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**Efimov effect in  $D$  spatial dimensions in  $AAB$  systems**D. S. Rosa,<sup>1</sup> T. Frederico,<sup>2</sup> G. Krein,<sup>1</sup> and M. T. Yamashita<sup>1</sup><sup>1</sup>*Instituto de Física Teórica, Universidade Estadual Paulista, Rua Dr. Bento Teobaldo Ferraz, 271 - Bloco II, 01140-070 São Paulo, SP, Brazil*<sup>2</sup>*Instituto Tecnológico de Aeronáutica, 12228-900 São José dos Campos, SP, Brazil*

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The existence of the Efimov effect is drastically affected by the dimensionality of the space in which the system is embedded. The effective spatial dimension containing an atomic cloud can be continuously modified by compressing it in one or two directions. In the present Rapid Communication we determine the dimensionality  $D$  for which the Efimov effect can exist for different values of the mass ratio  $\mathcal{A} = m_B/m_A$  for a general  $AAB$  system formed by two identical bosons  $A$  and a third particle  $B$  in the two-body unitary limit. In addition, we provide a prediction for the Efimov discrete scaling factor  $\exp(\pi/s)$  as a function of a wide range of values of  $\mathcal{A}$  and  $D$ , which can be tested in experiments that can be realized with currently available technology.

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**Introduction.** The Efimov effect [1] appears over an incredibly large variety of systems and spans a wide range of scales—atoms, molecules, exotic nuclei, condensed matter systems, and even the DNA—turning into a quite rich research area in physics [2]. Such a surprising phenomenon that can occur in a system of three particles with finite-range forces was discovered by Efimov in 1970 within the nuclear physics context, by showing that the system can have an infinite sequence of geometrically spaced energy levels when pairs of the three particles have an infinite two-body scattering length. Such a spectrum is closely related to the collapse of the three-body binding energy, discovered by Thomas in 1935 [3], by decreasing the range of the interaction with respect to its scattering length.

The Thomas collapse and the Efimov effect are intimately related to the dimension where the system is inserted. Both Thomas and Efimov considered only a three-dimensional environment ( $D = 3$ ). However, the flexibility of manipulating ultracold atomic traps brought the study of few-body systems to a new era: The Efimov effect was experimentally confirmed [4] after almost 40 years since the original prediction; the use of the Feshbach resonance technique allows a fine tuning of the two-body interactions [5]; and the possibility to compress and expand the atomic cloud by changing the lasers and magnetic fields can create effectively two- and one-dimensional situations [6].

Despite the many advances in theory and experiment, there is an almost unexplored issue: the effect of a continuous changing of the spatial dimension on the different observables. For integer dimensions, we know that the Efimov effect does not exist in  $D = 2$  or  $D = 1$ . More precisely, it has been demonstrated for identical bosons that the Efimov effect survives for  $2.3 < D < 3.8$  [7]. However, the possibilities in engineering atomic traps with heteronuclear species raises the question on how the spatial dimensionality affects mass-imbalanced three-body systems. Such systems provide a more favorable situation for the experimental investigation of the Efimov effect [8–15]. Notwithstanding the fact that an atomic cloud compressed in one or two directions—when it acquires,

respectively, a pancake or cigar shape—is rigorously always a  $D = 3$  system, once the cloud size in one direction starts to become much smaller than the other, the system effectively starts to be affected by the dimensional reduction. This means that at some point, the excited three-body bound Efimov states start disappearing one by one, reflecting that the system is entering into a  $D = 2$  regime.

In a broader context the continuous change in dimension affects the transition from few to many-body physics in cold atomic traps, as was discussed for Fermi gases in arbitrary dimensions [16], impurities forming polaritons within mass-imbalanced mixtures [17,18], and now also with the feasibility of creating Efimov molecules from a resonantly interacting Bose gas [19]. The change in dimension would also certainly affect systems formed by four mass-imbalanced atoms [20]. An experiment designed to investigate the response of a heteronuclear three-atom  $AAB$  system to the change of dimensionality in a cold atomic trap close to a Feshbach resonance is lacking.

