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BASIC HYPERGEOMETRIC FUNCTIONS AND ORTHOGONAL LAURENT POLYNOMIALS

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ABSTRACT. A three-complex-parameter class of orthogonal Laurent polynomials on the unit circle associated with basic hypergeometric or q-hypergeometric functions is considered. To be precise, we consider the orthogonality properties of the sequence of polynomials $\{{}_2\Phi_1(q^{-n}, q^{b+1}; q^{-c+b-n}; q, q^{-c+d-1}z)\}_{n=0}^{\infty}$, where $0 < q < 1$ and the complex parameters b, c and d are such that $b \neq -1, -2, \ldots$, $c-b+1 \neq -1, -2, \ldots$, $\mathcal{R}e(d) > 0$ and $\mathcal{R}e(c-d+2) > 0$. Explicit expressions for the recurrence coefficients, moments, orthogonality and also asymptotic properties are given. By a special choice of the parameters, results regarding a class of Szegő polynomials are also derived.

1. INTRODUCTION

Given the double sequence $\{\mu_n\}_{n=-\infty}^{\infty}$ of complex numbers, let the linear functional $\mathcal{M}$ on the space of Laurent polynomials be defined by

$$\mathcal{M}[z^{-n}] = \mu_n, \quad n = 0, \pm 1, \pm 2, \ldots \, . \tag{1.1}$$

The functional $\mathcal{M}$ can be referred to as a moment functional.

Let D_n, $n = 0, 1, \ldots$, be the associated Toeplitz determinants as given by:

$$D_{-1} = 1, \quad D_0 = \mu_0 \quad \text{and} \quad D_n = \begin{vmatrix} \mu_0 & \mu_{-1} & \cdots & \mu_{-n} \\ \mu_1 & \mu_0 & \cdots & \mu_{-n+1} \\ \vdots & \vdots & & \vdots \\ \mu_n & \mu_{n-1} & \cdots & \mu_0 \end{vmatrix}, \quad n \geq 1.$$

We consider the sequence of polynomials $\{Q_n\}$ that satisfies

$$\mathcal{M}[z^{-s}Q_n(z)] = \rho_n \, \delta_{n,s}, \quad 0 \leq s \leq n, \quad n \geq 1,$$

where Q_n, for any $n \geq 0$, is a monic polynomial of degree n. If the moment functional $\mathcal{M}$ is such that $D_n \neq 0$, $n \geq 0$, then we will refer to it as a semi-definite

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moment functional. In this case it is easily seen that the sequence of polynomials $\{Q_n\}$ exists uniquely and that

$$Q_n(z) = \frac{1}{D_{n-1}} \begin{vmatrix} \mu_0 & \mu_{-1} & \cdots & \mu_{-n} \\ \mu_1 & \mu_0 & \cdots & \mu_{-n+1} \\ \vdots & \vdots & & \vdots \\ \mu_{n-1} & \mu_{n-2} & \cdots & \mu_{-1} \\ 1 & z & \cdots & z^n \end{vmatrix} \quad \text{and} \quad \rho_n = \mathcal{M}[z^{-n}Q_n(z)] = \frac{D_n}{D_{n-1}}.$$

There have been different nomenclatures used with respect to such polynomials in recent years. The polynomials Q_n are related to *orthogonal Laurent polynomials* considered by, for example, Hendriksen and van Rossum [11] and Jones and Thron [15], in the sense that the Laurent polynomials

$$B_{2n}(z) = z^{-n}Q_{2n}(z), \quad B_{2n+1}(z) = z^{-n}Q_{2n+1}(z), \quad n \geq 0,$$

satisfy the orthogonality relations $\mathcal{M}[B_n(z)B_m(z)] = \delta_{n,m}\tilde{\rho}_n,\ \ n, m = 0, 1, 2, \ldots$.

With the monic polynomials $\{\hat{Q}_n\}$ given by

$$\hat{Q}_n(z) = \frac{1}{D_{n-1}} \begin{vmatrix} \mu_0 & \mu_1 & \cdots & \mu_n \\ \mu_{-1} & \mu_0 & \cdots & \mu_{n-1} \\ \vdots & \vdots & & \vdots \\ \mu_{-n+1} & \mu_{-n+2} & \cdots & \mu_1 \\ 1 & z & \cdots & z^n \end{vmatrix}, \quad n \geq 1,$$

we obtain the biorthogonality relations $\mathcal{M}[\hat{Q}_m(1/z)Q_n(z)] = \delta_{n,m}\rho_n,\quad n, m = 0, 1, 2, \ldots$. Hence, Zhedanov [32] calls such polynomials Laurent biorthogonal.

With respect to the moment functional $\mathcal{L}[z^n] = \mathcal{M}[z^{-n}] = \mu_n$, $n = 0, \pm1, \pm2, \ldots$, the reciprocal polynomials $Q_n^{\bullet}(z) = z^nQ_n(1/z)$ satisfy the orthogonality relations $\mathcal{L}[z^{-n+s}Q_n^{\bullet}(z)] = \delta_{n,s}\rho_n,\ \ 0 \leq s \leq n$. Polynomials satisfying such orthogonality relations have been referred to as L-orthogonal polynomials in some earlier contributions, including [1], of one of the present authors. We remark that Zhedanov [32] uses the definition $\mathcal{L}[z^n] = \mu_n$, $n = 0, \pm1, \pm2, \ldots$, for his moment functional and $\mathcal{L}[\hat{Q}_m(z)Q_n(1/z)] = \delta_{n,m}\rho_n,\ \ n, m = 0, 1, 2, \ldots$.

In a recent manuscript [17], $\{Q_n\}$ has been called a sequence of monic *Szegő type* polynomials when $\mathcal{M}$ is such that $D_n \neq 0$ and $\mu_{-n} = \mu_n$ for $n \geq 0$. In this case the Zhedanov [32] biorthogonality can be written as $\mathcal{M}[Q_m(1/z)Q_n(z)] = \delta_{n,m}\rho_n$, $n, m = 0, 1, 2, \ldots$.

However, if $\mathcal{M}$ is such that $D_n > 0$ and $\mu_{-n} = \overline{\mu}_n$, $n \geq 0$, then this moment functional is known as a positive definite moment functional and the sequence of polynomials $\{S_n\} = \{Q_n\}$ are known as monic *Szegő* polynomials. Now we must have $\mathcal{M}[f] = \int_{\mathcal{C}} f(z)d\mu(z)$, where $\mu(z) = \mu(e^{i\theta})$ is a positive measure on the unit circle $\mathcal{C} = \{z = e^{i\theta} : 0 \leq \theta \leq 2\pi\}$. Since the integration is along the unit circle, $\int_{\mathcal{C}} z^{-j}S_n(z)d\mu(z) = \int_{\mathcal{C}} \bar{z}^jS_n(z)d\mu(z)$ and the associated sequence of monic Szegő polynomials $\{S_n\}$ are usually defined by

$$\int_{\mathcal{C}} \overline{S_m(z)}\, S_n(z)d\mu(z) = \int_0^{2\pi} \overline{S_m(e^{i\theta})}\, S_n(e^{i\theta})d\mu(e^{i\theta}) = \kappa_n^{-2}\delta_{nj}, \quad m, n = 0, 1, 2, \ldots,$$

where $\kappa_n^{-2} = \|S_n\|^2 = \int_{\mathcal{C}} |S_n(z)|^2 d\mu(z)$.

