right)\right] I^{(5)kj}_{(0)}, \tag{3.86}$$

that comes from reducing the accelerations that appear at (3.73) by the 1PN acceleration (3.61) and the reverse (i.e., replacing the second time derivatives of the position, which appear in the 1PN acceleration expression, for the 2.5PN acceleration), which results in a 3.5PN correction to the total acceleration.

4. Spin effects in radiation reaction

In this last chapter, we present the main results of this thesis: the leading order spin contributions to the radiation reaction on a binary system of compact bodies. The motivation for achieving these results is that the 4PN acceleration is fundamental in the construction of the fully accurate waveform to this order, what enforces its practical relevance.

Firstly, we introduce an extension necessary to accomodate spin effects in the formalism for non-conservative system used in the previous chapter to investigate radiation reaction. In the second section, we show how we compute spin-orbit effects, which enters at 4PN order, in radiation reaction within the EFT approach. We calculate the spin-orbit radiation reaction acceleration in terms of the multipole moments and then write the final result explicitly in terms of the masses, relative velocity, separation and spins of the binary components. Moreover, the spin-orbit evolution due to radiation reaction is given and discussed. In order to grant our results clarity and consistency, the spin-orbit calculations are carried out in Newton-Wigner and also covariant spin supplementary conditions. In the third section we do the same for spin-spin effects in radiation reaction which appears at 4.5PN order. Furthermore, a non-trival consistency check, which takes into account the total emmited power in the far zone in order to prove equivalence with the energy loss induced by the radiation reaction acceleration computed within the EFT framework, is provided in the last section.

In this chapter we will use the following definitions:

$$\boldsymbol{S} \equiv \boldsymbol{S}_1 + \boldsymbol{S}_2, \tag{4.1}$$

$$\boldsymbol{\Sigma} \equiv \frac{m}{m_2}\boldsymbol{S}_2 - \frac{m}{m_1}\boldsymbol{S}_1, \tag{4.2}$$

$$\boldsymbol{\xi} \equiv \frac{m_2}{m_1}\boldsymbol{S}_1 + \frac{m_1}{m_2}\boldsymbol{S}_2, \tag{4.3}$$

$$\boldsymbol{\chi} \equiv \left(2 + \frac{3m_2}{2m_1}\right)\boldsymbol{S}_1 + \left(2 + \frac{3m_1}{2m_2}\right)\boldsymbol{S}_2. \tag{4.4}$$

We will also use

$$\boldsymbol{L} = \mu r \boldsymbol{n} \times \boldsymbol{v} \ , \quad \tilde{\boldsymbol{L}} \equiv \boldsymbol{L}/\mu, \tag{4.5}$$

with $m = m_1 + m_2$, $\nu \equiv m_1 m_2/m^2$, and $\mu = m\nu$. In addition, $\boldsymbol{r} \equiv \boldsymbol{x}_1 - \boldsymbol{x}_2$ is the relative position, $\boldsymbol{v} \equiv \boldsymbol{v}_1 - \boldsymbol{v}_2$ and $\boldsymbol{a} \equiv \boldsymbol{a}_1 - \boldsymbol{a}_2$ are the relative velocity and acceleration, respectively, and $\boldsymbol{n} \equiv \boldsymbol{r}/r$.

4.1. Spin algebra

In order to be suitable for the purposes of this thesis presented in the following sections, it was necessary to perform an extension of the formalism for non-conservative systems. To accomplish the treatment of spinning bodies, we have introduced an spin algebra in terms of the new Poisson Brackets (C.9) of the formalism for non-conservative systems. This was done through the investigation on how the angular momentum algebra was modified when the ± variables was considered.

From the angular momentum

$$\mathbf{L}^i = \epsilon^{ijk}\mathbf{r}_j\mathbf{p}_k, \tag{4.6}$$

we can write, in view of the relations (C.2) and (C.3) in appendix C, the two components $\mathbf{L}_1$ and $\mathbf{L}_2$ in terms of the ± variables, as it is shown bellow:

$$\mathbf{L}_{1i} = \epsilon_{ijk}\left(\mathbf{r}^j_+\mathbf{p}^k_+ + \frac{\mathbf{r}^j_+\mathbf{p}^k_-}{2} + \frac{\mathbf{r}^j_-\mathbf{p}^k_+}{2} + \frac{\mathbf{r}^j_-\mathbf{p}^k_-}{4}\right), \tag{4.7}$$

$$\mathbf{L}_{2i} = \epsilon_{ijk}\left(\mathbf{r}^j_+\mathbf{p}^k_+ - \frac{\mathbf{r}^j_+\mathbf{p}^k_-}{2} - \frac{\mathbf{r}^j_-\mathbf{p}^k_+}{2} + \frac{\mathbf{r}^j_-\mathbf{p}^k_-}{4}\right), \tag{4.8}$$

such that

$$\mathbf{L}_{+i} = \epsilon_{ijk}\left(\mathbf{r}^j_+\mathbf{p}^k_+ + \frac{\mathbf{r}^j_-\mathbf{p}^k_-}{4}\right), \tag{4.9}$$

$$\mathbf{L}_{-i} = \epsilon_{ijk}\left(\mathbf{r}^j_+\mathbf{p}^k_- + \mathbf{r}^j_-\mathbf{p}^k_+\right). \tag{4.10}$$

Regarding these two last expression, we can proceed to see how the angular momentum algebra changes. Let's start with the relation between two components $\mathbf{L}_+$:

$$\begin{aligned}\{\{\mathbf{L}_{+i}, \mathbf{L}_{+j}\}\} \quad = \quad & \epsilon_{ikl}\epsilon_{jmn}\left[\frac{\mathbf{r}^k_+\mathbf{p}^n_-}{4}\left\{\left\{\mathbf{p}^l_+, \mathbf{r}^m_-\right\}\right\} + \frac{\mathbf{p}^l_+\mathbf{r}^m_-}{4}\left\{\left\{\mathbf{r}^k_+, \mathbf{p}^n_-\right\}\right\}\right. \\ & \left. + \frac{\mathbf{r}^k_-\mathbf{p}^n_+}{4}\left\{\left\{\mathbf{p}^l_-, \mathbf{r}^m_+\right\}\right\} + \frac{\mathbf{p}^l_-\mathbf{r}^m_+}{4}\left\{\left\{\mathbf{r}^k_-, \mathbf{p}^n_+\right\}\right\}\right],\end{aligned} \tag{4.11}$$

what can be reduced as

$$\left\{\left\{\mathbf{L}^i_+, \mathbf{L}^j_+\right\}\right\} = \frac{1}{4}\epsilon^{ijl}\epsilon_{lkn}\left(\mathbf{r}^k_+\mathbf{p}^n_- + \mathbf{r}^k_-\mathbf{p}^n_+\right) = \frac{1}{4}\epsilon^{ijk}\mathbf{L}_{-k}. \tag{4.12}$$

In a similar way, the relation for two components $\mathbf{L}_-$ is given by

$$\left\{\left\{\mathbf{L}_-^i, \mathbf{L}_-^j\right\}\right\} = \epsilon_{ikl}\epsilon_{jmn}\left\{\left\{\mathbf{r}_+^k\mathbf{p}_-^l + \mathbf{r}_-^k\mathbf{p}_+^l, \mathbf{r}_+^m\mathbf{p}_-^n + \mathbf{r}_-^m\mathbf{p}_+^n\right\}\right\} = \epsilon^{ijk}\mathbf{L}_{-k}. \tag{4.13}$$

The remain relation is $\mathbf{L}_+$ with respect to $\mathbf{L}_-$:

$$\left\{\left\{\mathbf{L}_+^i, \mathbf{L}_-^j\right\}\right\} = \epsilon_{ikl}\epsilon_{jmn}\left\{\left\{\mathbf{r}_+^k\mathbf{p}_+^l + \frac{\mathbf{r}_-^k\mathbf{p}_-^l}{4}, \mathbf{r}_+^m\mathbf{p}_-^n + \mathbf{r}_-^m\mathbf{p}_+^n\right\}\right\} = \epsilon^{ijk}\mathbf{L}_{+k}. \tag{4.14}$$

The important relations for the work developed in this thesis are, in Euclidean metric,

$$\left\{\left\{\mathbf{S}_+^i, \mathbf{S}_+^j\right\}\right\} = -\frac{1}{4}\epsilon^{ijk}\mathbf{S}_-^k \tag{4.15}$$

$$\left\{\left\{\mathbf{S}_-^i, \mathbf{S}_-^j\right\}\right\} = -\epsilon^{ijk}\mathbf{S}_-^k, \tag{4.16}$$

$$\left\{\left\{\mathbf{S}_+^i, \mathbf{S}_-^j\right\}\right\} = -\epsilon^{ijk}\mathbf{S}_+^k. \tag{4.17}$$

For the covariant SSC, we need this additional relation:

$$\left\{\left\{\mathbf{S}_+^i, \mathbf{S}_-^{j0}\right\}\right\} = -\epsilon^{ijk}\mathbf{S}_+^{k0}. \tag{4.18}$$

At last, concerning (C.12), the equation of motion for spin is be given by

$$\dot{\mathbf{S}} = \left[\{\{\mathbf{S}_+, \mathcal{R}\}\}\right]_{p.l.}, \tag{4.19}$$

where the physical limit reads as $\mathbf{S}_- \to 0$ and $\mathbf{S}_+ \to \mathbf{S}$.

4.2. Spin-orbit effects

Spin-orbit effects appear on radiation-reaction for the first time when we take the following action into consideration:

$$\begin{aligned} S_{eff}^{SO} &= -\frac{1}{5}\int dt\,\Big[\Big[I_{(0)-}^{ij}(t)I_{(0)+}^{ij(5)}(t) + I_{S(0)-}^{ij}(t)I_{(0)+}^{ij(5)}(t) + I_{(0)-}^{ij}(t)I_{S(0)+}^{ij(5)}(t)\Big] \\ &\quad + \frac{16}{9}\left(J_{S(0)-}^{ij}(t)J_{(0)+}^{ij(5)}(t) + J_{(0)-}^{ij}(t)J_{S(0)+}^{ij(5)}(t)\right)\Big]. \end{aligned} \tag{4.20}$$

The first term gives rise to a reduced contribution when the equations of motion are calculated, since the leading order spin-orbit acceleration must be regarded at second order time derivatives of the position. The second and third terms are a combination of the leading order mass quadrupole moments (with and without spin) and the last two terms are the contributions coming from the leading order current quadrupole moments.

4.2.1. Multipole moments

In terms of the $\pm$-variables, the mass quadrupole moments necessary for the computation of spin-orbit effects in radiation reaction at 4PN order are given by

$$I^{ij}_{(0)-} = m\nu \left[\boldsymbol{r}^i_+\boldsymbol{r}^j_- + \boldsymbol{r}^i_-\boldsymbol{r}^j_+\right]_{\rm TF}, \tag{4.21}$$

$$I^{ij}_{(0)+} = m\nu \left[\boldsymbol{r}^i_+\boldsymbol{r}^j_+\right]_{\rm TF}, \tag{4.22}$$

$$\begin{aligned} I^{ij}_{S(0)-} = 2\nu\Bigg\{ & m\left[\left(\frac{\boldsymbol{S}^i_{(0)1+}}{m_1} - \frac{\boldsymbol{S}^i_{(0)2+}}{m_2}\right)\boldsymbol{r}^j_- + \left(\frac{\boldsymbol{S}^i_{(0)1-}}{m_1} - \frac{\boldsymbol{S}^i_{(0)2-}}{m_2}\right)\boldsymbol{r}^j_+\right] \\ & + \frac{1}{3}\epsilon^{ilk}\boldsymbol{\xi}^k_- \left(\boldsymbol{v}^l_+\boldsymbol{r}^j_+ - 2\boldsymbol{r}^l_+\boldsymbol{v}^j_+\right) \\ & + \frac{1}{3}\epsilon^{ilk}\boldsymbol{\xi}^k_+ \left(\boldsymbol{v}^l_+\boldsymbol{r}^j_- + \boldsymbol{v}^l_-\boldsymbol{r}^j_+ - 2\boldsymbol{r}^l_+\boldsymbol{v}^j_- - 2\boldsymbol{r}^l_-\boldsymbol{v}^j_+\right)\Bigg\}_{\rm STF}, \end{aligned} \tag{4.23}$$

$$I^{ij}_{S(0)+} = 2\nu \left[m\left(\frac{\boldsymbol{S}^i_{(0)1+}}{m_1} - \frac{\boldsymbol{S}^i_{(0)2+}}{m_2}\right)\boldsymbol{r}^j_+ + \frac{1}{3}\epsilon^{ilk}\boldsymbol{\xi}^k_+\left(\boldsymbol{v}^l_+\boldsymbol{r}^j_+ - 2\boldsymbol{r}^l_+\boldsymbol{v}^j_+\right)\right]_{\rm STF}, \tag{4.24}$$

while the current quadrupole moments read as

$$J^{ij}_{(0)-} = -\nu\delta m\left[\epsilon^{ikl}\left(\boldsymbol{r}^k_+\boldsymbol{v}^l_+\boldsymbol{r}^j_- + \boldsymbol{r}^k_+\boldsymbol{v}^l_-\boldsymbol{r}^j_+ + \boldsymbol{r}^k_-\boldsymbol{v}^l_+\boldsymbol{r}^j_+\right)\right]_{\rm STF}, \tag{4.25}$$

$$J^{ij}_{(0)+} = -\nu\delta m\left[\epsilon^{ikl}\boldsymbol{r}^k_+\boldsymbol{v}^l_+\boldsymbol{r}^j_+\right]_{\rm STF}, \tag{4.26}$$

$$J^{ij}_{S(0)-} = -\frac{3\nu}{2}\left[\boldsymbol{\Sigma}^i_+\boldsymbol{r}^j_- + \boldsymbol{\Sigma}^i_-\boldsymbol{r}^j_+\right]_{\rm STF}, \tag{4.27}$$

$$J^{ij}_{S(0)+} = -\frac{3\nu}{2}\left[\boldsymbol{\Sigma}^i_+\boldsymbol{r}^j_+\right]_{\rm STF}. \tag{4.28}$$

It is important to remember that whenever we choose to work with the Newton-Wigner SSC the mass quadrupoles are given by

$$I^{ij}_{S(0)-} = \frac{\nu}{3}\left[\epsilon^{ikl}\boldsymbol{\xi}^l_+\left(5\left(\boldsymbol{v}^k_+\boldsymbol{r}^j_- + \boldsymbol{v}^k_-\boldsymbol{r}^j_+\right) - 4\left(\boldsymbol{r}^k_+\boldsymbol{v}^j_- + \boldsymbol{r}^k_-\boldsymbol{v}^j_+\right)\right) + \epsilon^{ikl}\boldsymbol{\xi}^l_-\left(5\boldsymbol{v}^k_+\boldsymbol{r}^j_+ - 4\boldsymbol{v}^k_+\boldsymbol{r}^j_+\right)\right]_{\rm STF}, \tag{4.29}$$

$$I^{ij}_{S(0)+} = \frac{\nu}{3}\left[\epsilon^{ikl}\boldsymbol{\xi}^l_+\left(5\boldsymbol{v}^k_+\boldsymbol{r}^j_+ - 4\boldsymbol{r}^k_+\boldsymbol{v}^j_+\right)\right]_{\rm STF}, \tag{4.30}$$

since in this gauge the spin conditions must be applied before the equations of motion are derived.

4.2.2. Radiation reaction acceleration

Firstly, we present the computation for the radiation reaction acceleration at 4PN order in the Newton-Wigner SSC.

4.2.2.1. Newton-Wigner SSC

It is convenient to calculate the contributions coming from the mass and current quadrupoles and also the reduced part separately. We start with the Routhian for the mass moment contribution

$$\mathcal{R}^{\mathrm{RR}}_{\mathrm{mq}}[\boldsymbol{r}_\pm, S^{\mu\nu}_\pm] = -\frac{1}{5}\left[I^{ij}_{S(0)-}(t)I^{ij(5)}_{(0)+}(t) + I^{ij}_{(0)-}(t)I^{ij(5)}_{S(0)+}(t)\right]. \tag{4.31}$$

In order to compute the acceleration due to radiation reaction, the Routhian can be written in terms of the ±-variables related to position and velocity only whereas the spin variables can be taken in the ordinary fashoin, since ±-variables for spin would make no difference in the derivation of the Euler-Lagrange equations. Taking into account (general EL eq in the previous chap), (4.29) and (4.30), the equation of motion corresponding to the Routhain above is given by

$$\begin{aligned}\boldsymbol{a}^m_{\mathrm{RR(mq)}} = &-\frac{3}{5m}\left[\epsilon^{ikl}\boldsymbol{v}^k\boldsymbol{\xi}^l\delta^{jm} + \epsilon^{mil}\boldsymbol{\xi}^l\boldsymbol{v}^j\right]I^{ij(5)}_{(0)} - \frac{2}{5}\left[\boldsymbol{r}^i\delta^{jm}\right]I^{ij(5)}_{S(0)}\\ &-\frac{1}{15m}\left[5\epsilon^{mil}\boldsymbol{\xi}^l\boldsymbol{r}^j + 4\epsilon^{ikl}\boldsymbol{r}^k\boldsymbol{\xi}^l\delta^{jm}\right]I^{ij(6)}_{(0)}.\end{aligned} \tag{4.32}$$

After computing all the time derivatives of the multipoles and applying the leading order equation of motion, i.e. replacing the second order time derivatives of the position for the Newtonian acceleration, we find a final expression for the mass quadrupole contribution:

$$\begin{aligned}\boldsymbol{a}_{\mathrm{RR(mq)}} = \frac{m\nu}{15r^6}\Bigg\{&\left(\tilde{\boldsymbol{L}}\cdot\boldsymbol{\xi}\right)\left[15\dot{r}\boldsymbol{r}\left(42\frac{m}{r} - 51v^2 + 119\dot{r}^2\right) - 2r\boldsymbol{v}\left(97\frac{m}{r} - 81v^2 + 405\dot{r}^2\right)\right]\\ &-(\boldsymbol{r}\times\boldsymbol{\xi})\left[40\frac{m^2}{r} + 261mv^2 - 15r\dot{r}^2\left(41\frac{m}{r} - 207v^2\right) - 333rv^4 - 3360r\dot{r}^4\right]\\ &-\dot{r}r^2\left(\boldsymbol{v}\times\boldsymbol{\xi}\right)\left[596\frac{m}{r} - 1233v^2 + 1905\dot{r}^2\right]\Bigg\}.\end{aligned} \tag{4.33}$$