In this Rapid Communication we consider mass-imbalanced  $AAB$  systems formed by two identical bosons ( $A$ ) and a different particle ( $B$ ) interacting by a zero-range pairwise interaction. We start from the Skorniakov and Ter-Martirosyan (STM) equations [21] for the bound state of a system of three particles interacting pairwise through zero-range interactions. We demonstrate that the STM coupled integral equations generalized to  $D$  dimensions become scale invariant for large relative momenta  $q$ , so that the solutions are homogeneous functions of  $q$  and have a log-periodic behavior for large  $q$ , a characteristic feature of the wave functions of the Efimov states. For three identical particles, our result for the critical interval of  $D$  values where the Efimov state may be found agrees with that reported in Ref. [7].

Our solutions, for a wide range of values of mass ratio  $\mathcal{A} = m_B/m_A$ , provide predictions for the discrete Efimov scaling factor  $\exp(\pi/s)$ , which relates the three-body energies of consecutive excited states,  $E_3^{(N)}/E_3^{(N+1)} = \exp(2\pi/s)$ , where  $N = 0, 1, 2, \dots$  identifies the  $N$ th Efimov state, with  $N = 0$  being the ground state. Furthermore, the critical values of  $D$  where these bound states appear are determined. We show how

the asymmetric compression of the cloud can be associated with fractional dimensions.

*Methodology.* The Efimov effect is intimately related to the concept of universality. In the universal regime, the details of the short-range part of the potential are not important for the low-energy properties of a system. This peculiar situation can be achieved once the two-body scattering length  $a$  is much larger than the range  $r_0$  of the potential. Then, a natural way to treat this regime is to choose a zero-range potential. Our calculation of the critical values of  $D$ , for which the Efimov effect ceases to exist, is based on two key points of the method proposed by Danilov [22] to find the solution of STM integral equation. The first key point is the

dominance of the large momentum region for the solution of the STM equation due to the Thomas collapse. For a large momentum, the finite values of the two- and three-body energies are irrelevant when compared to the kinetic energies and can then be set to zero. The second key point is the emergence of invariance under a scale transformation of the STM integral equations in the limit of large momentum. These two points are enough for our purposes in the present work.

The set of coupled integral equations for the spectator functions,  $\chi_A(q)$  and  $\chi_B(q)$ , which generalizes the set of STM equations to  $D$  dimensions, reads (for simplicity, we take  $\hbar = m_A = 1$ )

$$\chi_A(q) = \tau_{AB} \left( E_3 - \frac{\mathcal{A}+2}{2(\mathcal{A}+1)} q^2 \right) \int d^D k \left( \frac{\chi_B(k)}{E_3 - q^2 - \frac{\mathcal{A}+1}{2\mathcal{A}} k^2 - \mathbf{k} \cdot \mathbf{q}} + \frac{\chi_A(k)}{E_3 - \frac{\mathcal{A}+1}{2\mathcal{A}} (k^2 + q^2) - \frac{1}{\mathcal{A}} \mathbf{k} \cdot \mathbf{q}} \right), \quad (1)$$

$$\chi_B(q) = 2\tau_{AA} \left( E_3 - \frac{\mathcal{A}+2}{4\mathcal{A}} q^2 \right) \int d^D k \frac{\chi_A(k)}{E_3 - \frac{\mathcal{A}+1}{2\mathcal{A}} q^2 - k^2 - \mathbf{k} \cdot \mathbf{q}}, \quad (2)$$

where  $E_3$  is the energy of the three-body system. The relative Jacobi momenta  $\mathbf{q}$  and  $\mathbf{k}$  are defined such that their origin is the center of mass of a given pair and point towards the remaining particle. The two-body transition amplitudes  $\tau_{AB}$  and  $\tau_{AA}$  are given by