With his publications [28] and [29], Szegő introduced these *orthogonal polynomials on the unit circle* in the early 20th century. Many interesting results on these polynomials can be found in his classical book [30], the first edition of which was published in 1939. Since then, these polynomials which bear the name of Szegő were extensively studied by many. We cite, for example, [5], [6], [7], [10], [19], [20], [22], [25] and [27] as some of the very recent contributions. The recent publications of the two excellent volumes [23] and [24] by Simon have given a boost to the interest in studying these polynomials. We also cite the recent book [13] by Ismail containing a nice chapter on these orthogonal polynomials on the unit circle.

Some information on the Szegő polynomials with respect to the measure $d\mu(e^{i\theta}) = [e^{-\theta}]^{\eta}[\sin^2(\theta/2)]^{\lambda}d\theta$, defined for $\eta, \lambda \in \mathbb{R}$ and $\lambda > -1/2$, are provided in [27]. It was shown that these Szegő polynomials are constant multiples of the hypergeometric polynomials ${}_2F_1(-n, b+1; b+\bar{b}+1; 1-z)$, $n \geq 1$, where $\eta = \mathcal{I}m(b)$ and $\lambda = \mathcal{R}e(b)$.

Results used in [27] have an important root in the paper [11] of Hendriksen and van Rossum, where these authors look at T-fractions and orthogonal Laurent polynomials originating from three-term contiguous relations satisfied by the hypergeometric functions ${}_2F_1(a, b; c; z)$.

In this paper, using a three-term contiguous relation satisfied by q-hypergeometric functions ${}_2\Phi_1(q^a, q^b; q^c; q, z)$, we obtain information on the three-parameter class of orthogonal Laurent polynomials $\{z^{-\lfloor n/2 \rfloor} Q_n^{(b,c,d)}(z)\}_{n=0}^{\infty}$ on the unit circle $\mathcal{C} = \{z = e^{i\theta} : 0 < \theta < 2\pi\}$, where the monic polynomials $Q_n^{(b,c,d)}$, $n \geq 0$, are given by

$$Q_n^{(b,c,d)}(z) = \frac{(q^{c-b+1}; q)_n}{(q^{b+1}; q)_n} q^{n(b-d+1)} {}_2\Phi_1(q^{-n}, q^{b+1}; q^{-c+b-n}; q, q^{-c+d-1}z),$$

with $0 < q < 1$ and the three complex parameters b, c and d are such that $b \neq -1, -2, \ldots,\ \ c-b+1 \neq -1, -2, \ldots$, $\mathcal{R}e(d) > 0$ and $\mathcal{R}e(c+2-d) > 0$. The orthogonality is with respect to the semi-definite moment functional $\mathcal{M}^{(b,c,d)}$ given by

$$\mathcal{M}^{(b,c,d)}[f(z)] = \frac{\tau^{(b,c)}}{2\pi} \int_0^{2\pi} f(e^{i\theta}) \frac{(q^{-b+d}e^{i\theta}; q)_\infty (q^{b-d+1}e^{-i\theta}; q)_\infty}{(q^d e^{i\theta}; q)_\infty (q^{c+2-d}e^{-i\theta}; q)_\infty} d\theta.$$

Here the constant $\tau^{(b,c)}$, defined in Theorem 3.5, is such that $\mathcal{M}^{(b,c,d)}[1] = 1$. By considering separately the real and imaginary parts of b, c and d, and neglecting $\mathcal{I}m(d)$, which induces only a rotation, we can also consider $\{z^{-\lfloor n/2 \rfloor} Q_n^{(b,c,d)}(z)\}_{n=0}^{\infty}$ as a five-real-parameter class of orthogonal Laurent polynomials.

The class of polynomials considered here is somewhat different and broader than the class of orthogonal Laurent polynomials $\{z^{-\lfloor n/2 \rfloor} P_n(-z, \alpha, \beta)\}_{n=0}^{\infty}$ that follows from Pastro [21], where

$$P_n(z, \alpha, \beta) = {}_2\Phi_1(\tilde{q}^{-n}, \tilde{q}^{\alpha}; \tilde{q}^{2-\beta-n}; \tilde{q}, \tilde{q}^{-\beta+3/2}z),$$

with $|\tilde{q}| < 1$, $\alpha > 1/2$ and $\beta > 1/2$. Pastro shows that the polynomials $P_n(-z, \alpha, \beta)$ are the Laurent biorthogonal polynomials with respect to the semi-definite moment functional given by

$$\tilde{\mathcal{M}}^{(\alpha,\beta)}[f(z)] = \int_0^{2\pi} f(e^{i\theta}) \frac{(\tilde{q}^{1/2}e^{i\theta}; \tilde{q})_\infty (\tilde{q}^{1/2}e^{-i\theta}; \tilde{q})_\infty}{(\tilde{q}^{\alpha-1/2}e^{i\theta}; \tilde{q})_\infty (\tilde{q}^{\beta-1/2}e^{-i\theta}; \tilde{q})_\infty} d\theta.$$

These Pastro polynomials can be considered as belonging to a class with the three real parameters α, β and, say, ϑ if one assumes the $\tilde{q}$ to be such that $\tilde{q} = |\tilde{q}|e^{i\vartheta}$. An explicit expression for the moments $\tilde{\mathcal{M}}^{(\alpha,\beta)}[z^{-n}]$ is found in Vinet and Zhedanov [31].

Note that the moment functional $\tilde{\mathcal{M}}^{(\alpha,\beta)}$ can only be made positive definite with the choice $-1 < \tilde{q} < 1$ and $\alpha = \beta > 1/2$. Thus with this choice, Pastro [21] recovers the class of real Szegő polynomials previously described by Askey [4, p. 806].

By a special choice of the parameters b, c and d we also obtain in the present manuscript information regarding the class of (complex and real) Szegő polynomials $S_n^{(\lambda,\eta,\phi)}$ characterized by the reflection coefficients

$$a_n^{(\lambda,\eta,\phi)} = \frac{(q^{\lambda+i\eta};q)_n}{(q^{\lambda+1-i\eta};q)_n}\, q^{n[\frac{1}{2}-i(\eta+\phi)]}, \quad n \geq 1,$$

where $\lambda, \eta, \phi \in \mathbb{R}$ and $\lambda > -1/2$. The parameter ϕ, which comes from $\mathcal{I}m(d)$, induces only a rotation and can be made equal to zero without any loss of generality.

The polynomials obtained by taking $\eta = \phi = 0$ and $\lambda > -1/2$, for example, coincide with the real Szegő polynomials of [21] and [4], obtained when $0 < \tilde{q} < 1$ and $\alpha = \beta > 1/2$.

The paper is organized as follows. In Section 2 we present some fundamental results on three-term recurrence relations, continued fractions and basic hypergeometric functions, which we will be using in later sections. In Section 3 we define the monic q-hypergeometric polynomials $Q_n^{(b,c,d)}(z)$ and obtain their orthogonality and asymptotic properties. In Section 4, in addition to discussing when the polynomials $Q_n^{(b,c,d)}(z)$ coincide with the Szegő polynomials $S_n^{(\lambda,\eta,\phi)}$ mentioned above, we also obtain explicitly the associated Szegő function.

2. Some preliminary results

Let $\{Q_n\}$ be the sequence of polynomials given by the three-term recurrence relation

$$Q_{n+1}(z) = \big(z+\beta_{n+1}\big)Q_n(z) - \alpha_{n+1} z Q_{n-1}(z), \quad n \geq 1, \tag{2.1}$$

with $Q_0(z) = 1$ and $Q_1(z) = z + \beta_1$.