Now, by considering the Routhian for the current quadrupole contribution,

$$\mathcal{R}^{\mathrm{RR}}_{\mathrm{cq}}[\boldsymbol{r}_\pm, S^{\mu\nu}_\pm] = -\frac{16}{45}\left[J^{ij}_{S(0)-}(t)J^{ij(5)}_{(0)+}(t) + J^{ij}_{(0)-}(t)J^{ij(5)}_{S(0)+}(t)\right], \tag{4.34}$$

we compute the following equation of motion:

$$\begin{aligned}\boldsymbol{a}^m_{\mathrm{RR(cq)}} = &\frac{8}{15m}\left[\boldsymbol{\Sigma}^i\delta^{jm}\right]J^{ij(5)}_{(0)} + \frac{16\delta m}{45m}\left[\epsilon^{ikl}\boldsymbol{r}^k\boldsymbol{v}^l\delta^{jm} + \epsilon^{imk}\left(2\boldsymbol{v}^k\boldsymbol{r}^j + \boldsymbol{r}^k\boldsymbol{v}^j\right)\right]J^{ij(5)}_{S(0)}\\ &+\frac{16\delta m}{45m}\left[\epsilon^{imk}\boldsymbol{r}^k\boldsymbol{r}^j\right]J^{ij(6)}_{S(0)},\end{aligned} \tag{4.35}$$

which can be written as

$$\begin{aligned}\boldsymbol{a}_{\text{RR(cq)}} = & -\frac{4\nu\delta m}{15r^6}\left\{\left(\tilde{\boldsymbol{L}}\cdot\boldsymbol{\Sigma}\right)\left[15\dot{r}\boldsymbol{r}\left(12\frac{m}{r}-15v^2+35\dot{r}^2\right)-4r\boldsymbol{v}\left(8\frac{m}{r}-9v^2+45\dot{r}^2\right)\right]\right.\\ & -(\boldsymbol{r}\times\boldsymbol{\Sigma})\left[22\frac{m^2}{r}-42mv^2+15r\dot{r}^2\left(8\frac{m}{r}-9v^2\right)+18rv^4+105r\dot{r}^4\right]\\ & \left.+15r^2\dot{r}\,(\boldsymbol{v}\times\boldsymbol{\Sigma})\left[4\frac{m}{r}-3v^2+7\dot{r}^2\right]\right\}. \end{aligned} \tag{4.36}$$

At last, the contribution

$$\mathcal{R}^{\text{RR}}_{\text{red}}[\boldsymbol{r}_\pm, S^{\mu\nu}_\pm] = -\frac{1}{5}\left[I^{ij}_{(0)-}(t)I^{ij(5)}_{(0)+}(t)+I^{ij}_{(0)-}(t)I^{ij(5)}_{(0)+}(t)\right] \tag{4.37}$$

provides the equation of motion that has to be conveniently reduced using the spin-orbit accelerations at second time derivatives of the position, in order to result in spin-orbit effects at 4PN order:

$$\begin{aligned}\boldsymbol{a}_{\text{RR(red)}} = & \frac{2m\nu}{5r^6}\left\{\left[\left(\tilde{\boldsymbol{L}}\cdot\boldsymbol{\chi}\right)\right]\left[\dot{r}\boldsymbol{r}\left(-32\frac{m}{r}-255v^2+455\dot{r}^2\right)+4r\boldsymbol{v}\left(-7\frac{m}{r}+15v^2-60\dot{r}^2\right)\right]\right.\\ & -3(\boldsymbol{r}\times\boldsymbol{\chi})\left[7mv^2+3r\dot{r}^2\left(\frac{m}{r}+45v^2\right)-15rv^4-140r\dot{r}^4\right]\\ & \left.+5r^2\dot{r}\,(\boldsymbol{v}\times\boldsymbol{\chi})\left[4\frac{m}{r}+33v^2-45\dot{r}^2\right]\right\}. \end{aligned} \tag{4.38}$$

Taking these results together, we can write the total 4PN radiation-reaction acceleration in the Newton-Wigner SSC:

$$\begin{aligned}\left(\boldsymbol{a}^{\text{SO(NW)}}_{\text{RR}}\right)^m = & -\frac{3}{5m}\left[\epsilon^{ikl}\boldsymbol{v}^k\boldsymbol{\xi}^l\delta^{jm}+\epsilon^{mil}\boldsymbol{\xi}^l\boldsymbol{v}^j\right]I^{ij(5)}_{(0)}-\frac{1}{15m}\left[5\epsilon^{mil}\boldsymbol{\xi}^l\boldsymbol{r}^j+4\epsilon^{ikl}\boldsymbol{r}^k\boldsymbol{\xi}^l\delta^{jm}\right]I^{ij(6)}_{(0)}\\ & -\frac{2}{5}\left[\boldsymbol{r}^i\delta^{jm}\right]I^{ij(5)}_{S(0)}-\frac{2}{5}\boldsymbol{r}^j\left[I^{mj(5)}_{(0)}\right]_S+\frac{8}{15m}J^{ij(5)}_{(0)}\left[\boldsymbol{\Sigma}^i\delta^{jm}\right] \qquad (4.39)\\ & +\frac{16\delta m}{45m}\left[\epsilon^{ikl}\boldsymbol{r}^k\boldsymbol{v}^l\delta^{jm}+\epsilon^{imk}\left(2\boldsymbol{v}^k\boldsymbol{r}^j+\boldsymbol{r}^k\boldsymbol{v}^j\right)\right]J^{ij(5)}_{S(0)}+\frac{16\delta m}{45m}\left[\epsilon^{imk}\boldsymbol{r}^k\boldsymbol{r}^j\right]J^{ij(6)}_{S(0)}, \end{aligned}$$

which yields, after using the convenient equations of motions,

$$\begin{aligned}\boldsymbol{a}^{\text{SO(NW)}}_{\text{RR}} = & \frac{2m\nu}{15r^6}\left\{3\dot{r}\boldsymbol{r}\left[4\left(\tilde{\boldsymbol{L}}\cdot\boldsymbol{S}\right)\left(14\frac{m}{r}-165v^2+315\dot{r}^2\right)-9\left(\tilde{\boldsymbol{L}}\cdot\boldsymbol{\xi}\right)\left(7\frac{m}{r}+40v^2-70\dot{r}^2\right)\right]\right.\\ & -r\boldsymbol{v}\left[8\left(\tilde{\boldsymbol{L}}\cdot\boldsymbol{S}\right)\left(29\frac{m}{r}-54v^2+225\dot{r}^2\right)+3\left(\tilde{\boldsymbol{L}}\cdot\boldsymbol{\xi}\right)\left(53\frac{m}{r}-93v^2+375\dot{r}^2\right)\right]\\ & -2(\boldsymbol{r}\times\boldsymbol{S})\left[22\frac{m^2}{r}+21mv^2+3r\dot{r}^2\left(49\frac{m}{r}+360v^2\right)-117rv^4-1155r\dot{r}^4\right] \qquad (4.40)\\ & +3(\boldsymbol{r}\times\boldsymbol{\xi})\left[8\frac{m^2}{r}-103mv^2+r\dot{r}^2\left(169\frac{m}{r}-1215v^2\right)+135rv^4+1260r\dot{r}^4\right]\\ & \left.+120r^2\dot{r}\,(\boldsymbol{v}\times\boldsymbol{S})\left[9v^2-13\dot{r}^2\right]-r^2\dot{r}\,(\boldsymbol{v}\times\boldsymbol{\xi})\left[88\frac{m}{r}-1269v^2+1755\dot{r}^2\right]\right\}. \end{aligned}$$

4.2.2.2. Covariant SSC

Now we reproduce the same calculations shown in the previous section but using the covariant SSC. The only new expression, in this case, comes from the mass quadrupole contribution because we use the source multipole moments ((**??**)) and (4.24) without applying the SSC before deriving the equation of motion:

$$
\begin{aligned}
\boldsymbol{a}^m_{\mathrm{RR(mq)}} = &-\frac{2}{5m}\left[\left(\frac{m}{m_1}\boldsymbol{S}^i_{(0)1}-\frac{m}{m_2}\boldsymbol{S}^i_{(0)2}\right)\delta^{jm}+\epsilon^{mik}\boldsymbol{\xi}^k\boldsymbol{v}^j+\epsilon^{ilk}\boldsymbol{v}^l\boldsymbol{\xi}^k\delta^{jm}\right]I^{ij(5)}_{(0)}\\
&-\frac{2}{15m}\left[\epsilon^{mik}\boldsymbol{\xi}^k\boldsymbol{r}^j+2\epsilon^{ilk}\boldsymbol{\xi}^k\boldsymbol{r}^l\delta^{jm}\right]I^{ij(6)}_{(0)}-\frac{2}{5}\left[\boldsymbol{r}^i\delta^{jm}\right]I^{ij(5)}_{S(0)}
\end{aligned}
\tag{4.41}
$$

which becomes, after applying the equations of motion and the covariant SSC,

$$
\begin{aligned}
\boldsymbol{a}_{\mathrm{RR(mq)}} = &-\frac{4m\nu}{15r^6}\left\{\left(\tilde{\boldsymbol{L}}\cdot\boldsymbol{\xi}\right)\left[15\dot{r}\boldsymbol{r}\left(-6\frac{m}{r}-3v^2+7\dot{r}^2\right)+2r\boldsymbol{v}\left(7\frac{m}{r}+9v^2-45\dot{r}^2\right)\right]\right.\\
&+(\boldsymbol{r}\times\boldsymbol{\xi})\left[8\frac{m^2}{r}+47mv^2+15r\dot{r}^2\left(-7\frac{m}{r}+39v^2\right)-63rv^4-630r\dot{r}^4\right]\\
&\left.+\dot{r}r^2(\boldsymbol{v}\times\boldsymbol{\xi})\left[116\frac{m}{r}-243v^2+375\dot{r}^2\right]\right\}.
\end{aligned}
\tag{4.42}
$$

The expressions for the current-quadrupole and reduced contributions remain formally the same as in the NW SSC, but for the latter contribution we input the spin-orbit acceleration in the covariant SSC to obtain

$$
\begin{aligned}
\boldsymbol{a}_{\mathrm{RR(red)}} = &-\frac{4m\nu}{15r^6}\left\{\dot{r}\boldsymbol{r}\left[\left(\tilde{\boldsymbol{L}}\cdot\boldsymbol{S}\right)\left(218\frac{m}{r}-675v^2+1435\dot{r}^2\right)+\left(\tilde{\boldsymbol{L}}\cdot\boldsymbol{\chi}\right)\left(-61\frac{m}{r}+720v^2-1400\dot{r}^2\right)\right]\right.\\
&+3r\boldsymbol{v}\left[2\left(\tilde{\boldsymbol{L}}\cdot\boldsymbol{S}\right)\left(-21\frac{m}{r}+25v^2-115\dot{r}^2\right)+5\left(\tilde{\boldsymbol{L}}\cdot\boldsymbol{\chi}\right)\left(7\frac{m}{r}-11v^2+47\dot{r}^2\right)\right]\\
&+15(\boldsymbol{r}\times\boldsymbol{S})\left[3mv^2+r\dot{r}^2\left(-5\frac{m}{r}+27v^2\right)-3rv^4-28r\dot{r}^4\right]\\
&+3(\boldsymbol{r}\times\boldsymbol{\chi})\left[3mv^2+r\dot{r}^2\left(17\frac{m}{r}+135v^2\right)-15rv^4-140r\dot{r}^4\right]\\
&\left.+15r^2\dot{r}(\boldsymbol{v}\times\boldsymbol{S})\left[4\frac{m}{r}-7v^2+11\dot{r}^2\right]-15r^2\dot{r}(\boldsymbol{v}\times\boldsymbol{\chi})\left[4\frac{m}{r}+13v^2-17\dot{r}^2\right]\right\}.
\end{aligned}
\tag{4.43}
$$

The total 4PN radiation-reaction acceleration in the covariant SSC is finally given by collecting all pieces together:

$$\begin{aligned}\boldsymbol{a}_{\rm RR}^{\rm SO(cov)} = \frac{2m\nu}{15r^6}\Big\{ & 3\dot{r}\boldsymbol{r}\left[4\left(\tilde{\boldsymbol{L}}\cdot\boldsymbol{S}\right)\left(14\frac{m}{r}-165v^2+315\dot{r}^2\right)+\left(\tilde{\boldsymbol{L}}\cdot\boldsymbol{\xi}\right)\left(\frac{m}{r}-540v^2+980\dot{r}^2\right)\right] \\ & -r\boldsymbol{v}\left[8\left(\tilde{\boldsymbol{L}}\cdot\boldsymbol{S}\right)\left(29\frac{m}{r}-54v^2+225\dot{r}^2\right)+9\left(\tilde{\boldsymbol{L}}\cdot\boldsymbol{\xi}\right)\left(31\frac{m}{r}-43v^2+175\dot{r}^2\right)\right] \\ & -2\left(\boldsymbol{r}\times\boldsymbol{S}\right)\left[22\frac{m^2}{r}+21mv^2+3r\dot{r}^2\left(49\frac{m}{r}+360v^2\right)-117rv^4-1155r\dot{r}^4\right] \\ & +\left(\boldsymbol{r}\times\boldsymbol{\xi}\right)\left[28\frac{m^2}{r}-205mv^2+9r\dot{r}^2\left(33\frac{m}{r}-295v^2\right)+297rv^4+2730r\dot{r}^4\right] \\ & +120r^2\dot{r}\left(\boldsymbol{v}\times\boldsymbol{S}\right)\left[9v^2-13\dot{r}^2\right]+r^2\dot{r}\left(\boldsymbol{v}\times\boldsymbol{\xi}\right)\left[68\frac{m}{r}+981v^2-1305\dot{r}^2\right]\Big\}. \end{aligned} \quad (4.44)$$

4.2.3. Spin evolution

To investigate how radiation reaction affects the spin of the compact bodies we had extended the formalism presented in the previous chapter for treating spinning bodies. Through (4.19) and regarding the Routhian corresponding to dissipative spin-orbit effects of leading order

$$\begin{aligned}\mathcal{R}_{\rm eff}^{\rm RR}[\boldsymbol{r}_\pm, S_\pm^{\mu\nu}] = & -\frac{1}{5}\Big[I^{ij}_{(0)-}(t)I^{ij(5)}_{(0)+}(t)+I^{ij}_{S(0)-}(t)I^{ij(5)}_{(0)+}(t)+I^{ij}_{(0)-}(t)I^{ij(5)}_{S(0)+}(t) \\ & +\frac{16}{9}\left(J^{ij}_{S(0)-}(t)J^{ij(5)}_{(0)+}(t)+J^{ij}_{(0)-}(t)J^{ij(5)}_{S(0)+}(t)\right)\Big], \end{aligned} \quad (4.45)$$

the spin evolution at 4PN order is expressed by

$$\dot{\boldsymbol{S}}^k_{a\,\rm RR} = -\frac{1}{5}\left[I^{ij(5)}_{(0)+}\{\!\{\boldsymbol{S}^k_{a+}, I^{ij}_{S(0)-}\}\!\}+\frac{16}{9}J^{ij(5)}_{(0)+}\{\!\{\boldsymbol{S}^k_{a+}, J^{ij}_{S(0)-}\}\!\}\right]_{\rm PL}. \quad (4.46)$$

The right-hand side depends on the choice for SSC. We may notice here that there is no need to write the Routhian (4.45) in terms of the ±-variables for the relative position or velocities (since they will play no role at the computation of the equations of motion via the new Poisson brackets), but only for the dynamical variables of spin. In what follows, we firstly present the result at the Newton-Wigner SSC. Later, we compute the same result at the covariant SSC and prove the equivalence between the two gauges.

4.2.3.1. Newton-Wigner SSC

Choosing the Newton-Wigner SSC means that we need to apply such condition prior to deriving the equation of motion. Then, for particle 1,

$$\begin{aligned}\left(\dot{\boldsymbol{S}}_{1\,\mathrm{RR}}^{\mathrm{SO(NW)}}\right)^k &= -\frac{\nu}{15}\left[I_{(0)}^{ij(5)}\left(5\epsilon^{iql}\boldsymbol{v}^q\boldsymbol{r}^j - 4\epsilon^{iql}\boldsymbol{r}^q\boldsymbol{v}^j\right)\left\{\!\!\left\{\boldsymbol{S}_{1+}^k,\boldsymbol{\xi}_-^l\right\}\!\!\right\} - 8J_{(0)}^{ij(5)}\boldsymbol{r}^j\left\{\!\!\left\{\boldsymbol{S}_{1+}^k,\boldsymbol{\Sigma}_-^i\right\}\!\!\right\}\right]_{\mathrm{PL}} \\ &= -\frac{\nu}{15}\left\{\frac{m_2}{m_1}I_{(0)}^{ij(5)}\left[-\boldsymbol{S}_1^i\left(5\boldsymbol{v}^k\boldsymbol{r}^j - 4\boldsymbol{r}^k\boldsymbol{v}^j\right) + \delta^{ik}\boldsymbol{S}_1^l\left(5\boldsymbol{v}^l\boldsymbol{r}^j - 4\boldsymbol{r}^l\boldsymbol{v}^j\right)\right]\right. \\ &\quad \left. -8\frac{m}{m_1}J_{(0)}^{ij(5)}\boldsymbol{r}^j\epsilon^{kiq}\boldsymbol{S}_1^q\right\}. \end{aligned} \tag{4.47}$$

The expression for particle 2 can be trivially obtained by interchanging $1 \leftrightarrow 2$ in this last result. We can reduce the derivatives on the multipoles of this expression by using the spin-independent equations of motion. Thus, we find

$$\dot{\boldsymbol{S}}_{1\,\mathrm{RR}}^{\mathrm{SO(NW)}} = \frac{4m\nu\dot{r}}{15r^4}\left(\boldsymbol{L}\times\boldsymbol{S}_1\right)\left[\left(-22\frac{m}{r} + 36v^2 - 60\dot{r}^2\right) + \frac{m_2}{m_1}\left(16\frac{m}{r} - 48v^2 + 75\dot{r}^2\right)\right]. \tag{4.48}$$

It is straightforward to show that this result is a total derivative which can be absorbed into a redefinition of the spin

$$\boldsymbol{S}_1 \to \boldsymbol{S}_1 - \frac{2m\nu}{15r^3}\left(\boldsymbol{L}\times\boldsymbol{S}_1\right)\left[\left(3\frac{m}{r} - 8v^2 + 24\dot{r}^2\right) + \frac{m_2}{m_1}\left(\frac{m}{r} + 12v^2 - 30\dot{r}^2\right)\right]. \tag{4.49}$$

At this point, we can conclude that radiation reaction, at 4PN order, exerts no influence on the spin of the compact bodies at the binary system.