$$\tau_{AB}^{-1} \left( E_3 - \frac{\mathcal{A}+2}{2(\mathcal{A}+1)} q^2 \right) = \int d^D k \left( \frac{1}{-|E_2^{AB}| - \frac{\mathcal{A}+1}{2\mathcal{A}} k^2} - \frac{1}{E_3 - \frac{\mathcal{A}+2}{2(\mathcal{A}+1)} q^2 - \frac{\mathcal{A}+1}{2\mathcal{A}} k^2} \right), \quad (3)$$

$$\tau_{AA}^{-1} \left( E_3 - \frac{\mathcal{A}+2}{4\mathcal{A}} q^2 \right) = \int d^D k \left( \frac{1}{-|E_2^{AA}| - k^2} - \frac{1}{E_3 - \frac{\mathcal{A}+2}{4\mathcal{A}} q^2 - k^2} \right), \quad (4)$$

where  $E_2^{AB}$  and  $E_2^{AA}$  are the two-body energies of the bound  $AB$  and  $AA$  systems, respectively. For large values of  $q$ , relevant for exploring the Efimov effect, the integrals in Eqs. (3) and (4) are determined by the region of large values of  $k$  [22] and, therefore, the energies  $E_3$ ,  $E_2^{AA}$ , and  $E_2^{AB}$  can be set to zero in those integrals—this is the first point noted by Danilov. In this situation, one can obtain closed forms for the amplitudes  $\tau_{AB}$  and  $\tau_{AA}$ .

$$\tau_{AB}^{-1} \left( -\frac{\mathcal{A}+2}{2(\mathcal{A}+1)} q^2 \right) = -q^{D-2} \left( \frac{\mathcal{A}+2}{2(\mathcal{A}+1)} \right)^{D/2-1} \left( \frac{2\mathcal{A}}{\mathcal{A}+1} \right)^{D/2} \Gamma(D/2-1) \Gamma(2-D/2) \frac{\pi^{D/2}}{\Gamma(D/2)}, \quad (5)$$

$$\tau_{AA}^{-1} \left( -\frac{\mathcal{A}+2}{4\mathcal{A}} q^2 \right) = -q^{D-2} \left( \frac{\mathcal{A}+2}{4\mathcal{A}} \right)^{D/2-1} \Gamma(D/2-1) \Gamma(2-D/2) \frac{\pi^{D/2}}{\Gamma(D/2)}, \quad (6)$$

where  $\Gamma(z)$  is the gamma function, defined for all complex numbers  $z$  except for the nonpositive integers. This restricts the validity of our results to the interval  $2 < D < 4$ . Their solutions are homogeneous functions, that is, the amplitudes  $\chi_A(q)$  and  $\chi_B(q)$  are given by

$$\chi_A(q) = C_A q^{r+is} \quad \text{and} \quad \chi_B(q) = C_B q^{r+is}, \quad (7)$$

where  $r$  and  $s$  are real numbers—this is the second point noted by Danilov. These solutions are the well-known log-periodic functions, associated with the infinitely many three-body bound states in the Efimov limit. Using Eq. (7) in Eqs. (1) and (2) leads to a complex homogeneous linear matrix equation for the coefficients  $C_A$  and  $C_B$ . The parameters  $r$  and  $s$  are found by solving the corresponding characteristic equation. The real part of the exponent is given by the ansatz  $r = 1 - D$  for all  $D$ , which removes any ultraviolet divergence. For  $D = 3$  and  $\mathcal{A} = 1$  one obtains  $s = 1.006\,23$ , a value that agrees with the well-known result from Efimov [1]. Moreover, for  $r = 1 - D$ , the characteristic equation reads

$$\mathcal{F}_D \left[ \mathcal{A} I_1(\mathcal{A}, s) + 2 \left( \frac{4\mathcal{A}}{\mathcal{A}+2} \right)^{D/2-1} \mathcal{F}_D I_2(\mathcal{A}, s) I_3(\mathcal{A}, s) \right] = \left( \frac{\mathcal{A}+2}{2(\mathcal{A}+1)} \right)^{D/2-1} \left( \frac{2\mathcal{A}}{\mathcal{A}+1} \right)^{D/2}, \quad (8)$$