Lemma 2.1. *Let $\beta_1 \neq 0$ and $\alpha_{n+1} \neq 0$ for $n \geq 1$. Given any sequence $\{h_n\}$ of arbitrary complex numbers h_n (or complex functions $h_n(z)$), let the sequence of functions $\{G_n(h_n;z)\}$ be such that* $G_1(h_1;z) = \dfrac{\beta_1}{z+\beta_1-h_1}$ *and*

$$G_n(h_n;z) = \frac{\beta_1}{z+\beta_1} - \frac{\alpha_2 z}{z+\beta_2} - \cdots - \frac{\alpha_n z}{z+\beta_n-h_n}, \quad n \geq 2.$$

Then

$$G_n(h_n;z) - G_n(0;z) = \frac{\beta_1\alpha_2\alpha_3\cdots\alpha_n h_n z^{n-1}}{Q_n(z)[Q_n(z)-h_nQ_{n-1}(z)]}\,.$$

Proof. Let the sequence of polynomials $\{R_n\}$ be such that

$$R_{n+1}(z) = \big(z+\beta_{n+1}\big)R_n(z) - \alpha_{n+1} z R_{n-1}(z), \quad n \geq 1,$$

with $R_0(z) = 0$, $R_1(z) = \beta_1$. Then from basic results on continued fractions (see, for example, [14, 18])

$$G_n(h_n;z) - G_n(0;z) = \frac{R_n(z)-h_nR_{n-1}(z)}{Q_n(z)-h_nQ_{n-1}(z)} - \frac{R_n(z)}{Q_n(z)}, \quad n \geq 1.$$

Hence,

$$G_n(h_n;z) - G_n(0;z) = \frac{h_n[R_n(z)Q_{n-1}(z) - Q_n(z)R_{n-1}(z)]}{Q_n(z)[Q_n(z) - h_nQ_{n-1}(z)]}, \quad n \geq 1.$$

Therefore, the lemma follows from $R_n(z)Q_{n-1}(z) - Q_n(z)R_{n-1}(z) = \beta_1\alpha_2\alpha_3 \cdots \alpha_n z^{n-1}$. □

As a particular case of this lemma, if one takes $h_n = \alpha_{n+1}z/(z+\beta_{n+1})$, then

$$G_{n+1}(0;z) - G_n(0;z) = \frac{\beta_1\alpha_2\alpha_3\cdots\alpha_{n+1}}{Q_n(z)Q_{n+1}(z)}\, z^n, \quad n \geq 1. \tag{2.2}$$

Lemma 2.2. *In the three-term recurrence relation* (2.1), *if*

$$\beta_n \neq 0 \quad \text{and} \quad \alpha_{n+1} \neq 0, \quad n \geq 1,$$

then there exists a semi-definite moment functional $\mathcal{M}$ such that the polynomials Q_n satisfy

$$\mathcal{M}[z^{-s}Q_n(z)] = \delta_{n,s}\rho_n, \quad 0 \leq s \leq n, \quad n \geq 1,$$

where $\rho_n = \dfrac{\alpha_2\cdots\alpha_{n+1}}{\beta_2\cdots\beta_{n+1}}$. *Moreover, the associated moments* $\mu_n = \mathcal{M}[z^{-n}]$, $n = 0, \pm1, \pm2, \ldots$ *are such that* $L_0(z) = \sum_{j=0}^{\infty}\mu_j z^j$, $L_\infty(z) = -\sum_{j=1}^{\infty}\mu_{-j}z^{-j}$, *where*

$$\begin{aligned} L_0(z) - G_n(0;z) &= \rho_n\frac{1}{Q_n(0)}z^n + O(z^{n+1}), \\ L_\infty(z) - G_n(0;z) &= \rho_n Q_{n+1}(0)\frac{1}{z^{n+1}} + O(\frac{1}{z^{n+2}}). \end{aligned} \tag{2.3}$$

Proof. First note that $Q_n(0) = \beta_1\beta_2\cdots\beta_n \neq 0$. Now from (2.2) by considering the expansions of $G_n(0;z)$ about the origin and infinity there exist power series $L_0(z) = \sum_{j=0}^{\infty}\mu_j z^j$ and $L_\infty(z) = -\sum_{j=1}^{\infty}\mu_{-j}z^{-j}$ such that (2.3) holds.

With respect to these power series coefficients, if we define the moment functional $\mathcal{M}$ by (1.1), then the lemma follows from the linear system on the coefficients of Q_n and R_n obtained from (2.3). □

For $a, b, c \in \mathbb{C}$, $c \neq 0, -1, -2, \ldots$ and $0 < |q| < 1$, the ${}_2\Phi_1$ q-hypergeometric or the ${}_2\Phi_1$ basic hypergeometric function (hypergeometric function with base q) may be defined by

$${}_2\Phi_1(q^a, q^b; q^c; q, z) = \sum_{n=0}^{\infty}\frac{(q^a;q)_n\,(q^b;q)_n}{(q^c;q)_n\,(q;q)_n}\, z^n,$$

for $|z| < 1$ and by analytic continuation for other values of $z \in \mathbb{C}$. Here, $(q^a;q)_0 = 1$ and $(q^a;q)_n = (1-q^a)(1-q^{a+1})\cdots(1-q^{a+n-1})$, $n \geq 1$.

For more information regarding q-hypergeometric functions, we refer to, for example, Andrews, Askey and Roy [2], Gasper and Rahman [8], Koekoek and Swarttouw [16] and Slater [26].

Two "distinct" q-hypergeometric functions ${}_2\Phi_1(q^{a_1}, q^{a_2}; q^{a_3}; q, z)$ and ${}_2\Phi_1(q^{\tilde{a}_1}, q^{\tilde{a}_2}; q^{\tilde{a}_3}; q, z)$ may be called contiguous if $|a_i - \tilde{a}_i| = 0$ or 1 for at least one $i \in \{1, 2, 3\}$. There are interesting relations between contiguous q-hypergeometric functions called contiguous relations.

Lemma 2.3. *If $c \neq 0, -1, -2, \dots$, then*

$$ {}_2\Phi_1(q^a, q^{b+1}; q^c; q, z) = \big(1 + \frac{1-q^{a-b}}{1-q^c}\, q^b z\big)\; {}_2\Phi_1(q^{a+1}, q^{b+1}; q^{c+1}; q, z) - \frac{(1-q^{a+1})\,(1-q^{c-b})}{(1-q^c)\,(1-q^{c+1})}\, q^b z\, {}_2\Phi_1(q^{a+2}, q^{b+1}; q^{c+2}; q, z). $$

Proof. From contiguous relations obtained by Heine (see [8, p. 22]), we consider the following:

$$ {}_2\Phi_1(q^{a+1}, q^{b+1}; q^{c+1}; q, z) = {}_2\Phi_1(q^{a+1}, q^b; q^c; q, z) + \frac{(1-q^{a+1})\,(1-q^{c-b})}{(1-q^c)\,(1-q^{c+1})}\, q^b z\; {}_2\Phi_1(q^{a+2}, q^{b+1}; q^{c+2}; q, z), $$

$$ {}_2\Phi_1(q^{a+1}, q^b; q^c; q, z) = {}_2\Phi_1(q^a, q^{b+1}; q^c; q, z) - \frac{(1-q^{a-b})}{(1-q^c)}\, q^b z\; {}_2\Phi_1(q^{a+1}, q^{b+1}; q^{c+1}; q, z), $$

which hold for $c \neq 0, -1, -2, \dots$. Substitution for ${}_2\Phi_1(q^{a+1}, q^b; q^c; q, z)$ in the first relation using the other gives the required result. □

We will be using the q-binomial theorem (see [16, Eq. (0.5.2)])

$$ {}_2\Phi_1(q^a, q^c; q^c; q, z) = {}_1\Phi_0(q^a; q, z) = \frac{(q^a z; q)_\infty}{(z; q)_\infty}, \tag{2.4} $$

which holds for $c \neq 0, -1, -2, \dots$ and $|z| < 1$, and the Heine transformation formula (see [16, Eq. (0.6.3)])

$$ {}_2\Phi_1(q^a, q^b; q^c; q, z) = \frac{(q^{a+b-c} z; q)_\infty}{(z; q)_\infty}\, {}_2\Phi_1(q^{-a+c}, q^{-b+c}; q^c; q, q^{a+b-c} z), \tag{2.5} $$

which holds for $c \neq 0, -1, -2, \dots$ and $|z| < \min\{1, |q^{c-a-b}|\}$. We will also be needing the polynomial identities (see [16, Eq. (0.6.19)])

$$ \begin{aligned} &{}_2\Phi_1(q^{-n}, q^b; q^c; q, z) \\ &= \frac{(q^b; q)_n}{(q^c; q)_n} q^{-n(n+1)/2} (-z)^n\, {}_2\Phi_1(q^{-n}, q^{-c-n+1}; q^{-b-n+1}; q, q^{c-b+n+1} z^{-1}), \end{aligned} \tag{2.6} $$

for $n \geq 0$, which hold when $c \neq 0, -1, -2, \dots$ and $b \neq -n+1, -n+2, -n+3, \dots$.