4.2.3.2. Covariant SSC

Now we proceed with the same calculations but for the covariant SSC. Differently from the previous case, the covariant condition must be applied after the brackets are computed:

$$\begin{aligned}\left(\dot{\boldsymbol{S}}_{1\,\mathrm{RR}}^{\mathrm{SO(cov)}}\right)^k &= -\frac{2\nu}{15}\left\{I_{(0)}^{ij(5)}\frac{m_2}{m_1}\left[3\boldsymbol{S}_1^k v^i\boldsymbol{r}^j - 4\boldsymbol{v}^k\boldsymbol{S}_1^i\boldsymbol{r}^j + 2\boldsymbol{S}_1^i\boldsymbol{r}^k\boldsymbol{v}^j + \delta^{ik}\boldsymbol{S}_1^l\left(\boldsymbol{v}^l\boldsymbol{r}^j - 2\boldsymbol{r}^l\boldsymbol{v}^j\right)\right]\right. \\ &\quad\left. -4\frac{m}{m_1}J_{(0)}^{ij(5)}\epsilon^{kil}\boldsymbol{r}^j\boldsymbol{S}_1^l\right\}, \end{aligned} \tag{4.50}$$

yielding, after reduction,

$$\begin{aligned}\dot{\boldsymbol{S}}_{1\,\mathrm{RR}}^{\mathrm{SO(cov)}} = \frac{4m\nu}{15r^4}\Bigg\{&\dot{r}\left(\boldsymbol{L}\times\boldsymbol{S}_1\right)\left[-22\frac{m}{r} + 36v^2 - 60\dot{r}^2 + \frac{m_2}{m_1}\left(16\frac{m}{r} - 48v^2 + 75\dot{r}^2\right)\right] \\ &- \frac{m_2^2}{m}\Bigg[-2\boldsymbol{S}_1\left(6mv^2 + r\dot{r}^2\left(2\frac{m}{r} + 99v^2\right) - 18rv^4 - 75r\dot{r}^4\right) \\ &+ \dot{r}\left(\boldsymbol{r}\left(\boldsymbol{S}_1\cdot\boldsymbol{v}\right) + \boldsymbol{v}\left(\boldsymbol{S}_1\cdot\boldsymbol{r}\right)\right)\left(2\frac{m}{r} + 54v^2 - 75\dot{r}^2\right) \\ &+6r\boldsymbol{v}\left(\boldsymbol{S}_1\cdot\boldsymbol{v}\right)\left(2\frac{m}{r} - 6v^2 + 15\dot{r}^2\right)\Bigg]\Bigg\}. \end{aligned} \tag{4.51}$$

It is well known that at leading order the spin evolution equation for the covariant SSC, due to the spin definition in this gauge, does not conserve the norm of the vector, as opposed to the Newton-Wigner case. Hence, it is not a surprise that the spin evolution at 4PN in the covariant SSC is not, at first, a total derivative. However, it is possible to show its equilavence with the spin evolution obtained in the Newton-Wigner case. In order to do so, we take into account the known transformation that links the spin variables at both SSC:

$$\boldsymbol{S}_{1(\mathrm{NW})} = \boldsymbol{S}_{1(\mathrm{cov})} + \frac{1}{2}\Big(\boldsymbol{v}_1(\boldsymbol{v}_1 \cdot \boldsymbol{S}_{1(\mathrm{cov})}) - \boldsymbol{S}_{1(\mathrm{cov})}\boldsymbol{v}_1^2\Big) + \cdots\,, \tag{4.52}$$

which implies

$$\dot{\boldsymbol{S}}_{1(\mathrm{NW})} = \dot{\boldsymbol{S}}_{1\,(\mathrm{cov})} - \frac{1}{2}\left(\frac{m_2}{m}\right)^2 \Big[2\,\boldsymbol{S}_{1\,(\mathrm{cov})}\,(\boldsymbol{a}\cdot\boldsymbol{v}) - \Big(\boldsymbol{S}_{1\,(\mathrm{cov})}\cdot\boldsymbol{v}\Big)\,\boldsymbol{a} - \boldsymbol{v}\,\Big(\boldsymbol{a}\cdot\boldsymbol{S}_{1\,(\mathrm{cov})}\Big)\Big] + \cdots\,. \tag{4.53}$$

For the purpose of proving the equivalence between the spin evolutions, this transformation have to be considered at 4PN order, such that the accelerations inside the square brackets must be regarded as

$$\Big(\boldsymbol{a}^{\mathrm{RR}}_{\mathrm{LO}}\Big)^i = -\frac{2}{5}\boldsymbol{r}^j I^{ij(5)}_{(0)}. \tag{4.54}$$

In this way, inserting (4.50) in the right-hand side of (4.53), we recover (4.47) as expected. And of course the same is valid for particle 2.

4.3. Spin-spin effects

We may now proceed with the second part of our work: the leading order spin-spin effects in radiation reaction. The effective action governing this actual dissipative phenomenon is given bellow:

$$S^{SS}_{eff} = -\frac{1}{5}\int dt\left[I^{ij}_{(0)-}(t)I^{ij(5)}_{(0)+}(t) + I^{ij}_{(0)-}(t)I^{ij(5)}_{S^2(0)+}(t) + I^{ij}_{S^2(0)-}(t)I^{ij(5)}_{(0)+}(t) + \frac{16}{9}J^{ij}_{S(0)-}(t)J^{ij(5)}_{S(0)+}(t)\right]. \tag{4.55}$$

In this case, there is no need to worry about supplementary conditions for spin, since only spatial components of the spin vector appear at the right-hand side of the expression above. The first term contributes at 4.5PN order, as desired, when spin-orbit as well as spin-spin leading order accelerations are considered for the second order time derivatives of the relative position. The second term is contribution to the mass quadrupole from finite size effects, but it will not affect the acceleration, only the spin evolution. The last term simply represents the current quadrupole with spin respecting this order. Taken together, these mass and current quadrupole contributions are responsable for the first appearance of spin-spin effects in radiation reaction of a binary system, as we shall present bellow.

4.3.1. Multipole moments

In view of the actual purpose, the necessary multipole moments were already given in the first section of this chapter, with the exception of the mass quadrupole moment from finite size effects [98, 100], which in terms of the $\pm$-variables reads as

$$I^{ij}_{S^2(0)+} = -\sum_a \frac{C^{(a)}_{ES^2}}{m_a}\left[\boldsymbol{S}^i_{a+}\boldsymbol{S}^j_{a+}\right]_{TF}, \tag{4.56}$$

$$I^{ij}_{S^2(0)-} = -\sum_a \frac{C^{(a)}_{ES^2}}{m_a}\left[\boldsymbol{S}^i_{a+}\boldsymbol{S}^j_{a-} + \boldsymbol{S}^i_{a-}\boldsymbol{S}^j_{a+}\right]_{TF}, \tag{4.57}$$

where ${C^{(a)}_{ES^2}}^{bh} = 1$ for black holes, while for neutron stars ${C^{(a)}_{ES^2}}^{ns} = 4-8$ depending on the equation of state and internal dynamics [91]. These mass quadrupoles, at (4.55), play no role in the derivation of the radiation reaction acceleration at 4.5PN order because the derivatives of the spin variables give rise to terms of higher order and also because there is no $\boldsymbol{r}_-$ or $\boldsymbol{v}_-$ to be regarded during the computation of the Euler-Lagrange equation. However, we will see later that they do have relevance for spin evolution at 4.5PN order.

4.3.2. Radiation reaction acceleration

The contribution from the current quadrupole for the leading order spin-spin effects in radiation reaction are derived from

$$\mathcal{R}^{\rm RR}_{\rm cq}[\boldsymbol{r}_\pm, S^{\mu\nu}_\pm] = -\frac{16}{45}\, J^{ij}_{S(0)-}(t)J^{ij(5)}_{S(0)+}(t)\,, \tag{4.58}$$

yielding

$$\boldsymbol{a}^m_{\rm RR(cq)} = \frac{8}{15m}\left[\boldsymbol{\Sigma}^i\delta^{jm}\right]\frac{d^5 J^{ij}_{S(0)}}{dt^5}\,. \tag{4.59}$$

This expression, after reduction using the lowest order acceleration, is explicitly written as

$$\begin{aligned}\boldsymbol{a}_{\rm RR(cq)} = \frac{2\nu}{r^7}\Bigg\{&3\left(\boldsymbol{\Sigma}\cdot\boldsymbol{\Sigma}\right)\left[-r\dot{r}\boldsymbol{r}\left(2\frac{m}{r} - 3v^2 + 7\dot{r}^2\right) + r^2\boldsymbol{v}\left(\frac{8}{15}\frac{m}{r} - \frac{3}{5}v^2 + 3\dot{r}^2\right)\right] \\ &+ r\boldsymbol{\Sigma}\left[-\dot{r}\left(\boldsymbol{\Sigma}\cdot\boldsymbol{r}\right)\left(2\frac{m}{r} - 3v^2 + 7\dot{r}^2\right) + r\left(\boldsymbol{\Sigma}\cdot\boldsymbol{v}\right)\left(\frac{8}{15}\frac{m}{r} - \frac{3}{5}v^2 + 3\dot{r}^2\right)\right]\Bigg\}.\end{aligned} \tag{4.60}$$

On the other hand, the reduced contribution is simply given by

$$a^m_{\mathrm{RR(red)}} = -\frac{2}{5}\left[r^i\delta^{jm}\right]\left[\frac{d^5 I^{ij}_{(0)}}{dt^5}\right]_{SS}, \tag{4.61}$$

which after reduction using the leading order spin-spin accelerations can be splitted into two pieces:

$$\begin{aligned} a^{s_1s_2}_{\mathrm{RR(red)}} = \frac{2}{5r^7}\Bigg\{ & r\Big[r\dot r\,(S_1\cdot S_2)\Big(-112\frac{m}{r}+480v^2-980\dot r^2\Big)+\frac{\dot r}{r}(S_1\cdot r)(S_2\cdot r)\Big(392\frac{m}{r}-3990v^2+10710\dot r^2\Big) \\ & +((S_1\cdot r)(S_2\cdot v)+(S_1\cdot v)(S_2\cdot r))\Big(-79\frac{m}{r}+555v^2-3465\dot r^2\Big)+960r\dot r\,(S_1\cdot v)(S_2\cdot v)\Big] \\ & +6v\Big[2r^2(S_1\cdot S_2)\Big(8\frac{m}{r}-10v^2+45\dot r^2\Big)+(S_1\cdot r)(S_2\cdot r)\Big(-51\frac{m}{r}+115v^2-735\dot r^2\Big) \\ & +205r\dot r\,((S_1\cdot r)(S_2\cdot v)+(S_1\cdot v)(S_2\cdot r))-40r^2(S_1\cdot v)(S_2\cdot v)\Big] \\ & +3rS_1\Big[\dot r\,(S_2\cdot r)\Big(17\frac{m}{r}+80v^2-140\dot r^2\Big)+r\,(S_2\cdot v)\Big(3\frac{m}{r}-15v^2+55\dot r^2\Big)\Big] \\ & +3rS_2\Big[\dot r\,(S_1\cdot r)\Big(17\frac{m}{r}+80v^2-140\dot r^2\Big)+r\,(S_1\cdot v)\Big(3\frac{m}{r}-15v^2+55\dot r^2\Big)\Big]\Bigg\}, \end{aligned} \tag{4.62}$$

which can be regarded as the "structureless" contribution, while the finite-size contribution, for the first time, is given by

$$\begin{aligned} a^{\mathrm{FS}}_{\mathrm{RR(red)}} = \frac{2}{5r^7}\sum_{a\neq b}\frac{C^{(a)}_{ES^2}m_b}{m_a}\Bigg\{ & r\Big[-2r\dot r S_a^2\Big(28\frac{m}{r}-120v^2+245\dot r^2\Big)+7\frac{\dot r}{r}(S_a\cdot r)^2\Big(28\frac{m}{r}-285v^2+765\dot r^2\Big) \\ & +(S_a\cdot r)(S_a\cdot v)\Big(-79\frac{m}{r}+555v^2-3465\dot r^2\Big)+480r\dot r\,(S_a\cdot v)^2\Big] \\ & +3v\Big[2r^2S_a^2\Big(8\frac{m}{r}-10v^2+45\dot r^2\Big)+(S_a\cdot r)^2\Big(-51\frac{m}{r}+115v^2-735\dot r^2\Big) \\ & +410r\dot r\,(S_a\cdot r)(S_a\cdot v)-40r^2(S_a\cdot v)^2\Big] \\ & +3rS_a\Big[\dot r\,(S_a\cdot r)\Big(17\frac{m}{r}+80v^2-140\dot r^2\Big)+r\,(S_a\cdot v)\Big(3\frac{m}{r}-15v^2+55\dot r^2\Big)\Big]\Bigg\}. \end{aligned} \tag{4.63}$$

Although the total spin acceleration at 4.5PN order is simply the sum of the contributions in (4.60)–(4.63), it is also instructive to decompose the final expression as

$$\begin{aligned} a^{s_1s_2}_{\mathrm{RR}} = \frac{2}{r^7}\Bigg\{ & r\Big[-r\dot r\,(S_1\cdot S_2)\Big(\frac{52}{5}\frac{m}{r}-78v^2+154\dot r^2\Big)+\frac{\dot r}{r}(S_1\cdot r)(S_2\cdot r)\Big(\frac{392}{5}\frac{m}{r}-798v^2+2142\dot r^2\Big) \\ & +((S_1\cdot r)(S_2\cdot v)+(S_1\cdot v)(S_2\cdot r))\Big(-\frac{79}{5}\frac{m}{r}+111v^2-693\dot r^2\Big)+192r\dot r\,(S_1\cdot v)(S_2\cdot v)\Big] \\ & +2v\Big[r^2(S_1\cdot S_2)\Big(8\frac{m}{r}-\frac{51}{5}v^2+45\dot r^2\Big)+3(S_1\cdot r)(S_2\cdot r)\Big(-\frac{51}{5}\frac{m}{r}+23v^2-147\dot r^2\Big) \\ & +123\,r\dot r\,((S_1\cdot r)(S_2\cdot v)+(S_1\cdot v)(S_2\cdot r))-24r^2(S_1\cdot v)(S_2\cdot v)\Big] \\ & +rS_1\Big[\dot r\,(S_2\cdot r)\Big(\frac{61}{5}\frac{m}{r}+45v^2-77\dot r^2\Big)+r\,(S_2\cdot v)\Big(\frac{19}{15}\frac{m}{r}-\frac{42}{5}v^2+30\dot r^2\Big)\Big] \\ & +rS_2\Big[\dot r\,(S_1\cdot r)\Big(\frac{61}{5}\frac{m}{r}+45v^2-77\dot r^2\Big)+r\,(S_1\cdot v)\Big(\frac{19}{15}\frac{m}{r}-\frac{42}{5}v^2+30\dot r^2\Big)\Big]\Bigg\}, \end{aligned} \tag{4.64}$$

for the $\mathcal{O}(\boldsymbol{S}_1\boldsymbol{S}_2)$ terms whereas the $\mathcal{O}(\boldsymbol{S}_a^2)$ contributions are given by

$$\begin{aligned}
\boldsymbol{a}^{s^2}_{\rm RR} = \frac{2}{r^7}\Bigg\{ & \left(\frac{m_2}{m_1}\boldsymbol{S}_1^2 + \frac{m_1}{m_2}\boldsymbol{S}_2^2\right)\left[-3r\dot{r}\boldsymbol{r}\left(2\frac{m}{r} - 3v^2 + 7\dot{r}^2\right) + r\boldsymbol{v}\left(\frac{8}{5}\frac{m}{r} - \frac{9}{5}v^2 + 9\dot{r}^2\right)\right] \\
& + \frac{m^2\nu}{m_1^2}r\boldsymbol{S}_1\left[-\dot{r}\left(\boldsymbol{S}_1\cdot\boldsymbol{r}\right)\left(2\frac{m}{r} - 3v^2 - 7\dot{r}^2\right) + r\left(\boldsymbol{S}_1\cdot\boldsymbol{v}\right)\left(\frac{8}{15}\frac{m}{r} - \frac{3}{5}v^2 + 3\dot{r}^2\right)\right] \\
& + \frac{m^2\nu}{m_2^2}r\boldsymbol{S}_2\left[-\dot{r}\left(\boldsymbol{S}_2\cdot\boldsymbol{r}\right)\left(2\frac{m}{r} - 3v^2 - 7\dot{r}^2\right) + r\left(\boldsymbol{S}_2\cdot\boldsymbol{v}\right)\left(\frac{8}{15}\frac{m}{r} - \frac{3}{5}v^2 + 3\dot{r}^2\right)\right]\Bigg\} \\
+ \frac{2}{5r^7}\sum_{a\neq b}\frac{C^{(a)}_{ES^2}m_b}{m_a}\Bigg\{ & \boldsymbol{r}\Bigg[-2r\dot{r}\boldsymbol{S}_a^2\left(28\frac{m}{r} - 120v^2 + 245\dot{r}^2\right) + 7\frac{\dot{r}}{r}\left(\boldsymbol{S}_a\cdot\boldsymbol{r}\right)^2\left(28\frac{m}{r} - 285v^2 + 765\dot{r}^2\right) \\
& \quad + \left(\boldsymbol{S}_a\cdot\boldsymbol{r}\right)\left(\boldsymbol{S}_a\cdot\boldsymbol{v}\right)\left(-79\frac{m}{r} + 555v^2 - 3465\dot{r}^2\right) + 480r\dot{r}\left(\boldsymbol{S}_a\cdot\boldsymbol{v}\right)^2\Bigg] \\
& + 3\boldsymbol{v}\Bigg[2r^2\boldsymbol{S}_a^2\left(8\frac{m}{r} - 10v^2 + 45\dot{r}^2\right) + \left(\boldsymbol{S}_a\cdot\boldsymbol{r}\right)^2\left(-51\frac{m}{r} + 115v^2 - 735\dot{r}^2\right) \\
& \quad + 410r\dot{r}\left(\boldsymbol{S}_a\cdot\boldsymbol{r}\right)\left(\boldsymbol{S}_a\cdot\boldsymbol{v}\right) - 40r^2\left(\boldsymbol{S}_a\cdot\boldsymbol{v}\right)^2\Bigg] \\
& + 3r\boldsymbol{S}_a\left[\dot{r}\left(\boldsymbol{S}_a\cdot\boldsymbol{r}\right)\left(17\frac{m}{r} + 80v^2 - 140\dot{r}^2\right) + r\left(\boldsymbol{S}_a\cdot\boldsymbol{v}\right)\left(3\frac{m}{r} - 15v^2 + 55\dot{r}^2\right)\right]\Bigg\}. \qquad (4.65)
\end{aligned}$$

4.3.3. Spin evolution

Taking into consideration (4.55) we obtain for the spin evolution

$$\begin{aligned}
\dot{\boldsymbol{S}}^m_{1\,\rm RR} &= -\frac{8\nu}{15}\frac{m}{m_1}\left[J^{ij(5)}_{S(0)+}\boldsymbol{r}^j\{\!\{\boldsymbol{S}^m_{1+},\boldsymbol{S}^i_{1-}\}\!\}\right]_{\rm PL} - \frac{2}{5}\frac{C^{(1)}_{ES^2}}{m_1}\left[I^{ij(5)}_{(0)+}\boldsymbol{S}^i_{1+}\{\!\{\boldsymbol{S}^m_{1+},\boldsymbol{S}^j_{1-}\}\!\}\right]_{\rm PL} \qquad (4.66)\\
&= \frac{8\nu}{15}\frac{m}{m_1}J^{ij(5)}_{S(0)}\epsilon^{mik}\boldsymbol{r}^j\boldsymbol{S}^k_1 + \frac{2}{5}\frac{C^{(1)}_{ES^2}}{m_1}\epsilon^{mjk}\boldsymbol{S}^i_1\boldsymbol{S}^k_1 I^{ij(5)}_{(0)},
\end{aligned}$$

which after reduction yields

$$\begin{aligned}
\dot{\boldsymbol{S}}_{1\,\rm RR} = \frac{2m\nu}{15r^6}\Bigg\{ & 6r^2\dot{r}\left(\boldsymbol{S}_1\times\boldsymbol{S}_2\right)\left(11\frac{m}{r} - 18v^2 + 30\dot{r}^2\right) \qquad (4.67)\\
& - 3r\left(\boldsymbol{S}_1\times\boldsymbol{v}\right)\left(\left(\boldsymbol{S}_2\cdot\boldsymbol{r}\right) - \frac{m_2}{m_1}\left(\boldsymbol{S}_1\cdot\boldsymbol{r}\right)\right)\left(8\frac{m}{r} - 9v^2 + 45\dot{r}^2\right) \\
& + \left(\boldsymbol{S}_1\times\boldsymbol{r}\right)\Bigg[15\dot{r}\left(\left(S_2\cdot\boldsymbol{r}\right) - \frac{m_2}{m_1}\left(\boldsymbol{S}_1\cdot\boldsymbol{r}\right)\right)\left(2\frac{m}{r} - 3v^2 + 7\dot{r}^2\right) \\
& \quad + 2r\left(\left(\boldsymbol{S}_2\cdot\boldsymbol{v}\right) - \frac{m_2}{m_1}\left(\boldsymbol{S}_1\cdot\boldsymbol{v}\right)\right)\left(8\frac{m}{r} - 9v^2 + 45\dot{r}^2\right)\Bigg]\Bigg\} \\
- \frac{4C^{(1)}_{ES^2}m_2}{5r^5}\Bigg\{ & 4\left(\frac{m}{r} - 3v^2 + 15\dot{r}^2\right)\left[\left(\boldsymbol{r}\times\boldsymbol{S}_1\right)\left(\boldsymbol{S}_1\cdot\boldsymbol{v}\right) + \left(\boldsymbol{v}\times\boldsymbol{S}_1\right)\left(\boldsymbol{S}_1\cdot\boldsymbol{r}\right)\right] \\
& + 15\frac{\dot{r}}{r}\left(\boldsymbol{r}\times\boldsymbol{S}_1\right)\left(\boldsymbol{S}_1\cdot\boldsymbol{r}\right)\left(3v^2 - 7\dot{r}^2\right) - 30r\dot{r}\left(\boldsymbol{v}\times\boldsymbol{S}_1\right)\left(\boldsymbol{S}_1\cdot\boldsymbol{v}\right)\Bigg\}.
\end{aligned}$$