where

$$\mathcal{F}_D = \frac{1}{\Gamma(D/2-1) \Gamma(2-D/2)}, \quad (9)$$

and

$$I_1(\mathcal{A}, s) = \int_0^\infty dz \frac{z^{is}}{z} \ln \left[ \frac{(z^2 + 1)(\mathcal{A} + 1) + 2z}{(z^2 + 1)(\mathcal{A} + 1) - 2z} \right], \quad (10)$$

$$I_2(\mathcal{A}, s) = \int_0^\infty dz \frac{z^{is}}{z} \ln \left[ \frac{2\mathcal{A}(z^2 + z) + \mathcal{A} + 1}{2\mathcal{A}(z^2 - z) + \mathcal{A} + 1} \right], \quad (11)$$

$$I_3(\mathcal{A}, s) = \int_0^\infty dz \frac{z^{is}}{z} \ln \left[ \frac{2\mathcal{A}(1 + z) + (\mathcal{A} + 1)z^2}{2\mathcal{A}(1 - z) + (\mathcal{A} + 1)z^2} \right], \quad (12)$$

which are the same integrals found in Ref. [23] for the  $D = 3$  problem. We note that the result  $r = 1 - D$  is exact. This can be proved by setting  $r = 1 - D + \epsilon$  and expanding the characteristic equation in a power series in  $\epsilon$ : For given values of  $D$  and  $\mathcal{A}$ , it can be verified analytically that the only possible solution occurs for  $\epsilon = 0$ .

**Results.** In cold-atom traps when a three-atom bound state crosses the continuum threshold, the atoms can recombine, forming a deeply bound two-atom molecule plus an atom. The recoil energy of the atom-molecule system is much larger than the depth of the ultracold trap in such a way that the three atoms are lost. The three-atom recombination peaks appear at two-body scattering lengths  $a_{-}^{(N)}$  separated by multiplicative factors of  $\exp(\pi/s)$ . Deviations from the  $D = 3$  limit, excluding range corrections, are associated with the response of the three-body system to the dimension changes between  $D = 2$  and  $D = 3$ . Note that in heteronuclear systems two scattering lengths can be distinguished: one for the  $AA$  subsystem, and another one for the  $AB$  subsystem. The present solution given by Eq. (7) corresponds to the limit of both scattering lengths tending to infinity.

Once  $\mathcal{A}$  is fixed, the imaginary part  $s$  of the exponent of  $q$  in Eq. (7) is the solution of Eq. (8). The boundaries of the region of values of  $D$  for which the Efimov effect survives are determined by the existence of nonzero values of  $s$ ; close to the threshold, the Efimov effect disappears as  $s \rightarrow 0$  and the energy gap between levels tends to infinity. The boundaries are shown in Fig. 1.

In experiments, it is possible to change the confining potential in order to squeeze one or two directions of the trap transforming the cloud in a quasi- $(D = 2)$  or quasi- $(D = 1)$  environment, respectively. Rigorously, as mentioned, all these

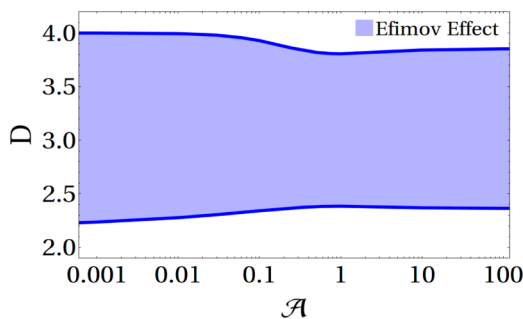


FIG. 1. Regions (in blue) where there is a real solution for the scaling factor  $s$ , solution to Eq. (8); outside this “dimensional band,” the Efimov effect does not exist. For  $\mathcal{A} = 1$  we reproduce exactly the result in Ref. [7], where the dimensional limits are given by  $2.3 < D < 3.8$ .

systems are in  $D = 3$ ; however, the three-body system embedded in the atomic cloud feels an effective dimension when compressed—as shown in previous works [24,25]—that makes the most excited Efimov states disappear one by one until reaching the expected number of bound states in  $D = 2$ .