3. q-ORTHOGONAL LAURENT POLYNOMIALS

From now on we restrict the value of q to be such that $0 < q < 1$. Then for any $b \in \mathbb{C}$ we have

$$ \overline{q^b} = q^{\bar{b}} \quad \text{and} \quad |q^b| = q^{\mathcal{R}e(b)}. $$

With $b, c, d \in \mathbb{C}$ and $c - b + 1 \neq 0, -1, -2, \dots$, let

$$ F_n^{(b,c,d)}(z) = \frac{{}_2\Phi_1(q^{n+1}, q^{-b}; q^{c-b+n+2}; q, q^d z)}{{}_2\Phi_1(q^n, q^{-b}; q^{c-b+n+1}; q, q^d z)}, \quad n \geq 0. $$

Then from Lemma 2.3,

$$ F_n^{(b,c,d)}(z) = \frac{1}{1 + g_{n+1}^{(b,c,d)} z - f_{n+2}^{(b,c,d)} z F_{n+1}^{(b,c,d)}(z)}, \quad n \geq 0, $$

where

$$g_n^{(b,c,d)} = \frac{1-q^{b+n}}{1-q^{c-b+n}}\, q^{-b+d-1}, \quad f_{n+1}^{(b,c,d)} = \frac{(1-q^n)\,(1-q^{c+n+1})}{(1-q^{c-b+n})\,(1-q^{c-b+n+1})}\, q^{-b+d-1},$$

for $n \geq 1$. This leads to the continued fraction expansion

$$F_0^{(b,c,d)}(z) = \frac{1}{1+g_1^{(b,c,d)}z} - \frac{f_2^{(b,c,d)}z}{1+g_2^{(b,c,d)}z} - \cdots - \frac{f_n^{(b,c,d)}z}{1+g_n^{(b,c,d)}z - f_{n+1}^{(b,c,d)}zF_n^{(b,c,d)}(z)}. \tag{3.1}$$

Also assuming $b \neq -1, -2, \ldots$, this can be written in the equivalent form

$$F_0^{(b,c,d)}(z) = \frac{\beta_1^{(b,c,d)}}{z+\beta_1^{(b,c,d)}} - \frac{\alpha_2^{(b,c,d)}z}{z+\beta_2^{(b,c,d)}} - \cdots - \frac{\alpha_n^{(b,c,d)}z}{z+\beta_n^{(b,c,d)}} - \frac{\alpha_{n+1}^{(b,c,d)}z\,F_n^{(b,c,d)}(z)}{\beta_{n+1}^{(b,c,d)}}, \tag{3.2}$$

where $\beta_n^{(b,c,d)} = 1/g_n^{(b,c,d)}$ and $\alpha_{n+1}^{(b,c,d)} = f_{n+1}^{(b,c,d)}/(g_n^{(b,c,d)}g_{n+1}^{(b,c,d)})$, $n \geq 1$.

Theorem 3.1. *With $b \neq -1, -2, \ldots$ and $c-b+1 \neq -1, -2, \ldots$, let the sequence of monic polynomials $\{Q_n^{(b,c,d)}\}$ be given by*

$$Q_{n+1}^{(b,c,d)}(z) = (z+\beta_{n+1}^{(b,c,d)})Q_n^{(b,c,d)}(z) - \alpha_{n+1}^{(b,c,d)}\,z\,Q_{n-1}^{(b,c,d)}(z), \quad n \geq 1, \tag{3.3}$$

with $Q_0^{(b,c,d)}(z) = 1$ and $Q_1^{(b,c,d)}(z) = z + \beta_1^{(b,c,d)}$, where

$$\beta_n^{(b,c,d)} = \frac{1-q^{c-b+n}}{1-q^{b+n}}\, q^{b-d+1}, \quad \alpha_{n+1}^{(b,c,d)} = \frac{(1-q^n)\,(1-q^{c+n+1})}{(1-q^{b+n})\,(1-q^{b+n+1})}\, q^{b-d+1}, \quad n \geq 1.$$

Then the polynomials $Q_n^{(b,c,d)}$ satisfy the orthogonality relations

$$\mathcal{M}^{(b,c,d)}[z^{-s}Q_n^{(b,c,d)}(z)] = \delta_{n,s}\rho_n^{(b,c)}, \quad 0 \leq s \leq n, \quad n \geq 1, \tag{3.4}$$

with respect to the semi-definite moment functional

$$\mathcal{M}^{(b,c,d)}[z^{-j}] = \frac{(q^{-b};q)_j}{(q^{c-b+2};q)_j}\, q^{jd}, \quad j = 0, \pm 1, \pm 2, \ldots .$$

Here,

$$\rho_n^{(b,c)} = \frac{\alpha_2^{(b,c,d)} \cdots \alpha_{n+1}^{(b,c,d)}}{\beta_2^{(b,c,d)} \cdots \beta_{n+1}^{(b,c,d)}} = \frac{(q;q)_n\,(q^{c+2};q)_n}{(q^{b+1};q)_n\,(q^{c-b+2};q)_n}.$$

Proof. We first prove the theorem for $c-b+1 \neq 0, -1, -2, \ldots$ and $b \neq -1, -2, \ldots$. With these restrictions $\beta_n^{(b,c,d)} \neq 0$ and $\alpha_{n+1}^{(b,c,d)} \neq 0$, $n \geq 1$, and hence from Lemma 2.2 there exists a semi-definite moment functional such that (3.4) holds.

To obtain the values of $\mu_j^{(b,c,d)} = \mathcal{M}^{(b,c,d)}[z^{-j}]$, $j = 0, \pm 1, \pm 2, \ldots$, let us consider the functions

$$G_n^{(b,c,d)}(z) = \frac{\beta_1^{(b,c,d)}}{z+\beta_1^{(b,c,d)}} - \frac{\alpha_2^{(b,c,d)}z}{z+\beta_2^{(b,c,d)}} - \cdots - \frac{\alpha_n^{(b,c,d)}z}{z+\beta_n^{(b,c,d)}}, \quad n \geq 1.$$

Then from Lemma 2.1 and the continued fraction expansion (3.2),

$$\begin{aligned} &F_0^{(b,c,d)}(z) - G_n^{(b,c,d)}(z) \\ &\qquad = \frac{\beta_1^{(b,c,d)}\alpha_2^{(b,c,d)}\cdots\alpha_n^{(b,c,d)}\alpha_{n+1}^{(b,c,d)} z^n F_n^{(b,c,d)}(z)}{Q_n^{(b,c,d)}(z)\big[\beta_{n+1}^{(b,c,d)}Q_n^{(b,c,d)}(z) - \alpha_{n+1}^{(b,c,d)}\, z\, F_n^{(b,c,d)}(z)\, Q_{n-1}^{(b,c,d)}(z)\big]} \\ &\qquad = \rho_n^{(b,c)}\frac{1}{Q_n^{(b,c,d)}(0)} z^n + O(z^{n+1}), \qquad \text{for} \quad n \geq 1. \end{aligned}$$

Since $F_0^{(b,c,d)}(z) = {}_2\Phi_1(q, q^{-b}; q^{c-b+2}; q, q^d z)$, from the latter part of Lemma 2.2,

$$\mu_j^{(b,c,d)} = \frac{(q^{-b};q)_j}{(q^{c-b+2};q)_j}\, q^{jd}, \quad j = 0, 1, 2, \ldots,$$

thus giving the results for the positive moments.