At this point, we can define

$$\boldsymbol{\Omega}^{RR}_1 \equiv \frac{m}{3r^6}\left(\tilde{\boldsymbol{S}}_2 - \frac{m_2}{m_1}\tilde{\boldsymbol{S}}_1\right)\times\boldsymbol{L}\,, \qquad (4.68)$$

such that

$$\left(\frac{d\tilde{\boldsymbol{S}}_1}{dt}\right)_{\rm RR} = \boldsymbol{\Omega}_1^{RR} \times \tilde{\boldsymbol{S}}_1\,, \tag{4.69}$$

where

$$\begin{aligned}\tilde{\boldsymbol{S}}_1 = \boldsymbol{S}_1 + \frac{m\nu}{15r^5}\Bigg\{&3r^2\left(\boldsymbol{S}_1\times\boldsymbol{S}_2\right)\left(3\frac{m}{r}-8v^2+24\dot{r}^2\right) \\ &-\frac{48r\dot{r}}{m\nu}\left[\boldsymbol{L}\left(\boldsymbol{S}_1\cdot\boldsymbol{S}_2\right)-\boldsymbol{S}_2\left(\boldsymbol{L}\cdot\boldsymbol{S}_1\right)+\frac{m_2}{m_1}\Big(\boldsymbol{S}_1\left(\boldsymbol{L}\cdot\boldsymbol{S}_1\right)-\boldsymbol{L}(\boldsymbol{S}_1\cdot\boldsymbol{S}_1)\Big)\right] \\ &+3\left(\boldsymbol{S}_1\times\boldsymbol{r}\right)\left[\left((\boldsymbol{S}_2\cdot\boldsymbol{r})-\frac{m_2}{m_1}\left(\boldsymbol{S}_1\cdot\boldsymbol{r}\right)\right)\left(\frac{m}{r}-2v^2+10\dot{r}^2\right)-2r\dot{r}\left((\boldsymbol{S}_2\cdot\boldsymbol{v})-\frac{m_2}{m_1}\left(\boldsymbol{S}_1\cdot\boldsymbol{v}\right)\right)\right] \\ &-2r^2\left(\boldsymbol{S}_1\times\boldsymbol{v}\right)\left((\boldsymbol{S}_2\cdot\boldsymbol{v})-\frac{m_2}{m_1}\left(\boldsymbol{S}_1\cdot\boldsymbol{v}\right)\right)+\frac{2m\nu}{5}\frac{C^{(1)}_{ES^2}}{m_1}\frac{d^4}{dt^4}\Big[\left(\boldsymbol{S}_1\cdot\boldsymbol{r}\right)\left(\boldsymbol{r}\times\boldsymbol{S}_1\right)\Big]\Bigg\}.\end{aligned} \tag{4.70}$$

We can interpret (4.68) as a precession frequency induced by radiation reaction effects on the spin evolution and from (4.69) and (4.70) we conclude that the spin equation describes precessing evolution with a constant norm.

4.4. Consistency test

Finally, we shall close this chapter proving the consistency of our results for spin-orbit and spin-spin effects in radiation reaction by showing the equivalence between the radiated power in the zone and the power loss calculated through the accelerations computed by us.

4.4.1. Schott terms

Firstly, we introduce the concept of Schott terms[1], which simply consists in a redefinition of the conserved energy.

We have to show that the energy flux computed with (3.27) and power lost due to radiation reaction,

$$\mathcal{P}_{\rm RR} \equiv m\nu\, \boldsymbol{a}_{\rm RR}\cdot\boldsymbol{v}, \tag{4.71}$$

are equivalent, but not the same, since one power is given in the far zone and the other is computed in the near zone with respect to the compact bodies. Hence, the difference is a local redefinition of the conserved energy, i.e. a total time derivative or Schott terms, that will not affect the radiated power in the far region. To illustrate, we consider the a general Routhian

[1] In appendix D, we comment the appearance of the Schott terms in the context of Electrodynamics.

with contributions from mass and current quadrupoles, implying

$$\mathcal{P}_{\rm RR} = -\frac{1}{5}\boldsymbol{v}\cdot\left\{\left[\frac{\partial}{\partial\boldsymbol{r}_-} - \frac{d}{dt}\frac{\partial}{\partial\boldsymbol{v}_-}\right]\left(I^{ij}_-(t)I^{ij(5)}_+(t) + \frac{16}{9}\,J^{ij}_-(t)J^{ij(5)}_+(t)\right)\right\}_{\rm PL}, \tag{4.72}$$

and taking a time average, we obtain

$$\begin{aligned}\langle\mathcal{P}_{\rm RR}\rangle = & -\frac{1}{5}\left\langle\left(\left[\boldsymbol{v}\cdot\frac{\partial}{\partial\boldsymbol{r}_-} + \boldsymbol{a}\cdot\frac{\partial}{\partial\boldsymbol{v}_-}\right]I^{ij}_-(t)\right)I^{ij(5)}_+(t)\right\rangle_{\rm PL} \\ & -\frac{16}{9}\left\langle\left(\left[\boldsymbol{v}\cdot\frac{\partial}{\partial\boldsymbol{r}_-} + \boldsymbol{a}\cdot\frac{\partial}{\partial\boldsymbol{v}_-}\right]J^{ij}_-(t)\right)J^{ij(5)}_+(t)\right\rangle_{\rm PL},\end{aligned} \tag{4.73}$$

where we integrated by parts the time derivative in the second term in both of the square brackets. But, since

$$\left(\left[\boldsymbol{v}\cdot\frac{\partial}{\partial\boldsymbol{r}_-} + \boldsymbol{a}\cdot\frac{\partial}{\partial\boldsymbol{v}_-}\right]I^{ij}_-(t)\right)_{\rm PL} = I^{ij(1)}, \tag{4.74}$$

and integrating by parts once more, it results

$$\langle\mathcal{P}_{\rm RR}\rangle = \frac{dE}{dt}, \tag{4.75}$$

We can say that the two expressions for $\mathcal{P}_{\rm RR}$ and dE/dt are related by a total time derivative,

$$\mathcal{P}_{\rm RR} = \frac{d\tilde{E}}{dt}, \quad \tilde{E} \equiv E - E_{\mathcal{S}}, \tag{4.76}$$

with E the mechanical energy of the system and $E_{\mathcal{S}}$ the Schott terms [101], which here are given by

$$E_{\mathcal{S}} = \frac{1}{5}\left(I^{ij(1)}I^{ij(4)} - I^{ij(2)}I^{ij(3)}\right) + \frac{16}{45}\left(J^{ij(1)}J^{ij(4)} - J^{ij(2)}J^{ij(3)}\right) + \cdots. \tag{4.77}$$

This equivalence can be proven to all orders.

4.4.2. Spin-orbit

Newton-Wigner SSC

Using (4.39), we start writing the power for the spin-orbit case in Newton-Wigner SSC:

$$\begin{aligned}\mathcal{P}^{\rm SO(NW)}_{\rm RR} = \frac{4m\nu}{15r^6}\Bigg\{&(\boldsymbol{L}\cdot\boldsymbol{S})\left[22\frac{m^2}{r} - 95mv^2 + 3r\dot{r}^2\left(77\frac{m}{r} - 270v^2\right) + 99rv^4 + 735r\dot{r}^4\right] \\ &-3\,(\boldsymbol{L}\cdot\boldsymbol{\xi})\left[4\frac{m^2}{r} - 25mv^2 + 4r\dot{r}^2\left(29\frac{m}{r} - 60v^2\right) + 21rv^4 + 315r\dot{r}^4\right]\Bigg\}.\end{aligned} \tag{4.78}$$

Comparing this with the result (3.46), we find

$$\left(\frac{dE}{dt}\right)_{\rm SO(NW)} - \mathcal{P}^{\rm SO}_{\rm RR} = \frac{4m\nu}{15r^6}\left\{(\boldsymbol{L}\cdot\boldsymbol{S})\left[2\frac{m^2}{r} + 169mv^2 + 15r\dot{r}^2\left(-19\frac{m}{r}+54v^2\right) - 99rv^4 - 735r\dot{r}^4\right]\right.$$
$$\left. + (\boldsymbol{L}\cdot\boldsymbol{\xi})\left[28\frac{m^2}{r} - 37mv^2 + 24r\dot{r}^2\left(13\frac{m}{r} - 30v^2\right) + 63rv^4 + 945r\dot{r}^4\right]\right\}. \tag{4.79}$$

It can be shown that the right-hand side of the expression above is a total derivative, after some algebra. Then, once the energy is redefined accounting the Schott terms,

$$E^{\rm SO(NW)}_{\mathcal{S}} = \frac{1}{5}\left[I^{ij(1)}_{(0)}I^{ij(4)}_{(0)} - I^{ij(2)}_{(0)}I^{ij(3)}_{(0)}\right]_S + m\nu\,(\boldsymbol{L}\cdot\boldsymbol{S})\left(\frac{88}{15}\frac{m\dot{r}}{r^5} - \frac{72}{5}\frac{v^2\dot{r}}{r^4} + 24\frac{\dot{r}^3}{r^4}\right) \tag{4.80}$$
$$+ m\nu\,(\boldsymbol{L}\cdot\boldsymbol{\xi})\left(-\frac{8}{3}\frac{m\dot{r}}{r^5} + \frac{129}{5}\frac{v^2\dot{r}}{r^4} - 39\frac{\dot{r}^3}{r^4}\right),$$

we may say that the power computed through radiation reaction and the power computed in the far zone are equal in face of this redefinition.

Covariant SSC

For the covariant case, the energy balance acquires an extra term due to the presence, in the binding energy, of $\boldsymbol{S}_{(0)}$, which cannot be regarded as constant in time as we do for the spatial components of the spin vector. Hence, at linear order in the spin we have

$$\left(\frac{dE}{dt}\right)_{\rm SO} = \left\langle\left[\frac{\partial E}{\partial \boldsymbol{r}^i}\boldsymbol{v}^i + \frac{\partial E}{\partial \boldsymbol{v}^i}\boldsymbol{a}^i + \sum_a \frac{\partial E}{\partial \boldsymbol{S}^i_{(0)a}}\dot{\boldsymbol{S}}^i_{(0)a}\right]\right\rangle$$
$$= m\nu\left\langle \boldsymbol{a}^{\rm RR}_{\rm SO(cov)}\cdot\boldsymbol{v}\right\rangle + \left\langle\sum_a \frac{\partial E}{\partial \boldsymbol{S}^i_{(0)a}}\dot{\boldsymbol{S}}^i_{(0)a}\right\rangle. \tag{4.81}$$

The covariant SSC preserves evolution, what implies, to the desired order,

$$\dot{\boldsymbol{S}}_{(0)a} = \boldsymbol{a}^{\rm RR}_{{\rm LO}a}\times\boldsymbol{S}_a, \tag{4.82}$$

where $\boldsymbol{a}^{\rm RR}_{{\rm LO}a}$ must be read as the 2.5PN acceleration due to radiation reaction.The leading order term for the spin-orbit energy is [45, 91, 102]

$$E_{\rm SO} = -\sum_a \boldsymbol{a}_{{\rm N}a}\cdot\boldsymbol{S}_{(0)a} + \cdots, \tag{4.83}$$

with $\boldsymbol{a}_{{\rm N}a}$ the Newtonian acceleration for each body, such that

$$\frac{\partial E_{\rm SO}}{\partial \boldsymbol{S}^i_{(0)a}} = -\boldsymbol{a}^i_{{\rm N}a} + \cdots. \tag{4.84}$$

Taking this into account, we have

$$\frac{\partial E}{\partial \boldsymbol{S}^i_{(0)a}}\dot{\boldsymbol{S}}^i_{(0)a} = -\sum_a \boldsymbol{a}_{\mathrm{N}a}\cdot\left(\boldsymbol{a}^{\mathrm{RR}}_{\mathrm{LO}a}\times\boldsymbol{S}_a\right) = -\frac{2\nu}{5}\int dt\epsilon^{ikl}\boldsymbol{a}^k_{\mathrm{N}}\boldsymbol{\xi}^l\boldsymbol{r}^j I^{ij(5)}_{(0)}. \tag{4.85}$$

Finally, to prove the equivalence between far zone and radiation-reaction computation, we have to show that

$$\left(\frac{d\tilde{E}}{dt}\right)_{\mathrm{SO(cov)}} = \left[m\nu\,\boldsymbol{a}^{\mathrm{SO(cov)}}_{\mathrm{RR}}\cdot\boldsymbol{v} - \frac{2\nu}{5}\epsilon^{ikl}\boldsymbol{a}^k_{\mathrm{N}}\boldsymbol{\xi}^l\boldsymbol{r}^j I^{ij(5)}_{(0)}\right], \tag{4.86}$$

For the right hand side, we have

$$\begin{aligned}\boldsymbol{a}^{\mathrm{SO(cov)}}_{\mathrm{RR}}\cdot\boldsymbol{v} = \frac{4}{15r^6}\Bigg\{&(\boldsymbol{L}\cdot\boldsymbol{S})\left[22\frac{m^2}{r} - 95mv^2 + 3r\dot{r}^2\left(77\frac{m}{r} - 270v^2\right) + 99rv^4 + 735r\dot{r}^4\right]\\ &+(\boldsymbol{L}\cdot\boldsymbol{\xi})\left[-14\frac{m^2}{r} - 37mv^2 - 3r\dot{r}^2\left(49\frac{m}{r} + 90v^2\right) + 45rv^4 + 105r\dot{r}^4\right]\Bigg\}\end{aligned} \tag{4.87}$$

and

$$\frac{2\nu}{5}\epsilon^{ikl}\boldsymbol{a}^k\boldsymbol{\xi}^l\boldsymbol{r}^j I^{ij(5)}_{(0)} = -\frac{8m^2\nu}{5r^6}(\boldsymbol{L}\cdot\boldsymbol{\xi})\left(2\frac{m}{r} - 6v^2 + 15\dot{r}^2\right). \tag{4.88}$$

Comparing with (3.45), we can show that the difference between the powers is a total time derivative, with the Schott term given by

$$\begin{aligned}E^{\mathrm{SO(cov)}}_{\mathcal{S}} = \frac{1}{5}\left[I^{ij(1)}_{(0)}I^{ij(4)}_{(0)} - I^{ij(2)}_{(0)}I^{ij(3)}_{(0)}\right]_S &+ m\nu(\boldsymbol{L}\cdot\boldsymbol{S})\left(\frac{88}{15}\frac{m\dot{r}}{r^5} - \frac{72}{5}\frac{v^2\dot{r}}{r^4} + 24\frac{\dot{r}^3}{r^4}\right)\\ &+ m\nu(\boldsymbol{L}\cdot\boldsymbol{\xi})\left(\frac{8}{5}\frac{m\dot{r}}{r^5} + 36\frac{v^2\dot{r}}{r^4} - 52\frac{\dot{r}^3}{r^4}\right),\end{aligned} \tag{4.89}$$

such that the expression in (4.86) is valid.

4.4.3. Spin-spin

From the radiation-reaction acceleration in the spin-spin case, we have

$$\mathcal{P}^{SS}_{\mathrm{RR}} \equiv \left(\boldsymbol{a}^{s_1s_2}_{\mathrm{RR}} + \boldsymbol{a}^{s^2}_{\mathrm{RR}}\right)\cdot\boldsymbol{v} = \mathcal{P}^{s_1s_2}_{\mathrm{RR}} + \mathcal{P}^{s^2}_{\mathrm{RR}}. \tag{4.90}$$

Using (4.64) and (4.65), we find

$$\begin{aligned}\mathcal{P}^{s_1s_2}_{\mathrm{RR}} = \frac{4m\nu}{15r^8}\Bigg\{&3r^2(\boldsymbol{S}_1\cdot\boldsymbol{S}_2)\left[40mv^2 + r\dot{r}^2\left(-26\frac{m}{r} + 420v^2\right) - 51rv^4 - 385r\dot{r}^4\right]\\ &+3(\boldsymbol{S}_1\cdot\boldsymbol{r})(\boldsymbol{S}_2\cdot\boldsymbol{r})\left[-153mv^2 + 28r\dot{r}^2\left(7\frac{m}{r} - 150v^2\right) + 345rv^4 + 5355r\dot{r}^4\right]\\ &-3r^2\dot{r}\left[(\boldsymbol{S}_1\cdot\boldsymbol{r})(\boldsymbol{S}_2\cdot\boldsymbol{v}) + (\boldsymbol{S}_1\cdot\boldsymbol{v})(\boldsymbol{S}_2\cdot\boldsymbol{r})\right]\left(9\frac{m}{r} - 1005v^2 + 1925\dot{r}^2\right)\\ &+r^3(\boldsymbol{S}_1\cdot\boldsymbol{v})(\boldsymbol{S}_2\cdot\boldsymbol{v})\left(19\frac{m}{r} - 486v^2 + 1890\dot{r}^2\right)\Bigg\}\end{aligned} \tag{4.91}$$

and

$$\begin{aligned}\mathcal{P}^{s^2}_{\rm RR} =& -\frac{2m^3\nu^2}{15r^8}\Bigg\{3r^2\left(\frac{\boldsymbol{S}_1^2}{m_1^2}+\frac{\boldsymbol{S}_2^2}{m_2^2}\right)\left[-8mv^2+30r\dot{r}^2\left(\frac{m}{r}-3v^2\right)+9rv^4+105r\dot{r}^4\right]\\&+15r^2\dot{r}\left(\frac{(\boldsymbol{S}_1\cdot\boldsymbol{r})(\boldsymbol{S}_1\cdot\boldsymbol{v})}{m_1^2}+\frac{(\boldsymbol{S}_2\cdot\boldsymbol{r})(\boldsymbol{S}_2\cdot\boldsymbol{v})}{m_2^2}\right)\left(2\frac{m}{r}-3v^2+7\dot{r}^2\right)\\&+r^3\left(\frac{(\boldsymbol{S}_1\cdot\boldsymbol{v})^2}{m_1^2}+\frac{(\boldsymbol{S}_2\cdot\boldsymbol{v})^2}{m_2^2}\right)\left(-8\frac{m}{r}+9v^2-45\dot{r}^2\right)\Bigg\}\\&-\frac{2}{5mr^8}\sum_{a\neq b}C^{(a)}_{ES^2}m_b^2\Bigg\{2r^2\boldsymbol{S}_a^2\left[-24mv^2+r\dot{r}^2\left(28\frac{m}{r}-255v^2\right)+30rv^4+245r\dot{r}^4\right]\\&+(\boldsymbol{S}_a\cdot\boldsymbol{r})^2\left[153mv^2-28r\dot{r}^2\left(7\frac{m}{r}-150v^2\right)+345rv^4-5355r\dot{r}^4\right]\\&+r^2\dot{r}(\boldsymbol{S}_a\cdot\boldsymbol{r})(\boldsymbol{S}_a\cdot\boldsymbol{v})\left(28\frac{m}{r}-2025v^2+3885\dot{r}^2\right)\\&-3r^3(\boldsymbol{S}_a\cdot\boldsymbol{v})^2\left(3\frac{m}{r}-55v^2+215\dot{r}^2\right)\Bigg\}.\end{aligned}\tag{4.92}$$