The physical reason behind the disappearance of the Efimov states close to the critical dimension can be easily understood considering the Born-Oppenheimer (BO) approximation, valid in the situation  $m_A \gg m_B$ . In the BO approximation, an effective potential coming from the exchange of the light particle between the two heavy ones can be extracted. The form of this potential is well known in  $D = 3$ , given by Ref. [26],  $-(s + 1/4)/R^2$ , where  $R$  is the separation distance between the heavy particles, and  $s$  is the imaginary part of the exponent of  $q$  in Eq. (7). The Efimov effect is due to the “fall to the center” for  $s > -1/4$ . For heteronuclear systems in  $D$  dimensions the effective potential is still proportional to  $-1/R^2$ , but the strength is now more complicated, depending on  $D$  and  $\mathcal{A}$ . For a given mass ratio, at the critical dimensions on either side of  $D = 3$ , i.e.,  $D > 3$  and  $D < 3$ , the Efimov effect disappears precisely at the critical strength  $-(D - 2)^2/4$  [27], reproducing the result for  $D = 3$  [26], where the fall to the center stops.

Figure 2 shows the value of the discrete scaling factor  $\exp(\pi/s)$  for a wide range of the mass ratio  $\mathcal{A}$  and of the band

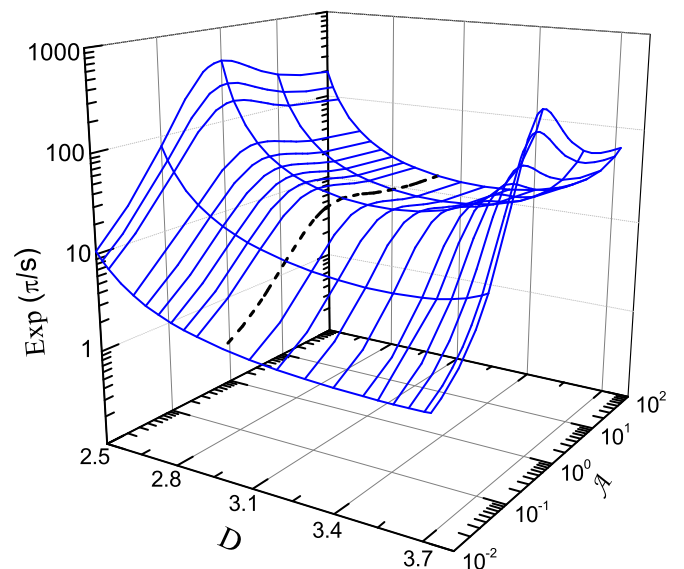


FIG. 2. Discrete scaling factor as a function of the mass ratio  $\mathcal{A} = m_B/m_A$ , and dimension  $D$ . The black dashed line shows the well-known situation of  $D = 3$ .

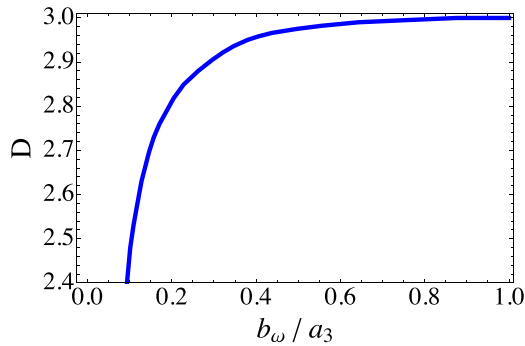


FIG. 3. Effective dimension  $D$  for  $\mathcal{A} = 6/133$  as a function of  $b_\omega/a_3$ , where  $a_3$  is the  $AB$  scattering length for  $D = 3$ .

of  $D$  values that includes the only allowed integer dimension ( $D = 3$ ) for which the Efimov effect exists. The black dashed line indicates the well-known result for  $D = 3$  [28]. The most symmetrical case, where  $\mathcal{A} = 1$ , presents the worst situation to observe consecutive Efimov excited states as for any  $D$  the scaling factor presents a maximum for this mass ratio.