From (3.1), by realizing that $g_n^{(c-b,c,c+2-d)} = \beta_n^{(b,c,d)}$ and $f_{n+1}^{(c-b,c,c+2-d)} = \alpha_{n+1}^{(b,c,d)}$, $n \geq 1$, we also obtain the continued fraction expansion

$$\begin{aligned} &\frac{\beta_1^{(b,c,d)}}{z} F_0^{(c-b,c,c+2-d)}(z^{-1}) = \\ &\frac{\beta_1^{(b,c,d)}}{z + \beta_1^{(b,c,d)}} - \frac{\alpha_2^{(b,c,d)} z}{z + \beta_2^{(b,c,d)}} - \cdots - \frac{\alpha_n^{(b,c,d)} z}{z + \beta_n^{(b,c,d)}} - \frac{\alpha_{n+1}^{(b,c,d)} F_n^{(c-b,c,c+2-d)}(z^{-1})}{1}. \end{aligned}$$

Hence, again from Lemma 2.1,

$$\begin{aligned} &\frac{\beta_1^{(b,c,d)}}{z} F_0^{(c-b,c,c+2-d)}(z^{-1}) - G_n^{(b,c,d)}(z) \\ &\qquad = \frac{\beta_1^{(b,c,d)}\alpha_2^{(b,c,d)}\cdots\alpha_n^{(b,c,d)}\alpha_{n+1}^{(b,c,d)} z^{n-1} F_n^{(c-b,c,c+2-d)}(z^{-1})}{Q_n^{(b,c,d)}(z)\big[Q_n^{(b,c,d)}(z) - \alpha_{n+1}^{(b,c,d)} F_n^{(c-b,c,c+2-d)}(z^{-1}) Q_{n-1}^{(b,c,d)}(z)\,\big]} \\ &\qquad = \rho_n^{(b,c)} Q_{n+1}^{(b,c,d)}(0)\frac{1}{z^{n+1}} + O(\frac{1}{z^{n+2}}), \qquad \text{for} \quad n \geq 1. \end{aligned}$$

Since $F_0^{(c-b,c,c+2-d)}(z^{-1}) = {}_2\Phi_1(q, q^{-c+b}; q^{b+2}; q, q^{c+2-d} z^{-1})$, from the latter part of Lemma 2.2,

$$\mu_{-j}^{(b,c,d)} = \frac{(q^{-c+b-1};q)_j}{(q^{b+1};q)_j}\, q^{j(c+2-d)}, \quad j = 1, 2, 3, \ldots .$$

Thus, using $(a,q)_n = (a;q)_\infty/(aq^n;q)_\infty$, for $n = 0, \pm1, \pm2, \ldots$, we also obtain the results for the negative moments. This concludes the theorem for $c - b + 1 \neq 0, -1, -2, \ldots$ and $b \neq -1, -2, \ldots$.

Now to extend the results for $c - b + 1 \neq -1, -2, \ldots$ and $b \neq -1, -2, \ldots$, we need to prove the theorem for $c - b + 1 = 0$ and $b \neq -1, -2, \ldots$.

If $b \neq -1, -2, \ldots$, then $\beta_1^{(b,b-1,d)} = 0$ and $\beta_{n+1}^{(b,b-1,d)} = \alpha_{n+1}^{(b,b-1,d)} \neq 0$ for $n \geq 1$. Hence, $Q_n^{(b,b-1,d)}(z) = z^n$, $n \geq 0$ and

$$\mathcal{M}^{(b,b-1,d)}[z^{-s}Q_n^{(b,b-1,d)}(z)] = \mathcal{M}^{(b,b-1,d)}[z^{n-s}] = \frac{(q^{-b};q)_{-n+s}}{(q;q)_{-n+s}}\, q^{(-n+s)d}.$$

Since

$$\frac{(q^{-b};q)_{-n+s}}{(q;q)_{-n+s}}\, q^{(-n+s)d} = 0 \ \text{ if } \ s < n \quad \text{and} \quad \rho_n^{(b,b-1)} = \frac{(q^{-b};q)_0}{(q;q)_0}\, q^{(0)d} = 1,$$

the validity of the theorem when $c-b+1=0$ and $b \neq -1,-2,\ldots$ is confirmed. This concludes the theorem. $\square$

The same explicit expression for the moments, when the moment functional is considered as in Pastro [21], is obtained in [31].

From the three-term recurrence relation (3.3) it follows that

$$Q_n^{(b,c,d)}(0) = \beta_1^{(b,c,d)}\beta_2^{(b,c,d)}\cdots\beta_n^{(b,c,d)} = \frac{(q^{c-b+1};q)_n}{(q^{b+1};q)_n}\, q^{n(b-d+1)}, \quad n \geq 1.$$

Theorem 3.2. *Let $b \neq -1,-2,\ldots$ and $c-b+1 \neq -1,-2,\ldots$. Then*

$$a) \quad \lim_{n\to\infty} \beta_n^{(b,c,d)} = q^{b-d+1}, \qquad \lim_{n\to\infty} \alpha_n^{(b,c,d)} = q^{b-d+1},$$

$$b) \quad \lim_{n\to\infty} q^{-n(b-d+1)} Q_n^{(b,c,d)}(0) = (1-q)^{-c+2b}\frac{\Gamma_q(b+1)}{\Gamma_q(c-b+1)},$$

$$c) \quad \lim_{n\to\infty} \rho_n^{(b,c)} = \frac{\Gamma_q(b+1)\,\Gamma_q(c-b+2)}{\Gamma_q(c+2)}.$$

Proof. Part $a)$ of this theorem is clear. To obtain parts $b)$ and $c)$ we use the definition

$$\Gamma_q(x) = \frac{(q;q)_\infty}{(q^x;q)_\infty}(1-q)^{1-x}$$

of the q-gamma function. $\square$

Theorem 3.3. *Let $b \neq -1,-2,\ldots$ and $c-b+1 \neq -1,-2,\ldots$. Then the monic polynomials $Q_n^{(b,c,d)}$, $n \geq 0$, given by the recurrence relation* (3.3) *have the explicit representation*

$$(3.5) \quad Q_n^{(b,c,d)}(z) = \frac{(q^{c-b+1};q)_n}{(q^{b+1};q)_n}\, q^{n(b-d+1)}\, {}_2\Phi_1(q^{-n}, q^{b+1}; q^{-c+b-n}; q, q^{-c+d-1}z).$$