This allows us to write the difference between the powers in the far zone (3.51) and (3.52) and that due to the radiation reaction, as it is shown bellow:

$$\begin{aligned}\left(\frac{dE}{dt}\right)_{s_1s_2}-\mathcal{P}^{s_1s_2}_{\rm RR} =& \frac{4m\nu}{5r^8}\Bigg\{r^2(\boldsymbol{S}_1\cdot\boldsymbol{S}_2)\left[-87mv^2+r\dot{r}^2\left(81\frac{m}{r}-420v^2\right)+51rv^4+385r\dot{r}^4\right]\\&+(\boldsymbol{S}_1\cdot\boldsymbol{r})(\boldsymbol{S}_2\cdot\boldsymbol{r})\left[321mv^2+r\dot{r}^2\left(-465\frac{m}{r}+4200v^2\right)-345rv^4-5355r\dot{r}^4\right]\\&+r^2\dot{r}\left[(\boldsymbol{S}_1\cdot\boldsymbol{r})(\boldsymbol{S}_2\cdot\boldsymbol{v})+(\boldsymbol{S}_1\cdot\boldsymbol{v})(\boldsymbol{S}_2\cdot\boldsymbol{r})\right]\left(66\frac{m}{r}-1005v^2+1925\dot{r}^2\right)\\&+6r^3(\boldsymbol{S}_1\cdot\boldsymbol{v})(\boldsymbol{S}_2\cdot\boldsymbol{v})\left(-5\frac{m}{r}+27v^2-105\dot{r}^2\right)\Bigg\},\end{aligned}\tag{4.93}$$

and

$$\begin{aligned}\left(\frac{dE}{dt}\right)_{s^2}-\mathcal{P}^{s^2}_{\rm RR} =& -\frac{2m^3\nu^2}{5r^8}\Bigg\{-3r^2\left(\frac{\boldsymbol{S}_1^2}{m_1^2}+\frac{\boldsymbol{S}_2^2}{m_2^2}\right)\left[-3mv^2+r\dot{r}^2\left(9\frac{m}{r}-30v^2\right)+3rv^4+35r\dot{r}^4\right]\\&+3m\dot{r}^2\left(\frac{(\boldsymbol{S}_1\cdot\boldsymbol{r})^2}{m_1^2}+\frac{(\boldsymbol{S}_2\cdot\boldsymbol{r})^2}{m_2^2}\right)-r^2\dot{r}\left(\frac{(\boldsymbol{S}_1\cdot\boldsymbol{r})(\boldsymbol{S}_1\cdot\boldsymbol{v})}{m_1^2}+\frac{(\boldsymbol{S}_2\cdot\boldsymbol{r})(\boldsymbol{S}_2\cdot\boldsymbol{v})}{m_2^2}\right)\left(12\frac{m}{r}-15v^2+35\dot{r}^2\right)\\&+3r^3\left(\frac{(\boldsymbol{S}_1\cdot\boldsymbol{v})^2}{m_1^2}+\frac{(\boldsymbol{S}_2\cdot\boldsymbol{v})^2}{m_2^2}\right)\left(\frac{m}{r}-v^2+5\dot{r}^2\right)\Bigg\}\\&-\frac{2}{5mr^8}\sum_{a\neq b}C^{(a)}_{ES^2}m_b^2\Bigg\{-2r^2\boldsymbol{S}_a^2\left[-48mv^2+3r\dot{r}^2\left(18\frac{m}{r}-85v^2\right)+30rv^4+245r\dot{r}^4\right]\\&+3(\boldsymbol{S}_a\cdot\boldsymbol{r})^2\left[-107mv^2+4r\dot{r}^2\left(39\frac{m}{r}-350v^2\right)+115rv^4+1785r\dot{r}^4\right]\\&+3r^2\dot{r}(\boldsymbol{S}_a\cdot\boldsymbol{r})(\boldsymbol{S}_a\cdot\boldsymbol{v})\left(-48\frac{m}{r}+675v^2-1295\dot{r}^2\right)+3r^3(\boldsymbol{S}_a\cdot\boldsymbol{v})^2\left(11\frac{m}{r}-55v^2+215\dot{r}^2\right)\Bigg\}.\end{aligned}\tag{4.94}$$

The right-hand side of the last two equations are a total derivative, what allows us to affirm

$$\left(\frac{dE}{dt}\right)_{SS}=\left\langle\mathcal{P}^{SS}_{\rm RR}\right\rangle\,.\tag{4.95}$$

Similarly to the spin-orbit case, we can also introduce Schott terms: $\tilde{E} = E - E_{\mathcal{S}}$, with

$$E_{\mathcal{S}} = -\frac{1}{5}\left[\frac{d^2 I^{ij}_{(0)}}{dt^2}\frac{d^3 I^{ij}_{(0)}}{dt^3} - \frac{dI^{ij}_{(0)}}{dt}\frac{d^4 I^{ij}_{(0)}}{dt^4}\right]_{SS} + \frac{2\nu}{5}\Big[2\Big(\boldsymbol{S}_1\cdot\boldsymbol{S}_2\Big)\left(\boldsymbol{a}\cdot\dot{\boldsymbol{a}} - \boldsymbol{v}\cdot\ddot{\boldsymbol{a}}\right) \tag{4.96}$$

$$+\frac{1}{3}\Big(\left(\boldsymbol{S}_1\cdot\dot{\boldsymbol{a}}\right)\left(\boldsymbol{S}_2\cdot\boldsymbol{a}\right) + \left(\boldsymbol{S}_1\cdot\boldsymbol{a}\right)\left(\boldsymbol{S}_2\cdot\dot{\boldsymbol{a}}\right) - \left(\boldsymbol{S}_1\cdot\ddot{\boldsymbol{a}}\right)\left(\boldsymbol{S}_2\cdot\boldsymbol{v}\right) - \left(\boldsymbol{S}_1\cdot\boldsymbol{v}\right)\left(\boldsymbol{S}_2\cdot\ddot{\boldsymbol{a}}\right)\Big)\Big]$$

$$-\frac{2m^2\nu^2}{15}\Big[3\left(\frac{\boldsymbol{S}_1^2}{m_1^2} + \frac{\boldsymbol{S}_2^2}{m_2^2}\right)\left(\boldsymbol{a}\cdot\dot{\boldsymbol{a}} - \boldsymbol{v}\cdot\ddot{\boldsymbol{a}}\right) + \frac{\left(\boldsymbol{S}_1\cdot\boldsymbol{a}\right)\left(\boldsymbol{S}_1\cdot\dot{\boldsymbol{a}}\right)}{m_1^2} + \frac{\left(\boldsymbol{S}_2\cdot\boldsymbol{a}\right)\left(\boldsymbol{S}_2\cdot\dot{\boldsymbol{a}}\right)}{m_2^2}$$

$$-\frac{\left(\boldsymbol{S}_1\cdot\boldsymbol{v}\right)\left(\boldsymbol{S}_1\cdot\ddot{\boldsymbol{a}}\right)}{m_1^2} - \frac{\left(\boldsymbol{S}_2\cdot\boldsymbol{v}\right)\left(\boldsymbol{S}_2\cdot\ddot{\boldsymbol{a}}\right)}{m_2^2}\Big], \tag{4.97}$$

where the explicitly written acceleration terms reads as the Newtonian acceleration, whereas the leading order spin-spin equations of motion are used for the evaluation of the derivatives in the mass-quadrupole terms. At last,

$$\left(\frac{d\tilde{E}}{dt}\right)_{SS} = \mathcal{P}^{SS}_{\mathrm{RR}}\,, \tag{4.98}$$

as expected. We conclude that, as well as for the spin-orbit case, the results calculated for spin-spin effects in radiation reaction through a EFT framework are consistent.

5. Conclusion and perspectives

In this thesis, we have computed the leading order spin effects on radiation reaction of a binary system of compact bodies through an EFT approach. We focused on the inspiral phase of the binary system, where the separation of scales provides a natural scenario for the implementation of perturbative methods. We have shown that within the EFT framework developed in [38], and extendend in [45] in order to incorporate spin, it is possible to investigate the conservative and dissipative dynamics systematically. We focused on the radiation sector of the theory, which is constructed by performing a multipole expansion at the level of the action. As we have seen, after the potential modes of the gravitational fields are integrated out, the resulting theory describes radiation gravitons coupled to moments of the two-particles which compose the binary. The power counting allowed us to determine which multipole moments contribute to the computation of the power loss to the orders of interest.

To investigate radiation reaction within the EFT approach, we made use of the formalism developed in [74]. One contribution of our work is the extension of this formalism to incorporate spin effects. We have constructed the spin algebra in terms of the new set of Poisson Brackets by investigating how the angular momentum algebra changes, in light of the doubling of degrees of freedom. Then, we computed the spin-orbit contributions to the acceleration and spin evolution to 4PN order. We performed the calculations both in the covariant and Newton-Wigner SSCs, and proved the equivalence between different choices. We found that there is no net effect on the spin evolution from radiation reaction at this order, which is consistent with the findings in [84]. Subsequently, we computed the spin-spin contributions to the acceleration and spin evolution to 4.5PN order. Unlike the spin-orbit case, we found that the spin precesses due to spin-spin radiation reaction effects at this order. Our results are consistent with the findings in [103]. However, we went a step further by extending the computations to all spin squared terms, including finite-size effects. This was not the case in [84,103]. To our knowledge, the calculation of finite size effects in radiation reaction had not been carried out until now. Moreover, to test the consistency of our results for the spin-orbit as well as for spin-spin effects, we explicitly showed that the induced power resulting from the radiation-reaction force is equivalent to the total radiated emission computed in the far zone, using the standard multipole expansion [57].

The results presented in this thesis increase the knowledge for the modeling of spinning binary systems and the construction of accurate waveforms, aiding in the extraction of precise information from future gravitational wave observations. In order to construct accurate waveforms to 4.5PN order, much work still remains to be done. While up to 3PN all the ingredients necessary to the gravitational waveform were obtained, many other contributions are still needed up to 4.5PN, most notably radiated multipole moments for non-rotating and spinning bodies, besides radiation reaction effects. These calculations are presently undergoing within the EFT approach.

A. Einstein-Infeld-Hoffmann Lagrangian

Here we give an example to show how works the machinery of integrating out the potential gravitons in order to derive the Einstein-Infeld-Hoffman, which is the first PN correction to the Newtonian physics.

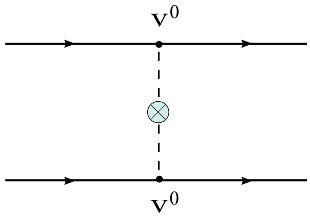


Figure A.1.

Firstly, let's calculate the contributions of second order in velocity coming the perturbation depending on the time derivatives of the potential graviton field in interaction action, as depicted in Fig.(A.1):

$$\begin{aligned} W_{[\partial_0 H \partial_0 H]} &= -\frac{i}{4}\int d^4x_1 d^4x_2 J^{\mu\nu}_{(v^0)}\left(x_1\right) J^{\alpha\beta}_{(v^0)}\left(x_2\right) \\ &\quad \left\langle T\left\{H_{\mu\nu}\left(x_1\right) H_{\alpha\beta}\left(x_2\right)\int d^4y T^{\rho\sigma\theta\gamma}\partial_0 H_{\rho\sigma}\left(y\right)\partial_0 H_{\theta\gamma}\left(y\right)\right\}\right\rangle . \end{aligned} \tag{A.1}$$

We have to write each of the two possible Wick's contractions for this diagram:

$$\begin{aligned} &\left\langle T\left\{H_{\mu\nu}\left(x_1\right) H_{\alpha\beta}\left(x_2\right)\int d^4y T^{\rho\sigma\theta\gamma}\partial_0 H_{\rho\sigma}\left(y\right)\partial_0 H_{\theta\gamma}\left(y\right)\right\}\right\rangle \\ = \;& T^{\rho\sigma\theta\gamma}\int d^4y \left\langle H_{\mu\nu}\left(x_1\right)\partial_0 H_{\rho\sigma}\left(y\right)\right\rangle \left\langle H_{\alpha\beta}\left(x_2\right)\partial_0 H_{\theta\gamma}\left(y\right)\right\rangle . \end{aligned} \tag{A.2}$$

We have, firstly,

$$\begin{aligned} \left\langle H_{\mu\nu}\left(x_1\right)\partial_0 H_{\rho\sigma}\left(y\right)\right\rangle &= \int \frac{d^3\mathbf{k}}{\left(2\pi\right)^3}\int \frac{d^3\mathbf{k}'}{\left(2\pi\right)^3} e^{i\mathbf{k}\cdot\mathbf{x}_1} e^{i\mathbf{k}'\cdot\mathbf{y}} \frac{d}{dt_y}\left\langle H_{\mu\nu}\left(\mathbf{k}, t_1\right) H_{\rho\sigma}\left(\mathbf{k}', t_y\right)\right\rangle \\ &= -i\left(2\pi\right)^3 P_{\mu\nu\rho\sigma}\int \frac{d^3\mathbf{k}}{\left(2\pi\right)^3}\int \frac{d^3\mathbf{k}'}{\left(2\pi\right)^3} e^{i\mathbf{k}\cdot\mathbf{x}_1} e^{i\mathbf{k}'\cdot\mathbf{y}} \frac{\delta^3\left(\mathbf{k}+\mathbf{k}'\right)}{\mathbf{k}^2}\frac{d}{dt_y}\delta\left(t_1 - t_y\right) . \end{aligned} \tag{A.3}$$

Similarly,

$$\langle H_{\alpha\beta}(x_2)\,\partial_0 H_{\theta\gamma}(y)\rangle = -i(2\pi)^3 P_{\alpha\beta\theta\gamma}\int\frac{d^3\mathbf{q}}{(2\pi)^3}\int\frac{d^3\mathbf{q}'}{(2\pi)^3}e^{i\mathbf{q}\cdot\mathbf{x}_2}e^{i\mathbf{q}'\cdot\mathbf{y}}\delta^3(\mathbf{q}+\mathbf{q}')\frac{1}{\mathbf{q}^2}\frac{d}{dt_y}\delta(t_2-t_y). \tag{A.4}$$

Having in mind the property of the Dirac delta

$$\frac{d}{dx}\delta(x-x') = \int\frac{d^dp}{(2\pi)^d}ipe^{ip(x-x')} = -\frac{d}{dx'}\delta(x-x'), \tag{A.5}$$

we use

$$\int dt_y\frac{d}{dt_y}\delta(t_1-t_y)\frac{d}{dt_y}\delta(t_2-t_y) = \frac{d}{dt_1}\frac{d}{dt_2}\delta(t_1-t_2) \tag{A.6}$$

to write

$$\begin{aligned}&\left\langle T\left\{H_{\mu\nu}(x_1)H_{\alpha\beta}(x_2)\int d^4yT^{\rho\sigma\theta\gamma}\partial_0 H_{\rho\sigma}(y)\,\partial_0 H_{\theta\gamma}(y)\right\}\right\rangle \\ &= -T^{\rho\sigma\theta\gamma}P_{\mu\nu\rho\sigma}P_{\alpha\beta\theta\gamma}\frac{d}{dt_1}\frac{d}{dt_2}\delta(t_1-t_2)\int d^3\mathbf{y}\int\frac{d^3\mathbf{k}}{(2\pi)^3}\int\frac{d^3\mathbf{q}}{(2\pi)^3}e^{i\mathbf{k}\cdot\mathbf{x}_1}e^{-i(\mathbf{k}+\mathbf{q})\cdot\mathbf{y}}e^{i\mathbf{q}\cdot\mathbf{x}_2}\frac{1}{\mathbf{k}^2\mathbf{q}^2}.\end{aligned} \tag{A.7}$$

Since

$$\begin{aligned}T^{\rho\sigma\theta\gamma}P_{\mu\nu\rho\sigma}P_{\alpha\beta\theta\gamma} &= \frac{1}{2}\left(\eta^{\rho\theta}\eta^{\sigma\gamma}+\eta^{\rho\gamma}\eta^{\sigma\theta}-\eta^{\rho\sigma}\eta^{\theta\gamma}\right)(\eta_{\mu\rho}\eta_{\nu\sigma}+\eta_{\mu\sigma}\eta_{\nu\rho}-\eta_{\mu\nu}\eta_{\rho\sigma})P_{\alpha\beta\theta\gamma} \\ &= \eta_{\mu\alpha}\eta_{\nu\beta}+\eta_{\mu\beta}\eta_{\nu\alpha}-\eta_{\mu\nu}\eta_{\alpha\beta} = 2P_{\mu\nu\alpha\beta},\end{aligned} \tag{A.8}$$

we have

$$\begin{aligned}&\left\langle T\left\{H_{\mu\nu}(x_1)H_{\alpha\beta}(x_2)\int d^4yT^{\rho\sigma\theta\gamma}\partial_0 H_{\rho\sigma}(y)\,\partial_0 H_{\theta\gamma}(y)\right\}\right\rangle \\ &= -2P_{\mu\nu\alpha\beta}\frac{d}{dt_1}\frac{d}{dt_2}\delta(t_1-t_2)\int\frac{d^3\mathbf{k}}{(2\pi)^3}\frac{e^{i\mathbf{k}\cdot(\mathbf{x}_1-\mathbf{x}_2)}}{\mathbf{k}^4},\end{aligned} \tag{A.9}$$

so that

$$W_{[\partial_0 H\partial_0 H]} = \frac{i}{2}\int d^4x_1d^4x_2J^{\mu\nu}_{(v^0)}(x_1)\,J^{\alpha\beta}_{(v^0)}(x_2)\,P_{\mu\nu\alpha\beta}\frac{d}{dt_1}\frac{d}{dt_2}\delta(t_1-t_2)\int\frac{d^3\mathbf{k}}{(2\pi)^3}\frac{e^{i\mathbf{k}\cdot(\mathbf{x}_1-\mathbf{x}_2)}}{\mathbf{k}^4}. \tag{A.10}$$

Writing the sources explicitly considering particle 1 at $J(x_1)$ and particle 2 at $J(x_2)$,

$$J^{\mu\nu}_{(v^0)}(x_1) = \frac{T^{\mu\nu}_{(v^0)}}{2m_{Pl}} = \frac{m_1}{2m_{Pl}}\eta^{\mu0}\eta^{\nu0}\delta\left(\mathbf{x}_1(t_1)-\mathbf{x}_{1(1)}(t_1)\right), \tag{A.11}$$

$$J^{\alpha\beta}_{(v^0)}(x_2) = \frac{T^{\alpha\beta}_{(v^0)}}{2m_{Pl}} = \frac{m_2}{2m_{Pl}}\eta^{\alpha0}\eta^{\beta0}\delta\left(\mathbf{x}_2(t_2)-\mathbf{x}_{2(2)}(t_2)\right), \tag{A.12}$$

we have

$$\begin{aligned}W_{[\partial_0 H\partial_0 H]} &= i\frac{m_1m_2}{8m_{Pl}^2}\int dt_1dt_2P_{0000}\frac{d}{dt_1}\frac{d}{dt_2}\delta(t_1-t_2)\int\frac{d^3\mathbf{k}}{(2\pi)^3}\frac{e^{i\mathbf{k}\cdot\left(\mathbf{x}_{1(1)}(t_1)-\mathbf{x}_{2(2)}(t_2)\right)}}{\mathbf{k}^4} \\ &= i\frac{m_1m_2}{16m_{Pl}^2}\int dt\mathbf{v}_{1i}\mathbf{v}_{2j}\frac{(r^2\delta_{ij}-\mathbf{r}^i\mathbf{r}^j)}{8\pi r^3},\end{aligned} \tag{A.13}$$

where $\mathbf{r} = \mathbf{x}_1 - \mathbf{x}_2$ and $r = |\mathbf{r}|$. Writing explicitly $m_{Pl}^{-2} = 32\pi G$, the result above can be written as

$$W_{[\partial_0 H \partial_0 H]} = \frac{i}{4} G m_1 m_2 \int dt \frac{\left[r^2 \mathbf{v}_1 \cdot \mathbf{v}_2 - \left(\mathbf{v}_1 \cdot \mathbf{r}_{12}\right)\left(\mathbf{v}_2 \cdot \mathbf{r}_{12}\right)\right]}{r^3}. \tag{A.14}$$

There are still other term identical to this one coming from the contribution of the particle 2 in $J\left(x_1\right)$ and particle 1 in $J\left(x_2\right)$, while the self-contributions are ignored because they become zero after dimensional regularization. Finally, we have

$$W_{[\partial_0 H \partial_0 H]} = \frac{i}{2} G m_1 m_2 \int dt \frac{\left[r_1^2 \mathbf{v} \cdot \mathbf{v}_2 - \left(\mathbf{v}_1 \cdot \mathbf{x}_{12}\right)\left(\mathbf{v}_2 \cdot \mathbf{x}_{12}\right)\right]}{r^3}. \tag{A.15}$$

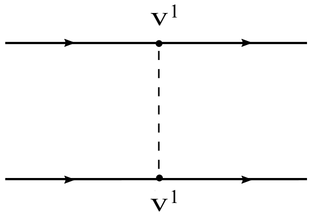


Figure A.2.