*Experimental possibilities.* A connection between our calculations and finite-energy situations can be made as follows. In a system confined by a squeezed harmonic trap with an oscillator length in the squeezed direction given by  $b_\omega$ , the two- and three-body energies, respectively,  $E_2^{AB}$  and  $E_3$ , can be calculated by solving the Faddeev equations in momentum space with a compactified dimension as detailed in Ref. [29]. From Fig. 2 of Ref. [29], one can extract  $s = s(b_\omega)$  that gives a relationship between  $b_\omega$  and  $D$  through Eq. (8). This relationship should be adequate for an experimental assessment of our prediction. To illustrate the principle, we consider the realistic case of a  ${}^6\text{Li}$ - ${}^{133}\text{Cs}$  mixture, for which  $\mathcal{A} = 6/133$ . The results for  $D$  are shown in Fig. 3 as a function of the ratio  $b_\omega/a_3$ , where  $a_3$  is the  $AB$  scattering length for  $D = 3$ , for the situation in which  $E_2^{AB}$  is kept fixed to its value for  $D = 3$  (dashed curves in Fig. 2 of Ref. [29]).

It is important to note that, for finite two- and three-body energies, when  $b_\omega$  is much larger than the size of the three-body system, all states experience the same effective dimension. Therefore, the discrete scaling factor can be used to infer the effective dimension during the squeezing process along the region where these trimer states exist. When  $b_\omega$  becomes of the order of the size of the state, this is no longer possible. In this situation, the three-body bound states turn into a virtual or resonant state, depending on whether the two-body state is bound or virtual, respectively. Likewise, the features of our

predictions for the scaling factor as a function of  $D$  and  $\mathcal{A}$  shown in Fig. 2 are not expected to be washed out away from the unitary limit and should be tested experimentally by measuring the ratio of the scattering lengths at the peaks of the atomic loss rate, associated with the appearance of three-body bound states.

Experimentally, it is not possible to reach the exact unitary limit, i.e., exact zero two-body binding energy. The discrete scaling symmetry away from the unitary limit has been well established for  $D = 3$  in homonuclear systems. For finite scattering lengths, deviations from the exact unitary symmetry have been observed in the  ${}^{133}\text{Cs}$ - ${}^{133}\text{Cs}$ - ${}^6\text{Li}$  system for  $a_{BB} > 0$  and  $a_{AB} < 0$  [13]. In the same way, deviations from our predictions are also expected for noninteger dimensions. In an experiment it is also not possible to access very highly excited states. However, the measurement of consecutive three-body bound states—revealed by the appearance of peaks in the atomic loss rate—can be observed by choosing systems with sufficiently large mass asymmetry, as in this case the gap between consecutive energy levels is reduced. Corrections due to the effect of finite two-body binding energy affects considerably only the scale factor extracted from the ground state, as shown in Ref. [30]. If higher excited states are considered to calculate  $s$ , the scale factor will be practically given by the values predicted in this Rapid Communication.

*Conclusion.* We have studied the dimensional limits for the occurrence of the Efimov effect for a general mass ratio  $\mathcal{A} = m_B/m_A$ . Our calculation is performed in the unitary limit and reproduces the well-known result for  $\mathcal{A} = 1$ , where the limit is given by  $2.3 < D < 3.8$  [7], and generalizes it for a wide range of  $\mathcal{A}$  and  $D$ . We also predict the numerical values of the discrete scaling factor  $\exp(\pi/s)$  as a function of  $\mathcal{A}$  and  $D$ , which can be tested in experiments with currently available technology.

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