Proof. From (3.3) it is easily verified that the reciprocal (or inverse) polynomials

$$Q_n^{*(b,c,d)}(z) = z^n\overline{Q_n^{(b,c,d)}(1/\bar{z})} \quad \text{and} \quad Q_n^{\bullet(b,c,d)}(z) = z^n Q_n^{(b,c,d)}(1/z), \quad n \geq 0,$$

satisfy the three-term recurrence relations

$$(3.6) \quad \begin{aligned} Q_{n+1}^{*(b,c,d)}(z) &= (1+\beta_{n+1}^{(\bar{b},\bar{c},\bar{d})}z)Q_n^{*(b,c,d)}(z) - \alpha_{n+1}^{(\bar{b},\bar{c},\bar{d})}\, z\, Q_{n-1}^{*(b,c,d)}(z), \\ Q_{n+1}^{\bullet(b,c,d)}(z) &= (1+\beta_{n+1}^{(b,c,d)}z)Q_n^{\bullet(b,c,d)}(z) - \alpha_{n+1}^{(b,c,d)}\, z\, Q_{n-1}^{\bullet(b,c,d)}(z), \end{aligned} \quad n \geq 1,$$

with $Q_0^{*(b,c,d)}(z) = Q_0^{\bullet(b,c,d)}(z) = 1$, $Q_1^{*(b,c,d)}(z) = 1 + \beta_1^{(\bar{b},\bar{c},\bar{d})}z$ and $Q_1^{\bullet(b,c,d)}(z) = 1 + \beta_1^{(b,c,d)}z$. This means that

$$\begin{aligned} Q_n^{*(b,c,d)}(z) &= {}_2\Phi_1(q^{-n}, q^{\bar{c}-\bar{b}+1}; q^{-\bar{b}-n}; q, q^{-\bar{d}+1}z), \\ Q_n^{\bullet(b,c,d)}(z) &= {}_2\Phi_1(q^{-n}, q^{c-b+1}; q^{-b-n}; q, q^{-d+1}z), \end{aligned} \quad n \geq 1,$$

which we can easily verify from Lemma 2.3. Hence, application of the transformation (2.6) in $Q_n^{\bullet(b,c,d)}$, for example, gives the required results of the theorem. $\square$

Note that by the application of the transform (2.5) in $Q_n^{*(b,c,d)}$, for example, we can also write that

$$\frac{(q^{-\bar{d}+1}z;q)_\infty}{(q^{\bar{c}-\bar{d}+2}z;q)_\infty}\,Q_n^{*(b,c,d)}(z) = {}_2\Phi_1(q^{-\bar{b}}, q^{-\bar{c}-n-1}; q^{-\bar{b}-n}; q, q^{\bar{c}-\bar{d}+2}z), \quad n \geq 1,$$

provided that $|z| < q^{-\mathcal{R}e(c-d+2)}$. This can also be directly verified from Lemma 2.3 and (3.6).

Theorem 3.4. *Let* $b \neq -1, -2, \dots, \;\; c-b+1 \neq -1,-2,\dots$ *and*

$$\sigma = \min\{q^{-\mathcal{R}e(c-d+2)}, q^{-\mathcal{R}e(b-d+1)}\}.$$

Then uniformly on compact subsets of $|z| < \sigma$,

$$\lim_{n\to\infty} Q_n^{*(b,c,d)}(z) = \frac{(q^{\bar{c}-\bar{d}+2}z;q)_\infty}{(q^{\bar{b}-\bar{d}+1}z;q)_\infty}.$$

Proof. Since

$$\lim_{n\to\infty} \frac{(q^{-\bar{c}-n-1};q)_j}{(q^{-\bar{b}-n};q)_j} q^{(\bar{c}-\bar{d}+2)j} = q^{(\bar{b}-\bar{d}+1)j},$$

using Lebesgue's dominated convergence theorem and then (2.4), we obtain

$$\lim_{n\to\infty} {}_2\Phi_1(q^{-\bar{b}}, q^{-\bar{c}-n-1}; q^{-\bar{b}-n}; q, q^{\bar{c}-\bar{d}+2}z) = {}_1\Phi_0(q^{-\bar{b}}; q, q^{\bar{b}-\bar{d}+1}z) = \frac{(q^{-\bar{d}+1}z;q)_\infty}{(q^{\bar{b}-\bar{d}+1}z;q)_\infty},$$

uniformly on compact subsets of $|z| < \sigma$. Thus, the result of the theorem follows. □

Theorem 3.5. *In addition to* $b \neq -1,-2,\dots$ *and* $c-b+1 \neq -1,-2,\dots$, *if one also assumes that*

$$\mathcal{R}e(c+2) > \mathcal{R}e(d) > 0,$$

then the polynomials $Q_n^{(b,c,d)}$, $n \geq 0$, *given by* (3.5), *satisfy the orthogonality relations*

$$\frac{\tau^{(b,c)}}{2\pi i}\int_{\mathcal{C}} z^{-s} Q_n^{(b,c,d)}(z)\,\frac{(q^{-b+d}z;q)_\infty (q^{b-d+1}/z;q)_\infty}{(q^d z;q)_\infty (q^{c-d+2}/z;q)_\infty}\,\frac{1}{z}\,dz = \delta_{n,s}\rho_n^{(b,c)}, \quad 0 \leq s \leq n.$$

Here, $\rho_n^{(b,c)}$ *are as in Theorem* 3.1 *and*

$$\tau^{(b,c)} = \frac{(q;q)_\infty (q^{c+2};q)_\infty}{(q^{c-b+2};q)_\infty (q^{b+1};q)_\infty}.$$

Proof. Let us consider the following identity of Ramanujan:

$$\sum_{-\infty}^{\infty} \frac{(\alpha;q)_n}{(\beta;q)_n} x^n = \frac{(q;q)_\infty\,(\frac{\beta}{\alpha};q)_\infty\,(\alpha x;q)_\infty\,(\frac{q}{\alpha x};q)_\infty}{(\beta;q)_\infty\,(\frac{q}{\alpha};q)_\infty\,(\frac{\beta}{\alpha x};q)_\infty\,(x;q)_\infty},$$

which holds for $|\beta\alpha^{-1}| < |x| < 1$. Simple proofs of this identity can be found in [3] and [12].

In our case, since $0 < q < 1$, with the assumptions of the theorem if we take $x = q^d z$, $\alpha = q^{-b}$ and $\beta = q^{c-b+2}$, then

$$\sum_{-\infty}^{\infty} \frac{(q^{-b};q)_n}{(q^{c-b+2};q)_n} q^{nd} z^n = \tau^{(b,c)} \frac{(q^{-b+d}z;q)_\infty (q^{b-d+1}/z;q)_\infty}{(q^d z;q)_\infty (q^{c+2-d}/z;q)_\infty}, \tag{3.7}$$

which holds for $|q^{c+2-d}| < |z| < |q^{-d}|$, where $|q^{c+2-d}| < 1$ and $|q^{-d}| > 1$.

Hence, multiplying by z^{-j-1} and integrating along the unit circle we obtain from Laurent's theorem

$$\frac{(q^{-b};q)_j}{(q^{c-b+2};q)_j}q^{jd}=\frac{\tau^{(b,c)}}{2\pi i}\int_{\mathcal{C}} z^{-j-1}\frac{(q^{-b+d}z;q)_\infty(q^{b-d+1}/z;q)_\infty}{(q^d z;q)_\infty(q^{c+2-d}/z;q)_\infty}dz, \quad j=0,\pm1,\pm2,\ldots.$$

Thus, the moment functional in Theorem 3.1 satisfies

$$\mathcal{M}^{(b,c,d)}[z^{-j}]=\frac{\tau^{(b,c)}}{2\pi i}\int_{\mathcal{C}} z^{-j-1}\frac{(q^{-b+d}z;q)_\infty(q^{b-d+1}/z;q)_\infty}{(q^d z;q)_\infty(q^{c+2-d}/z;q)_\infty}\,dz, \tag{3.8}$$

for $j=0,\pm1,\pm2,\ldots$, which completes the proof of the theorem. □

As a particular case, letting $b=0$ and $c+2\neq 0,-1,-2,\ldots$ we have $\beta_1^{(0,c,d)}=\frac{1-q^{c+1}}{1-q}\,q^{-d+1}$ and

$$\beta_{n+1}^{(0,c,d)}=\alpha_{n+1}^{(0,c,d)}=\frac{1-q^{c+n+1}}{1-q^{n+1}}\,q^{-d+1}, \quad n\geq 1.$$

Moreover, $\mu_0^{(0,c,d)}=1$,

$$\mu_j^{(0,c,d)}=0 \quad\text{and}\quad \mu_{-j}^{(0,c,d)}=\frac{(q^{-c-1};q)_j}{(q;q)_j}\,q^{j(c+2-d)}, \quad j=1,2,\ldots\ .$$

Furthermore, the following corollary holds.