From Fig.(A.2), we have:

$$W_{[J(v^1)J(v^1)]} = -\frac{1}{2!} \int d^4x_1 d^4x_2 J^{\mu\nu}_{(v^1)}\left(x_1\right) J^{\alpha\beta}_{(v^1)}\left(x_2\right) \left\langle T\left\{H_{\mu\nu}\left(x_1\right) H_{\alpha\beta}\left(x_2\right)\right\}\right\rangle. \tag{A.16}$$

We consider particle 1 in the source $J\left(x_1\right)$ and particle 2 in the source $J\left(x_2\right)$, such that

$$J^{\mu\nu}_{(v^1)}\left(x_1\right) J^{\alpha\beta}_{(v^1)}\left(x_2\right) = \frac{m_1 m_2 \mathbf{v}_{i1} \mathbf{v}_{i2}}{m_{Pl}^2} \delta_i^\mu \eta^{\nu 0} \delta_i^\alpha \eta^{\beta 0} \delta\left(\mathbf{x}\left(t_1\right) - \mathbf{x}_{1(1)}\left(t_1\right)\right) \delta\left(\mathbf{x}_2\left(t_2\right) - \mathbf{x}_{2(2)}\left(t_2\right)\right). \tag{A.17}$$

Regarding

$$\left\langle T\left\{H_{\mu\nu}\left(x_1\right) H_{\alpha\beta}\left(x_2\right)\right\}\right\rangle = -i P_{\mu\nu\alpha\beta} \delta\left(t_1 - t_2\right) \frac{1}{4\pi r}, \tag{A.18}$$

and noticing that $P_{i0i0} = \frac{1}{2}$, we have

$$W_{[J(v^1)J(v^1)]} = i \frac{m_1 m_2}{16\pi m_{Pl}^2} \int dt \frac{\mathbf{v}_1 \cdot \mathbf{v}_2}{r} = i2 G m_1 m_2 \int dt \frac{\mathbf{v}_1 \cdot \mathbf{v}_2}{r}. \tag{A.19}$$

But regarding the contribution of the particle 2 in $J\left(x_1\right)$ and particle 1 in $J\left(x_2\right)$, the total result is

$$W_{[J(v^1)J(v^1)]} = i4 G m_1 m_2 \int dt \frac{\mathbf{v}_1 \cdot \mathbf{v}_2}{r}. \tag{A.20}$$

Following very similar steps to those from above, we find for Fig.(A.3)

$$W_{[J(v^2)J(v^0)]} = \frac{3i}{2} G m_1 m_2 \int dt \frac{\mathbf{v}_1^2}{r} + \frac{3i}{2} G m_1 m_2 \int dt \frac{\mathbf{v}_2^2}{r}. \tag{A.21}$$

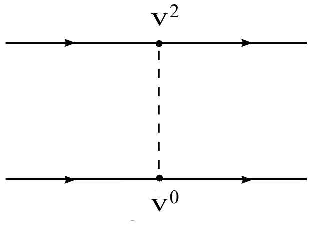


Figure A.3.

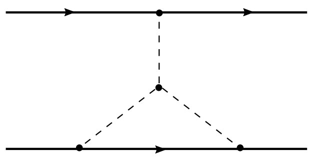


Figure A.4.

Using the three-graviton vertex (2.77), the diagram in Fig.(A.4) yields

$$W_{[(H_{00})^3]} = \frac{im_1 m_2^2}{16m_{Pl}^2} \int dt_1 dt_2 dt_3 \int \frac{d^3\mathbf{k}_1 d^3\mathbf{k}_2 d^3\mathbf{k}_3}{(2\pi)^9} e^{i\sum_i \mathbf{k}_i\cdot\mathbf{x}_i} \left\langle T\left\{H^{00}_{\mathbf{k}_1}(x_1)\, H^{00}_{\mathbf{k}_2}(x_2)\, H^{00}_{\mathbf{k}_3}(x_3)\right\}\right\rangle \tag{A.22}$$

$$= -i\int dt \frac{G^2 m_1 m_2^2}{r},$$

and taking into account the interchange of the particles, the final result is

$$W_{[(H_{00})^3]} = -i\int dt \frac{G^2 m_1 m_2 (m_1+m_2)}{r}. \tag{A.23}$$

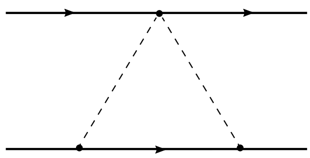


Figure A.5.

Finally, for the seagull diagram in Fig.(A.5), we have

$$W_{[H_{00}^2]} = \frac{i^3}{3!}\int d^4x_1 d^4x_2 d^4x_3 \left\langle T\left\{J^{\mu\nu}(x_1)\, J^{\alpha\beta}(x_2)\, J^{\rho\sigma}(x_3)\, H_{\mu\nu}(x_1)\, H_{\alpha\beta}(x_2)\, H_{\rho\sigma}(x_3)\right\}\right\rangle. \tag{A.24}$$

Considering the particle 1 at $J^{\mu\nu}(x_1)$ and the particle 2 at $J^{\alpha\beta}(x_2)$ and $J^{\rho\sigma}(x_3)$,

$$J^{\mu\nu}(x_1) = -\frac{m_1}{8m_{Pl}^2}\delta\left(\mathbf{x}_1 - \mathbf{x}_{(1)}\right)\eta^{\mu 0}\eta^{\nu 0}H_{00}(x_1), \tag{A.25}$$

$$J^{\alpha\beta}(x_2) = -\frac{m_2}{2m_{Pl}}\delta\left(\mathbf{x}_2 - \mathbf{x}_{(2)}\right)\eta^{\alpha 0}\eta^{\beta 0}, \tag{A.26}$$

$$J^{\rho\sigma}(x_3) = -\frac{m_2}{2m_{Pl}}\delta\left(\mathbf{x}_3 - \mathbf{x}_{(2)}\right)\eta^{\rho 0}\eta^{\sigma 0}, \tag{A.27}$$

this yields, concerning the symmetry factor 2 for this Wick contraction,

$$\begin{aligned} W_{[H_{00}^2]} &= -i\frac{m_1 m_2^2}{32m_{Pl}^4}\int dt_1 dt_2 dt_3 \left\langle T\left\{H_{00}(x_1)\,H_{00}(x_1)\,H_{00}(x_2)\,H_{00}(x_3)\right\}\right\rangle \\ &= i\frac{m_1 m_2^2}{32m_{Pl}^4}\frac{1}{64\pi^2}\int dt\frac{1}{r^2}, \end{aligned} \tag{A.28}$$

or

$$W[J]_{(S_{pp}\propto H_{oo}^2)} = i\frac{G^2 m_1 m_2^2}{2}\int dt\frac{1}{r^2}. \tag{A.29}$$

Putting together all the results of the diagrams, we end up with

$$L_{EIH} = \frac{1}{8}\sum_a m_a\mathbf{v}_a^4 + \frac{Gm_1m_2}{2r}\left[3\left(\mathbf{v}_1^2 + \mathbf{v}_2^2\right) - 7\left(\mathbf{v}_1\cdot\mathbf{v}_2\right) - \frac{\left(\mathbf{v}_1\cdot\mathbf{r}\right)\left(\mathbf{v}_2\cdot\mathbf{r}\right)}{r^2}\right] - \frac{G^2m_1m_2\left(m_1+m_2\right)}{2r^2}, \tag{A.30}$$

which is the well-known result obtained first by Einstein, Infeld and Hoffmann [104] using different methods.

B. Spin supplementary conditions

In this appendix, we provide a derivation of the leading order spin-orbit acceleration. Since it is a conservative effect, its computation throught the ordinary Euler-Lagrange equations is straightforward, but it can be taken as a good example on how to deal with the different supplementary conditions when deriving equations of motion. Moreover, we show how to link the result using the covariant SSC with the one using Newton-Wigner SSC throught a transformation of the centre of mass reference frame.

Covariant SSC

It is possible to use the covariant SSC without changing the canonical algebra if this condition is applied after the equations of motion are computed. In view of this, the spin-orbit potential has to be considered in its original form, without imposing any condition, i.e.

$$V_{SO} = m_2 \frac{\mathbf{r}^j}{r^3}\left[S_1^{j0} + S_1^{jk}\frac{(m_2+2m_1)}{m}\mathbf{v}^k\right] - m_1\frac{\mathbf{r}^j}{r^3}\left[S_2^{j0} - S_2^{jk}\frac{(m_1+2m_2)}{m}\mathbf{v}^k\right]. \tag{B.1}$$

which yields

$$\frac{\partial V_{SO}}{\partial \mathbf{r}_a} = \frac{(\delta^{aj} - 3\mathbf{n}^j\mathbf{n}^a)}{r^3}\left\{m_2\left[S_1^{j0} + S_1^{jk}\frac{(m_2+2m_1)}{m}\mathbf{v}^k\right] - m_1\left[S_2^{j0} - S_2^{jk}\frac{(m_1+2m_2)}{m}\mathbf{v}^k\right]\right\}, \tag{B.2}$$

$$\frac{d}{dt}\left(\frac{\partial V_{SO}}{\partial \mathbf{v}_a}\right) = \frac{m_2}{r^3}\left(\mathbf{v}^j - 3\mathbf{n}^j\dot{r}\right)\left[S_1^{ja}\frac{(m_2+2m_1)}{m}\right] + \frac{m_1}{r^3}\left(\mathbf{v}^j - 3\mathbf{n}^j\dot{r}\right)\left[S_2^{ja}\frac{(m_1+2m_2)}{m}\right]. \tag{B.3}$$

Putting them together into

$$m\nu\mathbf{a}^a_{SO} = -\frac{\partial V_{SO}}{\partial \mathbf{r}^a} + \frac{d}{dt}\left(\frac{\partial V_{SO}}{\partial \mathbf{v}^a}\right), \tag{B.4}$$

it results

$$\begin{aligned} m\nu\mathbf{a}^a_{SO} = &-\frac{1}{r^3}\left[m_2 S_1^{a0} + S_1^{ak}\frac{m_2(m_2+2m_1)}{m}\mathbf{v}^k - m_1 S_2^{a0} + S_2^{ak}\frac{m_1(m_1+2m_2)}{m}\mathbf{v}^k\right] + \\ &+ 3\frac{\mathbf{n}^j\mathbf{n}^a}{r^3}\left[m_2 S_1^{j0} + S_1^{jk}\frac{m_2(m_2+2m_1)}{m}\mathbf{v}^k - m_1 S_2^{j0} + S_2^{jk}\frac{m_1(m_1+2m_2)}{m}\mathbf{v}^k\right] + \\ &+ \frac{(\mathbf{v}^j - 3\mathbf{n}^j\dot{r})}{r^3}\left[S_1^{ja}\frac{m_2(m_2+2m_1)}{m} + S_2^{ja}\frac{m_1(m_1+2m_2)}{m}\right]. \end{aligned} \tag{B.5}$$

Now that the Euler-Lagrange equation has been computed, we can use the covariant SSC, i.e. $S_1^{j0} = \epsilon^{jkl}\mathbf{S}_1^l\mathbf{v}_1^k$ and $S_1^{jk} = \epsilon^{jkl}\mathbf{S}_1^l$ (with the same for particle 2), such that

$$m\nu\mathbf{a}_{SO}^a = -\frac{\epsilon^{akl}\mathbf{v}^k}{mr^3}\left[\left(m_2^2 + 2m_2\left(m_2+2m_1\right)\right)\mathbf{S}_1^l + \left(m_1^2 + 2m_1\left(m_1+2m_2\right)\right)\mathbf{S}_2^l\right] + \tag{B.6}$$
$$+ 3\frac{\mathbf{n}^j\mathbf{n}^a\epsilon^{jkl}\mathbf{v}^k}{mr^3}\left[\left(m_2^2 + m_2\left(m_2+2m_1\right)\right)\mathbf{S}_1^l + \left(m_1^2 + m_1\left(m_1+2m_2\right)\right)\mathbf{S}_2^l\right] +$$
$$+ \frac{3\mathbf{n}^j\dot{r}\epsilon^{ajl}}{mr^3}\left[m_2\left(m_2+2m_1\right)\mathbf{S}_1^l + m_1\left(m_1+2m_2\right)\mathbf{S}_2^l\right],$$

but since

$$m_2\left(m_2+2m_1\right) = m^2\nu\left(\frac{m_2}{m_1}+2\right), \tag{B.7}$$
$$m_1\left(m_1+2m_2\right) = m^2\nu\left(\frac{m_1}{m_2}+2\right), \tag{B.8}$$

we get

$$\mathbf{a}_{SO}^{(cov)a} = -\frac{\epsilon^{akl}\mathbf{v}^k}{r^3}\left[3\xi^l + 4\mathbf{S}^l\right] + \frac{6\epsilon^{jkl}\mathbf{n}^a\mathbf{n}^j\mathbf{v}^k}{r^3}\left[\xi^l + \mathbf{S}^l\right] + \frac{3\dot{r}\epsilon^{ajl}\mathbf{n}^j}{r^3}\left[\xi^l + 2\mathbf{S}^l\right], \tag{B.9}$$

which can be equivalently written as

$$\mathbf{a}_{SO}^{cov} = \frac{1}{r^5}\left[6\mathbf{r}\left(\mathbf{r}\times\mathbf{v}\right)\cdot\left(2\mathbf{S} + \frac{\delta m}{m}\mathbf{\Sigma}\right) - r^2\left(\mathbf{v}\times\left(7\mathbf{S} + 3\frac{\delta m}{m}\mathbf{\Sigma}\right)\right) + 3\left(\mathbf{r}\cdot\mathbf{v}\right)\left(\mathbf{r}\times\left(3\mathbf{S} + \frac{\delta m}{m}\mathbf{\Sigma}\right)\right)\right]. \tag{B.10}$$

Newton-Wigner SSC

For the case of Newton-Wigner SSC, which keeps the canonical algebra unchanged up to terms quadratic in the spin, the condition $S_1^{j0} = \frac{1}{2}\epsilon^{jkl}\mathbf{S}_1^l\mathbf{v}_1^k$ (same for particle 2) has be imposed at the level of the potential:

$$V_{SO} = m_2\frac{\epsilon^{jkl}\mathbf{r}^j}{r^3}\left[\frac{1}{2}\mathbf{S}_1^l\mathbf{v}_1^k + \mathbf{S}_1^l\frac{\left(m_2+2m_1\right)}{m}\mathbf{v}^k\right] - m_1\frac{\epsilon^{jkl}\mathbf{r}^j}{r^3}\left[\frac{1}{2}\mathbf{S}_2^l\mathbf{v}_2^k - \mathbf{S}_2^l\frac{\left(m_1+2m_2\right)}{m}\mathbf{v}^k\right]$$
$$= \frac{\epsilon^{jkl}}{m}\frac{\mathbf{r}^j}{r^3}\frac{3}{2}\left[m_2^2\mathbf{S}_1^l + m_1^2\mathbf{S}_2^l\right]\mathbf{v}^k + 2m\nu\frac{\mathbf{r}^j}{r^3}\epsilon^{jkl}\mathbf{v}^k\mathbf{S}^l. \tag{B.11}$$

Hence,

$$\frac{\partial V_{SO}}{\partial \mathbf{r}_a} = \frac{\epsilon^{jkl}\left(\delta^{aj} - 3\mathbf{n}^j\mathbf{n}^a\right)}{mr^3}\frac{3}{2}\left[m_2^2\mathbf{S}_1^l + m_1^2\mathbf{S}_2^l\right]\mathbf{v}^k + 2m\nu\frac{\left(\delta^{aj} - 3\mathbf{n}^j\mathbf{n}^a\right)}{r^3}\epsilon^{jkl}\mathbf{v}^k\mathbf{S}^l, \tag{B.12}$$

$$\frac{d}{dt}\left(\frac{\partial V_{SO}}{\partial \mathbf{v}_a}\right) = -\frac{1}{r^3}\left(\mathbf{v}^j - 3\mathbf{n}^j\dot{r}\right)\left\{\frac{\epsilon^{ajl}}{m}\frac{3}{2}\left[m_2^2\mathbf{S}_1^l + m_1^2\mathbf{S}_2^l\right] - 2m\nu\epsilon^{ajl}\mathbf{S}^l\right\}, \tag{B.13}$$

such that

$$\mathbf{a}_{SO}^{(NW)a} = 3\mathbf{n}^a\frac{\epsilon^{jkl}\mathbf{n}^j\mathbf{v}^k}{r^3}\left[\frac{3}{2}\left(\frac{m_2}{m_1}\mathbf{S}_1^l + \frac{m_1}{m_2}\mathbf{S}_2^l\right) + 2\mathbf{S}^l\right] - \frac{2\epsilon^{akl}\mathbf{v}^k}{r^3}\left[\frac{3}{2}\left(\frac{m_2}{m_1}\mathbf{S}_1^l + \frac{m_1}{m_2}\mathbf{S}_2^l\right) - 2\mathbf{S}^l\right] +$$
$$+ \frac{3}{r^3}\dot{r}\epsilon^{ajl}\mathbf{n}^j\left[\frac{3}{2}\left(\frac{m_2}{m_1}\mathbf{S}_1^l + \frac{m_1}{m_2}\mathbf{S}_2^l\right) - 2\mathbf{S}^l\right], \tag{B.14}$$

or

$$\boldsymbol{a}_{SO}^{NW} = \frac{1}{r^3}\left\{\frac{3}{2}\boldsymbol{n}\left[(\boldsymbol{n}\times\boldsymbol{v})\cdot\left(7\boldsymbol{S}+3\frac{\delta m}{m}\boldsymbol{\Sigma}\right)\right] - \boldsymbol{v}\times\left(7\boldsymbol{S}+3\frac{\delta m}{m}\boldsymbol{\Sigma}\right) + \frac{3}{2}\dot{r}\left[\boldsymbol{n}\times\left(7\boldsymbol{S}+3\frac{\delta m}{m}\boldsymbol{\Sigma}\right)\right]\right\}. \tag{B.15}$$