Corollary 3.5.1. *If $\mathcal{R}e(c+2)>\mathcal{R}e(d)>0$, then the sequence of polynomials $\{\mathcal{Q}_n^{(0,c,d)}\}$ given by*

$$\mathcal{Q}_n^{(0,c,d)}(z)=\frac{(q^{c+1};q)_n}{(q;q)_n}\,q^{n(-d+1)}\,{}_2\Phi_1(q^{-n},q;q^{-c-n};q,q^{-c+d-1}z), \quad n\geq 1,$$

apart from satisfying the three-term recurrence relation

$$\mathcal{Q}_{n+1}^{(0,c,d)}(z)=(z+\frac{1-q^{c+n+1}}{1-q^{n+1}}q^{-d+1})\mathcal{Q}_n^{(0,c,d)}(z)-\frac{1-q^{c+n+1}}{1-q^{n+1}}q^{-d+1}\,z\,\mathcal{Q}_{n-1}^{(0,c,d)}(z),$$

for $n\geq 1$, with $\mathcal{Q}_0^{(0,c,d)}(z)=1$ and $\mathcal{Q}_1^{(0,c,d)}(z)=z+\frac{1-q^{c+1}}{1-q}\,q^{-d+1}$, satisfies the orthogonality relations

$$\frac{1}{2\pi i}\int_{\mathcal{C}} z^{-s}\,\mathcal{Q}_n^{(0,c,d)}(z)\,\frac{(q^{-d+1}/z;q)_\infty}{(q^{c+2-d}/z;q)_\infty}\,\frac{1}{z}\,dz=\delta_{n,s}, \quad 0\leq s\leq n.$$

Moreover, uniformly on compact subsets of $|z|<\min\{q^{-\mathcal{R}e(c-d+2)},q^{-\mathcal{R}e(-d+1)}\}$,

$$\lim_{n\to\infty}\mathcal{Q}_n^{*(0,c,d)}(z)=\frac{(q^{\bar{c}-\bar{d}+2}z;q)_\infty}{(q^{-\bar{d}+1}z;q)_\infty}\,.$$

As another particular case, letting $c=b$ and $b+1\neq 0,-1,-2,\ldots$, we have

$$\beta_n^{(b,b,d)}=\alpha_{n+1}^{(b,b,d)}=\frac{1-q^n}{1-q^{b+n}}\,q^{b-d+1}, \quad n\geq 1.$$

Moreover,

$$\mu_j^{(b,b,d)}=\frac{(q^{-b};q)_j}{(q^2;q)_j}\,q^{jd}, \quad j=0,\pm1,\pm2,\ldots\ .$$

Furthermore, the following corollary can be stated.

Corollary 3.5.2. *If* $b+1 \neq 0$ *and* $\mathcal{R}e(b+2) > \mathcal{R}e(d) > 0$*, then the sequence of polynomials* $\{\mathcal{Q}_n^{(b,b,d)}\}$ *given by*

$$\mathcal{Q}_n^{(b,b,d)}(z) = \frac{(q;q)_n}{(q^{b+1};q)_n}\, q^{n(b-d+1)} \sum_{j=0}^{n} \frac{(q^{b+1};q)_j}{(q;q)_j} q^{-j(b-d+1)} z^j, \quad n \geq 1,$$

satisfies the orthogonality relations

$$\frac{1}{2\pi i}\frac{(1-q)}{(1-q^{b+1})} \int_{\mathcal{C}} z^{-s} \mathcal{Q}_n^{(b,b,d)}(z)\, \frac{(q^{-b+d}z;q)_\infty}{(q^d z;q)_\infty}\, \frac{z-q^{b-d+1}}{z^2}\, dz = \delta_{n,s}, \quad 0 \leq s \leq n.$$

Moreover, uniformly on compact subsets of $|z| < q^{-\mathcal{R}e(b-d+1)}$,

$$\lim_{n\to\infty} \mathcal{Q}_n^{*(b,b,d)}(z) = \frac{1}{(1-q^{\bar{b}-\bar{d}+1}z)}\,.$$

4. q-Szegő polynomials

From (3.8) the moment functional $\mathcal{M}^{(b,c,d)}$ is easily seen to be positive definite if $b \neq -1,-2,\ldots$, $c-b+1 \neq -1,-2,\ldots$, $\mathcal{R}e(c+2) > \mathcal{R}e(d) > 0$, $\overline{-b+d} = b-d+1$ and $\overline{d} = c+2-d$. That is, with the restrictions

$$c = b+\bar{b}-1, \quad d+\bar{d} = b+\bar{b}+1 \quad \text{and} \quad \mathcal{R}e(b) > -1/2,$$

the moment functional $\mathcal{M}^{(b,c,d)}$ is positive definite, and hence the polynomials $Q_n^{(b,c,d)}$ are the associated Szegő polynomials.

Hence, setting

$$b = \lambda - i\eta, \quad c = 2\lambda - 1 \quad \text{and} \quad d = \frac{1}{2}+\lambda+i\phi,$$

if $\lambda > -1/2$, our special case of Ramanujan identity (3.7) becomes

$$\sum_{-\infty}^{\infty} \frac{(q^{-\lambda+i\eta};q)_n}{(q^{\lambda+1+i\eta};q)_n} q^{n(\frac{1}{2}+\lambda+i\phi)} z^n = \tilde{\tau}^{(\lambda,\eta)}\, \frac{(q^{\frac{1}{2}+i(\eta+\phi)}z;q)_\infty\, (q^{\frac{1}{2}-i(\eta+\phi)}/z;q)_\infty}{(q^{\frac{1}{2}+\lambda+i\phi}z;q)_\infty\, (q^{\frac{1}{2}+\lambda-i\phi}/z;q)_\infty}\,,$$

which holds for $q^{\lambda+1/2} < |z| < q^{-\lambda-1/2}$, where

$$\tilde{\tau}^{(\lambda,\eta)} = \frac{(q;q)_\infty\, (q^{2\lambda+1};q)_\infty}{(q^{1+\lambda+i\eta};q)_\infty\, (q^{1+\lambda-i\eta};q)_\infty}.$$