Transformation

One goes from covariant SSC to NW SSC making the coordinate transformation in the relative position:

$$\mathbf{r}\to\mathbf{r}-\frac{(\mathbf{v}\times\xi)}{2m}, \tag{B.16}$$

which implies

$$\mathbf{v}\to\mathbf{v}+\frac{(\mathbf{r}\times\xi)}{2r^3}, \tag{B.17}$$

$$\mathbf{a}\to\mathbf{a}+\frac{(\mathbf{v}\times\xi)}{2r^3}-\frac{3}{2}\frac{\dot{r}\,(\mathbf{r}\times\xi)}{r^3}, \tag{B.18}$$

$$\dot{r}\to\dot{r}+\frac{\dot{r}}{2m^2\nu}\left(\mathbf{L}\cdot\xi\right), \tag{B.19}$$

$$\frac{1}{r}\to\frac{1}{r}+\frac{(\mathbf{L}\cdot\xi)}{2m^2\nu r^3}, \tag{B.20}$$

$$\frac{1}{r^3}\to\frac{1}{r^3}+\frac{3}{2m^2\nu r^5}\left(\mathbf{L}\cdot\xi\right). \tag{B.21}$$

We can test this taking the spin-orbit acceleration at covariant SSC and try to go to the spin-orbit acceleration at NW SSC. The transformation must be regarded in the Newtonian acceleration, since it gives the following 1.5PN contribution:

$$\mathbf{a}_0 = -\frac{m\mathbf{r}}{r^3}\to-\frac{m\mathbf{r}}{r^3}+\frac{(\mathbf{v}\times\xi)}{2r^3}-\frac{3}{2}\frac{\mathbf{r}\,(\mathbf{L}\cdot\xi)}{m\nu r^5}. \tag{B.22}$$

The general equation of motion reads

$$\mathbf{a}=\mathbf{a}_0+\mathbf{a}_1+\mathbf{a}_{1.5}+... \tag{B.23}$$

Thus, when the transformation is performed at both sides of this equation, there will be new terms at 1.5PN order:

$$\mathbf{a}+\frac{(\mathbf{v}\times\xi)}{2r^3}-\frac{3}{2}\frac{\dot{r}\,(\mathbf{r}\times\xi)}{r^3}=\mathbf{a}_0+\frac{(\mathbf{v}\times\xi)}{2r^3}-\frac{3}{2}\frac{\mathbf{r}\,(\mathbf{L}\cdot\xi)}{m\nu r^5}+\mathbf{a}_1+\mathbf{a}_{1.5}+..., \tag{B.24}$$

where $\mathbf{a}_{1.5}$ in this case is the spin-orbit acceleration at the covariant SSC, which is

$$\mathbf{a}_{SO}^{(cov)} = \frac{1}{r^5}\left[6\mathbf{r}\,(\mathbf{r}\times\mathbf{v})\cdot\left(2\mathbf{S}+\frac{\delta m}{m}\boldsymbol{\Sigma}\right) - r^2\left(\mathbf{v}\times\left(7\mathbf{S}+3\frac{\delta m}{m}\boldsymbol{\Sigma}\right)\right) + 3\,(\mathbf{r}\cdot\mathbf{v})\left(\mathbf{r}\times\left(3\mathbf{S}+\frac{\delta m}{m}\boldsymbol{\Sigma}\right)\right)\right]. \tag{B.25}$$

Then, the spin-orbit acceleration at NW SSC turns out to be

$$\mathbf{a}_{SO}^{NW} = \frac{3}{2}\frac{\dot{r}\left(\mathbf{r}\times\xi\right)}{r^3} - \frac{3}{2}\frac{\mathbf{r}\left(\mathbf{L}\cdot\xi\right)}{m\nu r^5} + \mathbf{a}_{SO}^{(cov)}, \tag{B.26}$$

which reproduces (2.102), as expected.

Similarly, but without concerning supplementary conditios, the spin1-spin2 and spin-spin accelerations can be derived from (2.99) and (2.100), yielding

$$\mathbf{a}_{S_1S_2} = -\frac{3}{m\nu r^4}\left[\mathbf{n}\left(\mathbf{S}_1\cdot\mathbf{S}_2\right) - 5\mathbf{n}\left(\mathbf{S}_1\cdot\mathbf{n}\right)\left(\mathbf{S}_2\cdot\mathbf{n}\right) + \mathbf{S}_1\left(\mathbf{S}_2\cdot\mathbf{n}\right) + \mathbf{S}_2\left(\mathbf{S}_1\cdot\mathbf{n}\right)\right], and$$

C. Non-conservative dynamics

Hamilton's principle of stationary action plays an important role in Physics, since it is a straight way to derive the equations of motion for a large variety of systems which share distinct levels of complexity. In view of its efficiency for describing the evolution of a physical system, this principle has been proven readly useful in the development of fundamental areas, such as Mechanics and Field Theory. An action principle offers the benefits of evidencing the set of symmetries of a given system, since the action might be invariant under those transformations, what is essencial to the understanding of the conserved quantities. It is also straighforward to introduce interactions and extra degrees of freedom by adding them through desired energy terms into the action. Yet, at the level of the action, problems involving constraints are handled in a natural way using, for example, Dirac's formalism.

However, based on a Lagrangian or Hamiltonian formulation, Hamilton's principle cannot deal with non-conservative but conservative processes. The subtlety that makes Hamilton's principle unsuitable for non-conservative systems is that the evolution and final configuration of such systems must be dictated by the initial conditions, whereas such a principle is formulated with boundary conditions. Working on this point, Galley [74] presented a formulation for Mechanics which is well suited for the treatment of generic conservative or non-conservative systems. This formalism was further developed in [75] to accomadate classical field theories and the treatment of thermodynamical and electromagnetic processes.

We summarize in this appendix the aspects of the generalized mechanics developed by Galley which we will make use in this work. For a more complete presentation, see [74] and [75].

Lagrangian formulation

There is a way to treat generic non-conservative systems properly by doubling the number of degrees of freedom: $\mathbf{q} \to (\mathbf{q}_1, \mathbf{q}_2)$ and $\dot{\mathbf{q}} \to (\dot{\mathbf{q}}_1, \dot{\mathbf{q}}_2)$. The new Lagrangian

$$\Lambda\left(\mathbf{q}_a, \dot{\mathbf{q}}_a\right) \equiv L\left(\mathbf{q}_1, \dot{\mathbf{q}}_1\right) - L\left(\mathbf{q}_2, \dot{\mathbf{q}}_2\right) + K\left(\mathbf{q}_a, \dot{\mathbf{q}}_a\right), \tag{C.1}$$

accounts for the conservative effects, given by the usual Lagrangian L, and the dissipative effects represented by K. The index $a = 1, 2$ can also represent $a = +, -$, respecting the relations

$$\mathbf{q}_{+} = \frac{\mathbf{q}_1 + \mathbf{q}_2}{2}, \tag{C.2}$$

$$\mathbf{q}_{-} = \mathbf{q}_1 - \mathbf{q}_2. \tag{C.3}$$

The equation of motion in this case is

$$\frac{d\pi_{\mp}}{dt} = \frac{\partial \Lambda}{\partial \mathbf{q}_{\pm}}, \tag{C.4}$$

but in the physical limit, $(\mathbf{q}_{-}, \dot{\mathbf{q}}_{-}) \to 0$ and $(\mathbf{q}_{+}, \dot{\mathbf{q}}_{+}) \to (\mathbf{q}, \dot{\mathbf{q}})$, it becomes

$$\frac{d\pi}{dt} = \left[\frac{\partial \Lambda}{\partial \mathbf{q}_{-}}\right]_{p.l.}, \tag{C.5}$$

where

$$\pi = \left[\frac{\partial \Lambda}{\partial \dot{\mathbf{q}}_{-}}\right]_{p.l.} = \frac{\partial L}{\partial \dot{\mathbf{q}}} + \left[\frac{\partial K}{\partial \dot{\mathbf{q}}_{-}}\right]_{p.l.}. \tag{C.6}$$

In the physical limit, only the Euler-Lagrange equation for the "+" variable survives. Hence, the variational principle becomes

$$\left[\frac{\delta S\left[\mathbf{q}_{\pm}\right]}{\delta \mathbf{q}_{-}}\right]_{p.l.} = 0. \tag{C.7}$$

Hamiltonian formulation

Similarly, we can define a new Hamiltonian

$$A\left(\mathbf{q}_{\pm}, \mathbf{p}_{\pm}\right) \equiv \mathbf{p}_{+} \cdot \dot{\mathbf{q}}_{-} + \mathbf{p}_{-} \cdot \dot{\mathbf{q}}_{+} - \Lambda\left(\mathbf{q}_{\pm}, \dot{\mathbf{q}}_{\pm}\right) \tag{C.8}$$

and consider a new Poisson Brackets (PB)

$$\{\{f, g\}\} \equiv \frac{\partial f}{\partial \mathbf{q}_{+}} \frac{\partial g}{\partial \mathbf{p}_{-}} - \frac{\partial f}{\partial \mathbf{p}_{-}} \frac{\partial g}{\partial \mathbf{q}_{+}} + \frac{\partial f}{\partial \mathbf{q}_{-}} \frac{\partial g}{\partial \mathbf{p}_{+}} - \frac{\partial f}{\partial \mathbf{p}_{+}} \frac{\partial g}{\partial \mathbf{q}_{-}}, \tag{C.9}$$

where $\mathbf{p}$ is the 'usual' conjugate momentum

$$\mathbf{p}_{\mp} = \frac{\partial L}{\partial \dot{\mathbf{q}}_{\pm}}. \tag{C.10}$$

Hamilton's equations, in the physical limit, become

$$\dot{\mathbf{q}} = \{\mathbf{q}, H\} - \left[\{\{\mathbf{q}_{+}, K\}\}\right]_{p.l.}, \tag{C.11}$$

$$\dot{\mathbf{p}} = \{\mathbf{p}, H\} - \left[\{\{\mathbf{p}_{+}, K\}\}\right]_{p.l.}. \tag{C.12}$$

The momentum $\mathbf{p}$ is not quite usual, because $\mathbf{p}_-$ is conjugate to $\mathbf{q}_+$ while $\mathbf{p}_+$ is conjugate to $\mathbf{q}_-$. This is an important point. With this new PB, new canonical relations follow as a consequence:

$$\left\{\left\{\mathbf{q}_+^i, \mathbf{q}_+^j\right\}\right\} = \left\{\left\{\mathbf{q}_-^i, \mathbf{q}_-^j\right\}\right\} = \left\{\left\{\mathbf{p}_+^i, \mathbf{p}_+^j\right\}\right\} = \left\{\left\{\mathbf{p}_-^i, \mathbf{p}_-^j\right\}\right\} = 0, \tag{C.13}$$

$$\left\{\left\{\mathbf{q}_+^i, \mathbf{q}_-^j\right\}\right\} = \left\{\left\{\mathbf{p}_+^i, \mathbf{p}_-^j\right\}\right\} = 0, \tag{C.14}$$

$$\left\{\left\{\mathbf{q}_+^i, \mathbf{p}_-^j\right\}\right\} = \left\{\left\{\mathbf{q}_-^i, \mathbf{p}_+^j\right\}\right\} = \delta^{ij}. \tag{C.15}$$

Harmonic Oscillator

Let us consider a harmonic oscillator to demonstrate the problem which arises from Hamilton's principle and its solution using the formalism presented in the last section. The action for a harmonic oscillator with amplitude $q\,(t)$, mass m and frequency ω coupled with strengh λ to another harmonic oscillator is given bellow:

$$S\,[q,Q] = \int_{t_i}^{t_f} dt \left[\frac{m}{2}\left(\dot{q}^2 - \omega^2 q^2\right) + \lambda q Q + \frac{M}{2}\left(\dot{Q}^2 - \Omega^2 Q^2\right)\right]. \tag{C.16}$$

The oscillator $q\,(t)$ is open to exchange energy with $Q\,(t)$ and consequently should be described by a non-conservative dynamics. Let us account for the effect of the latter oscillator with the former by integrating out $Q\,(t)$, what leads to the effective action

$$S_{eff}\,[q] = \int_{t_i}^{t_f} dt \left[\frac{m}{2}\left(\dot{q}^2 - \omega^2 q^2\right) + \lambda q Q^{(h)} + \frac{\lambda^2}{2M}\int_{t_i}^{t_f} dt' q\,(t)\, G_{ret}\,(t-t')\, q\,(t')\right], \tag{C.17}$$

where $Q^{(h)}$ is a homogeneous solution. The last term involves two integrals and the product $q\,(t)\, q\,(t')$, which is symmetric in $t \leftrightarrow t'$ and couples only to the time-symmetric part of the retarded Green function, what allows us to re-write it as

$$\frac{\lambda^2}{2M}\int_{t_i}^{t_f} dt dt' q\,(t) \left[\frac{G_{ret}\,(t-t') + G_{adv}\,(t-t')}{2}\right] q\,(t')\,. \tag{C.18}$$

Applying Hamilton's principle to (C.17) results in the equation of motion for $q\,(t)$:

$$m\ddot{q} + m\omega^2 q = \lambda Q^{(h)} + \frac{\lambda^2}{2M}\int_{t_i}^{t_f} dt' \left[\frac{G_{ret}\,(t-t') + G_{adv}\,(t-t')}{2}\right] q\,(t')\,. \tag{C.19}$$

The second term on the right side depends on the advanced Green function implying that solutions to the equation of motion above do not evolve causally nor are specified by initial data alone. In addition, the kernel of the integral in (C.19) is symmetric in time, which means that the integral describes conservative interactions between the oscillators. This tells us that (**??**) does not account for dissipation (a time-asymmetric process).

Now let us the formalism presented in last section to see if it get rid off this problem. Assume initial conditions $q(t_i) = q_i$, $\dot{q}(t_i) = v_i$, $Q(t_i) = Q_i$ and $\dot{Q}(t_i) = V_i$. The total system is closed implying that the Lagrangian has no dissipative part $K = 0$. Doubling the degrees of freedom and considering the initial conditions, we get the solutions

$$Q_+(t) = Q^{(h)}(t) + \frac{\lambda}{M} \int_{t_i}^{t_f} dt' G_{ret}(t - t') q_+(t'), \tag{C.20}$$

$$Q_-(t) = \frac{\lambda}{M} \int_{t_i}^{t_f} dt' G_{adv}(t - t') q_-(t'), \tag{C.21}$$

where $Q^{(h)}(t) = Q_i \cos\Omega(t - t_i) + V_i \sin\Omega(t - t_i)$. The "$+$" variable evolves foward in time and satisfies the initial conditions while the "$-$" variable evolves backward in time because of the equality condition at the final time. This is a general feature of the $\pm$ variables.

Substituting these solutions in (C.16) results

$$S_{eff}[q_\pm] = \int_{t_i}^{t_f} dt \left[m\left(\dot{q}_+\dot{q}_- - \omega^2 q_+ q_-\right) + \lambda q_- Q^{(h)} + \frac{\lambda^2}{M} \int_{t_i}^{t_f} dt' q_+(t) G_{ret}(t - t') q_+(t') \right]. \tag{C.22}$$

The quantity $q_+(t) q_+(t')$ is asymmetric in time and couples to the full retarded Green function. Applying (C.7) to (C.22) gives the equation of motion

$$m\ddot{q} + m\omega^2 q = \lambda Q^{(h)} + \frac{\lambda^2}{2M} \int_{t_i}^{t_f} dt' G_{ret}(t - t') q(t'). \tag{C.23}$$

We now see that the solutions to this equation of motion evolve causually from initial data, as it is desired.

D. Radiation reaction in Classical Electrodynamics

We provide a brief review on radiation reaction in Classical Electrodynamics with a historical background [105] and shortly discuss the appearance of the Schott terms, which is regarded when we test the consistency of the main results present in this thesis at the last chapter.

In 1897, after the discovery of electron by Thomson and Lorentz described the forces which act on the electron when it is accelerated:

$$m\mathbf{v} = \mathbf{F}_L + \frac{2}{3}e^2\dot{\mathbf{a}}, \tag{D.1}$$

where $\mathbf{F}_L = e\left(\mathbf{E} + \mathbf{v} \times \mathbf{B}\right)$. In that year, Larmor provided the formula for the radiation rate of an accelerated electron, which was generalized to the relativistic case by Heaviside

$$\frac{dE_R}{d\tau} = \frac{2}{3}e^2 a^\alpha a_\alpha, \tag{D.2}$$

In the former case, the derivatives are taken with respect to time t; in the latter case, they are taken with respect to the proper time τ. Further progress was made by Abraham, in 1903, when he derived the rate of momentum carried away from the charge by radiation, which relativistic generalization is given by

$$\frac{dp^\mu}{d\tau} = \frac{2}{3}e^2 a^\alpha a_\alpha v^\mu. \tag{D.3}$$

The motion of charged particles in external force fields necessarily involves the emission of radiation whenever the charges are accelerated. The emitted radiation carries energy, linear and angular momenta and so must influence the subsequent motion of the charged particles. Consequently, the motion of the sources of radiation is partially determined by the manner radiation is emitted.

An accelerated charged particle loses energy and momentum emmiting electromagnetic waves. Besides, the electromagnetic field of the particle also acts as a recoil force to the particle's moviment. These two facts are kown by the name of *radiation reaction*. As mentioned above, this phenomenum has been studied since the end of XIX century and until nowadays it finds place in the scientific research.

The radiation reaction is observed in special situations which satisfy an energetic criterion, i.e., for the radiation reaction be neglected is that the radiated energy be very small in comparison to the particle's kinetic energy.

a) A charge that was at rest is accelerated by an external force during a time interval T. Comparing the energy radiated away (Larmor formula) with the kinetic energy acquired during the acceleration, we find that the radiaction reaction can be neglected if

$$T \gg \frac{2}{3}\frac{e^2}{mc^3}. \tag{D.4}$$

If the charge is an electron, using CGS units for instance, we have $T \gg 7.10^{-24}s$. In this time, electromagnetic waves travel $10^{-15}m$. Then, the radiation reaction is expected to be detected if the external force acts on the particle during a time shorter than $10^{-24}s$.

b) A charged particle in periodic motion gives the same result as above.

We must observe that the electronic Compton wavelenght is of order $10^{-15}m$. This size gives the "limit" between classical and quantum effects. Then, time intervals shorter than $10^{-24}s$ involves distance even smaller than the Compton wave lenght of the eletron, what means, in my opinion, that the radiation reaction for the eletron should be treated only in a quantum approach.

A lots of works were done treating the radiation reaction for extended charged particles (taking the limit of its radius to zero) in a classical approach. One of the most well succeeded works on this subject was done by Dirac in 1938. He did a relativistic analysis of the worldline of an electron surrounded by a tube whose surface he used to calculate the flow of energy passing through it. Proceeding this way, he wrote the equation of motion for an accelerated electron, known as the Abraham-Lorentz-Dirac (ALD) equation:

$$m\dot{v}_\mu - \frac{2}{3}e^2\ddot{v}_\mu - \frac{2}{3}e^2\dot{\mathbf{v}}^2 v_\mu = ev_\nu F_\mu{}^\nu, \tag{D.5}$$

where $F_{\mu\nu}$ is the field that accelerates the electron. We can make some comments on the equation above:

- The first term is related to the kinetic energy of a particle with rest mass m.
- The second one is known as the *Schott term* and it comes from the recoil force on the charged particle generated by its own electromagnetic fields. A detailed analysis of the Schott term origin is given at [106].

- The last term at the left hand side is the *radiation term*, i.e., a reaction force due to the energy emmited by the charge in form of electromagnetic waves.

Bibliography

[1] B. P. Abbott et al. Gw170104: Observation of a 50-solar-mass binary black hole coalescence at redshift 0.2. *Phys. Rev. Lett.*, 118:221101, Jun 2017.

[2] J. N. Taylor, A. Wolszczan, and T. Damour. Experimental Constraints on Strong Field Relativistic Gravity. *Nature*, 355:132, 1993.

[3] B. P. Abbott et al. GW150914: The Advanced LIGO Detectors in the Era of First Discoveries. *Phys. Rev. Lett.*, 116(13):131103, 2016.

[4] F. Acernese et al. Advanced Virgo: a second-generation interferometric gravitational wave detector. *Class. Quant. Grav.*, 32(2):024001, 2015.

[5] Kentaro Somiya. "Detector configuration of KAGRA: The Japanese cryogenic gravitational-wave detector". *Class. Quant. Grav.*, 29:124007, 2012.

[6] C. S. Unnikrishnan. IndIGO and LIGO-India: Scope and plans for gravitational wave research and precision metrology in India. *Int. J. Mod. Phys.*, D22:1341010, 2013.

[7] M. Punturo et al. "The Einstein Telescope: A third-generation gravitational wave observatory". *Class. Quant. Grav.*, 27:194002, 2010.

[8] Pau Amaro-Seoane et al. "eLISA/NGO: Astrophysics and cosmology in the gravitational-wave millihertz regime". *GW Notes*, 6:4–110, 2013.

[9] H. Sana, S. E. de Mink, A. de Koter, N. Langer, C. J. Evans, M. Gieles, E. Gosset, R. G. Izzard, J.-B. Le Bouquin, and F. R. N. Schneider.

[10] Luis Lehner and Frans Pretorius. Numerical Relativity and Astrophysics. *Ann. Rev. Astron. Astrophys.*, 52:661–694, 2014.

[11] G. B. Cook et al. Boosted three-dimensional black hole evolutions with singularity excision. *Phys. Rev. Lett.*, 80:2512–2516, 1998.

[12] Sascha Husa et al. “Frequency-domain gravitational waves from non-precessing black-hole binaries. I. New numerical waveforms and anatomy of the signal". *Phys. Rev.*, D93(4):044006, 2016.

[13] Sebastian Khan et al. “Frequency-domain gravitational waves from non-precessing black-hole binaries. II. A phenomenological model for the advanced detector era". *Phys. Rev.*, D93(4):044007, 2016.

[14] T. Ohta, H. Okamura, K. Hiida, and T. Kimura. “Higher order gravitational potential for many-body system". *Prog. Theor. Phys.*, 51:1220–1238, 1974.

[15] A. Maheshwari, E. R. Nissimov, and I. T. Todorov. “Classical and Quantum Two-body Problem in General Relativity". *Lett. Math. Phys.*, 5:359, 1981.

[16] Thibault Damour and Gerhard Schaefer. “Lagrangians forn point masses at the second post-Newtonian approximation of general relativity". *Gen. Rel. Grav.*, 17:879–905, 1985.

[17] Thibault Damour and Gilles Esposito-Farese. “Tensor multiscalar theories of gravitation". *Class. Quant. Grav.*, 9:2093–2176, 1992.

[18] Thibault Damour and Gilles Esposito-Farese. “Testing gravity to second postNewtonian order: A Field theory approach". *Phys. Rev.*, D53:5541–5578, 1996.

[19] W.E. Caswell and G.P. Lepage. “Effective Lagrangians for Bound State Problems in QED, QCD, and Other Field Theories". *Phys.Lett.*, B167:437, 1986.

[20] Michael E. Luke, Aneesh V. Manohar, and Ira Z. Rothstein. “Renormalization group scaling in nonrelativistic QCD". *Phys. Rev.*, D61:074025, 2000.

[21] Aneesh V. Manohar and Mark B. Wise. “Heavy Quark Physics". *Camb. Monogr. Part. Phys. Nucl. Phys. Cosmol.*, 10:1–191, 2000.

[22] Suraj N. Gupta. “Gravitation and Electromagnetism". *Phys. Rev.*, 96:1683–1685, 1954.

[23] Robert H. Kraichnan. “Special-Relativistic Derivation of Generally Covariant Gravitation Theory". *Phys. Rev.*, 98:1118–1122, 1955.

[24] R. P. Feynman. “Quantum theory of gravitation". *Acta Phys. Polon.*, 24:697–722, 1963.

[25] Steven Weinberg. “Photons and Gravitons in s Matrix Theory: Derivation of Charge Conservation and Equality of Gravitational and Inertial Mass". *Phys. Rev.*, 135:B1049–B1056, 1964.

[26] Steven Weinberg. “Infrared photons and gravitons". *Phys. Rev.*, 140:B516–B524, 1965.

[27] Y. Iwasaki. “Fourth-order gravitational potential based on quantum field theory". *Lett. Nuovo Cim.*, 1S2:783–786, 1971. [Lett. Nuovo Cim.1,783(1971)].

[28] Y. Iwasaki. Quantum theory of gravitation vs. classical theory. - fourth-order potential. *Prog. Theor. Phys.*, 46:1587–1609, 1971.

[29] K. Hiida and H. Okamura. “Gauge transformation and gravitational potentials". *Prog. Theor. Phys.*, 47:1743–1757, 1972.

[30] M. J. Duff. “Quantum Tree Graphs and the Schwarzschild Solution". *Phys. Rev.*, D7:2317–2326, 1973.

[31] Walter D. Goldberger and Ira Z. Rothstein. “Dissipative effects in the worldline approach to black hole dynamics". *Phys.Rev.*, D73:104030, 2006.

[32] Rafael A. Porto. Absorption effects due to spin in the worldline approach to black hole dynamics. *Phys. Rev.*, D77:064026, 2008.

[33] Walter D. Goldberger and Andreas Ross. “Gravitational radiative corrections from effective field theory". *Phys. Rev.*, D81:124015, 2010.

[34] Andreas Ross. “Multipole expansion at the level of the action". *Phys. Rev.*, D85:125033, 2012.

[35] Walter D. Goldberger, Andreas Ross, and Ira Z. Rothstein. “Black hole mass dynamics and renormalization group evolution". *Phys. Rev.*, D89(12):124033, 2014.

[36] Chad R. Galley and Manuel Tiglio. “Radiation reaction and gravitational waves in the effective field theory approach". *Phys. Rev.*, D79:124027, 2009.

[37] Chad R. Galley and Manuel Tiglio. Radiation reaction and gravitational waves in the effective field theory approach. *Phys. Rev.*, D79:124027, 2009.

[38] Walter D. Goldberger and Ira Z. Rothstein. "An Effective field theory of gravity for extended objects". *Phys.Rev.*, D73:104029, 2006.

[39] James B. Gilmore and Andreas Ross. "Effective field theory calculation of second post-Newtonian binary dynamics". *Phys. Rev.*, D78:124021, 2008.

[40] Stefano Foffa and Riccardo Sturani. "Effective field theory calculation of conservative binary dynamics at third post-Newtonian order". *Phys. Rev.*, D84:044031, 2011.

[41] Stefano Foffa and Riccardo Sturani. "Dynamics of the gravitational two-body problem at fourth post-Newtonian order and at quadratic order in the Newton constant". *Phys. Rev.*, D87(6):064011, 2013.

[42] Chad R. Galley, Adam K. Leibovich, Rafael A. Porto, and Andreas Ross. "The tail effect in gravitational radiation-reaction: time non-locality and renormalization group evolution". *to appear in Phys. Rev. D*, 2015.

[43] Stefano Foffa, Pierpaolo Mastrolia, Riccardo Sturani, and Christian Sturm. Effective field theory approach to the gravitational two-body dynamics, at fourth post-Newtonian order and quintic in the Newton constant. 2016.

[44] S. Foffa and Riccardo Sturani. "Tail terms in gravitational radiation reaction via effective field theory". *Phys. Rev.*, D87(4):044056, 2013.

[45] Rafael A. Porto. Post-Newtonian corrections to the motion of spinning bodies in NRGR. *Phys.Rev.*, D73:104031, 2006.

[46] Rafael A. Porto. *"An effective field theory of gravity for spinning extended objects"*. PhD thesis, Carnegie Mellon U., 2007. http://search.proquest.com/docview/304883966.

[47] A.J Hanson and T Regge. The relativistic spherical top. *Annals of Physics*, 87(2):498 – 566, 1974.

[48] Rafael A. Porto and Ira Z. Rothstein. "The Hyperfine Einstein-Infeld-Hoffmann potential". *Phys.Rev.Lett.*, 97:021101, 2006.

[49] Rafael A. Porto. "New results at 3PN via an effective field theory of gravity". In *Proceedings, 11th Marcel Grossmann Meeting, MG11, Berlin, Germany, July 23-29, 2006. Pt. A-C*, pages 2493–2496, 2007.

[50] Rafael A. Porto and Ira Z. Rothstein. "Comment on 'On the next-to-leading order gravitational spin(1)-spin(2) dynamics' by J. Steinhoff et al." . 2007.

[51] Rafael A. Porto and Ira Z. Rothstein. "Spin(1)Spin(2) Effects in the Motion of Inspiralling Compact Binaries at Third Order in the Post-Newtonian Expansion". *Phys.Rev.*, D78:044012, 2008.

[52] Rafael A Porto and Ira Z. Rothstein. "Next to Leading Order Spin(1)Spin(1) Effects in the Motion of Inspiralling Compact Binaries". *Phys.Rev.*, D78:044013, 2008.

[53] Jan Steinhoff, Steven Hergt, and Gerhard Schaefer. "On the next-to-leading order gravitational spin(1)-spin(2) dynamics". *Phys.Rev.*, D77:081501, 2008.

[54] Steven Hergt and Gerhard Schaefer. "Higher-order-in-spin interaction Hamiltonians for binary black holes from Poincare invariance". *Phys. Rev.*, D78:124004, 2008.

[55] Steven Hergt, Jan Steinhoff, and Gerhard Schaefer. "Reduced Hamiltonian for next-to-leading order Spin-Squared Dynamics of General Compact Binaries". *Class. Quant. Grav.*, 27:135007, 2010.

[56] Jan Steinhoff, Gerhard Schaefer, and Steven Hergt. "ADM canonical formalism for gravitating spinning objects". *Phys. Rev.*, D77:104018, 2008.

[57] Luc Blanchet. "Gravitational Radiation from Post-Newtonian Sources and Inspiralling Compact Binaries". *Living Rev.Rel.*, 17:2, 2014.

[58] Michele Levi. "Binary dynamics from spin1-spin2 coupling at fourth post-Newtonian order". *Phys.Rev.*, D85:064043, 2012.

[59] Michele Levi and Jan Steinhoff. "Next-to-next-to-leading order gravitational spin-orbit coupling via the effective field theory for spinning objects in the post-Newtonian scheme". *JCAP*, 1601:011, 2016.

[60] Michele Levi and Jan Steinhoff. Next-to-next-to-leading order gravitational spin-squared potential via the effective field theory for spinning objects in the post-Newtonian scheme. *JCAP*, 1601:008, 2016.

[61] Michele Levi and Jan Steinhoff. Equivalence of ADM Hamiltonian and Effective Field Theory approaches at next-to-next-to-leading order spin1-spin2 coupling of binary inspirals. *JCAP*, 1412(12):003, 2014.

[62] Johannes Hartung, Jan Steinhoff, and Gerhard Schaefer. Next-to-next-to-leading order post-Newtonian linear-in-spin binary Hamiltonians. *Annalen Phys.*, 525:359–394, 2013.

[63] Sylvain Marsat, Alejandro Bohe, Guillaume Faye, and Luc Blanchet. Next-to-next-to-leading order spin-orbit effects in the equations of motion of compact binary systems. *Class. Quant. Grav.*, 30:055007, 2013.

[64] Johannes Hartung and Jan Steinhoff. Next-to-next-to-leading order post-Newtonian spin(1)-spin(2) Hamiltonian for self-gravitating binaries. *Annalen Phys.*, 523:919–924, 2011.

[65] Alejandro Bohe, Sylvain Marsat, Guillaume Faye, and Luc Blanchet. Next-to-next-to-leading order spin-orbit effects in the near-zone metric and precession equations of compact binaries. *Class. Quant. Grav.*, 30:075017, 2013.

[66] Chad R. Galley and B.L. Hu. Self-force on extreme mass ratio inspirals via curved spacetime effective field theory. *Phys. Rev.*, D79:064002, 2009.

[67] Julian S. Schwinger. Brownian motion of a quantum oscillator. *J. Math. Phys.*, 2:407–432, 1961.

[68] L. V. Keldysh. Diagram technique for nonequilibrium processes. *Zh. Eksp. Teor. Fiz.*, 47:1515–1527, 1964. [Sov. Phys. JETP20,1018(1965)].

[69] Bala R. Iyer and C. M. Will. PostNewtonian gravitational radiation reaction for two-body systems. *Phys. Rev. Lett.*, 70:113–116, 1993.

[70] L. Blanchet. Time asymmetric structure of gravitational radiation. *Phys. Rev.*, D47:4392–4420, 1993.

[71] Bala R. Iyer and C. M. Will. PostNewtonian gravitational radiation reaction for two-body systems: Nonspinning bodies. *Phys. Rev.*, D52:6882–6893, 1995.

[72] Luc Blanchet. Energy losses by gravitational radiation in inspiraling compact binaries to five halves postNewtonian order. *Phys. Rev.*, D54:1417–1438, 1996. [Erratum: Phys. Rev.D71,129904(2005)].

[73] Chad R. Galley and Adam K. Leibovich. Radiation reaction at 3.5 post-Newtonian order in effective field theory. *Phys. Rev.*, D86:044029, 2012.

[74] Chad R. Galley. Classical Mechanics of Nonconservative Systems. *Phys. Rev. Lett.*, 110(17):174301, 2013.

[75] Chad R. Galley, David Tsang, and Leo C. Stein. The principle of stationary nonconservative action for classical mechanics and field theories. 2014.

[76] Natália T. Maia, Chad R. Galley, Adam K. Leibovich, and Rafael A. Porto. Radiation reaction for spinning bodies in effective field theory I: Spin-orbit effects. *submitted to Phys. Rev. D.*, 2017.

[77] Natália T. Maia, Chad R. Galley, Adam K. Leibovich, and Rafael A. Porto. Radiation reaction for spinning bodies in effective field theory II: Spin-spin effects. *submitted to Phys. Rev. D.*, 2017.

[78] P. L. Bender et al. Lisa: A cornerstone mission for the observation of gravitational waves. *System and Technology Study Report ESA-SCI 11*, 2000.

[79] K. Riles. Gravitational Waves: Sources, Detectors and Searches. *Prog. Part. Nucl. Phys.*, 68:1–54, 2013.

[80] B. Barker and R. O'Connell. Lisa: A cornerstone mission for the observation of gravitational waves. *Gen. Rel. Grav.*, 5, 1975.

[81] Kip S. Thorne and James B. Hartle. Laws of motion and precession for black holes and other bodies. *Phys. Rev. D*, 31:1815–1837, Apr 1985.

[82] T. Damour and G. Schäeer. Higher-order relativistic periastron advances and binary pulsars. *Il Nuovo Cimento B (1971-1996)*, 101(2):127–176, 1988.

[83] Lawrence E. Kidder, Clifford M. Will, and Alan G. Wiseman. "Spin effects in the inspiral of coalescing compact binaries". *Phys. Rev.*, D47:4183–4187, 1993.

[84] Clifford M. Will. Post-newtonian gravitational radiation and equations of motion via direct integration of the relaxed einstein equations. iii. radiation reaction for binary systems with spinning bodies. *Phys. Rev. D*, 71:084027, Apr 2005.

[85] Lawrence E. Kidder. "Coalescing binary systems of compact objects to postNewtonian 5/2 order. 5. Spin effects". *Phys. Rev.*, D52:821–847, 1995.

[86] H. T. Cho. PostNewtonian approximation for spinning particles. *Class. Quant. Grav.*, 15:2465–2478, 1998.

[87] Hideyuki Tagoshi, Akira Ohashi, and Benjamin J. Owen. “Gravitational field and equations of motion of spinning compact binaries to 2.5 postNewtonian order". *Phys. Rev.*, D63:044006, 2001.

[88] Walter D. Goldberger and Ira Z. Rothstein. “Towers of Gravitational Theories". *Gen. Rel. Grav.*, 38:1537–1546, 2006. [Int. J. Mod. Phys.D15,2293(2006)].

[89] Eric Poisson. “Tidal deformation of a slowly rotating black hole". *Phys. Rev.*, D91(4):044004, 2015.

[90] Solomon Endlich and Riccardo Penco. “An effective field theory approach to tidal dynamics of astrophysical systems". *Phys. Rev.*, D93(6):064021, 2016.

[91] Rafael A. Porto. The effective field theorist's approach to gravitational dynamics. *Phys. Rept.*, 633:1–104, 2016.

[92] T. Damour and Bala R. Iyer. “Multipole analysis for electromagnetism and linearized gravity with irreducible cartesian tensors". *Phys. Rev.*, D43:3259–3272, 1991.

[93] Luc Blanchet. On the multipole expansion of the gravitational field. *Class. Quant. Grav.*, 15:1971–1999, 1998.

[94] K. S. Thorne. “Multipole Expansions of Gravitational Radiation". *Rev. Mod. Phys.*, 52:299–339, 1980.

[95] Benjamin Grinstein and Ira Z. Rothstein. “Effective field theory and matching in nonrelativistic gauge theories". *Phys. Rev.*, D57:78–82, 1998.

[96] Ira Z. Rothstein. “TASI lectures on effective field theories" . 2003.

[97] Walter D. Goldberger. “Les Houches lectures on effective field theories and gravitational radiation". In *Les Houches Summer School - Session 86: Particle Physics and Cosmology: The Fabric of Spacetime Les Houches, France, July 31-August 25, 2006*, 2007.

[98] Rafael A. Porto, Andreas Ross, and Ira Z. Rothstein. “Spin induced multipole moments for the gravitational wave flux from binary inspirals to third Post-Newtonian order". *JCAP*, 1103:009, 2011.

[99] K. G. Arun, Alessandra Buonanno, Guillaume Faye, and Evan Ochsner. Higher-order spin effects in the amplitude and phase of gravitational waveforms emitted by inspiraling compact binaries: Ready-to-use gravitational waveforms. *Phys. Rev.*, D79:104023, 2009. [Erratum: Phys. Rev.D84,049901(2011)].

[100] Rafael A. Porto, Andreas Ross, and Ira Z. Rothstein. “Spin induced multipole moments for the gravitational wave amplitude from binary inspirals to 2.5 Post-Newtonian order". *JCAP*, 1209:028, 2012.

[101] Chad R. Galley, Adam K. Leibovich, Rafael A. Porto, and Andreas Ross. Tail effect in gravitational radiation reaction: Time nonlocality and renormalization group evolution. *Phys. Rev.*, D93:124010, 2016.

[102] Rafael A. Porto. “Next to leading order spin-orbit effects in the motion of inspiralling compact binaries". *Class.Quant.Grav.*, 27:205001, 2010.

[103] Han Wang and Clifford M. Will. Post-Newtonian gravitational radiation and equations of motion via direct integration of the relaxed Einstein equations. IV. Radiation reaction for binary systems with spin-spin coupling. *Phys. Rev.*, D75:064017, 2007.

[104] Albert Einstein, L. Infeld, and B. Hoffmann. The Gravitational equations and the problem of motion. *Annals Math.*, 39:65–100, 1938.

[105] F. Rohrlich. The dynamics of a charged sphere and the electron. *American Journal of Physics*, 65(11):1051–1056, 1997.

[106] Dmitri V. Gal’tsov and Pavel Spirin. Radiation reaction reexamined: Bound momentum and Schott term. *Grav. Cosmol.*, 12:1–10, 2006.