This means that we can write

$$\begin{aligned}\mathcal{M}^{(\lambda-i\eta,\, 2\lambda-1,\, \lambda+i\phi+1/2)}[z^{-j}] &= \frac{(q^{-\lambda+i\eta};q)_j}{(q^{\lambda+1+i\eta};q)_j}\, q^{j(\frac{1}{2}+\lambda+i\phi)} \\ &= \int_{\mathcal{C}} z^{-j} d\mu^{(\lambda,\eta,\phi)}(z) = \int_0^{2\pi} e^{-ij\theta} \omega^{(\lambda,\eta,\phi)}(\theta)\, d\theta,\end{aligned}$$

for $j = 0, \pm 1, \pm 2, \ldots$, where $\omega^{(\lambda,\eta,\phi)}(\theta)d\theta = d\mu^{(\lambda,\eta,\phi)}(e^{i\theta})$, with

$$\frac{d\mu^{(\lambda,\eta,\phi)}(z)}{dz} = \frac{\tilde{\tau}^{(\lambda,\eta)}}{2\pi i}\, \frac{1}{z}\, \frac{(q^{\frac{1}{2}+i(\eta+\phi)}z;q)_\infty\, (q^{\frac{1}{2}-i(\eta+\phi)}/z;q)_\infty}{(q^{\frac{1}{2}+\lambda+i\phi}z;q)_\infty\, (q^{\frac{1}{2}+\lambda-i\phi}/z;q)_\infty}$$

and

$$\omega^{(\lambda,\eta,\phi)}(\theta) = \frac{\tilde{\tau}^{(\lambda,\eta)}}{2\pi}\, \frac{(q^{\frac{1}{2}+i(\eta+\phi)}e^{i\theta};q)_\infty\, (q^{\frac{1}{2}-i(\eta+\phi)}e^{-i\theta};q)_\infty}{(q^{\frac{1}{2}+\lambda+i\phi}e^{i\theta};q)_\infty\, (q^{\frac{1}{2}+\lambda-i\phi}e^{-i\theta};q)_\infty}. \tag{4.1}$$

As expected, $\omega^{(\lambda,\eta,\phi)}(\theta)$ is a positive weight function in $[0, 2\pi]$.

Adopting the notation $S_n^{(\lambda,\eta,\phi)}(z) = Q_n^{(\lambda-i\eta,\,2\lambda-1,\,\frac{1}{2}+\lambda+i\phi)}(z)$ we can write the following:

$$S_{n+1}^{(\lambda,\eta,\phi)}(z) = \Big(z + \frac{1-q^{\lambda+i\eta+n}}{1-q^{\lambda-i\eta+n+1}}\, q^{\frac{1}{2}-i(\eta+\phi)}\Big) S_n^{(\lambda,\eta,\phi)}(z) - \frac{(1-q^n)(1-q^{2\lambda+n})}{(1-q^{\lambda-i\eta+n})(1-q^{\lambda-i\eta+n+1})}\, q^{\frac{1}{2}-i(\eta+\phi)}\, z\, S_{n-1}^{(\lambda,\eta,\phi)}(z), \quad n \geq 1,$$

with $S_0^{(\lambda,\eta,\phi)}(z) = 1$ and $S_1^{(\lambda,\eta,\phi)}(z) = \Big(z + \frac{1-q^{\lambda+i\eta}}{1-q^{\lambda-i\eta+1}}\, q^{\frac{1}{2}-i(\eta+\phi)}\Big)$. Moreover,

$$S_n^{(\lambda,\eta,\phi)}(0) = \frac{(q^{\lambda+i\eta};q)_n}{(q^{1+\lambda-i\eta};q)_n}\, q^{n[\frac{1}{2}-i(\eta+\phi)]}, \quad n \geq 1. \tag{4.2}$$

Hence, in particular, using Theorems 3.3 and 3.5 we have

Theorem 4.1. *If $\lambda, \eta, \phi \in \mathbb{R}$ and $\lambda > -1/2$, then the polynomials*

$$S_n^{(\lambda,\eta,\phi)}(z) = \frac{(q^{\lambda+i\eta};q)_n\, q^{n[\frac{1}{2}-i(\eta+\phi)]}}{(q^{\lambda+1-i\eta};q)_n}\, {}_2\Phi_1(q^{-n}, q^{\lambda+1-i\eta};\, q^{-\lambda-n+1-i\eta};\, q, q^{\frac{1}{2}-\lambda+i\phi}z)$$

are the monic Szegő polynomials satisfying

$$\int_0^{2\pi} \overline{S_n^{(\lambda,\eta,\phi)}(e^{i\theta})}\, S_m^{(\lambda,\eta,\phi)}(e^{i\theta})\, \omega^{(\phi,\eta,\lambda)}(\theta) d\theta = [\kappa_n^{(\lambda,\eta)}]^{-2} \delta_{n,m}, \quad n, m = 0, 1, 2, \ldots,$$

with respect to the weight function $\omega^{(\lambda,\eta,\phi)}(\theta)$ given by (4.1). Here,

$$[\kappa_n^{(\lambda,\eta)}]^{-2} = \rho_n^{(\lambda-i\eta,2\lambda-1)} = \frac{(q;q)_n\, (q^{2\lambda+1};q)_n}{(q^{\lambda+1+i\eta};q)_n\, (q^{\lambda+1-i\eta};q)_n}.$$

Moreover, these polynomials satisfy the Szegő recurrence relation

$$S_n^{*(\lambda,\eta,\phi)}(z) = \overline{a_n^{(\lambda,\eta,\phi)}}\, z S_{n-1}^{(\lambda,\eta,\phi)}(z) + S_{n-1}^{*(\lambda,\eta,\phi)}(z), \quad n \geq 1,$$

where the reflection (or Verblunsky) coefficients $a_n^{(\lambda,\eta,\phi)} = S_n^{(\lambda,\eta,\phi)}(0)$ are given by (4.2).

Now using Theorem 3.4 we can state the following. Let $\lambda, \eta, \phi \in \mathbb{R},\ \lambda > -1/2$ and $\sigma = \min\{q^{-1/2}, q^{-\lambda-1/2}\}$. Then uniformly on compact subsets of $|z| < \sigma$,

$$\lim_{n\to\infty} S_n^{*(\lambda,\eta,\phi)}(z) = \frac{(q^{\lambda+\frac{1}{2}+i\phi}z;q)_\infty}{(q^{\frac{1}{2}+i(\eta+\phi)}z;q)_\infty}\,. \tag{4.3}$$

Moreover,

$$\sum_{n=1}^{\infty} |a_n^{(\lambda,\eta,\phi)}|^2 = |1-q^{\lambda+i\eta}|^2 \sum_{n=1}^{\infty} \frac{q^n}{|1-q^{n+\lambda+i\eta}|^2} \leq |1-q^{\lambda+i\eta}|^2 \sum_{n=1}^{\infty} \frac{q^n}{(1-q^{n+\lambda})^2} < \infty.$$

This last result means that the Szegő condition

$$\frac{1}{2\pi}\int_0^{2\pi} \log(\omega^{(\lambda,\eta,\phi)}(\theta)) d\theta > -\infty$$

holds and we can now give an expression for the associated Szegő function

$$D^{(\lambda,\eta,\phi)}(z) = \exp\left(\frac{1}{4\pi}\int_0^{2\pi} \frac{e^{i\theta}+z}{e^{i\theta}-z} \log(\omega^{(\lambda,\eta,\phi)}(\theta)) d\theta\right).$$

Theorem 4.2. *If $\lambda, \eta, \phi \in \mathbb{R}$ and $\lambda > -1/2$, then for $|z| < 1$,*

$$D^{(\lambda,\eta,\phi)}(z) = \sqrt{\frac{\Gamma_q(2\lambda+1)}{\Gamma_q(\lambda+1-i\eta)\,\Gamma_q(\lambda+1+i\eta)}}\,\frac{(q^{\frac{1}{2}+i(\eta+\phi)}z;q)_\infty}{(q^{\lambda+\frac{1}{2}+i\phi}z;q)_\infty}\,.$$

Proof. Since, $\kappa_n^{(\lambda,\eta)} S_n^{*(\lambda,\eta,\phi)} \to [D^{(\lambda,\eta,\phi)}(z)]^{-1}$ for $|z| < 1$ (see [23, p. 144]), the result follows from part c) of Theorem 3.2 and from (4.3). □

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